Algebraic Geometry: Filling in the Gaps

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Contents

Introduction

Spec **and the Structure Sheaf**

1.1 Spec **of a Ring**

Throughout this expository paper, we take our rings to be commutative with identity. We begin with the definition of Spec:

Definition 1.1.1. If R is a commutative ring, then $\operatorname{Spec} R$ is a priori a set defined by:

$$\operatorname{Spec} R = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R\}$$

Example 1.1.1. Let $R = \mathbb{Z}$, then we have that $\operatorname{Spec} \mathbb{Z}$ can be identified with the set of all prime numbers. Moreover, if R is a field, then $\operatorname{Spec} R$ is the singleton set consisting only of the zero ideal $\langle 0 \rangle$. If $R = \mathbb{R}[x]$ is the polynomial ring with real coefficients, then:

$$\operatorname{Spec} R = \{\langle 0 \rangle, \langle x - r \rangle, \langle x^2 + bx + c \rangle : r, b, c \in \mathbb{R}, b^2 - 4c < 0\}$$

Note that if $\phi : A \to B$ is a ring homomorphism between commutative rings A and B, we have that there is induced set map:

$$\begin{aligned} \psi : \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ \mathfrak{q} &\longmapsto \phi^{-1}(q) \end{aligned}$$

turning Spec into a contravariant functor from the category of commutative rings to the category of sets. We will shortly put a topology on $\operatorname{Spec} R$ so that the induced set map is actually a continuous map between topological spaces.

Definition 1.1.2. Let I be an ideal of a commutative ring R, then we define the set $\mathbb{V}(I)$ to be:

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$$

If $f \in A$, then we take $\mathbb{V}(f) := \mathbb{V}(\langle f \rangle)$, and clearly we have that:

$$\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} R : f \in \mathfrak{p}\}$$

Similarly for any set S, we define $\mathbb{V}(S) := \mathbb{V}(\langle S \rangle)$.

We now have the following:

Proposition 1.1.1. *Taking the closed sets of* $\operatorname{Spec} R$ *to be* $\mathbb{V}(I)$ *defines a topology on* $\operatorname{Spec} R$ *such that the induced map* $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ *from* $\phi : A \to B$ *is continuous.*

Proof. We need to check that the finite unions of closed sets are closed, that infinite intersections of closed sets are closed, and that $\emptyset$ and $\operatorname{Spec} R$ are closed. We begin with the latter, note that:

$$\mathbb{V}(\operatorname{Spec} R) = \{\mathfrak{p} \in \operatorname{Spec} R : \operatorname{Spec} R \subset \mathfrak{p}\} = \emptyset$$

and that:

$$\mathbb{V}(\langle 0 \rangle) = \{\mathfrak{p} \in \operatorname{Spec} R : 0 \in \mathfrak{p}\} = \operatorname{Spec} R$$

so the emptyset and $\operatorname{Spec} R$ are closed. Now suppose that I and J are two ideals, then:

$$\mathbb{V}(I) \cup \mathbb{V}(J) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \in \mathbb{V}(I) \text{ or } \mathfrak{p} \in \mathbb{V}(J)\}$$

We claim that this equal to $\mathbb{V}(I \cap J)$. Suppose that $\mathfrak{p} \in \mathbb{V}(I) \cup (J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, if $I \subset \mathfrak{p}$, then we have that $I \cap J \subset I \subset \mathfrak{p}$, and similarly for J, hence $\mathfrak{p} \in \mathbb{V}(I \cap J)$. If $\mathfrak{p} \in \mathbb{V}(I \cap J)$, then $I \cap J \subset \mathfrak{p}$; let $r \in I \cdot J$, then $r = i \cdot j$ for some $i \in I$ and $j \in J$. It follows that $r \in I \cap J$, so $I \cdot J \subset \mathfrak{p}$. Now suppose that $I \not\subset \mathfrak{p}$, then there exists at least one $i \notin \mathfrak{p}$. It follows that for all $j \in J$, that $i \cdot j \in \mathfrak{p}$, hence $J \subset \mathfrak{p}$. The same argument for J then implies that if $J \not\subset \mathfrak{p}$, then $I \subset \mathfrak{p}$. Note that if neither $I \subset \mathfrak{p}$, nor $J \subset \mathfrak{p}$, we have that there exists an $i \in I$, and a $j \in J$, such that $i, j \notin \mathfrak{p}$, but $i \cdot j \in \mathfrak{p}$ contradicting the fact that $\mathfrak{p}$ is prime. It follows that if $I \cap J \subset \mathfrak{p}$, then $I \cdot J \subset \mathfrak{p}$, and thus either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, implying that $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, hence $\mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J)$, as desired.

Now let I_α be an infinite family of ideals, we claim that:

$$\bigcap_\alpha \mathbb{V}(I_\alpha) = \mathbb{V}\left(\sum_\alpha I_\alpha\right)$$

where $\sum_\alpha I_\alpha$ is the smallest ideal containing I_α. In other words, it is the ideal generated by $\cup_\alpha I_\alpha$. Suppose that $\mathfrak{p} \in \cap_\alpha \mathbb{V}(I_\alpha)$, then we have that $I_\alpha \subset \mathfrak{p}$ for all α. Now since an $i \in \sum_\alpha I_\alpha$, can be written as the a finite sum $\sum_{j=1}^n r_j$, where each $r_j \in I_\alpha \subset \mathfrak{p}$, we have that $i \in \mathfrak{p}$, so $\bigcap_\alpha \mathbb{V}(I_\alpha) \subset \mathbb{V}\left(\sum_\alpha I_\alpha\right)$. Now suppose that $\mathfrak{p} \in \mathbb{V}\left(\sum_\alpha I_\alpha\right)$, then $\sum_\alpha I_\alpha \subset \mathfrak{p}$. It follows that for all α, $I_\alpha \subset \sum_\alpha I_\alpha$, so $I_\alpha \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(I_\alpha)$ for all α. It follows that $\mathfrak{p} \in \cap_\alpha \mathbb{V}(I_\alpha)$ implying the claim.

Let $\phi : A \to B$ be a ring homomorphism, and $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding set map. We need only show that for each $I \subset A$, that $\psi^{-1}(\mathbb{V}(I))$ is a closed set in $\operatorname{Spec} B$. We have that:

$$\begin{aligned}
\psi^{-1}(\mathbb{V}(I)) =& \{\mathfrak{q} \in \operatorname{Spec} B : \psi(\mathfrak{q}) \in \mathbb{V}(I)\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \phi^{-1}(\mathfrak{q}) \in \mathbb{V}(I)\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : I \subset \phi^{-1}(\mathfrak{q})\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \phi(I) \subset \mathfrak{q}\} \\
=& \{\mathfrak{q} \in \operatorname{Spec} B : \langle \phi(I) \rangle \subset \mathfrak{q}\} \\
=& \mathbb{V}(\langle \phi(I) \rangle) \\
=& \mathbb{V}(\phi(I))
\end{aligned}$$

so ψ is a continuous map. □

The above topology is called the **Zariski topology** on $\operatorname{Spec} R$. We also have the following helpful lemma:

Lemma 1.1.1. *Let R be a commutative ring, then the following relations hold:*

a) $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

b) $J \subset I \Longrightarrow \mathbb{V}(J) \supset \mathbb{V}(I)$

c) $\mathbb{V}(I) \subset \mathbb{V}(J) \Longleftrightarrow \sqrt{I} \supset \sqrt{J}$

Proof. First note that the radical of I is defined by:

$$\sqrt{I} = \{r \in R : \exists n \in \mathbb{Z}^+, r^n \in I\} = \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p}$$

If $\mathfrak{p} \in \mathbb{V}(I)$, then we have that $I \subset \mathfrak{p}$. Suppose that $r \in \sqrt{I}$, then we have that $r^n \in I$, so $r^n \in \mathfrak{p}$. We can write $r^n = r^{n-1} \cdot r$, so either $r^{n-1} \in \mathfrak{p}$ or $r \in \mathfrak{p}$. If $r \in \mathfrak{p}$, then we are done. If $r^{n-1} \in \mathfrak{p}$, then we repeat the process until we come to conclusion that $r^2 \in \mathfrak{p}$, implying $r \in \mathfrak{p}$, so $\sqrt{I} \subset \mathfrak{p}$, hence $\mathbb{V}(I) \subset \mathbb{V}(\sqrt{I})$. If $\mathfrak{p} \in \mathbb{V}(\sqrt{I})$, then we have that $\sqrt{I} \subset \mathfrak{p}$, however clearly $I \subset \sqrt{I}$, so $I \subset \mathfrak{p}$, hence $\mathbb{V}(\sqrt{I}) \subset \mathbb{V}(I)$, implying *a*).

Now suppose that $J \subset I$, and let $\mathfrak{p} \in \mathbb{V}(I)$. It follows that $I \subset \mathfrak{p}$, so $J \subset I \subset \mathfrak{p}$, implies that $\mathfrak{p} \in \mathbb{V}(J)$, so $\mathbb{V}(J) \supset \mathbb{V}(I)$, hence we have $b)$.

Finally suppose $\mathbb{V}(I) \subset \mathbb{V}(J)$. By definition:

$$\sqrt{J} = \bigcap_{\mathfrak{p} \in \mathbb{V}(J)} \mathfrak{p} \qquad \text{and} \qquad \sqrt{I} = \bigcap_{\mathfrak{p} \in \mathbb{V}(I)} \mathfrak{p}$$

Suppose that $r \in \sqrt{J}$, then $r \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathbb{V}(J)$. Since all $\mathfrak{p} \in \mathbb{V}(I)$ lie in $\mathbb{V}(J)$ as well, it follows that $r \in \sqrt{I}$ hence $\sqrt{J} \subset \sqrt{I}$. If $\sqrt{J} \subset \sqrt{I}$, then by $b)$ and $a)$ we have that $\mathbb{V}(I) \subset \mathbb{V}(J)$ implying $c)$.

□

We want to develop a basis for the Zariski topology on $\operatorname{Spec} R$.

Definition 1.1.3. For each $r \in R$, define the **distinguished open** to be:

$$U_f = \mathbb{V}(f)^c = \operatorname{Spec} R \setminus \mathbb{V}(f)$$

Lemma 1.1.2. *The set of distinguished opens form a basis for the Zariski topology on* $\operatorname{Spec} R$.

Proof. Suppose that $U \subset \operatorname{Spec} R$ is an open subset, then for some I we have that:

$$\begin{aligned} U =& \mathbb{V}(I)^c \\ =& \mathbb{V}\left(\sum_{i \in I} \langle i \rangle\right)^c \\ =& \left(\bigcap_{i \in I} \mathbb{V}(i)\right)^c \\ =& \bigcup_{i \in I} U_i \end{aligned}$$

so any open set is the arbitrary union of distinguished opens, hence the distinguished opens generate the Zariski topology on $\operatorname{Spec} R$. □

Note that if $\mathfrak{q} \in U_f \cap U_g$, then $f \notin \mathfrak{q}$ and $g \notin \mathfrak{q}$, so $fg \notin \mathfrak{q}$, hence $\mathfrak{q} \in U_{fg}$. We thus have that the intersection of two distinguished opens is again a distinguished open. We also have the following lemma, akin to Lemma 1.1.1:

Lemma 1.1.3. *For all $f, g \in R$, the distinguished opens satisfy:*

a) $U_{f^n} = U_f$

b) $U_f \subset U_g \iff \sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$

c) $U_f \subset U_g \iff \exists m \in \mathbb{Z}^+, r \in R, f^m = r \cdot g$

Proof. Suppose that $\mathfrak{q} \in U_{f^n}$, then $f^n \notin \mathfrak{q}$, however this implies that both f^{n-1} and f are not in $\mathfrak{q}$, so $\mathfrak{q} \in U_f$. Now suppose that $\mathfrak{q} \in U_f$, then $f \notin \mathfrak{q}$, so $f^2 \notin \mathfrak{q}$. Assume that $f^n \notin \mathfrak{q}$, then $f^{n+1} = f^n \cdot f \notin \mathfrak{q}$, so $f^n \notin \mathfrak{q}$ by induction. This then implies $a)$.

Suppose that $U_f \subset U_g$, then we have that:

$$\mathbb{V}(f)^c \subset \mathbb{V}(g)^c \implies \mathbb{V}(f) \supset \mathbb{V}(g)$$

It follows from Lemma 1.1.1 that $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$. Suppose that $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, then again from Lemma 1.1.1, we have that $\mathbb{V}(g) \subset \mathbb{V}(f)$, taking compliments we thus have shown $b)$.

For $c)$, we see that if $U_f \subset U_g$, then $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, implying that $f \in \sqrt{\langle g \rangle}$, so there exists some $m \in \mathbb{Z}^+$, and some $r \in R$ such that $f^m = r \cdot g$. Conversely, if we have that $f^m = r \cdot g$, then we have that $f \in \sqrt{\langle\langle\rangle\, g\rangle}$. So suppose that $a \in \sqrt{\langle f \rangle}$, then $a^k = p \cdot f$ for some $k \in \mathbb{Z}^+$, and some $p \in R$. It follows that:

$$(a^k)^m = p^m \cdot f^m = (p^m \cdot r) \cdot g \in \langle g \rangle$$

so $\sqrt{\langle f \rangle} \subset \sqrt{\langle g \rangle}$, and by $b)$ we have that $U_f \subset U_g$, implying $c)$. $\square$

We now want to show that each U_f is actually homeomorphic to Spec of some ring. We begin with the following definition:

Definition 1.1.4. Let A be a commutative ring, S be a multiplicatively closed set, then the **localization** of A by S, denoted $S^{-1}A$, is a ring equipped with a morphism $\pi : A \to S^{-1}A$, such that for all $s \in S$ $\pi(s)$ is invertible in $S^{-1}A$, and for any homomorphism $\phi : A \to B$ where $\phi(s)$ is a unit for all $s \in S$, there exists a unique homomorphism $\theta : S^{-1}A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \\ S^{-1}A \arrow[ur, "\theta"'] & \end{tikzcd}$$

Our first goal is to show that such a ring exists.

Proposition 1.1.2. *Let A be a ring, and S be a multiplicatively closed set. Then $S^{-1}A$ exists, and is unique up to unique isomorphism.*

Proof. We define an equivalence relation on the set $A \times S$ by:

$$(a, s) \sim (b, t) \iff \exists u \in S, u(at - sb) = 0$$

It is clear that this relation is symmetric, reflexive, and with some work transitive, hence it it indeed defines an equivalence relation. We claim that $A \times S/\sim$ has the structure of a ring. We define addition by:

$$[a, s] + [b, t] = [at + bs, ts]$$

We check that this well defined. Suppose that $[f, v] = [a, s]$, then we need to show that:

$$[ft + bv, tv] = [at + bs, ts]$$

so we need to find a u such that:

$$u(ft^2s + bvts - at^2v + btsv) = u(ft^2s - at^2v) = 0$$

Note that there exists a w such that $w(fs - av) = 0$, hence with $u = w$ we have that:

$$w(ft^2s - at^2v) = t^2(w[fs - av]) = t^2 \cdot 0 = 0$$

so addition is well defined. It is then clear that for any s, the zero element is given by $[0, s]$, and that any $[a, s]$, has inverse given by $[-a, s]$, so $A \times S/\sim$ is an abelian group. We define a ring structure on $A \times S/\sim$ by:

$$[a, s] \cdot [b, t] = [ab, st]$$

We again wish to check that this well defined, so let $[f, v] = [a, s]$, then:

$$[f, v] \cdot [b, t] = [fb, vt]$$

We again want to find a u such that:

$$u \cdot (fbst - abvt) = 0$$

Let $u = w$, then we have that:

$$wfbst - wabvt = bt(wfs - wav) = 0$$

so multiplication is well defined. It is then clear that the multiplicative identity is given by $[1, 1]$ which is then clearly equivalent to $[s, s]$ for any $s \in S$.

Let the map $\pi : A \to A \times S/\sim$ be given by:

$$a \longmapsto [a, 1]$$

This is then clearly a ring homomorphism, and we see that for:

$$\theta \circ \pi = \phi$$

we must have that:

$$\theta([a, 1]) = \phi(a)$$

for all $a \in A$. We thus define θ by:

$$\theta([a, s]) = \phi(a) \cdot \phi(s)^{-1}$$

where $\phi(s)^{-1}$ exists, as $\phi(S)$ is a set of units in B. This uniquely determines θ, so long as it is well defined. We check that this well defined, let $[a, s] = [f, v]$, then $wav = wfs$, so we have that:

$$\phi(w) \cdot \phi(a) \cdot \phi(v) = \phi(w) \cdot \phi(f) \cdot \phi(s)$$

Since $\phi(w), \phi(s), \phi(v)$ are all units, we then have that by multiplying both sides by $\phi(w)^{-1}$, $\phi(s)^{-1}$, and $\phi(v^{-1})$:

$$\theta([a, s])\phi(a) \cdot \phi(s)^{-1} = \phi(f) \cdot \phi(v)^{-1} = \theta([f, v])$$

To see this is a ring homomorphism, we note that:

$$\theta([a, s]) + \theta([b, t]) = \phi(a)\phi(s)^{-1} + \phi(b) \cdot \phi(t)^{-1}$$

while:

$$\theta([at + bs, st]) = \phi(at + bs) \cdot \phi(st)^{-1} = \phi(a) \cdot \phi(s)^{-1} + \phi(b) \cdot \phi(t)^{-1}$$

so θ respects addition. Moreover,

$$\theta([a, s]) \cdot \theta([b, t]) = \phi(a) \cdot \phi(b) \cdot \phi(s)^{-1} \cdot \phi(t)^{-1}$$

while:

$$\theta([ab, st]) = \phi(ab)\phi(st)^{-1} = \phi(a) \cdot \phi(b) \cdot \phi(s)^{-1} \cdot \phi(t)^{-1}$$

so θ respects multiplication as well, and is thus a ring homomorphism such that $\theta \circ \pi = \phi$. It follows that $A \times S/\sim$ satisfies Definition 1.1.4, and so $S^{-1}A$ exists, and is unique up to unique isomorphism as it is defined by a universal property[1]. □

Note that the localization of A by S is easily seen to mimic multiplication and addition of fractions, it is for the purpose that going forward we denote the equivalence classes $[a, s]$ by:

$$\frac{a}{s}$$

Moreover, if $f \in A$, we denote by A_f the localization of A by the multiplicatively closed subset $\{1, f, f^2, \dots\}$, and if $\mathfrak{p}$ is a prime ideal of A, we denote by $A_\mathfrak{p}$ the localization of A by the multiplicatively closed subset $(A - \mathfrak{p})$. Moreover, A_f can be thought of as the polynomial ring:

$$A_f = A[1/f]$$

We have the following lemma:

[1]Note that $\pi(S)$ is a set of units in $S^{-1}A$, so one can apply the universal property to any other object satisfying said property and get a unique isomorphism between the two.

Lemma 1.1.4. *Let A be a commutative ring, and $f, g \in A$, then there exist unique isomorphisms:*

$$(A_f)_g \cong A_{fg} \cong (A_g)_f$$

where in the first and third terms g and f are really the equivalence classes $g/1$ and $f/1$. Moreover, if $\sqrt{\langle g \rangle} = \sqrt{\langle f \rangle}$, then there exists a unique isomorphism:

$$A_f \cong A_g$$

Proof. Clearly we need only prove that $(A_f)_g \cong A_{fg}$, as the proof of the other isomorphism will be identical. We first note that the map the natural map $\pi_{fg} : A \to A_{fg}$ maps:

$$f \longmapsto \frac{f}{1}$$

This is clearly a unit in A_{fg} where $(f/1)^{-1}$ is given by g/fg. It follows that there exists a unique map $\omega : A_f \to A_{fg}$ given by:

$$\frac{a}{f^k} \longmapsto \frac{a}{1} \cdot \frac{g^k}{f^k g^k} = \frac{ag^k}{f^k g^k}$$

Now suppose that $\phi : A_f \to B$ is any map such that $g/1$ is a unit in B, we want to show that there exists a unique map $\theta : A_{fg} \to B$ such that:

$$\theta \circ \omega = \phi$$

However, note that:

$$\omega \circ \pi_f(a) = \frac{a}{1}$$

so:

$$\omega \circ \pi_f = \pi_{fg}$$

Moreover, we obtain a unique map $\psi : A \to B$ such that both f and g are units in B, defined by:

$$\psi = \phi \circ \pi_f$$

We thus have the following diagram:

$$\begin{tikzcd}
A \arrow[d, "\pi_f"] \arrow[dr, "\psi"] \arrow[dd, bend right, "\pi_{fg}"'] & \\
A_f \arrow[r, "\phi"] \arrow[d, "\omega"] & B \\
A_{fg} \arrow[ur, dotted, "\exists!\theta"'] &
\end{tikzcd}$$

where θ is the unique homomorphism such that $\theta \circ \pi_{fg} = \psi$. We need to check that θ satisfies:

$$\theta \circ \omega = \phi$$

Let $\frac{a}{f^k} \in A_f$, then by definition we have that:

$$\phi(a/f^k) = \psi(a) \cdot \psi(f^k)^{-1}$$

Meanwhile:

$$\theta \circ \omega(a/f^k) = \theta(ag^k/(g^k f^k)) = \psi(ag^k) \cdot \psi(g^k f^k)^{-1} = \psi(a) \cdot \psi(f^k)^{-1}$$

so θ is the unique map which satisfies $\theta \circ \omega = \phi$. It follows that A_{fg} satisfies the universal property of the localization of A_f by $g/1$, then $(A_f)_g$ is uniquely isomorphic to A_{fg}.

Note that if $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$, then there exists elements $u, v \in A$ and $m, n > 0$ such that:

$$f^m = ug \qquad \text{and} \qquad g^n = vf$$

It follows that $\pi_g(f)$ is a unit in A_g and that $\pi_f(g)$ is a unit in A_f with inverses given by v/g^n and u/f^m respectively. We thus have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\pi_g"'] \arrow[d, "\pi_f"] & A_g \arrow[dl, shift left, "\theta_g"] \\
A_f \arrow[ur, shift left, "\theta_f"] &
\end{tikzcd}$$

Let $a/f^k \in A_f$, then:

$$\theta_g(a/f^k) = \frac{av^k}{g^{nk}}$$

and then:

$$\theta_f \circ \theta_g(a/f^k) = \frac{av^k u^{nk}}{f^{nkm}}$$

so we need to find a K such that:

$$f^K(av^k u^{nk} f^k - f^{nkm} a) = 0$$

However we see that:

$$av^k u^{nk} f^k = ag^{nk} u^{nk} = af^{nkm}$$

so $K = 0$ will do, and we see that $\theta_f \circ \theta_g = \mathrm{Id}$. The same argument shows that $\theta_g \circ \theta_f = \mathrm{Id}$, so $A_f \cong A_g$ are isomorphic as desired. $\square$

Proposition 1.1.3. *Let A be a commutative ring, and $f \in A$, then the distinguished open set U_f is homeomorphic to* $\operatorname{Spec} A_f$.

