Algebraic Geometry: Filling in the Gaps

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May 4, 2025

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Introduction

Properties of Schemes and their Morphisms

3.1 Closed Embeddings

In this chapter we will broadly discuss some topological, and algebraic properties of schemes and subschemes, along with their morphisms. Reader be warned: this chapter may feel like whiplash. Recall that in Definition 1.3.7 we defined what an open embedding is; we now define a similar class of morphisms:

Definition 3.1.1. Let $f : X \to Y$ be a morphism of schemes, then f is a **closed embedding**[41] if $f(X) \subset Y$ is closed, f is a homeomorphism onto it's image, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is surjective.

Example 3.1.1. Let A be a ring and $I \subset A$ be an ideal. We claim that the natural map $g : \operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection map $\pi : A \to A/I$ is a closed embedding. First note that if $\mathfrak{p} \subset A/I$ is a prime ideal, then we have that $I \subset \pi^{-1}(\mathfrak{p})$. Indeed, we have that $\ker \pi = I$, so $\pi^{-1}(0) = I$, and $\pi^{-1}(0) \subset \pi^{-1}(\mathfrak{p})$. It follows that we get a induced continuous map $g : \operatorname{Spec} A/I \to \mathbb{V}(I)$. However, we have already shown in Proposition 2.1.3 that there is a homeomorphism $f : \mathbb{V}(I) \to \operatorname{Spec} A/I$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})$. We see that $f \circ g(\mathfrak{p}) = \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$, so $f \circ g = \mathrm{Id}$. We want to show that $\pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ as well. Note that:

$$\pi^{-1}(\pi(\mathfrak{p})) = \{a \in A : [a] \in \pi(\mathfrak{p})\}$$

while:

$$\pi(\mathfrak{p}) = \{[a] \in A/I : a \in \mathfrak{p}\}$$

If $a \in \mathfrak{p}$, then clearly we have that $[a] \in \pi(\mathfrak{p})$ so $a \in \pi^{-1}(\pi(\mathfrak{p}))$ implying that $\mathfrak{p} \subset \pi^{-1}(\pi(\mathfrak{p}))$. If $a \in \pi^{-1}(\pi(\mathfrak{p}))$ then $[a] \in \pi(\mathfrak{p})$, so $a + i \in \mathfrak{p}$ for some $i \in I$. We have that $I \subset \mathfrak{p}$, so $i \in \mathfrak{p}$, hence $a + i - i = a \in \mathfrak{p}$, implying that $\pi^{-1}(\pi(\mathfrak{p})) \subset \mathfrak{p}$. It follows that $g \circ f(\mathfrak{p}) = \pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ so $g \circ f = \mathrm{Id}$ as well. We thus have that g is a homeomorphism onto the closed subspace A/I.

We now check that the morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow g_*\mathscr{O}_{\operatorname{Spec} A/I}$ is surjective, and it suffices to check that $g^\sharp_{U_h}$ is surjective for every distinguished open U_h, as then the induced morphism on stalks will always be surjective. Note that:

$$g_*\mathscr{O}_{\operatorname{Spec} A/I}(U_h) = \mathscr{O}_{\operatorname{Spec} A/I}(U_{[h]}) \cong (A/I)_{[h]}$$

Note that that $g^\sharp_{U_h}$ is given by:

$$\begin{aligned} g^\sharp_{U_h} : A_h &\longrightarrow (A/I)_{[h]} \\ a/h^k &\longmapsto [a]/[h]^k \end{aligned}$$

which is clearly surjective so $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a closed embedding as desired.

With this example in mind, we wish to show that every closed embedding is locally of this form.

Lemma 3.1.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is a closed embedding if and only if for every open affine $U = \operatorname{Spec} A \subset Y$ there exists an ideal $I \subset A$ such that $f^{-1}(U) = \operatorname{Spec} A/I \subset X$, and $f|_{f^{-1}(U)}$ comes from the projection (up to isomorphism).*

[41] This is sometimes referred to in the literature as a closed immersion.

Proof. Let $f : X \to Y$ be a closed immersion, and let $I_{X/Y}$ be the sheaf of ideals on Y given by $\ker f^\sharp$. If $U = \operatorname{Spec} A \subset Y$ is an affine open then $I = I_{X/Y}(U)$ is an ideal of A and thus determines a closed subset $\mathbb{V}(I) \subset U$. Let $V = f^{-1}(U)$ then we have an induced morphism of schemes $f|_V : V \to U$ which must be a homeomorphism onto its image, so we simply need to show that $f(V) = \mathbb{V}(I)$. By Proposition 2.1.2, we have this morphism of schemes is uniquely determined by the morphism $(f|_V)^\sharp_U : \mathscr{O}_U(U) = A \to \mathscr{O}_V(V)$, which we denote by ψ going forward. If $x \in V$, then we have that:

$$f|_V(x) = \psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$$

where π_x is the morphism $\mathscr{O}_V(V) \to (\mathscr{O}_V)_x$. We have that I is the kernel of ψ, and so $I \subset f(x)$ as $0 \in \pi_x^{-1}(\mathfrak{m}_x)$ hence $\psi^{-1}(0) \subset \psi^{-1}(\pi_x^{-1}(x))$. It follows that $f|_V : V \to U$ has image in $\mathbb{V}(I)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(I)$, we want to show that $\mathfrak{p} \in f(V)$; since $f|_V$ is a closed embedding, we have that the stalk map:

$$(f|_V)^\sharp_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow ((f|_V)_*\mathscr{O}_V)_{\mathfrak{p}}$$

is surjective with kernel $I_{\mathfrak{p}}$. If $\mathfrak{p} \notin f(V)$ then we clearly have that $((f|_V)_*\mathscr{O}_V)_{\mathfrak{p}}$ is zero, implying that $I_{\mathfrak{p}} = A_{\mathfrak{p}}$. However, $I \subset \mathfrak{p}$, so this means that $\mathfrak{m}_{\mathfrak{p}} = A_{\mathfrak{p}}$ as $I_{\mathfrak{p}} \subset \mathfrak{m}_{\mathfrak{p}}$. This is clearly a contradiction, so we have that if $I \subset \mathfrak{p}$ then $\mathfrak{p} \in f(V)$ as desired. It follows that $f|_V : V \to U$ is a homeomorphism onto $\mathbb{V}(I)$.

Note that $\mathbb{V}(I) \cong \operatorname{Spec} A/I$, so we can freely identify the two. Let $g : V \to \operatorname{Spec} A/I$ be the homeomorphism induced by $f|_V : V \to U$. We note that for all $x \in V$, we have that $f|_V(x) = g(x)$. If $W \subset U$ is open, we have that $W \cap \mathbb{V}(I)$ is open in $\operatorname{Spec} A/I$, and we thus have that:

$$(f|_V)^{-1}(W) = (f|_V^{-1})(W) \cap (f|_V)^{-1}(\mathbb{V}(I)) = f|_V^{-1}(W \cap \mathbb{V}(I)) = g^{-1}(W \cap \mathbb{V}(I))$$

It follows that for any open set $Z = W \cap \mathbb{V}(I) \subset \mathbb{V}(I)$:

$$g_*\mathscr{O}_V(Z) = (f|_V)_*\mathscr{O}_V(W)$$

In particular, if U_g is an affine open of $\operatorname{Spec} A$, then:

$$g_*\mathscr{O}_V(U_{[g]}) = (f|_V)_*\mathscr{O}_V(U_g)$$

We thus define a morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ on a basis of affine opens by noting that for each U_g we have a morphism:

$$(f|_V)^\sharp_{U_g} : A_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

whose kernel is precisely I_g. It follows that we get a unique morphism:

$$g^\sharp_{U_{[g]}} : \mathscr{O}_{\operatorname{Spec} A/I}(U_{[g]}) = A_g/I_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

which is trivially injective on each distinguished open. Moreover, these maps then clearly commute with the restriction maps, since localization commutes with taking quotients, as we have shown earlier. It follows that $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ is an injective morphism of sheaves, and is surjective on stalks because $(f|_V)^\sharp$ is. Since it is injective and surjective on stalks, we have that $g^\sharp$ is an isomorphism, implying that $f^{-1}(U) \cong \operatorname{Spec} A/I$ as schemes as desired. It follows that $f|_V : V \to U$ is now a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$, such that the kernel of $\psi : A \to A/I$ is precisely I, hence up to isomorphism ψ is the projection map as desired.

Now suppose that for every affine open $U = \operatorname{Spec} A \subset Y$ we have that $f^{-1}(U) \cong \operatorname{Spec} A/I$, for some ideal I. Then with $V = f^{-1}(U)$, we have that $f|_V : V \to U$ is a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection. By Example 3.1.1, we have that $f|_V$ is a a closed immersion for all V. Since locally we have that $f^\sharp$ comes from the projection, we have that the stalk map $(f^\sharp)_y : (\mathscr{O}_Y)_y \to (f_*\mathscr{O}_X)_y$, is surjective. It follows that $f^\sharp$ is surjective by Proposition 1.2.8. Moreover, since each $f|_V$ is a homeomorphism onto it's image for all U, we have that $f : X \to Y$ must also be a homeomorphism onto it's image. Let $\{U_i\}$ be an open cover of Y, and $V_i = f^{-1}(U_i)$ then $f(X) \cap U_i = f|_{V_i}(V_i)$ which is closed in U_i. It follows that $U_i \smallsetminus f|_{V_i}(V_i)$ is open in Y. We claim that:

$$Y \smallsetminus f(X) = \bigcup_i U_i \smallsetminus f|_{V_i}(V_i)$$

Indeed, suppose that $y \in Y \smallsetminus f(X)$, then for all i, we have that there is no $x \in V_i$ such that $f|_{V_i}(x) = y$. It follows that $y \in U_i \smallsetminus f|_{V_i}(V_i)$ for all i, hence $Y \smallsetminus f(X) \subset \bigcup_i U_i \smallsetminus f|_{V_i}(V_i)$. Now suppose that:

$$y \in \bigcup_i U_i \smallsetminus f|_{V_i}(V_i)$$

then for all i we have that there so no x such that $f|_{V_i}(x) = y$, hence there is no $x \in X$ such that $f(x) = y$ so $y \in Y \smallsetminus f(X)$ giving us the other inclusion. Since $Y \smallsetminus f(X)$ is the union of open sets, it is open, implying that $f(X)$ is closed, f is a homeomorphism onto its image, and $f^\sharp$ is surjective, hence f is a closed embedding implying the claim. ☐

We have the following obvious corollaries:

Corollary 3.1.1. *If $X \to \operatorname{Spec} A$ is a closed embedding then $X \cong A/I$ for some I.*

Corollary 3.1.2. *A morphism $f : X \to Y$ is a closed embedding if and only if there exists an affine cover $\{U_i\}$ of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a closed embedding.*

We can properly define closed subschemes now:

Definition 3.1.2. Let X be a scheme, then a **closed subscheme** of X is an equivalence class of closed immersions $f : Z \to X$, where two closed immersions f and g are equivalent if and only if there is an isomorphism $F : Z \to Z$ such that $f \circ F = g$.

The clunky nature of the definition of above can be best explained by noting that for $X = \operatorname{Spec} \mathbb{C}[x]$, we have that $\mathbb{V}(x) = \mathbb{V}(x^2)$ as $\sqrt{\langle x^2 \rangle} = \sqrt{x}$, but $\operatorname{Spec} \mathbb{C}[x]/\langle x \rangle \ncong \operatorname{Spec} \mathbb{C}[x]/\langle x^2 \rangle$. So even though the two topological spaces agree, and both are the same from a topological embedding point of view, the two closed subschemes are not isomorphic. In particular, there are a multitude of scheme structures one can put on a closed subspace of any scheme X, with the induced reduced subscheme structure being just one of many.

Example 3.1.2. Let $X = \operatorname{Proj} A$ for a graded ring A, and Z a closed subscheme of X. Furthermore, suppose that the irrelevant ideal satisfies[42]:

$$A_+ = \sqrt{\langle g_1, \ldots, g_n \rangle} \tag{3.1.1}$$

for some $g_i \in A_+^{\text{hom}}$. Note that this condition is equivalent to $\operatorname{Proj} A$ being quasi-compact; indeed, suppose that $\operatorname{Proj} A$ is quasi-compact then there clearly exists a finite covering of X by projective distinguished opens $\{U_{g_i}\}$. Since $\mathbb{V}(A_+) = \emptyset$, we have that:

$$\mathbb{V}(A_+) = \left(\bigcup_{i=1}^{n} U_{g_i} \right)^c = \bigcap_{i=1}^{n} \mathbb{V}(\langle g_i \rangle) = \mathbb{V}(\langle g_1, \ldots, g_n \rangle)$$

so (3.1.1) follows immediately. Now suppose that (3.1.1) holds, then X is equal to the union of U_{g_i}, which is finite, hence X is a finite union of quasi-compact schemes and is thus quasi-compact[43].

With the quasi-compactness assumption on X, we wish to show that Z is of the form $\operatorname{Proj} A/I$ for some homogenous ideal $I \subset A$. Supposing (2.4.1), we have an open cover of X given by $\{U_{g_i}\}$, and thus we obtain a finite open cover of Z by $\{V_i = f^{-1}(U_{g_i})\}$. Since f is a closed embedding, each $V_i = \operatorname{Spec}(A_{g_i})_0/I_i$; our goal is to construct I out of these I_i. Let $m_i = \deg g_i$, for each i, and define:

$$J_{i,d} = \begin{cases} \{0\} & \text{if } m_i \nmid d \\ \{a \in A_d : a/g_i^{d/m_i} \in I_i\} & \text{if } m_i \mid d \end{cases}$$

Note that $\deg(a/g_i^{d/m_i}) = d - d/m_i \cdot m_i = 0$, so $a/g_i^{d/m_i} \in (A_{g_i})_0$. We set:

$$J_i = \bigoplus_d J_i$$

[42]Note that A_+ is radical, as if $f \in \sqrt{A_+}$, then for some n, $f^n \in A_+$. If f has a degree zero part then f^n has a degree zero part hence $f^n \notin A_+$. It follows that f is a sum of positively graded elements, and thus $f \in A_+$.

[43]In general topology this is the same as say if X is a finite union of compact spaces then X is compact. This setting just feels weird as for Hausdorff spaces compact sets are closed.

It is clear that J_i is a homogenous ideal for each i, hence we set:

$$I = \bigcap_{i=1}^{n} J_i$$

We want to show that $f(Z) = \mathbb{V}(I)$, and it suffices to show that $f|_{V_i}(V_i) = \mathbb{V}(I) \cap U_{g_i}$ for all i. If $\pi_i : A \to A_{g_i}$ is the localization map, and $\imath_i : (A_{g_i})_0 \to A_{g_i}$ is the inclusion, then we set:

$$(I_{g_i})_0 = \imath_i^{-1}\left(\langle \pi_i(I) \rangle\right)$$

Let $\phi : U_{g_i} \to (\operatorname{Spec} A_{g_i})_0$ be the homeomorphism from Proposition 2.2.2 given by $\mathfrak{p} \mapsto \mathfrak{p}_{g_i} \mapsto (\mathfrak{p}_{g_i})_0$; we first claim that:

$$\mathbb{V}(I) \cap U_{g_i} = \phi^{-1}(\mathbb{V}((I_{g_i})_0)) \subset U_{g_i}$$

Let $\mathfrak{p} \in \mathbb{V}(I) \cap U_{g_i}$, then $\mathfrak{p}$ is a homogenous prime ideal such that $I \subset \mathfrak{p}$, and $g_i \notin \mathfrak{p}$. Since $\mathfrak{p} \in U_{g_i}$, we have that $\phi(\mathfrak{p}) = (\mathfrak{p}_{g_i})_0 \subset (A_{g_i})_0$. Since $I \subset \mathfrak{p}$, we have that $I_{g_i} \subset \mathfrak{p}_{g_i}$, hence $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$, so $\mathfrak{p} \in \phi^{-1}(\mathbb{V}((I_{g_i})_0))$.

Now suppose that $\mathfrak{p} \in \phi^{-1}(\mathbb{V}(I_{g_i})_0) \subset U_{g_i}$, then $\mathfrak{p} \in U_{g_i}$ vacuously, so we need to show that $\mathfrak{p} \in \mathbb{V}(I)$. By definition, $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$; in A_{g_i}, we have that $(\mathfrak{p}_{g_i})_0$ corresponds to $\sqrt{(\mathfrak{p}_{g_i})_0 A_f}$, so we have that $\sqrt{(I_{g_i})_0 A_f} \subset \sqrt{(\mathfrak{p}_{g_i})_0 A_f}$ as well. It thus suffices to show that $I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$, as then:

$$I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right) \subset \pi_i^{-1}\left(\sqrt{(\mathfrak{p}_{g_i})_0 A_f}\right) = \mathfrak{p}$$

Furthermore, as I is homogenous, we need only check that every homogenous element of I lies in $\pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$. Let $a \in I$ be homogenous of degree d; if $a \in \ker \pi_i$ then we are done, otherwise, we have that $a^{m_i}/g_i^d \in (I_{g_i})_0$. It follows that $a^{m_i}/1 \in (I_{g_i})_0$, hence $a^{m_i}/1 \in (I_{g_i})_0 A_f$, so $a/1 \in \sqrt{(I_{g_i})_0 A_f}$ by definition[44].

It now suffices to show that $f|_{V_i}(V_i) = \phi^{-1}(\mathbb{V}(I_{g_i}))$. Since $f|_{V_i}(V_i) \subset U_{g_i}$, we have that $f|_{V_i}$ is a homeomorphism onto the closed subset $\mathbb{V}(I_i) \subset \operatorname{Spec}(A_{g_i})_0$. Therefore, it suffices to check that $\mathbb{V}(I_i) = \mathbb{V}((I_{g_i})_0)$, and in particular that $I_i = (I_{g_i})_0$ for all i. Now note that the only elements in A_{g_i} which have degree zero are those of the form a/g_i^n where a is homogenous and satisfying $\deg a = n \cdot m_i$. Let $a/g_i^n \in (I_{g_i})_0$, then $a/g_i^n \in I_{g_i}$, so $a/1 \in I_{g_i}$ as well. It follows that $a \in I \cap A_{n \cdot m_i}$, hence $a/g_i^n \in I_i$ for all i, so $(I_{g_i})_0 \subset I_i$ as desired.

Now let $a/g_i^n \in I_i$, and $l = \operatorname{lcm}(m_1, \ldots, m_n)$. We have that there exists a $k \leq r \in \mathbb{N}$ such that:

$$n = k \cdot l + r \Rightarrow n + (k - r) = (k + 1)l$$

so by taking $a/g_i^n = ag^{k-r}/g^{k-r+n}$, we may assume that l divides n. Since $\ker f^\sharp$ is a sheaf of ideals, if $I_{ij} = \ker f^\sharp_{U_{g_i} \cap U_{g_j}}$, we have that $a|_{U_{g_i} \cap U_{g_j}} \in I_{ij}$. Recall that $U_{g_i} \cap U_{g_j} = U_{g_i g_j} = \operatorname{Spec}(A_{g_i g_j})_0$, hence we have that:

$$a/g^n|_{U_{g_i} \cap U_{g_j}} = ag_j^n/(g_i g_j)^n \in I_{ij} \subset (A_{g_i g_j})_0$$

Moreover, we also have that

$$U_{g_i g_j} \cong \operatorname{Spec}((A_{g_i})_0)_h$$

where $h = g_j^{m_i}/g_i^{m_j}$. The ring homomorphism

$$f^\sharp_{U_{g_i}} : (A_{g_i})_0 \to (A_{g_i})_0/I_i$$

determines a morphism of affine schemes which on all distinguished opens of $\operatorname{Spec}(A_{g_i})_0$ of the form U_b, has kernel given by $(I_i)_b$. The morphism determined by $f^\sharp_{U_i}$ must agree with f on all open subsets of U_{g_i}, hence we have that I_{ij} is naturally isomorphic to the ideal $(I_i)_h$, via the unique isomorphism

[44] Note that we have now shown that for any homogenous ideal I, $\mathbb{V}(I) \cap U_h = \mathbb{V}((I_h)_0) \subset U_h$

$((A_{g_i})_0)_h \cong (A_{g_ig_j})_0$ from Lemma 2.2.7. Similarly, with $h^{-1} = g_i^{m_j}/g_j^{m_i}$, we must have that $(I_j)_{h^{-1}}$ is naturally isomorphic to I_{ij} via the same isomorphism. Any element in $(I_j)_{h^{-1}}$ can be written as:

$$\frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}}\right)^{-e} \tag{3.1.2}$$

where $b/g_j^k \in I_j$. Recall that we took n to be divisible by l, so $n = m_i \cdot p$ and $n = m_j \cdot q$ for some p and q. Hence, under the isomorphism $(I_i)_h \cong (I_j)_{h^{-1}}$ we have that:

$$\frac{a}{g_i^{m_j \cdot q}} \longmapsto \frac{a}{g_j^{m_i \cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i \cdot q}}\right)^{-1}$$

So for an element of the form (3.1.2) we must have that:

$$\frac{a}{g_i^{m_j \cdot q}} \longmapsto \frac{a}{g_j^{m_i \cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i \cdot q}}\right)^{-1} = \frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}}\right)^{-e}$$

We thus have that by the definition of localization we have that:

$$\frac{g_i^{m_j \cdot e} a}{g_j^{m_i \cdot e + m_i \cdot q}} \in I_j$$

We can take e large enough so that $e'_j = m_j \cdot e$ is divisible by l, hence we can write that:

$$\frac{g_i^{e'_j} a}{g_j^{(e'+n) \cdot (m_i/m_j)}} \in I_j$$

Do this for all j, and let $e' = \max(e'_1, \ldots, e'_n)$, then $g_i^{e'} a \in J_j$ for all j. It follows that $g_i^{e'} a \in I$, hence:

$$\frac{g_i^{e'} a}{1} \in I_{g_i}$$

so $a/1 \in I_{g_i}$, giving us that $a/g^n \in (I_{g_i})_0$. It follows that $I_i = (I_{g_i})_0$ so $f(Z) = \mathbb{V}(I)$ as desired.

We now show that $\mathbb{V}(I)$ is homeomorphic to $\operatorname{Proj} A/I$. Let $\pi : A \to A/I$ be the projection map, where A/I has the induced grading, and $\mathfrak{p} \in \operatorname{Proj} A/I$. The prime ideal $\pi^{-1}(\mathfrak{p})$ is homogenous, as if $a \in \pi^{-1}(\mathfrak{p})$ then we write a as:

$$a = \sum_d a_d \tag{3.1.3}$$

where $a_d \in A_d$. It follows that $\pi(a) \in \mathfrak{p}$, and since $\mathfrak{p}$ is homogenous each $\pi(a_d)$ is in $\mathfrak{p}$ so each $a_d \in \pi^{-1}(\mathfrak{p})$. Each $\pi^{-1}(\mathfrak{p})$ contains I so this defines a map $F : \operatorname{Proj} A/I \to \mathbb{V}(I)$. Via the bijection between prime ideals of A/I and prime ideals of A containing I it follows that this map is a bijection, so it suffices to check that this is continuous and open.

We can do this on the distinguished basis for $\operatorname{Proj} A/I$ and the basis $\{\mathbb{V}(I) \cap U_g\}_{g \in A_+^{\mathrm{hom}}}$ for $\mathbb{V}(I)$. Let U_g be the projective distinguished open in $\operatorname{Proj} A$, then

$$F^{-1}(V(I) \cap U_g) = F^{-1}(V(I)) \cap F^{-1}(U_g) = F^{-1}(U_g)$$

I claim that this is equal to $U_{[g]}$. Suppose $[g] \notin \mathfrak{p} \subset A/I$, then for all $i \in I$ we must have that $g+i \notin \pi^{-1}(\mathfrak{p})$ hence $g \notin \pi^{-1}(\mathfrak{p})$. It follows that $\mathfrak{p} \in U_g$ so $U_{[g]} \subset U_g$. Now let $\mathfrak{p} \in f^{-1}(U_g)$, then $g \notin \pi^{-1}(\mathfrak{p})$, but this implies that $[g] \notin \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$ so $\mathfrak{p} \in U_{[g]}$. Therefore $f^{-1}(U_g) = U_{[g]}$ and f is continuous.

To show that F is open we claim that $F(U_{[g]}) = V(I) \cap U_g$, but this is now clear as $F : \operatorname{Proj} A/I \to V(I)$ is bijective, so since $F^{-1}(V(I) \cap U_g) = U_{[g]}$ we get that $F(F^{-1}(V(I) \cap U_g) = V(I) \cap U_g = U_{[g]}$. It follows that f is a continuous open bijective map and thus a homeomorphism.

Now note that the structure sheaf $\mathscr{O}_{\operatorname{Proj} A/I}$ satisfies:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = ((A/I)_{[g]})_0$$

However, recall that there is a unique surjective homomorphism

$$\begin{aligned} A_g &\longrightarrow (A/I)_{[g]} \\ a/g^k &\longmapsto [a]/[g]^k \end{aligned}$$

which commutes with localization maps, and clearly preserves grading. It follows, that we have a unique surjective homomorphism commuting with the isomorphisms from Lemma 2.2.7:

$$\begin{aligned} (A_g)_0 &\longrightarrow ((A/I)_{[g]})_0 \\ a/g^k &\longrightarrow [a]/[g]^k \end{aligned}$$

where $\deg a = k \cdot \deg g$. Note that clearly $(I_g)_0$ maps to zero under this map, so we have unique surjective homomorphism:

$$\begin{aligned} \phi : (A_g)_0/(I_g)_0 &\longrightarrow ((A/I)_{[g]})_0 \\ [a/g^k] &\longrightarrow [a]/[g]^k \end{aligned}$$

Now suppose that $\phi([a/g^k]) = 0$, then we have that $[a]/[g]^k = 0 \in ((A/I)_{[g]})_0 \subset (A/I)_{[g]}$. It follows that there an M such that $[g^M \cdot a] = 0 \in A/I$, hence $g^M a \in I$. We thus have that $g^M a/1 \in I_g$, so $g^M a/g^{M+k} = a/g^k \in (I_g)_0$. By the naturality[45] of these isomorphisms it follows that up to a unique sheaf isomorphism:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = (A_g)_0/(I_g)_0$$

Now equip $\mathbb{V}(I)$ with the sheaf $\mathscr{O}_{\mathbb{V}(I)} = F_* \mathscr{O}_{\operatorname{Proj} A/I}$, and note that this endows $\mathbb{V}(I)$ with the structure of a scheme isomorphic to $\operatorname{Proj} A/I$[46].

Let $\tilde{f}$ be restriction of the codomain to $\mathbb{V}(I)$. In particular, we have that:

$$\tilde{f} : Z \longrightarrow \mathbb{V}(I)$$

Since $I_i = (I_{g_i})_0$, we define a sheaf morphism on the open cover $\{\mathbb{V}(I) \cap U_{g_i}\}$ as the identity map:

$$\tilde{f}^\sharp_{\mathbb{V}(I)\cap U_{g_i}} : \mathscr{O}_{\mathbb{V}(I)}(\mathbb{V}(I) \cap U_{g_i}) = (A_{g_i})_0/(I_{g_i})_0 \longrightarrow \mathscr{O}_Z(V_i) = (A_{g_i})_0/I_i$$

These then agrees on overlaps $U_{g_i} \cap U_{g_j}$ as $((I_{g_i})_0)_h \cong I_{ij} \cong ((I_{g_j})_0)_{h^{-1}}$ via the natural isomorphisms which glue $\operatorname{Proj} A$ together. It follows that this defines a sheaf isomorphism:

$$\tilde{f}^\sharp : \mathscr{O}_{\mathbb{V}(I)} \longrightarrow \mathscr{O}_Z$$

hence $(\tilde{f}, \tilde{f}^\sharp)$ determines a scheme isomorphism $Z \to \mathbb{V}(I)$. Since $\mathbb{V}(I) \cong \operatorname{Proj} A/I$ as schemes, we thus have that $Z \cong \operatorname{Proj} A/I$ as desired.

We now briefly show that the condition that $\operatorname{Proj} A$ be quasi-compact is extremely necessary. Indeed take:

$$\mathbb{P}^\infty_k = \operatorname{Proj} k[x_1, x_2, \dots]$$

for any field k. Let:

$$Z = \coprod_{i=1}^{\infty} X_i = \operatorname{Spec} k[x_1/x_i, x_2/x_i, \dots, \hat{x}_i/x_i, \dots]/\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

where the ideal:

$$\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

[45]Note that $(A_g)_0/(I_g)_0$ does not depend on the class representative g, as for any homogeneous i of degree equal to g, $[a/(g+i)^k] = [a/g^k]$.

[46]This is not the reduced scheme structure, rather one induced by the sheaf of ideals determined by I itself. If $\mathbb{V}(I)$ was equipped with the reduced structure, then as schemes $\mathbb{V}(I) \cong \operatorname{Proj} A/\sqrt{I}$.

is generated by all ith fold products of elements in $\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$. Note that each X_i is is a singleton set as

$$\sqrt{\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i} = \langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$$

Denote each point by $0^i \in X_i \subset Z$, and define a closed embedding by:

$$\begin{aligned} f : Z &\longrightarrow \mathbb{P}^\infty_k \\ 0^i &\longmapsto [0, \dots, 0, 1, 0 \dots, 0, \dots] \end{aligned}$$

where the 1 is in the ith position. If we take the homogenous ideal $I = \langle x_i x_j : i \neq j\rangle$, then clearly for all k:

$$(I_{x_k})_0 = \langle x_1/x_k, x_2, x_k, \dots \hat{x}_k/x_k, \dots\rangle$$

So under the identification $\mathbb{V}(I) \cap U_{x_i} = \mathbb{V}((I_{x_i})) \subset U_{x_i}$, we discern that $\mathbb{V}(I) \cap U_{x_i}$ contains only the point $[0, \dots, 0, 1, 0 \dots, 0, \dots]$, where the 1 is again in the ith position. Clearly we then have that for all U_{x_i}, $f(Z) \cap U_{x_i} = \mathbb{V}(I) \cap U_{x_i}$, hence $f(Z) = \mathbb{V}(I)$, and f has closed image.

We set:

$$I_i = \langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

and define a sheaf morphism on the affine open cover $\{U_{x_i}\}_{i=1}^\infty$ via the canonical projections:

$$\begin{aligned} f^\sharp_{U_{x_i}} : k[\{x_j/x_i\}^\infty_{j=1, j\neq i}] &\longrightarrow k[\{x_j/x_i\}^\infty_{j=1, j\neq i}]/I_i \\ g &\longmapsto [g] \end{aligned}$$

and note that there is nothing to glue as $f^{-1}(U_{x_i} \cap U_{x+j})$ is the empty set. This sheaf homomorphism is clearly surjective on stalks so $Z \hookrightarrow \mathbb{P}^\infty_k$ is a closed embedding.

We claim that there is no homogenous ideal I such that $Z \cong \operatorname{Proj} A/I$. Indeed, suppose there was. Then by the work above we would have that for all x_i,

$$(I_{x_i})_0 = I_i$$

Let $f \in I$ be a nonzero homogenous element of degree d, then $f/1 \in I_{x_i}$, and $f/x_i^d \in (I_{x_i})_0$ for all i.[47] For the above to be true, we must then have that $f/x_i^d \in I_i$ for all i as well. However, if $k > d$, for f/x_k^d to lie in I_k, we must have that f/x_k^d is a sum of k fold products of elements of the form x_i/x_k, an obvious contradiction if f is nonzero, hence $f = 0$. Since this can be done for arbitrary degree d, as there is no upper bound on the ideals I_i, we have that Z cannot possibly be isomorphic to $\operatorname{Proj} A/I$, as no homogenous ideal can agree with I_i on the affine open cover.

With the above example in mind, we can classify all projective schemes over some fixed ring B:

Theorem 3.1.1. *X is a projective scheme over B if and only if it is a closed subscheme of $\mathbb{P}^n_B$ for some n.*

Proof. Suppose that X is a projective scheme over B, then by Definition 2.2.7, we have that:

$$X = \operatorname{Proj} A$$

where A is a graded ring, satisfying $A_0 = B$, and is finitely generated in degree one as a B algebra. Since A is finitely generated in degree one, let $a_1, \dots, a_n$ be a generating set of degree one elements; this defines a surjection

$$\phi : B[x_0, \dots, x_n] \to A$$

which preserves grading. It follows that $\ker \phi$ is a homogenous ideal, and that $A \cong B[x_0, \dots, x_n]/\ker \phi$, hence:

$$X = \operatorname{Proj}(B[x_0, \dots x_n]/\ker \phi)$$

[47] Note that f/x_i^d cannot be zero as k is a field, so localization maps are injective.

As a scheme, X is canonically isomorphic to $\mathbb{V}(\ker\phi) \subset \mathbb{P}^n_B$[48], hence X determines a closed subscheme of $\mathbb{P}^n_B$.

If X is a closed subscheme of $\mathbb{P}^n_B$, then since $\mathbb{P}^n_B$ is quasicompact, we have that by Example 3.1.2 $X \cong \operatorname{Proj} B[x_0, \dots, x_n]/I$ for some homogenous ideal I. If I contains the irrelevant ideal, then X is the empty scheme and thus isomorphic to $\operatorname{Proj} B$, where B has the trivial grading, so X is trivially a projective B scheme. If I does not contain the irrelevant ideal, then $B[x_0, \dots, x_n]/I$ is a graded, finitely generated in degree one, B-algebra, hence X is projective B scheme as desired. □

Example 3.1.3. Recall from Example 2.3.4 that locally the morphism:

$$f : \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$$

is given by scheme morphisms:

$$U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$$

The U_{v_i} cover $\mathbb{P}^3_{\mathbb{C}}$, and $f^{-1}(U_{v_i}) = U_{x_l} \times_{\mathbb{C}} U_{y_n}$. We claim that this morphism is a closed embedding, and by Corollary 3.1.2 it suffices to check that each $U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$ is a closed embedding. By Corollary 3.1.2, it suffices to check that $U_{x_l} \times_{\mathbb{C}} U_{y_n} \cong \operatorname{Spec} \mathbb{C}[\{v_k/v_i\}_{k \neq i}]/I$ for some ideal I. We check this in case of $i = 0$. Note that the morphism of affine schemes comes from the ring homomorphism:

$$\begin{aligned} \phi_0 : \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0] &\longrightarrow \mathbb{C}[x_1/x_0, y_1/y_0] \\ v_i/v_0 &\longmapsto \begin{cases} x_1/x_0 & \text{if } i = 1 \\ y_1/y_0 & \text{if } i = 2 \\ (x_1/x_0)\cdot(y_1/y_0) & \text{if } i = 3 \end{cases} \end{aligned}$$

This is clearly surjective, hence $\mathbb{C}[x_1/x_0, y_1/y_0] \cong \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0]/\ker\phi_0$, and it follows that the induced morphism is a closed embedding. The kernel of this homomorphism is:

$$I = \langle v_1/v_0 \cdot v_2/v_0 - v_3/v_0 \rangle$$

and so the homogenous ideal cutting out $f(\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})$ is given by $J = \langle v_1 v_2 - v_3 v_0 \rangle$. It follows that as schemes,

$$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1 \cong \operatorname{Proj} \mathbb{C}[v_0, v_1, v_2, v_3]/\langle v_1 v_2 - v_3 v_0 \rangle$$

Example 3.1.4. Let $Z \subset X$ be a closed subset of a scheme X, and equip Z with the induced reduced closed subscheme structure, then we have that the inclusion map $\imath : Z \to X$ is a homeomorphism onto its image. We want to define a sheaf morphism $\imath^\sharp : \mathscr{O}_X \to \imath_* \mathscr{O}_Z$. Recall that if $I_{Z/X}$ is the sheaf of ideals associated to the closed subset Z then $\mathscr{O}_Z = \imath^{-1}\mathscr{O}_X/I_{Z/X}$. By Corollary 1.3.4, we have that there is a canonical morphism:

$$\mathscr{O}_X/I_{Z/X} \longrightarrow \imath_*\imath^{-1}\mathscr{O}_X/I_{Z/X}$$

which is surjective. There is a surjective morphism $\mathscr{O}_X \to \mathscr{O}_X/I_{Z/X}$, so we define $\imath^\sharp$ to the be composition of these sheaf morphisms. It follows that $(\imath, \imath^\sharp)$ is a closed embedding as desired.

We now go to our next result regarding closed embeddings which is an analogue of Lemma 2.3.7:

Lemma 3.1.2. *Let $f : X \to Z$ be a closed embedding, and let $g : Y \to Z$ be any morphism. Then the base change $X \times_Z Y \to Y$ is also a closed embedding.*

Proof. We have the following Cartesian square:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

[48] Note that $\mathbb{V}(\ker\phi)$ is not necessarily equipped with the reduced subscheme structure, but instead equipped with scheme structure determined by the sheaf of ideals induced by $\ker\phi$. This only coincides with the reduced structure if $\ker\phi$ satisfies $\sqrt{\ker\phi} = \ker\phi$.

