Fractional Programming for Communication Systems—Part I: Power Control and Beamforming

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***Abstract*—Fractional programming (FP) refers to a family of optimization problems that involve ratio term(s). This two-part paper explores the use of FP in the design and optimization of communication systems. Part I of this paper focuses on FP theory and on solving continuous problems. The main theoretical contribution is a novel quadratic transform technique for tackling the multiple-ratio concave–convex FP problem—in contrast to conventional FP techniques that mostly can only deal with the single-ratio or the max-min-ratio case. Multiple-ratio FP problems are important for the optimization of communication networks, because system-level design often involves multiple signal-to-interference-plus-noise ratio terms. This paper considers the applications of FP to solving continuous problems in communication system design, particularly for power control, beamforming, and energy efficiency maximization. These application cases illustrate that the proposed quadratic transform can greatly facilitate the optimization involving ratios by recasting the original nonconvex problem as a sequence of convex problems. This FP-based problem reformulation gives rise to an efficient iterative optimization algorithm with provable convergence to a stationary point. The paper further demonstrates close connections between the proposed FP approach and other well-known algorithms in the literature, such as the fixed-point iteration and the weighted minimum mean-square-error beamforming. The optimization of discrete problems is discussed in Part II of this paper.**

***Index Terms*—Fractional programming (FP), quadratic transform, power control, beamforming, energy efficiency.**

I. Overview

OPTIMIZATION is a key aspect of communication system design [3], [4]. This two-part work explores the application of fractional programming (FP) in the design and optimization of communication systems. FP refers to a family of optimization problems containing ratio term(s). Its history can be traced back to an early paper on economic expansion [5] by von Neumann in 1937; it has since been studied extensively in broad areas in economics, management science, information theory, optics, graph theory, and computer science [6]–[8]. For example, FP has recently been applied in [9]–[12] to solve the energy efficiency maximization problem for wireless communication systems.

The aim of this two-part paper is to extend the use of FP to address a broader range of optimization problems in communication system design, in particular on power control, beamforming, and user scheduling, which often cannot be directly expressed in ratio forms. We focus on communication systems in which the data rate is computed as $\log(1 + \text{SINR})$, where SINR is the signal-to-interference-plus-noise ratio. The prominent role played by "SINR" in communication systems makes FP an invaluable tool for network design and optimization. The discussion throughout the paper focuses on wireless cellular networks, but it can be readily adapted to many other networks (e.g., the optical network or the digital subscriber lines).

Although an extensive literature already exists for FP, most of them specialize in *single-ratio* problems. For example, prior works on communication system design [9]–[12] that rely on classical FP techniques have had to limit the system model to the scenario involving only one single ratio. Although multiple-ratio problems are dealt with in [13], they are limited to specific forms (e.g., the max-min problem). System-level communication network design, however, often has to deal with multiple ratios, because the overall system performance is typically a function of multiple fractional parameters (e.g., SINRs) from multiple interfering links. Solving *multiple-ratio* FP is, however, NP-hard [14]. The state-of-the-art methods for finding the globally optimal solution all require exponential running time (e.g., using branch-and-bound search [15]–[17]). In fact, as pointed out in [16] and [18], the solution to a general FP problem consisting of more than 20 ratio terms is already beyond the reach of known approaches within reasonable time. As to finding a stationary-point solution of the multiple-ratio problem, only general-purpose techniques such as successive convex approximation are known.