Proof. We have a ring homomorphism $\pi : A \to A_f$, which induces a continuous map $\psi : \operatorname{Spec} A_f \to \operatorname{Spec} A$. We first want to show that $\operatorname{im} \psi = U_f$. Suppose that $\mathfrak{p} \in \operatorname{im} \psi$, then $\mathfrak{p}$ is of the form $\pi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} A_f$. Note that:

$$\pi^{-1}(\mathfrak{q}) = \{a \in A : \pi(a) \in \mathfrak{q}\}$$

If $f \in \pi^{-1}(q)$, then we have that $\pi(f) = f/1 \in \mathfrak{q}$, implying that $1 \in \mathfrak{q}$ so it follows that that $f \notin \pi^{-1}(q)$, hence $\mathfrak{p} \in U_f$. Now suppose that $\mathfrak{p} \in U_f$, we want to show that there exists a prime ideal $\mathfrak{q} \in A_f$ such that $\pi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Define $\mathfrak{q}$ by:

$$\mathfrak{q} = \left\{ \frac{p}{f^k} \in A_f : p \in \mathfrak{p}, k \geq 0 \right\}$$

We see that this is an ideal, $-p \in \mathfrak{p}$, $0 \in p$, and for any $b/f^m \in A_f$, we have that $f^{m+k} \in S$, and $pb \in \mathfrak{p}$, hence $bp/(f^{k+m}) \in \mathfrak{q}$. It is prime as if:

$$\frac{a}{f^k} \cdot \frac{b}{f^m} \in \mathfrak{q}$$

then we have that:

$$\frac{ab}{f^{k+m}} = \frac{p}{f^n}$$

for some $p \in \mathfrak{p}$ an $n > 0$. This implies that there exists a $j \geq 0$ such that:

$$f^j(abf^n - pf^{k+m}) = 0$$

We then see that:

$$abf^{j+n} = pf^{k+m_j}$$

implying that $abf^{j+n} \in \mathfrak{p}$. We have that $f^{j+n} \notin \mathfrak{p}$, so $ab \in \mathfrak{p}$, implying either a or b is in $\mathfrak{p}$, so $\mathfrak{q}$ is a prime ideal. It is then clear that:

$$\pi^{-1}(\mathfrak{q}) = \mathfrak{p}$$

as if $a \in \pi^{-1}(\mathfrak{q})$, then we have that $a/1 \in \mathfrak{q}$, so:

$$\frac{a}{1} = \frac{p}{f^k} \Longrightarrow af^k = p$$

so $a \in \mathfrak{p}$. If $a \in \mathfrak{p}$, then we have $a/1 \in \mathfrak{q}$, and $\pi(a) = a/1$, so $a \in \pi^{-1}(\mathfrak{q})$.

The map ψ is then a continuos surjection onto it's image by definition, so we define an inverse map $\eta : U_f \to \operatorname{Spec} A_f$, by:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{f^k} \in A_f : p \in \mathfrak{p}, k \geq 0 \right\}$$

which as we have just shown is a prime ideal in A_f. We check that these are inverses, let $\mathfrak{p} \in U_f$, then our argument showing that $\operatorname{im} \psi = U_f$ demonstrates:

$$\psi \circ \eta(\mathfrak{p}) = \mathfrak{p}$$

Now suppose that $\mathfrak{q} \in \operatorname{Spec} A_f$, we have that:

$$\eta \circ \psi(\mathfrak{q}) = \eta(\pi^{-1}(\mathfrak{q})) = \left\{ \frac{p}{f^k} \in A_f : p \in \pi^{-1}(q), k \geq 0 \right\} := I$$

Let $p/f^k \in \mathfrak{q}$, then $p/1 \in \mathfrak{q}$, so $p \in \pi^{-1}(\mathfrak{q})$. It follows that $p/f^k \in I$. Now suppose that $p/f^k \in I$, then $p \in \pi^{-1}(q)$, so $p/1 \in \mathfrak{q}$, hence $p/f^k \in \mathfrak{q}$. It follows that $I = \mathfrak{p}$, the two are inverses of one another.

We need to show that η is continuous, it suffices to check that on basis open sets. First note that $U_{a/f^k} = U_{a/1} \subset \operatorname{Spec} A_f$, as if $\mathfrak{q} \in U_{a/f^k}$, then we have that $a/f^k \notin \mathfrak{q}$. Since $f/1 \notin \mathfrak{q}$, we have that $a/f^k \cdot f^k = a/1 \notin \mathfrak{q}$, hence $\mathfrak{q} \in U_{a/1}$. Moreover, if $\mathfrak{q} \in U_{a/1}$, then $a/1 \notin \mathfrak{q}$, and since $f \notin \mathfrak{q}$, we have that $a \cdot 1/f^k \notin \mathfrak{q}$, so $\mathfrak{q} \in U_{a/f^k}$. It thus suffices to check this on distinguished opens of the form U_g for some $g/1 \in A_f$. We see that:

$$\eta^{-1}(U_g) = \{\mathfrak{p} \in U_f : \eta(\mathfrak{p}) \in U_g\}$$

We claim that:

$$\eta^{-1}(U_g) = U_{fg} = U_f \cap U_g \subset \operatorname{Spec} A$$

and would thus be open in the subspace topology on U_f. Let $\mathfrak{p} \in U_{fg}$, then $\mathfrak{p} \in U_f \cap U_g$, so neither g nor f lie in $\mathfrak{p}$. Now, we see that:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{f^k} : p \in \mathfrak{p}, k \geq 0 \right\}$$

Since $g \notin \mathfrak{p}$, it follows that $g/1 \notin \eta(\mathfrak{p})$, hence $\eta(\mathfrak{p}) \in U_g \subset A_f$. If $\mathfrak{p} \in \eta^{-1}(U_g)$, then we have that $g/1 \notin \eta(\mathfrak{p})$, implying that $g \notin \mathfrak{p}$, so $\mathfrak{p} \in U_f \cap U_g = U_{fg}$. It follows that η is a continuous map, and in particular, the inverse ψ, hence U_f is homeomorphic to $\operatorname{Spec} A_f$, as desired. $\square$

1.2 Some Category Theory: Sheaves, Stalks, Germs, and all that

In this section we go over the basics of sheaf theory, and attempt to take a categorical approach wherever possible. We begin by fixing a topological space X, and a category denoted $\mathscr{C}(X)$, whose objects are open sets of X, and morphisms are inclusion maps $\imath_U^V : U \to V$, whenever $U \subset V$. Note that this puts a partial order on $\mathscr{C}(X)$, where $U < V \Leftrightarrow U \supset V$[2].

Definition 1.2.1. A **pre sheaf** is a contravariant functor $\mathscr{F} : \mathscr{C}(X) \to D$ where D is generally one of the following categories: Set, Ab, or Ring[3]. We call the object $\mathscr{F}(U)$ **sections** over U, and the induced maps $\theta_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$, the **restriction maps**. A **sheaf**, is then presheaf such that:

i) Let U_i be an open cover for U, then if $s, t \in \mathscr{F}(U)$ such that $\theta_{U_i}^U(s) = \theta_{U_i}^U(t)$ for all i, then $s = t$.

ii) If U_i is an open cover of U, and there exists $s_i \in \mathscr{F}(U_i)$ such that:

$$\theta_{U_i \cap U_j}^{U_i}(s_i) = \theta_{U_i \cap U_j}^{U_j}(s_j)$$

for all i and j, then there exists a section $s \in \mathscr{F}(U)$ such that $\theta_{U_i}^U(s) = s_i$.

We have the following example:

Example 1.2.1. If X is a topological space, then let $\mathscr{F} = C^0$ assign to each open set of X the ring of continuous real valued functions. This obviously defines a presheaf on X, where the restriction maps are given by $f \in C^0(V) \mapsto f \circ \imath_U^V \in C^0(U)$. Now suppose that U_i is an open cover of U, and $f \circ \imath_{U_i}^U = 0$ for all U_i. Well this implies that $f(p) = 0$ for all $p \in U$, as all $p \in U$ lie in U_i for some i, and $f \circ \imath_{U_i}^U(p) = 0 = f(p)$ by definition, so sheaf axiom one is satisfied[4]. Now suppose that U_i covers U, and $f_i \in C^0(U_i)$ satisfy $f \circ \imath_{U_i \cap U_j}^{U_i} = f_j \circ \imath_{U_i \cap U_j}^{U_i}$. We define a map f by:

$$f(p) = f_i(p)$$

when $p \in U_i$. If $p \in U_i \cap U_j$, then since f_i and f_j agree on the overlap we have that $f_i = f_j$. We show that this is continuous, Let $W \subset \mathbb{R}$ be open, then

$$\begin{aligned} f^{-1}(W) =& \{p \in U : f(p) \in W\} \\ =& \bigcup_i \{p \in U_i : f_i(p) \in W\} \\ =& \bigcup_i f_i^{-1}(W) \end{aligned}$$

however, each f_i is continuos, hence $f \in C^0(U)$, and satisfies $\theta_{U_i}^U(f) = f_i$.

Example 1.2.2. Let X be a smooth manifold. A similar argument shows that the contravariant functor $\mathscr{F} = C_X^\infty$, which assigns to each open set of X the ring of smooth functions $C^\infty(U)$, is a sheaf. Moreover though, if $E \to X$ is a smooth vector bundle over X, then the $\mathscr{F} = \Gamma$, which is the functor that assigns to each open set of X the ring of smooth local sections of E is also a sheaf. Indeed, the restriction maps are just composition of with the inclusions, and sheaf axiom one is satisfied in the same as in Example 1.2.1. Now suppose that U_i is an open cover of U, and $\phi_i \in \Gamma(U_i)$ are smooth sections such that $\theta^{U_i} U_i \cap U_j(\phi_i) = \theta^{U_i} U_i \cap U_j(\phi_i)$. We let ψ_i be a partition of unity subordinate to the open cover U_i, then we see that:

$$\xi_i = \begin{cases} \psi_i \phi_i & \forall x \in U_i \\ 0 & \forall x \notin U_i \end{cases}$$

defines an element $\xi_i \in \Gamma(U)$. We define $\phi \in \Gamma(U)$ by:

$$\phi = \sum_i \xi_i$$

[2]The reason for the reverse inclusion is to due the contravariant nature of a presheaf/sheaf.

[3]By Ring we always mean commutative rings.

[4]If addition is well defined, and a group operation in the objects of your target category, sheaf axiom one is equivalent to the case where s restricted U_i is zero for all i implies that s is zero.

Then this satisfies $\theta^U_{U_k}(\phi) = \phi_k$, as for all $p \in U_k$, we have that for some n:

$$\phi(p) = \sum_{i:U_i \cap U_k \neq \emptyset} \psi_i(p)\phi_i(p) = \sum_{j=1}^{n} \psi_j(p)\phi_j(p)$$

since all ϕ_j agree with ϕ_k on $U_j \cap U_k$, and $p \in U_k$, this becomes:

$$\phi(p) = \phi_k(p) \cdot \sum_j \psi_j(p) = \phi_k(p)$$

as a partition of unity always sums to one. It follows that $\phi \circ \imath^U_{U_k} = \phi_k$ as desired, so Γ is a sheaf.

In the case where are sheafs are literally rings/groups of maps to another topological or smooth space, the sheaf axioms encode a sort of locality condition that mimics continuity, and smoothness. When we turn to studying schemes, and locally ringed spaces in generality, it will be good to keep this picture in mind. At times we write $s|_U$, for $\theta^V_U(s)$ when it is understood that $s \in \mathscr{F}(V)$.

Definition 1.2.2. Let X be a topological space, and $\mathscr{F} : \mathscr{C}(X) \to D$ a pre sheaf, the **stalk** of $\mathscr{F}$ at $x \in X$, denoted $\mathscr{F}_x$ is an object in D satisfying the following conditions:

a) For all $U \subset V$ where $x \in U$ and V, there exist morphisms $\psi_U : \mathscr{F}(U) \to \mathscr{F}_x$, $\psi_V : \mathscr{F}(V) \to \mathscr{F}_x$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(V) \arrow[rr, "\theta^V_U"] \arrow[dr, "\psi_V"] & & \mathscr{F}(U) \arrow[dl, "\psi_U"'] \\ & \mathscr{F}_x & \end{tikzcd}$$

b) If G is another object in D, equipped with morphisms $\phi_U : \mathscr{F}(U) \to G$, $\phi_V : \mathscr{F}(V) \to G$ for U, V where $x \in U, V$, such that a similar diagram commutes, then there exists a unique map $\phi_x : \mathscr{F}_x \to G$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(V) \arrow[rr, "\theta^V_U"] \arrow[dr, "\psi_V"] \arrow[ddr, bend right, "\phi_V"'] & & \mathscr{F}(U) \arrow[dl, "\psi_U"'] \arrow[ddl, bend left, "\phi_U"] \\ & \mathscr{F}_x \arrow[d, "\exists! \phi_x"] & \\ & G & \end{tikzcd}$$

Elements of $\mathscr{F}_x$ are called **germs**

The reader familiar with category theory will notice that this definition is equivalent to the definition of the colimit, or direct limit. In other words, we have that:

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$$

As always, when defining something by a universal property, it is important to check that such an object exists.

Proposition 1.2.1. *Let X be a topological space, and $\mathscr{F}$ a presheaf, then for all $x \in X$ the stalk $\mathscr{F}_x$ exists.*

Proof. We work in the category $D = \text{Ring}$, as the proof in this case will imply the others. Define $\mathscr{F}_x$ as the following set:

$$F = \{(U, s) : x \in U, s \in \mathscr{F}(U)\}$$

modulo the equivalence relation

$$(U, s) \sim (V, t)$$

if and only there exists a $W \in U \cap V$ such that $x \in W$ and:

$$s|_W = t|_W$$

One easily checks that this is indeed an equivalence relation on F, thus we set:

$$\mathscr{F}_x = F/\sim$$

We first check that $\mathscr{F}_x$ is indeed a ring. We define addition on $\mathscr{F}_x$ by:

$$[U, s] + [V, t] = [U \cap V, s|_{U\cap V} + t|_{U\cap V}]$$

We need to check that this well defined. Suppose that $[Z, f] = [U, s]$, then we need to show that:

$$[Z \cap V, s|_{Z\cap V} + t|_{Z\cap V}] = [U \cap V, s|_{U\cap V} + t|_{U\cap V}]$$

Well, consider $W = U \cap Z \cap V$, and note that by the functorial properties of the restriction maps we have that:

$$(s|_{Z\cap V} + t|_{Z\cap V})|_W = s|_W + t|_W = (s|_{U\cap V} + t|_{U\cap V})|_W$$

so addition is well defined. Moreover the zero element is given by $[U, 0]$ for any open set U which contains x. Indeed, we have clearly have that:

$$[U, 0] + [V, s] = [U \cap V, s|_{U\cap V}] = [V, s]$$

The inverse of any element $[U, s]$ is then easily seen to be $[U, -s]$, so $\mathscr{F}_x$ is indeed an abelian group. We define a ring structure in the same:

$$[U, s] \cdot [V, t] = [U \cap V, s|_{U\cap V} \cdot t|_{U\cap V}]$$

and the same argument demonstrates that this well defined, and that $[U, 1]$ is the multiplicative identity of $\mathscr{F}_x$, so $\mathscr{F}_x$ is a ring.

For all open sets U, we define a map $\psi_U : \mathscr{F}(U) \to \mathscr{F}_x$ by:

$$s \longmapsto [U, s]$$

Let $V \cap U$, and $\theta^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ be the restriction map, then we have that:

$$\psi_V \circ \theta^U_V(s) = [V, s|_V]$$

However, we see that $U \cap V = V$, so tautologically we have that:

$$[U, s] = [V, s|_V]$$

it follows that property a) of Definition 1.2.2 is satisfied. Now suppose that for all open U we have ring homomorphisms $\phi_U : \mathscr{F}(U) \to G$, such that $\phi_U = \phi_V \circ \theta^U_V$, then we see that if $\phi_x : \mathscr{F}_x \to G$ exists it must satisfy:

$$\phi_x \circ \psi_U = \phi_U$$

so we define ϕ_x by:

$$\phi_x([U, s]) = \phi_U(s) \tag{1.2.1}$$

We need to check that this well defined; let $[U,s]=[V,t]$, then there exists a $W\subset U\cap V$ such that $s|_W=t|_W$. It follows that:

$$\phi_W(s|_W)=\phi_W(t|_W)$$

however:

$$\phi_W(s|_W)=\phi_U(s)$$

and:

$$\phi_W(t|_W)=\phi_V(t)$$

so $\phi_U(s)=\phi_V(t)$ hence $\phi_x([U,s])=\phi_x([V,t])$. We check that ϕ_x is a ring homomorphism, let $[U,s]$ and $[V,t]\in\mathscr{F}_x$, then:

$$\phi_x([U,s]+[V,t])=\phi_{U\cap V}(s|_{U\cap V}+t|_{U\cap V})$$

while:

$$\phi_x([U,s])+\phi_x([V,t])=\phi_U(s)+\phi_V(t)=\phi_{U\cap V}(s|_{U\cap V})+\phi_{U\cap V}(t|_{U\cap V})$$

so by the fact $\phi_{U\cap V}$ is a ring homomorphism, we have that ϕ_x respects addition. The same argument shows that ϕ_x respects multiplication, and sends 0 and 1 to 0 and 1 respectively so ϕ_x is a ring homomorphism. It is unique, as any other ring homomorphism that makes the diagram in $b)$ commute must satisfy (1.1). It follows that $F/\sim$ satisfies the properties of Definition 1.2.2, so $\mathscr{F}_x$ exists and is unique up to unique isomorphism. $\square$

Definition 1.2.3. Let X be a topological space, and $\mathscr{F}$ and $\mathscr{G}$ be sheaves (pre sheaves) on X. A **morphism of (pre) sheaves** is a natural transformation $F:\mathscr{F}\to\mathscr{G}$. In particular, the a morphism of (pre) sheaves is a family of morphisms $F_U:\mathscr{F}(U)\to\mathscr{G}(U)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F_U"] \arrow[d, "\theta^U_V"'] & \mathscr{G}(U) \arrow[d, "\theta^U_V"] \\
\mathscr{F}(V) \arrow[r, "F_V"'] & \mathscr{G}(V)
\end{tikzcd}$$

A **isomorphism of sheaves** (**presheaves**) is a natural transformation in which every morphism F_U is an isomorphism. We denote the category of presheaves on X by $\mathrm{PSh}(X)$.

Lemma 1.2.1. *Let $F:\mathscr{F}\to\mathscr{G}$ be a morphism of presheaves or sheaves, then there exists a unique map on stalks $F_x:\mathscr{F}_x\to\mathscr{G}_x$.*

Proof. Clearly, we need only define maps $\phi_U:\mathscr{F}(U)\to\mathscr{G}_x$, satisfying $\phi_U=\phi_V\circ\theta^U_V$, then by the universal property of the colimit, we will have a unique map F_x. We define ϕ_U by:

$$\phi_U(s)=[U,F_U(s)]$$

We see that this is clearly a ring homomorphism by our previous work, and that:

$$\phi_V(s|_V)=[V,F_V(s|_v)]=[V,F_U(s)|_V]=[U,F_(s)]$$

implying the claim. $\square$

Note that we have that:

$$F_x([U,s])=[U,F_U(s)]$$

If $s\in\mathscr{F}(U)$, we often denotes its image in $\mathscr{F}_x$ as s_x. Moreover, if it is not understood which stalk $[U,s]$ belongs to, we write $[U,s]_x$. Importantly this lemma implies the following:

Corollary 1.2.1. *Let $F : \mathscr{F} \to \mathscr{G}$ and $G : \mathscr{G} \to \mathscr{H}$ be morphisms of (pre) sheaves, then for all $x \in X$:*

$$(G \circ F)_x = G_x \circ F_x$$

Proof. We have that $G \circ F$ is a morphism $\mathscr{F} \to \mathscr{H}$, so there exists a unique map $(G \circ F)_x : \mathscr{F}_x \to \mathscr{H}_x$ such that:

$$(G \circ F)_x([U, s]) = [U, (G \circ F)_U(s)] = [U, G_U \circ F_U(s)] = G_x([U, F_U(s))) = G_x \circ F_x([U, s])$$

implying the claim. □

Lemma 1.2.2. *If $\mathscr{F}$ is a sheaf, then the natural homomorphism:*

$$\mathscr{F}(U) \longrightarrow \prod_{x \in U} \mathscr{F}_x$$
$$s \longmapsto (s_x)$$

is injective.

Proof. Suppose that $s, t \in \mathscr{F}(U)$ and $(s_x) = (t_x)$. Then for each x we have that:

$$[U, s]_x = [U, t]_x$$

implying that there exists $W_x \subset U$ such that:

$$s|_{W_x} = t|_{W_x}$$

We then obtain an open cover $\{W_x\}$ of U such that $s|_{W_x} = t|_{W_x}$, so sheaf axiom one implies the claim. □

Proposition 1.2.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X , then F is an isomorphism if and only if $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for all $x \in X$. Moreover, if $F, G : \mathscr{F} \to \mathscr{G}$ are morphisms such that $F_x = G_x$ for all $x \in X$, then $F = G$.*

Proof. If $F : \mathscr{F} \to \mathscr{G}$ is an isomorphism, the there exists an inverse natural transformation given F^{-1} : $\mathscr{F} \to \mathscr{G}$. Let $[U, s] \in \mathscr{F}_x$, then:

$$F_x^{-1} \circ F_x([U, s]) = F_x^{-1}([U, F_U(s)]) = [U, F_U^{-1} \circ F_U(s)] = [U, s]$$

Similarly if $[U, s] \in \mathscr{G}_x$, then we have that:

$$F_x \circ F_x^{-1}([U, s]) = [U, s]$$

so we have that F_x is an isomorphism.

For the converse, note that since the target category of our functors $\mathscr{F}$ and $\mathscr{G}$ is either Set, Ab, or Ring, it suffices to check that F_U is injective and surjective for all U. Note that we get an induced isomorphism:

$$\prod_{x \in U} \mathscr{F}_x \longrightarrow \prod_{x \in U} \mathscr{G}_x$$
$$(s_x) \longmapsto (F_x(s_x)) \tag{1.2.2}$$

as F_x is an isomorphism for all x. Suppose that $F_U(s) = F_U(t)$, then we have that by definition of the stalk map $(F_U(s))_x = F_x(s_x) = F_x(t_x) = (F_U(t))_x$ for all $x \in U$. Since $F_x(s_x) = F_x(t_x)$ for all U, we have that $(s_x) = (t_x)$ so by Lemma 1.2.2 F_U is injective.

Now let $g \in \mathscr{G}(U)$, then by the isomorphism (1.2), we have that there exists a unique sequence $(s_x) \in \prod_{x \in U} \mathscr{F}_x$ such that $(F_x(s_x)) = (g_x)$. Write $[V_x, f^x]_x$ for each s_x in the sequence, and without loss of generality let $V_x \subset U$[5]. Then note that:

$$F_x([V_x, f^x]_x) = [V_x, F_{V_x}(f^x)]_x = [U, g]_x$$

[5]We can always further restrict to $U \cap V_x$ to make this true.

so there exists a $W_x \subset V_x$ such that $F_{V_x}(f^x)|_{W_x} = g|_{W_x}$. Cover U by $\{W_x\}$, then we have sections $f^x|_{W_x} \in \mathscr{F}(W_x)$. We see that:

$$F_{W_x \cap W_y}(f^x|_{W_x \cap W_y}) = g|_{W_x \cap W_y} = F_{W_x \cap W_y}(f^y|_{W_x \cap W_y})$$

and since F is injective, it follows that:

$$f^x|_{W_x \cap W_y} = f^x|_{W_x \cap W_y}$$

so we have a global section $f \in \mathscr{F}(U)$ by sheaf axiom two. We see that:

$$F_U(f)|_{W_x} = F_{W_x}(f|_{W_x}) = F_{W_x}(f^x|_{W_x}) = g|_{W_x}$$

so by sheaf axiom one $F_U(f) = g$, implying that F_U is surjective for all U, and thus F is a natural isomorphism as desired.