Let $\{U_i = \operatorname{Spec} A_i\}$ be an affine open cover of Z, and choose an affine open cover $\{V_{ij} = \operatorname{Spec} B_{ij}\}$ of Y such that $g(V_{ij}) \subset U_i$. Note that $f^{-1}(U_i) \cong \operatorname{Spec} A_i/I_i$ for some ideal I_i. We have that:

$$\pi_Y^{-1}(V_{ij}) \cong X \times_Z V_{ij}$$

We claim that this is isomorphic to $f^{-1}(U_i) \times_{U_i} V_i$. Indeed, we need to show that the following diagram is cartesian:

$$\begin{tikzcd}
f^{-1}(U_i) \times_{U_i} V_i \arrow[r, "\pi_Y"] \arrow[d, "\pi_X \circ \imath"'] & V_{ij} \arrow[d, "g|_{V_{ij}}"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

where $\imath : f^{-1}(U_i) \to X$ is the inclusion, is Cartesian. Let Q be any scheme with maps $p_X : Q \to X$ and $p_{V_{ij}} : Q \to V_{ij}$ which make the relevant diagram commute. Since $g(V_{ij}) \subset U_i$, we have that $g \circ p_{V_{ij}}(Q) \subset U_i$. Since $f \circ p_X = g \circ p_{V_{ij}}$, we have that $f \circ p_X(Q) \subset U_i$ as well, and thus $p_X(Q) \subset f^{-1}(U_i)$. Since $X \times_Z V_{ij}$ is a fibre product we have a unique morphism $\phi : Q \to X \times_Z V_{ij}$ such that $\pi_X \circ \imath \circ \phi = p_X$. We thus have that $\pi_X \circ \imath \circ \phi(Q) = p_X(Q) \subset f^{-1}(U_i)$. We see that $(\pi_X \circ \imath)^{-1}(f^{-1}(U_i)) \subset f^{-1}(U_i) \times_Z V_i$, hence $\phi(Q) \subset p_X(Q)$. Since both $f(f^{-1}(U_i)) \subset U_i$, and $g(V_{ij}) \subset U_i$, we have that this $f^{-1}(U_i) \times_Z V_{ij} = f^{-1}(U_i) \times_{U_i} V_{ij}$, and it follows that $\phi(Q) \subset f^{-1}(U_i) \times_{U_i} V_{ij}$, so ϕ factors uniquely through the open embedding $f^{-1}(U_i) \times_{U_i} V_{ij} \to X \times_Z V_{ij}$, and we have a unique morphism $Q \to f^{-1}(U_i) \times_{U_i} V_{ij}$. It follows that $f^{-1}(U_i) \times_{U_i} V_{ij} \cong f^{-1}(U_i) \times_{U_i} V_{ij}$ as desired. We thus have the following chain of isomorphisms:

$$\begin{aligned}
\pi_Y^{-1}(V_{ij}) &\cong X \times_Z V_{ij} \\
&\cong f^{-1}(U_i) \otimes_{U_i} V_{ij} \\
&\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} \operatorname{Spec} B_{ij} \\
&\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} B_{ij}
\end{aligned}$$

Let $\phi : A_i \to B_{ij}$ be the ring homomorphism making B_{ij} an A_i algebra, and set $J = \langle \phi(I_i) \rangle$, then we have that:

$$A_i/I_i \otimes_{A_i} B_{ij} \cong B_{ij}/J$$

hence:

$$\pi_Y^{-1}(V_i) \cong \operatorname{Spec} B_{ij}/J$$

so $\pi_Y : X \times_Z Y \to Y$ is a closed embedding by Corollary 3.1.2. □

These two lemmas each provide an example of properties of morphisms we are about to study, namely being *local on target* and *stable under base change*. More precisely, let $f : X \to Z$ be a morphism of schemes and P a property morphisms of schemes, then P is local on target if for any affine cover of $\{U_i\}$ of Z such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ satisfies P for all i we have that f satisfies P, and if $f : X \to Y$ satisfies P, then for all affine opens U, $f|_{f^{-1}(U)}$ satisfies P as well. In other words, a property of a morphism of schemes is called local on target if it can be checked affine locally. Let $g : Y \to Z$ be any other morphism of schemes, and let $f : X \to Z$ be a morphism satisfying P, then P is stable under base change if $X \times_Z Y \to Y$ also satisfies the property.

There is a third property of morphisms important to study, and that is notion of begin *closed under composition*. In particular if P is a property of morphisms, then P is closed under composition if for all $f : X \to Y$ and $g : Y \to Z$ satisfying P, then $g \circ f$ satisfies P as well.

Lemma 3.1.3. *Closed embeddings are closed under composition.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be closed embeddings, and let $U = \operatorname{Spec} A$ be an open affine. Then since g is a closed embedding, there is some $I \subset A$ such that:

$$(g \circ f)^{-1}(U) = f^{-1}(\operatorname{Spec} A/I)$$

Since g is a closed embedding, there is some $J \subset A/I$ such that:

$$(g \circ f)^{-1}(U) = \operatorname{Spec}(A/I)/J$$

Let I' be the kernel of the morphism $A \to A/I \to (A/I)/J$, then $(A/I)/J \cong A/I'$, so $g \circ f$ is a closed embedding by Lemma 3.1.1. □

We end the section with the following general result:

Theorem 3.1.2. *Let P be a property of a morphism of Z-schemes $f : X \to Y$ such that P is closed under composition and stable under base change. Then if $f : X \to Y$ and $g : X' \to Y'$ both satisfy P then the induced morphism $f \times g : X \times_Z X' \to Y \times_Z Y'$ satisfies property P.*

Proof. Let h and h' be the morphisms making Y and Y' Z-schemes, and q and q' the morphisms making X and X' Z-schemes. We have that $f \times g$ comes from the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[dr, "f\times g"] \arrow[drr, bend left, "g\circ\pi_{X'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h'"] \\
& Y \arrow[r, "h"] & Z
\end{tikzcd}$$

It is clear that $f \times g = \mathrm{Id} \times g \circ f \times \mathrm{Id}$, so it suffices to show that $f \times \mathrm{Id}$ and $\mathrm{Id} \times g$ both satisfy proper P. We have the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[r, "f\times\mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_Y"] & X' \arrow[d, "q'"] \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

The right square is Cartesian, and since $h \circ f = q$, and $\pi_{X'} \circ f \times \mathrm{Id} = \pi_{X'}$ the outer diagram is Cartesian, so the left square is also Cartesian. Since the left square is Cartesian, it follows that $f \times \mathrm{Id}$ is the base change of f, and thus satisfies property P. Now note that we also have the following commutative diagram:

$$\begin{tikzcd}
Y \times_Z X' \arrow[r, "\mathrm{Id}\times g"] \arrow[d, "\pi_{X'}"] & Y \times_Z Y' \arrow[r, "\pi_Y"] \arrow[d, "\pi_{Y'}"] & Y \arrow[d, "h"] \\
X' \arrow[r, "g"] & Y' \arrow[r, "h'"] & Z
\end{tikzcd}$$

The right square is Cartesian, and the outer square satisfies $h' \circ g = q'$, and $\pi_Y \circ \mathrm{Id} \times g = \pi_Y$, so it is Cartesian as well. It follows that the left square is Cartesian, and that $\mathrm{Id} \times g$ is the base change of g, so $\mathrm{Id} \times g$ satisfies property P as well. Since P is closed under composition, we have that $f \times g$ satisfies property P as well. □

3.2 Reduced, Irreducible, and Integral Schemes

In the following sections, we will study some algebraic and topological properties schemes may have, and the interplay between them. We begin with the following definition:

Definition 3.2.1. Let X be a scheme, then X is **irreducible** if it is irreducible as a topological space as in Definition 1.4.3. We also have that X is **reduced** if $\mathscr{O}_X(U)$ has no nilpotents for all $U \subset X$, and is **integral** if $\mathscr{O}_X(U)$ is an integral domain for all $U \subset X$.

We first check that being reduced is an inherently local property.

Lemma 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is reduced

b) There exists an affine open cover $\{U_i\}$ such that each U_i is reduced

c) Every stalk $(\mathscr{O}_X)_x$ is reduced

Proof. Clearly $a \Rightarrow b$, so we first show that $b \Rightarrow c$. Let $x \in U_i = \operatorname{Spec} A$, then $(\mathscr{O}_X)_x \cong A_{\mathfrak{p}}$ so it suffices to check that $A_{\mathfrak{p}}$ has no nilpotents. Let $a/g \in A_{\mathfrak{p}}$ where $g, h \notin \mathfrak{p}$. Then if $(a/g)^k = 0$ for some k there exists a $c \in \mathbb{A} - \mathfrak{p}$ such that:

$$c \cdot a^k = 0$$

We see that $c \neq 0$, so since A has not nilpotents we have that either $a = 0$ hence $a/g = 0$, implying the claim.

Now we show that $c \Rightarrow a$. Let U be an open set of X, and $s \in \mathscr{O}_X(U)$ such that $s^k = 0$ for some k. Then for every $x \in U$ we have that $(s^k)_x = s_x^k = 0$ implying that $s_x = 0$ for all s. However, the map:

$$\mathscr{O}_X(U) \longrightarrow \prod_{x \in U} (\mathscr{O}_X)_x$$

is injective so $s = 0$, hence $\mathscr{O}_X(U)$ has no nilpotents. □

Example 3.2.1. Let X be a scheme, and Y a closed subset of X, then Y equipped with the induced reduced closed subscheme structure is irreducible. Indeed, let $\{U_i = \operatorname{Spec} A\}$ be an affine open cover of X, then $U_i \cap Y \cong \operatorname{Spec} A_i/I_i$ determines an affine open cover of Y. Each I_i is radical, hence we have that if $[a] \in A_i/I_i$ satisfies $[a]^k = 0$ then $a^k \in I_i$, implying that $a \in I_i$. It follows that $[a] = 0$, so A_i/I_i is reduced. We thus have an affine open cover of Y such that each affine scheme is reduced, so by Lemma 3.2.1 we have that Y is reduced as well. In particular, X is a closed subset of X, and thus there is a reduced scheme X_{red}, such that the underlying topological space is X, and it's structure sheaf is $\mathscr{O}_{X_{\text{red}}} = \mathscr{O}_X/I$, where I is the sheaf of ideals corresponding to X. In particular, the stalks $\mathscr{O}_{X_{\text{red},x}}$ are isomorphic to $\mathscr{O}_{X,x}/\sqrt{\langle 0 \rangle}$, and on any affine open, $\mathscr{O}_{X_{\text{red}}}(U) \cong \mathscr{O}_X(U)/\sqrt{\langle 0 \rangle}$.

We now show some properties of X being irreducible. We need the following definition:

Definition 3.2.2. Let X be a topological space, then a **generic point** is a point $\eta \in X$ which is dense, i.e. $\bar{\{\eta\}} = X$.

Lemma 3.2.2. *Let X be an irreducible topological space, then every non empty open subset of X is irreducible when equipped with the subspace topology. Moreover, a topological space is irreducible if and only if the intersection of every two non empty open sets is non empty.*

Proof. Suppose that X is irreducible, then by Lemma 1.4.4 we have that X is connected and every open subset of X is dense. Let U be a non empty open subset of X, then we claim that U is irreducible when equipped with the subspace topology. Indeed, suppose that $U = Y_1 \cup Y_2$ for two proper closed subsets of U. Then $Y_1 = Z_1 \cap U$ and $Y_2 = Z_2 \cap U$, then we have that $U = (Z_1 \cup Z_2) \cap U$, implying that $U \subset Z_1 \cup Z_2$ hence U is contained in the closed subset $Z_1 \cup Z_2$. However, $\bar{U} = X$, so $X = Z_1 \cup Z_2$ implying that X is reducible. The claim follows from the contrapositive.

Now let U and V be two nonempty open subsets of X such that $U \cap V = \emptyset$. Then $U^c \cup V^c = X$, so X is reducible. By the contrapositive we have that if X is irreducible then $U \cap V \neq \emptyset$.

Suppose that $U \cap V \neq \emptyset$ for every open set, and let $Z_1, Z_2 \subset X$ be two proper closed subsets. We see that since $Z_1^c \cap Z_2^c \neq \emptyset$ that $Z_1 \cup Z_2 \neq X$, so X is irreducible □

Lemma 3.2.3. *Let X be a scheme, then X is irreducible if and only if X has unique generic point η.*

Proof. Suppose that X is reducible, then $X = Z_1 \cup Z_2$ for two closed proper subsets of X. It follows that every $x \in X$ lies in Z_1 or Z_2 so the closure of every point is contained in Z_1 or Z_2. We thus have that X has no generic points, let alone a unique one. By the contrapositive, we have that if X has a unique generic point, then X is irreducible.

Now let X be irreducible, by Lemma 3.2.2 we have that $U = \operatorname{Spec} A$ is a irreducible topological space as well. We claim that the nilradical:

$$I = \{a \in A : \exists k \in \mathbb{N}, a^k = 0\}$$

is prime. Let $f, g \in A$ then if $f, g \in I$ we have that $U_f = U_{f^k} = U_0 = \emptyset$ and similarly for g. Similarly, if $U_f = U_0$ then there is some k such that $f^k = 0$ so we have that a distinguished open is empty if and only if the element lies in I. Now suppose that $U_f \cap U_g$ is not empty, the fact that $U_f \cap U_g$ is not empty implies that $fg \notin I$. It follows by the contrapositive that if $fg \in I$ then either f or g are in I so I is prime. The closure of the singleton set $\{I\} \in \operatorname{Spec} A$ is given by $\mathbb{V}(I)$ and we claim that this is equal to $\operatorname{Spec} A$. We need only show that $I \subset \mathfrak{p}$ for any prime in A, however this is clear as $0 \in \mathfrak{p}$, and for any $f \in I$ we have that $f^k = 0 \in \mathfrak{p}$, hence $f \in \mathfrak{p}$, so $I \subset \mathfrak{p}$. We show that I is unique, suppose that $\mathfrak{q}$ is prime, and satisfies $\mathfrak{q} \subset \mathfrak{p}$ for every prime. Then $\mathbb{V}(\mathfrak{q}) = \operatorname{Spec} A$, so $\mathfrak{q} = \sqrt{\mathfrak{q}} = \sqrt{I} = I$ implying uniqueness.

We now claim that the point $x \in X$ corresponding to $I \in \operatorname{Spec} A$ is actually a generic point of X. Indeed, suppose that $\overline{\{x\}} = V$ for some closed subset of X, then we have that:

$$V = \bigcap_{Z \ni x} Z$$

where $Z \subset X$ is closed. In the subspace topology, since x is a generic point, we have that:

$$U = \bigcap_{Y \ni x} Y$$

where $Y \subset U$ is closed. The subsets of U which are closed are of the form $Z \cap U$ where Z is closed in X, hence we have that:

$$U = \bigcap_{Z \ni x} Z \cap U = V \cap U$$

hence $U \subset V$. However, the only closed set of X which contains U is X itself, so $V = X$ and $\{x\}$ is generic.

To show uniqueness, note that x lies in every open set of $U \subset X$, as other wise, $x \in U^c$, which is closed and thus contradicts the fact that $\{x\}$ is dense. Now suppose that U is any open affine, and $y \in X$ is a generic point not equal to x. Then y is clearly a generic point of every open affine, so $y, x \in U$ are both generic points. But then $x = y$ as every irreducible affine scheme only has one generic point, implying the claim. □

Note that if the nilradical of a ring A is prime then its vanishing locus is the whole of $\operatorname{Spec} A$, so $\operatorname{Spec} A$ contains a generic point, and is thus irreducible. In particular, $\operatorname{Spec} A$ is irreducible if and only if the nilradical is prime.

Lemma 3.2.4. *Let X be a scheme, which is not the empty scheme. Then the following are equivalent:*

a) X is irreducible

b) There exists an affine open covering $\{U_i\}$ of X such that U_i is irreducible for all i, and $U_i \cap U_j \neq \emptyset$ for all i and j.

c) Every nonempty open affine $U \subset X$ is irreducible.

Proof. Note that Lemma 3.2.2 implies that $a \Rightarrow b, c$. We show that $b \Rightarrow a$. Suppose that $X = Z_1 \cup Z_2$, then we have that since each U_i is irreducible $U_i \subset Z_1$ or Z_2. Indeed suppose otherwise, then $U_i \cap (Z_1 \cup Z_2) = (U_i \cap Z_1) \cup (U_i \cap Z_2)$ which are both closed in the subspace topology, thus $U_i \cap Z_j$ must equal Z_j for at least one j. Without loss of generality suppose that $U_i \subset Z_1$ and take any other U_j. Then $U_i \cap U_j$ is non empty and in dense in U_j. Since $U_i \subset Z_1$, we have that $U_i \cap U_j \subset Z_1 \cap U_j$, which is closed in U_j. It follows that the closure of $U_i \cap U_j$ is contained in Z_1, thus $U_j \subset Z_1$. We thus have that $\bigcup U_i = X \subset Z_1$, so $X = Z_1$, implying that X is irreducible.

For $c \Rightarrow a$, let $U \cap V$ be empty for some open affines, then $U \cup V$ is affine as it is trivially a disjoint union, and thus the coproduct in the category of schemes, and finite coproducts of affine schemes are affine by Example 2.1.3. However, irreducible spaces are connected, and $U \cup V$ is an affine open so is not irreducible contradicting c. It follows that the intersection of every open affine is non trivial, and since the open affines generate the topology on X we must have that the intersection of every open set is non empty, thus by Lemma 3.2.2 we have that X is irreducible. □

Example 3.2.2. Note that any disconnected scheme is not irreducible, we now give an example of a connected but reducible scheme. We first note that an affine scheme $\operatorname{Spec} A$ is connected if and only if it only has no nontrivial idempotents. Indeed, suppose that A has a nontrivial idempotent a, then $a \cdot a = a$. Note that $\langle a \rangle + \langle 1 - a \rangle = A$, implying that

$$\mathbb{V}(a) \cap \mathbb{V}(1-a) = \mathbb{V}(1) = \emptyset$$

Since $\langle a\rangle$ and $\langle 1-a\rangle$ are coprime, we have that $\langle a\rangle \cap \langle 1-a\rangle = \langle a\rangle \cdot \langle 1-a\rangle = \langle 0\rangle$. We thus have that:

$$\mathbb{V}(a) \cup \mathbb{V}(1-a) = \mathbb{V}(0) = \operatorname{Spec} A$$

But this then implies that $\mathbb{V}(a)^c = \mathbb{V}(1-a)$, so we have that $\operatorname{Spec} A$ is the union of two open disjoint sets, and thus disconnected. It follows by the contrapositive that if $\operatorname{Spec} A$ is connected, then there are no nontrivial idempotents.

Now suppose $\operatorname{Spec} A$ is disconnected, then there exist open sets such that $U \cap V = \emptyset$, and $U \cup V = \operatorname{Spec} A$. It follows that U and V are both also closed so $U = \mathbb{V}(I)$ and $V = \mathbb{V}(J)$ for two radical ideals I and J. Now we have that $I + J = A$, and $I \cap J = \{0\}$ so by the Chinese remainder theorem there is an isomorphism:

$$A \to A/I \times A/J$$

It follows that A is a product of two rings A/I and A/J so $\operatorname{Spec} A$ is the disjoint union of two affine schemes. It follows that $(1, 0)$ is a nontrivial idempotent of A, hence disconnected and affine implies the existence of an idempotent, and the claim follows from contradiction.

We thus wish to find a ring with no nontrivial idempotents and a nilradical which is not prime[49]. Consider $\mathbb{Z}[x]/\langle 2x\rangle$, then the nilradical contains $[0]$ but $[2]\cdot[x] = 0$ so the nilradical is not prime. It follows that $\operatorname{Spec} \mathbb{Z}[x]/\langle 2x\rangle$ is reducible, but there are no non trivial idempotents. Indeed, if $[p] \in \mathbb{Z}[x]/\langle 2x\rangle$ satisfies $[p]^2 = [p]$ then we have that $p^2 - p \in \langle 2x\rangle$, but the only way this can be true if $p^2 - p$ is divisible by $2x$ or is just actually equal to zero. This is only satisfied if $[p] = 0$ or if $p = 1$, hence $[p] = 1$. It follows that $\mathbb{Z}[x]/\langle 2x\rangle$ has no non trivial idempotents, and is thus connected but not irreducible.

We now turn to proving results regarding integral schemes. We have our first theorem of the section:

Theorem 3.2.1. *Let X be a scheme, then X is integral if and only if it is reduced and irreducible.*

Proof. Suppose that X is integral, then X is automatically reduced. Moreover, every open affine of X corresponds to $\operatorname{Spec} A$ where A is an integral domain, so every open affine is irreducible by Lemma 1.4.5. It follows by Lemma 3.2.4 that X is irreducible as well.

Now suppose that X is irreducible and reduced, then every open affine is irreducible and reduced, so we have that for each affine open $\operatorname{Spec} A$, A is an integral domain. Indeed, this implies that the generic point of A is the zero ideal, hence $\{0\}$ is a prime ideal implying that A is an integral domain. We now claim that the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ is injective whenever V is an open affine contained in U. Note that if $V \subset U$ is an open affine, then V is an open affine in X and is thus an integral scheme. Furthermore, for any affine scheme $\operatorname{Spec} A$ where A is an integral domain, the restriction maps $A \to \mathscr{O}_{\operatorname{Spec} A}(U)$ are injective, as for any cover of U by distinguished opens the localization maps $A \to A_g$ are injective. It follows that if $f \in A$ satisfies $f|_U = 0$, then $f|_{U_g} = 0$ so $f = 0$ as well. Now let W be an affine scheme such that $W \subset U$, then $f|_{W\cap V} = 0$ but this implies that $f|_W = 0$, as $f|_{W\cap V} = f|_W|_{W\cap V} = 0$. It follows that if $\{W_i, V\}$ is an open cover of U by affine schemes such that $f|_V = 0$ for then $f|_{W_i} = 0$ for all i as well. We thus have that $f = 0 \in \mathscr{O}_X(U)$ by sheaf axiom one. It follows that $\mathscr{O}_X(U)$ can be identified as a subring of the integral domain $\mathscr{O}_X(V)$, hence $\mathscr{O}_X(U)$ is an integral domain implying the claim. □

We now have the obvious corollary:

Corollary 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is integral

b) There exists an affine cover $\{U_i\}$ of X such that $U_i \cap U_j \neq \emptyset$ and U_i is integral for all i.

c) Every open affine $U \subset X$ is integral

Proof. We have that $a \Rightarrow b$ as if X is integral then X is irreducible by Theorem 3.2.1, so there exists an affine open cover of X such that each U_i is irreducible and $U_i \cap U_j \neq \emptyset$. Since every affine open is reduced we have the claim by Theorem 3.2.1 as well.

For $a \Rightarrow c$, we see that every open set $\mathscr{O}_X(U)$ is an integral domain, so if $U = \operatorname{Spec} A$ is integral, we have that $\mathscr{O}_X(U) = A$ is an integral domain implying that $\operatorname{Spec} A$ irreducible by Lemma 1.4.5. Every affine open of U is reduced, so U is reduced, and irreducible implying that U is integral again by Theorem 3.2.1.

[49] As by the affine case in Lemma 3.2.3, if the nilradical is not prime then X is not irreducible.

For $b \Rightarrow a$, note that each U_i is reduced and irreducible by Theorem 3.2.1, so by Lemma 3.2.4 we have that X irreducible, and by Lemma 3.2.1 we have that X is reduced. By Theorem 3.2.1, X is integral.

For $c \Rightarrow a$,the same argument holds. □

Example 3.2.3. We claim that $\mathbb{P}^n_A$ is integral if A is integral domain. Indeed, we have an affine open cover by:

$$U_{x_i} = A[\{x_j/x_i\}_{j\neq i}]$$

such that $U_{x_i} \cap U_{x_j} = U_{x_ix_j} \neq \emptyset$. Each of these is integral, so we have that $\mathbb{P}^n_A$ is integral as well.

Proposition 3.2.1. *Let X and Y be integral schemes over an algebraically closed field k. If X is locally of finite type then $X \times_k Y$ is an integral scheme.*

Proof. It suffices to prove that for any affine opens $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} B \subset Y$, that $A \otimes_k B$ is an integral domain. We first claim that the natural map:

$$A \longrightarrow \prod_{\mathfrak{m}\in|\operatorname{Spec} A|} A/\mathfrak{m}$$

is injective. Indeed, we can write $A \cong k[x_1,\dots,x_n]/I$ for some prime ideal I, and some $n \in \mathbb{N}$. The maximal ideals of A are then precisely the maximal ideals of $k[x_1,\dots,x_n]$ such that $I \subset \mathfrak{m}$. Suppose that $[f] \in A \mapsto (0_{\mathfrak{m}}) \in \prod_{\mathfrak{m}\in|\operatorname{Spec} A|} A/\mathfrak{m}$. Then we have that $f \in \mathfrak{m}$ for every $I \subset \mathfrak{m}$. By Hilbert's strong Nullstellensatz we have that there exists a k such that $f^k \in I$, but since I is prime we have that $f \in I$ hence $[f] = 0$.

Now note that for every $\mathfrak{m} \in |\operatorname{Spec} A|$, we have that $A/\mathfrak{m} \cong k$ as k is algebraically closed. For each $\mathfrak{m}$, let $\phi_{\mathfrak{m}}$ be the unique isomorphism $A \to k$ with kernel $\mathfrak{m}$, then we have the following chain of maps:

$$A \otimes_k B \longrightarrow A/\mathfrak{m} \otimes_k B \longrightarrow B$$

given on simple tensors by:

$$a \otimes b \longmapsto \phi_{\mathfrak{m}}([a]) \cdot b$$

Let:

$$x = \sum_i a_i \otimes b_i \qquad \text{and} \qquad y = \sum_i c_i \otimes d_i$$

be such that $x \cdot y = 0$. By the bilinearity of the tensor product, and the fact that A and B are both vector spaces, we can take $\{b_i\}$ and $\{d_i\}$ to be linearly independent sets over k. We see that for every $\mathfrak{m} \in |\operatorname{Spec} A|$:

$$x \cdot y \longmapsto (\phi_{\mathfrak{m}}([a_i])b_i) \cdot (\phi_{\mathfrak{m}}([c_i)]d_i) = 0$$

Since B is an integral domain, we have that it follows that either:

$$(\phi_{\mathfrak{m}}([a_i])b_i) = 0 \qquad \text{or} \qquad (\phi_{\mathfrak{m}}([c_i])d_i) = 0$$

Suppose the first summation is zero, then since $\{b_i\}$ is linearly independent, we have that $\phi_{\mathfrak{m}}([a_i]) = 0$ for all a_i. This implies that each $a_i \in \mathfrak{m}$ for all $\mathfrak{m}$. By the injectivity of the map $A_i \to \prod A_{\mathfrak{m}}$, it follows that each $a_i = 0 \in A$, hence:

$$x = \sum_i 0 \otimes b_i = 0$$

The same argument demonstrates that if the second sum is equal to zero, then $y = 0$, thus if $x \cdot y = 0$, we have that either $x = 0$ or $y = 0$ so $A \otimes_k B$ is an integral domain. □

3.3 Normal Schemes

Recall that if A is an integral domain, and $\eta = \langle 0 \rangle$ is the zero ideal, then $A_\eta = \mathrm{Frac}(A)$, that is the localization at the zero prime ideal is the field of fractions. This can be seen easily by noting that a), A_η is easily seen to be a field, and b), that the constructions of $\mathrm{Frac}(A)$ is identical to A_η. Further recall that if $A \subset B$, then B is an A algebra, and we say that $b \in B$ is *integral over* A, if there exists a monic polynomial $p \in A[x]$ such that $p(b) = 0$. We set the integral closure of A to be:

$$\bar{A} = \{b \in B : b \text{ is integral over } A\}$$

We now have the following definition:

Definition 3.3.1. Let A be an integral domain, then A is an **integrally closed domain** if $\bar{A} = A$, where A is being viewed as a subring of $\mathrm{Frac}(A)$[50].

We have the following example:

Example 3.3.1. The integers are an integrally closed domain. Indeed, note that $\mathrm{Frac}\,\mathbb{Z} = \mathbb{Q}$, clearly $\mathbb{Z} \subset \bar{\mathbb{Z}}$ as for any element in $a \in \mathbb{Z}$ we have that $x - a$ has a a root. Now let $a/b \in \bar{\mathbb{Z}}$, such that a and b have greatest common divisor equal to 1. Then their must exist some monic polynomial:

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_i \in \mathbb{Z}$, such that $p(a/b) = 0$. It follows that:

$$a^n/b^n + a_{n-1}a^{n-1}/b^{n-1} + \cdots a_1 a/b + a_0 = 0$$

Multiplying throughout by b^n we obtain that:

$$a^n + a_{n-1}a^{n-1}b + \cdots a_1 a b^{n-1} + a_0 b^n = 0$$

however, since $a_{n-1}a^{n-1}b + \cdots a_1 a b^{n-1} + a_0 b^n$ is divisible by b, we must have that a^n is divisible by b. Since a and b both have unique factorizations into primes, it follows that a is divisible by b, a clear contradiction, implying the claim.

As a counter example, take $\mathbb{C}[x, y]/\left\langle x^2 - y^3 \right\rangle$. We first claim that this ring is isomorphic to $\mathbb{C}[t^2, t^3]$. Indeed, consider the ring homomorphism $\mathbb{C}[x, y] \to \mathbb{C}[t^2, t^3]$ given by $x \mapsto t^3$ and $y \mapsto t^2$, then we see that $x^2 - y^3 \mapsto t^6 - t^6 = 0$, so there is a unique ring homomorphism given by $[x] \mapsto t^3$ and $[y] \mapsto t^2$. We define an inverse by sending $t^3 \mapsto x$ and $t^2 \mapsto y$, and composing with the projection. This is easily seen to be an isomorphism, and $\mathbb{C}[t^2, t^3]$ is obviously an integral domain. It's field of fractions is the localization at the zero ideal, which contains $\mathbb{C}[t, t^{-1}]$, as $t^2 \cdot t^{-3} = t^{-1}$ and $t^3 \cdot t^{-2} = t$. However, t is integral over $\mathbb{C}[t^2, t^3]$ as it is the root of the polynomial $(\mathbb{C}[t^2, t^3])[\alpha]$ given by $\alpha^2 - t^2$.

We now develop a scheme theoretic analogue of the above construction:

Definition 3.3.2. Let X be a scheme, then X is **normal** if for all $x \in X$, the stalk $(\mathscr{O}_X)_x$ is an integrally closed domain.

We have the following (non)examples:

Example 3.3.2. We claim that $\mathbb{P}^n_{\mathbb{C}}$ is a normal scheme. Indeed, the $U_i = \mathrm{Spec}(\mathbb{C}[x_0, \ldots, x_n]_{x_i})_0$ cover $\mathbb{P}^n_{\mathbb{C}}$, so suppose $x \in U_i$. Then x corresponds to a prime ideal $\mathfrak{p}$ of the ring $\mathbb{C}[\{x_j/x_i\}_{j \neq i}]$. Any polynomial ring is a unique factorization domain, and so is it's localization at $\mathfrak{p}$, so the argument that $\mathbb{Z}$ is an integrally closed domain holds pretty much verbatim for $\mathbb{C}[\{x_j/x_i\}_{j \neq i}]_{\mathfrak{p}}$, hence $\mathbb{P}^n_{\mathbb{C}}$ is normal.

As a counter example take $X = \mathrm{Spec}\,\mathbb{C}[t^2, t^3]$, and consider the maximal ideal $\mathfrak{m} = \left\langle t^2, t^3 \right\rangle$. Then the stalk at $\mathfrak{m}$ does not invert t^2 or t^3, hence the same argument as in Example 3.3.1 demonstrates that X is not a normal scheme.

We now wish to describe a process in which we take an integral scheme X and normalize it. We first need the following definition:

Definition 3.3.3. Let $f : X \to Y$ be a morphism of schemes. Then f is **dominant** if $f(X)$ is a dense subset of Y.

We need the following lemma:

Lemma 3.3.1. *Let $f : X \to Y$ be a morphism of integral schemes, then the following are equivalent:*

[50] Recall that the localization map for an integral domain is injective, so A is indeed a subring.

a) f is dominant.

b) f takes the generic point of X to the generic point of Y.

c) For every open affines $U \subset X$, $V \subset Y$, such that $f(U) \subset V$, the ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective.

d) For all $x \in X$ the map of local rings $(\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is injective.

Proof. Let $f : X \to Y$ be dominant and by Lemma 3.2.3 let $\eta_X \in X$ and $\eta_Y \in Y$ be the unique generic points. It follows that since f is dominant that $f(X) \subset Y$ is a dense subset. We first note that $f(X)$ is an irreducible subspace, as if $Z_1, Z_2 \subset f(X)$ are closed such that $Z_1 \cup Z_2 = f(X)$, then we can write $Z_1 = W_1 \cap f(X)$, and $Z_2 = W_2 \cap f(X)$, hence $f(X) = (W_1 \cup W_2) \cap f(X)$, but then $f(X) \subset W_1 \cup W_2$, so $W_1 \cup W_2 = X$ as $f(X)$ is dense. Since Y is irreducible we must have that $W_1 = Y$ or $W_2 = Y$, either way it follows that $Z_1 = f(X)$ or $Z_2 = f(X)$. It follows that $f(X)$ must contain a unique generic point point η, and this point must also be a generic point for Y, so $\eta = \eta_Y$.

We now claim that $f(\eta_X) = \eta_Y$. Note that for any subset U we have that $f(\bar{U}) \subseteq \overline{f(U)}$. Indeed, if f is continuous, then $f^{-1}(\overline{f(U)})$ is closed, and since $f(U) \subset \overline{f(U)}$, we have that $f^{-1}(f(U)) \subset f^{-1}(\overline{f(U)})$, so $U \subset \overline{f}^{-1}(\overline{f(U)})$ implying that $\bar{U} \subset f^{-1}(\overline{f(U)})$, and finally that $f(\bar{U}) \subset \overline{f(U)}$. It follows that $f(X) = f(\bar{\eta_X}) \subset \overline{f(\eta_X)}$ which must be equal to Y as $f(X)$ is dense. It follows that $f(\eta_X)$ is dense, hence $f(\eta_X)$ must be η_Y. We thus have that $a \Rightarrow b$. Clearly if f takes the generic point of X to the generic point of Y then $f(X)$ is dense in Y so $b \Rightarrow a$ as well.

To see that $b \Rightarrow c$, let $U \subset X$, and $V \subset Y$ be affine opens such that $f(U) \subset V$. Then we have an induced morphism of affine schemes $f|_U : U \to V$. Since $\mathscr{O}_X(U)$ and $\mathscr{O}_Y(V)$ are integral domains, and $\eta_X \in U$ and $\eta_Y \in V$ both correspond to the zero ideal, we have that by b), $f|_U$ must come from a ring homomorphism ϕ satisfying $\phi^{-1}(\langle 0 \rangle) = \langle 0 \rangle$, hence $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective. If this holds for all such open affine, then f must take the generic point to the generic point so $c \Rightarrow b$ as well.

For $c \Rightarrow d$, let $x \in U$ and $f(x) \in V$. Then writing $U = \operatorname{Spec} A$, and $V = \operatorname{Spec} B$, we let $x = \mathfrak{p}$, and $f(x) = \phi^{-1}(\mathfrak{p})$, where $\phi : B \to A$ is the ring homomorphism inducing $f|_U$. The map $(\mathscr{O}_Y)_{f(x)} \to (f_*\mathscr{O}_X)_{f(x)}$ is clearly injective, so it suffices to check that $(f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$ is injective. Let $U_g \subset \operatorname{Spec} B$, and take $[U_g, s]_{\phi^{-1}(\mathfrak{p})} \in (f_*\mathscr{O}_X)_{f(x)} \cong ((f|_U)_*\mathscr{O}_{\operatorname{Spec} A})_{\phi^{-1}(p)}$, then we have that this maps to $[f|_U^{-1}(U_g), s]_{\mathfrak{p}} = [U_{\phi(g)}, s]_{\mathfrak{p}}$, where $\phi(g) \neq 0$. If this is zero, then there exists some distinguished open $U_h \subset U_{\phi(g)}$ such that $s|_{U_h} = 0$, but the restriction maps on an integral affine scheme are injective, so this implies $s = 0$, hence $[U_g, s] = 0$, hence $c \Rightarrow d$ as desired. To see that $d \Rightarrow c$, it suffices to reduce to the case of affine schemes, let $\phi : B \to A$ be the ring homomorphism inducing $\operatorname{Spec} A \to \operatorname{Spec} B$. The stalk map $(\mathscr{O}_{\operatorname{Spec} B})_{\phi^{-1}(\mathfrak{p})} \to (\mathscr{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$ is then the localization of the map $B \to A_{\mathfrak{p}}$ at $\phi^{-1}(\mathfrak{p})$, which exists as $\phi(\phi^{-1}(\mathfrak{p})) \subset \mathfrak{p}$. We have the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A \arrow[d] \\
B_{\phi^{-1}(\mathfrak{p})} \arrow[r] & A_{\mathfrak{p}}
\end{tikzcd}$$

Since A and B are integral domains the vertical arrows are injective, and by hypothesis the bottom arrow is injective. It follows that if $\phi(b) = 0$, then $\phi(b)/1 \in A_{\mathfrak{p}}$ is zero, implying that $b/1 \in B_{\phi^{-1}(\mathfrak{p})}$ is zero hence $b \in B$ is zero. Therefore ϕ is injective as desired, so $c \Rightarrow d$. $\square$

We have the following definition:

Definition 3.3.4. Let X be an integral scheme, then the **normalization of X** is the scheme $\tilde{X}$, equipped with a morphism $N : \tilde{X} \to X$, such that for every normal integral scheme Z, and every dominant $f : Z \to X$ the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\exists! \tilde{f}"'] & X \\
\tilde{X} \arrow[ur, "N"'] &
\end{tikzcd}$$

where

As with every object defined this way we must show that such an object exists and is unique up to unique isomorphism. We do so now:

Theorem 3.3.1. *Let X be an integral scheme, then it's normalization, $\tilde{X}$ exists, and is unique up to unique isomorphism.*

Proof. If such an object exists it is obviously unique to up to unique isomorphism, as the morphism N we construct will be dominant so we need only check the universal property.

First consider the case where $X = \operatorname{Spec} A$ is affine, then we take $\tilde{X} = \operatorname{Spec} \tilde{A}$, where $\tilde{A}$ is the integral closure of A in $\operatorname{Frac}(A)$. This comes with a canonical injection map $A \to \tilde{A}$, so we get a dominant morphism $N : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$. Now let Z be a normal integral scheme, and $f : Z \to \operatorname{Spec} A$ be a dominant morphism, then for every affine open $U \subset Z$, we have that $f|_U : U \to \operatorname{Spec} A$, is induced by an injective ring map. The homomorphism $A \to \mathscr{O}_Z(U)$ is given by the ring homomorphism $A \to \mathscr{O}_Z(Z)$ composed with restriction to $\mathscr{O}_Z(U)$. This second map is injective, and since the composition is injective, we must have that $A \to \mathscr{O}_Z(Z)$ is injective as well.