This paper addresses the multiple-ratio FP problem from a new perspective. Our main theoretic contribution is a novel technique called *quadratic transform* that introduces some suitable auxiliary variables, then recasts the original problem to a form amenable to iterative optimization. Specifically, this new technique decouples the numerator and the denominator of each ratio term, similar to the classical *Dinkelbach's transform* (but works with multiple ratios as opposed to mostly single ratio for the classic method). This decoupling feature of the proposed quadratic transform is particularly suited for the coordinated resource optimization problem across multiple cells in a wireless cellular network. For instance, the intercell power spectrum optimization problem is a challenging nonconvex problem, because the transmit power levels of the different links strongly

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2. Dinkelbach's Transform

(1) FP problem → $B(x) > 0$

(2) Concave−convex FP problem → $A(x)$: concave, ≥ 0 → maximize problem

$B(x)$: convex > 0

max일 때 x에 대해서도 / 무조건 해가 존재 concave여서

모든 problem에 적용가능하지만 해를 무조건 구하는 건 아니다.

- $\max_{x}$ or $\min_{x}$ $\dfrac{A(x)}{B(x)}$ ⇒ 분수 꼴인 optimize problem을 분리해서 풀겠다

subject to $x \in \mathcal{X}$

라그랑주 dual : boundary

FP

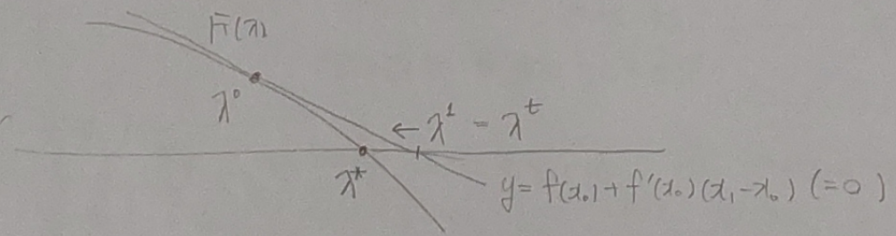
- $\dfrac{A(x)}{B(x)}$가 변수 λ에 관한 식으로 표현 된다. $\dfrac{A(x)}{B(x)} = \lambda^*$ → 특정 λ^*에 대해 $A(x) - B(x)\lambda^* = 0$은 성립한다

$F(\lambda) = A(x) - \lambda B(x)$ ($F(\lambda^*) = A(x) - \lambda^* B(x) = 0$) 인 λ^*를 찾는 게 → Solving FP

- 위 식을 토대로 $\min_{x \in \mathcal{X}} \dfrac{A(x)}{B(x)} = \lambda^*$, $F(\lambda) = \min_{x \in \mathcal{X}} \left[A(x) - \lambda B(x) \right]$ 설정

- $F(\lambda)$는 λ에 대해서 concave function (Concave−Convex FP problem에 대해선 x에 대해서도 concave)

- $F'(\lambda) = -B(x) < 0$ 이므로 감소 함수 +) Newton method → $f(x) = 0$의 근사값 찾기



$$\Rightarrow \lambda_{t+1} = \lambda_t - \frac{F(\lambda_t)}{-B(x_t)} = \lambda_t + \frac{A(x_t)}{B(x_t)} - \lambda_t = \frac{A(x_t)}{B(x_t)}$$

- $F(\lambda)$는 non-increasing이므로 $\lambda^* \leq \lambda_t$

Solve : x_0 초기화 → $\dfrac{A(x_0)}{B(x_0)} = \lambda_0$ 초기화 / $\min_{x \in \mathcal{X}} \left[f(x) - \lambda g(x) \right] = F(\lambda)$ / if $|F(\lambda_k)| < \varepsilon$, stop

$F(\lambda) = 0$이 되는 $A(x) - \lambda B(x)$를 가지고 minimize하면, 그게 곧 $\min \dfrac{A(x)}{B(x)}$가 된다.