Now suppose that F and G are morphisms of sheaves such that $F_x = G_x$ for all $x \in X$. It suffices to show that $F_U(s) = G_U(s)$ for all $U \subset X$, and $s \in \mathscr{F}(U)$. The hypotheses tells us that for all $x \in U$ we have that $F_x(s_x) = G_x(s_x)$ for all $x \in U$. For each $x \in U$ there is thus an open neighborhood $V_x \subset U$ of x such that $F_{V_x}(s|_{V_x}) = G_{V_x}(s|_{V_x})$. We have thus obtained an open cover $\{V_x\}_{x \in U}$ of U such that $F_U(s)|_{V_x} = G_U(s)|_{V_x}$ so sheaf axiom implies that $F_U(s) = G_U(s)$. It follows that $F = G$. $\square$

One can easily check that presheaves with values in Ab form an abelian category, as one easily define the kernel and cokernel of a presheaf morphism to be the functor on X that takes U to $\ker F_U$ and $\operatorname{coker} F_U$. This does not work with sheaves, however there is a workaround.

Definition 1.2.4. Let $\mathscr{F}$ be a presheaf on X, then sheafification of $\mathscr{F}$, denoted $\mathscr{F}^\sharp$, is the a sheaf equipped with a morphism $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^\sharp$, such that for all morphisms $\phi : \mathscr{F} \to \mathscr{G}$, where $\mathscr{G}$ is a sheaf, then there exists a unique morphism $\phi^\sharp : \mathscr{F}^\sharp \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\phi"] \arrow[d, "\mathrm{sh}"] & \mathscr{G} \\ \mathscr{F}^\sharp \arrow[ur, "\phi^\sharp"] & \end{tikzcd}$$

As always, we check that such a construction exists, and is thus unique up to unique isomorphism.

Proposition 1.2.3. *Let $\mathscr{F}$ be a presheaf on X, then the sheafification of $\mathscr{F}^\sharp$ exists.*

Proof. We define $\mathscr{F}^\sharp$ on open sets by:

$$\mathscr{F}^\sharp(U) = \left\{ (s_x) \in \prod_{x \in U} \mathscr{F}_x : \forall p \in U, \exists V_p \subset U, \text{ and a } f \in \mathscr{F}(V_p), \text{such that } f_q = s_q, \forall q \in V_p \right\}$$

All this is saying, is that for each $p \in U$, we can find an open neighborhood of p, and a section on that open neighborhood such that the germ of that section at every point agrees with s_q. With that in mind, we quickly check that this is a subring of $\prod_{x \in U} \mathscr{F}_x$. Clearly, $\mathscr{F}^\sharp(U)$ contains the zero section and the multiplicative identity. Moreover, if $(s_x) \in \mathscr{F}^\sharp(U)$, then it's inverse $(-s_x) \in \mathscr{F}^\sharp(U)$, as we just take $-f$ to cover $(-s_x)$ for each $V_p \subset U$. It is closed under addition, and multiplication as if $(s_x), (t_x) \in \mathscr{F}^\sharp(U)$, then we have that:

$$(s_x) + (t_x) = (s_x + t_x) \qquad \text{and} \qquad (s_x) \cdot (t_x) = (s_x \cdot t_x)$$

Let $p \in U$, then there exists W_p, Z_p, $s^p \in \mathscr{F}(W_p)$ and $t^p \in \mathscr{F}(Z_p)$ such that $s^p_q = s_q$ and $t^p_q = t_q$ for all $q \in W_p$ and Z_p respectively. We then see that:

$$(s^p|_{W_p \cap Z_p} + t^p|_{W_p \cap Z_p})_q = (s_q) + (t_q) = (s_q + t_q)$$

and similarly for multiplication. It follows that for all p there exists sections on small enough neighborhoods that agree with addition or multiplication of two elements in $\mathscr{F}^\sharp(U)$, so $\mathscr{F}^\sharp(U)$ is a subring of $\prod_{x\in U}\mathscr{F}_x$, and thus a ring.

We check that $U \mapsto \mathscr{F}^\sharp(U)$ is a contravariant functor. Define restriction maps θ_V^U in the obvious way:

$$\begin{aligned} \theta_V^U : \prod_{x\in U}\mathscr{F}_x &\longrightarrow \prod_{x\in V}\mathscr{F}_x \\ (s_x) &\longmapsto (s_x) \end{aligned}$$

which is clearly a ring homomorphism, as we essentially just toss out the elements in the (s_x) where $x \in\notin U$. Restricting the restriction maps to $\mathscr{F}^\sharp(U)$, it is clear that θ_V^U has image in $\mathscr{F}^\sharp(V)$ as restricting sections commutes with the map from sections to stalks. It is then clear that:

$$\theta_U^U = \mathrm{Id} \text{ and } \theta_W^V \circ \theta_V^U = \theta_W^U$$

so $\mathscr{F}^\sharp$ is a presheaf.

To see that $\mathscr{F}^\sharp$ is a sheaf, let $\{U_i\}$ be an open cover of U, and $(s_x) \in \mathscr{F}^\sharp(U)$ such that $(s_x)|_{U_i} = 0$. Then clearly by the definition of the restriction map, $s_x = 0$ for all $x \in U$, so $(s_x) = 0$ and sheaf axiom one is satisfied. Now suppose that we have sections $(s_x^i) \in \mathscr{F}^\sharp(U_i)$ such that $(s_x^i)|_{U_i\cap U_j} = (s_x^j)|_{U_i\cap U_j}$, then we define a section $(s_x) \in \mathscr{F}^\sharp(U)$ by:

$$(s_x) = (s_x^i)$$

whenever $x \in U_i$. If $x \in U_i \cap U_j$, then since $(s_x^i)|_{U_i\cap U_j} = (s_x^j)|_{U_i\cap U_j}$ implies that for all $p \in U_i \cap U_j$ we have $s_p^i = s_p^j$, it is clear that this assignment is well defined. Moreover, (s_x) lies in $\mathscr{F}^\sharp(U)$, as for all $p \in U$, there exists a U_i such that $p \in U_i$, and $(s_x^i) \in \mathscr{F}^\sharp(U_i)$ with $s_x^i = s_x$ for all $x \in U_i$, so there must exist a section f on each open neighborhood of $x \in U_i$ such that $f_q = s_q^i = s_q$, hence $(s_x) \in \mathscr{F}^\sharp(U)$. Moreover, we have that by construction $(s_x)|_{U_i} = (s_x^i)$. It follows that $\mathscr{F}^\sharp$ is a sheaf.

We define the natural transformation $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^\sharp$ by:

$$\begin{aligned} \mathrm{sh}_U : \mathscr{F}(U) &\longrightarrow \mathscr{F}^\sharp(U) \\ s &\longmapsto (s_x) \end{aligned}$$

which has image in $\mathscr{F}^\sharp$ essentially by construction, i.e. take $V_p = U$ for all $p \in U$, then $s \in \mathscr{F}(U)$ satisfies $s_q = s_q$ tautologically. Moreover, this clearly commutes with restriction maps, and is thus a natural transformation.

We construct the natural transformation $\phi^\sharp$ for all U as follows; let $(s_x) \in \mathscr{F}^\sharp(U)$ then for all $p \in U$ there exists V_p and $f^p \in \mathscr{F}(V_p)$ such that $[V_p, f^p]_q = s_q$ for all $q \in V_p$. We thus obtain an open cover U by $\{V_p\}$ and section $\phi_{V_p}(f^p) \in \mathscr{G}(V_p)$. Now consider overlaps $W = V_x \cap V_y$, then:

$$\phi_{V_x}(f^x)|_W = \phi_W(f^x|_W)$$

By the universal property of the colimit, we have a unique map $\phi_q : \mathscr{F}_q \to \mathscr{G}^q$ for all q such that:

$$\phi_q(f_q^x) = [V_x, \phi_{V_x}(f^x)]_q = [W, \phi_W(f^x|_W)]_q = (\phi_W(f^x|_W))_q$$

However, for all $q \in W$, we have that $f_q^x = f_q^y$, hence:

$$(\phi_W(f^x|_W))_q = \phi_q(f_q^x) = \phi_q(f_q^y) = (\phi_W(f^y|_W))$$

implying that:

$$(\phi_{V_x}(f^x)|_W)_q = (\phi_{V_y}(f^y)|_W)_q$$

for all $q \in W$. However, $\mathscr{G}$ is a sheaf, so by Lemma 1.2.2, we have that $\phi_{V_x}(f^x)|_W = \phi_{V_y}(f^y)|_W$. So by sheaf axiom two, the $\phi_{V_x}(f^x)$ glue together to form a unique global section $g \in \mathscr{G}(U)$. We thus define $\phi_U^\sharp$ by:

$$\phi_U^\sharp((s_x)) = g$$

This is well defined, since if we had some other set of functions on e^p on some other open cover Z_p, repeating the same process yields a section $h \in \mathscr{G}(U)$. For all $q \in U$, we then have that:

$$h_q = \phi_q(s_q) = g_q$$

so by Lemma 1.2.2, it follows that $g = h$. This is clearly a ring homomorphism as if $(s_x), (t_x) \in \mathscr{F}^\sharp(U)$, then we have that:

$$\phi^\sharp_U(s_x) + \phi^\sharp_U(t_x) = g + h$$

where $g = \phi^\sharp_U(s_x)$ and $h = \phi^\sharp_U(t_x)$. Now suppose that:

$$\phi^\sharp_U(s_x + t_x) = f$$

for some $f \in \mathscr{G}(U)$. Then we have that:

$$f_q = \phi_q(s_q + t_q) = \phi_q(s_q) + \phi_q(t_q) = g_q + h_q = (g+h)_q$$

Since this holds for all q, we have again by Lemma 1.2.2 that:

$$\phi^\sharp_U(s_x + t_x) = \phi^\sharp_U(s_x) + \phi^\sharp_U(t_x)$$

The same argument shows that $\phi^\sharp_U$ respects multiplication.

Finally, we check that that $\phi^\sharp \circ \mathrm{sh} = \phi$. Let $s \in \mathscr{F}(U)$, then $\mathrm{sh}_U(s) = (s_x)$. Since $\phi^\sharp$ is independent of the choice of cover we use to obtain a section, chose the trivial cover U with $s \in \mathscr{F}(U)$, then clearly:

$$\phi^\sharp_U \circ \mathrm{sh}_U(s) = \phi_U(s)$$

so $\mathscr{F}^\sharp$ satisfies the universal property, implying the claim. □

Importantly if $\mathscr{F}$ is already a sheaf, we have that $\mathscr{F}^\sharp$ is uniquely isomorphic to $\mathscr{F}$.

Lemma 1.2.3. *Suppose that $\mathscr{F}$ is a sheaf, then $\mathscr{F} \cong \mathscr{F}^\sharp$*

Proof. We simply check that $\mathscr{F}, \mathrm{Id}$ satisfies the universal property of sheafification. Let ϕ by a morphism $\mathscr{F} \to \mathscr{G}$, then we have the following diagram:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\phi"'] \arrow[d, "\mathrm{Id}"'] & \mathscr{G} \\
\mathscr{F} \arrow[ur, "\exists!\phi^\sharp?"'] &
\end{tikzcd}$$

Clearly, for this diagram to commute we must have that $\phi^\sharp = \phi$, but that morphism exists, and is unique so $\mathscr{F}, \mathrm{Id}$ satisfies the universal property of sheafification and is thus uniquely isomorphic to $\mathscr{F}^\sharp$. □

Lemma 1.2.4. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ are presheaves, and $\phi : \mathscr{F} \to \mathscr{G}$ a morphism between them. Then there exists unique isomorphisms $\mathrm{sh}_q : \mathscr{F}_q \to \mathscr{F}^\sharp_q$, $\mathrm{sh}_q : \mathscr{G}_q \to \mathscr{G}^\sharp_q$[6], such that $\phi^\sharp_q \circ \mathrm{sh}_q = \mathrm{sh}_q \circ \phi_q$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{sh}"'] \arrow[d, "\phi"'] \arrow[dr, dashed, "\mathrm{sh} \circ \phi"] & \mathscr{F}^\sharp \\
\mathscr{G} \arrow[r, "\mathrm{sh}"'] & \mathscr{G}^\sharp
\end{tikzcd}$$

[6]Abuse of notation alert! We are using the same notation to refer to two different sheafification map.

So there exists a unique morphism $(\phi)^\sharp$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[rr, "\mathrm{sh}"'] \arrow[d, "\phi"'] \arrow[drr, dashed, "\mathrm{sh}\circ\phi"] & & \mathscr{F}^\sharp \arrow[d, "\phi^\sharp"] \\ \mathscr{G} \arrow[rr, "\mathrm{sh}"'] & & \mathscr{G}^\sharp \end{tikzcd}$$

It follows that $\mathrm{sh}\circ\phi = \phi^\sharp \circ \mathrm{sh}$, so we need only show that the unique map $\mathrm{sh}_q : \mathscr{F}_q \to \mathscr{F}_q^\sharp$ is an isomorphism. We have that:

$$\mathrm{sh}_q([U,s]) = [U, \mathrm{sh}_U(s)] = [U,(s_x)]$$

Suppose that $\mathrm{sh}_q([U,s]) = \mathrm{sh}_q([V,t])$, then we have that there exists a $W \ni q \subset U \cap V$, such that:

$$(s_x)|_W = (t_x)|_W$$

implying that for all $x \in W$ $s_x = t_x$. Since $q \in W$, it follows that $s_q = t_q$ so $[U,s] = [U,t]$. Now let $[U,(s_x)] \in \mathscr{F}_q^\sharp$, and take $(s_x) \in \mathscr{F}^\sharp(U)$. It follows that there exists an open neighborhood V_q of q, and a section $f \in \mathscr{F}(U)$ such that $f_x = s_x$ for $x \in V_q$. We see that:

$$\mathrm{sh}_q([V_x, f]) = [V_x,(f_x)] = [V_x,(s_x)|_{V_x}] = [U,(s_x)]$$

so sh_q is surjective. It follows that sh_q is an isomorphism, as the $\mathscr{F}_q$ and $\mathscr{F}_q^\sharp$ are either sets, groups, or rings. □

Example 1.2.3. Let X be a topological space, and denote by $\mathbb{Z}$ the constant presheaf which assigns to each open non empty set the abelian group $\mathbb{Z}$, and to $\emptyset$ the trivial group. The restriction maps are either the identity, or the trivial morphism. This is not necessarily a sheaf, as if U open is the disjoint union of two open sets U_1, U_2, then we have that $s \in \mathbb{Z}(U_1)$, and $t \in \mathbb{Z}(U_2)$, such that $s \neq t$, it follows that $s|_{U_1\cap U_2} = t|_{U_1\cap U_2}$, but clearly s and t can't glue together to form a section of U restricting to s and t.

We want to find the sheafification of $\mathbb{Z}$. Define $\mathbb{Z}^\sharp$ by:

$$\mathbb{Z}^\sharp(U) = \{\text{locally constant functions } s : U \to \mathbb{Z}\}$$

i.e. if U is connected then $s : U \to \mathbb{Z}$ is a constant function, and if U is disconnected then s is constant on each connected component. The restriction maps are just the restriction of the function s to a smaller domain. This is then clearly a sheaf, as if U_i is a cover for U, and each $s|_{U_i} = 0$, then at each point in U $s(p) = 0$ so $s = 0$. Moreover, if we have $s_i \in U_i$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then the same construction in Example 1.2.1 gives a section on U that restricts to s_i.

We need only show that $\mathbb{Z}^\sharp$ satisfies the universal property of sheafification. Define sh on each open set by:

$$\mathrm{sh}_U(a) = s_a$$

where $s_a : U \to \mathbb{Z}$ is the constant function $s_a(p) = a$. This clearly commutes with restriction maps, hence defines a natural transformation $\mathbb{Z} \to \mathbb{Z}^\sharp$. Let $\phi : \mathbb{Z} \to \mathscr{G}$ be any morphism, where $\mathscr{G}$ is a sheaf. We see that $\phi^\sharp$ must satisfy:

$$\phi^\sharp \circ \mathrm{sh} = \phi$$

Let $s \in \mathbb{Z}^\sharp(U)$, then note that $s^{-1}(a)$ is open in U as s is locally constant. Indeed, if s is locally constant, then for each $x \in U$ there exists an open neighborhood of x such that s is constant. The preimage $s^{-1}(a)$ is then the union of all such open neighborhoods which is certainly open. Moreover, we see that

$s^{-1}(a) \cap s^{-1}(b) = \emptyset$ for all $a \neq b$, and that $\{s^{-1}(a)\}_{a \in \mathbb{Z}}$ forms an open cover of U. For each $a \in \mathbb{Z}$, we choose the section $a \in \mathbb{Z}(s^{-1}(a))$, and then see that:

$$\phi_{s^{-1}(a)}(a)|_{s^{-1}(a) \cap s^{-1}(b)} = \phi_{s^{-1}(b)}(b)|_{s^{-1}(a) \cap s^{-1}(b)}$$

as the restrictions map to the empty set. It follows that since $\phi_{s^{-1}(a)}(a)$ glue together to give a global section $g \in \mathscr{G}(U)$. We thus define $\phi^\sharp$ by:

$$\phi^\sharp(s) = g$$

We thus see that if $a \in \mathbb{Z}(U)$, then $\mathrm{sh}(a) = s_a$:

$$\phi^\sharp(s_a) = \phi(a)$$

as $s_a^{-1}(a) = U$. It follows that $\phi^\sharp$ is unique, and well defined by the same argument in Proposition 1.2.3, so $\mathbb{Z}^\sharp$ is the sheafification of $\mathbb{Z}$. Going forward, we call $\mathbb{Z}^\sharp$ the **constant sheaf with values in** $\mathbb{Z}$[7], and denote by $\mathbb{Z}$.

We now go out of our way to explicitly explain the kernel sheaf, cokernel, sheaf, and the image sheaf. We work entirely with sheafs of abelian groups, though similar objects can be defined in the category of rings, the resulting sheafs just don't necessarily stay in the category of rings. The zero sheaf, will be denoted by 0, and is the sheaf that sends every open set to the trivial group, and the trivial transformation will be denote 0.

Definition 1.2.5. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheafs, then the **kernel sheaf**, denoted $(\ker F, \iota)$ is a sheaf equipped with a natural transformation $\iota : \ker F \to \mathscr{F}$ such that $F \circ \iota = 0$, and for all $\psi : \mathscr{H} \to \mathscr{F}$ such that $F \circ \psi = 0$, there exists a unique $\theta : \mathscr{H} \to \ker \mathscr{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{H} \arrow[rr, "\psi"] \arrow[rrr, bend left, "0"] \arrow[dr, dashed, "\exists!\theta"'] & & \mathscr{F} \arrow[r, "F"] & \mathscr{G} \\
& \ker F \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

Proposition 1.2.4. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then* $\ker F$ *exists and is unique up to unique isomorphism.*

Proof. We define $\ker F$ by:

$$(\ker F)(U) = \ker F_U$$

This is easily seen to be a presheaf. We check that it is a sheaf. Let U_i cover U, and $s \in \ker F_U$ such that $s|_{U_i} = 0$. However, each $s|_{U_i} \in \ker F_{U_i} \subset \mathscr{F}(U_i)$, and $s \in \ker F_U \subset \mathscr{F}(U)$ hence $s = 0$. Now suppose we have $s_i \in \ker F_{U_i}$ such that:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

These glue together to form an $s \in \mathscr{F}(U)$, however we need to check that $s \in \ker F_U$. Note that:

$$F_U(s)|_{U_i} = F_{U_i}(s|_{U_i}) = 0$$

and since $\mathscr{G}$ is a sheaf we have that $F(s) = 0$, so $s \in \ker F_U$. It follows that $\ker F$ is a sheaf.