We want to show that the ring homomorphism $A \to \mathscr{O}_Z(Z)$ factors through the inclusion $A \to \tilde{A}$. We first show that $\mathscr{O}_Z(Z)$ is integrally closed. Let $a \in \operatorname{Frac}(\mathscr{O}_Z(Z))$ be integral over $\mathscr{O}_Z(Z)$, and let $\operatorname{Spec} B \subset Z$ be an affine open. Then since Z is integral, we have that $\mathscr{O}_Z(Z) \subset B$[51], so $\operatorname{Frac}(\mathscr{O}_Z(Z)) \subset \operatorname{Frac}(B)$. It follows that $a \in \operatorname{Frac}(B)$, and that a is integral over B. Let $I = \{b \in B : ab \in B\}$, if $I = B$, then $a \in B$ so we are done. If $I \neq B$, then $I \subset \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset B$. We see that a is integral over $B_{\mathfrak{p}}$, and thus $a \in B_{\mathfrak{p}}$ as Z is normal. However, there then exists an $s \in B \setminus \mathfrak{p}$ such that $s \cdot a \in B$, implying that $s \in I$, contradicting the fact that $s \in B \setminus \mathfrak{p}$, so $I = B$. It follows that $a \in B = \mathscr{O}_Z(V)$. Cover Z with affine opens V_i, and the same argument shows that $b \in \mathscr{O}_Z(V_i)$ for all i. For all affine opens $V_{ijk} \subset V_i \cap V_j$, we have that we can identify $\mathscr{O}_Z(V_i)$ and $\mathscr{O}_Z(V_j)$ as subrings of $\mathscr{O}_Z(V_{ijk})$, so $b \in \mathscr{O}_Z(V_i)$ and $b \in \mathscr{O}_Z(V_j)$ both map to the same element in $\mathscr{O}_Z(V_{ijk})$[52]. Since the affine opens form a basis for the topology on Z, and thus determine a sheaf on a base, it follows that $b \in \mathscr{O}_Z(Z)$ so $\mathscr{O}_Z(Z)$ is indeed integrally closed.

It follows that since A injects into $\mathscr{O}_Z(Z)$, and $\mathscr{O}_Z(Z)$ is integrally closed, that $\tilde{A}$ injects into $\mathscr{O}_Z(Z)$ as well, thus we have a morphism $\tilde{A} \to Z$. Since A injects into $\tilde{A}$ we clearly have the following commutative diagram in the category of rings:

$$\begin{tikzcd} \mathscr{O}_Z(Z) & & A \arrow[ll] \arrow[dll] \\ & & \\ \tilde{A} \arrow[uu] & & \end{tikzcd}$$

which yields the following commutative diagram in the category of schemes:

$$\begin{tikzcd} Z \arrow[rr] \arrow[dd] & & \operatorname{Spec} A \\ & & \\ \operatorname{Spec} \tilde{A} \arrow[urr] & & \end{tikzcd}$$

implying the result for affine integral schemes.

Now let X be an integral scheme, and $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover for X. Then we have isomorphisms $\beta_{ij} : U_{ij} \subset \operatorname{Spec} A_i \to \operatorname{Spec} A_j$ which agree on triple overlaps. For each i, set $\tilde{U}_i = \operatorname{Spec} \tilde{A}_i$, and let $N_i : \operatorname{Spec} \tilde{A}_i \to \operatorname{Spec} A_i$ be the normalization map. Finally set $\tilde{U}_{ij} \subset \operatorname{Spec} \tilde{A}_i$ to be $N_i^{-1}(U_{ij})$. We claim that $\tilde{U}_{ij}$ satisfies the universal property of the normalization of U_{ij}. Indeed, we have a morphism $N_i|_{\tilde{U}_{ij}} : \tilde{U}_{ij} \to U_{ij}$ which must be dominant as it sends the unique generic point of $\tilde{U}_{ij}$ to U_{ij}. Now let $f : Z \to U_{ij}$ be any dominant morphism from an integrally closed scheme Z, then the composition $\imath \circ f : Z \to \operatorname{Spec} A_i$ is dominant, and there is a unique morphism $g : Z \to \operatorname{Spec} \tilde{A}_i$ such that $N \circ g = \imath \circ f$. But this implies that $g(Z) \subset \tilde{U}_{ij}$, so g factors through the inclusion map$\tilde{U}_{ij} \to \operatorname{Spec} \tilde{A}_i$ implying that $\tilde{U}_{ij}$ is indeed the normalization of U_{ij}.

[51]Via the inclusion map.

[52]If one is unconvinced, then they can write out the restriction maps themselves, and find that this must be true by examining the induced injective maps $\operatorname{Frac}(\mathscr{O}_Z(Z)) \to \operatorname{Frac}(\mathscr{O}_Z(V_i))$.

We want to show that there exist scheme isomorphisms $\phi_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ji}$ which agree on triple overlaps. Fix the notation $N_i|_{\tilde{U}_{ij}} = N_{ij}$, and note that we have a dominant morphism $\beta_{ij} \circ N_{ij} : \tilde{U}_{ij} \to U_{ji}$. It follows that there is a unique morphism ϕ_{ij} such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_{ij} \arrow[r, "\beta_{ij}\circ N_{ij}"] \arrow[d, "\phi_{ij}"'] & U_{ji} \\
\tilde{U}_{ji} \arrow[ur, "N_{ji}"'] &
\end{tikzcd}$$

Similarly, we have a morphism $\phi_{ji} : \tilde{U}_{ji} \to \tilde{U}_{ji}$ such that a similar diagram commutes. We thus claim that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_{ij} \arrow[r, "N_{ij}"] \arrow[d, "\phi_{ji}\circ\phi_{ij}"'] & U_{ij} \\
\tilde{U}_{ij} \arrow[ur, "N_{ij}"'] &
\end{tikzcd}$$

Indeed, note that $N_{ij} \circ \phi_{ji} = \beta_{ji} \circ N_{ji}$, so:

$$\begin{aligned}
N_{ij} \circ \phi_{ji} \circ \phi_{ij} =& \beta_{ji} \circ N_{ji} \circ \phi_{ij} \\
=& \beta_{ji} \circ \beta_{ij} \circ N_{ij} \\
=& N_{ij}
\end{aligned}$$

so the diagram commutes. But the identity map also makes this diagram commutes so $\phi_{ji} \circ \phi_{ij} = \mathrm{Id}$, and similarly $\phi_{ij} \circ \phi_{ji}$ is the identity, implying that they are isomorphisms. It is easily seen by a similar argument that these morphisms agree on triple overlaps, as the β_{ij} agree on triple overlaps so the $\tilde{U}_i$ glue together to form an integral normal scheme $\tilde{X}$.

It follows that the N_i then also glue together to form a dominant morphism $N : \tilde{X} \to X$, such that $N|_{\tilde{U}_i} = N_i$. Given $f : Z \to X$ with f dominant and Z integral normal, we obtain an open cover of Z by $V_i = f^{-1}(U_i)$. Each of these schemes is normal integral, and the restriction is clearly dominant, so we obtain gives unique morphisms $V_i \to \tilde{U}_i$ which which clearly agree on $V_i \cap V_j$. These maps then glue to yield a unique dominant morphism $Z \to \tilde{X}$ such that the relevant diagram commutes, so $\tilde{X}$ is indeed the normalization of X. □

3.4 Noetherian Schemes

We now turn to defining another important class of schemes, called Noetherian schemes, which again have an interesting interplay between the algebraic properties of their structure sheaf, and the topological properties of the total space. To begin, we review some commutative algebra:

Definition 3.4.1. Let A be a commutative ring, then A is **Noetherian** if every strictly increasing chain of ideals:

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

terminates. In other words, there exists some m such that $I_m = I_{m+k}$ for all $k \geq 0$.

Example 3.4.1. Any field is obviously Noetherian, any finite ring is also obviously Noetherian. Mildly more interestingly, $\mathbb{Z}$ is Noetherian. Indeed, every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$, so suppose we have the following infinite chain of ideals:

$$\langle n_1 \rangle \subset \langle n_2 \rangle \subset \cdots$$

We see that if $\langle n_1 \rangle \subset \langle n_2 \rangle$, then $n_1 \in \langle n_2 \rangle$, hence $n_1 = a \cdot n_2$ for some $a \in \mathbb{Z}$. It follows that n_2 divides n_1. If this is chain is infinite, then n_1 has infinitely many divisors, which is absurd implying the claim.

We have the following useful lemma which makes the example above a bit more immediate:

Lemma 3.4.1. *Let A be a ring, then A is Noetherian if and only if every ideal of A is finitely generated.*

Proof. Suppose that every ideal of A is finitely generated, and that:

$$I_1 \subset I_2 \subset \cdots$$

is a strictly increasing chain of ideals, and let:

$$I = \bigcup_i I_i$$

We claim that I is an ideal (no generating set needed!). Indeed, we see that if $0 \in I$, and that if $a, b \in I$ then $a \in I_i$ and $b \in I_j$ for some i and j. Without loss of generality suppose that $i \leq j$, then $a_i \in I_j$ so $a + b \in I_j$ hence $a + b \in I$. We see that I clearly contains all of it's inverses so I is a subgroup. Now let $a \in I$, and $b \in A$, then $a \in I_i$ for some i, and $a \cdot b \in I_i$ so $a \cdot b \in I$ as well implying that I is an ideal.

Since I is finitely generated, let $I = \langle a_1, \ldots, a_n \rangle$ for some $n \in \mathbb{Z}$. We have that each a_i lies in some I_{j_i} for some j_i, so let $j_k = \max(j_1, \cdots j_n)$, then since $I_{j_i} \subset I_{j_k}$ for all $i \in \{1, \ldots, n\}$ we must have that I_{j_k} contains each a_i. Let $j_k = m$, then it follows that $I \subset I_m$, so $I = I_m$ as $I_m \subset I$ by definition. For any $l \geq m$, we have that $I_m = I \subset I_l$ so the chain clearly terminates, and A is Noetherian.

Conversely, let $I \subset A$ be any ideal with minimal generating set $\{a_i\}_{i \in J}$ where J is a totally ordered set that is not finite. For any $j \in J$ we set $I_j = \{a_i\}_{i \leq j}$, and note that for any $j < k$, we have that $I_j \subset I_k$ and that this inclusion is strict. Indeed, if $I_j = I_k$ then for all $j < l \leq k$, we have that $a_l \in I_j$, implying that $a_l = \sum_{i \leq j} b_i a_i$ hence a_l is not a generating element of I, a contradiction, so $I_j \subset I_k$. We can label the initial segment of J with natural numbers regardless of it's cardinality, hence:

$$I_1 \subset I_2 \subset \cdots$$

is an infinite strictly increasing chain of ideals, so A is not Noetherian. The claim then follows by the contrapositive. □

We also have the following collection results:

Lemma 3.4.2. *Let A be a Noetherian ring then:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Noetherian.

b) If $I \subset A$ is an ideal then A/I is Noetherian.

Proof. Let $I_S \subset S^{-1}A$ be an ideal, then we first claim that

$$I_S = S^{-1}I := \left\{ \frac{a}{s} : a \in I, s \in S \right\}$$

for some $I \subset A$. In particular, let $I = \pi^{-1}(I_S)$ where $\pi : A \to S^{-1}A$ is the localization map. Indeed, we have that:

$$S^{-1}\pi^{-1}(I_S) = \left\{ \frac{a}{s} : a \in \pi^{-1}(I_S), s \in S \right\}$$

Suppose that $a/s \in I_S$, then we have that $a/1 \in I_S$, so $a \in \pi^{-1}(I_S)$. It follows that $a/s \in S^{-1}\pi^{-1}$ giving us one inclusion. Now suppose that $a/s \in S^{-1}\pi^{-1}$, then $a \in \pi^{-1}$, so $a/1 \in I_S$ by definition. It follows that $a/s \in I$, by $a/1 \cdot 1/s = a/s$, hence $I = S^{-1}\pi^{-1}(I_S)$ implying the claim.

Since A is Noetherian, it follows that $\pi^{-1}(I_S)$ is finitely generated. In particular, since any of ideal of $S^{-1}A$ is generated by elements of the form $a/1$ as $1/s$ is invertible, we clearly see that $S^{-1}I$ is finitely generated as well. By the above paragraph, it follows that I_S is finitely generated, hence $S^{-1}A$ is Noetherian by Lemma 3.4.1 implying b).

Now let $I \subset A$ be an ideal. We see that if J is an ideal of A/I, then J is of the form $\pi(\pi^{-1}(J))$ as the quotient map $\pi : A \to A/J$ is surjective. We see that $\pi^{-1}(J)$ is finitely generated as A is Noetherian, so J itself must be finitely generated as well. Indeed suppose that $\{a_1, \ldots, a_n\}$ are generating elements of $\pi^{-1}(J)$, and let $[j] \in J$. We see that $j \in \pi^{-1}(J)$ can be written as $\sum_i b_i a_i$, hence $[j] = \sum_i [b_i][a_i]$, so $\{[a_1], \ldots, [a_n]\}$ generates J. It follows that every ideal of A/I is finitely generated, hence A/I is Noetherian by Lemma 3.4.1 implying b). □

The following results are some of the most famous results in commutative algebra, the first of which is known as the Hilbert Basis theorem.

Theorem 3.4.1. *Let A be a ring, then $A[x_1, \ldots, x_n]$ is Noetherian if and only if A is Noetherian.*

Proof. We see that if $A[x_1, \ldots, x_n]$ is Noetherian, then $A[x_1, \ldots, x_n]/\langle x_1, \ldots, x_n\rangle \cong A$ must be Noetherian by Lemma 3.4.3.

Now suppose that A is Noetherian, since we trivially have that $A[x, y] \cong (A[x])[y]$, it suffices by an induction argument to show that $A[x]$ is Noetherian. Let $I \subset A[x]$ be an ideal, we will show that I is finitely generated. We have a partial order on I, by writing:

$$f = a_n x^n + \cdots + a_1 x^1 + a_0 \qquad g = b_k x^k + \cdots + b_1 x^1 + b_0$$

and saying that $f \leq g$ if and only if $n \leq k$, we call n and k the degree of f and g respectively, and write it as $\deg f$. Choose an element of least degree $f_0 \in I$, i.e. an element f_0 such that there is no g in I satisfying $\deg g < \deg f$. If $\langle f_0\rangle = I$ we are done, if not, then we choose an element f_2 in $I \setminus \langle f_0\rangle$ of least degree. We perform this recursively obtaining a sequence[53] $\langle f_0, f_1, \ldots\rangle \subset I$. For each f_i, let $a_{\deg f_i}$ be the leading coefficient of f_i, and consider the ideal $J = \langle a_{\deg f_0}, \ldots\rangle \subset A$. Then, since A is Noetherian, we know that the sequence:

$$\langle a_{\deg f_0}\rangle \subset \langle a_{\deg f_0}, a_{\deg f_1}\rangle \subset \cdots$$

terminates, so for some $m \geq 0$, we have that this chain must terminate with $\langle a_{\deg f_0}, \ldots, a_{\deg f_m}\rangle$ implying that $J = \langle a_{\deg f_0}, \ldots, a_{\deg f_m}\rangle$. We claim that $I = \langle f_0, \ldots, f_m\rangle$. Suppose otherwise, then by construction $f_{m+1} \notin \langle f_0, \ldots, f_m\rangle$, but $a_{\deg f_{m+1}} \in J$, so we can write:

$$a_{\deg f_{m+1}} = \sum_{i=0}^{m} a_{\deg f_i} b_i$$

for some $b_i \in A$. Define g by:

$$g = \sum_i b_i f_i x^{\deg f_{m+1} - \deg f_i}$$

Note that this clearly lies in $\langle f_0, \ldots, f_m\rangle$, but this element as the same degree as f_{m+1} with $a_{\deg g} = a_{\deg f_{m+1}}$. We thus see that $f_{m+1} - g$ has degree strictly less than f_{m+1}, and that $f_{m+1} - g \notin \langle f_0, \ldots, f_m\rangle$, so $f_{m+1} - g$ is the minimal element of $I \setminus \langle f_0, \ldots, f_m\rangle$, a contradiction. It follows that $I = \langle f_0, \ldots, f_m\rangle$, so every ideal of $A[x]$ is finitely generated and thus by Lemma 3.4.1 we have that $A[x]$ is Noetherian. ☐

We now have the following obvious corollary:

Corollary 3.4.1. *Let A be a Noetherian and B be any finitely generated A algebra, then B is Noetherian.*

To prove our second famous result, we need to extend the idea of a Noetherian ring to modules.

Definition 3.4.2. Let M be an A module, then M is Noetherian if for every strictly increasing chain of submodules:

$$N_1 \subset N_2 \subset \cdots$$

terminates.

We prove the following analogue Lemma 3.4.1

Lemma 3.4.3. *Let M be an A module, then the following hold:*

a) *M is Noetherian if and only if every submodule is finitely generated.*

b) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then M_2 is Noetherian if and only if M_1 and M_3 are.*

c) *If A is Noetherian, and M is finitely generated then M is Noetherian.*

[53] This is equivalent to using the axiom of dependent choice.

Proof. We begin with a). Suppose that every submodule of M finitely generated, and consider the following sequence of submodules:

$$N_1 \subset N_2 \subset \cdots$$

Let:

$$N' = \bigcup_i N_i$$

Then this has finitely many generators $(m_1, \ldots, m_n)$ for some n, and each must lie in N_i for some i, so choose the largest such i, and call it k. We have that $(m_1, \ldots, m_n) \subset N_k$ essentially by construction, $N' = N_k$. It follows that for any $l \geq k$, we have that $N_l \subset N' = N_k$, so for all $l \geq k$ we have that $N_l = N_k$, so M is Noetherian.

Now suppose that M is Noetherian, and let N be a submodule which is not finitely generated. Let $\{m_i\}_{i\in I}$ be the minimal generating set of N where I is a totally ordered set of any cardinality. For any $j \in I$ let N_j be the submodule generated by the elements $\{m_i\}_{i\leq j}$, then for any $k < j$, we have that $N_k \subseteq N_j$. If $N_k = N_j$ then for each $k < l \leq j$, we have that m_l can be written as a linear combination of $\{m_i\}_{i\leq k}$, hence m_l is not a generating element. It follows that N_k is a strict subset of N_j for each $k < j$. Since we can write the initial segment of any totally ordered set as the natural numbers, we have that:

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

is a strictly increasing chain of ideals which does not terminate, hence M is not Noetherian, a contradiction. It follows that every submodule of M (including M itself) must be finitely generated, thus we have a).

Now suppose that we have an exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is an exact sequence of A modules. If M_2 is Noetherian, then we have that $M_3 \cong M_2/\ker g$ so M_3 is Noetherian, as every submodule of M_3 must be finitely generated. Moreover, every submodule of M_1 is a submodule of M_2, so we have that every submodule of M_1 is finitely generated hence M_1 is also Noetherian.

Let M_1 and M_3 be Noetherian modules, and consider the following chain of strictly increasing submodules of M_2:

$$N_1 \subset N_2 \subset \cdots$$

Then we have that:

$$f^{-1}(N_1) \subset f^1(N_2) \cdots \qquad \text{and} \qquad g(N_1) \subset g_2(N_2) \subset \cdots$$

are are strictly increasing chains of ideals in M_1 and M_3 respectively. Since M_1 and M_3 it follows that there exist n_1 and n_3 such that $f^{-1}(N_{n_1})$ and $g(N_{n_3})$ make the above chains terminate. Without loss of generality let $n_3 > n_2$, and denote n_3 by n. Then we claim that for all $k > n$, $N_k = N_n$. Indeed consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}(N_n) \arrow[r, "f"] \arrow[d] & N_n \arrow[r, "g"] \arrow[d] & g(N_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^{-1}(N_k) \arrow[r, "f"] & N_k \arrow[r, "g"] & g(N_k) \arrow[r] & 0
\end{tikzcd}$$

The vertical arrows are inclusion maps, so the leftmost and rightmost arrows are the identities. We want to show that the middle arrow is the identity as well, and it suffices to show that $N_k \subset N_n$. Let $m \in N_k$, and consider $g(m) \in g(N_k)$. Since the right most arrow is the identity, we have that $g(m) \in g(N_n)$, since g is surjective there exists an element $l \in N_n$ which maps to $g(m)$. Let $\imath : N_n \to N_k$ denote the inclusion map, then since:

$$g(\imath(l)) = g(m)$$

It follows that $\imath(l) - m \in \ker g$, but the kernel of g is the image of f, so we have that there exists an $\eta \in f^{-1}(N_k)$ such that $f(\eta) = \imath(l) - m$. Since the left most arrow is the identity, we have that $\eta \in f^{-1}(N_k)$, so $f(\eta) \in N_n$. It follows that $\imath(l) - m \in N_n \subset N_l$ hence $m \in N_n$ as well so $N_k = N_n$, and the middle arrow is the identity. We thus have that N_n makes the chain terminate implying b).

To prove c), let A be a Noetherian ring, and suppose that M is finitely generated. Then M is a quotient of the free module A^n for some n, and so it suffices to check that A^n is a Noetherian A-module. Note that every submodule of A is by definition of an ideal, as it is a subgroup and swallows multiplication, so A is Noetherian as a module over itself as well. We proceed by induction, suppose that A^n is Noetherian, then we have the following short exact sequence:

$$0 \longrightarrow A \xrightarrow{f} A^{n+1} \xrightarrow{g} A^n \longrightarrow 0$$

Since A^n is Noetherian and A are Noetherian, it follows by b) that A^{n+1} is Noetherian implying c) as desired.

□

We are now in position to prove the following result, known as the Artin-Tate lemma:

Theorem 3.4.2. *Let $A \subset B \subset C$ be rings where A is Noetherian, C is finitely generated over A, and C is a finite B module. Then B is finitely generated as an A algebra.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the generators of C as an A algebra, and let $\{y_1, \ldots, y_m\}$ be the generators of C as a B module. Then we have that for some $b_{ij}, b_{ijk} \in B$ that:

$$x_i = \sum_j b_{ij} y_j \qquad \text{and} \qquad y_i y_j = \sum_k b_{ijk} y_k \tag{3.4.1}$$

Let B_0 be the A algebra generated by $\{b_{ij}, b_{ijk}\}$. By Corollary 3.4.1, we have that B_0 is Noetherian, and we have that $A \subseteq B_0 \subseteq B$.

It is clear that C is a B_0-algebra, so we claim that C is finite over B_0, i.e. is a finitely generated B_0 module. Every element $c \in C$ can be written as:

$$c = \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

By making repeated use of the equations in (2.4.1) we can rewrite this in terms of a linear combination of y_k and elements of B_0, hence C is a finitely generated B_0 module. It follows from Lemma 3.4.2 part c) that C is a Noetherian B_0 module, so every submodule of C is finitely generated. We thus have that B is a finitely generated B_0 module, and thus a finitely generated A algebra as desired. □

After our brief detour into commutative algebra, we are now ready to dive back into scheme theory. It should be no surprise that the class of schemes we are about to study are intimately related to Noetherian rings. We begin with the following definition:

Definition 3.4.3. Let X be a topological space, then X is Noetherian if every decreasing sequence of closed subsets:

$$Y_1 \supset Y_2 \supset \cdots$$

terminates. In other words there exits an integer m such that for all $k \geq m$ we have $Y_m = Y_k$.

Example 3.4.2. Let A be a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space. Indeed, any descending sequence of closed subsets can be written uniquely as a sequence of the vanishing locus of radical ideals I_k:

$$\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$$

This then corresponds to an increasing sequence of ideals:

$$I_1 \subset I_2 \subset \cdots$$

which must terminate as A is Noetherian. It follows that the chain $\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$ must terminate as well.

Note that not every affine scheme which is a Noetherian topological space comes from a Noetherian ring. Indeed consider the infinite polynomial ring $A = k[x_1, x_2, \dots]/\left\langle x_1^2, x_2^2, \dots \right\rangle$ over a field k. Every prime ideal must contain the nilpotents $[x_i]$ for all i, so the only prime is given by $\mathfrak{p} = \langle [x_1], [x_2], \dots \rangle$ implying that $\operatorname{Spec} A$ is a single point and thus Noetherian. It clear that A is not Noetherian as $\mathfrak{p}$ is not finitely generated.

Lemma 3.4.4. *Let X be a Noetherian topological space, then every non empty closed subset $Z \subset X$ can be expressed uniquely as $Z = Z_1 \cup \cdots \cup Z_n$ where each Z_n is an irreducible closed subspace, and for all i, j we have that $Z_i \not\subset Z_j$.*

Proof. Suppose there exists a closed subset Y_1 that cannot be expressed as a finite union of irreducible closed subspaces. If Y_1 contains another such closed subset Y_2, then we have that $Y_1 \supset Y_2$. We can repeat this process ad infinitum, but since X is Noetherian, we must have that this chain eventually terminates for some Y_r. Now since this chain terminates, it follows that every proper closed subset of Y_r can be written as the finite union of irreducible closed subspaces. We see that Y_r is not irreducible as other wise it is trivially a finite of union of irreducible closed subspaces, hence $Y_r = W_1 \cup W_2$ for proper closed subsets of Y_r. However, W_1 and W_2 can be written as a finite union of irreducible closed subsets, a contradiction. It follows that every closed subset of X can thus be written as a finite union of irreducible closed subsets, and by discarding those that satisfy $W_i \subset W_j$, we have that every closed subset of X can be written as a finite union of irreducible closed subspaces none of which fully contain each other.

To show uniqueness, suppose that:

$$Z = Z_1 \cup \cdots \cup Z_n = Y_1 \cup \cdots \cup Y_m$$

where Z_i and Y_j are irreducible closed subspaces none of which contain the other. It follows that for any $Z_1 \subset Y_1 \cup \cdots \cup Y_m$, so $Z_1 = (Y_1 \cap Z_1) \cup \cdots \cup (Y_m \cap Z_1)$, but then for some i we have that $Z_1 = Y_i \cap Z_1$ as Z_1 is irreducible. It follows that $Z_1 \subset Y_i$, and similarly for some j we have that $Y_i \subset Z_j$, but then $j = 1$ as we have that $Z_1 \subset Y_i \subset Z_j$ and Z_1 is only contained in Z_1. It follows that $Y_i = Z_1$, so repeating this process for all $1 \leq i \leq n$ we have that the lists are the same, implying the claim. □

Definition 3.4.4. Let X be a scheme, then X is **locally Noetherian** if there exists a cover $\{U_i\}$ of X by affine schemes such that each U_i is the spectrum of a Noetherian ring. Moreover, X is **Noetherian** if it can be covered by finitely many such affine schemes.

Example 3.4.3. Lemma 3.4.3 demonstrates that every affine scheme $\operatorname{Spec} A$ where A is Noetherian is Noetherian.

Lemma 3.4.5. *A topological space X is Noetherian if and only if every subspace of X is quasi-compact. In particular, X is quasi-compact, and every subspace of X is Noetherian.*

Proof. Let $Y \subset X$ be a subset equipped with subspace topology, and $\{U_i \cap Y\}_{i \in I}$ be an open cover of Y. Consider the set:

$$\mathscr{U} = \{\text{finite unions of elements in } \{U_i\}\}$$

and equip this set with the partial order given by $V \leq W$ if and only if $V \subset W$. Consider an ascending chain of elements in $\mathscr{U}$:

$$V_1 \subset V_2 \subset \cdots$$

then we obtain the descending chain of closed subsets of X:

$$V_1^c \supset V_2^c \supset \cdots$$

which must terminate for some m as X is Noetherian. By Zorn's lemma, there must then be a maximal element of $\mathscr{U}$, call it W. Then we have that for some $\{i_1, \dots, i_n\}$

$$W = U_{i_1} \cup \cdots \cup U_{i_n}$$

and moreover that:

$$W \cap Y = (U_{i_1} \cap Y) \cup \cdots \cup (U_{i_n} \cap Y)$$

Suppose that $Y \not\subset W$, then there is a $y \in Y$ such that $y \notin W$. However, $\{U_i \cap\}_{i \in I}$ covers Y, so for some $k \in I$, we have that $y \in U_k$. It follows that $W \subset W \cup U_k$, contradicting the fact that W is maximal, hence $Y \subset W$. Therefore, $Y = W \cap Y$, and the set $\{U_{i_j} \cap Y\}_{j=1}^{n}$ is a finite subcover, so Y is quasi-compact. In particular, we have that X is quasi-compact.

Now suppose that every subspace of X is compact, and let:

$$V_1 \supset V_2 \supset \cdots$$

be a descending chain of closed subsets of X. Then we obtain an ascending chain of open sets:

$$U_1 \subset U_2 \subset \cdots$$

by letting $U_i = V_1^c$. Consider the open subspace:

$$U = \bigcup_{i=1}^{\infty} U_i$$

which has an open cover given by $\{U_i\}_{i \in \mathbb{N}}$. Since U is quasi-compact, this subspace has an open cover given $U_{i_1} \cup \cdots U_{i_n}$ for some $\{i_1, \cdots i_n\} \subset \mathbb{N}$. Via reordering we can assume that $U_{i_1} \subset \cdots \subset U_{i_n}$, so $U = U_{i_n}$. We claim that the ascending chain stabilize with U_{i_n}. Indeed suppose that $m > i_n$, then $U_{i_n} \subset U_m$, however, by construction, $U_m \in U$, so $U_m = U_{i_n}$. By taking compliments again we obtain that the descending chain of closed subsets:

$$V_1 \supset V_2 \supset \cdots$$

stabilizes so X is Noetherian.

Now finally let Y be a subspace of a Noetherian topological space X. Let $W \subset Y$, then the subspace topology on W induced from Y is the same as the one induced from X. That is, $U \subset W$ is open in the subspace topology if and only $U = Y \cap V$ for some open set $V \subset Y$. However, V is open in Y if and only if $V = X \cap Z$ for some Z open in X. It follows that U is open in W if and only if $U = Y \cap V = X \cap Y \cap Z = X \cap Z$, hence the topologies are equivalent. Since W is quasi-compact as a subspace of X, it follows that W is quasi-compact as subspace Y, hence Y must be Noetherian by argument above. □

Proposition 3.4.1. *Let X be a Noetherian scheme, then X is a Noetherian topological space*

Proof. By Example 3.4.2 we have that X is the union of finitely many Noetherian topological spaces, so it suffices to prove that any such topological space is Noetherian. Let $\{U_i\}_{i \in I}$ be the finite cover of X by Noetherian affine schemes, and suppose that:

$$Y_1 \supset Y_2 \supset \cdots$$

is a descending chain of closed subsets. Then for each i we have that there exists an m_i such that the following chain terminates at m_i:

$$Y_1 \cap U_i \supset Y_2 \cap U_i \supset \cdots \supset Y_{m_i} \cap U_i$$

Take $\max\{m_i\}_{i \in I}$ which exists as I is a finite set, and let m be the maximum number. Then we claim that for any $k \geq m$ we have $Y_m = Y_k$. Indeed, we can write:

$$Y_m = \bigcup_i Y_m \cap U_i = \bigcup_i Y_k \cap U_i = Y_k \tag{3.4.2}$$

as the $\{U_i\}$ cover X, implying the claim. □

As Example 3.4.2 shows, the converse does not hold. We continue to prove topological properties of Noetherian schemes:

Lemma 3.4.6. *Let X be a Noetherian scheme, then X has finitely many irreducible components. In particular, X a finite number of connected components, each of which is the finite union of irreducible components.*

Proof. Note that any irreducible component is closed. Indeed, $Z \subset X$ is irreducible then clearly so is $\bar{Z}$, so it follows that Z is maximal that $Z = \bar{Z}$ as Z is by definition a subset of $\bar{Z}$. Since X is a Noetherian topological space by Proposition 3.4.1, and by Lemma 3.4.4 we have that every closed subset of X can be written as the finite union of irreducible closed subsets it follows that:

$$X = Z_1 \cup \cdots \cup Z_n$$

where each Z_i irreducible. Let $\{Y_i\}$ be the set of irreducible components, then since each Z_i must be contained in one of these irreducible components, it follows that:

$$X = \bigcup_i Y_i$$

However, this is a decomposition of X into irreducible closed subspaces, each of which is not contained in the other as they are all maximal. It follows that each Y_i must be equal to some Z_j for some i and j by the uniqueness part of Lemma 3.4.4, hence there can only be finitely many irreducible components.

Let $\{X_i\}$ be the set of connected components, then since each X_i is closed we have that by Lemma 3.4.4:

$$X_i = Z_{1_i} \cup \cdots \cup Z_{n_i}$$

for irreducible closed subsets of X_i. It follows that:

$$X = \bigcup_i Z_{1_i} \cup \cdots \cup Z_{n_i}$$

which must be a finite union as X is Noetherian, implying there are only finitely many X_i. It follows that each X_i must be a finite union of irreducible components Y_i of X by uniqueness of the decomposition of X into irreducible components, again by Lemma 3.4.4, implying the claim. □

It turns out we can check the locally Noetherian condition affine locally (hence the name):

Proposition 3.4.2. *Let X be a scheme, then X is locally Noetherian if and only if every open affine is Noetherian.*

Proof. If every open affine is Noetherian, then clearly X is locally Noetherian.

Suppose that $\{U_i = \operatorname{Spec} A_i\}$ is an affine open cover of X with A_i Noetherian for all i, and $V = \operatorname{Spec} B \subset X$ be any open affine. Then we obtain an open cover of V by $\{V \cap U_i\}$, and for each of these there is an open cover by distinguished opens $U_f \subset \operatorname{Spec} B$ and $U_g \subset \operatorname{Spec} A_i$. Since the schemes $U_i \cap V \subset \operatorname{Spec} A_i$ and $U_i \cap V \subset \operatorname{Spec} B$ are clearly isomorphic (just take the identity), it follows that V admits a cover of distinguished opens all of which are Noetherian schemes. In particular, we have that there exists a finite set of elements $\{f_i\}$ of B which generate the unit ideal $\langle 1 \rangle$ such that for all i B_{f_i} is a Noetherian ring.

Now let $I \subset B$ be an ideal, and let $\pi_i : A \to A_{f_i}$ be the localization map. If I_{f_i} is the localized ideal in B_{f_i} then we claim that:

$$I = \bigcap_i \pi_i^{-1}(I_{f_i})$$

For each i we have that $I \subset \pi_i^{-1}(\pi_i(I))$ so it follows that $I \subset \bigcap_i \pi_i^{-1}(I_{f_i})$. Now let $b \in \bigcap_i \pi_i^{-1}(I_{f_i})$, then for each i we have that $\pi_i(b) \in I_{f_i}$, so we have that for some $a_i \in I$, and some integer m_i:

$$\frac{b}{1} = \frac{a_i}{f_i^{m_i}}$$

It follows that there exists an M_i such that $f_i^{M_i} b \in I$. Let M be the maximum of all such M, then since $\langle \{f_i\} \rangle = \langle 1 \rangle$, we have that $\langle \{f_i^M\} \rangle = \langle 1 \rangle$ so we there exist c_i in B such that:

$$1 = \sum_i c_i f_i^M$$

hence:

$$b = \sum_i c_i f_i^M b \in I$$

Now suppose that:

$$I_1 \subset I_2 \subset \cdots$$

is an increasing chain of ideals, then for each i we have that:

$$I_{1_{f_i}} \subset I_{2_{f_i}} \subset \cdots$$

terminates for some m_{f_i}. Let m be the maximum of all such m_{f_i}, then for all $k > m$ and all i, we have that $I_{k_{f_i}} = I_{m_{f_i}}$. It follows that for all $k > m$ we have that:

$$I_k = \bigcap_i \pi_i^{-1}(I_{k_{f_i}}) = \bigcap_i \pi_i^{-1}(I_{m_{k_{f_i}}}) = I_m$$

so the chain in B terminates with I_m, implying that B is Noetherian, and that V is Noetherian. □

We have the following corollary:

Corollary 3.4.2. *Let X be a scheme, then X is Noetherian if and only if it is quasi-compact, and for every affine open $\mathscr{O}_X(U)$ is a Noetherian ring.*

The condition that X is Noetherian is in a sense a finiteness condition that allows us to prove some striking results. Often times we will restrict to the case where we deal with Noetherian or locally Noetherian schemes, as they are easier to work with, and the condition is actually quite a reasonable one. As an example, note that we showed that X is a reduced scheme if and only if all of it's stalks have no nontrivial nilpotents. The astute reader will recognize that we did not have a similar equivalent condition for a scheme to be integral. As the following theorem shows, we can deduce such a result if we work with sufficiently nice schemes:

Theorem 3.4.3. *Let X be a connected and Noetherian scheme, then X is integral if and only if the stalk $(\mathscr{O}_X)_x$ is an integral domain.*

Proof. Note that if X is integral then stalks are integral domains.

Conversely, suppose that X is a connected Noetherian scheme, such that all the stalks are integral domains. Then all the stalks also contain no nontrivial nilpotents hence X is reduced by Lemma 3.2.1.

By Theorem 3.2.1 need only show that X is irreducible. As X is connected we have only one connected component, and by Lemma 3.4.6 we have that X has finitely many irreducible components. Let X have a decomposition into:

$$Z_1 \cup \cdots \cup Z_n$$

where each Z_i is an irreducible component. We see that if $Z_1 \cap Z_j = \emptyset$ for all j then Z_1 is open as its compliment is the finite union of closed subset. Since Z_1 is irreducible and thus connected, it follows that either $n = 1$ and $Z_1 = X$ so we are done, or that Z_1 and $Z_2 \cup \cdots \cup Z_n$ are disjoint open sets that cover X so X is disconnected. It follows that if $n \neq 1$, every irreducible component of X must intersect with at least one other irreducible component.

Suppose that $n \neq 1$, then there exist irreducible components Z and Y such that $Z \cap Y \neq \emptyset$. Let $x \in Z \cap Y$ and let $U = \operatorname{Spec} A$ be a affine open containing x. Note that if for all $x \in Z \cap Y$ and all $U = \operatorname{Spec} A$ we have that $Z \cap \operatorname{Spec} A = Y \cap \operatorname{Spec} A$, we can conclude that $Y = Z$, so without loss of generality assume that $Z \cap \operatorname{Spec} A \neq Y \cap \operatorname{Spec} A$. By Lemma 3.2.2, we have that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible closed subsets of $\operatorname{Spec} A$. We claim that they are irreducible components, indeed, suppose that there was an irreducible closed subset $S \subset \operatorname{Spec} A$ such that $Z \cap \operatorname{Spec} A \subset S$, then the closure of S in X is an irreducible closed subset of X containing the closure of $Z \cap \operatorname{Spec} A$, however this is equal to $\bar{Z} = Z$ contradicting the fact that Z is irreducible. It follows that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible components of $\operatorname{Spec} A$.

