L2S4 S8 (01/04/2023)

45

$$A = \begin{pmatrix} 2a+3 & -2(a+1) \\ a+1 & -a \end{pmatrix}$$

1) $P(\lambda) = \begin{vmatrix} 2a+3-\lambda & -2(a+1) \\ a+1 & -a-\lambda \end{vmatrix} = (-a-\lambda)(2a+3-\lambda) + 2(a+1)^2$

$= \lambda^2 - \lambda\left[(2a+3)-a\right] - a(2a+3) + 2(a+1)^2$

$= \lambda^2 - \lambda\left[a+3\right] - 3a + 4a + 2 = \lambda^2 - \lambda(a+3) + a + 2$

Polynôme 2^{nd} degré en λ $\quad \Delta = (a+3)^2 - 4(a+2) = a^2 + 2a + 1 = (a+1)^2$

• Si $a+1=0 \iff a=-1$; 1 seule vp $\lambda_0 = \frac{a+3}{2}$

Alors $A(-1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

Q 3

• Si $a \neq -1$ 2 vp $\lambda_1 = \frac{a+3-a-1}{2} = 1$ $\quad \lambda_2 = \frac{a+3+a+1}{2} = a+2$.

2) a. Si $a \neq -1$, A a 2 vp $\neq$ et est diagonalisable. Alors $a+2 \neq 1$

2. b) $\underline{a \neq -1}$ vp 1, $a+2$

SEP pour $\lambda = 1$. On résout $(A - I_2)X = 0 \iff \begin{bmatrix} 2a+2 & -2(a+1) \\ a+1 & -a-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} (2a+2)x - 2(a+1)y = 0 \\ (a+1)x - (a+1)y = 0 \end{cases} \iff \begin{cases} 2(a+1)(x-y) = 0 \\ (a+1)(x-y) = 0 \end{cases} \underset{\underline{a \neq -1}}{\iff} \begin{cases} x = y \\ x = y \end{cases} \iff x = y$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ est un $\vec{vp}$ pour $\lambda = 1$.

SEP pour $\lambda = a+2$ On résout $(A - (a+2)I_2)X = 0$ $\begin{bmatrix} a+1 & -2(a+1) \\ a+1 & -2a-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} (a+1)x - 2(a+1)y = 0 \\ (a+1)x - 2(a+1)y = 0 \end{cases} \underset{\underline{a \neq -1}}{\iff} x = 2y$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{vp}$ pour $\lambda = a+2$

$\underset{\underline{\text{énoncé}}}{D = P^{-1} A P}$ $D = S^{-1} A S$ où $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

46

1)

$$\begin{pmatrix} 2 & 1 & 1 \\ -3 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \qquad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ -2 & -2 & -1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ -2 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad BA = \begin{pmatrix} 2 & 1 & 1 \\ -3 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$AB = BA$

2) $$P(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & -1 \\ -2 & -2 & -1-\lambda \end{vmatrix} \underset{L_3 \leftarrow L_3 + L_1 + L_2}{=} \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & -1 \\ -1-\lambda & -1-\lambda & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda) \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 1 & 1 \end{vmatrix} \underset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - C_1}}{=} (-1-\lambda) \begin{vmatrix} 1-\lambda & \lambda & \lambda \\ 0 & -\lambda & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

DVLP % L_3

$$P(\lambda) = (-1-\lambda) \begin{vmatrix} \lambda & \lambda \\ -\lambda & -1 \end{vmatrix} = (-1-\lambda)(-\lambda + \lambda^2) = (-1-\lambda)\lambda(-1+\lambda)$$

vp : $\lambda = 0$ mult. 1 ; $\lambda = 1$, mult. 1.
$\lambda = -1$ mult 1

Pour $\lambda = 0$ $\vec{vp}$ On résout $AX = \vec{0} \iff$

$$\begin{cases} x + y + z = 0 \\ -z = 0 \\ -2x - 2y - z = 0 \end{cases} \iff \begin{cases} x = -y \\ z = 0 \\ y \in \mathbb{R} \end{cases} \qquad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$ est 1 $\vec{vp}$ pour $\lambda = 0$.