Define $\imath : \ker F \to \mathscr{F}$ by $\imath_U(s) = s$, i.e. $\imath_U$ is just the natural inclusion of abelian groups. It is clear that $\phi \circ \imath = 0$. Let $\psi : \mathscr{H} \to \mathscr{F}$ be a morphism such that $F \circ \psi = 0$, then we need a morphism θ such that for all U :

$$\iota_U \circ \theta_U = \psi_U$$

[7]We can also use the same construction to obtain the constant sheaf with values in any set, abelian group, or ring

We note that since $\phi_U \circ \psi_U = 0$, so ψ_U has in $\ker F_U$. We thus define:

$$\theta_U(s) = \psi_U(s)$$

with restricted target. This is readily seen to be a natural transformation, and is unique and well defined, hence $\ker F$ satisfies the universal property of a sheaf kernel. It follows that $\ker F$ is unique up to unique isomorphism, implying the claim. $\square$

We have a similar definition for the cokernel, but with arrows reversed:

Definition 1.2.6. Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then the **sheaf cokernel**, denoted $(\operatorname{coker} F, \pi)$ is a sheaf equipped with a morphism $\pi : \mathscr{G} \to \operatorname{coker} F$, such that $\pi \circ F = 0$, and for all morphisms $\psi : \mathscr{G} \to \mathscr{H}$ such that $\psi \circ F = 0$, there exists a unique morphism $\theta : \operatorname{coker} F \to \mathscr{H}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "F"] \arrow[rrr, bend left, "0"] & \mathscr{G} \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & \mathscr{H} \\
& & \operatorname{coker} F \arrow[ur, dashed, "\exists!\theta"'] &
\end{tikzcd}$$

Proposition 1.2.5. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism, then the sheaf cokernel $(\operatorname{coker} F, \pi)$ exists and is unique up to unique isomorphism.*

Proof. Note that in the category of abelian groups, the cokernel of F_U is given by:

$$\mathscr{G}(U)/\operatorname{im} F_U$$

Using this assignment as the cokernel is problematic, as $\operatorname{coker} F$ will then fail be to a sheaf. In particular, the gluing property does not always hold. We thus define the presheaf:

$$\operatorname{coker}^p F : U \longmapsto \operatorname{coker} F_U = \mathscr{G}(U)/\operatorname{im} F_U$$

with π^p to be the natural transformation defined as the projection map $\mathscr{G}(U) \to \mathscr{G}(U)/\operatorname{im} F_U$ for all U, and define the cokernel sheaf to be:

$$\operatorname{coker} \mathscr{F} = (\operatorname{coker}^p F)^\sharp$$

with $\pi = \operatorname{sh} \circ \pi^p$. Suppose that $\operatorname{coker}^p F$ satisfies the universal property of the cokernel in the category of presheaves, then for all morphisms $\psi : \mathscr{G} \to \mathscr{H}$, we obtain the following commutative diagram by the universal property of sheafification:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "F"] \arrow[rrr, bend left, "0"] & \mathscr{G} \arrow[rr, "\psi"] \arrow[dr, "\pi^p"'] & & \mathscr{H} \\
& & \operatorname{coker}^p F \arrow[ur, "\theta"] \arrow[r, "\operatorname{sh}"] & \operatorname{coker} F \arrow[u, "\theta^\sharp"']
\end{tikzcd}$$

It would then follow that $(\operatorname{coker} F, \pi)$ satisfies the universal property of cokernels in the category of sheaves. We now show that coker^p is a presheaf, and π^p is a natural transformation. First note that for all $U \subset V$ we have the following diagram:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "\theta^U_V"] \arrow[d, "\pi^p_U"] & \mathscr{G}(V) \arrow[d, "\pi^p_V"] \\
\mathscr{G}(U)/\operatorname{im} F_U & \mathscr{G}(V)/\operatorname{im} F_V
\end{tikzcd}$$

Note that $\pi_V^p \circ \theta_V^U$ is a morphism $\mathscr{G}(U) \to \mathscr{G}(V)/\operatorname{im} F_V$. Suppose that $g \in \operatorname{im} F_U$, then $g = F_U(s)$ for some $s \in \mathscr{F}(U)$, and we see that

$$\pi_V^p \circ \theta_V^U(F_U(s)) = \pi_V^p(F_V(s|_V)) = 0$$

so $\operatorname{im} F_V \subset \ker \theta_V^U \circ \pi_V^p$. It follows that there exists a unique map which we also denote by $\theta_V^U : \mathscr{G}(U)/\operatorname{im} F_U \to \mathscr{G}(V)/\operatorname{im} F_V$, such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\theta_V^U"] \arrow[d, "\pi_U^p"] & \mathscr{G}(V) \arrow[d, "\pi_V^p"] \\ \mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta_V^U"] & \mathscr{G}(V)/\operatorname{im} F_V \end{tikzcd}$$

This implies that if coker^p is a presheaf, then π_U^p is a natural transformation. We need to check that $\theta_U^U = \mathrm{Id}$. Examine the diagram:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\theta_U^U"] \arrow[d, "\pi_U^p"] & \mathscr{G}(U) \arrow[d, "\pi_U^p"] \\ \mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta_U^U"] & \mathscr{G}(U)/\operatorname{im} F_U \end{tikzcd}$$

The top θ_U^U is the identity, so the only way for the bottom θ_U^U to make the diagram commute is for $\theta_U^U = \mathrm{Id}$ as well. We need to show that $\theta_W^V \circ \theta_V^U = \theta_W^U$; examine the diagram:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\theta_V^U"] \arrow[d, "\pi_U^p"] & \mathscr{G}(V) \arrow[r, "\theta_W^V"] \arrow[d, "\pi_V^p"] & \mathscr{G}(W) \arrow[d, "\pi_V^p"] \\ \mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta_V^U"] & \mathscr{G}(V)/\operatorname{im} F_V \arrow[r, "\theta_W^V"] & \mathscr{G}(W)/\operatorname{im} F_W \end{tikzcd}$$

Erase the middle to obtain:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\theta_W^U"] \arrow[d, "\pi_U^p"] & \mathscr{G}(V) \arrow[d, "\pi_W^p"] \\ \mathscr{G}(U)/\operatorname{im} F_U \arrow[r, "\theta_W^V \circ \theta_V^U"] & \mathscr{G}(W)/\operatorname{im} F_W \end{tikzcd}$$

Then since $\theta_W^V \circ \theta_V^U$ makes this diagram commute, we must have that $\theta_W^V \circ \theta_V^U = \theta_W^U$, so $\operatorname{coker}^p F$ is a presheaf.

To see that this satisfies the universal property of the presheaf cokernel, let $\psi : \mathscr{G} \to \mathscr{H}$ be a morphism of presheaves such that $\psi \circ F = 0$, then we want to find a morphism $\theta : \operatorname{coker}^p F \to \mathscr{H}$ such that for all U:

$$\theta_U \circ \pi_U^p = \psi_U$$

We define θ_U by:

$$\theta_U([g]) = \psi_U(g)$$

and note that this well defined, as if $[h] = [g]$, then we have that $h = g + F_U(s)$ for some $s \in \mathscr{F}(U)$. It follows that since $\psi \circ F = 0$:

$$\psi_U(h) = \psi_U(g + F_U(s)) = \psi_U(g)$$

It is clear that the assignment $U \to \theta_U$ then defines a natural transformation θ, as ψ is a natural transformation. It follows that $(\operatorname{coker}^p F, \pi^p)$ is the cokernel in the category of presheaves, implying that $(\operatorname{coker} F, \pi)$ is the cokernel in the category of sheaves. $\square$

Corollary 1.2.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then* $(\ker F)_x = \ker F_x$ *and* $(\operatorname{coker} F)_x \cong \operatorname{coker} F_x$

Proof. We first that $\imath_x : (\ker F)_x \to \mathscr{F}_x$ is an inclusion map, so $(\ker F)_x \subset \mathscr{F}_x$. It thus suffices to show that $(\ker F)_x = \ker F_x$ as both are subgroups of $\mathscr{F}_x$. Let $s_x \in (\ker F_x)$, then $s_x = [U, s]$ for some $s \in \ker F_U$, it follows that $F_x(s_x) = [U, F_U(s)] = [U, 0] = 0$, so $(\ker F)_x \subset \ker F_x$. Now let $s_x \in \ker F_x$, and let $s_x = [U, s]$ for some $s \in \mathscr{F}(U)$. It follows that $[U, F_U(s)] = [U, 0]$, so there exists a V such that $F_V(s|_V) = 0$, hence $s|_V \in \ker F_V$. We have that $[U, s] = [V, s|_V]$, and $[V, s|_V] \in (\ker F)_x$, hence $s \in (\ker F)_x$, implying equality.

For the other statement, we need only show that $(\operatorname{coker}^p F)_x \cong \operatorname{coker} F_x$, then since sheafification provides an isomorphism $\mathrm{sh}_x : (\operatorname{coker}^p F)_x \to (\operatorname{coker} F)_x$ we will have the claim. Note that we have a map $\pi^p_x : \mathscr{G}_x \to (\operatorname{coker}^p F)_x$, which satisfies $\pi^p_x \circ F_x = 0$, so let $\phi : \mathscr{G}_x \to A$ be an morphism such that $\phi \circ F_x = 0$. Note that for $[U, g] \in \mathscr{G}_x$, we have that:

$$[U, g] \mapsto [U, [g]]$$

We thus define a homomorphism $\theta : (\operatorname{coker}^p F)_x \to A$ by:

$$\theta([U, [g]]) = \psi([U, g])$$

We need to check that this independent of the choice of g, let $[U, [h]] = [U, [g]]$, then there exists an open set $V \subset U$, such that:

$$[h]|_V = [g]|_V \Rightarrow h|_V = g|_V + s$$

where $s \in \operatorname{im} F_V$. Since ψ itself must be well defined, we have that:

$$\theta([U, [h]]) = \psi([U, h]) = \psi([V, h|_V]) = \psi([V, g|_V] + [V, s]) = \psi([V, g|_V]) = \psi([U, g])$$

It follows that $(\operatorname{coker}^p F)_x$ then satisfies the universal property of the cokernel of F_x, hence there is a unique isomorphism $(\operatorname{coker}^p F)_x \cong \operatorname{coker} F_x$, and thus a unique isomorphism $(\operatorname{coker} F)_x \cong \operatorname{coker} F_x$.

□

Now that we know kernels and cokernels exist, we wish to show that the category of sheaves of abelian groups over a topological space X is an abelian category. We need the following terminology:

Definition 1.2.7. A **additive category** is a category with a 0 object[8], finite products and coproducts, and each set $\operatorname{Hom}(A, B)$ for objects A and B has an abelian group structure such that the composition maps are bilinear.

Note that in any additive category the (finite) product and coproduct are naturally isomorphic.

Lemma 1.2.5. *Let X be a topological space, then the category of sheaves with values in abelian groups is additive.*

Proof. We first note that the trivial sheaf which assigns $\{0\}$ to each open set is easily seen to be a zero object.

We define the product of two sheaves $\mathscr{G}$ and $\mathscr{F}$ to be:

$$(\mathscr{G} \times \mathscr{F})(U) = \mathscr{G}(U) \times \mathscr{F}(U)$$

It is clear that this defines a sheaf, and moreover there clearly exist natural formations $\pi_{\mathscr{G}} : \mathscr{G} \times \mathscr{F} \to \mathscr{G}$ and $\pi_{\mathscr{F}} : \mathscr{G} \times \mathscr{F} \to \mathscr{F}$ such that $(\pi_{\mathscr{G}})_U$ and $(\pi_{\mathscr{F}})_U$ are the natural projections in the category of abelian

[8] A zero object is one that is both an initial and final object in the category, i.e. for every object A there exist unique morphisms $0 \to A$ (initial), and $A \to 0$ (final)

groups. Let $\mathscr{H}$ be another sheaf with morphisms $\phi_{\mathscr{F}} : \mathscr{H} \to \mathscr{F}$ and $\phi_{\mathscr{G}} : \mathscr{H} \to \mathscr{G}$, then we want to show that there exists a unique $\psi : \mathscr{H} \to \mathscr{G} \times \mathscr{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathscr{H} \arrow[ddl, bend right, "\phi_{\mathscr{G}}"'] \arrow[d, dashed, "\exists!\psi"] \arrow[ddr, bend left, "\phi_{\mathscr{G}}"] & \\
& \mathscr{G} \times \mathscr{F} \arrow[dl, "\pi_{\mathscr{G}}"'] \arrow[dr, "\pi_{\mathscr{F}}"] & \\
\mathscr{G} & & \mathscr{F}
\end{tikzcd}$$

In the category of abelian groups, we have that ψ_U would be given by:

$$\psi_U(h) = (\phi_{\mathscr{G}}(h), \phi_{\mathscr{F}}(h))$$

so assignment $U \mapsto \psi_U$ is the natural transformation which makes the above diagram commute, demonstrating that $\mathscr{F} \times \mathscr{G}$ is indeed the product. Since the product are coproduct are the same in abelian groups, it follows that the same argument with the arrows reversed shows that $\mathscr{G} \times \mathscr{F}$ is the coproduct in the category of sheaves as well.

We have that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is the set of all natural transformations $\mathscr{F} \to \mathscr{G}$. We define addition in this set by:

$$(\phi + \psi)_U = \phi_U + \psi_U \in \mathrm{Hom}(\mathscr{F}(U), \mathscr{G}(U))$$

We see that this is a natural transformation, as:

$$\begin{aligned}
\theta^U_V \circ (\phi + \psi)_U =& \theta^U_V \circ (\phi_U + \psi_U) \\
=& \theta^U_V \circ \phi_U + \theta^U_V \psi_U \\
=& \phi_V \circ \theta^U_V + \psi_V \circ \theta^U_V \\
=& (\phi_V + \psi_V) \circ \theta^U_V \\
=& (\psi + \phi)_V \circ \theta^U_V
\end{aligned}$$

So addition makes sense. Note that natural transformation $U \mapsto 0_U$ [9], which we suggestively denote by 0, is the 0 element in this set. Indeed, we have that for all U:

$$(\phi + 0)_U = \phi_U + 0_U = \phi_U$$

so $\phi + 0 = \phi$. We see that for any $\phi \in \mathrm{Hom}(\mathscr{F}, \mathscr{G})$, we can define $-\phi$ by:

$$(-\phi)_U = -\phi_U$$

which is clearly a natural transformation by the same argument above. It follows that for all U:

$$(\phi - \phi)_U = \phi_U - \phi_U = 0_U$$

so $\phi - \phi = 0$. We thus see that $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ is indeed an abelian group. Let $\mathscr{H}$ be another sheaf, and consider $\mathrm{Hom}(\mathscr{G}, \mathscr{H})$, we want to show that for all $\theta \in \mathrm{Hom}(\mathscr{G}, \mathscr{H})$ we have that:

$$\theta \circ (\phi + \psi) = \theta \circ \phi + \theta \circ \psi \in \mathrm{Hom}(\mathscr{F}, \mathscr{H})$$

We see that $\theta \circ (\phi + \psi)$ for all U:

$$(\theta \circ (\phi + \psi))_U = \theta_U \circ (\phi + \psi)_U = \theta_U \circ (\phi_U + \psi_U) = \theta_U \circ \phi_U + \theta_U \circ \psi_U = (\theta \circ \phi)_U + (\theta \circ \psi)_U$$

[9] 0_U being the trivial morphism in $\mathrm{Hom}(\mathscr{F}(U), \mathscr{G}(U))$

so $\theta \circ (\phi + \psi) = \theta \circ \phi + \theta \circ \psi$. The same argument in the other direction demonstrates that for all $\theta \in \mathrm{Hom}(\mathscr{H}, \mathscr{F})$, we have that:

$$(\phi + \psi) \circ \theta = \phi \circ \theta + \psi \circ \theta \in \mathrm{Hom}(\mathscr{H}, \mathscr{G})$$

so composition is bilinear, implying the claim. $\square$

Definition 1.2.8. Let $\phi : A \to B$ be a morphism in an additive category, then ϕ is **monomorphism** if for all $\theta : Z \to A$, we have that $\phi \circ \theta = 0 \Rightarrow \theta = 0$. A morphism ϕ is an **epimorphism** if for all $\theta : B \to Z$ we have that $\theta \circ \phi = 0 \Rightarrow \theta = 0$. If we are not in an additive category, then ϕ is a monomorphism if for all $\theta_1, \theta_2 : Z \to A$ we have that $\phi \circ \theta_1 = \phi \circ \theta_2 \Rightarrow \theta_1 = \theta_2$. Similarly, ϕ is an epimorphism if for all $\theta_1, \theta_2 : B \to Z$, we have that $\theta_1 \circ \phi = \theta_2 \circ \phi \Rightarrow \theta_1 = \theta_2$. [10]

Lemma 1.2.6. *Let $\phi : A \to B$ a morphism in an additive category with kernels and cokernels. Then, the morphism $\iota : \ker \phi \to A$ is a monomorphism, and the morphism $\pi : B \to \operatorname{coker} \phi$ is an epimorphism.*

Proof. Suppose that $\theta : Z \to \ker \phi$ such that $\imath \circ \theta = 0$, then we have the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, "\imath\circ\theta"] \arrow[dr, "\theta"'] \arrow[rrr, bend left, "0"] & & A \arrow[r, "\phi"] & B \\
& \ker\phi \arrow[ur, "\imath"'] & &
\end{tikzcd}$$

Where $\theta \circ \iota = 0$ so $\phi \circ \theta \circ \iota = 0$. By the universal property of kernels, it follows that θ is the unique map that makes this commute. However, $\iota \circ \theta = 0$, so $\theta = 0$ also makes this map commute, hence by uniqueness $\theta = 0$, and ϕ is a monomorphism.

Suppose that $\theta : \operatorname{coker} \phi \to Z$ such that $\theta \circ \pi = 0$, then we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[rrr, bend left, "0"] & B \arrow[rr, "\theta\circ\pi"] \arrow[dr, "\pi"'] & & Z \\
& & \operatorname{coker}\phi \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Again by the universal property, since θ makes the map we commute we have that it must be unique. However, since $\theta \circ \pi = 0$, clearly $\theta = 0$ makes this map commute as well so by uniqueness $\theta = 0$, and π is an epimorphism. $\square$

Applying Lemma 1.2.6 to the the category sheaves, demonstrates that $(\ker F, \imath)$ and $(\operatorname{coker} F, \pi)$ are monomorphisms and epimorphisms for all natural transformations F.

Proposition 1.2.6. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves with values in abelian groups, then the following are equivalent:*

a) F is a monomorphism.

b) For all $x \in X$, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective.

c) $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U

Similarly the following are equivalent:

d) F is an epimorphism.

e) For all $x \in X$, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective.

We need the following lemma:

[10] In the category of abelian groups, a morphism is mono if and only if it is injective, and epi if and only if it is surjective.

Lemma 1.2.7. *Let A be an abelian group, and $x \in X$, then the assignment:*

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

is a sheaf such that $(x_*A)_x = A$ and is 0 for all other points. This is often referred to as the ***skyscraper sheaf****.*

Proof. Define restriction maps by $\theta_V^U = \mathrm{Id}$ if $x \in V$, and U, and 0 otherwise. This is clearly a presheaf, and $x \notin U$ there are no sheaf axioms to check. Suppose $x \in U$, and let U_i be an open cover for U, such that for $s \in A$, we have that $s|_{U_i} = 0$. It follows that $s = 0$, because for at least one i we have that $x \in U_i$ and $\theta_{U_i}^U = \mathrm{Id}$. Now suppose that we have $s_i \in U_i$, such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. If $x \notin U_i$ or U_j, then we have that both restrictions are zero, if $x \in U_i \cap U_j$ then we must have that $s_i = s_j$, hence $s = s_i$ for any U_i containing x restricts to each s_i. It follows that (x_*A) is a sheaf.

If $y \neq x$, then any element $[U, s] \in (x_*A)_y$ is equal to $[V, 0]$ for some smaller V not containing x, so A_y must be the zero group as every element is the zero element. We show that A satisfies the universal property of the direct limit. Let $\phi_U : (x_*A)(U) \to G$ be maps which commute with restriction, and let $\psi_U : (x_*A)(U) \to A$ be the identity, which also commutes with restriction as U contains x. We define $F : A \to G$ by:

$$F(a) = \phi_U(a)$$

which is well defined because for all $\phi_U : (x_*A)(U) \to G$, we must have that $\phi_V = \phi_U$ as the restriction maps are the identity. It follows that A satisfies the universal property of direct limit and is thus the stalk of (x_*A) at x. □

We now prove the proposition:

Proof. Note that $c) \Rightarrow a)$, as if $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U, then F_U is a monomorphism for all U. It follows that if $\theta : \mathscr{H} \to \mathscr{F}$ satisfies $F \circ \theta = 0$, then for all U we have that $\theta_U = 0$, so θ is the trivial morphism.

We now show that $a) \Rightarrow b)$. Take the natural morphism $\imath : \ker F \to \mathscr{F}$, and note that $F \circ \imath = 0$, so $\imath = 0$. However, we have that on each open set $\imath_U(s) = s = 0$, so for all $s \in \ker F_U$, we have that $s = 0$. It follows that $\ker F_U = 0$ so $\ker F$ is the trivial sheaf, and $(\ker F)_x$ is the trivial group, but by Corollary 1.2.2 $(\ker F)_x = \ker F_x$ so $\ker F_x = 0$ and F_x is injective.

We now show that $b) \Rightarrow c)$. Suppose that for all x, $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective, then we have an induced injection:

$$\begin{aligned} \prod_{x \in U} \mathscr{F}_x &\longrightarrow \prod_{x \in U} \mathscr{G}_x \\ (s_x) &\longmapsto (F_x(s_x)) \end{aligned}$$

Suppose that $s, t \in \mathscr{F}(U)$, such that $F_U(s) = F_U(t)$, then we we have that by the definition of the stalk map $F_x(s_x) = F_x(t_x)$ for all $x \in U$. However, the map above is injective so $(s_x) = (t_x)$ implying that $s = t$ by Lemma 1.2.2. We thus have that:

$$a) \Longrightarrow b) \Longrightarrow c) \Longrightarrow a)$$

implying the first part of the claim.

We now show that $d) \Rightarrow e)$. Let $\mathscr{H}$ be the skyscraper sheaf $x_*(\mathscr{G}_x / \operatorname{im} F_x)$, and note that the map $\psi_U : \mathscr{G}(U) \to \mathscr{H}(U)$:

$$\psi_U(g) = \begin{cases} [g_x] & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

trivially commutes with restriction maps, and thus defines a natural transformation. Vacuously we have that $\psi \circ F = 0$, as if $x \in U$, we have that:

$$\psi_U(F(s)) = [F(s)_x] = [F_x(s_x)] = 0$$

and if $x \notin U$, we have that $\psi = 0$ anyways. However, since F is an epimorphism, this implies that $\psi = 0$ so $\psi_x = 0$. Note however, that ψ_x is the map defined by:

$$\begin{aligned} \psi_x : \mathscr{G}_x &\longrightarrow \mathscr{G}_x / \operatorname{im} \mathscr{F}_x \\ g_x &\longmapsto [g_x] \end{aligned}$$

which is clearly a surjection, hence $\mathscr{G}_x / \operatorname{im} F_x = 0$, implying that $\operatorname{im} F_x = \mathscr{G}_x$, and thus the claim.

To show that $e) \Rightarrow d)$, suppose that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is a surjection for all $x \in X$. Let ϕ be any other an morphism $\mathscr{G} \to \mathscr{H}$ such that $\phi \circ F = 0$. This implies that on stalks:

$$\phi_x \circ F_x = 0$$

But F_x is a surjection, and thus an epimorphism, so on the level of stalks we have that $\phi_x = 0$ for all $x \in U$. Now examine the commutative diagram:

$$\begin{tikzcd} \mathscr{G}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathscr{H}(U) \arrow[d] \\ \prod_{x \in U} \mathscr{G}_x \arrow[r, "0"] & \prod_{x \in U} \mathscr{H}_x \end{tikzcd}$$

If $g \in \mathscr{G}(U)$ we have that:

$$(\phi_U(g)_x) = 0$$

however, the downward maps are injections, hence we must have that $\phi_U(g) = 0$ for all $g \in \mathscr{G}(U)$, and all U, thus $\phi = 0$, so F is an epimorphism.

□

Definition 1.2.9. A category is **abelian**, if it is additive, kernels and cokernels exist, and every monomorphism and epimorphism are the kernel and cokernel of some morphism.[11]

Theorem 1.2.1. *Let X be a topological space, then the category of sheaves with values in abelian groups is an abelian category.*

Proof. We need only show that every monomorphism is the kernel of some morphism, and that every epimorphism is the cokernel of some morphism.

Suppose that $F : \mathscr{F} \to \mathscr{G}$ is a monomorphism, we want to show that $(\mathscr{F}, F)$ is the kernel of some morphism $\psi : \mathscr{G} \to \mathscr{H}$. Well, take $\mathscr{H}$ to be $\operatorname{coker} F$, and ψ to be the projection π. We note that $\pi \circ F = 0$, indeed $\pi = \operatorname{sh} \circ \pi^p$, so for all open sets U, we have that:

$$\pi_U \circ F_U = \operatorname{sh}_U \circ \pi_U^p \circ F_U = \operatorname{sh}_U \circ 0_U = 0$$

so $\pi \circ F$ is the trivial morphism. Now let $\phi : \mathscr{H} \to \mathscr{G}$, such that $\pi \circ \phi = 0$, then we want to obtain the following commutative diagram:

$$\begin{tikzcd} \mathscr{H} \arrow[r, "\phi"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists ! \psi"'] & \mathscr{G} \arrow[r, "\pi"] & \operatorname{coker} F \\ & \mathscr{F} \arrow[u, "F"'] & \end{tikzcd}$$

[11] Often times people refer to the morphisms $\imath$ and π as the kernel and cokernel, so when we see every monomorphism (epimorphism) is a kernel (cokernel) of some morphism we are saying every monomorphism (epimorphism) can be written as the inclusion (projection) map $\imath$ (π) induced by the kernel (cokernel) of some morphism.