Now let x correspond to the prime ideal $\mathfrak{p} \subset A$, $Z \cap \operatorname{Spec} A = \mathbb{V}(I)$, and $Y \cap \operatorname{Spec} A = \mathbb{V}(J)$ for radical ideals $I \neq J \subset A$. We claim that I and J are minimal prime ideals over $\langle 0 \rangle$, in the sense that *a*) they are prime ideals, and *b*) for every prime ideal we have that if $\mathfrak{q} \subset I$ then $I = \mathfrak{q}$. Let $a, b \in A$ such that $a \cdot b \in I$, then we have that:

$$U_{ab} \cap \mathbb{V}(I) = (U_a \cap \mathbb{V}(I)) \cap (U_b \cap \mathbb{V}(I)) = \emptyset$$

Since $\mathbb{V}(I)$ is irreducible, it follows that either $U_a \cap \mathbb{V}(I)$ or $U_b \cap \mathbb{V}(I)$ are empty, hence either $a \in I$ or $b \in I$ so I and J are both prime. To see that they are minimal, suppose that there exists a prime ideal $\mathfrak{q} \subset I$, then $\mathbb{V}(I) \subset \mathbb{V}(\mathfrak{q})$, but by reversing the argument above we have that $\mathbb{V}(\mathfrak{q})$ is an irreducible closed subset so it follows that $\mathbb{V}(I) = \mathbb{V}(\mathfrak{q})$ as $\mathbb{V}(I)$ is maximal. We thus have that $I = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$ so I and J are both minimal prime ideals over $\langle 0 \rangle$.

We see that A is not an integral domain. Indeed, if A were an integral domain, then $\langle 0 \rangle$ is the unique minimal prime ideal over $\langle 0 \rangle$. In particular, there is a bijection between prime ideals which are contained in $\mathfrak{p}$ and prime ideals of $A_\mathfrak{p}$, hence we must have that there $I_\mathfrak{p}$ and $I_\mathfrak{q}$ are minimal primes of $A_\mathfrak{p}$. It follows that $A_\mathfrak{p} \cong (\mathscr{O}_X)_x$ is not an integral domain, a contradiction, hence we must have that $n = 1$, implying that X is irreducible, so X is reduced and irreducible and thus by Theorem 3.2.1 an integral scheme as desired. □

3.5 Morphisms of Finite Type

Recall that in Definition 2.3.4 we defined what it meant for a k-scheme to be locally of finite type. We now extend this definition to arbitrary schemes as follows:

Definition 3.5.1. Let $f : X \to Y$ be a morphism of schemes. Then f is **locally of finite type** if there exists an affine open cover $\{V_i = \operatorname{Spec} B_i\}$ of Y, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i algebra. The morphism is of **finite type** if the cover of $f^{-1}(V_i)$ is finite.

We have the following obvious examples:

Example 3.5.1. Let A be a finitely generated B algebra, then $\operatorname{Spec} A \to \operatorname{Spec} B$ is obviously of finite type. Let X be a k-scheme of locally finite type, and $f : X \to \operatorname{Spec} k$ the morphism making X a k-scheme, then f is also trivially locally of finite type. If we can take X to be Noetherian k-scheme of locally finite type, then we also have that f is of finite type.

We now show that being locally of finite type is local on target:

Proposition 3.5.1. *Morphisms of locally finite type are:*

a) Local on target.

b) Stable under base change.

c) Closed under composition.

Moreover, morphisms of finite type are closed under composition as well.

Proof. Clearly we have that if for every affine open $V \subset Y$ the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is locally of finite type then f is.

Now suppose that $f : X \to Y$ is a morphism of locally finite type. Let $\{V_i = \operatorname{Spec} B_i\}$ be an open cover for Y, and for each i, let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. Let $V = \operatorname{Spec} B$ be any affine open, then we can write:

$$V = \bigcup_i V_i \cap V$$

hence:

$$\begin{aligned} f^{-1}(V) &= \bigcup_i f^{-1}(V_i) \cap f^{-1}(V) \\ &= \bigcup_{i,j} U_{ij} \cap f^{-1}(V) \end{aligned}$$

Now note that $U_{ij} \cap f^{-1}(V) \subset U_{ij} \cong \operatorname{Spec} A_{ij}$, thus there exist elements $f_{ijk} \in A_{ij}$ such that:

$$U_{ij} \cap f^{-1}(V) = \bigcup_k U_{f_{ijk}}$$

We note that $U_{f_{ijk}} \cong \operatorname{Spec}(A_{ij})_{f_{ijk}}$, hence doing this for all i and j we have obtained an affine open cover:

$$f^{-1}(V) = \bigcup_{i,j,k} U_{f_{ijk}} = \bigcup_{i,j,k} \operatorname{Spec}(A_{ij})_{f_{ijk}}$$

It thus suffices to show that if A is a finitely generated B algebra, then A_f is also a finitely generated B algebra for all $f \in A$. Let $\pi : A \to A_f$ be the localization map, and $\phi : B \to A$ be the map making A a finitely generated B algebra. The map $\pi \circ \phi$ which takes $b \mapsto \phi(b)/1$ is then the map making A_f a B algebra. Let $\{a_1, \ldots, a_n\}$ be the generators of A as a B algebra, then any element $a \in A$ can be written as:

$$a = \sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}$$

We claim that $\{a_1/1, \ldots, a_n/1, 1/f\}$ is a generating set for A_f. Indeed, we see that any element in A_f, can be written as a/f^k, hence:

$$\begin{aligned} a/f^k =& (1/f^k) \cdot a/1 \\ =& (1/f^k) \cdot \frac{\sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}}{1} \\ =& \sum_{i_1 \cdots i_n} \frac{1}{f^k} \cdot \frac{\phi(b_{i_1 \cdots i_n})}{1} \cdot \frac{a_1^{i_1}}{1} \cdots \frac{a_n^{i_n}}{1} \end{aligned}$$

implying a).

Let $\{V_i = \operatorname{Spec} B_i\}$ be a cover of Z by affine opens, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ a cover of X by affine opens such that $f(U_{ij}) \subset V_i$. Moreover, let $\{W_{ij} = \operatorname{Spec} C_{ij}\}$ be a cover of Y of affine opens such that $g(W_{ij}) \subset V_i$. It follows that $\pi_Y^{-1}(W_{ij}) \cong X \times_{V_i} W_{ij} \cong f^{-1}(V_i) \times_{V_i} W_{ij}$. Now $f^{-1}(V_i) \times_{V_i} W_{ij}$ admits an open affine cover of the form $U_{ik} \times_{V_i} W_{ij} = \operatorname{Spec}(A_{ik} \otimes_{B_i} C_{ij})$. We then need only show that $A_{ik} \otimes_{B_i} C_{ij}$ is a finitely generated C_{ij} algebra. However, this is then clear, as if $\{a_1, \ldots, a_n\}$ are the generators of A_{ik} as a B_i algebra, then $\{a_1 \otimes 1, \ldots, a_n \otimes 1\}$ are generators of $A_{ik} \otimes_{B_i} C_{ij}$ as C_{ij} algebra. Indeed, we can write any element ω in $A_{ik} \otimes B_i C_{ij}$ as a sum of trivial tensors:

$$\omega = \sum_i \alpha_i \otimes c_i = \sum_i (\alpha_i \otimes 1) \cdot (1 \otimes c_i)$$

Each α_i can be written as the finite sum:

$$\alpha_i = \sum_{j_1 \cdots j_n} b_{ij_1 \cdots j_n} a_1^{j_1} \cdots a_n^{j_n}$$

hence:

$$\begin{aligned} \omega =& \sum_i \sum_{j_1 \cdots j_n} (b_i j_1 \cdots j_n a_1^{j_1} \cdots a_n^{j_n} \otimes 1) \cdot (1 \otimes c_i) \\ =& \sum_i \sum_{j_1 \cdots j_n} (a_1 \otimes 1)^{j_1} \cdots (a_n \otimes 1)^{j_n} \cdot (1 \otimes b_{ij_1 \cdots j_n} c_i) \end{aligned}$$

By collecting terms, and relabeling we obtain that:

$$\omega = \sum_{i_1 \cdots i_n} (a_1 \otimes 1)^{i_1} \cdots (a_n \otimes 1)^{i_n} \cdot (1 \otimes c_{i_1 \cdots i_n})$$

implying that $A_{ik} \otimes_{B_i} C_{ij}$ is indeed a finitely generated C_{ij} algebra, and thus b).

Let $\{W_i = \operatorname{Spec} C_i\}$ be an open affine cover for Z. Since g is (locally) of finite type, there exists an open affine cover $g^{-1}(W_i)$, $\{V_{ij} = \operatorname{Spec} B_{ij}\}_j$, such that each B_{ij} is a a finitely generated C_i algebra. By the same logic, there exists an affine open cover of each $f^{-1}(V_{ij})$, $\{U_{ijk} = \operatorname{Spec} A_{ijk}\}_k$, such that each

A_{ijk} is a finitely generated B_{ij} algebra. Now note that for each i:

$$\begin{aligned}\bigcup_{jk} U_{ijk} =& \bigcup_{ij}\left(\bigcup_k U_{ijk}\right)\\ =& \bigcup_j f^{-1}(V_{ij})\\ =& f^{-1}\left(\bigcup_j V_{ij}\right)\\ =& f^{-1}(g^{-1}(W_i))\end{aligned}$$

hence for each i, the $\{U_{ijk}\}_{jk}$ form an affine open cover of $(g\circ f)^{-1}(W_i)$. It now suffices to show that each A_{ijk} is a finitely generated C_i algebra. Each A_{ijk} is a finitely generated B_{ij} algebra, so let $\{a_1,\dots,a_n\}$ generate A_{ijk} as a B_{ijk} algebra. Moreover, we have that each B_{ij} is a finitely generated C_i algebra so let $\{b_1,\dots,b_m\}$ generate B_{ij} as a C_i algebra. We claim that $\{a_1,\dots,a_n,b_1,\dots,b_m\}$[54] generates A_{ijk} as C_i algebra. Indeed, let $a\in A$, then:

$$a=\sum_{l_1\cdots l_n} b_{l_1\cdots l_n} a_1^{l_1}\cdots a_n^{l_n}$$

We can write:

$$b_{l_1\cdots l_n}=\sum_{\lambda_1\cdots\lambda_m} c_{l_1\cdots l_n\lambda_1\cdots\lambda_m} b_1^{\lambda_1}\cdots b_m^{\lambda_m}$$

hence:

$$a=\sum_{l_1\cdots l_n\lambda_1\cdots\lambda_m} c_{l_1\cdots l_n\lambda_1\cdots\lambda_m} b_1^{\lambda_1}\cdots b_m^{\lambda_m} a_1^{l_1}\cdots a_n^{l_n}$$

implying c).

For the last claim, if g and f are of finite type, then every cover can be taken to be finite, hence $\{U_{ijk}\}_{jk}$ is a finite cover of $(g\circ f)^{-1}(W_i)$, so $g\circ f$ is of finite type as well. ☐

Example 3.5.2. Let $f: X\to Y$ be a closed embedding, then f is of finite type. Indeed, for every affine open $U=\operatorname{Spec} A\subset Y$, we have that $f^{-1}(U)=\operatorname{Spec} A/I$, so admits a finite cover of affine opens of X. It remains to show that A/I is a finitely generated A algebra, however this clear as any $[a]\in A$ can be written as $a\cdot[1]=[a\cdot 1]=[a]$, hence A/I is finitely generated over A by $[1]$.

Let $\imath: U\to X$ be an open embedding, then $\imath$ is locally of finite type. Indeed, let $\{V_i=\operatorname{Spec} B_i\}$ be an affine open cover of X, then $\imath^{-1}(V_i)=U\cap V_i$ and $\imath|_{U\cap V_i}: U\cap V_i\to V_i$ is an open embedding into an affine scheme. We can cover each $U\cap V_i$ with $U_{f_{ij}}\subset\operatorname{Spec} B_i$ for some $f_{ij}\in B$. It follows that $\{U_{f_{ij}}\}_j$ is a cover for $\imath^{-1}(V_i)$, and that $\imath|_{U_{f_{ij}}}$ is the given by the localization map $\pi_{ij}: B_i\to(B_i)_{f_{ij}}$. Consider the morphism:

$$\begin{aligned}\phi: B_i[x]&\longrightarrow(B_i)_{f_{ij}}\\ x&\longmapsto 1/f_{ij}\end{aligned}$$

Let $b/f_{ij}^n\in(B_i)_{f_{ij}}$, then $bx^n\mapsto b/f_{ij}^n$ so $(B_i)_{f_{ij}}$ is finitely generated by $\{1,1/f_{ij}\}$ as a B_i algebra. If X is Noetherian, then we can take $\imath$ to be of finite type.

Locally finite type schemes over a field have the following useful property:

Proposition 3.5.2. *Let X and Y be schemes locally of finite type over k, and $f: X\to Y$ a morphism of k-schemes*[55]*. Then if $x\in|X|$ we have that $f(x)\in|Y|$, i.e. f takes closed points to closed points.*

Proof. Let $x\in|X|$, and choose affine open $V=\operatorname{Spec} B\subset Y$ such that $f(x)\in V$. There then exists an affine open $U=\operatorname{Spec} A\subset X$, containing x which maps into V. It follows that $f|_U: U\to V$ is a morphism of affine schemes; let it be induced by the ring map $\phi: B\to A$. Since x is closed, x corresponds to a

[54] Here it understood that by b_l we mean the image of b_l in A_{ijk} under the homomorphism making A_{ijk} a B_{ij} algebra.
[55] I.e the relevant diagram commutes.

maximal idea $\mathfrak{m}$ of A; by Lemma 2.3.12 it suffices to show that $\phi^{-1}(\mathfrak{m})$ is also maximal. Note that since f is a morphism of k schemes, that ϕ is a morphism of k algebras. Since A is finitely generated we have that $A/\mathfrak{m} \cong k_x$ is a finite field extension of k by Zariski's lemma[56]. There is an induced map:

$$\pi \circ \phi : B \to A/\mathfrak{m} \cong k_x$$

where $\pi : A \to A/\mathfrak{m}$ is the projection. The kernel of this morphism is:

$$(\pi \circ \phi)^{-1}(0) = \phi^{-1}(\pi^{-1}(0)) = \phi^{-1}(\mathfrak{m})$$

Moreover, the image of this morphism is the k algebra $B/\phi^{-1}(\mathfrak{m})$ which now obviously sits in the following inclusions:

$$k \subset B/\phi^{-1}(\mathfrak{m}) \subset k_x$$

Since k_x is a finite field extension, and thus a finite dimensional vector space over k, and $B/\phi^{-1}(\mathfrak{m})$ is an integral domain, it follows that $B/\phi^{-1}(\mathfrak{m})$ is a field,[57] implying that $\phi^{-1}(\mathfrak{m})$ is maximal as desired. □

3.6 Separated Z-Schemes

In the category of topological spaces, direct products exist, and a space is Hausdorff if and only if the map $\Delta : X \to X \times X$ has closed image. In the category of schemes, the topological spaces we are dealing with are almost never dealing with Hausdorff spaces and we do not have product. Indeed, consider the affine plane $\mathbb{A}^n_{\mathbb{C}}$, then this space is modeled off of $\mathbb{C}^n$, but is certainly not Hausdorff, as the unique generic point is contained in every open set. Moreover, we have that $\mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^m_{\mathbb{C}} \cong \mathbb{A}^{n+m}_{\mathbb{C}}$, so fibre products mildly behave like direct products, but $\mathbb{A}^{n+m}_{\mathbb{C}}$ has many more points than the naive cartesian product[58].

However, if we restrict ourself to the category of Z-schemes, then fibre product, $X \times_Z Y$, does satisfy the universal property of the direct product. Indeed, this is true essentially by constriction, if $f_X : X \to Z$ and $f_Y : Y \to Z$ are Z-schemes, then their fibre product is a Z-scheme. If $f_Q : Q \to Z$ is a Z-scheme, and $p_X : Q \to X$ and $p_Y : Q \to Y$ are morphisms of Z-schemes, then we automatically have $f_X \circ p_X = Q_X$ and $f_Y \circ p_Y = Q_Y$, so there exists a unique morphism $Q \to X \times_Z Y$ of Z-schemes which satisfies the direct product diagram. With this in mind, we wish to develop an analogue to a scheme being Hausdorff, which mimics the definition of Hausdorff in the category of topological spaces, leading us to the next definition:

Definition 3.6.1. Let X be a Z-scheme, then X is **separated over Z**, or alternatively a separated **Z**-scheme, if the diagonal map $\Delta : X \to X \times_Z X$ has closed image. A morphism $f : X \to Z$ is **separated** if $\Delta : X \to X \times_Z X$ is a closed embedding.

The notion of separatedness is our analogue of Hausdorff in the category of schemes, and we will spend the next few pages discussing the implications of such a result.

Example 3.6.1. Let $X = \mathbb{A}^n_k$, then we claim that $\mathbb{A}^n_k$ is separated over k. Indeed, we have that $X \times_k X \cong \operatorname{Spec} k[x_1, \dots, x_n] \otimes_k k[x_1, \dots, x_n]$, and that the diagonal morphism is induced by the ring homomorphism given on simple tensors by $\phi : f \otimes g \mapsto fg$. This is a surjective ring homomorphism, so if $I = \ker \phi$, we have that $\Delta(X) = \mathbb{V}(I) \subset X \times_k X$. It follows that X is separated over k.

We would actually like to show that the notion of being separated over a scheme Z is the same as the morphism $f : X \to Z$ being a separated morphism. To do so we will need to show that the diagonal map is a closed embedding if it has closed image[59] We need the following definition:

Definition 3.6.2. A morphism $f : X \to Y$ is a **locally closed immersion**[60] if f factors as a closed embedding followed by an open embedding. In other words we have the following commutative diagram for some open subset $U \subset Y$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & U \arrow[ur, "\imath"'] & \end{tikzcd}$$

where g is a closed embedding, and $\imath$ is the inclusion.

[56]See Theorem 6.1.3
[57]We used a similar argument in Lemma 2.3.12.
[58]Which is not even a scheme!
[59]The other direction is immediate.
[60]In the literature this is sometimes called a locally closed embedding, or simply an immersion.

We want to show every diagonal map is a locally closed immersion.

Lemma 3.6.1. *Let $f : X \to Z$ be a morphism, then $\Delta : X \to X \times_Z X$ is a locally closed immersion.*

Proof. Let $\{V_i\}$ be an affine open cover for Z, and for each i let: $\{U_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. We have that $\{U_{ij} \times_{V_i} U_{ik}\}_{i,j,k}$ is an open affine cover for $X \times_Z X$, and claim that:

$$U = \bigcup_{ij} U_{ij} \times_{V_i} U_{ij}$$

contains the image of Δ. However, this is clear because $\Delta|_{U_{ij}}$ has image in $U_{ij} \times_{V_i} U_{ij}$, so Δ has image in U, and we have that Δ factors as:

$$X \longrightarrow U \longrightarrow X \times_Z X$$

The second morphism is clearly an open embedding, so we need only show that the morphism with restricted image, which we denote by g, is a closed embedding. This is also clear, as if $U_{ij} = \operatorname{Spec} A_{ij}$, and $V_i = \operatorname{Spec} B$, then $g|_{U_{ij}}$ is induced by the ring homomorphism $A_{ij} \otimes_{B_i} A_{ij} \to A_{ij}$ which is surjective, and is thus a quotient map. By Corollary 3.1.2 we have that g is a closed embedding, implying the claim. $\square$

We now prove the following more general statement:

Proposition 3.6.1. *Let $f : X \to Y$ be a locally closed embedding, then f is a closed embedding if and only if $f(X)$ has closed image in Y.*

Proof. Suppose that f is a closed embedding, then f trivially has closed image. Moreover, every closed embedding is a locally closed immersion as Y is trivially an open subscheme of Y.

Now let f be a locally closed immersion, and factor as $\imath \circ g$ where $g : X \to U$ is a closed embedding, and $\imath$ is the inclusion map into Y. Suppose $f(X)$ has closed image in Y, then by Corollary 3.1.2 we need to find an open cover of Y such that f restricts to a closed embedding. Note that:

$$Y = U \cup f(X)^c$$

as $f(X) \subset U$. We have that $f|_{f^{-1}U} : f^{-1}(U) = X \to U$ is a closed embedding as it is equal to g, and that $f^{-1}(f(X)^c)$ is the empty scheme $\emptyset$, so $f|_\emptyset$ is the empty map which is also trivially a closed embedding, implying the claim. $\square$

We now have the following corollary:

Corollary 3.6.1. *Let $f : X \to Z$ be a morphism of schemes, then f is separated if and only if X is separated over Z.*

Before calculating some examples, the following lemma will prove useful:

Lemma 3.6.2. *Let $X \to Z$ be a Z scheme, and $U_i, U_j \subset X$ open subschemes mapping into $V \subset Z$. Then $\Delta^{-1}(U_i \times_V U_j) = U_i \cap U_j$.*

Proof. We have that $U_i \times_V U_j$ is an open subscheme of $X \times_Z X$, and that X is an $X \times_Z X$ scheme via the diagonal map. Lemma 2.3.7 then tells us that $(U_i \times_V U_j) \times_{X \times_Z X} X$ is canonically an open subscheme of X given by $\Delta^{-1}(U_i \times_V U_j)$.

We see that we have the following canonical isomorphism:

$$(U_i \times_V U_j) \times_{X \times_Z X} X \cong (U_i \times_{X \times_Z X} X) \times_X (X \times_{X \times_Z X} U_j)$$

Now $U_i \times_{X \times_Z X} X$ makes the following square cartesian:

$$\begin{tikzcd}
U_i \times_{X \times_Z X} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
U_i \arrow[r, "\Delta|_{U_i}"] & X \times_Z X
\end{tikzcd}$$

We claim that U_i satisfies the universal property of $U_i \times_{X \times_Z X} X$ with projections given by the identity map $U_i \to U_i$, and the inclusion map $\imath : U_i \to X$. Let Q be a scheme such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "f_X"] \arrow[dr, bend right, "f_{U_i}"'] & & \\
& U_i \arrow[r, "\imath"] \arrow[d, "\mathrm{Id}"] & X \arrow[d, "\Delta"] \\
& U_i \arrow[r, "\Delta|_{U_i}"] & X \times_Z X
\end{tikzcd}$$

Then in particular, we have that since $\Delta|_{U_i} = \Delta \circ \imath$:

$$\Delta \circ \imath \circ f_{U_i} = \Delta \circ f_X$$

Let $\pi_1 : X \times_Z X \to X$ be the projection onto the first factor, then:

$$\pi_1 \circ \Delta \circ \imath \circ f_{U_i} = \pi_1 \circ \Delta \circ f_X$$

However, $\pi_1 \circ \Delta = \mathrm{Id}_X$, hence:

$$\imath \circ f_{U_i} = f_X$$

It follows that the choosing as f_{U_i} as the middle morphism makes the diagram commute, and it is unique because any other g morphism needs to satisfy $\mathrm{Id} \circ g = f_{U_i}$. Therefore there is a unique isomorphism $U_i \times_{X \times_Z X} X \cong U_i$.

It follows that:

$$(U_i \times_V U_j) \times_{X \times_Z X} X \cong U_i \times_X U_j \cong U_i \cap U_j$$

implying the claim.

□

We now list some examples (and non-examples) of separated schemes and morphisms:

Example 3.6.2. Every morphism of affine schemes is separated. Indeed, let $\operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine scheme, then $\operatorname{Spec} A \times_B \operatorname{Spec} A = \operatorname{Spec} A \otimes_B A$, and the diagonal morphism is given by $a_1 \otimes a_2 \mapsto a_1 \dot{a}_2$ which is surjective, so Δ is a closed embedding. In particular, $\operatorname{Spec} A$ is separated over $\operatorname{Spec} B$.

Example 3.6.3. We claim that $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$ is separated over A. We construct the map $\mathbb{P}^n_A \to \operatorname{Spec} A$ given on the open cover $\{U_{x_i} = \operatorname{Spec}(A[x_0, \ldots, x_n]_{x_i})_0\}$ by the morphism of affine schemes induced by the ring homomorphisms $A \hookrightarrow (A[x_0, \ldots, x_n]_{x_i})_0$. We have an open cover of $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ by $\{U_{x_i} \times_A U_{x_j}\}_{i,j}$. Now note that $\Delta^{-1}(U_{x_i} \times_A U_{x_j})$ is equal to the intersection:

$$U_{x_i} \cap U_{x_j} = (\operatorname{Spec}(A[x_0, \ldots, x_n]_{x_i})_0)_{x_j/x_i} \cong \operatorname{Spec} A[\{x_k/x_i\}_{k \neq i}, x_i/x_j]$$

We have that:

$$U_{x_i} \times_A U_{x_j} = \operatorname{Spec} A[\{z_k/z_i\}_{k \neq i}, \{y_k/y_j\}_{k \neq j}]$$

We have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is induced by the ring homomorphism which makes the following diagram of rings commute:

$$\begin{tikzcd}
A[\{x_k/x_i\}_{k \neq i}, x_i/x_j] & & \\
& A[\{z_k/z_i\}_{k \neq i}, \{y_k/y_j\}_{k \neq j}] \arrow[ul, "\Delta^\sharp"'] & A[\{x_k/x_j\}_{k \neq j}] \arrow[l, "\imath_j"'] \arrow[ull, bend right, "\phi"'] \\
& A[\{x_k/x_i\}_{k \neq i}] \arrow[u, "\imath_i"] \arrow[uul, bend left, "\psi"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

where ψ is the inclusion, ϕ is the morphism $x_k/x_j \mapsto x_k/x_i \cdot x_i/x_j$ for $j \neq i$, and $x_i/x_j \mapsto x_i/x_j$. The maps $\imath_i$ and $\imath_j$ take x_k/x_i and x_k/x_j to z_k/z_i and y_k/y_j respectively. It follows that $\Delta^\sharp(z_k/z_i) = x_k/x_i$, and that $\Delta^\sharp(y_i/y_j) = x_i/x_j$ so $\Delta^\sharp$ is indeed surjective. Therefore, on the open cover $U_{x_i} \times U_{x_j}$ we have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is a closed embedding, so Δ is a closed embedding thus $\mathbb{P}^n_A$ is separated over $\operatorname{Spec} A$.

Example 3.6.4. As a generalization of the preceding example, we let A now be any graded ring, and claim that $\operatorname{Proj} A$ is separated over $\operatorname{Spec} A_0$. Note that for any $f \in A_+^{\mathrm{hom}}$, the morphism $\operatorname{Proj} A \to \operatorname{Spec} A_0$ is induced by the obvious morphism $A_0 \to (A_f)_0$. The maps obviously agree on overlaps as any element in the image of this map is over the form $a/1$.

Now, $\operatorname{Proj} A \times_{A_0} \operatorname{Proj} A$, has an open cover by $U_f \times_{A_0} U_g \cong \operatorname{Spec}(A_f)_0 \otimes_{A_0} (A_g)_0$ for all $f, g \in A_+^{\mathrm{hom}}$. By Lemma 3.6.2, we have that $\Delta^{-1}(U_f \times_{A_0} U_g) = U_f \cap U_g = U_{fg}$, hence it suffices to check that the morphism of affine schemes $U_{fg} \to U_f \times_{A_0} U_g$ is a closed embedding. By replacing f with $f^{\deg g}$ and $g^{\deg f}$, we may assume that f and g are of the same degree. Now the diagonal morphism $U_{fg} \to U_f \times_{A_0} U_g$ must come from a ring homomorphism $\Delta^\sharp : (A_{fg})_0 \longrightarrow (A_f)_0 \otimes_{A_0} (A_g)_0$. In particular, this homomorphisms must make the following diagram commute:

$$\begin{tikzcd}
(A_{fg})_0 & & \\
& (A_f)_0 \otimes (A_g)_0 \arrow[ul, "\Delta^\sharp"'] & (A_g)_0 \arrow[l] \arrow[ull, bend right, "\theta_g"'] \\
& (A_f)_0 \arrow[u] \arrow[uul, bend left, "\theta_f"] & A_0 \arrow[l] \arrow[u]
\end{tikzcd}$$

where in this case the restriction maps $\theta_f : (A_f)_0 \to (A_{fg})_0$ are given by $a/f^k \mapsto a \cdot g^k/(fg)^k$. Note that this clearly lands in the degree 0 part of A_{fg}. It follows that $\Delta^\sharp$ must be given on simple tensors by:

$$a/f^k \otimes b/g^l \longmapsto \frac{abf^l g^k}{(fg)^{k+l}}$$

Now let $a/(fg)^k \in (A_{fg})_0$, then $\deg a = 2 \cdot k \cdot \deg f$, so consider the element:

$$\frac{a}{f^{2k}} \otimes \frac{f^k}{g^k} \in (A_f)_0 \otimes (A_g)_0$$

Under $\Delta^\sharp$, we have that this maps to:

$$\frac{ag^{2k}f^{2k}}{(fg)^{3k}} = \frac{a}{(fg)^k}$$

It follows that $\Delta^\sharp$ is surjective, and so $\Delta|_{U_{fg}}$ is a closed embedding for all all f and g, hence Δ is a closed embedding and $\operatorname{Proj} A$ is separated over $\operatorname{Spec} A_0$.

We have the following non example:

Example 3.6.5. Let Z be the scheme obtained by gluing $X = \operatorname{Spec} \mathbb{C}[x]$ and $Y = \operatorname{Spec} \mathbb{C}[y]$ along the affine open U_x and U_y via the isomorphism induced by $x \mapsto y$. We claim that Z is not separated over $\operatorname{Spec} \mathbb{C}$. If ψ_X and ψ_Y are the open embeddings $X \to Z$ and $Y \to Z$ respectively, we have that Z has an open cover given by $\psi_X(X)$ and $\psi_Y(Y)$. It follows that $Z \times_{\mathbb{C}} Z$ has an open cover given by:

$$\{\psi_X(X) \times_{\mathbb{C}} \psi_X(X), \psi_Y(Y) \times_{\mathbb{C}} \psi_Y(Y), \psi_X(X) \times_{\mathbb{C}} \psi_Y(Y), \psi_Y(Y) \times_{\mathbb{C}} \psi_X(X)\}$$

Each of these is isomorphic to the affine plane $\mathbb{A}^2_{\mathbb{C}}$, so we need to determine how these schemes glue together. We label these schemes by X_1, X_2, X_3 and X_4 in the order which they appear, and set:

$$X_i = \operatorname{Spec} \mathbb{C}[x_i] \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}[y_i] = \operatorname{Spec} \mathbb{C}[x_i, y_i]$$

Then X_1 and X_2 are glued on $U_{x_1} \cap U_{y_1}$ and $U_{x_2} \cap U_{y_2}$, X_1 and X_3 are glued along U_{y_1} and U_{y_3}, X_1 and X_4 are glued along U_{x_1} and U_{x_4}, X_2 and X_3 are glued along U_{x_2} and U_{x_3}, X_2 and X_4 are glued on U_{y_2}

and U_{y_4} and X_3 and X_4 are glued along $U_{x_3} \cap U_{y_3}$ and $U_{x_4} \cap U_{y_4}$. All of these morphisms are induced by the by isomorphism $x_i, y_i \mapsto x_j, y_j$.

It follows that $Z \times_{\mathbb{C}} Z$ is the affine plane with four origins, and doubled axis. The diagonal $\Delta(Z)$ is equal to $\Delta(\psi_X(X)) \cup \Delta(\psi_Y(Y))$, which via the above identification is contained in $X_1 \cup X_2$[61]. In particular, geometrically $\Delta(Z) \cap X_1$ and $\Delta(Z) \cap X_2$ is the diagonal in $\mathbb{A}^2_{\mathbb{C}}$, while $\Delta(Z) \cap X_3$ and $\Delta(Z) \cap X_4$ is the diagonal of $\mathbb{A}^2_{\mathbb{C}}$ minus the origin. Therefore, $\Delta(Z) \cap X_i$ is not closed for all i, hence by definition of the topology on $Z \times_{\mathbb{C}} Z$, we have that Z is not separated.

Note that this also shows that Z is not an affine scheme by Example 3.6.2.

We now show that every open and closed embedding is also a separated morphism:

Proposition 3.6.2. *Let $f : X \to Z$ be a closed or open embedding, then X is separated over Z.*

Proof. First suppose that $f : X \to Z$ is a closed embedding, then there exist an open affine cover $\{V_i = \operatorname{Spec} B_i\}$ of Z such that $U_i = f^{-1}(V_i) = \operatorname{Spec} B_i/I_i$ for some ideal I. It follows that $X \times_Z X$ admits an open affine cover of the form:

$$\{U_i \times_{V_i} U_i = \operatorname{Spec}(B_i/I_i \otimes_{B_i} B_i/I_i)\}$$

Since $B_i/I_i \otimes_{B_i} B_i/I_i \cong B_i/I_i$ we have that $U_i \times_{V_i} U_i \cong U_i$ so $X \times_Z X \cong X$ and the diagonal map is just the identity. In particular, one can also see this by noting the $f(X) \times_Z f(X) \cong f(X) \cap f(X) \cong f(X) \cong X$.

Now suppose that $f : X \to Z$ is an open embedding, then $X \cong U$ for some open subscheme of Z. We have that $X \times_Z X \cong U \times_Z U \cong U \times_U U = U$, so again the diagonal map is just the identity, implying the claim. □

Recall that morphisms/topological properties of schemes are generally considered 'nice' if they are either local on target or stable under base change. We want to see that separated morphisms fall into this category as well:

Proposition 3.6.3. *Let $f : X \to Z$ be a morphism of schemes, then the following hold:*

a) Separated morphisms are local on target.

b) Separated morphisms are stable under base change.

c) Separated morphisms are closed under composition.

Proof. To show a), we first assume that f is separated. It follows that $\Delta : X \to X \times_Z X$ is a closed embedding, so $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is also a closed embedding. It follows that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is a separated morphism as well.

Now suppose that we have affine open cover $\{V_i = \operatorname{Spec} B_i\}$ such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated. This then implies that $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is a closed embedding. Let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$, then we have that $\{U_{ij} \times_{V_i} U_{ik}\}$ is an affine open cover for $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$, and that $\Delta|^{-1}_{f^{-1}(V_i)}(U_{ij} \times_{V_i} U_{ik}) = U_{ij} \cap U_{ik}$. Since this is a closed embedding, we thus have that $U_{ij} \cap U_{ik}$ is affine and of the form $\operatorname{Spec} A_{ij} \otimes_{B_i} A_{ik}/I$ for some ideal I. Doing this for all i, we obtain an open affine cover of $X \times_Z X$ such Δ restricts to a closed embedding on $\Delta^{-1}(U_{ij} \times_{V_i} U_{ik})$ so it follows that Δ itself is a closed embedding.

To show b), suppose that $f : X \to Z$ is separated, and let $g : Y \to Z$ be any morphism. We want to show that $X \times_Z Y$ is separated over Y. We have that:

$$\begin{aligned}(X \times_Z Y) \times_Y (X \times_Z Y) &\cong (X \times_Z Y) \times_Y (Y \times_Z X) \\ &\cong ((X \times_Z Y) \times_Y Y) \times_Z X \\ &\cong (X \times_Z Y) \times_Z X \\ &\cong (X \times_Z X) \times_Z Y\end{aligned}$$

The diagonal map $\Delta : X \times_Z Y \to (X \times_Z Y) \times_Y (X \times_Z Y)$ is then the map induced by $\Delta_X : X \to X \times_Z X$ and the identity on Y, composed with the above chain of isomorphisms. In other words we have the

[61] Abuse of notation alert! Technically, each X_i is a copy of $\mathbb{A}^2_{\mathbb{C}}$ which we glue together to get $Z \times_{\mathbb{C}} Z$, so only their images under the canonical open embeddings are contained in $Z \times_{\mathbb{C}} Z$. We employ this abuse so as to not clutter the page with notation.

following diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\pi_Y"] \arrow[ddr, bend right, "\Delta_X \circ \pi_X"'] & & \\
& (X \times_Z X) \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{X \times_Z X}"] & Y \arrow[d, "g"] \\
& X \times_Z X \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}$$

The claim then follows from Theorem 3.1.2.

We write $\Delta : X \to X \times_Z X$ for the diagonal map we wish to prove is a closed embedding, and $\Delta_X : X \to X \times_Y X$, $\Delta_Y : Y \to Y \times_Z Y$ for the diagonal maps we know to be closed embeddings. From Theorem 2.3.1 we have the following Cartesian square:

$$\begin{tikzcd}
X \times_Y X \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"] & X \times_Z X \arrow[d, "f \times f"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_Z Y
\end{tikzcd}$$

where ψ is the map coming from the following diagram[62]:

$$\begin{tikzcd}
X \times_Y X \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\
& X \arrow[r, "g \circ f"] & Z
\end{tikzcd}$$

It follows that ψ is closed embedding as it is the base change of the closed embedding Δ_Y. We claim that $\Delta = \psi \circ \Delta_X$. Indeed, Δ comes from the following diagram:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\
& X \arrow[r, "g \circ f"] & Z
\end{tikzcd}$$

Now $\pi_X \circ \psi \circ \Delta_X = \pi_X \circ \Delta_X = \mathrm{Id}_X$, so $\psi \circ \Delta_X$ makes the above diagram commute. It follows that Δ is the composition of a closed embeddings, and thus a closed embedding, hence $g \circ f$ is separated. $\square$

Note that the above implies that if X is a separated Z scheme, and Z is separated over $\operatorname{Spec} \mathbb{Z}$ then X is separated over $\operatorname{Spec} \mathbb{Z}$. In particular, every separated k scheme is separated over $\operatorname{Spec} \mathbb{Z}$, and every affine scheme, and scheme of the form $\operatorname{Proj} A$ is separated over $\operatorname{Spec} \mathbb{Z}$. In particular, the following lemma demonstrates that any scheme which is separated over $\operatorname{Spec} \mathbb{Z}$ is separated over any other scheme.

