Correction examen 2021 L2S3

Exo 1

1) $S = \sum_{n \geq 0} \frac{1}{3^n}$ On reconnait une série géométrique de raison $1/3$. $-1 < 1/3 < 1 \Rightarrow S$ CV.

$$S = \frac{1}{1-q} = \frac{1}{1-1/3} = \frac{3}{2}$$

2) $S = \sum_{n \geq 0} \frac{2^n}{n!} = e^2$. On reconnait une $\sum$ expo qui CV vers e^2.

3) $S = \sum_{n \geq 0} \frac{n^n}{n!}$. D'Alembert $\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)}{(n+1)} \left(\frac{n+1}{n}\right)^n$

$$\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \xrightarrow[\text{cours}]{n \to +\infty} e > 1 \text{ donc } S \text{ DV.}$$

4) $S = \sum_{n \geq 0} \left(\frac{n}{2n+1}\right)^n$ Cauchy $(u_n)^{1/n} = \frac{n}{2n+1} \xrightarrow[n \to +\infty]{} \frac{1}{2} < 1$

$\forall n \geq 0$, $u_n \geq 0$ et S CV.

5. $S = \sum_{n \geq 1} \frac{n}{n^2 - 1}$. $u_n > 0$, $\forall n \geq 1$. $u_n \underset{n \to +\infty}{\sim} \frac{n}{n^2} = \frac{1}{n}$

étant donné que $\sum_{n \geq 1} \frac{1}{n}$ DV, $\sum u_n$ DV aussi.

Exo 2

$$I_1 = \int_{-\frac{1}{2}}^{0} (2x+1)^3 \, dx = \left[\frac{(2x+1)^4}{4 \times 2} \right]_{-\frac{1}{2}}^{0} = \frac{1}{8}.$$

$$I_2 = \int_0^1 \underbrace{(4x-4)}_{u'} \underbrace{e^{x^2 - 2x}}_{e^u} \, dx = \left[2e^{x^2 - 2x} \right]_0^1 = 2e^{-1} - 2.$$

$u = x^2 - 2x$

$u' = 2x - 2$

Exo 3

1) a, b ? tels que $\dfrac{1}{x(x+1)} = \dfrac{a}{x} + \dfrac{b}{x+1}$

$\dfrac{a}{x} + \dfrac{b}{x+1} = \dfrac{ax+a+bx}{x(x+1)} = \dfrac{(a+b)x+a}{x(x+1)}$. On identifie et $\begin{cases} a+b = 0 \\ a = 1 \end{cases}$

$a = 1 \quad b = -1$. $\qquad \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$

2) $I = \displaystyle\int_1^e \frac{1}{x(x+1)}\,dx = \int_1^e \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \Big[\ln|x| - \ln|x+1|\Big]_1^e$

$I = \ln e - \ln e+1 - \ln 1 + \ln 2 = 1 + \ln\left(\dfrac{2}{e+1}\right)$.

3) $I_2 = \int_1^e \frac{\ln x}{(x+1)^2}\,dx$ IPP $u' = \frac{1}{(x+1)^2}$ $v = \ln x$

$u = -\frac{1}{x+1}$ $v' = \frac{1}{x}$

$$I_2 = \left[- \frac{\ln x}{x+1} \right]_1^e - \int_1^e \frac{-1}{x(x+1)}\,dx$$

$$= \frac{-1}{e+1} + I = -\frac{1}{e+1} + 1 + \ln\left(\frac{2}{e+1}\right)$$

Exo 4

$$f(x,y) = x^3 + x^2 - xy + y^2 + 4$$

1) $\dfrac{\partial f}{\partial x} = 3x^2 + 2x - y$ $\dfrac{\partial f}{\partial y} = -x + 2y$

2) $A(-0,99 ; 0,1)$ Soit $B(-1 ; 0)$

$A(-1 + 0,01 ; 0 + 0,1)$

$$f(-0,99 ; 0,1) \simeq f(-1;0) + 0,01 \frac{\partial f}{\partial x}(-1;0) + 0,1 \frac{\partial f}{\partial y}(-1;0)$$

$$\simeq 4 + 0,01 \times 1 + 0,1 \times 1 = 4,11.$$

3) Les points critiques sont solutions de

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 2x - y = 0 \\ -x + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 2x - \frac{1}{2}x = 0 \\ y = \frac{1}{2}x \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{1}{2}x \\ 3x^2 + \frac{3}{2}x = 0 \end{cases} \Leftrightarrow \begin{cases} y = \frac{1}{2}x \\ x\left(x + \frac{1}{2}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ ou } \begin{cases} x = -\frac{1}{2} \\ y = -\frac{1}{4} \end{cases}$$

4) $\dfrac{\partial^2 f}{\partial x^2} = 6x + 2$ $\qquad \dfrac{\partial^2 f}{\partial x \partial y} = -1 = \dfrac{\partial^2 f}{\partial y \partial x}$ $\qquad \dfrac{\partial^2 f}{\partial y^2} = 2$

Schwarz

5) En $O(0,0)$: min local

$r = 2 > 0$

$s = -1$ $\qquad \delta = 3 > 0$

$t = 2$

En $A(-1/2 ; -1/4)$ pt col

$r = -1$

$s = -1$ $\qquad \delta = -3 < 0$

$t = 2$

Exo 5

$f(x,y) = x e^{y}$ $\qquad 2x + y + 4 = 0$

$\iff y = -2x - 4$

On substitue : soit $g(x) = f(x; -2x-4) = x e^{-2x-4}$.

$g'(x) = e^{-2x-4} + x(-2) e^{-2x-4} = (1-2x) e^{-2x-4}$

$g'(x)$ du signe de $1-2x$.

x	$-\infty$		$1/2$		$+\infty$
$1-2x$		$+$	0	$-$	
$g'(x)$		$+$	0	$-$	
g		$\nearrow$		$\searrow$	

$g\left(\frac{1}{2}\right) = \frac{1}{2} e^{-5}$

Sous la contrainte $2x+y+4=0$

f admet un max local

en $x = 1/2$ $\quad y = -5$

valant $\frac{1}{2} e^{-5}$.