We move to level of stalks, and since $(\operatorname{coker} F)_x \cong \mathscr{G}_x/\operatorname{im} F_x$ we have the following diagram:

$$\begin{tikzcd}
\mathscr{H}_x \arrow[rr, "\phi_x"] \arrow[dr, dashed, "\exists!\psi_x"'] \arrow[rrrr, bend left, "0"] & & \mathscr{G}_x \arrow[rr, "\tilde{\pi}_x"] & & \mathscr{G}_x/\operatorname{im} F_x \\
& \mathscr{F}_x \arrow[ur, "F_x"'] & & &
\end{tikzcd}$$

where $\tilde{\pi}_x$ is π_x composed with the isomorphism $(\operatorname{coker} F)_x \to \mathscr{G}_x/\operatorname{im} F_x$, and by the uniqueness of the quotient map it follows that $\tilde{\pi}_x$ is the quotient map. We see that for all $h_x \in \mathscr{H}_x$:

$$\tilde{\pi}_x \circ \phi_x(h_x) = 0$$

so $\phi_x(h_x) \in \ker \tilde{\pi}_x = \operatorname{im} F_x$, and we have that $\operatorname{im} \phi_x(h_x) \subset \operatorname{im} F_x$. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathscr{H}(U) \arrow[rr, "\phi_U"] \arrow[ddd] & & \mathscr{G}(U) \arrow[ddd] \\
& \mathscr{F}(U) \arrow[ur, "F_U"] \arrow[d] & \\
& \prod_{x\in U} \mathscr{F}_x \arrow[dr, "(F_x)"] & \\
\prod_{x\in U} \mathscr{H}_x \arrow[rr, "(\phi_x)"] & & \prod_{x\in U} \mathscr{G}_x
\end{tikzcd}$$

Take an $h \in \mathscr{H}(U)$, then we have that:

$$\phi_U(h)_x = \phi_x(h_x) \in \operatorname{im} F_x$$

Since (F_x) is an injection, we see that there exists a unique $(s_x) \in \prod_{x\in U} \mathscr{F}_x$, such that:

$$(F_x(s_x)) = (\phi_U(h)_x)$$

For each $x \in U$, we thus have that there exists an open neighborhood $V_x \subset U$, and a section $s^x \in \mathscr{F}(V_x)$ such that:

$$[V_x, F_{V_x}(s^x)] = [U, \phi_U(h)]$$

implying that there is a $W_x \subset V_x \cap U = V_x$ such that:

$$F_{V_x}(s^x)|_{W_x} = \phi_U(h)|_{W_x}$$

The set of all such W_x's and $s^x|_{W_x}$'s covers U, and we see that for $V_x \cap V_y \neq \emptyset$:

$$F_{W_x\cap W_y}(s^x|_{W_x\cap W_y}) = \phi_U(h)|_{W_x\cap W_y} = F_{W_x\cap W_y}(s^y|_{W_x\cap W_y})$$

however $F_{W_x\cap W_y}$ is injective by Proposition 1.2.6, so we have that:

$$s^x|_{W_x\cap W_y} = (s^x|_{W_x})|_{W_x\cap W_y} = (s^y|_{W_x})|_{W_x\cap W_y} = s^y|_{W_x\cap W_y}$$

The s^x's then glue together to form an $s \in \mathscr{F}(U)$ such that:

$$F_U(s)_x = \phi_U(h)_x$$

for all $x \in U$. Since $F_U(s)$ and $\phi_U(h)$ both lie in $\mathscr{G}(U)$, and they agree on all stalks we must have that $\phi_U(h) = F_U(s)$. It follows that for all U $\operatorname{im} \phi_U \subset \operatorname{im} \mathscr{F}_U$. We now define a morphism $\psi : U \to \psi_U$ by:

$$\psi_U(h) = s$$

where $s \in \mathscr{F}(U)$ is the unique section such that that $\phi_U(h) = F_U(s)$. To see that this commutes with restriction maps, we need to show that $\theta_V^U \circ \psi_U = \psi_V \circ \theta_V^U$. In particular, we need:

$$\psi_V(\theta_V^U(h)) = \theta_V^U(s)$$

Take $h|_V$, then $\psi_V(h|_V) = f \in \mathscr{F}(V)$, where $F_V(f) = \phi_V(h|_V)$. Now note that:

$$F_V(s|_V) = (F_U(s))|_V = (\phi_U(h))|_V = \phi_V(h|_V)$$

so $F_V(s|_V) = F(f)$, and thus $f = s|_V$ implying the claim. To see that this is actually a group homomorphism, let $h, g \in \mathscr{H}(U)$ such that $\psi_U(h) = s$ and $\psi_U(g) = t$. We need to show that:

$$\psi_U(h+g) = s + t$$

Well, let $\psi_U(h+g) = f$ be the unique $f \in \mathscr{F}(U)$ such $\phi_U(h+g) = F_U(f)$. However, we see that $\phi_U(h+g) = \phi_U(h) + \phi_U(g) = F_U(s) + F_U(t)$, hence:

$$F_U(f) = F_U(s) + F_U(t) \implies f = s + t$$

It follows that $\psi_U(h+g) = s + t$, and is thus a group homomorphism. In particular, this implies $(\mathscr{F}, F)$ satisfies the universal property of the kernel of the cokernel of F and is thus the kernel of some morphism.

Now let $F : \mathscr{F} \to \mathscr{G}$ be an epimorphism. We claim that $(\mathscr{G}, F)$ is the cokernel of $\iota : \ker F \to \mathscr{F}$. We first note that clearly:

$$F \circ \iota = 0$$

as $\operatorname{im} \iota_U = \ker F_U$ for all U. Let $\phi : \mathscr{F} \to \mathscr{H}$ be an morphism such that $\phi \circ \iota = 0$, we want to that there exists a unique ψ such that:

$$\begin{tikzcd}
\ker F \arrow[r, "\iota"] \arrow[rrr, bend left, "0"] & \mathscr{F} \arrow[rr, "\phi"] \arrow[dr, "F"] & & \mathscr{H} \\
& & \mathscr{G} \arrow[ur, dashed, "\exists!\psi"] &
\end{tikzcd}$$

As before, we exploit the fact that $(\ker F)_x = \ker F_x$, and move to the level of stalks:

$$\begin{tikzcd}
\ker F_x \arrow[r, "\iota_x"] \arrow[rrr, bend left, "0"] & \mathscr{F}_x \arrow[rr, "\phi_x"] \arrow[dr, "F_x"] & & \mathscr{H}_x \\
& & \mathscr{G}_x \arrow[ur, dashed, "\exists!\psi_x"] &
\end{tikzcd}$$

Since $\mathscr{F}$ is an epimorphism, we have that F_x is surjective by Proposition 1.2.6. We see that since $\imath_x$ is the inclusion map of the kernel of F_x, that $\operatorname{im} \imath_x = \ker F_x$. We also have that $\operatorname{im} \imath_x \subset \ker \phi_x$, hence we define a unique map ψ_x by:

$$\psi_x(g_x) = \phi_x(s_x)$$

where s_x is any element in $F_x^{-1}(g_x)$. This is well defined, as if s_x' is any other element in $F_x^{-1}(g_x)$, we have that:

$$\phi_x(s_x) - \phi_x(s_x') = \phi_x(s_x - s_x')$$

but $s_x, s'_x \in F_x^{-1}(g_x)$, so $s_x - s'_x \in \ker F_x = \operatorname{im} \imath_x$. It follows that $s_x - s'_x \in \ker \phi_x$, so $\phi_x(s_x) = \phi_x(s'_x)$ as desired. We now examine the following diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F"] \arrow[rr, bend left, "\phi_U"] \arrow[d] & \mathscr{G}(U) \arrow[r, dotted, "\exists!\psi_U"] \arrow[d] & \mathscr{H}(U) \arrow[d] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r, "(F_x)"] \arrow[rr, bend right, "(\phi_x)"'] & \prod_{x\in U} \mathscr{G}_x \arrow[r, "(\psi_x)"] & \prod_{x\in U} \mathscr{H}_x
\end{tikzcd}$$

where we want to define ψ_U such that this commutes. Take an element $g \in \mathscr{G}(U)$, then we have a unique corresponding element $(g_x) \in \prod_{x\in U} \mathscr{G}_x$. This then maps to $(\psi_x(g_x)) \in \prod_{x\in U} \mathscr{H}_x$, which is equal to $(\phi_x(s_x))$ for some $(s_x) \in (F_x)^{-1}(g_x)$. We want to find a section $h \in \mathscr{H}(U)$ such that $h_x = \psi_x(g_x)$, as we can then define ψ_U by $\psi_U(g) = h$. For each x we have that:

$$\phi_x(s_x) = \psi_x(g_x) \in \mathscr{H}_x$$

so in particular, there exists an open neighborhood V_x of x, and a section $s^x \in \mathscr{F}(s^x)$ such that:

$$[V_x, \phi_{V_x}(s^x)] = \psi_x(g_x)$$

Cover U with all such V_x, then we want to show that:

$$\phi_{V_x}(s^x)|_{V_x\cap V_y} = \phi_{V_y}(s^y)|_{V_x\cap V_y}$$

However, note that for all $p \in V_x \cap V_y$, we have that:

$$(\phi_{V_x}(s^x)|_{V_x\cap V_y})_p = \phi_{V_x}(s^x)_p = \psi_p(g_p) = \phi_{V_y}(s^y)_p = (\phi_{V_y}(s^y)|_{V_x\cap V_y})_p$$

so we must have that sections agree on overlaps. It follows that that $\phi_{V_x}(s^x)$'s glue together to form a section $h \in \mathscr{H}(U)$ such that $h_x = \psi_x(g_x)$ for all $x \in U$. We thus define ψ_U to be:

$$\psi_U(g) = h$$

It is then clear that h is independent of our choice of (s_x) as ψ_x is independent of that choice, and moreover that it is independent of our choice of cover of U, as any other choice will have to agree on stalks. This is also clearly a group homomorphism, and is compatible with restriction maps; indeed if $g, g' \in \mathscr{G}(U)$, and we have that $\psi_U(g) = h$ and $\psi_U(g') = h'$, then we see that for all $x \in U$:

$$(\psi_U(g) + \psi_U(g'))_x = h_x + h'_x = \psi_x(g_x) + \psi_x(g'_x) = \psi_x((g+g')_x) = \psi_U(g+g')_x$$

Since they agree on stalks we must have that they are equal. Moreover, we want to show that:

$$\psi_U(g)|_V = h|_V$$

However, if we again take stalks, we see that for all $x \in V$,

$$(\psi_U(g)|_V)_x = \psi_U(g)_x = \psi_x(g_x) = h_x = (h|_V)_x$$

so the two must again agree. Finally, we check that $\psi \circ F = \phi$. Let $s \in \mathscr{F}(U)$, then we have $F_U(s) \in \mathscr{G}(U)$, which maps down to sequence $(F_U(s)_x) = (F_x(s_x))$, where each s_x clearly lies in $F_x^{-1}(F_x(s_x))$. We thus have that:

$$\psi_x(F_x(s_x)) = \phi_x(s_x) = \phi_U(s)_x$$

for all x by definition of ψ_x. It follows that from the defining property of ψ_U:

$$\psi_U(F_U(s))_x = \psi_x(F_x(s_x)) = \phi_U(s)_x$$

for all x, hence $\psi_U(F_U(s)) = \phi_U(s)$. We thus have that $(\mathscr{G}, F)$ satisfies the universal property of the cokernel of the kernel of F, and is thus a cokernel as desired. $\square$

We now briefly discuss the image sheaf, so that we can talk of exact sequences of abelian categories.

Definition 1.2.10. let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism. The **image sheaf**, denoted $\operatorname{im} F$ is the sheafification of the presheaf $\operatorname{im}^p F$ defined by:

$$(\operatorname{im}^p F)(U) = \operatorname{im} F_U \subset \mathscr{G}(U)$$

In general, we note that $\operatorname{im}^p F$ is not a sheaf, hence why we take the sheafification. We also have the following definition:

Definition 1.2.11. Let $\mathscr{F}$ be a sheaf over X, then a **subsheaf** $\mathscr{G}$ of $\mathscr{F}$ is a sheaf on X such that $\mathscr{G}(U) \subset \mathscr{F}(U)$ for all U, and the restriction maps on $\mathscr{G}$ are given by the restriction of $\theta^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ to $\mathscr{G}(U)$ for all $V \subset U$.

Proposition 1.2.7. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. There exists a natural map $\iota : \operatorname{im} F \to \mathscr{G}$, such that $\ker \iota = 0$, and $\iota(\operatorname{im} F) = \operatorname{im}^p \iota$ is a subsheaf of $\mathscr{G}$.*

Proof. First note that we have a clear inclusion morphism $\imath^p : \operatorname{im}^p F \to \mathscr{G}$, which is injective on all U. By the universal property of sheafification, we thus have a unique map $\imath : \operatorname{im} F \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{im}^p F \arrow[r, "\imath^p"] \arrow[d, "\mathrm{sh}"] & \mathscr{G} \\
\operatorname{im} F \arrow[ur, "\imath"] &
\end{tikzcd}$$

It thus suffices to check that $\ker \iota_U = 0$ for all U. Let $(s_x) \in (\operatorname{im} F)(U)$, and suppose that:

$$\iota((s_x)) = 0$$

Since $(s_x) \in (\operatorname{im} F)(U)$, we have that for each x there exists V_x, and an $s^x \in \mathscr{F}(V_x)$ such that $s^x_q = s_q$ for all $q \in V_x$. Moreover, we have that by our work in Proposition 1.2.3, that:

$$\iota((s_x))|_{V_x} = \iota^p(s^x) = 0$$

However, this implies that $s^x = 0$ for each x, hence $s^x_p = 0 = s_p$ for all $p \in V_x$, and all $x \in V_x$. It follows that $(s_x) = 0$, so the $\ker \iota = 0$.

We have that $\iota(\operatorname{im} F)$ is a sub presheaf, by defining:

$$\iota(\operatorname{im} F)(U) = \iota_U(\operatorname{im} F_U) \subset \mathscr{G}(U)$$

We define restriction maps, $\theta^U_V : \iota(\operatorname{im} F)(U) \to \iota(\operatorname{im} F)(V)$, by restricting $\theta^U_V : \mathscr{G}(U) \to \mathscr{G}(V)$ to the subgroup $\iota(\operatorname{im} F_U)$. It follows if $g \in \iota_U(\operatorname{im} F_U)$, then $g = \iota_U((s_x))$, so $g|_V = \iota_V((s_x)|_V)$, thus θ^U_V has image in $\iota(\operatorname{im} F)(V)$. The restriction maps are then compatible with one another, as they are compatible on $\mathscr{G}$.

To show this is a sheaf, let $\{U_i\}$ be an open cover U, and $g \in \iota(\operatorname{im} F)(U)$, such that $g|_{U_i} = 0$ for all U_i. Well, since $g \in \iota(\operatorname{im} F)(U) \subset \mathscr{G}(U)$, and $\mathscr{G}$ is a sheaf, we must have that $g = 0$. Now suppose that we have $g_i \in \iota(\operatorname{im} F)(U_i)$ such that $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$ for all i, j. Then we must have have that there exists a $g \in \mathscr{G}(U)$ such that $g|_{U_i} = g_i$. We need to show that $g \in \iota(\operatorname{im} F)(U)$. For each i write $g_i = \iota_{U_i}((s_x)_i)$, then we have that:

$$\iota_{U_i \cap U_j}((s_x)_i|_{U_i \cap U_j}) = \iota_{U_i \cap U_j}((s_x)_j|_{U_i \cap U_j})$$

which implies that:

$$(s_x)_i|_{U_i \cap U_j} = (s_x)_j|_{U_i \cap U_j}$$

as ι_U is injective for all U. It follows that the $(s_x)_i$ glue together to form a global section $(s_x) \in (\operatorname{im} F)(U)$, such that $(s_x)|_{U_i} = (s_x)_{U_i}$. We see that for all U_i:

$$(g - \iota_U((s_x)))|_{U_i} = g_i - \iota_{U_i}((s_x)_i) = 0$$

hence $g - \iota_U((s_x)) = 0$, implying that $g \in \iota_U(\operatorname{im} F_U)$. It follows that $\iota(\operatorname{im} F)$ is a subsheaf of $\mathscr{G}$ as desired. ☐

Definition 1.2.12. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups or rings, then F is **injective** if $\ker F$ is the trivial sheaf, and F is **surjective** if $\iota(\operatorname{im} F) = \mathscr{G}$.

Proposition 1.2.8. *Let $F : \mathscr{F} \to \mathscr{G}$ be a sheaf morphism of abelian groups, then F is surjective if and only if* $\operatorname{coker} F$ *is the trivial sheaf. Moreover, if F is surjective if and only if $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for all x.*[12]

Proof. Suppose that $\operatorname{coker} F$ is the trivial sheaf, then we have that $\operatorname{im} F_x = \mathscr{G}_x$ for all $x \in X$. Since ι is a monomorphism, we have that $\imath_x : (\operatorname{im} F)_x \to \mathscr{G}_x$ is an injection. Moreover, since $(\operatorname{im}^p F)_x = \operatorname{im} F_x = \mathscr{G}_x$, we have that $\imath^p_x : (\operatorname{im}^p F)_x \to \mathscr{G}_x$ is an isomorphism. Since sh_x is an isomorphism, and:

$$\imath_x \circ \mathrm{sh}_x = \imath^p_x$$

we must have that $\imath_x$ is a surjection as well, and thus an isomorphism. Since $\imath_x$ is an isomorphism for all, we must have that $\iota(\operatorname{im} F) = \mathscr{G}$ as desired.

Now suppose that F is surjective, that $\iota(\operatorname{im} F) = \mathscr{G}$. Then the stalk maps are isomorphisms, so we once again have that $(\operatorname{im} F)_x \cong \mathscr{G}_x$, implying that $\iota_x((\operatorname{im} F)_x) = \operatorname{im} F_x = \mathscr{G}_x$. The stalks of $\operatorname{coker} F$ are then isomorphic to $\mathscr{G}_x / \operatorname{im} F_x \cong \{0\}$, hence very section of $\operatorname{coker} F$ must be trivial, implying the claim.

Now suppose that $F : \mathscr{F} \to \mathscr{G}$ is surjective, then $\imath(\operatorname{im} F) = \mathscr{G}$. In particular, we have that $\imath_x(\operatorname{im}(F)_x) = \mathscr{G}_x$, however by the commutative diagram in Proposition 1.2.7, this implies that:

$$\imath_x(\mathrm{sh}_x((\operatorname{im}^p F)_x)) = \imath^p_x((\operatorname{im}^p F)_x) = \mathscr{G}_x$$

Since $\imath^p$ is an honest to god inclusion map, $\imath^p_x$ is an honest to god inclusion map, and it follows that $(\operatorname{im}^p F)_x = \mathscr{G}_x$, hence $\operatorname{im}(F_x) = \mathscr{G}_x$.

Now supposing that F_x is surjective for all $x \in X$. Then the map:

$$\imath^p : \operatorname{im}^p F \to \mathscr{G}$$

is an isomorphism on stalks. It follows that $\imath : \operatorname{im} F \to \mathscr{G}$ is then an isomorphism on stalks, hence $\imath$ is an isomorphism so $\operatorname{im} F = \mathscr{G}$. ☐

Definition 1.2.13. A sequence of sheaf morphisms:

$$\cdots \longrightarrow \mathscr{F}_{i-1} \xrightarrow{\quad F_{i-1} \quad} \mathscr{F}_i \xrightarrow{\quad F_i \quad} \mathscr{F}_{i+1} \longrightarrow \cdots$$

is called **exact** if $\ker F_i = \iota(\operatorname{im} F_{i-1})$ for all i.

Proposition 1.2.9. *Let:*

$$0 \longrightarrow \mathscr{F}_{i-1} \xrightarrow{\quad F_{i-1} \quad} \mathscr{F}_i \xrightarrow{\quad F_i \quad} \mathscr{F}_{i+1} \longrightarrow 0$$

be a sequence of sheaf morphisms. Then the sequence is exact if and only if the induced sequence of stalks:

$$\cdots \longrightarrow (\mathscr{F}_{i-1})_x \xrightarrow{\quad (F_{i-1})_x \quad} (\mathscr{F}_i)_x \xrightarrow{\quad (F_i)_x \quad} (\mathscr{F}_{i+1})_x \longrightarrow \cdots$$

is exact for all $x \in X$.

Proof. Suppose that the sequence is exact, then we need to show that for all $x \in X$, $\ker(F_i)_x = \operatorname{im}(F_{i-1})_x$. Let $s_x = [U, s] \in \ker(F_i)_x$, then we have that:

$$[U, (F_i)_U(s)] = [U, 0]$$

[12] Our proof of this second fact will hold for sheafs with values in Set, and Ring.

It follows that there exists an open neighborhood V_x of x such that:

$$(F_i)_V(s|_V) = 0$$

however, if $(F_i)_V(s|_V) = 0$, we have that by exactness $s|_V \in \imath(\operatorname{im} F_{i-1})(V)$, so there exists an $(s_x) \in (\operatorname{im} F_{i-1})(V)$ such that $\iota((s_x)) = s|_V$. It follows that $s|_V$ is then section such that for each open neighborhood of x, W_x, $s|_{W_x} = \iota^p(s^x)$, implying that $s_x = (s|_V)_x = \iota^p_x(s^x_x) \in \operatorname{im}(F_{i-1})_x$, hence $s_x \in (\operatorname{im} F_{i-1})_x$. Now suppose that $s_x \in \operatorname{im}(F_{i-1})_x$, then we have that there exists an $f_x \in (\mathscr{F}_{i-1})_x$ such that $(F_{i-1})_x(f_x) = s_x$. Hence for some U and V, and some $f \in \mathscr{F}_{i-1}(U)$, $s \in \mathscr{F}_i(V)$ we have that:

$$[U, (F_{i-1})_U(f)] = [V, s]$$

so there exists a open subset $x \in W \subset U \cap V$, such that:

$$(F_{i-1})_W(f|_W) = s|_W$$

It follows that $s|_W \in (\operatorname{im}^p F_{i-1})(W)$, and by the universal property we have that:

$$\iota_W \circ \operatorname{sh}_W(s|_W) = \iota^p_W(s|_W) = s|_W$$

Taking stalks, we find that:

$$\iota_x \circ \operatorname{sh}_x(s_x) = s_x$$

so $s_x \in \iota(\operatorname{im}(F_{i-1}))_x$. We thus see that:

$$(F_{i-1})_x(s_x) = [W, F_W \circ \iota_W \circ \operatorname{sh}_W(s|_W)] = [W, 0] = 0$$

so $s_x \in \ker(F_{i-1})_x$. It follows that that $\ker(F_{i-1})_x = (\operatorname{im} F_{i-1})_x$, so the sequence of stalks is exact.