Lemma 3.6.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated, then so is f.*

[62] Abuse of notation alert! We are once again using the notation π_X to refer to multiple maps.

Proof. Let $\psi : X \times_Y X \to X \times_Z X$ be the morphism from Proposition 3.6.3, then by our work in that proposition we know that ψ is the base change of a closed embedding and is thus a closed embedding. Moreover, we know that the diagonal map $\Delta : X \to X \times_Z X$ factors as:

$$\begin{tikzcd} X \arrow[r, "\Delta_Y"] \arrow[rr, bend right, "\Delta"] & X \times_Y X \arrow[r, "\psi"] & X \times_Z X \end{tikzcd}$$

Now $\Delta(X) \subset X \times_Z X$ is closed by assumption, and $\psi(\Delta_Y(X)) = \Delta(X)$, hence since ψ is injective, we have that:

$$\psi^{-1}(\Delta(X)) = \psi^{-1}(\psi(\Delta_Y(X))) = \Delta_Y(X)$$

hence $\Delta_Y(X)$ is closed in $X \times_Y X$. The claim follows from Corollary 3.6.1. □

Now we have the following corollary:

Corollary 3.6.2. *Let X be separated over* $\operatorname{Spec} \mathbb{Z}$*, then any morphism $f : X \to Y$ is separated.*

Proof. Let $g : Y \to \operatorname{Spec} \mathbb{Z}$ be the unique morphism from Y to $\operatorname{Spec} \mathbb{Z}$. Then $g \circ f$ is the unique morphism from $X \to \operatorname{Spec} \mathbb{Z}$ which is separated by assumption. It follows from the proceeding lemma that f is separated. □

One less than ideal artifact of the topology on a scheme is that the intersection of two affine opens need not be affine. Indeed, let X be the affine plane over $\mathbb{C}$ with doubled origin, then their are two copies of $\mathbb{A}^2_{\mathbb{C}}$ contained in X, but their intersection is two copies of the zero ideal $\langle 0 \rangle$ which is manifestly not affine, i.e. no ring has two copies of the zero ideal as a prime spectrum. We now show that separated morphisms provide a solution to this problem:

Proposition 3.6.4. *Let $f : X \to Z$ be a separated morphism, and let $V =$* $\operatorname{Spec} B$ *$\subset Z$ be an open affine. Then for every open affine $U_i =$* $\operatorname{Spec} A_i$ *$\subset X$ which maps into V, we have that $U_i \cap U_j$ is an open affine.*

Proof. Let $\Delta : X \to X \times_Z X$, and then note that $\Delta(f^{-1}(V))$ is contained in $f^{-1}(V) \times_V f^{-1}(V)$. We see that if U_i and U_j are as above, we have that $\Delta^{-1}(U_i \times_V U_j) = U_i \cap U_j$, but Δ is a closed embedding so $U_i \cap U_j$ is of the form $\operatorname{Spec}(A_i \otimes_B A_j)/J$ hence $U_i \cap U_j$ is indeed an open affine. □

We have the following obvious corollary:

Corollary 3.6.3. *Let X be separated over an affine scheme* $\operatorname{Spec} A$*. Then the intersection of every affine open in X is an affine open.*

Note that this text in algebraic geometry has never once mentioned the notion of a variety, largely because the author was first introduced to algebraic geometry through the language of schemes. However, we now have sufficient language to give the definition of a variety, which are often the most geometric feeling schemes. We note that the definition of a variety varies wildly throughout the literature, and will change in this text when we discuss Abelian varieties.

Definition 3.6.3. Let X be a scheme, then X is a **variety over k** if X is of finite type over a field k, reduced, and separated over $\operatorname{Spec} k$.

Note that every variety is immediately quasi-compact as it is the finite union of affine schemes. Each of these affine schemes is Spec of a finitely generated k-algebra thus every variety is locally Noetherian. In particular, by Corollary 3.4.2 every variety is Noetherian.

Example 3.6.6. The n-dimensional affine plane $\mathbb{A}^n_{\mathbb{C}}$, and projective space $\mathbb{P}^n_{\mathbb{C}}$ are varieties. In general, the closed points of 'nice enough' varieties over $\mathbb{C}$, when equipped with the standard topology induced by that on $\mathbb{C}^n$ have the structure of smooth manifolds. We will make this notion precise later in the text.

Example 3.6.7. Let X be a variety, then every closed subset of $Z \subset X$ is a variety when equipped with the induced reduced subscheme structure. Every reduced closed subscheme of X is isomorphic to such a Z, so every reduced closed subscheme of X is a variety.

Let U an open subscheme of X, then U is a variety. Indeed, open embeddings are separated by Proposition 3.6.2, and are locally of finite type by Example 3.5.2. Since X is Noetherian the open embedding $\imath : U \to X$ is of finite type, thus U. Finally U is reduced as being reduced is a local property.

Suppose that Y is a reduced locally closed subscheme of X, i.e. there exists a morphism $\imath : Y \to X$ such that $\imath$ is a locally closed immersion. Then $\imath$ factors as an open embedding in to a reduced closed subscheme $Z \subset X$, followed by the closed embedding $Z \hookrightarrow X$. It follows that Y is a variety as it is an open subscheme of the variety Z.

We have the following result:

Theorem 3.6.1. *Let X be a reduced projective k-scheme, then X is a variety. In particular, any closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure is a variety.*

Proof. By Theorem 3.1.1 a projective k scheme is closed subscheme of $\mathbb{P}^n_k$ for some k, hence there exists some closed embedding $X \hookrightarrow \mathbb{P}^n_k$. Since closed embeddings are separated by Proposition 3.6.2, and separated morphisms are closed under composition by Proposition 3.6.3, we have that the natural morphism $X \hookrightarrow \mathbb{P}^n_k \to \operatorname{Spec} k$ making a X a k scheme is separated. Moreover, by Example 3.5.2, $X \hookrightarrow \mathbb{P}^n_k$ is separated, so Proposition 3.6.3 implies that X is separated over k. Since X is assumed to be reduced, we have that X naturally carries the structure of a scheme of variety over k.

Let $X \subset \mathbb{P}^n_k$ be any closed subset, then equipped with the induced reduced subscheme structure, we have that the above discussion applies to X as well, hence X is a variety. □

With Theorem 3.6.1, and Example 3.6.6 in mind we employ the following definitions:

Definition 3.6.4. A scheme X is a **projective variety** if it is a reduced closed subscheme of $\mathbb{P}^n_k$ for some n. In particular, every projective variety is isomorphic to a closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure. A scheme X is a **quasi-projective variety** if it is an open subscheme of $\mathbb{P}^n_k$ for some n.

We note that $\mathbb{A}^n_k$ is a quasi-projective variety, and that every reduced closed subscheme of $\mathbb{A}^n$ is quasi-projective variety. Moreover, every projective variety is quasi-projective, as they $Z \to Z \hookrightarrow X$ is a locally closed immersion. We therefore end this discussion, by remarking that most varieties one comes across in nature are quasi-projective, and the construction of a variety that is not quasi-projective was a research area of great interest until Nagata provided such an example in 1950's.

3.7 Proper Z-Schemes

A compact topological space X is generally one where every open cover has a finite subcover. Throughout this text, we have called this property quasi-compactness, largely because this definition is not restrictive enough. Indeed, the analogue of the complex vector space $\mathbb{C}^n$ in algebraic geometry is $\mathbb{A}^n_{\mathbb{C}}$. Under the usual definition of compactness, $\mathbb{A}^n_{\mathbb{C}}$ is compact as every affine scheme is quasi-compact, but $\mathbb{C}^n$ is most definitely not. Given this, instead we follow the lead of our separatedness condition, and define our analogue of compactness relative to a base scheme.

In topology, a proper map $f : X \to Y$ is one in which the inverse image of a compact set is compact. This is the correct way of thinking of 'relative compactness' in the setting of topological spaces. However, in this sense, when working with schemes, almost every morphism is proper. Indeed, if we deal with Noetherian schemes, which are Noetherian topological spaces, every subset of a scheme is compact, so every map between Noetherian topological spaces is proper in the topological sense. This is not very a helpful condition, so, following our treatment of separatedness, we analyze an equivalent definition of proper maps.

Recall that if X and Y are locally compact Hausdorff spaces, then $f : X \to Y$ being proper is equivalent to f being universally closed. That is, in topology, if $g : Z \to Y$ is another continuous map, then there exists a fibre product:

$$X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}$$

equipped with the subspace topology. The map f is then universally closed if f is closed, and the projection $X \times_Y Z \to Z$ is also closed for every topological space Z. It is easy to check that these two descriptions of properness are equivalent in the setting of locally compact, Hausdorff spaces.

In the setting of schemes, the definition of universally closed is the same:

Definition 3.7.1. Let $f : X \to Z$ be a closed morphism of schemes, i.e. f takes closed subsets to closed subsets[63]. Then f is **universally closed** if for every Z-scheme Y the projection $X \times_Z Y \to Y$ is also closed. In other words a closed morphism is universally closed if it closed under base change.

[63]Note that this does not mean that f is a closed embedding!

Now, we know what the analogue of Hausdorff is in the category of Z-schemes, so we need a good analogue of what it means for a Z-scheme to be locally compact. However, we have already encountered such an analogue, indeed if X if of finite type over Z, i.e. if $f : X \to Z$ is of finite type, then this morally feels like X being locally compact in the usual sense. This motivates our definition of proper morphisms and 'compactness' in the category of schemes:

Definition 3.7.2. Let $f : X \to Z$ be a morphism of schemes. Then f is a **proper morphism** if f separated, of finite type, and universally closed. We call any Z-scheme $f : X \to Y$ a **proper Z-scheme**, or **proper over Z** if f is proper.

So our usual analogues of compactness, and proper maps in algebraic geometry are proper morphisms and proper Z-schemes respectively We wish to show that proper morphisms are local on target, stable under base change, and closed under composition. It clearly suffices to prove the following:

Lemma 3.7.1. *Universally closed morphism are:*

a) Local on target.

b) Stable under base change.

c) Closed under composition.

Proof. Let $f : X \to Z$ be a universally closed morphism, and $g : Y \to Z$ be any morphism of schemes. Let $\{V_i = \operatorname{Spec} C_i\}$ be an affine open cover of Z, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, $\{W_{ik} = \operatorname{Spec} B_{ik}\}$ be affine open covers of X and Y such that U_{ij} and W_{ik} map into V_i. We want to show that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is universally closed. First note that $f|_{f^{-1}(V_i)}$ is indeed a closed map, as if $S \subset f^{-1}(V_i)$ is a closed subset then $S = T \cap f^{-1}(V_i)$ for some closed $T \subset X$. We have that:

$$f|_{f^{-1}(V_i)}(S) = f(T) \cap V_i$$

so since f is closed, it follows that the restriction is too. Now note that $f^{-1}(V_i) \times_{V_i} Y \cong f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$. We already know that $\pi_Y : X \to_Z Y \to Y$ is a closed map, so it's restriction to the open set $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$ must now also be a closed map, hence $f|_{f^{-1}(V_i)}$ is again universally closed.

Now suppose that $f|_{f^{-1}(V_i)}$ is a universally closed map for all i. We first claim that f is closed. We have that $f(T) \cap V_i$ is closed for all i, hence:

$$Y \setminus f(T) = \bigcup_i V_i \setminus (f(T) \cap V_i)$$

which is an infinite union of open sets and thus open. It follows that if $f|_{f^{-1}(V_i)}$ is universally closed for all i, then $\pi_Y|_{f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)}$ is closed for all i, so the same argument above shows that π_Y is closed, implying a).

To show b), we need to show that $\pi_Y : X \times_Z Y \to Y$ is universally closed. Let $h : Y' \to Y$ be a Y scheme, and note that:

$$\begin{aligned}(X \times_Z Y) \times_Y Y' &\cong X \times_Z (Y \times_Y Y') \\ &\cong X \times_Z Y'\end{aligned}$$

The map $\pi_{Y'} : X \times_Z Y' \to Y'$ is closed, and is equal to the map $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ composed with the above isomorphisms, hence $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ is closed as well, so π_Y is also universally closed.

To show c), let $f : X \to Y$, and $g : Y \to Z$ be universally closed maps. We see that $g \circ f$ is a closed map, so we need only show that it is universally closed. Let Y' be a Z-scheme, then we need to show that $\pi_{Y'} : X \times_Z Y' \to Y'$ is a closed map. We have the following commutative diagram:

$$\begin{tikzcd} X \times_Z Y' \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $f \times \mathrm{Id}$ comes from the following diagram:

$$\begin{tikzcd}
X \times_Z Y' \arrow[dr, "f\times \mathrm{Id}"] \arrow[drr, bend left, "\pi_{Y'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The right and outer squares are cartesian, so it follows as that the left square is cartesian as well. We have that f is universally closed, so $f \times \mathrm{Id}$ must be a closed map. It follows that $\pi_{Y'} = \pi_{Y'} \circ f \times \mathrm{Id}$ is the composition of closed maps and is thus closed. Therefore we have that $g \circ f$ is universally closed as desired. ☐

We now have the following corollary:

Corollary 3.7.1. *Proper morphisms are local on target, stable under base change, and closed under composition.*

Proof. This follows because separated maps, universally closed maps, and maps of finite type are all local on target, stable under base change, and closed under composition. ☐

Note that if a scheme is proper over a field, i.e. $X \to \operatorname{Spec} k$ is proper for a field k, then X is in a sense 'compact'. We now demonstrate that $\mathbb{A}^n_{\mathbb{C}}$ is not proper over $\mathbb{C}$:

Example 3.7.1. Clearly the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is closed, separated, and of finite type. We need to show that this morphism is not universally closed. Consider the scheme morphism $\pi : \mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. This morphism comes from (up to isomorphism) the ring homomorphism $\mathbb{C}[y] \hookrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}]$. Consider the closed subset $\mathbb{V}(x_1 \cdots x_{n+1} - 1)$, we claim that:

$$\mathbb{C}[x_1, \ldots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \cong \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n}$$

which is an integral domain. Indeed, note that there is a map:

$$\mathbb{C}[x_1, \ldots, x_{n+1}] \longrightarrow \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n}$$

given by $x_i \mapsto x_i$ for $i \leq n$, and $x_{n+1} \mapsto 1/(x_1 \ldots x_n)$. This map clearly factors through the quotient hence we have well defined map:

$$\phi : \mathbb{C}[x_1, \ldots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \to \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n}$$

Now note that there is map:

$$\mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$$

given by the composition of the inclusion map with the map with the projection map. We have that $[x_1 \cdots x_n]$ is invertible in $\mathbb{C}[x_1, \ldots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$ so the there is a well defined map:

$$\psi : \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n} \longrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$$

These are then clearly inverses of one another, so we have that the two rings are isomorphic. As the localization of an integral domain is an integral domain, it follows that $x_1 \cdots x_{n+1} - 1$ is irreducible.

The induced projection map then takes $\langle x_1 \cdots x_{n+1} - 1 \rangle \subset \mathbb{V}(x_1 \cdots x_{n+1} - 1)$ to the zero ideal, which is the generic point in $\mathbb{A}^1_{\mathbb{C}}$. It follows that $\pi(\mathbb{V}(x_1 \cdots x_{n_1} - 1))$ cannot be closed, so the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not universally closed.

We now see that all closed embedding's are proper:

Example 3.7.2. Let $f : X \to Z$ be a closed embedding, then f is separated, of finite type and closed. We need only show that f is universally closed, but closed embeddings are stable under base change, so $\pi : X \times_Z Y \to Y$ is a closed embedding as well. It follows that π must be a closed map, hence f universally closed, and thus proper.

For our first nontrivial example we show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, however, we need to be able to characterize the scheme-theoretic fibre of a scheme morphism. In other words, for $f : X \to Y$, we would like to know how to make sense of the preimage of $f^{-1}(y)$ for $y \in Y$ as a scheme.

First note that in the category of topological spaces, if $f : X \to Y$ is continuous map, then we can naturally identify $f^{-1}(p)$ with $\{y\} \times_Y X$, where $\{y\} \hookrightarrow Y$ is the inclusion map. In the category of schemes, we can naturally equip $\{y\}$ with a scheme structure given by $\operatorname{Spec} k_y$, where k_y is the residue field. We define a scheme morphism $g : \operatorname{Spec} k_y \to Y$ by first defining the topological map to be $\eta = \langle 0 \rangle \mapsto y$, and the sheaf morphism $g^\sharp : \mathscr{O}_Y \to g_* \mathscr{O}_{\operatorname{Spec} k_y}$ by first noting that if $y \in U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(g^{-1}(U)) = k_y$, and if $y \notin U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(\emptyset) = \{0\}$. We thus define $g^\sharp$ on open sets by:

$$g^\sharp_U(s) = \begin{cases} 0 \in \{0\} & \text{if } y \notin U \\ [s_y] \in k_y & \text{if } y \in U \end{cases}$$

which trivially commutes with restriction maps[64]. This then motivates our following definition:

Definition 3.7.3. Let $f : X \to Y$ be a scheme, then the for any $y \in Y$, the **scheme theoretic fibre** over y, denoted X_y is given by $\operatorname{Spec} k_y \times_Y X$.

Note that this naturally has the structure of a scheme, so it is mainly important to show that there is a natural identification with elements in the fibre over y and elements in $\operatorname{Spec} k_y \times_Y X$.

Lemma 3.7.2. *Let $f : X \to Y$ be a morphism of schemes. Then there is a natural identification between $\operatorname{Spec} k_y \times_Y X$ with the fibre $f^{-1}(y)$.*

Proof. We have the following diagram:

$$\begin{tikzcd} X_y \arrow[r, "\pi_X"] \arrow[d, "\pi_y"] & X \arrow[d, "f"] \\ \operatorname{Spec} k_y \arrow[r, "g"] & Y \end{tikzcd}$$

We first want to show that the image of π_X is the fibre $f^{-1}(y)$, and then demonstrate that π_X is a homeomorphism onto it's image. Note that if suffices to check this on an affine open cover of X_y, so let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover of Y, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ an open cover of X such that $f(U_{ij}) \subset V_i$ for all i and j. It follows that:

$$X_y = \bigcup_{ij} \operatorname{Spec} k_y \times_{V_i} U_{ij}$$

We will show that $\pi_X|_{\operatorname{Spec} k_y \times_{V_i} U_{ij}} = \pi_{U_{ij}}$ is a homeomorphism onto $U_{ij} \cap f^{-1}(y) = f|^{-1}_{U_{ij}}(y)$. Moreover, supposing that $y \in V_i$ as otherwise $\operatorname{Spec} k_y \times_{V_i} U_{ij}$ is clearly empty, we can write y as a prime ideal $\mathfrak{p} \subset B_i$, so $k_y = k_\mathfrak{p} = B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$. Suppressing the i and j notation for clarity, we have the following diagram:

$$\begin{tikzcd} \operatorname{Spec} B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \otimes_B A \arrow[r, "\pi_U"] \arrow[d, "\pi_y"] & \operatorname{Spec} A \arrow[d, "f|_U"] \\ \operatorname{Spec} B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \arrow[r, "g"] & \operatorname{Spec} B \end{tikzcd}$$

where it is understood that g is now the morphism $\operatorname{Spec} B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \to \operatorname{Spec} B$ induced by the localization map followed by the projection to the residue field. Now note that:

$$\operatorname{Spec} B_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \otimes_B A \cong \operatorname{Spec}(B_\mathfrak{p} \otimes_B A)/\langle \mathfrak{m}_\mathfrak{p} \otimes 1 \rangle$$

and that A is a B algebra via the ring homomorphism $\phi : B \to A$ inducing $f|_U$. We define $A_\mathfrak{p}$ to be $\phi(B \setminus \mathfrak{p})^{-1}A$, and claim that:

$$B_\mathfrak{p} \otimes_B A \cong A_\mathfrak{p}$$

[64]Note that by Corollary 1.3.1 we have that g is a monomorphism, as the stalk map $g_\eta : (\mathscr{O}_Y)_{g(\eta)} \to (\mathscr{O}_{\operatorname{Spec} k_y})_\eta$ is always the projection $(\mathscr{O}_Y)_y \to k_y$

We have a map:

$$\beta : B_{\mathfrak{p}} \otimes_B A \longrightarrow A_{\mathfrak{p}}$$

given on simple tensors by $b/s \otimes a \mapsto \phi(b) \cdot a/\phi(s)$. Moreover, we have a ring homomorphism $A \to B_{\mathfrak{p}} \otimes A$ given by $a \mapsto 1 \otimes a$. For all $\phi(s) \in \phi(B \setminus \mathfrak{p})$, we have $1 \otimes \phi(s)$ is invertible, as $1 \otimes \phi(s) = s/1 \otimes 1$, which has inverse given by $1/s \otimes 1$. It follows that there is ring homomorphism:

$$\begin{aligned} \alpha : A_{\mathfrak{p}} &\to B_{\mathfrak{p}} \otimes A \\ a/\phi(s) &\longmapsto 1/s \otimes a \end{aligned}$$

These maps are clearly inverses of each other so we have that:

$$B_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}$$

Now note that under the map β we have that:

$$\beta(\langle \mathfrak{m}_{\mathfrak{p}} \otimes 1 \rangle) = \{a/\phi(s) \in A_{\mathfrak{p}} : a \in \langle \phi(\mathfrak{p}) \rangle\} = \langle \phi(\mathfrak{p})/1 \rangle \subset A_{\mathfrak{p}}$$

so it follows that we have the following isomorphism:

$$\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes A \cong \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$$

The projection π_U is now induced by the ring homomorphism $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$, and the projection π_y is given by the composition $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}/\left\langle \phi(q)/1 \right\rangle$. In the category of commutative rings, we thus have the following commutative diagram:

$$\begin{tikzcd} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle & A \arrow[l, "\pi \circ \pi_l"'] \\ B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \arrow[u, "\imath"] & B \arrow[l, "\nu"] \arrow[u, "\phi"'] \end{tikzcd}$$

where π_l is the localization map, and π is the quotient map. Let $\mathfrak{p} \in \operatorname{Spec} B$, then:

$$f|_U^{-1}(\mathfrak{p}) = \left\{\mathfrak{q} \in \operatorname{Spec} A : \phi^{-1}(\mathfrak{q}) = \mathfrak{p}\right\}$$

However, clearly from the commutativity of the first diagram, we have that $\pi_U(\operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle) \subset f^{-1}|_U(\mathfrak{p})$, so we need to define an inverse map $\eta : f|_U^{-1}(\mathfrak{p}) \to \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle$.

Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, implying that $\phi(\mathfrak{p}) \subset \mathfrak{q}$. We first show that:

$$\langle \pi_l(\mathfrak{q}) \rangle = \{a/s \in A_{\mathfrak{p}} : a \in \mathfrak{q}\}$$

is a prime ideal. This is clearly an ideal by construction, so suppose that $a/s, c/t \in A_{\mathfrak{q}}$ such that $ac/st \in \langle \pi_l(\mathfrak{q}) \rangle$. It follows that $ac/st = d/r$ such that $d \in \mathfrak{q}$, hence there exists some $u \in \phi(B \setminus \mathfrak{p})$ such that:

$$u(acr - dst) = 0$$

Note that $u, r, s, t \in \phi(B \setminus \mathfrak{p})$, hence $u, r, s, t \notin \phi(\mathfrak{p}) \subset \mathfrak{q}$. We thus have that $acru \in \mathfrak{q}$, so $ac \in \mathfrak{q}$, so either a or c are in $\mathfrak{q}$. Note that $\langle \pi_l(\mathfrak{q}) \rangle$ is not all of $A_{\mathfrak{q}}$, as otherwise we have that $\mathfrak{q} \cap \phi(B \setminus \mathfrak{p}) \neq \emptyset$, which would imply that $\phi^{-1}(\mathfrak{q}) \cap \phi^{-1}(\phi(B \setminus \mathfrak{p})) \neq \emptyset$, so $\mathfrak{p} \cap B \setminus \mathfrak{p} \neq \emptyset$ which is a clear contradiction.

Let $\psi = \pi \circ \pi_l$; since $\langle \pi_l(\mathfrak{q}) \rangle$ clearly contains $\langle \phi(\mathfrak{p})/1 \rangle$, we have that $\pi(\langle \pi_l(\mathfrak{q}) \rangle)$ is a prime ideal of $A_{\mathfrak{p}}$. We thus define $\eta(\mathfrak{q}) \in \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$ by:

$$\eta(\mathfrak{q}) = \{[a/s] : a \in \mathfrak{q}\}$$

which clearly then satisfies $\eta(\mathfrak{q}) = \langle \psi(\mathfrak{q}) \rangle = \pi(\langle \pi_l(\mathfrak{q}) \rangle)$. Let $U_{[a/1]}$ be a distinguished open of $\operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$, then we see that:

$$\begin{aligned} \eta^{-1}(U_{[a/1]}) =& \{\mathfrak{q} \in f|_U^{-1}(\mathfrak{p}) : [a/1] \notin \langle \psi(\mathfrak{q}) \rangle\} \\ =& \{\mathfrak{q} \in f_U^{-1}(\mathfrak{p}) : a \notin \mathfrak{q}\} \\ =& U_a \cap f|_U^{-1}(\mathfrak{p}) \end{aligned}$$

which is open in $f_U^{-1}(\mathfrak{p})$. Since $U_a \cap f|_U^{-1}(\mathfrak{p})$ form a basis we have that η is indeed continuous.

We see that $\psi^{-1}(\eta(\mathfrak{q})) = \mathfrak{q}$, so $\pi_U \circ \eta = \mathrm{Id}$. Now let $\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}} / \langle \phi(\mathfrak{q}) \rangle$, then:

$$\eta(\psi^{-1}(\mathfrak{q})) = \{[a/s] : a \in \psi^{-1}(\mathfrak{q})\}$$

Suppose that $[a/s] \in \mathfrak{q}$, then $[a/1] \in \mathfrak{q}$, and $a \in \psi^{-1}(\mathfrak{q})$. Now suppose that $[a/s]$ satisfies $a \in \psi^{-1}(\mathfrak{q})$, then $[a/1] \in \mathfrak{q}$ so $[a/s] \in \mathfrak{q}$ as well. It follows that $\eta(\psi^{-1}(\mathfrak{q})) = \mathfrak{q}$ hence $\eta \circ \pi_U = \mathrm{Id}$, and π_U is a homeomorphism onto $f|_U^{-1}(\mathfrak{p})$.

Since the above argument holds for all affine opens $\operatorname{Spec} k_y \times_{V_i} U_{ij}$, it follows that $\pi_X : X_s \to X$ is a homeomorphism onto $f^{-1}(y)$ implying the claim.

$\square$

We can now show that $\mathbb{P}^n_A$ is proper.

Example 3.7.3. Note that $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A$, so if $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper, we have that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, as proper morphisms are stable under base change.

We have already shown that $\mathbb{P}^n_{\mathbb{Z}}$ is separated, and it is clearly of finite type, so we need only show that $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed. Let $g : Y \to \operatorname{Spec} \mathbb{Z}$ be any $\mathbb{Z}$ scheme, then we want to show that $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y \to Y$ is closed. As we have shown, being closed is local on target, so it suffices to show that for any open affine $U = \operatorname{Spec} A \subset Y$ that $\pi : \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A \cong \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map.

Let $Z = \mathbb{V}(I) \subset \mathbb{P}^n_A$ where $I = \langle g_1, g_2, \dots \rangle$ is a homogenous ideal. We need to determine the primes $\mathfrak{p} \in \operatorname{Spec} A$ which lie in $\pi(Z)$. In other words, by the preceding lemma, we want to know for which $\mathfrak{p}$, the fiber $\pi^{-1}(\mathfrak{p}) \cap Z \cong \operatorname{Spec} k_{\mathfrak{p}} \times_A Z = Z_{\mathfrak{p}}$ is non empty. We have that $k_{\mathfrak{p}} = \operatorname{Frac}(A/\mathfrak{p})$ which is an A algebra, therefore, $Z_{\mathfrak{p}} \subset (\mathbb{P}^n_A)_{\mathfrak{p}}$, and $(\mathbb{P}^n_A)_{\mathfrak{p}} = \mathbb{P}^n_A \times_A \operatorname{Spec} k_{\mathfrak{p}} \cong \mathbb{P}^n_{k_{\mathfrak{p}}}$. It follows that $Z_{\mathfrak{p}}$ is a closed subset of $\mathbb{P}^n_{k_{\mathfrak{p}}}$, and that locally

$$Z_{\mathfrak{p}} \cap \operatorname{Spec} k_{\mathfrak{p}} \times_A U_{x_i} = \operatorname{Spec} k_{\mathfrak{p}} \otimes_A (A[x_0, \dots, x_n]_{x_i})/(I_{x_0})_0 \cong \operatorname{Spec}(k_{\mathfrak{p}}[x_0, \dots, x_n]_{x_i})_0/J$$

where J is the ideal generated by the image of $(I_{x_0})_0$ under the map $(A[x_0, \dots, x_n]_{x_i})_0 \to (k_{\mathfrak{p}}[x_0, \dots, x_n]_{x_i})_0$. Hence, $Z_{\mathfrak{p}} = \mathbb{V}(I_{\mathfrak{p}})$, where $I_{\mathfrak{p}} = \langle [g_1], [g_2], \dots \rangle$, and $[g_i]$ is the image of the map:

$$A[x_0, \dots, x_n] \longrightarrow k_{\mathfrak{p}}[x_0, \dots, x_n]$$

induced by the projection $\pi : A \to A/\mathfrak{p}$, followed by the inclusion $A/\mathfrak{p} \hookrightarrow \operatorname{Frac}(A/\mathfrak{p})$. It follows that $Z_{\mathfrak{p}}$ is non empty if and only if $\mathbb{V}(I_{\mathfrak{p}}) \neq \mathbb{V}(\langle x_0, \dots, x_n \rangle)$, hence $\sqrt{I_{\mathfrak{p}}} \not\supset \langle x_0, \dots, x_n \rangle$. Equivalently for all $n > 0$, we have that:

$$\langle x_0, \dots, x_n \rangle^n \not\subset \langle [g_1], [g_2], \dots \rangle$$

If $S = k_{\mathfrak{p}}[x_0, \dots, x_n]$, then non containment is equivalent to the map:

$$\begin{aligned} \bigoplus_i (A[x_0, \dots, x_n])_{d - \deg g_i} &\longrightarrow S_d \\ f_i &\longmapsto [f_i g_i] \end{aligned}$$

not begin surjective for all d. Let $d_0 = \dim_{k_{\mathfrak{p}}} S_d$[65], then this gives us a matrix with coefficients in A, d_0 rows, and potentially infinite columns. All of the $d_0 \times d_0$ minors of this matrix must have determinant zero in $k_{\mathfrak{p}}$, so the determinants lie in $\mathfrak{p}$, and therefore the ideal generated by these determinants, $\tilde{J}$, is contained in $\mathfrak{p}$. It follows that the fibre $Z_{\mathfrak{p}} = \pi^{-1}(\mathfrak{p}) \cap Z$ is non empty if and only $\mathfrak{p}$ lies in $\mathbb{V}(\tilde{J})$.

Now if $\mathfrak{p} \in \pi(Z)$, then $\pi^{-1}(\mathfrak{p}) \subset Z$, hence $Z_{\mathfrak{p}}$ is nonempty so $\mathfrak{p} \in \mathbb{V}(\tilde{J})$, and if $\mathfrak{p} \in \mathbb{V}(\tilde{J})$ then the fibre $Z_{\mathfrak{p}}$ is non empty, so $\mathfrak{p} \in \pi(Z)$. Therefore, $\pi(Z) = \mathbb{V}(\tilde{J})$, hence π is closed map, and $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper as desired.

We have the following corollary:

Corollary 3.7.2. *Let $Z \subset \mathbb{P}^n_A$ be a closed subscheme, then Z is proper over* $\operatorname{Spec} A$.

Proof. The map $Z \to \operatorname{Spec} A$ is given by the closed embedding $\imath : Z \to \mathbb{P}^n_A$, followed by the canonical morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$ from Example 2.3.1, then by Example 3.7.3, we have that this map is proper. By Example 3.7.2, closed embeddings are proper, and by Corollary 3.7.1 proper morphisms are closed under composition. It follows that $Z \to \operatorname{Spec} A$ is proper. $\square$

Recall that if $f : X \to Y$ is a continuous map between Hausdorff topological with X compact, then f is proper. We wish to prove the algebraic geometry analogue of this result; i.e. if X and Y are S-schemes, with X proper over S and Y separated over S, then any morphism $f : X \to Y$ is proper as well. This will follow from the following lemma:

Lemma 3.7.3. *Let X and X' be Y-schemes, and Y a separated Z-scheme. Then the map $X \times_Y X' \to X \times_Z X'$ is a closed embedding.*

[65] Note this that this is finite, as $\dim_{k_{\mathfrak{p}}}$ is equal to the partitions of d.

Proof. This follows from Theorem 2.3.1 as the following diagram is Cartesian:

$$\begin{tikzcd} X \times_Y X' \arrow[r] \arrow[d] & X \times_Z X' \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

It follows that the morphism $X \times_Y X' \to X \times_Z X'$ is the base change of $\Delta_Y : Y \to Y \times_Z Y$, which is a closed embedding. Since closed embeddings are stable under base change the claim follows. □

We can now prove the desired result:

Theorem 3.7.1. *Let X and Y be Z-schemes, with Y separated over Z, and $f : X \to Y$ a Z-scheme morphism. Then the following hold:*

a) If X is universally closed over Z, then f is universally closed.

b) If X is proper over Z, then f is proper.

Proof. Let g and h be the morphisms which make X and Y Z-Schemes respectively. Let α be the unique morphism making the following the diagram commute:

$$\begin{tikzcd} X \arrow[dr, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "h"] \\ & X \arrow[r, "g"] & Z \end{tikzcd}$$

It follows that f factors as:

$$X \xrightarrow{\alpha} X \times_Z Y \xrightarrow{\pi_Y} Y$$

We see that π_Y is the base change of a universally closed morphism, and is thus universally closed. It thus suffices to show that α is a universally closed. With $X' = Y$, we claim that, up to isomorphism, α is the top horizontal map making the diagram in Lemma 3.7.3 commute. Indeed, if $X' = Y$ then $X \times_Y Y$ is uniquely isomorphic to X, with projections given by $\mathrm{Id} : X \to X$ and $f : X \to Y$. X then fits into the following Cartesian diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f \times \mathrm{Id}"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

By our work in Theorem 2.3.1, the horizontal map is then precisely the one defining α, so by Lemma 3.7.3 α is a closed embedding. Since closed embedding are universally closed by Example 3.7.2, we have proven *a*).

Now suppose that X is proper over Z, then π_Y is the base change of a proper map and is thus proper. In particular α is a closed embedding which is proper by Example 3.7.2, so the same argument guarantees that f is a proper map implying *b*). □

Example 3.7.4. Let X be a projective variety, then X is proper by Corollary 3.7.2 so any k morphism $X \to Y$ with Y separated over k is proper. In particular, every k morphism from X to a variety Y is proper.

3.8 Affine Morphisms

In this section we introduce affine morphisms, though it will more fruitful to study special types of affine morphisms as in the next section.

Definition 3.8.1. Let $f : X \to Y$ be a morphism of schemes, then f is affine if for every open affine $V \subset Y$, we have that $f^{-1}(V)$ is also affine.

Example 3.8.1. Any closed embedding is an affine morphism. Any open embedding is an affine morphism.

We prove the following structure result regarding schemes:

Lemma 3.8.1. *Let X be a scheme, and $\mathscr{O}_X(X) = A$. Suppose that $g_1, \ldots, g_n \in A$ generate the unit ideal, and that X_{g_i} is affine for each i, then $X \cong \operatorname{Spec} A$.*

Proof. Recall that:

$$X_{g_i} = \{x \in X : (g_i)_x \notin \mathfrak{m}_x\} \tag{3.8.1}$$

is an open set in X. Moreover, since the g_i generate the unit ideal in A, we have that for every affine open $U \subset X$, $g_i|_U$ generate the unit ideal of $\mathscr{O}_X(U)$. It follows that the distinguished open $U_{g_i|_U} \subset U$ cover U, however by our work in Proposition 2.1.2 we know that:

$$U_{g_i|_U} = X_{g_i} \cap U$$

It follows that X_{g_i} cover X as if $x \in X$, then there is an open affine U containing x, and thus an i such that $x \in U_{g_i|_U}$, hence $x \in X_{g_i}$ and $\bigcup_i X_{g_i} \subset X$.