Pour $\lambda = -1$ ⚠ On cherche 2 $\vec{vp}$ $\neq$. On résout $(A + I)X = \vec{0}$

$$\iff \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} 2x + y + z = 0 \\ y - z = 0 \\ -2x - 2y = 0 \end{cases}$$

$L_3 \leftarrow L_3 + L_1$

$$\iff \begin{cases} 2x + y + z = 0 \\ y - z = 0 \\ -y + z = 0 \end{cases} \underset{L_2 \iff L_3}{\iff} \begin{cases} 2x = -y - z \\ y = z \end{cases} \iff \begin{cases} 2x = 2z \\ y = z \end{cases} \quad \begin{cases} x = z \\ y = z \\ z \in \mathbb{R} \end{cases}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{vp}$ pour $\lambda = -1$

$\underline{\lambda = 1}$ vp On résout $(A - I_3)X = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} y + z = 0 \\ -y - z = 0 \\ -2(x+y+z) = 0 \end{cases} \iff \begin{cases} y = -z \\ x = 0 \end{cases}$$

$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\vec{vp}$ pour $\underline{\lambda = 1}$

$\exists$ vp $\neq$, A diagonalisable

$\underline{vp}$

$-1, 0, 1$

1) $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ $\longleftrightarrow$ $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ou

vp : $0, -1, 1$

2) $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

où $A = PDP^{-1}$ $(S = P)$

$$S^{-1} \text{ où } [S|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \underset{L_2 \leftarrow L_1 + L_2}{\Leftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$L_2 \leftrightarrow L_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \end{array}\right] \underset{L_3 \leftarrow L_3 - 2L_2}{\Leftrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -2 \end{array}\right]$$

$L_3 \leftarrow \frac{1}{2} L_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right]$$

$L_2 \leftarrow L_2 + L_3$

$L_1 \leftarrow L_1 - L_2$

$L_1 \leftarrow L_1 - 2L_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 0 & -1 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & -1 \end{matrix}}_{S^{-1}}$$

3)

$$B' = S^{-1} \cdot B \cdot S = \dots = \underline{\text{diagonal}}$$

47

$$A = \begin{bmatrix} 0 & 3 & 2 \\ -2 & 5 & 2 \\ 2 & -3 & 0 \end{bmatrix} \quad 1) \; p(\lambda) = \begin{vmatrix} -\lambda & 3 & 2 \\ -2 & 5-\lambda & 2 \\ 2 & -3 & -\lambda \end{vmatrix}$$

$L_2 \leftarrow L_2 - L_1$

$$- \begin{vmatrix} -\lambda & 3 & 2 \\ \lambda-2 & 2-\lambda & 0 \\ 2 & -3 & -\lambda \end{vmatrix} = (\lambda-2) \begin{vmatrix} -\lambda & 3 & 2 \\ 1 & -1 & 0 \\ 2 & -3 & -\lambda \end{vmatrix} = (\lambda-2) \begin{vmatrix} 3-\lambda^{+} & 3^{-} & 2 \\ 0 & -1^{+} & 0 \\ -1 & -3 & -\lambda \end{vmatrix}$$

$C_1 \leftarrow C_1 + C_2$

DVLP % L_2

$$p(\lambda) = (\lambda-2) \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = (\lambda-2)\left(-\lambda(3-\lambda) + 2\right) = -(\lambda-2)^2(\lambda-1)$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2) \qquad \Delta = 9-8 = 1$$

$\lambda_1 = 2$

$\lambda_2 = \frac{3-1}{2} = 1$

$\lambda_3 = \frac{3+1}{2} = 2$

2 a) → sp : 1 mult 1

2 mult 2

$\det(A) = p(0) = 4$

2.d) Pour $\lambda = 1$ $\quad (A - I)X = \vec{0} \iff \begin{bmatrix} -1 & 3 & 2 \\ -2 & 4 & 2 \\ +2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{cases} -x + 3y + 2z = 0 \\ -2x + 4y + 2z = 0 \\ +2x - 3y - z = 0 \end{cases}$$

$\begin{pmatrix} +1 \\ 1 \\ -1 \end{pmatrix}$ est solu ϑ. est un $\overrightarrow{vp}$.

Pour $\lambda = 2$ $\quad \begin{bmatrix} -2 & 3 & 2 \\ -2 & 3 & 2 \\ 2 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad -2x + 3y - 2z = 0$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ et $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ sont 2 $\overrightarrow{vp}$ libres

3.b) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ -1 & -1 & 0 \end{bmatrix}$ et $D = P^{-1} A P$.

2.c) $A^n = ?$ $A = P D P^{-1}$

Alors $A^n = P D^n P^{-1}$

Par récurrence

I. si $\underline{n=0}$ $A^0 = I_3$ $P D^0 P^{-1} = P I_3 P^{-1} = P P^{-1} = I_3$

si $\underline{n=1}$ $A = P D P^{-1}$ (con.)

II s'il existe $n \in \mathbb{N}$ $A^n = P D^n P^{-1}$

Alors $A^{n+1} = (P D^n P^{-1}) \cdot (P D P^{-1})$

$= P D^n \underbrace{P^{-1} P}_{I_3} D P^{-1} = P D^{n+1} P^{-1}$. □

Vrai par récurrence

$A^n = P D^n P^{-1}$

Vrai $\forall A$ diagonalisable