Now suppose the sequence of stalks is exact, we want to show that $(\ker F_i)(U) = \imath(\operatorname{im} \mathscr{F}_{i-1})(U)$. Note that in the last section, we have implicitly shown that $\imath(\operatorname{im} F_{i-1})_x = \operatorname{im}(F_{i-1})_x$. Let $s \in (\ker F_i)(U) = \ker(F_i)_U$, then we have that for each $x \in U$, $s_x \in \ker(F_i)_x$, hence each $s_x \in \operatorname{im}(F_{i-1})_x = \imath(\operatorname{im} F_{i-1})_x$. It follows that there is an open cover of U, by U_x, such that $s|_{U_x} \in \imath(\operatorname{im} F_{i-1})(U)$, which all vacuously agree on overlaps. We thus have that $s|_{U_x}$ glue together to $s \in \imath(\operatorname{im} F_{i-1})(U)$, implying that $(\ker F_i)(U) \subset \imath(\operatorname{im} F_{i-1}(U))$. Now let $s \in \imath(\operatorname{im} F_{i-1})(U)$, then for all $x \in U$, we have that $s_x \in \imath(\operatorname{im} F_{i-1})_x = \operatorname{im} F_x$, so by exactness each $s_x \in \ker(F_i)_x = \ker(F_i)_x$. It follows by the same argument that $s \in (\ker F_i)(U)$, hence $(\ker F_i)(U) = \imath(\operatorname{im} F_{i-1})$ implying the claim. □

We also have the following result:

Proposition 1.2.10. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups, then F is an isomorphism if and only if it is injective an surjective.*

Proof. Suppose that F is an isomorphism, then in particular, F_U is an isomorphism for all U. It follows that $\ker F_U = (\ker F)(U) = \{0\}$ so $\ker F$ is the trivial sheaf, implying that F is injective. To show that F is surjective, by Proposition 1.2.8 we need only show that $\operatorname{coker} F$ is the trivial sheaf. Since F is an isomorphism, we have that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism, so $\operatorname{im} \mathscr{F}_x = \mathscr{G}_x$ for all $x \in X$. Since the stalks of $\operatorname{coker} F$ are isomorphic to $\mathscr{G}_x / \operatorname{im} \mathscr{F}_x = 0$ it follows that $(\operatorname{coker} F)(U)$ is trivial for all U, thus F is surjective.

Now suppose that F is injective and surjective. Since F is a sheaf, Proposition 1.2.2 we need only check that F_x is an isomorphism for all x. Since F is injective, we have that $(\ker F)_x = 0$, so by Corollary 1.2.1 we have that $\ker F_x = 0$. Since F is surjective, we have that $\imath(\operatorname{im} F)_x = \mathscr{G}_x$, but $\imath(\operatorname{im} F)_x = \operatorname{im} F_x$, hence F_x is a a surjection, implying that F_x is an isomorphism for all x, hence F is an isomorphism. □

We now discuss the process of 'gluing together' sheaves. First some notation, if $\mathscr{F}$ is a sheaf on X, then we can obtain an induced sheaf on any open set $U \subset X$, denote $\mathscr{F}|_U$, by setting $\mathscr{F}_U(V) = \mathscr{F}(V)$ for all open subsets of U. Since any open subset of U is open in X, this assignment makes sense, and clearly determines a sheaf.

Theorem 1.2.2. *Let $\{U_i\}_{i\in I}$ be an open cover for a topological space X, and $\mathscr{F}_i$ be a sheaf on each U_i such that there exist isomorphisms $\phi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ which satisfy the **cocycle condition** for all i, j, i.e.*

$$\phi_{jk} \circ \phi_{ij} = \phi_{ik} \qquad \textit{and } \phi_{ii} = Id$$

on $U_i \cap U_j \cap U_k$. Then there exists a sheaf $\mathscr{F}$ on X such that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$ for all i.

Proof. Let $V \subset X$ be an open set, we define $\mathscr{F}(V)$ to be the set:

$$\mathscr{F}(V) = \left\{ (s_i) \in \prod_{i\in I} \mathscr{F}_i(V\cap U_i) : \forall i, j \in I, \phi_{ij}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j} \right\}$$

where it is understood that ϕ_{ij} is the group isomorphism $(\phi_{ij})_{V\cap U_i\cap U_j}$. We check that this is indeed a subgroup of $\prod_i \mathscr{F}(V\cap U_i)$. Clearly, $0 \in \mathscr{F}(V)$, so we need only check that $\mathscr{F}(V)$ is closed under addition and contains inverses. Let $(s_i), (t_i) \in \mathscr{F}(V)$, then we have the sequence $(s_i + t_i) \in \prod_i \mathscr{F}(V \cap U_i)$. We want to show that this sequence lies in $\mathscr{F}(V)$; note that ϕ_{ij} and restriction maps are homomorphisms, so we have that for each i and j:

$$\begin{aligned}\phi_{ij}([s_i + t_i]|_{V\cap U_i\cap U_j}) =& \phi_{ij}(s_i|_{V\cap U_i\cap U_j}) + \phi_{ij}(t_i|_{V\cap U_i\cap U_j})\\ =& s_j|_{V\cap U_i\cap U_j} + t_j|_{V\cap U_i\cap U_j}\\ =& [s_j + t_j]|_{V\cap U_i\cap U_j}\end{aligned}$$

hence $(s_i + t_i) \in \mathscr{F}(V)$. The same argument demonstrates that $(-s_i) \in \prod_i \mathscr{F}(V \cap U_i)$ is contained in the $\mathscr{F}(V)$, hence $\mathscr{F}(V)$ is a subgroup. Now let $W \subset V$, we define restriction maps θ^V_W by:

$$\theta^V_W((s_i)) = (\theta^{V\cap U_i}_{W\cap U_i}(s_i))$$

where where $\theta^{V\cap U_i}_{W\cap U_i}$ is the restriction map $\mathscr{F}_i(V\cap U_i) \to \mathscr{F}_i(W\cap U_i)$. It is then clear that $\mathscr{F}$ is a presheaf, as the restriction maps clearly satisfy $\theta^V_V = \mathrm{Id}$, and $\theta^W_Z \circ \theta^V_W = \theta^V_Z$. We first verify that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$. We define a morphism $F_j : \mathscr{F}|_j \to \mathscr{F}|_{U_j}$ on open sets $V \subset U_j$ by:

$$s \longmapsto (\phi_{ji}(s|_{V\cap U_i}))$$

We first check $F_j(s) \in \mathscr{F}|_{U_j}(V) = \mathscr{F}(V)$. We see for all k and l that by the cocycle condition:

$$\begin{aligned}\phi_{kl}(\phi_{jk}(s|_{V\cap U_k})|_{V\cap U_k\cap U_l}) =& \phi_{kl}(\phi_{jk}(s|_{V\cap U_k\cap U_l}))\\ =& \phi_{jl}(s|_{V\cap U_k\cap U_l})\end{aligned}$$

which is the lth component of our element, hence F has image in $\mathscr{F}|_{U_i}$. This map clearly commutes with restriction as ϕ_{ji} commutes restriction, hence F is a natural transformation, and thus indeed a morphism. We define an inverse morphism F_j^{-1} given on open sets by $(s_i) \mapsto s_j$, which is again clearly a natural transformation. We check that this is an inverse, let $s \in \mathscr{F}_j(V) = \mathscr{F}_j(V \cap U_j)$, then we have that:

$$F_j^{-1} \circ F_j(s) = \phi_{jj}(s|_{V\cap U_j}) = s|_{V\cap U_j} = s$$

While:

$$F_j \circ F_j^{-1}((s_i)) = (\phi_{ji}(s_j|_{V\cap U_j}))$$

However, we have that each $s_i \in \mathscr{F}(V \cap U_i)$:

$$s_i = s_i|_{V\cap U_i} = s_i|_{V\cap U_i\cap U_j} = \phi_{ji}(s_j|_{V\cap U_i\cap U_j}) = \phi_{ji}(s_j|_{V\cap U_j})$$

hence $F_j \circ F_j^{-1} = \mathrm{Id}$, and $F_j^{-1} \circ F_j = \mathrm{Id}$. It follows that $\mathscr{F}_j \cong \mathscr{F}|_{U_j}$. We can now show that $\mathscr{F}$ is a sheaf, take the sequence $(s_i) \in \prod_i \mathscr{F}_i(V \cap U_i)$ that satisfies $s_i|_{V_k\cap U_i} = s_i^k$ for each i and k. Moreover we see that if $s \in \mathscr{F}_i|_{U_i\cap U_j}(V)$:

$$\begin{aligned}F_j|_{U_i\cap U_j} \circ \phi_{ij}(s) =& (\phi_{jk}(\phi_{ij}(s)|_{V\cap U_k}))\\ =& (\phi_{jk} \circ \phi_{ij}(s)|_{V\cap U_k})\\ =& (\phi_{ik}(s)|_{V\cap U_k})\\ =& F_i(s)\end{aligned}$$

We now check that $\mathscr{F}$ is a sheaf; let V_k be an open cover of V, and $(s_i) \in \mathscr{F}(V)$ such that $(s_i)|_{V_k} = 0$ for all k. We see that $(s_i)|_{V_k} = 0$, implies that for each i we have that:

$$s_i|_{V_k \cap U_i} = 0$$

for all k. Since $\{V_k \cap U_i\}_k$ is an open cover for $V \cap U_i$, it follows that that $s_i = 0$ as $\mathscr{F}_i$ is a sheaf. This holds for all i so $(s_i) = 0$. Now suppose that we have sections $(s_i^k) \in \mathscr{F}(V_k)$ such that:

$$(s_i^k)_{V_k \cap V_m} = (s_i^m)|_{V_k \cap V_m}$$

implying that for all i:

$$s_i^k|_{V_k \cap V_m \cap U_i} = s_i^m|_{V_k \cap V_m \cap U_i}$$

It follows that since $\{V_k \cap U_i\}$ is an open cover of $V \cap U_i$, and $\mathscr{F}_i$ is a sheaf, that there exists a section $s_i \in \mathscr{F}_i(V \cap U_i)$ such that $s_i|_{V_k \cap U_i} = s_i^k$. We thus have a sequence $(s_i) \in \prod_i \mathscr{F}_i(V \cap U_i)$, such that $(s_i)|_{V_k} = (s_i^k)$. We want to show that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}$$

However, note that $s_i \in \mathscr{F}_i(V \cap U_i)$, hence we have that:

$$s_i|_{V \cap U_i \cap U_j} = \theta^{V \cap U_i}_{V \cap U_i \cap U_j}(s_i)$$

If we further restrict to $V_k \cap U_i \cap U_j$, then we have that:

$$(s_i|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j} = \theta^{V \cap U_i}_{V_k \cap U_i \cap U_j}(s_i) = \theta^{V_k \cap U_i}_{V_k \cap U_i \cap U_j} \circ \theta^{V \cap U_i}_{V_k \cap U_i}(s_i) = \theta^{V_k \cap U_i}_{V_k \cap U_i \cap U_j}(s_i^k) = s_i^k|_{V_k \cap U_i \cap U_j}$$

And we know that for all k:

$$\phi_{ij}(s_i^k|_{V_k \cap U_i \cap U_j}) = s_j^k|_{V_k \cap U_i \cap U_j}$$

Since ϕ_{ij} is a natural transformation, we thus have that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j} = (s_j|_{V \cap U_i \cap U_j})|_{V_k \cap U_i \cap U_j}$$

Since $V_k \cap U_i \cap U_j$ covers $V \cap U_i \cap U_j$, we have by sheaf axiom one that:

$$\phi_{ij}(s_i|_{V \cap U_i \cap U_j}) = s_j|_{V \cap U_i \cap U_j}$$

hence $(s_i) \in \mathscr{F}(V)$ as desired. □

Proposition 1.2.11. *Let U_i be an open cover for the topological space X, $\mathscr{F}_i$ be a sheaf on each U_i, and $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ isomorphism which satisfy the cocycle condition, then the sheaf $\mathscr{F}$ induced by gluing the of $\mathscr{F}_i$'s satisfies the following universal property: for all sheafs $\mathscr{G}$, and collection of morphisms $\psi_i : \mathscr{F}_i \to \mathscr{G}|_{U_i}$ such that $\psi_j|_{U_i \cap U_j} \circ \phi_{ij} = \psi_i|_{U_i \cap U_j}$, there exists a unique $\psi : \mathscr{F} \to \mathscr{G}$ such that the following diagram commutes for all i:*

$$\begin{tikzcd}
\mathscr{F}|_{U_i} \arrow[r, "\psi|_{U_i}"] \arrow[d, "F_i^{-1}"'] & \mathscr{G}|_{U_i} \\
\mathscr{F}_i \arrow[ur, "\psi_i"'] &
\end{tikzcd}$$

Proof. Let $(s_i) \in \mathscr{F}(V)$, then we see that $(\psi_i(s_i)) \in \prod_i \mathscr{G}(V \cap U_i)$, where we again suppress the notation $(\psi_i)_{V \cap U_i}$. However, note that $\{V \cap U_i\}$ cover V, hence since:

$$\begin{aligned}
\psi_i(s_i)|_{V \cap U_i \cap U_j} =& \psi_i(s_i|_{V \cap U_i \cap U_j}) \\
=& \psi_i|_{U_i \cap U_j}(\phi_{ji}(s_j|_{V \cap U_i \cap U_j})) \\
=& \psi_j|_{U_i \cap U_j}(s_j|_{V \cap U_i \cap U_j}) \\
=& \psi_j(s_j|_{V \cap U_i \cap U_j})
\end{aligned}$$

we have that the sections $\psi_i(s_i) \in \mathscr{G}(V \cap U_i)$ glue to a unique section $g \in \mathscr{G}(V)$, which satisfies $g|_{V \cap U_i} = \psi_i(s_i)$ for all i. We thus define θ on open sets by:

$$\psi_V((s_i)) = g$$

We check that this commutes with restrictions. Let $W \subset V$, then we want to show that:

$$\psi_W((s_i)|_W) = g|_W$$

We see that $(s_i)|_W$ is equal to $(s_i|_{W\cap U_i})$, so $\psi_W((s_i)|_W)$ is the section unique such that:

$$\psi_W((s_i)|_W)|_{W\cap U_i} = \psi_i(s_i|_{W\cap U_i})|_{W\cap U_i} = \psi_i(s_i)|_{W\cap U_i}$$

However, we have that:

$$(g|_W)|_{W\cap U_i} = g|_{W\cap U_i} = (g|_{U_i})|_{W\cap U_i} = (\psi_i(s_i))|_{W\cap U_i}$$

so sheaf axiom one implies that the assignment $V \mapsto \psi_V$ is indeed a natural transformation and thus a morphism as desired. We now show that the diagram commutes; let $V \subset U_j$, and $(s_i) \in \mathscr{F}|_{U_j}(V) = \mathscr{F}(U)$. Then we have that:

$$\psi_j \circ F_j^{-1}(s_i) = \psi_j(s_j)$$

while:

$$(\psi|_{U_j})_V(s) = \psi_V((s_i)) = g$$

where for all i, we have that $g|_{V\cap U_i} = \psi_i(s_i)$. We see that $V \cap U_j = V$, so $g = g|_{V\cap U_j} = \psi_j(s_j)$, so $\mathscr{F}$ satisfies universal property as desired. □

We now show that $\mathscr{F}$ is unique up to unique isomorphism, and that we can always glue a sheaf back together.

Corollary 1.2.3. *Let U_i be an open cover for X, and $\mathscr{F}_i$ sheafs on U_i equipped with isomorphisms $\phi_{ij} : \mathscr{F}_i \to \mathscr{F}_j$ which satisfy the cocycle condition. Then the sheaf $\mathscr{F}$ induced by the gluing of $\mathscr{F}_i$ is unique up to unique isomorphism. In particular, if $\mathscr{F}$ is a sheaf on X, and U_i is any open cover of X, then $\mathscr{F}$ is the sheaf induced by gluing of $\mathscr{F}|_{U_i}$ together.*

Proof. Let $\mathscr{F}$ be the sheaf induced by the gluing of $\mathscr{F}_i$, and $\mathscr{G}$ be any other sheaf which satisfies the universal property outlined in Proposition 1.2.11, i.e. $\mathscr{G}$ is a sheaf with isomorphisms $G_i : \mathscr{F}_i \to \mathscr{G}|_{U_i}$, such that $G_j|_{U_i\cap U_j} \circ \phi_{ij} = G_i|_{U_i\cap U_j}$, and that for any collection of morphisms $\psi_i : \mathscr{F}_i \to \mathscr{H}|_{U_i}$ there exists a unique $\psi : \mathscr{G} \to \mathscr{H}$ that makes the diagram commute for all i. In particular, we have that we get unique maps $\psi_{\mathscr{G}} : \mathscr{G} \to \mathscr{F}$, and $\psi_{\mathscr{F}} : \mathscr{F} \to \mathscr{G}$, such that:

$$G_i \circ F_i^{-1} = \psi_{\mathscr{F}}|_{U_i} \qquad \text{and} \qquad F \circ G^{-1} = \psi_{\mathscr{G}}|_{U_i}$$

On any open set V, we have that $V \cap U_i$ is an open cover. Let $s \in \mathscr{F}(V)$, then $s|_{V\cap U_i} \in \mathscr{F}(V \cap U_i)$, and we have that:

$$\begin{aligned}
(\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_V(s)|_{V\cap U_i} =&(\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_{V\cap U_i}(s|_{V\cap U_i})\\
=&(\psi_{\mathscr{G}}|_{U_i} \circ \psi_{\mathscr{F}}|_{U_i})_{V\cap U_i}(s|_{V\cap U_i})\\
=&(F \circ G \circ G_i^{-1} \circ F_i^{-1})_{V\cap U_i}(s|_{V\cap U_i})\\
=&s|_{V\cap U_i}
\end{aligned}$$

so by sheaf axiom one we have that $(\psi_{\mathscr{G}} \circ \psi_{\mathscr{F}})_V = \mathrm{Id}$. The same argument shows that $\psi_{\mathscr{F}} \circ \psi_{\mathscr{G}} = \mathrm{Id}$, then $\mathscr{F} \cong \mathscr{G}$, so $\mathscr{F}$ is unique up to unique isomorphism.

Now let $\mathscr{F}$ be a sheaf, and U_i an open cover of X. We see that by setting $\mathscr{F}_i = \mathscr{F}|_{U_i}$ we have natural isomorphisms $\mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}|_{U_i\cap U_j}$ given by the identity map $s \mapsto s$ on all open sets. This makes sense as unraveling the notation we have that for any open set $V \subset U_i \cap U_j$:

$$\mathscr{F}_i|_{U_i\cap U_j}(V) = \mathscr{F}_i(V) = \mathscr{F}|_{U_i}(V) = \mathscr{F}(V) = \mathscr{F}|_{U_j}(V) = \mathscr{F}_j(V) = \mathscr{F}_j|_{U_i\cap U_j}(V)$$

It suffices to show that $\mathscr{F}$ satisfies the universal property in Proposition 1.2.11, where the maps F_i^{-1} : $\mathscr{F}|_{U_i} \to \mathscr{F}_i$ are the identity maps. Let $\psi_i : \mathscr{F}_{\to}\mathscr{G}|_{U_i}$ be any collection of morphisms such that $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$, and take $s \in \mathscr{F}(V)$. We have define $\psi(s)$ to be the unique section $g \in \mathscr{G}(V)$ such that $g|_{V \cap U_i} = \psi_i(s|_{V \cap U_i})$. This section exists as $V \cap U_i$ cover V, and for all i and j we have that:

$$\begin{aligned} \psi_i(s|_{V \cap U_i})|_{V \cap U_i \cap U_j} &= \psi_i|_{U_i \cap U_j}(s|_{V \cap U_i \cap U_j}) \\ &= \psi_j|_{U_i \cap U_j}(s|_{V \cap U_i \cap U_j}) \\ &= \psi_j(s|_{V \cap U_i \cap U_j}) \\ &= \psi_j(s|_{V \cap U_j})|_{V \cap U_i \cap U_j} \end{aligned}$$

hence by sheaf axiom two, the sections glue together to form g. The same argument in Proposition 1.2.11 demonstrates that this a natural transformation, and that $\psi|_{U_i} = \psi_i$, so $\mathscr{F}$ satisfies the universal property as desired. □

In the process of proving Corollary 1.2, we have obtained the following corollary as well:

Corollary 1.2.4. *If $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is a collection of morphisms such that $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$ for all i and j then there exists a unique map $\psi : \mathscr{F} \to \mathscr{G}$ such that $\psi|_{U_i} = \psi_i$. In particular, ψ is an isomorphism if and only if ψ_i is an isomorphism for all i. Moreover, if $\psi : \mathscr{F} \to \mathscr{G}$ is a morphism such that $\psi|_{U_i} : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is an isomorphism then ψ is an isomorphism.*

Proof. We need only prove the last statement, in particular we need only show that $\psi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism. However for all $x \in X$, if $x \in U_i$, we have that $\psi_x = (\psi|_{U_i})_x$, as if $[U, s] \in \mathscr{F}_x$, then $[U, s] = [U_i, s|_{U_i}]$, so

$$\psi_x([U, s]) = \psi_x([U_i, s|_{U_i}]) = [U_i, \psi(s|_{U_i})] = [U_i, \psi|_{U_i}(s|_{U_i})] = (\psi|_{U_i})_x([U, s])$$

implying the claim. □

1.3 Locally Ringed Spaces

We recall the definition of a local ring:

Definition 1.3.1. A commutative ring R is a **local ring** if there exists a unique maximal ideal. A **local domain** is an integral domain that is local.

Example 1.3.1. Let A be a commutative ring and $\mathfrak{p}$ a prime ideal, then $A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1}A$ is a local ring. Indeed, consider the ideal $\mathfrak{m}$ defined by:

$$\mathfrak{m} = \left\{ \frac{p}{a} : p \in \mathfrak{p} \right\}$$

i.e. any element of $\mathfrak{m}$ can be written as the equivalent class $[(p, a)]$ where $p \in \mathfrak{p}$. We check that this is an ideal, clearly $\mathfrak{m}$ is closed under addition, contains inverses, and contains the zero element. It is also clear that $\mathfrak{m}$ swallows multiplication so $\mathfrak{m}$ is an ideal. We check this is maximal, suppose for the sake of contradiction that we have an ideal $J \subset A_{\mathfrak{p}}$ such that $\mathfrak{m} \subset J$. Then there must be some $a/s \in J$ where $a \notin \mathfrak{p}$, but it if $a \notin \mathfrak{p}$, then we have that $a \in A - \mathfrak{p}$, hence:

$$\frac{a}{s} \cdot \frac{s}{a} = 1$$

so $J = A_{\mathfrak{p}}$, so $\mathfrak{m}$ is indeed maximal. Now suppose that J is another maximal ideal not equal to $\mathfrak{m}$, then J contains an element a/s such that $a \notin \mathfrak{p}$, so the same argument shows that $J = A_{\mathfrak{p}}$. It follows that $A_{\mathfrak{p}}$ is a local ring.

We now define locally ringed spaces:

Definition 1.3.2. Let $(X, \mathscr{O}_X)$ be a topological space X, equipped with a sheaf of rings $\mathscr{O}_X$. Then $(X, \mathscr{O}_X)$ is a **locally ringed space** if the stalk of $\mathscr{O}_X$ at x, denoted $(\mathscr{O}_X)_x$ or $\mathscr{O}_{X,x}$, is a local ring for all $x \in X$. We denote the unique maximal ideal of the stalk a locally ringed space as $\mathfrak{m}_x$, and the sheaf $\mathscr{O}_X$ is called the **structure sheaf of** X.