Set $X_{g_i} = \operatorname{Spec} A_i$, and $X_{ij} = X_{g_i} \cap X_{g_j}$. Since each X_{g_i} is affine, by our work in Proposition 2.1.2, we have that each X_{ij} is a distinguished open in both $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$, thus:

$$\operatorname{Spec}(A_i)_{g_j|_{X_i}} \cong X_{ij} \cong \operatorname{Spec}(A_j)_{g_i|_{X_j}}$$

The rings $\mathscr{O}_X(X_{g_j})$ and $\mathscr{O}_X(X_{ij})$ have canonical $\mathscr{O}_X(X)$ module structures given by by the restriction maps $\theta^X_{X_{g_j}}$ and $\theta^X_{X_{ij}}$. There is a natural map

$$\alpha : \mathscr{O}_X(X) \longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j})$$
$$s \longmapsto (s|_{X_{g_j}})$$

where by $(s|_{X_{g_j}})$ we mean $(s|_{X_{g_0}}, \ldots, s|_{X_{g_n}})$. This map is an injection as the X_{g_j} cover X. We define another map:

$$\beta : \bigoplus_j \mathscr{O}_X(X_{g_j}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj})$$
$$(s_j) \longmapsto (s_{kj})$$

where:

$$(s_{kj}) = \left(s_k|_{X_{kj}} - s_j|_{X_{kj}}\right)$$

Note that $\beta \circ \alpha = 0$, as:

$$\beta((s|_{X_{g_j}})) = ((s|_{X_{g_k}})|_{X_{kj}} - (s|_{X_{g_j}})|_{X_{kj}}) = (s|_{X_{kj}} - s|_{X_{kj}}) = 0$$

Similarly, if $\alpha((s_j)) = 0$, then we have sections $s_j \in \mathscr{O}_X(X_{g_j})$ such that for all k and j:

$$s_j|_{X_{kj}} = s_k|_{X_{kj}}$$

It follows by the sheaf axioms, that there exists an $s \in \mathscr{O}_X(X)$ such that $s|_{X_{g_j}} = s_j$. We have thus shown that $\ker \beta = \operatorname{im} \alpha$, and so we have the following sequence of $\mathscr{O}_X(X)$ modules:

$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj})$$

hence the following exact sequence of A modules:

$$0 \longrightarrow A \longrightarrow \bigoplus_j A_j \longrightarrow \bigoplus_{k<j} A_{kj}$$

We can localize[66] the sequence at g_i, to obtain the exact sequence:

$$0 \longrightarrow A_{g_i} \longrightarrow \bigoplus_j (A_j)_{g_i|_{X_{g_j}}} \longrightarrow \bigoplus_{k<j} (A_{kj})_{g_i|_{X_{kj}}}$$

Note that first morphism, which we denote α_i, is induced by the unique ones which makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\theta^X_{X_{g_j}}"] \arrow[d, "\pi_{g_i}"] & A_j \arrow[d, "\theta^{X_{g_j}}_{X_{ij}}"] \\
A_{g_i} \arrow[r, "(\alpha_i)_j"] & (A_j)_{g_i|_{X_j}}
\end{tikzcd}$$

where $(\alpha_i)_j$ is the jth component of the map α_i. Moreover, the second morphism is given by:

$$\beta_i : \bigoplus_j \mathscr{O}_X(X_{ji}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj} \cap X_i)$$
$$(s_j) \longmapsto (s_k|_{X_{ki} \cap X_j} - s_j|_{X_{ji} \cap X_k})$$

Finally, note that $(A_i)_{g_i|_{X_i}}$ is A_i as $g_i|_{X_i}$ is invertible in A_i, so the map $(\alpha_i)_i$ is given by the localization of the restriction map $\theta^X_{X_i}$. We wish to show that $(\alpha_i)_i$ is an isomorphism.

Let $a/g_i^k \in A_{g_i}$ satisfy $(\alpha_i)_i(a/g_i^k) = 0$, then, since g_i maps to an invertible element, we have that $a/1$ also maps to zero. We claim that $\alpha_i(a/1) = 0$; indeed, we have that $(\alpha_i)_i(a/1) = 0$ by assumption, and that:

$$\begin{aligned}
(\alpha_i)_j(a/1) =& (\alpha_i)_j(\pi_{g_i}(a)) \\
=& (a|_{X_{f_j}})|_{X_{ij}} \\
=& a|_{X_{ij}} \\
=& (a|_{X_{f_i}})|_{X_{ij}}
\end{aligned}$$

Since $\theta^{X_{g_i}}_{X_{ii}} = \theta^{X_{g_i}}_{X_{g_i}} = \mathrm{Id}$, it follows that:

$$a|_{X_{f_i}} = (\alpha_i)_i(\pi_{g_i}(a)) = (\alpha_i)_i(a/1) = 0$$

hence $(\alpha_i)_j(a/1) = 0$ for all $j \neq i$ as well. By exactness, have that $a/1 = 0$, hence $(\alpha_i)_i$ is injective.

Now let $s \in A_i$; then $s|_{X_{ij}} \in (A_j)|_{f_i|_{X_j}}$ for all j, hence we have an element $(s_j) \in \bigoplus_j (A_j)|_{f_i|_{X_j}}$. It follows that:

$$\beta_i((s_j)) = (s_k|_{X_{ki} \cap X_j} - s_j|_{X_{ji} \cap X_k})$$

but:

$$s_k|_{X_{ki} \cap X_j} = s|_{X_{ik}}|_{X_{ij} \cap X_j} = s|_{X_{ij} \cap X_j}$$

and similarly for j, hence $\beta_i((s_j)) = 0$. It follows by exactness that there exists some $a/g_i^k \in A_{g_i}$ such that $\alpha_i(a/g_i^k) = (s_j)$, hence $(\alpha_i)_i$ is surjective. Therefore, we have $A_{g_i} \cong A_i$, and so $X_{g_i} \cong \operatorname{Spec} A_{g_i}$.

By Proposition 2.1.2, there is a natural map $f' : X \to \operatorname{Spec} A$ induced by the identity map $A \to \mathscr{O}_X(X)$. Furthermore, since the g_i generate the unit ideal in A, we know that U_{g_i} cover $\operatorname{Spec} A$. The morphism:

$$f'|_{X_{g_i}} : X_{g_i} \longrightarrow U_{g_i}$$

[66] We take this on a faith for the moment. A precise proof is given in greater generality in Lemma 5.3.1.

is the one induced by the ring homomorphism:

$$\begin{aligned} A_{g_i} &\longrightarrow \mathscr{O}_X(X_{g_i}) \\ a/g_i^k &\longmapsto a|_{X_{g_i}} \cdot (g_i|_{X_{g_i}})^{-k} \end{aligned}$$

however this is precisely $(\alpha_i)_i$, which we just showed was an isomorphism. Since f' restricts to an isomorphism on the inverse image of an open cover of $\operatorname{Spec} A$, we have that $X \cong \operatorname{Spec} A$ □

Proposition 3.8.1. *Affine morphisms are local on target.*

Proof. Suppose that $f : X \to Y$ is an affine morphism, and let $V \subset Y$ be an affine open. We wish to show that the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as well. Well, let $W \subset V$ be an affine open, then, in particular, W is an affine open in Y, and $(f|_{f^{-1}(V)})^{-1}(W) = f^{-1}(W)$ which is affine by assumption. It follows that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as desired.

Let $f : X \to Y$ be a morphism, and let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism. By assumption, each $f^{-1}(V_i)$ is affine so set $f^{-1}(V_i) = \operatorname{Spec} A_i$, and let $V = \operatorname{Spec} B \subset Y$ be an arbitrary open affine of Y. We have that:

$$V = \bigcup_i V_i \cap V$$

By Lemma 2.1.1, each $V_i \cap V$ can be covered by open affines:

$$V_i \cap V = \bigcup_j U_{ij}$$

where U_{ij} is a distinguished open affine in V_i and V. Hence:

$$V = \bigcup_{ij} U_{ij}$$

and each U_{ij} satisfies:

$$(f|_{f^{-1}(V)})^{-1}(U_{ij}) = f^{-1}(U_{ij}) = (f|_{f^{-1}(V_i)})^{-1}(U_{ij})$$

But $f|_{f^{-1}(V_i)} : \operatorname{Spec} A_i \to \operatorname{Spec} B_i$ is a morphism of affine schemes, and U_{ij} is a distinguished open, hence $(f|_{f^{-1}(V_i)})^{-1}(U_{ij})$ is a distinguished open of $\operatorname{Spec} A_i$ and thus an affine open of $f^{-1}(V)$. It follows that $V = \operatorname{Spec} B$ admits a cover of distinguished opens U_{ij} such that $f^{-1}(U_{ij}) \subset f^{-1}(V)$ is an affine open.

We have thus reduced the result to the following problem: if $f : X \to \operatorname{Spec} B$ is a morphism of schemes such that there is a cover of $\operatorname{Spec} B$ by distinguished opens $\{U_{b_i}\}_{i=0}^n$ with $f^{-1}(U_{b_i})$ affine, then X is an affine scheme. Let $\phi : B \to \mathscr{O}_X(X)$ be the unique ring homomorphism inducing f; by our work in Proposition 2.1.2, we know that $f^{-1}(U_{b_i}) = X_{\phi(b_i)}$. Since b_i generate the unit ideal in in B, we have that $\phi(b_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. Therefore, by Lemma 3.8.1 we have that X is affine, hence if $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism, then f is an affine morphism. □

Corollary 3.8.1. *Morphisms between affine schemes are affine. In particular, affine morphisms are local on target.*

Proof. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes. Let $V \subset \operatorname{Spec} B$ be an open affine scheme, then we would like to show that $f^{-1}(V)$ is an affine scheme.

Set $V = \operatorname{Spec} C$, and set $X = f^{-1}(V)$. Then we have have morphism $g : X \to \operatorname{Spec} C$ given by $f|_{f^{-1}(V)}$. We can cover $\operatorname{Spec} C$ with distinguished opens U_{c_i} which are also distinguished opens of $\operatorname{Spec} B$, hence $g^{-1}(U_{c_i})$ are open affines, as f is a morphism of affine schemes. In particular, if $\phi : C \to \mathscr{O}_X(X)$ is the unique morphism inducing g, then $g^{-1}(U_{c_i}) = X_{\phi(c_i)}$. Since c_i generate the unit ideal in C, $\phi(c_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. It follows by Lemma 3.8.1 that X is affine, hence morphisms between affine schemes are affine.

Now let $f : X \to Y$ be an affine morphism; then for any open affine cover $\{U_i\}$, we have that $f^{-1}(U_i)$ is open affine by definition. The restricted morphism is then a morphism of affine schemes, and thus an affine morphism by the discussion above. Conversely, if $f^{-1}(U_i)$ is an affine scheme, then $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a morphism of affine schemes, and thus an affine morphism. It follows by Proposition 3.8.1 that f is an affine morphism. □

We of course need to also check that affine morphisms are stable under base change, and that the composition of affine morphisms is affine:

Proposition 3.8.2.

a) *Affine morphisms are stable under base change.*

b) *The composition of affine morphisms is again affine.*

Proof. For *a*), let $f : X \to Z$ be an affine morphism, and $g : Y \to Z$ be any morphism. Let $\{V_i\}$ be an affine cover of Z, then $\{W_i = f^{-1}(V_i)\}$ is an affine cover for X, and we can obtain an affine open cover of $\{U_{ij}\}$ of Y such that $g(U_{ij}) \subset V_i$ for all j. We need only show that $\pi_Y^{-1}(U_{ij})$ is an affine scheme; indeed we claim that $\pi_Y^{-1}(U_{ij}) \cong W_i \times_{V_i} U_{ij}$, which is manifestly an affine scheme. For ease of notation, set $S = \pi_Y^{-1}(U_{ij})$, then $\pi_Y|_S(S) \subset U_{ij}$, and we have that:

$$f \circ \pi_X|_S(S) = g \circ \pi_Y|_S(S) \subset V_i$$

It follows that:

$$\pi_X|_S(S) \subset f^{-1}(V_i) = W_i$$

We thus have unique morphisms $\pi_{U_{ij}} : S \to U_{ij}$ and $\pi_{W_i} : S \to W_i$ such that $\imath_{U_{ij}} \circ \pi_{U_{ij}} = \pi_y|_S$ and $\imath_{W_i} \circ \pi_{W_i} = \pi_X|_S$. Moreover, these morphisms make the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"'] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\ W_i \arrow[r, "f|_{w_i}"] & V_i \end{tikzcd}$$

Now suppose that we have morphisms $p_{W_i} : Q \to W_i$ and $p_{U_{ij}} : Q \to U_{ij}$ which make the relevant diagram commute. Then by composing with open embeddings, we obtain a unique morphism $\phi : Q \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "\imath_{U_{ij}} \circ p_{U_{ij}}"] \arrow[ddr, bend right, "\imath_{W_i} \circ p_{W_i}"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

We first claim that $\phi(Q) \subset S$. Indeed, we have that

$$\pi_Y(\phi(Q)) = \imath_{U_{ij}} \circ p_{U_{ij}}(Q) \subset U_{ij}$$

hence:

$$\phi(Q) \subset \pi_Y^{-1}(U_{ij}) = S$$

Therefore, there exists a unique map $\psi : Q \to S$ such that $\imath_S \circ \psi = \phi$. We need to check that that ψ makes the relevant diagram commute. We see that:

$$\begin{aligned} \imath_{U_{ij}} \circ (\pi_{U_{ij}} \circ \psi) =& \pi_Y|_S \circ \psi \\ =& \pi_Y \circ \imath_S \circ \psi \\ =& \pi_Y \circ \phi \\ =& \imath_{U_{ij}} \circ p_{U_{ij}} \end{aligned}$$

and similarly that:

$$\imath_{W_i} \circ (\pi_{W_i} \circ \psi) = \imath_{W_i} \circ p_{W_i}$$

Since open embeddings are monomorphisms, it follows that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\psi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\
& W_i \arrow[r, "f|_{W_i}"] & V_i
\end{tikzcd}$$

so S satisfies the universal property of $W_i \times_{V_i} U_{ij}$ and is thus affine. It follows by Corollary 3.8.1 that π_Y is an affine morphism, as desired.

For b), let $f : X \to Y$, and $g : Y \to Z$ be affine morphisms, then clearly we have that for any affine open $U \subset Z$, that $f^{-1}(g^{-1}(U))$ is affine; it follows that $g \circ f$ is an affine morphism implying the claim. $\square$

3.9 Finite and Integral Morphisms

In this section, we discuss finite and integral morphisms of schemes. Recall that a morphism of rings $\phi : B \to A$ is finite if it makes A a finitely generated B module. That is, there is a finite set $\{a_1, \ldots, a_n\}$ such that any a can be written as:

$$a = \sum_{i=1}^{n} \phi(b_i) a_i$$

for some $b_i \in B$. Often times the notation ϕ is suppressed and we write $b_i \cdot a_i$. Furthermore, a morphism $\phi : B \to A$, is integral if every element of A is integral over B. That is, every $a \in A$ is the root of some monic polynomial in $\phi(B)[x]$. If a finite morphism, or an integral morphisms is injective, i.e. an inclusion of rings, then they are called finite extensions, or integral extensions respectively. In either case, we will often suppress the notation $\phi(p)$ for a polynomial in $\phi(B)[x]$, and simply write $p \in B[x]$ with evaluation on A understood to be the one induced by ϕ.

Definition 3.9.1. Let $f : X \to Y$ be a morphism of schemes, then f is **finite** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from a finite morphism of rings. Similarly, f is **integral** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from an integral morphism of rings.

Note that finite and integral morphisms are examples of affine morphisms. We need to show that finite morphisms are closed under composition before moving forward:

Lemma 3.9.1. *Let $f : X \to Y$ and $g : Y \to Z$ be finite morphisms. Then $g \circ f$ is a finite morphism.*

Proof. This statement clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are finite, then $\psi \circ \phi$ is finite. Suppose ψ and ϕ are finite, then there exists $\{b_1, \ldots, b_m\}$ and $\{a_1, \ldots, a_n\}$ which generate B as a C-module and A as a B-module. Let $a \in A$, then there exist β_i such that:

$$a = \sum_{i=1}^{n} \phi(\beta_i) \cdot a_i$$

There exist c_{ij} such that each β_i satisfies:

$$\beta_i = \sum_{j=1}^{m} \psi(c_i j) \cdot b_j$$

hence:

$$a = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(\psi(c_{ij})) \cdot \phi(b_j) a_i$$

hence the set $\{\phi(b_j) \cdot a_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ generates A as a C module, which is finite, so $\psi \circ \phi$ is finite. $\square$

We also demonstrate the following relationship between integral and finite morphisms:

Proposition 3.9.1. *Let $f : X \to Y$ be a finite morphism, then f is integral. If $f : X \to Y$ is integral and locally of finite type, then f is finite.*

Proof. For the first statement, it suffices to show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is finite then it is integral. This then reduces to the case that if $\phi : B \to A$ is a finite morphism then it is integral.

Suppose $\phi : B \to A$ is finite, then A is a finitely generated b module, hence there exists $a_1, \dots, a_n \in A$ such that for all $a \in A$ there are $b_1, \dots, b_n \in B$ satisfying:

$$a = b_1 a_1 + \cdots + b_n a_n$$

We want to show that any $a \in A$ is the root of a monic polynomial in $\phi(B)[x]$. First note that we have a surjective map of B-modules:

$$\begin{aligned} \pi : B^{\oplus n} &\longrightarrow A \\ (b_1, \dots, b_n) &\longmapsto \sum_{i=1}^{n} a_i b_i \end{aligned}$$

and that for any $a \in A$ we have B-module endomorphism $\psi_a \in \operatorname{End}_B(A)$ given by $s \mapsto a \cdot s$. For each i we have:

$$a \cdot a_i = \sum_{ij} b_{ij} a_j$$

for some $b_{ij} \in B$. This gives us an $n \times n$ matrix T with coefficients in B given by:

$$T = \begin{pmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{pmatrix}$$

The following diagram then commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "T"] & A \arrow[d, "\psi_a"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

Let $p \in B[x]$, and consider $p(T)$ and $p(\psi_a)$, where in the latter polynomial p is technically a polynomial in $\phi(B)[x]$ as p is acting on elements of a via the ring homomorphism ϕ. The following diagram then also commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "p(T)"] & A \arrow[d, "p(\psi_a)"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

as it would commute for any endomorphism of A and it's induced matrix T. Suppose that $p(T)$ is the zero morphism, and let $a \in A$. Then there exists $(b_1, \dots, b_n) \in B^{\oplus n}$ such that $\pi(b_1, \dots, b_n) = a'$ so:

$$p(\psi_a)(a') = p(\psi_a) \circ \pi(b_1, \dots, b_n) = \pi \circ p(T)(b_1, \dots, b_n) = 0$$

so $p(\psi_a)$ is also zero. If $p(\psi_a) = 0$, then $p(a)$ is also zero as the ring homomorphism $a \mapsto \psi_a$ is injective; it thus suffices to show that there exists a polynomial $p \in B[x]$ such that $p(T) = 0$.

Note that if B is a field then this holds by the Cayley-Hamilton theorem. Consider the surjection:

$$\begin{aligned} F : \mathbb{Z}[x_{ij}] &\longrightarrow B \\ x_{ij} &\longrightarrow b_{ij} \end{aligned}$$

and the inclusion:

$$G : \mathbb{Z}[x_{ij}] \longrightarrow \mathbb{Q}(x_{ij})$$

where $\mathbb{Q}(X_{ij})$ is the field of fractions $\mathrm{Frac}(\mathbb{Z}[x_{ij}])$. We have an induced ring homomorphism:

$$F' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_B(B^n)$$

which is given by[67]:

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & \ddots & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} F(p_{11}) & \cdots & F(p_{n1}) \\ \vdots & \ddots & \vdots \\ F(p_{1n}) & \cdots & F(p_{nn}) \end{pmatrix}$$

and a similar inclusion:

$$G' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_{\mathbb{Q}(x_{ij})}(\mathbb{Q}(x_{ij})^n)$$

Let:

$$T' = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}$$

then $F'(T') = T$. Since $\mathbb{Q}(x_{ij})$ is a field, there is a monic polynomial $q \in \mathbb{Q}(x_{ij})[y]$ such that $q(G'(T')) = 0$. This polynomial is given by $\det(y \cdot I_n - G'(T'))$, where I_n is the $n \times n$ identity matrix. Since each component $G'(T')_{ij} \in \mathbb{Z}[x_{ij}] \subset \mathbb{Q}(x_{ij})$, it follows that $q \in (\mathbb{Z}[x_{ij}])[y] \subset \mathbb{Q}(x_{ij})[y]$, and $q(T') = 0$. We have an induced ring homomorphism $F'' : (\mathbb{Z}[x_{ij}])[y] \to B[y]$, and it follows that:

$$0 = F'(q(T')) = F''(q)(F'(T')) = F''(q)(T)$$

so $p = F''(q)$ is a monic polynomial in $B[y]$ which has T as a root. By our earlier discussion it follows that $p(a) = 0$, and since $a \in A$ was arbitrary the map $\phi : B \to A$ is integral, implying the claim.

For the second statement, it also suffices to show that if $\phi : B \to A$ is an integral morphism which makes A a finitely generated B-algebra then ϕ is finite. Let $\{a_1, \dots, a_n\}$ generate A as a B algebra. Then morphism:

$$\begin{aligned} B[x_1, \dots, x_n] &\longrightarrow A \\ x_i &\longrightarrow a_i \end{aligned}$$

is surjective. Moreover, for all $a \in A$, there exists a monic $p \in B[y]$ such that $p(0) = a$. Let $p_i \in B[y]$ satisfy $p_i(0) = a_i$, and let $d_i = \deg(p_i)$, then we claim that the set:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

generate A as a B module. Let $a \in A$, then we have that:

$$a = \sum_{i_1 \cdots i_n} b_{i_1 \cdots i_n} a_1^{i_1} \cdots a_n^{i_n}$$

for some $b_{i_1 \cdots i_n}$, then we need only show that each $i_j \leq d_i - 1$. We prove this by induction on n; if $n = 1$ then we have that a can be written as:

$$a = \sum_i b_i a_1^i$$

Now $\phi(p)(a_1) = 0$, so:

$$a_1^{d_1} = -(b_{d_1 - 1} a_1^{d_1 - 1} + \cdots + b_0) \tag{3.9.1}$$

[67]Since both $\mathbb{Z}[x_{ij}]^n$ and B^n are free modules of rank n, their endomorphism rings are $n \times n$ matrices with coefficients in their respective rings.

We need only show that any $a_1^{d_1+m}$ for $m \geq 0$ is in the B-span of $\{a_1^i : 0 \leq i \leq d_i - 1\}$. The base case $m = 0$ is proven, now suppose $m-1$th case so that:

$$\begin{aligned} a_1^{d_1+m} =&(a_1^{d_1+m-1}) \cdot a_1 = a_1 \left(b'_{d_1-1} \cdot a_1^{d_1-1} + \cdots + b'_0\right) \\ =&a_1^{d_1} b'_{d_1-1} + \cdots + a_1 \cdot b'_0 \end{aligned}$$

Since $a_1^{d_1}$ can be written as in (3.9.1), when $n = 1$ we have that A is a finitely generated B module. Now supposing the $n-1$th case, we have that the sub algebra $A' \subset A$ generated by $\{a_1, \ldots, a_{n-1}\}$ is a finite B module. Since $\phi(B) \subset A'$, we have that A is integral over A', and A is clearly finitely generated over A' by a_n, hence A by the $n = 1$ case we have that A is a finite A' module. By Lemma 3.9.1, it follows that A is a finitely generated module with generators given by:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

as desired. □

Corollary 3.9.1. *Let $\phi : B \to A$ be a ring homomorphism, and $a_1, a_2 \in A$ be integral over B. Then, $a_1 + a_2$, $a_1 \cdot a_2$, and $b \cdot a_i$ are integral elements over B.*

Proof. Let $A' \subset B$ be the B algebra generated by a_1 and a_2. The same induction argument in the second part of Proposition 3.9.1 then shows that A' is a finite B module[68], and thus A' is integral over B implying the claim. □

Example 3.9.1. Consider the map $\mathbb{Q} \to \bar{\mathbb{Q}}$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. This is integral by construction, but is not finite as $\bar{\mathbb{Q}}$ is not a finite dimensional $\mathbb{Q}$-vector space. Indeed, suppose that $\bar{\mathbb{Q}}$ is n dimensional as a $\mathbb{Q}$ vector space, and consider the polynomial $x^{n+1} - 2$; this polynomial as $n+1$ roots over $\mathbb{C}$ all of which must lie in $\bar{\mathbb{Q}} \smallsetminus \mathbb{Q}$. These roots are all linearly independent hence $\bar{\mathbb{Q}}$ contains an $n+1$ dimensional $\mathbb{Q}$-linear subspace, and $\mathbb{Q} \to \bar{\mathbb{Q}}$ can't be finite.

Note that when dealing with varieties over a fixed field k, then every morphism is of finite type[69]. Indeed, let A and B be finitely generated k algebras, with generating sets $\{b_1, \ldots, b_n\}$ and $\{a_1, \ldots, a_m\}$. if $\phi : B \to A$ is a morphism, then consider the induced morphism:

$$\begin{aligned} \phi' : B[x_1, \ldots, x_m] &\longrightarrow A \\ x_i &\longmapsto a_i \end{aligned}$$

which on B acts by ϕ. This map is surjective as $k \subset B$, hence if:

$$a = p(a_1, \ldots, a_m)$$

for some $p \in k[x_1, \ldots, x_m]$, then $p \in B[x_1, \ldots, x_n]$, and $\phi'(p) = a$. It follows that A is a finitely generated B algebra so any morphism of varieties must be of finite type. It follows that in this setting we have that integral morphisms and finite morphisms between varieties are the same.

We now proceed with the rest of our standard results:

Proposition 3.9.2. *The following hold:*

a) *Integral morphisms are stable under composition.*
b) *Finite and integral morphisms are stable under base change.*
c) *Finite and integral morphism are local on target.*

Proof. As in Lemma 3.9.1, a) clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are integral, then $\psi \circ \phi$ is integral. Suppose that ϕ and ψ are integral; let $a \in A$, then there exists a monic polynomial $p \in B[x]$ such that $p(a) = 0$. Set:

$$p(x) = b_0 + b_1 x + \cdots + x^n$$

and let $B' \subset B$ be the C algebra generated by $\{b_0, \ldots, b_n\}$. Note that B' is integral over C as B is integral over C hence B' is a finite C module. Let $A' \subset A$ be the B' algebra generated by a, then A' is

[68]This is because the argument only uses that the generators are integral.

[69]Not locally of finite type, because every variety is quasi-compact, hence we can take every open cover to be finite

obviously finitely generated over B', and integral over B' by Corollary 3.9.1, so Proposition 3.9.1 show that A' is finite over B'. We thus have the composition:

$$C \to B' \to A'$$

is a composition of finite morphisms and is thus finite. It follows by Proposition 3.9.1 that $C \to A'$ is integral, thus there exists a monic polynomial $q \in C[y]$ such that $q(a) = 0$. Since $a \in A$ was arbitrary we have that $C \to A$ is integral as well.

We now have that b) reduces as: if $\phi : B \to A$ is finite/integral, and and $\psi : B \to C$ is any morphism, then the induced map $C \to A \otimes_B C$ is finite/integral. Suppose that ϕ is finite, and let $\{a_1, \dots, a_n\}$ generate A as a finite B-module. We claim that $S = \{a_1 \otimes 1, \dots, a_n \otimes 1\}$ generates $A \otimes_B C$ as a C module. Indeed, since $A \otimes_B C$ is generated as an abelian group by simple tensors, it suffices to show that any $a \otimes c$ lies in the C span of S. Well, for some $b_i \in B$:

$$\begin{aligned} a \otimes c &= \left(\sum_i a_i b_i \right) \otimes c \\ &= \sum_i (a_i b_i) \otimes c \\ &= \sum_i a_i \otimes (b_i c) \\ &= \sum_i (a_i \otimes 1) \cdot (1 \otimes b_i c) \end{aligned}$$

as desired[70].

Now suppose that ψ is an integral morphism, then by Corollary 3.9.1 we need only show that $a \otimes 1$ is integral over C. We know there exists a monic polynomial $p \in B[x]$, so consider it's image in $C[x]$, which we also denote by p. This polynomials image $A \otimes_B C[x]$ is given by:

$$(1 \otimes p)(x) = (1 \otimes b_0) + \cdots (1 \otimes b_n)x^n$$

then:

$$\begin{aligned} (1 \otimes p)(a \otimes 1) =& (1 \otimes b_0) + \cdots (1 \otimes 1)(a^n \otimes 1) \\ =& a \otimes b_0 + \cdots + a^n \otimes b_n \\ =& b_0 \otimes 1 + a^n \otimes 1 \\ =& (p(a)) \otimes 1 \\ =& 0 \end{aligned}$$

so $C \to A \otimes_B C$ is an integral, implying b).

For c), suppose that $f : X \to Y$ is an integral/finite morphism, and U is any affine open of Y, and set $V = f^{-1}(U)$. We need to show that $f|_V : V \to U$ is integral/finite. Note that by Proposition 3.8.1, we have that $f|_V : V \to U$ is an affine morphism, and that any open affine ove U is an affine open of Y, hence $(f|_V)|_{(f|_V)^{-1}(U)} = f|_{f^{-1}(U)}$ must come from an integral/finite morphism of rings by the definition of integral/finite morphisms. It follows that $f|_V$ is integral/finite.

Let $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover of Y, and $\{V_i = f^{-1}(U_i) = \operatorname{Spec} B_i\}$ be the corresponding open cover of X. Suppose that each $f : X \to Y$ is a morphism with each $f|_{V_i}$ integral/finite, and let $U = \operatorname{Spec} A \subset Y$ be an affine open of Y. Since f is affine by Proposition 3.8.1 we know that $f^{-1}(U) = \operatorname{Spec} B$ is affine. By Lemma 2.1.1, we can cover $\operatorname{Spec} A$ with open sets which are simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} A_i$ for some i, hence there exists a distinguished open cover $\{U_{a_j}\}$ of $\operatorname{Spec} A$ such that the induced morphism $f^{-1}(U_{a_j}) \to U_{a_j}$ is integral/finite. Let $\phi : A \to B$ be the ring homomorphism induces $f|_{f^{-1}(U)}$, then we have reduced the problem to the following situation: let $\{a_1, \dots, a_n\} \subset A$ generate the unit ideal, and the induced map $\phi_j : A_{a_j} \to B_{\phi(a_j)}$ be integral/finite, then ϕ is integral/finite.

[70]Recall that the canonical C module structure on $A \otimes_B C$ is given by $\dot{c(a} \otimes c') = (1 \otimes c) \cdot (a \otimes c')$.

First suppose that each ϕ_j is finite; then there exist $s_{1_j}, \ldots, s_{n_j} \in B_{\phi(a_j)}$ which generate $B_{\phi(a_j)}$ as an A_{a_j} module. We can write each s_{i_j} as:

$$s_{i_j} = \frac{b_{i_j}}{\phi(a_j)^{k_{i_j}}}$$

for some $b_{i_j} \in B$, some $k_{i_j} \in \mathbb{N}$. Since $1/a_j \in A_{a_j}$, it follows that we can take our generators to be of the form:

$$s_{i_j} = \frac{b_{i_j}}{1}$$

for all i_j. This gives us a finite set $\{b_{i_j}\} \subset B$, which we claim generates B as an A module; let $N = |\{b_{i_j}\}$, and consider the morphism of A modules:

$$\begin{aligned} \psi : A^{\oplus N} &\longrightarrow B \\ (a_{i_j}) &\longmapsto \sum_i \sum_j a_{i_j} b_{i_j} \end{aligned}$$

Let $\pi : B \to C$ be cokernel of this map, then since cokernels commute with localization[71], we have that the induced map $\pi_{a_j} : B_{\phi(a_j)} \to C_{\pi(\phi(a_j))}$ is the cokernel of:

$$A_{a_j}^{\oplus N} \longrightarrow B_{\phi(a_j)}$$

which is surjective hence $C_{\pi(\phi(a_j))} = 0$ for all j. Now let $c \in C$, then $c/1 \in C_{\pi(\phi(a_j))} = 0$, hence there exists some m_j such that $\pi(\phi(a_j))^{m_j} c = 0$. The a_j generate the unit ideal, so $a_j^{m_j}$ generate the unit ideal as well, hence $1 = \sum_j a_j^{m_j} \alpha_j$, therefore:

$$c = 1 \cdot c = \sum_j \pi(\phi(a_j^{m_j} \alpha_j)) \cdot c = 0$$

hence $C = 0$ and so ψ is surjective.[72]

Now suppose that each ϕ_j is integral. Let $b \in B$, then for all j, $b/1 \in B_{\phi(a_j)}$ is the root of a monic polynomial $p_j \in A_{a_j}[x]$. Note that $A_{a_j}[x] = (A[x])_{a_j}$; let:

$$p_j = x^{n_j} + \frac{b_{n_j-1}}{a_j^{k_{n_j-1}}} x^{n_j-1} + \cdots + \frac{b_0}{a_j^{k_0}}$$

There exists a M_j such that:

$$a_j^{M_j} p_j = a_j^{M_j} x^{n_j} + \frac{b'_{n_j-1}}{1} x^{n_j-1} + \cdots + \frac{b'_0}{1}$$

There thus exists a $p'_j \in A[x]$ such that $p'_j/1 \in (A[x])_{a_j}$ is equal to $a_j^{M_j} p_j$. Since $\phi(a_j)^{M_j} p_j(b) = 0$ it follows that there is an L_j such that $\phi(a_j)^{L_j+M_j} p_j(b) = 0$. Moreover, if we set $q_j = a_j^{L_j} \cdot p'_j$, then $q_j(b) = 0$, and $q_j/1 = a_j^{M_j+L_j} p_j$. Let N be the maximum degree of the q_j, and let $m_j = N - n_j$. Now again, we have that the set $\{a_1^{K_1}, \ldots, a_n^{K_n}\}$ generates the unit ideal, hence there are h_j such that:

$$1 = \sum_j h_j a_j^{K_j}$$

so we define $q \in A[x]$ by:

$$q = \sum_j h_j x^{m_j} q_j$$

[71] See Lemma 5.3.3 parti *iv*).

[72] If this feels like like there is some sheaf business going on here, that's because there is!

Note that each $x^{m_j}q_j$ has degree N, and that the degree N term of q is given by:

$$q = \sum_j h_j x^{m_j} a_j^{K_j} x^{n_j} = x^N \sum_j h_j a_j^{K_j} = x^N$$

hence q is a monic polynomial in $A[x]$. We claim that $q(b) = 0$, however this is clear as $q_j(b) = 0$ for all j. It follows that $\phi : A \to B$ is integral, implying the claim.

□

Our goal is to now further justify the the nomenclature 'finite morphism' in the sense that we wish to prove that these maps have finite fibres. Let $f : X \to Y$ be a finite morphism, and recall that the scheme theoretic fibre of $y \in Y$ is given by:

$$X_y = \operatorname{Spec} k_y \times_Y X$$

Note that if $U = \operatorname{Spec} A \subset Y$ is an affine scheme containing y then we have the following isomorphism:

$$X_y \cong \operatorname{Spec} k_y \times_U f^{-1}(U)$$

If f is finite then it is affine as well, and so with $f^{-1}(U) = \operatorname{Spec} B$, it suffices to show that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

is a finite topological space which ultimately amounts to showing that $k_y \otimes_A B$ has finitely many prime ideals. To do so we will need to develop the theory of Artinian rings, a class of rings which satisfy a condition dual to the Noetherian one.

Definition 3.9.2. Let A be a commutative ring, then A is commutative if every strictly decreasing chain of ideals:

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

terminates.

One quickly sees that being Artinian is a much less reasonable finiteness condition than being Noetherian. Indeed, let $A = \mathbb{Z}$, then the following chain never terminates:

$$2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots$$

so $\mathbb{Z}$ is not Artinian. Furthermore, in contrast to Theorem 3.4.1, we have that $A[x_1, \dots, x_n]$ is never Artinian as the following chain never terminates:

$$\langle x_i \rangle \supset \langle x_i^2 \rangle \supset \cdots \supset \langle x_i^n \rangle \supset \cdots$$

Example 3.9.2. Let $A = k^n$ with the ring structure given the canonical product ring structure. Then we have that every ideal is a vector subspace and the length of any chain of ideals is bounded above by $n+1$, hence must be finite. It follows that A is Artinian (and Noetherian). Moreover, any finite k-algebra is Artinian, and any ring that is finite as a set is also Artinian, i.e. $\mathbb{Z}/n\mathbb{Z}$.

The following is an analogue of Lemma 3.4.2:

Lemma 3.9.2. *Let A be a Artinian, then the following hold:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Artinian.

b) If $I \subset A$ is an ideal then A/I is Artinian.

Proof. For *a*) let:

$$J_1 \supset J_2 \supset \cdots$$

be a strictly descending chain of ideals in $S^{-1}A$. If $\pi : A \to S^{-1}A$ is the localization map, then we have that:

$$\pi^{-1}(J_1) \supset \pi^{-1}(J_2) \supset \cdots$$

is chain of ideals in A. For some n this must terminate, hence for all $m \geq n$ we have that $\pi^{-1}(J_m) = \pi^{-1}(J_n)$. It now suffices to show that $\langle \pi(\pi^{-1}(J_m)) \rangle = J_m$ for any m. Clearly, we have the inclusion $\langle \pi(\pi^{-1}(J_m)) \rangle \subset J_m$; let $a/s \in J_m$, then $a/1 \in J_m$,and $a \in \pi^{-1}(J_m)$. It follows that $a/1 \in \pi(\pi^{-1}(J_m))$, hence $a/s \in \langle \pi(\pi^{-1}(J_m)) \rangle$ implying the equality.

For b), we employ the same argument; however since $\pi : A \to A/I$ is surjective we automatically have the equality $\langle \pi(\pi^{-1}(J_m)) \rangle = J_m$. □

The above gives us the following strange result:

Proposition 3.9.3. *Let A be Artinian, then every $\mathfrak{p} \in \operatorname{Spec} A$ is maximal. In particular, A is an integral domain if and only if it is a field.*

Proof. Let A be Artinian, and $\mathfrak{p} \in \operatorname{Spec} A$, then by Lemma 3.9.2 we have that $A/\mathfrak{p}$ is an Artinian integral domain. Let $[a] \in A/\mathfrak{p}$ be nonzero and consider the following chain:

$$\langle [a] \rangle \supset \langle [a]^2 \rangle \cdots$$

which must stabilize, hence for some n we have that $\langle [a]^n \rangle = \langle [a]^{n+1} \rangle$. This implies that $[a]^n \in \langle [a]^{n+1} \rangle$ so there exists $[b] \in A/\mathfrak{p}$ such that $[a]^{n+1}[b] = [a]^n$, thus:

$$[a]^n([a] \cdot [b] - [1]) = 0 \Rightarrow [a] \cdot [b] - 1 = 0$$

as $[a]$ is assumed nonzero. It follows that $[b] = [a]^{-1}$ hence every nonzero element of $A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field implying that $\mathfrak{p}$ is maximal. In particular, if A is an integral domain then $\langle 0 \rangle$ is prime and thus maximal so A is a field. □

We now need the following general lemma:

Lemma 3.9.3. *Let A be a commutative ring, and $\mathfrak{q}, \mathfrak{p}_i \in \operatorname{Spec} A$ for $1 \leq i \leq n$. Then $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$ if and only if for some i we have $\mathfrak{p}_i \subset \mathfrak{q}$.*

Proof. We proceed by induction, the base case $n = 1$ is trivial, and if $\mathfrak{p}_i \subset \mathfrak{q}$ for some i, then clearly we have that $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$. Assuming the $n-1$th case, we have that:

$$\left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right) \cap \mathfrak{p}_n \subset \mathfrak{q}$$

If $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \subset \mathfrak{q}$, we are done by induction, so assume that $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \not\subset \mathfrak{q}$. Let $a \in \mathfrak{p}_n$, then by assumption there exists some $b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right)$ such that $b \notin \mathfrak{q}$. It follows that $a \cdot b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right) \cap \mathfrak{p}_n$ which lies in $\mathfrak{q}$, however $\mathfrak{q}$ is prime hence either $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$, thus again by assumption we have that $a \in \mathfrak{q}$. It follows that $\mathfrak{p}_n \subset \mathfrak{q}$.