- $F(\lambda_k) = 0$ 이 되는 λ_k를 찾아야 한다 (Newton method)

그 $\lambda_k = \lambda^*$를 대입한 $F(\lambda^*)$는 $\frac{A(x)}{B(x)}$가 된다

즉, $F(\lambda^*) = A(x) - \lambda^* B(x) = \frac{A(x)}{B(x)}$

$F(\lambda)=0$을 찾고, $A(x) - \lambda^* B(x)$를 $\underset{max}{min}$ 하는 것이 곧 $\frac{A(x)}{B(x)}$를 $\underset{max}{min}$ 하는 것과 같다

Dinkelbach ① x에 대해서 $A(x) - \lambda B(x) = 0$ 이 되는 λ^*를 찾는다

② λ^*에 대해 $A(x) - \lambda^* B(x) = \frac{A(x)}{B(x)}$를 만족한다

③ $[A(x) - \lambda^* B(x)]$를 $\underset{max}{min}$ 하는 것이 $\frac{A(x)}{B(x)}$를 $\underset{max}{min}$ 하는 것이다

④ 그냥 min, max는 constraint가 $x \in X$ 뿐이다.

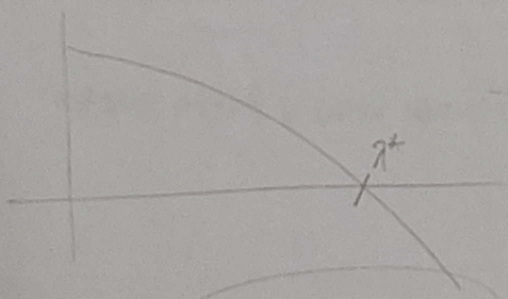
+) FP problem는 고려할 때 $A(x)$의 조건 ≥ 0이 없으므로 Iterative algorithm을

돌리는 데 해가 안 나올 수도 있다 $F(\lambda) = A(x) - \lambda B(x)$ $\lambda = \frac{A(x)}{B(x)}$

? >0

<0

≥ 0



A가 ≥ 0면

$F(\lambda) = 0$인 해가 하나 이상 보장이 된다

반면, Concave-Convex FP problem의 경우 $A(x) \geq 0$ 이므로 closed form이 존재

$\lambda = 0$일 때 $F(\lambda) \geq 0$임을 보장 → λ^*을 무조건 찾을 수 있다.

$F(\lambda)$는 $\frac{A(x)}{B(x)}$와 동일시 할 수 있고,

$F(\lambda)$는 concave function 이므로 (x)

→ x에 대해서도

$\lambda^* = \min_{x} \frac{f(x)}{g(x)}$, $F(\lambda) = \min_{x} f(x) - \lambda g(x)$ / $F(\lambda_k) = 0$인 λ_k를 찾는 것이 FP solve 하는 것

2. Quadratic Transform.

$g(x,y) = 2y\sqrt{A(x)} - y^2 B(x)$ / $\max_{x} \sum_{m=1}^{M} \frac{A_m(x)}{B_m(x)}$

1. y에 대해, $g(x,y)$는 concave function이다 (C_4)

2. $y^* = \arg\max_{y} g(x,y)$; $\underbrace{g(x,y^*) = \frac{A(x)}{B(x)}}$ (C_3) ⇒ multiple ratio 가능 Σ.

   Dinkelbach's Method는 $g(x,y^*) = 0$ 일 때 $g(x,y^*) = \frac{A(x)}{B(x)}$ 이다.

3. x^*는 y^*와 함께 $g(x,y)$를 maximize 하는 경우에만 $\frac{A(x)}{B(x)}$를 최대화 한다. (C_2)

4. 보조 변수 y 사용 (C_1)

◦ Dinkelbach's method는 y_m를 $y_m = \frac{A_m}{B_m}$ 으로 iterative하게 update해야 하는데

Σ를 보장하지 못한다

① nondecreasing function 적용 가능. (ex) log)

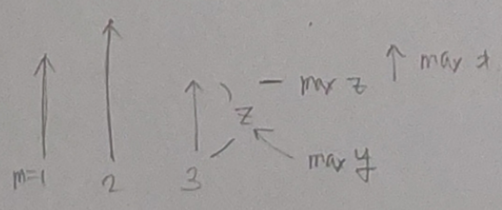
$$\max_{x} \sum_{m=1}^{M} f_m\left(2y_m\sqrt{A_m(x)} - y_m^2 B_m(x)\right)$$