Example 1.3.2. Let (M, C^∞) the data of a smooth manifold M with the sheaf of C^∞ functions on M. The stalk $(C^\infty)_x$ is the set of equivalence classes $[U, f]$, where $x \in U$. Consider the set:

$$\mathfrak{m}_x = \{[U, f] : f(x) = 0\}$$

Note that if $[U, f] \in \mathfrak{m}_x$ then clearly ever $[V, g] = [U, f]$ also satisfies $g(x) = 0$, as f and g have to agree on an open set containing x. It follows that $\mathfrak{m}_x$ is well defined. We see that $\mathfrak{m}_x$ is clearly a subgroup of (C^∞_x), so we check that $\mathfrak{m}_x$ is an ideal. If $[U, g] \in (C^\infty)_x$ and $[V, f] \in \mathfrak{m}_x$, then we have that $[U \cap V, f \cdot g]$ satisfies $(f \cdot g)(x) = f(x)g(x) = 0$, so $\mathfrak{m}_x$ is an ideal.

We show that $\mathfrak{m}_x$ is maximal; define a map $\psi : (C^\infty)_x \to \mathbb{R}$ by:

$$\psi([U, f]) = f(x)$$

This well defined for the same reason that $\mathfrak{m}_x$ is well defined, and satisfies $\ker \psi = \mathfrak{m}_x$ essentially by definition. It is also clearly a ring morphism, and surjective as the constant function maps $f(x) = a$ maps to $[M, f]$ under the map $C^\infty(M) \to (C^\infty)_x$, which maps to a under ψ. It follows that ψ descends to an isomorphism $(C^\infty)_x/\mathfrak{m}_x \to \mathbb{R}$. Since the quotient space is a field, it follows that $\mathfrak{m}_x$ is maximal.

To see that $\mathfrak{m}_x$ is unique, suppose that J is any other maximal ideal not equal to $\mathfrak{m}_x$. Then there must be some $[U, f] \in J$ such that $f(x) \neq 0$. However, this implies that there exists an open neighborhood V_x of x such that $f(y) \neq 0$ for all $y \in V_x$. The function $g(x) = f(x)^{-1}$ is then smooth on V_x, and we see that:

$$[U, f] \cdot [V_x, g] = [V_x, 1]$$

which is the unit element of $(C^\infty)_x$, hence $J = (C^\infty)_x$, and $\mathfrak{m}_x$ is unique.

Since every stalk in a locally ringed space has a unique maximal ideal, we can associate to each stalk a unique field as follows:

Definition 1.3.3. Let $(X, \mathscr{O}_X)$ be a locally ringed space, then for all $x \in X$ the **residue field** k_x is given by:

$$k_x = (\mathscr{O}_X)_x/\mathfrak{m}_x$$

For each open $U \subset X$, and all $x \in U$ we have the **evaluation map** $\mathrm{ev}_x : \mathscr{O}_X(U) \to k_x$ given by:

$$s \longmapsto [s_x]$$

where $[s_x]$ is the image of s_x under the projection $(\mathscr{O}_X)_x \to k_x$. We say that an element of s vanishes at s_x if $s \in \ker \mathrm{ev}_x$.

Definition 1.3.4. Let $(X, \mathscr{F})$ be the data of a topological space, and a sheaf on X, and let $f : X \to Y$ be a continuous map. Then $f_*\mathscr{F}$ is the sheaf on Y defined by:

$$(f_*F)(U) = F(f^{-1}(U))$$

We call $f_*\mathscr{F}$ the **pushforward** or **direct image sheaf**.

Proposition 1.3.1. *Let $(X, \mathscr{F})$ be the data of a topological space, and a sheaf on X, and let $f : X \to Y$ be a continuous map. Then $f_*\mathscr{F}$ is indeed a sheaf on Y.*

Proof. We first show that $f_*\mathscr{F}$ is presheaf. Define restriction functions $\theta^U_V : (f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$ by:

$$\theta^U_V = \theta^{f^{-1}(U)}_{f^{-1}(V)}$$

Note that this makes sense, as if $V \subset U$, then we have that $f^{-1}(V) \subset f^{-1}(U)$. It follows that for $W \subset V \subset U$:

$$\theta^V_W \circ \theta^U_V = \theta^{f^{-1}(V)}_{f^{-1}(W)} \circ \theta^{f^{-1}(U)}_{f^{-1}(V)} = \theta^{f^{-1}(U)}_{f^{-1}(W)} = \theta^U_W$$

It is clear that $\theta^U_U = \mathrm{Id}$, so $f_*\mathscr{F}$ is a presheaf. Now let $s \in (f_*\mathscr{F})(U)$, and U_i be a cover for U, such that $s|_{U_i} = 0$. Then this we have that $s \in \mathscr{F}(f^{-1}(U))$, and $s|_{f^{-1}(U_i)} = 0$ for all i. We that:

$$f^{-1}(U) = \bigcup_i f^{-1}(U_i)$$

so it follows that $s = 0$, as $\mathscr{F}$ is a sheaf. The same argument demonstrates that $f_*\mathscr{F}$ satisfies sheaf axiom two, so $f_*\mathscr{F}$ is a sheaf. □

Proposition 1.3.2. *Let $(X, \mathscr{F})$ be a sheaf, and $f : X \to Y$ be a continuous map between topological spaces. Then for all $p \in X$, there exists a natural morphism of stalks $(f_*)_p : (f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$.*

Proof. Let $p \in X$, for all U containing $f(p)$ we define maps $\phi_U : (f_*\mathscr{F})(U) \to \mathscr{F}_p$ by first noting that $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$, hence $p \in f^{-1}(U)$, and it thus makes sense to set:

$$s \mapsto [f^{-1}(U), s]_p$$

Since the restriction maps θ^U_V are $\theta^{f^{-1}(U)}_{f^{-1}(V)}$, it follows that $\phi_V \circ \theta^U_V = \phi_U$, hence by the universal property of the colimit, there exists a unique map:

$$(f_*)_p : (f_*\mathscr{F})_{f(p)} \to \mathscr{F}_p$$

such that:

$$(f_*)_p \circ \psi_U = \phi_U \tag{1.3.1}$$

for all U containing $f(p)$, implying the claim.

□

Note that if $s_{f(p)} \in (f_*\mathscr{F})_{f(p)}$, then by (1.3), we have that:

$$(f_*)_p(s_{f(p)}) = [f^{-1}(U), s]$$

for any $s \in (f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$, such that $f(p) \in U$, and $s_{f(p)} = [U, s]$.

Let (M, C^∞_M) and (N, C^∞_N) be smooth manifolds equipped with the structure sheaf of smooth functions on M and N respectively. If $F : M \to N$ is a smooth map, then we obtain a map of sheaves $F^\sharp : C^\infty_N \to F_*C^\infty_M$ given on open sets by:

$$\begin{aligned} F^\sharp_U : C^\infty_N(U) &\longrightarrow (F_*C^\infty_M)(U) = C^\infty_M(F^{-1}(U)) \\ f &\longmapsto f \circ F \end{aligned} \tag{1.3.2}$$

When $U = N$, $F^\sharp$ is the standard pull back map $f^* : \mathbb{C}^\infty(N) \to \mathbb{C}^\infty(M)$. In fact, one can show that F is smooth if and only if F induces a morphism on the sheaves of smooth functions. Indeed, if F is smooth then (1.4) is clearly a morphism of sheaves. Now suppose that F is a set map such that $F^\sharp : C^\infty_N \to f_*C^\infty_M$ is a morphism of sheaves. Let (U, ϕ) be a coordinate chart for N, where:

$$\phi = (x^1, \ldots, x^n)$$

It follows that for each i,

$$x^i \circ F : F^{-1}(U) \to \mathbb{R}$$

is a smooth map, hence:

$$\phi \circ F : F^{-1}(U) \to \mathbb{R}^n$$

is smooth. Letting (ψ, V) be any chart contained in $F^{-1}(U)$, we see that the composition:

$$\phi \circ F \circ \psi^{-1} : \psi(V) \to \phi(F(V))$$

is smooth a smooth map $\mathbb{R}^m \to \mathbb{R}^n$, hence F is smooth.

Our next goal is to extend this picture of smooth maps in differential geometry as the data of a continuous map between manifolds, and a sheaf morphisms between the sheaves of C^∞ functions to the general setting of ringed and locally ringed spaces.

Definition 1.3.5. Let $(X, \mathscr{O}_X)$, and $(Y, \mathscr{O}_Y)$ be ringed spaces, a **morphism of ringed spaces** $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is the data of a continuous map $f : X \to Y$, and a sheaf morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$. We generally refer to a morphism of ringed spaces only by the map on the underlying topological spaces.

Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$, and $(Z, \mathscr{O}_Z)$ be locally ringed spaces, and consider morphisms $f : X \to Y$ and $g : Y \to Z$, then it makes sense to take the composition of topological maps $g \circ f : X \to Z$. However, we see that $g^\sharp : \mathscr{O}_Z \to g_*\mathscr{O}_Y$, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ are morphisms of sheaves over different topological spaces, so it doesn't make sense to compose them. Indeed, $(g \circ f)^\sharp$ should me a morphism of sheaves over Z, $\mathscr{O}_Z \to (g \circ f)_*\mathscr{O}_X$. Well, note that $(g \circ f)_*\mathscr{O}_X = g_*(f_*\mathscr{O}_X)$, and that we obtain an induced map $g_*f^\sharp : g_*\mathscr{O}_Y \to g_*(f_*\mathscr{O}_X)$ given on open sets $V \subset Z$ by:

$$\begin{aligned} g_*f^\sharp : (g_*\mathscr{O}_Y)(V) = \mathscr{O}_Y(g^{-1}(V)) &\longrightarrow \mathscr{O}_X(f^{-1}(g^{-1}(V))) \\ s &\longmapsto f^\sharp_{g^{-1}(V)}(s) \end{aligned}$$

which clearly defines a morphism. We thus define the composition $g \circ f$ to be the data of the topological composition, along with the morphism of sheaves on Z given by $g_*f^\sharp \circ g^\sharp$. Obviously, this makes locally ringed spaces a category.

Lemma 1.3.1. *Let $(X, \mathscr{O}_x)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces, and $f : X \to Y$ a morphism between them. Then for all $x \in X$, there is an induced map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ satisfying $f^\sharp_U(s)_x = f_x(s_{f(x)})$ for all $x \in f^{-1}(U)$. If $g : Y \to Z$ is another morphism of ringed spaces, then the stalk map $(g \circ f)_x : (\mathscr{O}_Z)_{g(f(x))} \to (\mathscr{O}_X)_x$ is equal to $f_x \circ g_{f(x)}$. In particular, if $f, g : X \to Y$ are two morphisms of ringed spaces, such that the topological maps agree, and such that $f_x = g_x$ for all $x \in X$, then $f^\sharp = g^\sharp$.*

Proof. There is an induced map $f^\sharp_{f(x)} : (\mathscr{O}_Y)_{f(x)} \to (f_*\mathscr{O}_X)_{f(x)}$, and by Proposition 1.3.2, an induced map $(f_*)_x : (f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$, hence we define f_x by the composition:

$$f_x = (f_*)_x \circ f^\sharp_{f(x)}$$

which is indeed a map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$.

We note that for $s_{f(x)} = [U, s] \in (\mathscr{O}_Y)_{f(x)}$, hence we have that:

$$f_x(s_{f(x)}) = (f_*)_x([U, f^\sharp_U(s)]) = [f^{-1}(U), f^\sharp_U(s)] = f^\sharp_U(s)_x$$

Let $g : Y \to Z$ be another morphism of ringed spaces, then we have that:

$$(g \circ f)^\sharp = f_*g^\sharp \circ f^\sharp$$

so:

$$(g \circ f)_x = (g \circ f)_{*x} \circ (g_*f^\sharp \circ g^\sharp)_{g(f(x))}$$

Let $s_{g(f(x))} = [U, s]_{g(f(x))}$, for some open $U \subset Z$ and $s \in \mathscr{O}_Z(U)$, then we have that:

$$\begin{aligned} (g \circ f)_x(s_{f(x)}) =& (g \circ f)_{*x}([U, (g_*f^\sharp)_U \circ g^\sharp_U(s)]) \\ =& (g \circ f)_{*x}([U, f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_{g(f(x))}) \\ =& [f^{-1}(g^{-1}(U)), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_x \end{aligned}$$

Unraveling our definitions, we see that:

$$\begin{aligned} [f^{-1}(g^{-1}(U)), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_x =& (f_*)_x([g^{-1}(U), f^\sharp_{g^{-1}(U)} \circ g^\sharp_U(s)]_{f(x)}) \\ =& (f_*)_x \circ (f^\sharp)_{f(x)}([g^{-1}(U), g^\sharp_U(s)]_{f(x)}) \\ =& f_x([g^{-1}(U), g^\sharp_U(s)]_{f(x)}) \\ =& f_x \circ (g_*)_{f(x)}([U, g^\sharp_U(s)]_{g(f(x))}) \\ =& f_x \circ g_{f(x)}([U, s]_{g(f(x))}) \end{aligned}$$

hence:

$$f_x \circ g_{f(x)} = (f \circ g)_x$$

as desired.

The proof of the final statement is left until the introduction the inverse image sheaf. □

We can now adequately define morphisms of locally ringed spaces:

Definition 1.3.6. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces, then a **morphism of locally ringed spaces** is a morphisms of ringed spaces such that for all $x \in X$, the induced map on stalks $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ satisfies:

$$f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$$

where $\mathfrak{m}_{f(x)}$ and $\mathfrak{m}_x$ are the unique maximal ideals of $(\mathscr{O}_Y)_{f(x)}$ and $(\mathscr{O}_X)_x$ respectively. An **isomorphism of locally ringed spaces** is a morphism where f is a homeomorphism and $f^\sharp$ is an isomorphism.[13].

Lemma 1.3.2. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of rings, then $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an epimorphism if and only if F is an epimorphism..*

Proof. Suppose that F is an epimorphism, and let $\phi_1, \phi_2 : \mathscr{G}_x \to R$ be any two morphisms of rings such that:

$$\phi_1 \circ F_x = \phi_2 \circ F_x$$

Now note there exists maps $\phi^i_U : \mathscr{G}(U) \to R$, given by $\phi_i \circ \psi_U$, where $\psi_U : \mathscr{G}(U) \to \mathscr{G}_x$ is the usual ring homomorphism, such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\phi^i_U"'] & & \mathscr{G}(V) \arrow[dl, "\phi^i_V"] \\
& R &
\end{tikzcd}$$

implying that $\phi_i : \mathscr{G}_x \to R$ are the unique maps which make the following diagram commute:

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi^i_U"'] & & \mathscr{G}(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi^i_V"] \\
& \mathscr{G}_x \arrow[d, "\phi_i"] & \\
& R &
\end{tikzcd}$$

It thus suffices to check that $\phi_1 \circ \psi_U = \phi_2 \circ \psi_U$ for all U containing x, as then $\phi^1_U = \phi^2_U$ so by uniqueness $\phi_1 = \phi_2$. Note that F is a an epimorphism, and that we have that:

$$F_x \circ \psi_U = \psi_U \circ F_U \tag{1.3.3}$$

where ψ_U on the left hand side is the usual morphism $\mathscr{F}(U) \to \mathscr{F}_x$. Now consider the skyscraper sheaf $x_*(R)$ along with the morphism:

$$\tilde{\phi}^i : \mathscr{G} \longrightarrow x_*(R)$$

[13]Note that in category of topological spaces f is a monomorphism (epimorphism) if and only if it is injective (surjective). In the category of sheaves of rings monomorphisms and epimorphisms Proposition 1.2.6 still partially applies; the argument for $a) - c)$ is the same, as well as for $e) \Rightarrow d)$, we will prove a modified version of $d) \Rightarrow e)$ shortly. In particular the kernel sheaf is a sheaf of ideals, while the image sheaf is a sheaf of rings, and the cokernel sheaf is the zero sheaf.

defined by:

$$\tilde{\phi}^i_U(s) = \begin{cases} \phi_i \circ \psi_U(s) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

This commutes with restrictions, and thus defines a morphism of sheaves. We see that for all U not containing x we trivially have that $\tilde{\phi}^1_U \circ F_U = \tilde{\phi}^2_U \circ F_U$, while if U contains x then:

$$\begin{aligned} \tilde{\phi}^1_U \circ F_U(s) =& \phi_1 \circ \psi_U \circ F_U(s) \\ =& \phi_1 \circ F_x \circ \psi_U(s) \\ =& \phi_2 \circ F_x \circ \psi_U(s) \\ =& \tilde{\phi}^2_U \circ F_U(s) \end{aligned}$$

hence $\tilde{\phi}^1 \circ F = \tilde{\phi}^2 \circ F$ implying that $\tilde{\phi}^1 = \tilde{\phi}^2$. Thus on opens we must have that:

$$\phi_1 \circ \psi_U = \phi_2 \circ \psi_U$$

for all U containing x, implying the claim.

For the other direction, let F_x be an epimorphism for all x, and suppose that $\phi_i : \mathscr{G} \to \mathscr{H}$ are morphisms of sheaves of rings such that:

$$\phi_1 \circ F = \phi_2 \circ F$$

Then we have that:

$$(\phi_1)_x \circ F_x = (\phi_2)_x \circ F_x$$

however F_x is an epimorphism so $(\phi_1)_x = (\phi_2)_x$ for all $x \in X$. It follows that $\phi_1 = \phi_2$ and so F is an epimorphism. □

We also wish to extend Proposition 1.2.10 to the case of sheaves of rings:

Lemma 1.3.3. *Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of rings, then F is an isomorphism if and only if it is injective and surjective.*

Proof. If F is injective and surjective then the same argument as in Proposition 1.2.10 holds.

Let F be an isomorphism, then in particular $\ker F_U = 0$ for all 0, so F is injective. Moreover, we have that $\operatorname{im} F_U = \mathscr{G}_U$ for all U, so it follows that $(\operatorname{im}^p F)$ is actually a sheaf. It follows that $\operatorname{sh} : (\operatorname{im}^p F) \to \operatorname{im} F$ and $\imath^p : \operatorname{im}^p F \to \mathscr{G}$ are both isomorphisms, so we have that:

$$\imath \circ \operatorname{sh} = \imath^p \Rightarrow \imath \circ \operatorname{sh} \circ \operatorname{sh}^{-1} = \imath^p \circ \operatorname{sh}^{-1} \Rightarrow \imath = \imath^p \circ \operatorname{sh}^{-1}$$

hence $\imath$ must be an isomorphism. It follows that $\imath(\operatorname{im} F) = \mathscr{G}$, implying the claim. □

Lemma 1.3.4. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, and $U \subset X$ be an open set. Then, $(U, \mathscr{O}_X|_U)$ is a locally ringed space equipped with monomorphism $\iota : U \to X$.*

Proof. It is clear that $(U, \mathscr{O}_X|_U)$ is a locally ringed space, and moreover that the inclusion map $\iota : U \to X$ is an injection and thus a monomorphism in the category of topological spaces. We thus need to describe a map $\iota^\sharp : \mathscr{O}_X \to \iota_*(\mathscr{O}_X|_U)$. Let $V \subset X$ be open, and note that:

$$\iota^{-1}(V) = V \cap U$$

as if $x \in V \cap U$, then we have $x \in U$ and $x \in V$, hence $\iota(x) = x \in V$, so $x \in \iota^{-1}(V)$. If $x \in \iota^{-1}(V)$, then we have that $x \in U$, and $\iota(x) = x \in V$, so $x \in V$ and U. We thus define $\iota^\sharp$ on open sets as:

$$\begin{aligned} \iota^\sharp_V : \mathscr{O}_X(V) &\longrightarrow (\iota_* \mathscr{O}_X|_U)(V) = \mathscr{O}_X|_U(\iota^{-1}(V)) = \mathscr{O}_X(U \cap V) \\ s &\longmapsto s|_{U \cap V} \end{aligned}$$

We note that this commutes with restriction maps, as if $W \subset V$ then:

$$\iota_W^\sharp(\theta_W^V(s)) = \theta_{W\cap U}^W \circ \theta_W^V(s) = \theta_{W\cap U}^V(s)$$

while:

$$\theta_W^V \circ \iota_V^\sharp(s) = \theta_{W\cap U}^{V\cap U} \circ \theta_{V\cap U}^V(s) = \theta_{W\cap U}^V(s)$$

so this is a morphism of sheaves. We check that $\iota_x^\sharp$ is a surjection for all x. Let $s_x \in (\iota_*(\mathscr{O}_X|_U))$, then for some $V \subset X$, and some $s \in \mathscr{O}_X(U \cap V)$, we have that $s_x = [V, s]$. However, we have that $[U \cap V, s] \in (\mathscr{O}_X)_x$, so trivially:

$$\iota_x^\sharp([U \cap V, s]) = [U \cap V, s]$$

We want to show that $[V, s] = [U \cap V, s]$. Let $W = U \cap V$, then we have that:

$$\theta_{U\cap V}^V(s) = \theta_{U\cap V}^{U\cap V}(s) = s$$

so $\iota_x^\sharp$ is surjective for all x, and thus an epimorphism. It follows by Lemma 1.3.2 that $\iota^\sharp$ is an epimorphism as well.