□

Proposition 3.9.4. *Let A be Artinian, then $\operatorname{Spec} A$ is a finite topological space and carries the discrete topology*[73].

Proof. Suppose that $\operatorname{Spec} A$ has infinitely many maximal ideals, then we can choose some infinite sequence $\{\mathfrak{m}_i\}_{i=1}^{\infty}$ of pairwise distinct maximal ideals. Consider the following chain:

$$\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \cdots$$

We claim that this chain is strictly decreasing and never stabilizes, implying A is not Artinian. Suppose:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1}$$

then we have that:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1} \subset \mathfrak{m}_{n+1}$$

[73] Recall that in the discrete topology every subset is open

It follows that one of the $\mathfrak{m}_i$ is contained $\mathfrak{m}_{n+1}$ by Lemma 3.9.3, hence $\mathfrak{m}_i = \mathfrak{m}_{n+1}$ as these are all maximal ideals. However this is impossible as all maximal ideals are pairwise distinct by assumption, so A is not Artinian.

Supposing A is Artinian, we have by the above that $\operatorname{Spec} A$ has only finitely many maximal ideals. Since every prime ideal is maximal, by Proposition 3.9.3 we have that $\operatorname{Spec} A$ is a finite topological space equal to $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ where each $\mathfrak{m}_i$ is a maximal ideal. We see that $\mathbb{V}(\mathfrak{m}_i) = \{\mathfrak{m}_i\}$ so the singleton sets are closed, hence every subset of $\operatorname{Spec} A$ is closed, so every subset of $\operatorname{Spec} A$ is open implying that $\operatorname{Spec} A$ carries the discrete topology. □

We can now show that finite morphisms have finite fibres as initially discussed:

Corollary 3.9.2. *let $f : X \to Y$ be a finite morphism, then for all $y \in Y$, the fibre $X_y = \operatorname{Spec} k_y \times_Y X$ is a finite topological space.*

Proof. From our earlier discussion, if $U = \operatorname{Spec} A \subset Y$ contains y, and $\operatorname{Spec} B = f^{-1}(U)$, then we have that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

Since f is finite, we have that B is a finite A algebra, hence by Proposition 3.9.2 we have that $k_y \otimes_A B$ is a finite k_y algebra. Example 3.9.2 then implies that $k_y \otimes_A B$ is Artinian, hence $\operatorname{Spec}(k_y \otimes_A B)$ is a finite topological space with the discrete topology by Proposition 3.9.4 as desired. □

3.10 Finite Morphisms are Proper

We now end our discussion on integral and finite morphisms by connecting them to the other classes of morphisms discusses in this chapter. In particular we wish to show that integral morphisms are precisely those morphisms which are affine and universally closed, and finite morphisms are precisely those morphisms which are affine and proper. To do so, as usual, we will need to prove a slew of results from commutative algebra. Namely, this section could just as easily be called Lying Over, Going Up, and Nakayama's Lemma as our desired results will be applications of these lemmas.

We begin with Nakayama's Lemma; it comes in many flavors, and we prove five of them:

Lemma 3.10.1. *Let A be a ring, $I \subset A$ an ideal, and M a finitely generated A module. The following then hold:*

a) If $IM = M$ then there exists and $a \in A$ such that $[a] = [1] \in A/I$, and $a \cdot M = 0$.

b) If $IM = M$, and

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then $M = 0$.

c) Let N' and N be A-modules with $M, N \subset N'$, and suppose that I is contained in all maximal ideals of A as in b). Then if $N' = N + IM$, $N' = N$.

d) Let $f : N \to M$ be an A module morphism and suppose I is contained in all maximal ideals of A. Then if $\bar{f} : N/IN \to M/IM$ is surjective, f is surjective.

e) Suppose I is contained in all maximal ideals of A, and let $\pi : M \to M/IM$ be the natural surjection. If the image $\{f_1, \ldots, f_n\} \subset M$ generates M/IM then $\{f_1, \ldots, f_n\}$ generate M.

Proof. We start with a); note that:

$$IM = \{i \cdot m : i \in I, m \in M\}$$

Choose generators $f_1, \ldots, f_n$ of M, then we claim that the map:

$$\begin{aligned} \alpha : I^n &\longrightarrow M \\ (b_1, \ldots, b_n) &\longmapsto \sum_i b_i f_i \end{aligned}$$

is surjective. Let $m \in M$, then since $IM = M$ we have that $m = i \cdot n$ for some $i \in I$ some $n \in N$. However, $n = \sum_i a_i f_i$ as the f_i generate M, hence $m = \sum_i (ia_i) f_i$, and each $ia_i \in I$ implying the initial claim. In particular, we can write each generator as:

$$f_i = \sum_j c_{ij} f_j$$

for some $c_{ij} \in I$. Consider the matrix with coefficients in A given by:

$$S = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

which determines a morphism $A^n \to A^n$. Let $\beta : A^n \to M$ be the natural surjection[74] and set:

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is in the ith position, and note that $\beta(e_i) = f_i$. Then:

$$\beta \circ S(e_i) = \beta \left(\sum_j c_{ij} e_j \right) = \sum_j c_{ij} f_j = f_i$$

Now observe that for all i:

$$\beta \circ (\mathrm{Id} - S)(e_i) = 0$$

hence $\beta \circ (\mathrm{Id} - S)$ is identically zero. Define $a \in A$ by:

$$a = \det(\mathrm{Id} - S) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(\delta_{1\sigma(1)} - c_{1\sigma(1)}) \cdots (\delta_{n\sigma(n)} - c_{n\sigma(n)})$$

where S_n is the symmetric group. Note that $[a] = [1] \in A/I$ as if σ is not the identity then under the projection $\pi : A \to A/I$, we have

$$\pi(\delta_{i\sigma(i)} - c_{i\sigma(i)}) = \pi(c_{i\sigma(i)}) = 0$$

and if σ is the identity, then:

$$\pi(\delta ii - c_{ii}) = \pi(1 - c_{ii}) = [1]$$

Moreover, recall that for any matrix T there exists an adjugate matrix $\mathrm{adj}(T)$ satisfying:

$$\mathrm{adj}(T) \cdot T = T \cdot \mathrm{adj}(T) = \det(T) \cdot \mathrm{Id}$$

hence for all i, we have that:

$$\begin{aligned} a \cdot f_i =& a \cdot \beta(e_i) \\ =& \beta(a \cdot e_i) \\ =& \beta \circ (\det(\mathrm{Id} - S)\mathrm{Id})(e_i) \end{aligned}$$

Now note that for any matrix T, we have that:

$$\beta \circ T(e_i) = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix}$$

[74] Defined the same as α, just on all of A^n.

hence:

$$(\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

and so:

$$\beta \circ (\det(\mathrm{Id} - S))(e_i) = \mathrm{adj}(\mathrm{Id} - S) \cdot (\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

implying that $a \cdot f_i = 0$ as desired. In particular, since a annihilates each generator, we have that $a \cdot M = 0$, implying a).

For b) suppose in addition that:

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then with a as defined in a), we claim that a is invertible. Indeed, there exists $i \in I$ such that $a = 1 + i$, and this $i \in \mathfrak{m}$ for all $\mathfrak{m} \in |\operatorname{Spec} A|$. Consider the ideal $\langle 1 + i \rangle$, then if this ideal is not all of A, there must be some $\mathfrak{m} \in |\operatorname{Spec} A|$ such that $\langle 1 + i \rangle \subset \mathfrak{m}$. However, $\imath \in \mathfrak{m}$ as well so $1 \in \mathfrak{m}$ which is a contradiction. It follows that $\langle i + 1 \rangle = A$ hence a invertible. Let $m \in M$, then $a \cdot m = 0$ by construction, but:

$$0 = a^{-1} \cdot (a \cdot m) = m$$

hence $M = 0$ implying b).

For c), suppose that $N' = N + IM$, then note this implies that $N' = N + M$ as if $n' \in N'$ then we have $n' = n + i \cdot m$ for $n \in N$, $i \in I$, and $m \in M$. However $i \cdot m \in M$ hence $N' \subset N + M$. Since N and M are submodules of N' it follows that $N' = N + M = N + IM$. In particular, we have that N'/N is finitely generated, as if $\{f_1, \ldots, f_k\}$ generate M, then we claim that $\{[f_1], \ldots, [f_k]\}$ generate N'/N. Indeed, let $[n'] \in N'/N$, then any class representative n' can be written as $n + m$ for $n \in N$ and $m \in M$. Any $m \in M$ can be written as:

$$m = \sum_i a_i f_i$$

hence:

$$[n'] = \sum_i a_i [f_i] + [n] = \sum_i a_i [f_i]$$

Moreover we claim that $I(N'/N) = N'/N$; clearly we have $I(N'/N) \subset (N'/N)$, so let $[n'] \in N'/N$. Then any class representative n' can be written as $n + i \cdot m$, hence $[n'] = [i \cdot m] = i \cdot [m]$ so $[n'] \in I(N'/N)$. It follows by b) that $N'/N = 0$, implying the claim.

For d), let $f : N \to M$ be an A module homomorphism. Note that $\bar{f} : N/IN \to M/IM$ is induced by $\pi \circ f : N \to M/IM$ and factors uniquely through the quotient as $IN \subset \ker(\pi \circ f)$. Obviously, we have that $M = \mathrm{im}(f) + IM$, and M is finitely generated hence by c) we have that $\mathrm{im}(f) = M$ and f is surjective, as desired.

For e), I be as in b), and consider the natural projection $\pi : M \to M/IM$. We have that M/IM is finitely generated by $\{[f_1], \ldots, [f_n]\}$. Let $N \subset M$ be the submodule generated by $f_1, \ldots, f_n$, then we claim that $M = N + IM$. Let $m \in M$, and consider $[m]$. Then:

$$[m] = \sum_i a_i [f_i] = \left[\sum_i a_i f_i \right]$$

It follows that there exists $\beta \in IM$ such that:

$$m = \sum_i a_i f_i + \beta$$

hence $m \in N + \mathfrak{m}M$. Since M is finitely generated, it follows by *c*) that $M = N$ hence $\{f_1, \dots, f_n\}$ generate M. $\square$

We need the following lemma for both Lying Over and Going Up

Lemma 3.10.2. *Let $\phi : B \to A$ be an integral morphism, $I \subset A$, $J \subset B$ ideals, and $T \subset B$ a multiplicatively closed set. Then the following hold:*

a) *The morphism $B \to A/I$ is integral.*

b) *The morphism $B/J \to A/\langle \phi(J) \rangle$ is integral.*

c) *The morphism $T^{-1}B \to \phi(T)^{-1}A$ is integral.*

Proof. To show *a*), recall that the composition of integral morphisms is integral, so it suffices to show that $\pi : A \to A/I$ is integral. Let $[a] \in A/I$, then $p(x) = x - a \in A[x]$ is a monic polynomial with $[a]$ as a root, hence π is integral.

For *b*) let $J \subset B$ be an ideal, then the morphism $\psi : B/J \to A/\langle \phi(J) \rangle$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\ B/J \arrow[r, "\psi"] & A/\langle \phi(J) \rangle \end{tikzcd}$$

Let $[a] \in A/\langle \phi(J) \rangle$, then $a \in \pi_A^{-1}([a])$, and there is a polynomial $p \in B[x]$:

$$p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0$$

of which a is a root. There is then a polynomial $q \in B/J[x]$ given by:

$$q(x) = x^n + [b_{n-1}]x^{n-1} + \dots + b_0$$

We see that:

$$\begin{aligned} q([a]) =& [a]^n + [b_{n-1}][a]^{n-1} + \dots + b_0 \\ =& [a^n + b_{n-1}a^{n-1} + \dots + b_0] \\ =& [p(a)] \\ =& 0 \end{aligned}$$

hence ψ is integral.

For *c*), the morphism $\psi : T^{-1}B \to \phi(T)^{-1}A$ is the unique one which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\ T^{-1}B \arrow[r, "\psi"] & \phi(T)^{-1}A \end{tikzcd}$$

It suffices to show that $a/1$ and $1/\phi(t)$ are the roots of monic polynomials in $T^{-1}B[x]$ by Corollary 3.9.1. Let $a/1 \in \phi(T)^{-1}A$, then there exists $a \in A$ which maps to $a/1$ under π_A. Let $p \in B[x]$ be given by:

$$p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0$$

and satisfy $p(a) = 0$. Define $q \in T^{-1}B[x]$ by:

$$q(x) = x^n + \frac{b_{n-1}}{1}x^{n-1} + \dots + \frac{b_0}{1}$$

then:

$$q(a) = \frac{p(a)}{1} = 0$$

as desired. For $1/\phi(t)$ we claim that:

$$q(x) = x - \frac{1}{t} \in T^{-1}B[x]$$

satisfies $q(1/\phi(t)) = 0$. However, this clear as:

$$q(1/\phi(t)) = \frac{1}{\phi(t)} - \psi\left(\frac{1}{t}\right)$$

which by the definition of ψ reduces to:

$$\frac{1}{\phi(t)} - \frac{1}{\phi(t)} = 0$$

as desired. □

As the following example shows,

Example 3.10.1. If $S \subset A$ is multiplicatively closed, and $\phi : B \to A$ is a morphism of rings, then the natural map $\psi : B \to S^{-1}A$ given by $\psi = \pi \circ \phi$ is not in general integral even if ϕ is. Indeed, if this were true then every localization would be integral as the identity map is integral; as a counter example take the localization map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$, then since $\mathbb{C}[t]$ is an integrally closed domain it follows that if $\alpha \in \mathrm{Frac}(\mathbb{C}t)$ is integral then $\alpha \in \mathbb{C}[t]$. However $t^{-1} \notin \mathbb{C}[t]$ hence t^{-1} can't be integral over $\mathbb{C}[t]$ so the map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$ is not integral.

With our many flavours of Nakayama's lemma at hand, as well as Lemma 3.10.2 we can now prove the Lying Over, and Going Up result, beginning with the former:

Lemma 3.10.3. *Let $\phi : B \to A$ be an integral extension of rings, then induced map on schemes $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is surjective.*

Note that this is called 'Lying Over' because it implies that for any $\mathfrak{p} \in \operatorname{Spec} B$ we can find a prime $\mathfrak{q} \in \operatorname{Spec} A$ which maps to it.

Proof. Given $\mathfrak{p} \in \operatorname{Spec} B$, we simply need to show that the fibre $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A \times_B \operatorname{Spec} k_{\mathfrak{p}}$ is non empty. By Lemma 3.7.2, we have that:

$$f^{-1}(\mathfrak{p}) \cong \operatorname{Spec} A_{\mathfrak{p}} / \langle \phi(\mathfrak{p})/1 \rangle$$

where $A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1}A$. It follows that $f^{-1}(\mathfrak{p})$ is empty if and only if $\langle \phi(\mathfrak{p})/1 \rangle = A_{\mathfrak{p}}$, as the only ring without a maximal ideal is the 0 ring.

The localization map $\phi_{\mathfrak{p}} : B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is an integral morphism by Lemma 3.10.2. In particular, if $b/s \in B_{\mathfrak{p}}$, and $\phi(b)/\phi(s) = 0 \in A_{\mathfrak{p}}$, then there exists some $\phi(t) \in \phi(B \setminus \mathfrak{p})$ such that

$$\phi(bt) = 0$$

This implies $b \cdot t = 0$, but then $b/s = 0 \in B_{\mathfrak{p}}$. It follows that $\phi_{\mathfrak{p}}$ is injective as well. Let $\mathfrak{m}_{\mathfrak{p}}$ be the unique maximal ideal in $B_{\mathfrak{p}}$, then by the commutativity of the diagram:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_{\mathfrak{p}}"] & A \arrow[d, "\pi_{\mathfrak{p}}"] \\ B_{\mathfrak{p}} \arrow[r, "\phi_{\mathfrak{p}}"] & A_{\mathfrak{p}} \end{tikzcd}$$

we have that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = \langle \phi(\mathfrak{p})/1 \rangle$. Indeed, suppose that $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$, then by definition, we have that $a \in \phi(\mathfrak{p})$, and $s \in \phi(B \setminus \mathfrak{p})$. There is then a unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$ such that $\phi_{\mathfrak{p}}(b/t) = \phi(b)/\phi(t) = a/s$. Similarly, if $a/s \in \langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle$, then $a/s = \phi(b)/\phi(t)$ for some unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$, hence $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$.

The condition that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = A_{\mathfrak{p}}$ is now more aptly written as $\mathfrak{m}_{\mathfrak{p}} \cdot A_{\mathfrak{p}} = A_{\mathfrak{p}}$. For the sake of contradiction, suppose this holds, then we have that $1 \in A_{\mathfrak{p}}$ can be written as:

$$1 = \sum_{i=1}^{n} m_i \cdot g_i \tag{3.10.1}$$

with $m_i \in \mathfrak{m}_{\mathfrak{p}}$, and $g_i \in A_{\mathfrak{p}}$. Take the subalgebra $A' \subset A_{\mathfrak{p}}$ generated by $\{g_1, \ldots, g_n\}$, then A' is integral over $B_{\mathfrak{p}}$ and finitely generated, hence a a finite $B_{\mathfrak{p}}$ module by Proposition 3.9.1. We then have that (3.10.1) implies $1 \in \mathfrak{m}_{\mathfrak{p}} \cdot A'$, hence $\mathfrak{m}_{\mathfrak{p}} \cdot A' = A'$. However, $\mathfrak{m}_{\mathfrak{p}}$ is the only maximal ideal of $B_{\mathfrak{p}}$, so by Nakayama's lemma[75], we have that $A' = 0$, contradicting the injectivity of $\phi_{\mathfrak{p}}$. □

Going Up is now a borderline immediate consequence of Lying Over:

Lemma 3.10.4. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be an integral morphism, and $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$. Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $f(\mathfrak{q}) = \mathfrak{p}$, then there exists $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.*

Note that this is called 'Going Up' as it implies that we can lift chains of prime ideals.

Proof. Let $\phi : B \to A$ be the ring homomorphism which induces f; in particular, ϕ makes A integral over B. With $\mathfrak{p}, \mathfrak{p}'$, and $\mathfrak{q}$ as stated, consider the induced map $\phi' : B \to A/\mathfrak{q}$. Note that by Lemma 3.10.2 this map is integral. We claim that $\ker \phi' = \mathfrak{p}$. Indeed, let $b \in \mathfrak{p}$, then $\phi'(p) = [\phi(b)]$, but $\phi(b) \in \mathfrak{q}$ as $\phi(\mathfrak{p}) \subset \mathfrak{q}$. Now suppose that $\phi'(b) = 0$, then $\phi(b) \in \mathfrak{q}$, so $b \in \phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It follows that the map $B/\mathfrak{p} \to A/\mathfrak{q}$ is injective; in particular it is integral by Lemma 3.10.2 as $(A/\mathfrak{q})/\phi'(\mathfrak{p}) = (A/\mathfrak{q})/\langle 0 \rangle = A/\mathfrak{q}$. By Lying Over we have that that the induced map $\operatorname{Spec} A/\mathfrak{q} \to \operatorname{Spec} B/\mathfrak{p}$[76] is surjective, so there exists a prime $\mathfrak{q}'$ containing $\mathfrak{q}$ which maps to $\mathfrak{p}'$ as desired. □

Example 3.10.2. Let $N : \tilde{X} \to X$ be the normalization map of an integral scheme X. We claim that N is integral, and surjective. First note by the proof in Theorem 3.3.1, where we define N on an affine cover $\operatorname{Spec} A_i$, of X, that $N^{-1}(\operatorname{Spec} A_i) \cong \operatorname{Spec} \bar{A}_i$, so N is affine by Proposition 3.8.1. On this open cover, N is given by the $A \hookrightarrow \bar{A}$ which is integral extension by definition, hence N is integral by Proposition 3.9.2. In particular, by Lemma 3.10.3 we have that $\operatorname{Spec} \bar{A}_i \to \operatorname{Spec} A_i$ is surjective for all i, hence N is surjective.

If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism satisfying:

> For any $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ with $f(\mathfrak{q}) = \mathfrak{p}$, there exists a $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.

we say that Going Up holds for f. In particular, Going Up is equivalent to f being a closed map:

Proposition 3.10.1. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism, then f is closed if and only if Going Up holds for f.*

Proof. Suppose that $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is closed, and let $\phi : B \to A$ be the ring homomorphism inducing f. Let $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $f(\mathfrak{q}) = \mathfrak{p}$. Consider $\mathbb{V}(\mathfrak{q})$, then $f(\mathbb{V}(\mathfrak{q}))$ is closed, and contains $\mathfrak{p}$, hence $f(\mathbb{V}(\mathfrak{q}))$ contains the closure of $\mathfrak{p}$, $\mathbb{V}(\mathfrak{p})$. Since $\mathfrak{p}'$ is contained in $\mathbb{V}(\mathfrak{p})$, we have that $\mathfrak{p}' \in f(\mathbb{V}(\mathfrak{q}))$ hence there exists some $\mathfrak{q}' \in \mathbb{V}(\mathfrak{q})$ such that $f(\mathfrak{q}') = \mathfrak{q}$. It follows that Going Up holds for f.

Now suppose that going up holds for f, and let $\mathbb{V}(I) \subset \operatorname{Spec} A$ be a closed subset. Note that since $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is integral, we have that Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} A$, thus clearly Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} B$. It thus suffices to show that if Going Up holds for $f : \operatorname{Spec} A \to \operatorname{Spec} B$ then $f(\operatorname{Spec} A)$ has closed image.

Let $Z = f(\operatorname{Spec} A)$, and let $\mathfrak{p} \in \bar{Z}$. Then for any open set containing $\mathfrak{p}$ we must have that $U \cap Z \neq \emptyset$, as other wise U^c is a closed set containing Z, and thus contains $\bar{Z}$. However, $\mathfrak{p} \notin U^c$ so $\mathfrak{p} \notin \bar{Z}$, a contradiction. Hence, for all $g \notin \mathfrak{p}$, we have that $U_g \cap Z \neq \emptyset$. In particular, since $U_g \cap Z = f(U_{\phi(g)})$, we have that $U_{\phi(g)}$ is not empty for $g \notin \mathfrak{p}$.

This implies that $A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1} A$ is not the zero ring. Indeed, if $A_{\mathfrak{p}}$ is the zero ring that $1 = 0$, hence there would exist some $g \in B \setminus \mathfrak{p}$ such that $\phi(g) = 0$, but that would imply that $U_{\phi(g)}$ is empty, a contradiction. We now consider the composition:

$$\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A \to \operatorname{Spec} B$$

[75] Part $b)$ of Lemma 3.10.1.

[76] Which is topologically equivalent to the map $f|_{\mathbb{V}(\mathfrak{q})}$.

where the first map is induced by the localization map $\pi : A \to A_{\mathfrak{p}}$. Now let $\tilde{\mathfrak{q}} \in \operatorname{Spec} A_{\mathfrak{p}}$, and consider $\mathfrak{p}' = f(\pi^{-1}(\tilde{\mathfrak{q}}))$; we claim that $\mathfrak{p}' \subset \mathfrak{p}$. Suppose the contrary, then there exists a $g \in \mathfrak{p}'$ such that $g \notin \mathfrak{p}$. It follows that $\phi(g)/1 \in \tilde{\mathfrak{q}}$, but if $g \notin \mathfrak{p}$, then $\phi(g) \in \phi(B \setminus \mathfrak{p})$, so $\tilde{\mathfrak{q}} = A_{\mathfrak{p}}$, a contradiction.

In particular, we have shown that there exists $\mathfrak{p}' \subset \mathfrak{p} \in \operatorname{Spec} B$, and $\mathfrak{q}' = \pi^{-1}(\tilde{\mathfrak{q}})$ satisfying $f(\mathfrak{q}') = \mathfrak{p}'$. Since Going Up holds for f, it follows that there exists a $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $\mathfrak{q}' \subset \mathfrak{q}$ and $f(\mathfrak{q}) = \mathfrak{p}$. Therefore, if $\mathfrak{p} \in \bar{Z}$, we have $\mathfrak{p} \in Z$, so Z is closed, implying the claim. □

Lemma 3.10.5. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be induced by $\phi : B \to A$. Then, the closure of the image, $\operatorname{cl}(f(\operatorname{Spec} A))$ is equal to $\mathbb{V}(\ker \phi)$.*

Proof. Set $Z = \operatorname{cl}(f(\operatorname{Spec} A))$. First, let $\mathfrak{p} \in f(\operatorname{Spec} A)$, then $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} A$. Since $0 \in \mathfrak{q}$, we have that $\phi^{-1}(0) = \ker \phi \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(\ker \phi)$. It follows that $f(\operatorname{Spec} A) \subset \mathbb{V}(\ker \phi)$ hence $Z \subset \mathbb{V}(\ker \phi)$. Now by definition:

$$Z = \bigcap_{f(\operatorname{Spec} A) \subset \mathbb{V}(I)} \mathbb{V}(I)$$

If $f(\operatorname{Spec} A) \subset \mathbb{V}(I)$, then $I \subset \phi^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Let $b \in I$, then $\phi(b) \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, so $\phi(b) \in \sqrt{\langle 0 \rangle}$, i.e. there exists some n such that $\phi(b)^n = 0$. This however, implies that $b \in \sqrt{\ker \phi}$, hence $I \subset \sqrt{\ker \phi}$, and we have that $\mathbb{V}(I) \supset \mathbb{V}(\ker \phi)$. It follows that $\mathbb{V}(\ker \phi) = Z$ as desired. □

Lemma 3.10.6. *Let $f : X \to Z$ be a surjective*[77] *morphism of schemes, and $g : Y \to Z$ any other morphism. Then the base change $X \times_Z Y \to Y$ is surjective.*

Proof. Let $y \in Y$, then we need to show that the fibre:

$$\pi_Y^{-1}(y) = \operatorname{Spec} k_y \times_Y (Y \times_Z X)$$

is not empty. Note that:

$$\operatorname{Spec} k_y \times_Y (Y \times_Z X) \cong \operatorname{Spec} k_y \times_Z X$$

Let $z = g(y)$, then we also have that:

$$f^{-1}(z) \times_{k_z} \operatorname{Spec} k_y \cong (X \times_Z k_z) \times_{k_z} \operatorname{Spec} k_y \cong g^{-1}(y)$$

where the morphism making $\operatorname{Spec} k_y$ a k_z scheme comes from composing the stalk map $g_y : (\mathscr{O}_Z)_z \to (\mathscr{O}_Y)_y$ with the projection $\pi_z : (\mathscr{O}_Y)_y \to k_y$. Since g is a morphism of locally ringed spaces, this gives rise to a field morphism $k_z \to k_y$, which we take to induce the structural morphism of $\operatorname{Spec} k_y$ as a k_z scheme.

Now since $f^{-1}(z)$ is not empty, we have that there is a non empty affine open $U = \operatorname{Spec} A \subset f^{-1}(z)$. It thus suffices to show that $\operatorname{Spec} A \otimes_{k_z} k_y$ is nonempty. We claim that $A \otimes_{k_z} k_y$ is a nonzero ring, indeed since $A \neq 0$ we have that A is a non zero k_z vector space. Any k_z basis then extends to a k_y basis for $A \otimes_{k_z} k_y$ of the same cardinality, hence $A \otimes_{k_z} k_y$ cannot be the zero vector space. Since every ring has a maximal ideal, it follows that $\pi_Y^{-1}(y)$ is non empty implying the claim. □

We now prove the first major result of the section:

Theorem 3.10.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is integral if and only if f is affine, and universally closed.*

Proof. Suppose f is integral, then f is automatically affine, so it suffices to show f is universally closed. Since f is integral, it's base change is integral by Proposition 3.9.2, so it suffices to show that an integral morphism is closed. Clearly, it then suffices to show this in the case $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} B$, but this follows from the fact Going Up holds for integral morphism, and Proposition 3.10.1.

Now suppose that f is affine and universally closed. It again clearly suffices to show that f is integral in the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so let $\phi : B \to A$ be the morphism inducing f. We want

[77] Set theoretically.

to show that for all $a \in A$, there exists a monic polynomial $p \in B[x]$ such that p lies in the kernel of the map $\mathrm{ev}_a : B[x] \to A$, given by sending x to a. Consider the composition:

$$\psi : B[x] \to A[x] \to A[x] / \langle ax - 1 \rangle$$

Let $\beta \in \ker \psi$, then with $\beta = \sum_i b_i x^i$, there exists some polynomial $q \in A[x]$ such that:

$$\sum_i \phi(b_i) x^i = (ax - 1)q$$

Let $q = \sum_i c_i x^i$, then in particular we must have that:

$$\phi(b_i) = a \cdot c_{i-1} - c_i$$

If $\deg q = d$, we claim that:

$$p = \sum_{i=0}^{d} b_i x^{d+1-i} \in \ker \mathrm{ev}_a$$

We rewrite the sum as follows:

$$\begin{aligned} \sum_{i=0}^{d} \phi(b_i) x^{d+1-i} =& \sum_{i=0}^{d} (a \cdot c_{i-1} - c_i) x^{d+1-i} \\ =& a \sum_{i=0}^{d} c_{i-1} x^{d+1-i} - x \sum_{i=0} c_i x^{d-i} \end{aligned}$$

Now note that the $c_{-1} = 0$, so we we can rewrite the first sum as:

$$\begin{aligned} \sum_{i=0}^{d} \phi(b_i) x^{d+1-i} =& a \sum_{i=0}^{d} c_i x^{d-i} - x \sum_{i=0} c_i x^{d-i} \\ =& (a - x) \cdot \sum_{i=0} c_i x^{d-i} \end{aligned}$$

which certainly maps to zero under the morphism $B[x] \to A$ sending x to a, hence $p \in \ker \mathrm{ev}_a$. Moreover, if $b_0 = 1$ then p is monic, which would imply A is integral over B. It thus suffices to show that $\ker \psi$ contains a β satisfying $b_0 = 1$.

We claim this is equivalent to $\operatorname{Spec} B[x] / (\ker \psi + \langle x \rangle)$ being empty. Certainly, if $\beta \in \ker \psi$ with $b_0 = 1$ then $\ker \psi + \langle x \rangle = B[x]$. If $\ker \psi + \langle x \rangle = B[x]$, then that means $1 \in \ker \psi + \langle x \rangle$ hence:

$$1 = \beta + xg$$

for $\beta \in \ker \psi$, and $g \in B[x]$. However this clearly implies that $b_0 = 1$.

Note that the morphism $\varphi : B[x] \to A[x]$ is induced by the following diagram:

$$\begin{tikzcd} A \otimes_{\mathbb{Z}} \mathbb{Z}[X] & & \\ & B \otimes_{\mathbb{Z}} \mathbb{Z}[X] \arrow[ul, "\exists ! \varphi"] & \mathbb{Z}[X] \arrow[l, "\imath_{\mathbb{Z}[x]}"] \arrow[ull, bend right, "\mathrm{Id}"'] \\ & B \arrow[u, "\imath_B"] \arrow[uul, bend left, "\imath_A \circ \phi"] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

By Theorem 3.1.1, we have that φ induces a unique morphism $f' : \operatorname{Spec} A[x] \to \operatorname{Spec} B[x]$ which is universally closed. We claim that $f'(\mathbb{V}(ax - 1)) = \mathbb{V}(\ker \psi)$; note that $f'|_{\mathbb{V}(ax-1)}$ is induced by ψ, hence the closure of the image of $f'|_{\mathbb{V}(ax-1)}$ is equal to $\mathbb{V}(\ker \psi)$ by Lemma 3.10.5. However, f' is a closed map, so it's restriction to any closed set is a closed map, hence $f'(\mathbb{V}(ax - 1)) = \mathbb{V}(\ker \psi)$ as desired.

We claim that:

$$(B[x]/\ker\psi)\otimes_{B[x]} B \cong B[x]/(\ker\psi + \langle x\rangle)$$

Indeed, the morphism $\mathrm{ev}_0 : B[x] \to B$ is what makes B a $B[x]$ algebra, hence by our work in Lemma 3.1.2:

$$(B[x]/\ker\psi)\otimes_{B[x]} B \cong B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

It thus suffices to show that:

$$B/\langle \mathrm{ev}_0(\ker\psi)\rangle \cong B[x]/(\ker\psi + \langle x\rangle)$$

Consider the composition:

$$B \hookrightarrow B[x] \to B[x]/(\ker\psi + \langle x\rangle)$$

and note that if $b \in \langle \mathrm{ev}_0(\ker\psi)\rangle$, then:

$$b = \sum_i b_i p_i(0)$$

where $p_i \in \ker\psi$. If we consider b as an element in $b[x]$, then b is in $\ker\psi + \langle x\rangle$ as it is given by:

$$\sum_i b_i p_i - \sum_i b_i(p_i - p_i(0))$$

where clearly each $p_i - p_i(0) \in \langle x\rangle$. It follows that this factors through the quotient to give us a well defined homomorphism:

$$F : B/\langle \mathrm{ev}_0(\ker\psi)\rangle \longrightarrow B[x]/(\ker\psi + \langle x\rangle)$$

Now consider the composition:

$$B[x] \to B \to B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

If $p \in \ker\psi + \langle x\rangle$, then p can be written as:

$$p = q + xp'$$

where $q \in \ker\psi$, and $p' \in B[x]$. It follows that $q(0) \in \langle \mathrm{ev}_0(\ker\psi)\rangle$ hence this map also factors through the quotient to yield a well defined homomorphism:

$$G : B[x]/(\ker\psi + \langle x\rangle) \longrightarrow B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

Now let $[p] \in B[x]/(\ker\psi + \langle x\rangle)$, then:

$$G([p]) = [p(0)] \in B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

while:

$$F([p(0)]) = [p(0)] \in B[x]/(\ker\psi + \langle x\rangle)$$

However:

$$[p] - [p(0)] \in \langle x\rangle$$

so $F \circ G = \mathrm{Id}$. Clearly $G \circ F = \mathrm{Id}$, so the two are isomorphic as desired. It follows that the following diagram is Cartesian:

$$\begin{tikzcd}
\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} B[x]/\ker\psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

Moreover, we claim that the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle \arrow[r] \arrow[d] & \operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A[x]/\langle ax-1\rangle \arrow[r] & \operatorname{Spec} B[x]/\ker\psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

The right square is Cartesian, so we need only show the left square commutes, but this is equivalent to the following diagram commuting:

$$\begin{tikzcd}
B \otimes_{B[x]} A[x]/\langle ax-1\rangle & B[x]/(\ker\psi + \langle x\rangle) \arrow[l, "\imath_B"'] \\
A[x]/\langle ax-1\rangle \arrow[u, "\imath_A"] & B[x]/\ker\psi \arrow[l] \arrow[u]
\end{tikzcd}$$

However, $0 \in B$ is equal to $\mathrm{ev}_0(x)$, hence in $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$:

$$0 \otimes [a] = \mathrm{ev}_0(x) \otimes 1 = 1 \otimes [ax] = 1 \otimes 1$$

so $0 = 1$, and $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is the zero ring. It follows that the left square trivially commutes, so by Lemma 2.3.4 the left square is Cartesian. Now note, that the morphism:

$$\operatorname{Spec} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/\ker\psi$$

is surjective as it is given by $f'|_{\mathbb{V}(ax-1)}$ with restricted image, so by Lemma 3.10.6 we have that the morphism:

$$\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$$

is also surjective. However, $\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is empty, hence $\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$ is also empty, so by our earlier remarks $\operatorname{Spec} A \to \operatorname{Spec} B$ is integral as desired. □

We now proceed with showing that all finite morphisms are proper, though much of the leg work has already been covered. We first need the following immediate result:

Lemma 3.10.7. *Let $f : X \to Y$ be affine, then f is separated.*

Proof. Since the property of being separated is local on target, and f is affine, it suffices to show this in the case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. However this clear by Example 3.6.2, hence f is separated. □

The above borderline immediately implies the following:

Theorem 3.10.2. *Let $f : X \to Y$ be a morphism. Then f is finite if and only if it is affine and proper.*

Proof. Suppose f is finite, then f is automatically affine, and integral. It follows that f is separated by Lemma 3.10.7, and universally closed by Theorem 3.10.1. Moreover, f is of finite type as every finite morphism is automatically finite[78]. It follows that f is affine and proper.

Now suppose f is affine and proper, then f is affine and universally closed so it is integral by Theorem 3.10.1. Since f is of finite type, we then obtain that f is finite by Proposition 3.9.1, implying the claim. □

3.11 Quasicompact and Quasiseparated Morphisms

In this section we discuss quasicompact and quasiseparated morphisms. These can be thought of weaker versions of the finite type and separated hypothesis, but, as we will see, a quasicompact-quasiseparated hypothesis, often denoted qcqs, will be extremely fruitful. The definition of a quasicompact morphism is obvious:

[78] In particular if A is finitely generated as B module, then it is finitely generated as a B algebra by the same generating set.

Definition 3.11.1. Let $f : X \to Y$ be a morphism of schemes, then f is **quasicompact** if for every quasicompact open set $U \subset Y$, $f^{-1}(U)$ is quasicompact. If X is a Z-scheme, we say that X is a **quasicompact Z-scheme**, or is **quasicompact over Z** if the structural morphism $f : X \to Z$ is quasicompact.