② Max min ratio

$$\max_{x\in X} \min_{m} \left[\frac{A_m(x)}{B_m(x)}\right] \longrightarrow \max_{x,y,z} z$$

s.t. $x \in X$

$$2y_m\sqrt{A_m(x)} - y_m^2 B_m(x) \geq z, \ \forall m$$



$$\max \sum \frac{A(x)}{B(x)} \quad + \max \sum_i \left[2y A(x) - y^2 \sqrt{B(x)} \right]$$

or

①

or

② 이는 objective function은 x에 대해서 concave이므로 maximize problem은 풀 수 있다 (단, concave-convex FP)

* Dinkelbach's transform은 왜 Σ를 풀 수 없는가?

$y[t+1] = \frac{A(x[t])}{B(x[t])}$ 이고 결국 $\underset{\text{Min}}{\text{Max}} \frac{A(x)}{B(x)}$ 를 푸는 것이다 (y를 구하는 행위 자체가)

Iterative하게 y를 구하는 것은 $\underset{\min}{\max} \sum \frac{A}{B}$ 를 구하는 게 아닌,

$\max \frac{A_1}{B_1} + \max \frac{A_2}{B_2} + \cdots + \max \frac{A_m}{B_m}$ 이므로 $\max \sum \frac{A}{B}$와 다르다.

<Closed-Form FP>

$$f(p) = \sum_{i \in B} w_i R_i = \sum w_i \log(1 + \underbrace{\mathrm{SINR}}_{r\,(\text{보조 변수})})$$

$$\frac{|h_{i,i}|^2 p_i = A(x)}{\sum_{j \neq i} |h_{i,j}|^2 p_j + \sigma^2 = B(x)}$$

s.t $r_m \le \dfrac{A_m(x)}{B_m(x)}$

↓

$$f_r^{CF}(p, r) = \sum_{i \in B} w_i \log(1 + r_i) - \sum_{i \in B} w_i r_i + \underbrace{\sum_{i \in B} \frac{w_i (1 + r_i) |h_{i,i}|^2 p_i}{\sum_{j \in B} |h_{i,j}|^2 p_j + \sigma^2}}_{\text{Sum of ratio term}}$$

(1) $r^* = \dfrac{A}{B}$일 때 $f(p) = f_r^{CF}$이다.

(C_3)

(2) r가 고정되어 있을 때 f_r^{CF}의 마지막 항만이

p_i 최적화에 관여한다.

Sum of ratio term

$\dfrac{A}{A+B}$ 꼴

$y = \dfrac{\sqrt{w_i (1 + r_i) |h_{i,i}|^2 p_i}}{\sum |h_{i,j}|^2 p_j + \sigma^2}$ 일 때 둘이 같다.

$$f_q^{CF}(p, r, y) = \sum_{i \in B} 2 y_i \sqrt{w_i (1 + r_i) |h_{i,i}|^2 p_i} - \sum_{i \in B} y_i^2 \left(\sum_{j \in B} |h_{i,j}|^2 p_j + \sigma^2 \right)$$

$$+ \underbrace{\mathrm{const}(r)}_{r\text{가 고정됐을 때의 상수항}}$$

(1) p와 y에 대해서 f_q^{CF}를 iterative하게 maximize 하기 위해

Closed-form update를 한다

$$p_i^* = \min \left\{ p_{max}, \frac{y_i^2 w_i (1 + r_i) |h_{i,i}|^2}{\left\{ \sum_{j \in B} y_j^2 |h_{j,i}|^2 \right\}^2} \right\}$$

Algorithm (1) p 초기화 & r^* 구하기 → 이제 p 최적화 가능

repeat { (2) y update

(3) r update

(4) p update }

(5) f_q^{CF} 수렴 → Stop