Note that obviously a scheme is quasicompact as a topological if and only if it is a quasicompact $\mathbb{Z}$ scheme. Moreover, if Z is a quasicompact topological space, then every quasicompact Z scheme is quasicompact as topological space. We prove the standard results about morphisms:

Lemma 3.11.1. *Quasicompact morphisms are:*

a) Closed under composition.

b) Stable under base change.

c) Local on target.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasicompact morphism, and suppose that $U \subset Z$ is a quasicompact open subset of Z. Then, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ which is a quasicompact open subset of as g and f are both quasicompact. It follows that $g \circ f$ is a quasicompact morphism proving a).

Let $f : X \to Z$ be a quasicompact morphism, and $g : Y \to Z$ any morphism. We need to show that $\pi_Y : X \times_Z Y \to Y$ is a quasicompact morphism. Let $U \subset Y$ be quasicompact, then we want to show that $\pi_Y^{-1}(U) = X \times_Z U$ is also quasicompact. Since U is quasicompact, we know that $g(U)$ is quasicompact, and there thus exist finitely many open affines $V_1, \ldots, V_n \subset Z$ such that:

$$g(U) \subset \bigcup_{i=1}^{n} V_i = V$$

Note that V is quasicompact as it is a finite union of quasicompact spaces. It follows that:

$$\pi_Y^{-1}(U) = f^{-1}(V) \times_V U$$

where we know that $f^{-1}(V)$ is quasicompact because f is quasicompact. Let $\{W_{ij}\}$ and $\{U_{ik}\}$ be finite affine open covers of $f^{-1}(V)$ and U such that $f(W_{ij}) \subset V_i$ and $g(U_{ik}) \subset V_i$ respectively. It follows that $\pi_Y^{-1}(U)$ is a finite union of affine schemes, and thus a finite union of quasicompact spaces, and so must be quasicompact, proving b).

For iii), let $f : X \to Y$ be a quasicompact morphism, $U \subset Y$ be an open subset of Y. We need to check that $f|_{f^{-1}(U)}$ is quasicompact; let $V \subset U$ be an open quasicompact subset, then V is an open quasicompact subset of Y, and so $f|_{f^{-1}(U)}^{-1}(V) = f^{-1}(V)$ is a quasicompact open subset of $f^{-1}(U)$, and so $f|_{f^{-1}(U)}$ is quasicompact. Now suppose that U_i an open cover of Y such that $f|_{f^{-1}(U_i)}$ is a quasicompact morphism for each i. If V is a quasicompact open set, then finitely many of the U_i cover it, and without loss generality we can assume it is the first n of the U_i. Now cover each $V \cap U_i$ open affine schemes W_{ij}; the union of all such W_{ij} is equal to V, hence there is a finite subcover of the W_{ij}. Note that each W_{ij} is quasicompact because every affine scheme is quasicompact. It follows that:

$$f^{-1}(V) = f^{-1}\left(\bigcup_{ij} W_{ij}\right) = \bigcup_{ij} f^{-1}(W_{ij})$$

Now, $f^{-1}(W_{ij}) = (f|_{f^{-1}(U_i)})^{-1}(W_{ij})$ and is thus quasicompact by hypothesis. It follows that $f^{-1}(V)$ is a finite union of quasicompact spaces, and hence quasicompact implying c). ☐

The quasiseparated condition is a hair stranger than the quasicompact one, and best viewed as weakening of the separated separated condition.

Definition 3.11.2. Let $f : X \to Z$ be a morphism of schemes, then f is **quasiseparated** if the diagonal map $\Delta : X \to X \times_Z X$ is quasicompact. A Z-scheme X is a **quasiseparated Z scheme** if the structural morphism is quasiseparated. A scheme is **quasiseparated** if it is quasiseparated as $\mathbb{Z}$ scheme.

As with the quasicompact condition we prove the standard results about morphisms:

Lemma 3.11.2. *Quasiseparated morphisms are:*

a) Closed under composition.

b) Stable under base change.

c) *Local on target.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasiseparated morphisms. We write $\Delta : X \to X \times_Z X$ for the diagonal map we wish to show is quasicompact, and $\Delta_X : X \to X \times_Y X$, $\Delta_Y : Y \to Y \times_Z Y$ for the morphisms we know to be quasicompact. We emulate the proof of Proposition 3.6.3; from Theorem 2.3.1 we have the following cartesian square:

$$\begin{tikzcd} X \times_Y X \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"] & X \times_Z X \arrow[d, "f \times f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_Z Y \end{tikzcd}$$

where ψ is the morphism coming from the following diagram:[79]

$$\begin{tikzcd} X \times_Y X \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_X"] & & \\ & X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \times f"] \\ & X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

We claim that $\Delta = \psi \circ \Delta_X$; however this is obvious as:

$$\pi_X \circ \psi \circ \Delta_X = \mathrm{Id}_X \circ \pi_X \circ \Delta_X = \mathrm{Id}_X$$

so $\psi \circ \Delta_X$ makes the diagram defining Δ commute. Since ψ is the base change of Δ_Y and thus quasicompact, and Δ_X is quasicompact by assumption, we have that Δ is quasicompact by Lemma 3.11.1. This proves a).

For b), let $f : X \to Z$ be quasiseparated, and $g : Y \to Z$ another morphism. We need to show that $\pi_Y : X \times_Z Y \to Y$, and so need to show that the diagonal morphism:

$$X \times_Z Y \longrightarrow (X \times_Z Y) \times_Y (X \times_Z Y)$$

is quasicompact. Note that:

$$\begin{aligned} (X \times_Z Y) \times_Y (X \times_Z Y) &\cong X \times_Z (Y \times_Y X) \times_Z X \\ &\cong X \times_Z X \times_Z Y \end{aligned}$$

It follows that the diagonal map is, up to isomorphism, equal to $\Delta_X \times \mathrm{Id}_Y$ which is quasicompact by Theorem 3.1.2, probing b).

For c), suppose $f : X \to Y$ is quasiseparated, and $U \subset Y$ is an open subset. The diagonal morphism $f^{-1}(U) \to f^{-1}(U) \times_U f^{-1}(U)$ is given by $\Delta|_{f^{-1}(U)}$, which is quasicompact by Lemma 3.11.1. If f is a morphism such that for an open cover $\{U_i\}$ of Y, we have that each $f|_{f^{-1}(U_i)}$ is quasiseparated, then

$$\Delta|_{f^{-1}(U_i)} : f^{-1}(U_i) \longrightarrow f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$$

Since the $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ cover $X \times_Y X$ it follows by Lemma 3.11.1 that Δ is quasicompact, hence c). ☐

There is an equivalent, more topological formulation of a morphism of schemes being quasiseparated which mimics Proposition 3.6.4. We explore this with the following result:

Proposition 3.11.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is quasiseparated if and only if for every quasicompact open set $U \subset Y$, and quasicompact opens V_1 and V_2 of X which map into U, we have that $V_1 \cap V_2$ is quasicompact.*

[79] Abuse of notation alert! As always we are using π_X to refer to multiple maps.

Proof. Suppose that f is quasiseparated, and let U, V_1 and V_2 be as stated. Since $V_1 \times_U V_2$ is quasicompact, and f is quasiseparated, we have that $\Delta^{-1}(V_1 \times_U V_2)$ is quasicompact. Lemma 3.6.2 then implies that $V_1 \cap V_2$ is quasicompact.

Now suppose that for ever quasicompact open set $U \subset Y$, and every quasicompact opens V_1 and V_2 which map into U we have that $V_1 \cap V_2$ is quasicompact. We need to show that the diagonal morphism Δ is quasicompact. Let $W \subset X \times_Y X$ be a quasicompact open subset; then there is a finite open cover of W by affine schemes of the form $V_{ij} \times_{U_i} V_{ik}$, where V_{ij} and V_{ik} map into U_i. Note that U_i, V_{ij} and V_{ik} are quasicompact for all i, j, and k. We have that:

$$\Delta^{-1}(W) = \bigcup_{ijk} \Delta^{-1}(V_{ij} \times_{U_i} V_{ik}) = \bigcup_{ijk} V_{ij} \cap V_{ik}$$

which is a finite union of quasicompact sets, and is thus quasicompact. □

Note that the above implies that any scheme is quasiseparated over $\mathbb{Z}$, or any other quasicompact space for that matter, if the intersection of any two open quasicompact sets is quasicompact. We fix some nomenclature: by a scheme X qcqs over Z, we mean a scheme which is quasicompact and quasiseparated over Z. If we do not make the base scheme explicit, we will always mean $\operatorname{Spec}\mathbb{Z}$, which will in turn imply that X is a quasicompact scheme such that every intersection of quasicompact opens is again quasicompact.

With the above work on the quasiseparated condition, we may now shed light on why the qcqs hypothesis will be so fruitful: it will allow us to consider finite open covers who's intersections also admit finite open covers. Our first example of this fruitfulness will be in next section on the valuative criterion, but we will again employ this hypothesis in our chapter on $\mathscr{O}_X$ modules. For the moment, we explore some examples and non examples:

Example 3.11.1. A separated morphism is quasiseparated. In particular, since closed embeddings, open embeddings, proper morphisms, and affine morphisms[80] are separated, they are also quasiseparated.

Moreover, morphisms of finite type, closed embeddings, open embeddings, proper morphisms, and affine morphisms are all quasicompact.

Note that every Noetherian scheme is a Noetherian topological space, so every subspace is quasicompact. It follows that Noetherian schemes are qcqs.

Example 3.11.2. Any scheme that is of the form $\operatorname{Proj} A$ where the irrelevant ideal A_+ is not finitely generated up to radical is quasiseparated[81] but not quasicompact.

Example 3.11.3. Let Z be the scheme obtained by gluing $X = \operatorname{Spec}\mathbb{C}[x]$ nad $Y = \operatorname{Spec}\mathbb{C}[y]$ along the affine open U_x and U_y via the isomorphism induced by $x \mapsto y$. In particular, Z is the line with a double origin from Example 2.1.4. In Example 3.6.4 we demonstrated that Z is not separated, however, it is quasiseparated. Indeed, if $U_1, U_2 \subset Z$ are quasicompact, and do not contain any copy of either origin then the lie in an open subscheme isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ which is is obviously quasiseparated over $\operatorname{Spec}\mathbb{C}$, and thus over $\operatorname{Spec}\mathbb{Z}$. It follows that $U_1 \cap U_2$ is quasicompact. Now if U_1 contains a copy of one origin, and U_2 does not, then U_1 and U_2 are again contained in a single copy of $\mathbb{A}^1_{\mathbb{C}}$, and so their intersection is again quasicompact. The same is true if U_1 and U_2 contain the same copy of the origin. Now suppose that U_1 and U_2 contain different copies of the origin, then their intersection contains no copy of the origin, and so:

$$U_1 \cap U_2 = (U_1 \cap U_x) \cap (U_2 \cap U_y)$$

where by U_x and U_y we actually mean their image under the open embeddings. We have thus reduced this case to the original one where neither U_1 nor U_2 contain any copy of origin, hence $U_1 \cap U_2$ is quasicompact. It follows that every intersection of quasicompact opens is quasicompact and so Z si quasiseparated. Since Z is obviously quasicompact, Z is qcqs.

Example 3.11.4. Let $X = \operatorname{Spec} k[x_0, \dots]$ and $Y = \operatorname{Spec} k[y_0, \dots]$, and Z be the scheme glued along the open sets $\mathbb{V}(\langle x_0, \dots \rangle)^c$ and $\mathbb{V}(\langle y_0, \dots \rangle)^c$ induced by the map $x_i \mapsto y_i$. In particular, Z is quasicompact, but not quasiseparated because $X \cup Y \subset Z$ is the infinite plane with the origin removed. This is equal to infinite union of the distinguished opens U_{x_i} which has not finite subcover, and so cannot be quasicompact.

[80] And thus finite and integral morphisms
[81] As it is separated over $\operatorname{Spec} A_0$, and thus over $\operatorname{Spec}\mathbb{Z}$ by Example 3.6.4.

We end our discussion of qcqs schemes with the following result:

Lemma 3.11.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. The following hold:*

a) If $g \circ f$ is quasiseparated then so is f.

b) If $g \circ f$ is quasicompact, and g is quasiseparated, then f is quasicompact.

Proof. For a), assume that $g \circ f$ is quasiseparated, and let $V \subset Y$ be a quasicompact open subset, then in particular it's image in Z is quasicompact, hence by taking a finite open cover the image by affine opens, we can find a quasicompact open subset W which V maps ito. Now consider quasicompact opens $U_1, U_2 \subset X$ mapping into V. Since V maps into W, and $g \circ f$ is quasiseparated, we have that $U_1 \cap U_2$ is quasicompact, hence by Proposition 3.11.1 f must quasiseparated as well.

For b), note that f factors as:

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X \times f"] \arrow[rr, bend right, "f"] & X \times_Z Y \arrow[r, "\pi_Y"] & Y \end{tikzcd}$$

The morphism π_Y is the base change of the quasicompact morphism $g \circ f$, and thus quasicompact. Note that the first morphism satisfies $\pi_X \circ 1 \times f = \mathrm{Id}$. in particular, π_X is quasiseparated as it is the base change of $g : Y \to Z$. Up to isomorphism, $1 \times f$ comes from the morphism $X \times_Y Y \to X \times_Z Y$ which fits into the following diagram:

$$\begin{tikzcd} X \times_Y Y \arrow[r, "1 \times f"] \arrow[d, "\pi_Y"] & X \times_Z Y \arrow[d, "f \times \mathrm{Id}_Y"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_Z Y \end{tikzcd}$$

By Theorem 2.3.1 this square is cartesian, and so $1 \times f$ is quasicompact, as desired. It follows that f is the composition of quasicompact morphisms, and thus quasicompact itself. □

Corollary 3.11.1. *Let X be a scheme. If X is quasiseparated then any morphism $f : X \to Y$ is quasiseparated. If X is quasicompact, and Y is quasiseparated, then $f : X \to Y$ is also quasicompact. In particular, any morphism of qcqs schemes is qcqs.*

Proof. Let X be a quasiseparated, and $g : Y \to \operatorname{Spec} \mathbb{Z}$ the unique morphism. Then $g \circ f$ is the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ so the claim follows from Lemma 3.11.3. The same argument demonstrates the other two claims. □

3.12 The Valuative Criterion for Being Universally Closed

In the next sections we discuss a group of results, colloquially known as the valuative criteria, which will allow us to test whether a morphism is universally closed, or separated by examining lifting properties of certain commutative diagrams. We begin with the definition of a valuation ring:

Definition 3.12.1. A ring A is a **valuation ring** if A is an integral domain, A is local, and for every $a \in \mathrm{Frac}(A)^\times$, at least one of a or a^{-1} lies in A.

It is unimportant why this is called a valuation ring, but we discuss it briefly for historical reason. If we have a field k, then a valuation is a surjective map:

$$\nu : k^\times \longrightarrow \Gamma$$

where Γ is an ordered abelian group[82] satisfying:[83]

$$\nu(xy) = \nu(x) + \nu(y) \qquad \text{and} \qquad \nu(x+y) \geq \min\{\nu(x), \nu(y)\}$$

The ring associated to the valuation is the union:

$$A_\nu = \{0\} \cup \{a \in k^\times : \nu(a) \geq 0\}$$

[82] I.e. the integers. In particular, any ordered abelian group has to be compatible with addition and thus cannot have torsion.

[83] The following axioms, along with the fact that Γ should be torsion free, imply that $\nu(1)$ and $\nu(-1)$ are zero.

It is easy to see from the definition of that any ring associated to a valuation is a valuation ring with field of fractions k. In particular, the unique maximal ideal of A_ν is given by:

$$\mathfrak{m}_\nu = \{0\} \cup \{a \in k^\times : \nu(a) > 0\}$$

If $\nu(a) = 0$, then we have that:

$$0 = \nu(a \cdot a^{-1}) = \nu(a) + \nu(a^{-1}) = \nu(a^{-1})$$

so every element out side of $\mathfrak{m}_\nu$ is a unit in A_ν. It follows that $\mathfrak{m}_\nu$ is maximal, and in fact unique because any other maximal ideal not contained in $\mathfrak{m}_\nu$, and thus equal to $\mathfrak{m}_\nu$, would contain units. The following proof that these are equivalent ways of thinking of valuation rings is due to Krull:

Lemma 3.12.1. *Let k be a field, then there is a bijection:*

$$\left\{\begin{matrix} \text{valuations from } k \text{ up to} \\ \text{order preserving isomorphisms of } \Gamma \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{valuation rings with} \\ \text{fraction field } k \end{matrix}\right\}$$

Proof. The assignment $\nu \mapsto A_\nu$ is one direction of this bijection. Let A be a valuation ring with field of fractions k, and set $\Gamma = k^\times/A^\times$. This is an abelian group, and we make the unfortunate notational choice of setting:

$$[x] + [y] = [x \cdot y]$$

We give it a total order via:

$$[x] \leq [y] \Leftrightarrow y/x \in A \smallsetminus \{0\}$$

with equality if $a/b = 1$. Note that this well defined, if we choose different class representatives $x \cdot a_1$ and $y \cdot a_2$, then $ya_1/xa_2 = (y/x) \cdot (a_1/a_2)$, but $a_i \in A^\times$ hence $(y/x) \cdot (a_1/a_2) \in A^\times$. We check that this is a total order; it is clearly reflexive, and if we have that $[x] \leq [y]$ and $[y] \leq [z]$, then $x/y, y/z \in A^\times$, hence $x/z \in A^\times$ so $[x] \leq [z]$. If $[x] \leq [y]$ and $[y] \leq [x]$, then x/y and y/x lie in A, and hence $x/y \in A^\times$, so $[x] - [y] = [1] = 0 \in \Gamma$, implying that $[x] = [y]$. Finally we check that for all $[x]$ and $[y]$ we have either $[x] \leq [y]$ or $[y] \leq [x]$, however this follows from Definition 3.12.1, as if $x/y \notin A$, then $y/x \in A$ and vice versa.

We define ν_A to be the quotient map $k^\times \to k^\times/A^\times$. This satisfies $\nu_A(xy) = \nu_A(x) + \nu_A(y)$ by construction, hence we need to show that $[x+y] \geq \min\{[x], [y]\}$. Without loss of generality, suppose that $[x] \leq [y]$, then we have that $(x+y)/x = 1 + y/x \in A$ as $[x] \leq [y]$, implying the claim.

All that remains to be shown is that the assignments $\phi : \nu \mapsto A_\nu$ and $\psi : A \mapsto \nu_A$ are inverses of each other. We first show that for a valuation $\nu : k^\times \to \Gamma$:

$$\ker \nu = A_\nu^\times$$

for any valuation ν. We know that anything in the kernel of ν is a unit in $A_\nu^\times$. Suppose that $a \in A_\nu^\times$, then in particular, $a \notin \mathfrak{m}_\nu$ hence $a \in A_\nu^\times$. It follows that up to $\Gamma \cong k^\times/A_n^\times$, and that ν is up to isomorphism the quotient morphism. It is obvious that this isomorphism is order preserving, hence we have that $\psi \circ \phi$ is the identity.

Now suppose that A is a valuation ring, then we want to show that:

$$A = \{a \in k^\times : [a] \geq [1] \in k^\times/A^\times\}$$

However, this is obvious as if $a \in A$, then $a/1 \in A$ hence $[a] \geq [1]$, and if $[a] \geq [1]$ then $a/1 \in A$. This implies that $\phi \circ \psi = \mathrm{Id}$, and hence the claim. ☐

While this is historically where valuation rings come from, the main connection to algebraic geometry comes from a different but further equivalent characterization. We need the following definition:

Definition 3.12.2. Let A and B be local subrings of a field k, then B **dominates** A if $A \subset B$, and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$. We say that a local integral domain is **maximally dominant** if for all local subrings $B \subset \mathrm{Frac}(A)$, if B dominates A or A dominates B then $B = A$.

We first show that any maximally dominant ring is integrally closed:

Lemma 3.12.2. *Let A be maximally dominant, then A is integrally closed.*

Proof. We need to show that:

$$A = \{x \in k : x \text{ is integral over } A\}$$

Suppose that $x \in k$ is integral over A, and let B be the subring of k generated by A and x. Then the inclusion $\imath : A \to B$ is integral, hence by Lemma 3.10.3 hence there is a prime ideal $\mathfrak{p} \in \operatorname{Spec} B$ lying over the unique maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$. We have that $B_\mathfrak{p}$ is naturally a subring of k, and we claim that $B_\mathfrak{p}$ dominates A. Indeed, we need only show that:

$$\mathfrak{m} = A \cap \mathfrak{m}_\mathfrak{p}$$

In particular, if $\pi : B \to B_\mathfrak{p}$ is the localization map, then since π and $\imath$ are both naturally viewed as inclusions, we have that:

$$\mathfrak{m} = \imath^{-1}(\mathfrak{p}) = A \cap \mathfrak{p} = A \cap \pi^{-1}(\mathfrak{m}_\mathfrak{p}) = A \cap B \cap \mathfrak{m}_\mathfrak{p} = A \cap \mathfrak{m}_\mathfrak{p}$$

so $B_\mathfrak{p}$ dominates A. It follows that either $B_\mathfrak{p} = A$ or k. If $B_\mathfrak{p} = A$ then $x \in A$, hence every element integral over A lies in A, and A is integrally closed. □

We now show that a ring is a valuation ring if and only if it is maximally dominant:

Lemma 3.12.3. *A ring is maximally dominant if and only if it is a valuation ring.*

Proof. Suppose that A is a valuation ring with maximal ideal $\mathfrak{m}$, and set $k = \operatorname{Frac}(A)$. Let $B \subset k$ be a local ring with maximal ideal $\mathfrak{m}_B$, and suppose that B dominates A. Then we have that:

$$\mathfrak{m} = A \cap \mathfrak{m}_B$$

Let $b \in B$, if $b \notin A$ then $b^{-1} \in A$ by definition. However, if $b \notin A$ then $b^{-1} \in \mathfrak{m}$ as well as b^{-1} is not a unit in A. It follows that $b^{-1} \in \mathfrak{m}_B$ a contradiction, hence $b \in A$. It follows that $B \subset A$ and thus $A = B$, hence A is maximally dominant.

Suppose that A is maximally dominant, and let $x \in K$. We want to show that at least one of x or x^{-1} are in A. Assume without loss of generality that $x \notin A$, and let B be the subring of k generated by x and A. If there is a prime ideal $\mathfrak{p}$ of B lying over A then $B_\mathfrak{p}$ dominates A by our work in Lemma 3.12.2, implying that $B_\mathfrak{p} = A$. But then $B \subset B_\mathfrak{p} = A \subset B$ hence $A = B$ so no such $\mathfrak{p}$ can exist. Suppose that $\langle \mathfrak{m} \rangle \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} B$, then $\imath^{-1}(\mathfrak{p}) \in \operatorname{Spec} A$, and $\imath^{-1}(\langle \mathfrak{m} \rangle) = \mathfrak{m}$, hence $\mathfrak{m} \subset \imath^{-1}(\mathfrak{p})$, implying that $\imath^{-1}(\mathfrak{p}) = \mathfrak{m}$, which as we just mentioned is impossible. It follows that $\mathbb{V}(\langle \mathfrak{m} \rangle) \subset \operatorname{Spec} B$ is the emptyset, and thus $\langle \mathfrak{m} \rangle = B$, so we can write:

$$1 = \sum_i m_i b_i$$

for $m_i \in \mathfrak{m}$ and $b_i \in B$. Since any b_i can be written as a sum of polynomials in x with coefficients in A, and $m_i \cdot a \in \mathfrak{m}$ for all $a \in A$, we have that we can we replace the m_i and b_i to be of the form:

$$1 = \sum_{n=0}^{d} m_n x^n$$

Multiply both sides by x^{-d} to obtain that:

$$\begin{aligned} x^{-d} =& m_0 x^{-d} + m_1 x^{-d+1} + \cdots + m_{d-1} x^{-1} + m_d \\ =& m_0 x^{-d} + \sum_{n=1}^{d} m_n (x^{-1})^{d-n} \end{aligned}$$

We thus have that:

$$(1 - m_0) x^{-d} - \sum_{n=1}^{d} m_n (x^{-1})^{d-n} = 0$$

So the polynomial $p \in A[y]$ given by:

$$p(y) = (1 - m_0)y^d - \sum_{n=1}^{d} m_n (y)^{d-n}$$

satisfies $p(x^{-1}) = 0$. Moreover, we have that $1 - m_0 \notin \mathfrak{m}$ hence it is a unit, and so $p(y)/(1 - m_0)$ is a monic polynomial which has x^{-1} as a zero. Therefore x^{-1} is integral over A, and by Lemma 3.12.2 lies in A. □

We now begrudgingly introduce the following terminology to describe some very simple phenomena. Recall that on a scheme there are non-closed points, i.e. elements $x \in X$ such that $\overline{\{x\}} \neq \{x\}$. We say that y is a *specialization* of x, or x is *generalization* of y if $y \in \overline{\{x\}}$. We denote by $x \rightsquigarrow y$ y is a specialization of x, or x is a generalization . In essence, a specialization is a choice of point in side the closure another point. Now note that if A is a valuation ring, then there is unique specialization of the generic point to the unique maximal ideal, $\eta \rightsquigarrow \mathfrak{m}$. In fact, we will show that a choice of a specialization in a scheme X is equivalent to morphism $\operatorname{Spec} A \to X$ for some valuation ring A. Once we show this, we will demonstrate the connection between valuation rings, and a morphism being universally closed.

We need the following lemma:

Lemma 3.12.4. *Let k be a field, and $B \subset k$ a local subring, then there is a valuation ring with fraction field k dominating B.*

Proof. We partially order the set L_B of local subrings of k which dominate B by:

$$A_1 < A_2 \Leftrightarrow A_2 \text{ dominates } A_1$$

Let $\{A_i\}_{i \in I}$ be a totally ordered subset of L_B, then we set:

$$C = \bigcup_{i \in I} A_i$$

If $A_i < A_j$ then $A_i \subset A_j$ hence the above set is naturally a ring. Moreover, it the ideal:

$$\mathfrak{m}_C = \bigcup_i \mathfrak{m}_{A_i}$$

is an ideal of C. It is maximal because if $\mathfrak{m}_C \subset I$, then $I \cap A_i$ is an ideal containing $\mathfrak{m}_{A_i}$, and is thus equal to $\mathfrak{m}_{A_i}$ or A_i. If $I \cap A_i = A_i$, then $1 \in I$ and $I = C$, otherwise it follows that $\mathfrak{m}_C = C$ hence $\mathfrak{m}_C$ is maximal. Let $\mathfrak{n}$ be any other maximal ideal of C, then $\mathfrak{n} \cap A_i$ is a prime ideal of A_i and so contained in $\mathfrak{m}_{A_i}$ for all i. It follows that $\mathfrak{n} \subset \mathfrak{m}_A$ and thus $\mathfrak{n} = \mathfrak{m}_C$, so C is a local ring. It is obvious that C dominates every A_i by definition, and also dominates B.

By Zorn's lemma, it follows that there exists a maximal element of L_B, hence we need to show that if A is a maximal element of L_B then it has fraction field k. In particular, by the contrapositive, it suffices to show that if A does not have fraction field $\operatorname{Frac}(A) \subsetneq k$, then there exists a local ring $C \neq A$ which dominates A. Let $t \in k$ but not in $\operatorname{Frac}(A)$, then let $A[t]$ denote the A algebra generated by t.[84] This is still a subring of K, and the ideal $\langle t, \mathfrak{m} \rangle$ is obviously maximal; it is unique because every other elment of $A[t]$ is invertible. It follows that $A[t]_{\langle t, \mathfrak{m} \rangle}$ is a local ring not equal to A contained in k which dominates A.

Now suppose that t is not transcendental, then there is a polynomial $p \in A[x]$ such that $p(t) = 0$. If:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Then if we instead take the polynomial:

$$q(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

we have that $a_n t$ satisfies:

$$q(a_n t) = a_n^{n-1} p(t) = 0$$

so there exists some a such that at is integral over A. Let A' be the subring generated by A and at, then $A \hookrightarrow A'$ is an integral extension so by Lemma 3.10.3 there exists a $\mathfrak{p} \subset A'$ lying over $\mathfrak{m}$. We take $C = A'_{\mathfrak{p}}$, then C dominates A by the same argument in Lemma 3.12.2, and is obviously not equal to A. □

[84] This is a free A algebra as there are no polynomial relations in t.

In Proposition 2.1.2, we were able to construct a bijection between ring homomorphisms $A \to \mathscr{O}_X(X)$ and scheme morphisms $X \to \operatorname{Spec} A$. In order to show that certain valuation rings map into schemes, we will need construct morphism $\operatorname{Spec} A \to X$ when A is a local ring.

Lemma 3.12.5. *Let X be a scheme, and A local ring. Then the following hold:*

a) Then there is bijection between $\operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$ and pairs (x, ϕ) where $x \in X$ and $\phi : \mathscr{O}_{X,x} \to A$ is a local ring morphism.

b) $x \rightsquigarrow y$ if and only if x is in the image of $\operatorname{Spec} \mathscr{O}_{X,y} \to X$.

Proof. Let $f \in \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$, and let $x = f(\mathfrak{m})$. Then we have the induced local ring homomorphism $f_{\mathfrak{m}} : \mathscr{O}_{X,x} \to A_{\mathfrak{m}}$, but $A_{\mathfrak{m}} = A$ as every element outside of $\mathfrak{m}$ is already invertible. It follows that $(f(\mathfrak{m}), f_{\mathfrak{m}})$ a a pair of the desired form.

Now let (x, ϕ) be an aforementioned pair. Let $\operatorname{Spec} B$ be an affine open containing x, and identify x with $\mathfrak{p}$. Then ϕ is of the form $B_{\mathfrak{p}} \to A$, and so we get a morphism $B \to A$ by precomposing with the localization map. This yields a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$, which gives a morphism $f_{(x,\phi)} : \operatorname{Spec} A \to X$ by post composing with the open embedding.

By construction, the stalk map $f_{(x,\phi)_{\mathfrak{m}}} : \mathscr{O}_{X,x} \cong B_{\mathfrak{p}} \to A$ is equal to ϕ up to isomorphism. Moreover, we have that $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}_{\mathfrak{p}}$, and $\pi^{-1}(\mathfrak{m}_{\mathfrak{p}}) = \mathfrak{p} = x$, hence $f_{(x,\phi)}(\mathfrak{m}) = x$.

To complete the proof, we want to show that $f_{\mathfrak{m}}$ induces the same scheme morphism f. We first show that $\operatorname{im} f$ is contained in any affine open containing $f(\mathfrak{m})$. Suppose that $f(\mathfrak{m}) \in \operatorname{Spec} B \subset X$, then $f^{-1}(\operatorname{Spec} B)$ contains $\mathfrak{m}$. It follows that $f^{-1}(\operatorname{Spec} B)$ is a union of distinguished opens U_a, one of which must satisfy $a \notin \mathfrak{m}$, but then a is invertible and $U_a = \operatorname{Spec} A$. It follows that the construction of $f_{(f(\mathfrak{m}),f_{\mathfrak{m}})}$ is independent of the chosen affine open containing $f(\mathfrak{m})$. Choose such an open affine $\operatorname{Spec} B$, then f and $f_{(f(\mathfrak{m}),f_{\mathfrak{m}})}$ come from ring morphisms $\phi, \psi : B \to A$. The stalk maps are the unique ones coming from localization, and so satisfy:

$$\phi = f_{\mathfrak{m}} \circ \pi_{\mathfrak{p}} \qquad \text{and} \qquad \psi = f_{(f(\mathfrak{m}),f_{\mathfrak{m}})} \circ \pi_{\mathfrak{p}}$$

Since $f_{\mathfrak{m}} = f_{(f(\mathfrak{m}),f_{\mathfrak{m}})}$ by construction, we have that $\phi = \psi$, and thus $f = f_{(f(\mathfrak{m}),f_{\mathfrak{m}})}$ implying *a*).

For *b*), first note that if $x \rightsquigarrow y$ in any topological space X, and $f : X \to Y$ is continuous map, then $f(x) \rightsquigarrow f(y)$ as

$$y \in \overline{\{x\}} \Rightarrow f(y) \in f(\overline{\{x\}})$$

but a map is continuous if and only $f(\bar{V}) = \overline{f(V)}$ for all subsets $V \subset X$. By the work above, if we let $\operatorname{Spec} A$ be any affine open containing y, and set $y = \mathfrak{p} \in \operatorname{Spec} A$, then the map $\operatorname{Spec} \mathscr{O}_{X,y} \to X$ comes from the morphism $\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A$ post composed with an open embedding. Now suppose that x is in the image of $\operatorname{Spec} A_{\mathfrak{p}}$, then there is some $\mathfrak{q} \in \operatorname{Spec} A_{\mathfrak{p}}$ such that $\mathfrak{q} \rightsquigarrow \mathfrak{m}_{\mathfrak{p}}$. It follows that $x \rightsquigarrow y$.

If $x \rightsquigarrow y$, then let $U = \operatorname{Spec} A$ be an affine open containing y. Suppose that $x \notin U$, then $x \in X \setminus U$, which is a closed subset. It follows that $\overline{\{x\}} \subset X \setminus U$, hence $y \in X \setminus U$ a contradiction. Therefore, we have tha $x \in U$ as well. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals of A associated to y and x respectively. We have that $\mathfrak{q} \subset \mathfrak{p}$, so if π is the localization map then the prime ideal $\langle \pi(\mathfrak{q}) \rangle \subset A_{\mathfrak{p}}$ is contained in $\mathfrak{m}_{\mathfrak{p}}$. The morphism $\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A$ coming from π then sends $\mathfrak{m}_{\mathfrak{p}}$ to $\mathfrak{p}$, and $\langle \pi(\mathfrak{q}) \rangle$ to $\mathfrak{q}$, hence x is in the image of $\operatorname{Spec} \mathscr{O}_{X,y} \to X$ implying *b*). □

We now prove the aforementioned and desired result regarding valuation rings:

Proposition 3.12.1. *Let X be a scheme, and $x \rightsquigarrow y$ specialization of points. Then the following hold:*

a) There exists a valuation ring A, and a morphism $f : \operatorname{Spec} A \to X$ such that $f(\eta) = x$ and $f(\mathfrak{m}) = y$.

b) Given any field extension k/k_x, we can find an f such that the induced extension k_η/k_x is isomorphic to the given one.

Proof. Fix $x \rightsquigarrow y$ and a field extension k/k_x. By part *a*) of the previous lemma, we have a morphism:

$$\operatorname{Spec} \mathscr{O}_{X,y} \to X$$

which for any $\operatorname{Spec} C$ containing $y = \mathfrak{p}$, comes from the morphism $\operatorname{Spec} C_{\mathfrak{p}} \to \operatorname{Spec} A$ given by the localization map π. As mentioned, we have that $x = \mathfrak{q}$ is in the image of $\operatorname{Spec} C_{\mathfrak{p}} \to \operatorname{Spec} C$. In particular,

$\mathfrak{q}_\mathfrak{p} = \langle \pi(\mathfrak{q}) \rangle$ maps to $\mathfrak{q}$, and there is a morphism $C_\mathfrak{p} \to (C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ given by localizing further. However, everything invertible in $(C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ is invertible in $C_\mathfrak{q}$ and vice versa, so there is a canonical isomorphism $(C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}} \cong C_\mathfrak{q}$. It follows that there is a canonical morphism $C_\mathfrak{p} \to k_x = C_\mathfrak{q}/\mathfrak{m}_\mathfrak{q}$. In particular, the above implies that the morphism $\operatorname{Spec} k_x \to X$ taking η to x factors as:

$$\operatorname{Spec} k_x \to \operatorname{Spec} O_{X,y} \to X$$

which now takes η to x, and $\mathfrak{m}_y$ to y. We thus have a morphism of local rings:

$$\mathscr{O}_{X,y} \to k_x \to K$$

Let B be the image of $\mathscr{O}_{X,y}$ in k,[85] and A be any valuation ring which dominates B and satisfies $\operatorname{Frac}(A) = k$. Such an A exists by Lemma 3.12.4.

Since A dominates B, we have that $B \subset A$, hence there is a morphism:

$$\phi : \mathscr{O}_{X,y} \to k_x \hookrightarrow A$$

We let $f : \operatorname{Spec} A \to X$ be the induced morphism. Since $k_\eta = A_{\langle 0 \rangle} = k$, we have that k/k_x is isomorphic to k_η/k essentially by construction. We need to check that $f(\eta) = x$, and $f(\mathfrak{m}) = y$. In particular, we have that $\phi^{-1}(\eta)$ is the kernel of the morphism $\beta : C_\mathfrak{p} \to k_x$. We have the following commutative diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_\mathfrak{q} \arrow[r] & k_x \\ C_\mathfrak{p} \arrow[ur, dashed, "\alpha"] & & \end{tikzcd}$$

The kernel of $C_\mathfrak{q} \to k_x$ is:

$$\mathfrak{m}_\mathfrak{q} = \left\{ \frac{q}{a} \in C_\mathfrak{q} : q \in \mathfrak{q} \right\}$$

We obviously then have that:

$$\alpha^{-1}(\mathfrak{m}_\mathfrak{q}) = \mathfrak{q}_\mathfrak{p} = \left\{ \frac{q}{a} \in C_\mathfrak{q} : q \in \mathfrak{q} \right\}$$

It follows that η maps to $\mathfrak{q}_\mathfrak{p}$ which as discussed maps to $\mathfrak{q} = x$. Now let $\beta : C_\mathfrak{p} \to B$, and $\imath : B \to A$. We have that $\phi : C_\mathfrak{p} \to A$ factors as $\imath \circ \beta$, and that $\imath^{-1}(\mathfrak{m}) = \mathfrak{m}_B = \langle \beta(\mathfrak{m}_\mathfrak{p}) \rangle$. It follows that $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}_\mathfrak{p}$ which maps to $\mathfrak{p} = y$. Given $x \rightsquigarrow y$, we have thus found a valuation ring A satisfying a) and b).

□

3.13 The Valuative Criteria for Being Separated

[85]The image of a local ring is always a local ring.