Now let $(f, f^\sharp)$ and $(g, g^\sharp)$ be morphisms $Y \to U$, $\mathscr{O}_X|_U \to f_*\mathscr{O}_Y$, such that:

$$\iota \circ f = \iota \circ g$$

and that:

$$(\iota \circ f)^\sharp = (\iota \circ g)^\sharp$$

Clearly since ι is a monomorphism, we have that the topological maps are the same, so we need only show that $f^\sharp = g^\sharp$. By Lemma 1.3.1, we need only show that $f_y = g_y$ for all $y \in Y$. Note that since $(\iota \circ f)^\sharp = (\iota \circ g)^\sharp$, we have that:

$$(\iota \circ f)_y = f_y \circ \iota_{f(y)} = g_y \circ \iota_{g(y)} = (\iota \circ g)_y$$

It follows that:

$$(f_y \circ (\iota_*)_{f(y)}) \circ \iota_{\iota(f(y))}^\sharp = (g_y \circ (\iota_*)_{f(y)}) \circ \iota_{\iota(f(y))}^\sharp$$

however, $\iota^\sharp$ is an epimorphism so we have that:

$$f_y \circ (\iota_*)_{f(y)} = g_y \circ (\iota_*)_{f(y)}$$

It suffices to check that $(\iota_*)_{f(y)} : (\imath_*(\mathscr{O}_X|_U))_{\iota(f(y))} \to (\mathscr{O}_X|_U)_{f(y)}$ is an epimorphism. We show a stronger result, i.e. that $(\iota_*)_{f(y)}$ is surjective. Let $[V, s]_{f(y)} \in (\mathscr{O}_X|_U)_{f(y)}$; then $f(y) \in V$, and $s \in \mathscr{O}_X|_U(V) = \mathscr{O}_X(V)$. It follows that $V \subset U$, so $\imath_*(\mathscr{O}_X|_U)(V) = \mathscr{O}_X(U \cap V) = \mathscr{O}_X(V)$, hence there is an element $[V, s]_{\iota(f(y))}$ in $(\imath_*(\mathscr{O}_X|_U))_{\iota(f(y))}$. We see that:

$$(\iota_*)_{f(y)}([V, s]_{\iota(f(y))}) = [\iota^{-1}(V), s]_{f(y)} = [V, s]_{f(y)}$$

so $(\iota_*)_{f(y)}$ is surjective, and thus an epimorphism. It follows that $f_y = g_y$ for all $y \in Y$, hence $f^\sharp = g^\sharp$ implying the claim. □

We then have the obvious corollary:

Corollary 1.3.1. *Let $f : X \to Y$ be a morphism of ringed spaces, then if the topological map f is injective, and for all $x \in X$ the map $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an epimorphism then f is a monomorphism.*

Clearly, ι is an isomorphism onto its image, as we have that $\iota : U \to U \subset X$ is a homeomorphism, and $\iota^\sharp : \mathscr{O}_X|_U \to (\iota_*\mathscr{O}_X|_U)$ is the identity map. Importantly if $f : X \to Y$ is any morphism, then there exists a restricted map $f|_U : U \to Y$, where the topological map is the standard restriction, and:

$$(f|_U)^\sharp : \mathscr{O}_Y \longrightarrow (f|_U)_*(\mathscr{O}_X|_U)$$

is defined on open sets by:

$$\begin{aligned}(f|_U)^\sharp_V : \mathscr{O}_Y(V) &\longrightarrow ((f|_U)_*(\mathscr{O}_X|_U))(V) = \mathscr{O}_X|_U(f|_U^{-1}(V)) = \mathscr{O}_X(f^{-1}(V)\cap U)\\ s &\longmapsto \theta^{f^{-1}(V)}_{f^{-1}(V)\cap U}\circ f^\sharp_V(s)\end{aligned}$$

where here $\theta^{f^{-1}(V)}_{f^{-1}(V)\cap U}$ is the restriction map on $\mathscr{O}_X$. We see that this commutes with restriction maps as:

$$\begin{aligned}(f|_U)^\sharp_W(\theta^V_W(s)) =&\theta^{f^{-1}(W)}_{f^{-1}(W)\cap U}\circ f^\sharp_W(\theta^V_W(s))\\ =&\theta^{f^{-1}(W)}_{f^{-1}(W)\cap U}\circ \theta^{f^{-1}(V)}_{f^{-1}(W)}\circ f^\sharp_V(s)\\ =&\theta^{f^{-1}(V)}_{f^{-1}(W)\cap U}\circ f^\sharp_V(s)\\ =&\theta^{f^{-1}(V)\cap U}_{f^{-1}(W)\cap U}\circ \theta^{f^{-1}(V)}_{f^{-1}(V)\cap U}\circ f^\sharp_V(s)\\ =&\theta^{f|_U^{-1}(V)}_{f|_U^{-1}(W)}\circ (f|_U)^\sharp_V(s)\\ =&\theta^V_W\circ (f|_U)^\sharp_V(s)\end{aligned}$$

as desired. We can also look at the image restricted analogue, $\tilde{f} : X \to V$, where V is any open set containing $\operatorname{im} f$, and the structure sheaf is $\mathscr{O}_Y|_V$. In this case $\tilde{f}$ is the same as the original topological map, and $\tilde{f}^\sharp$ satisfies:

$$\begin{aligned}\tilde{f}^\sharp_W : \mathscr{O}_Y|_V(W) = \mathscr{O}_Y(W) &\longrightarrow (f_*\mathscr{O}_X(W)) = \mathscr{O}_X(f^{-1}(W))\\ s &\longmapsto f^\sharp_W(s)\end{aligned}$$

We thus have the following definition:

Definition 1.3.7. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be locally ringed spaces and f a morphism between them. Then f is an **open embedding** if there exists some open $V \subset Y$ such that $\tilde{f} : X \to V$ is an isomorphism of locally ringed spaces.

Now note that Theorem 1.2.2, Proposition 1.2.11, Corollary 1.2, and Corollary 1.2.4 also carry over immediately to the case of sheaves of rings. We want to be able to glue morphisms of ringed spaces together.

Proposition 1.3.3. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, U_i an open cover of X, and $f_i : U_i \to Y$ morphisms which agree on overlaps, i.e. $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Then there exists a morphism $f : X \to Y$, such that $f|_{U_i} = f_i$ for all i*

Proof. First note that we can glue together continuous maps by defining $f : X \to Y$ as follows:

$$f(x) = f_i(x)$$

whenever $x \in U_i$. This is well defined as if $x \in U_i \cap U_j$ then we have that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Moreover, it is continuous as each f_i is continuous, and the arbitrary union of open sets is open. It is easy to see that $f|_{U_i} = f_i$ for all i.

For each i we have a morphism:

$$f_i^\sharp : \mathscr{O}_Y \longrightarrow f_{i*}(\mathscr{O}_X|_{U_i})$$

Now note that for any $V \subset Y$ we have that:

$$(f_{i*}(\mathscr{O}_X|_{U_i}))(V) = \mathscr{O}_X|_{U_i}(f_i^{-1}(V)) = \mathscr{O}_X(f_i^{-1}(V))$$

We thus define $f^\sharp$ on open sets as:

$$f^\sharp_V(s) = t$$

where t is the unique element in $\mathscr{O}_X(f^{-1}(V))$ such that:

$$t|_{f_i^{-1}(V)} = (f_i^\sharp)_V(s)$$

for all i. First note that:

$$f^{-1}(V) = \bigcup_i U_i \cap f^{-1}(V) = \bigcup_i f|_{U_i}^{-1}(V) = \bigcup_i f_i^{-1}(V)$$

so we need only show that forall i and j:

$$(f_i^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)} = (f_j^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)}$$

However note that by our hypothesis:

$$\begin{aligned}
(f_i^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)} =& \theta^{f_i^{-1}(V)}_{f_i^{-1}(V)\cap f_j^{-1}(V)} \circ (f_i^\sharp)_V(s) \\
=& (f_i^\sharp)_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=& (f_i^\sharp|_{U_i\cap U_j})_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=& (f_j^\sharp|_{U_i\cap U_j})_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=& (f_j^\sharp)_{f_i^{-1}(V)\cap f_j^{-1}(V)}(s|_{f_i^{-1}(V)\cap f_j^{-1}(V)}) \\
=& (f_j^\sharp)_V(s)|_{f_i^{-1}(V)\cap f_j^{-1}(V)}
\end{aligned}$$

hence by sheaf axiom 2 we have that t exists, so $f_V^\sharp$ is well defined. It is clear that this defines a sheaf morphism, so we need only check that $f^\sharp|_{U_i}$ is equal to f_i. We have that the restriction is a morphism:

$$f^\sharp|_{U_i} : \mathscr{O}_Y \longrightarrow (f|_{U_i})_*(\mathscr{O}_X|_{U_i})$$

though $f|_{U_i}$ is equal to f_i hence we have that the restriction is actually a morphism:

$$f^\sharp|_{U_i} : \mathscr{O}_Y \longrightarrow (f_i)_*(\mathscr{O}_X|_{U_i})$$

On an open set $V \subset Y$, we have that:

$$(f_i)_*(\mathscr{O}_X|_{U_i})(V) = \mathscr{O}_X(f_i^{-1}(V)) = \mathscr{O}_X(f^{-1}(V)\cap U)$$

so we have that for $s \in \mathscr{O}_Y(V)$:

$$\begin{aligned}
(f^\sharp|_{U_i})_V(s) =& \theta^{f^{-1}(V)}_{f^{-1}(V)\cap U} \circ f_V^\sharp(s) \\
=& \theta^{f^{-1}(V)}_{f_i^{-1}(V)}(t) \\
=& t|_{f_i^{-1}(V)} \\
=& (f_i^\sharp)_V(s)
\end{aligned}$$

It follows that since f_i is a morphism of locally ringed spaces, and the stalk maps are inherently local, that $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ must be a morphism of local rings, i.e. $f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$, so f is a morphism of locally ringed spaces as desired.

□

Recall our discussion regarding the composition of morphisms of locally ringed spaces; let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a topological space X, and let $F : \mathscr{F} \to \mathscr{G}$ be a morphism between. Let $f : X \to Y$ be a continuous map, then further recall that $f_*\mathscr{F}$ and $f_*\mathscr{G}$ are sheaves on Y, and we have a morphism between them defined on open sets by $V \subset U$:

$$\begin{aligned}
(f_*F)_V : (f_*\mathscr{F})(V) &\longrightarrow (f_*\mathscr{G})(V) \\
s &\longmapsto F_{f^{-1}(V)}(s)
\end{aligned}$$

It follows easily that this then defines a covariant functor from the category of sheaves on X to the category of sheaves on Y, which we denote $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ respectively.

Definition 1.3.8. Let F be a covariant functor from the category $\mathcal{C}$ to the category $\mathcal{D}$. Then a is covariant functor $G : \mathcal{D} \to \mathcal{C}$, **left adjoint to** F if for all objects $C \in \mathcal{C}$, and all objects $D \in \mathcal{D}$ there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}}(G(D), C) \cong \mathrm{Hom}_{\mathcal{D}}(D, F(C))$$

We want to find a way to take sheaf on Y and 'pull it back' to X given a topological map $f : X \to Y$. In light of Definition 1.3.8, the following is probably the most natural construction:

Definition 1.3.9. Let $f : X \to Y$ be a topological map, then the **inverse image functor** from $\mathrm{Sh}(Y)$ to $\mathrm{Sh}(X)$, denoted f^{-1}, is the left adjoint of the direct image functor f_*

While Definition 1.3.9 is elegant enough, we must show that such a functor exists. In particular, we need to *a*) define a sheaf $f^{-1}(\mathscr{F})$ for every sheaf $\mathscr{F}$ on Y, *b*) define a morphism $(f^{-1}F) \in \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}(\mathscr{F}), f^{-1}(\mathscr{G}))$ for every morphism $F \in \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}, \mathscr{G})$, and *c*) show that for every sheaf $\mathscr{G}$ on X, and every sheaf $\mathscr{F}$ on Y there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}(\mathscr{F}), \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*(\mathscr{G}))$$

We will prove these statements separately with the following series of results.

Proposition 1.3.4. *Let $f : X \to Y$ be continuous map, and let $\mathscr{F}$ be a sheaf on Y. Then there exists an induced sheaf $f^{-1}(\mathscr{F})$ on X such that for all $x \in X$, $(f^{-1}\mathscr{F})_x$ is uniquely isomorphic to $\mathscr{F}_{f(x)}$.*

Proof. For every open set $U \subset X$, define $f_p^{-1}(\mathscr{F})(U)$ to be:

$$(f_p^{-1}\mathscr{F})(U) = \varinjlim_{V \supset f(U)} \mathscr{F}(V)$$

That is, let I be the partially ordered set:

$$I = \{V \text{ is open in } Y : f(U) \subset V\}$$

where $V_i < V_j$ if $V_j \subset V_i$. Then $f(_p^{-1}\mathscr{F})(U)$ is the unique set/group/ring, equipped with morphisms $\psi_i : \mathscr{F}(V_i) \to (f_p^{-1}\mathscr{F})(U)$ satisfying $\psi_j \circ \theta_{V_j}^{V_i} = \psi_i$, such that for another set/group/ring A, equipped with morphisms $\phi_i : \mathscr{F}(V_i) \to A$ which satisfy the same property, then there exists a unique morphism $\phi : (f_p^{-1}\mathscr{F})(U) \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}(V_i) \arrow[rr, "\theta_{V_j}^{V_i}"] \arrow[dr, "\psi_{V_i}"] \arrow[ddr, "\phi_{V_i}"'] & & \mathscr{F}(V_j) \arrow[dl, "\psi_{V_j}"'] \arrow[ddl, "\phi_{V_j}"] \\
& (f_p^{-1}\mathscr{F})(U) \arrow[d, "\exists!\phi"] & \\
& A &
\end{tikzcd}$$

The same argument as in Proposition 1.2.1 demonstrates that $(f_p^{-1}\mathscr{F})(U)$ must be given by:

$$F = \{(V, s) : V \in I, s \in \mathscr{F}(V)\}$$

modulo the equivalence relation $(V_i, s) \sim (V_j, t)$ if and only there exists a $(W_{ij} \in I) \subset V_i \cap V_j$ such that:

$$s|_W = t|_W$$

We check that this defines a presheaf; suppose that $U_j \subset U_i \subset X$, and let $[V, s]_i \in (f_p^{-1}\mathscr{F})(U_i)$. This implies that $f(U_i) \subset V$, and hence $f(U_j) \subset V$. We thus define $\theta_{U_j}^{U_i}$ by:

$$\theta_{U_j}^{U_i}([V, s]_i) = \psi_V^j(s)$$

where ψ_V^j is the map $\mathscr{F}(V) \to (f_p^{-1}\mathscr{F})(U_j)$. It is clear that $\theta_{U_i}^{U_i} = \mathrm{Id}$, hence we check that:

$$\theta_{U_k}^{U_j} \circ \theta_{U_j}^{U_i} = \theta_{U_k}^{U_i}$$

Let $[V, s] \in (f_p^{-1}\mathscr{F})(U_i)$, then we see that:

$$\begin{aligned}\theta_{U_k}^{U_j} \circ \theta_{U_j}^{U_i}([V, s]_i) =& \theta_{U_k}^{U_j}(\psi_V^j(s)) \\ =& \theta_{U_k}^{U_j}([V, s]_j) \\ =& \psi_V^k(s) \\ =& \theta_{U_k}^{U_i}([V, s]_i)\end{aligned}$$

We thus need only show that $\theta_{U_j}^{U_i}$ is well defined, as if so it is clearly a set/group/ring homomorphism. Let $[W, t]_i = [V, s]_i$, then there exists a subset $Z \subset W \cap V$ such that $f(U_i) \subset Z$ and:

$$t|_W = s|_W$$

We see that:

$$\theta_{U_j}^{U_i}([W, t]) = \psi_W(t) = \psi_Z(t|_Z) = \psi_W(s|_Z) = \psi_V(s) = \theta_{U_j}^{U_i}([V, s])$$

so these are indeed restriction making, making the assignment $U \mapsto f_p^{-1}(U)$ a presheaf.

Note that this not necessarily a sheaf; indeed if $X = \{x_1, x_2\}$, $Y = \{y\}$, both equipped with the discrete topology, and $f : X \to Y$ is the continuous map $x_1 \mapsto y$, $x_2 \mapsto y$, then clearly for every non trivial sheaf $\mathscr{F}$ on Y, $f^{-1}\mathscr{F}$ will fail the gluing axiom as:

$$f_p^{-1}\mathscr{F}(X) = f_p^{-1}\mathscr{F}(\{x_1\}) = f_p^{-1}\mathscr{F}(\{x_2\}) = \mathscr{F}(Y)$$

In particular, $f^{-1}\mathscr{F}$ is the constant presheaf, which we have already shown is not a sheaf.

To complete the proof we simply take:

$$f^{-1}\mathscr{F} = (f_p^{-1}\mathscr{F})^\sharp$$

i.e the sheafification of $f_p^{-1}\mathscr{F}$. The stalks of the sheafification are uniquely isomorphic to the stalks of the presheaf, so we need only show that $(f_p^{-1}\mathscr{F})_x$ is uniquely isomorphic to $\mathscr{F}_{f(x)}$. We first describe a map $\phi_V : \mathscr{F}(V) \to (f_p^{-1}\mathscr{F})_x$, for all V containing $f(x)$. Let $s \in \mathscr{F}(V)$, then we first map s to the equivalence class $[V, s] \in (f_p^{-1}\mathscr{F})(U)$ for any U such that $f(U) \subset V$, and then map $[V, s]$ to the equivalence class $[U, [V, s]] \in (f_p^{-1}\mathscr{F})_x$. We check that this is well defined, i.e. independent of our choice of U. If U' is any other open subset such that $f(U') \subset V$, then we need to show that:

$$[U, [V, s]] = [U', [V, s]']$$

Consider the intersection $W = U \cap U'$, then:

$$[V, s]|_{U \cap U'} = \psi_V(s)$$

where ψ_V is the map $\mathscr{F}(V) \to (f_p^{-1}]\mathscr{F})(U_i \cap U_j)$. We also have that:

$$[V, s]'|_{U_i \cap U_j} = \psi_V(s)$$

hence the map is well defined. We see that if $W \subset V$, then for some U open such that $f(U) \subset W$:

$$\phi_W(s|_W) = [U, [W, s|_W]] = [U, [V, s]]$$

hence the maps commute with restriction. It follows that there is a unique map $\phi : \mathscr{F}_{f(x)} \to (f_p^{-1}\mathscr{F})_x$, such that:

$$\phi([V, s]) = \phi_V(s) = [U, [V, s]]$$

where U is any open set such that $f(U) \subset V$. Suppose that $\phi([V,s]) = 0$, then there exists a $W \subset U$ containing x such that:

$$[V,s]|_W = 0 \Rightarrow \psi_V(s) = 0$$

where ψ_V is the map $\mathscr{F}(V) \to (f_p^{-1}\mathscr{F})(W)$. However, this implies that $[V,s] \in (f_p^{-1}\mathscr{F})(W)$ is zero, hence there exists an open subset $Z_x \subset V$ where $f(W) \subset Z_x$ and:

$$s|_{Z_x} = 0$$

It follows that $f(x) \in V$ and $f(x) \in Z_x$, hence we have that:

$$[V,s] = [Z_x, s|_{V_x}] = [Z_x, 0] = 0 \tag{1.3.4}$$

so ϕ is injective. Now let $[U,[V,s]] \in (f^{-1}\mathscr{F})_x$, we want to find a $[Z,t] \in \mathscr{F}_x$ such that $\phi([Z,t]) = [U,[V,s]]$. Note that $[U,[V,s]] \in (f_p^{-1}\mathscr{F})_x$, implies that $x \in U$, and $f(x) \in f(U) \subset V$. Choose the class $[V,s] \in \mathscr{F}_{f(x)}$, then $\phi([V,s]) = [W,[V,s]]$, for any W such that $f(W) \subset V$. Clearly $W = U$ works, hence $\phi([V,s]) = [U,[V,s]]$ so ϕ is surjective and thus an isomorphism as desired. □

Note that if $\mathscr{F}$ is a locally ringed space, then we clearly have that $f^{-1}\mathscr{F}$ is a locally ringed space from the Proposition 1.3.4. We now proceed with the results:

Proposition 1.3.5. *Let $f : X \to Y$ be a continuous map and $\mathscr{F}$ an object in* $\mathrm{Sh}(Y)$. *Then the assignment $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ defines a covariant functor* $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

Proof. Let $F : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on Y; we first define a morphism $f_p^{-1}F : f_p^{-1}\mathscr{F} \to f_p^{-1}\mathscr{G}$. Let $U \subset X$ be an open set, then we define $f_p^{-1}F$ on U by:

$$\begin{aligned}(f_p^{-1}F)_U : (f^{-1}\mathscr{F})(U) &\longrightarrow (f^{-1}\mathscr{G})(U)\\ [V,s] &\longmapsto [V, F_V(s)]\end{aligned}$$

We first check this is well defined, let $[W,t] = [V,s]$, then we want to show that:

$$[V, F_V(s)] = [W, F_V(t)]$$

Note that by assumption there exists a $Z \subset W \cap V$ such that:

$$s|_Z = t|_Z$$

and $f(U) \subset Z$. Note that we have:

$$F_V(s)|_Z = F_Z(s|_Z) = F_Z(t|_Z) = F_V(t)|_Z$$

so $[V, F_V(s)] = [W, F_V(t)]$, and the map is well defined. We check that this commutes with restrictions; let $[V,s] \in (f^{-1}\mathscr{F})(U_i)$, and suppose that $U_j \subset U_i$, then:

$$\begin{aligned}(f_p^{-1}F)_{U_i}([V,s]_i)|_{U_j} =& [V, F_v(s)]_i|_{U_j}\\ =& \psi_V^j(F_V(s))\\ =& [V, F_V(s)]_j\\ =& (f^{-1}F)_{U_j}([V,s]_j)\\ =& (f^{-1}F)_{U_j}(\psi_V^j(s))\\ =& (f^{-1}F)_{U_j}([V,s]|_{U_j})\end{aligned}$$

where ψ_V^j is the map $\mathscr{G}(V) \to (f_p^{-1}\mathscr{G})(U_j)$, and the subscripts describe which image set, $f(U_i)$, or $f(U_j)$ we are taking the colimit over. It follows that $f_p^{-1}F$ is a morphism of presheaves. By the universal property of sheafification we have the following diagram:

$$\begin{tikzcd} f_p^{-1}\mathscr{F} \arrow[r, "f_p^{-1}F"] \arrow[d, "\mathrm{sh}"] \arrow[dr, "\mathrm{sh}\circ f_p^{-1}F"] & f_p^{-1}\mathscr{G} \arrow[d, "\mathrm{sh}"] \\ f^{-1}\mathscr{F} \arrow[r, "\exists! f^{-1}F"] & f^{-1}\mathscr{G} \end{tikzcd}$$

so there exists a unique morphism $f^{-1}F : f^{-1}\mathscr{F} \to f^{-1}\mathscr{G}$. It is clear from the universal property that if $f_p^{-1}(F \circ G) = f_p^{-1}F \circ f_p^{-1}G$, and $f_p^{-1}\mathrm{Id} = \mathrm{Id}$, then the same will be true for $f^{-1}(F \circ G)$ and $f^{-1}\mathrm{Id}$. From our definition of the $f_p^{-1}F$ inverse on open sets however, both the statements in the presheaf case are clear, hence they hold in the sheafification case. It follows that f^{-1} is a functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. □

We can now finally prove the main claim:

Theorem 1.3.1. *Let $f : X \to Y$ be a map of topological spaces, then the functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is left adjoint to $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, in the sense that for all object $\mathscr{G}$ of $\mathrm{Sh}(X)$, and all objects $\mathscr{F}$ of $\mathrm{Sh}(Y)$ there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$$

Proof. Let $\mathscr{F} \in \mathrm{Sh}(Y)$, and $\mathscr{G} \in \mathrm{Sh}(X)$, and suppose that $F : f^{-1}\mathscr{F} \to \mathscr{G}$ is a sheaf morphism; we want to define a sheaf morphism $\tilde{F} : \mathscr{F} \to f_*\mathscr{G}$. Let V be an open set of Y, then we want $\tilde{F}$ to be a map on open sets:

$$\tilde{F}_V : \mathscr{F}(V) \longrightarrow \mathscr{G}(f^{-1}(V))$$

Note that $f^{-1}(V) \subset X$, and that $f(f^{-1}(V)) \subset V$, hence there exists a map $\psi_V : \mathscr{F}(V) \to f_p^{-1}\mathscr{F}(f^{-1}(V))$, which takes the section $s \in \mathscr{F}(V)$ to the equivalence class $[V, s] \in f_p^{-1}\mathscr{F}(f^{-1}(V))$. We then have the following chain of maps:

$$\mathscr{F}(V) \xrightarrow{\quad \psi_V \quad} (f_p^{-1}\mathscr{F})(f^{-1}(V)) \xrightarrow{\ \mathrm{sh}_{f^{-1}(V)}\ } (f^{-1}\mathscr{F})(f^{-1}(V)) \xrightarrow{\ F_{f^{-1}(V)}\ } \mathscr{G}(f^{-1}(V))$$

We define $\tilde{F}_V$ as this composition for all $V \subset Y$. We check that this composition is compatible with restriction maps. Let $V_j \subset V_i \subset Y$, then:

$$\tilde{F}_{V_i}(s)|_{V_j} = \theta^{V_i}_{V_j} \circ F_{f^{-1}(V_i)} \circ \mathrm{sh}_{f^{-1}(V_i)} \circ \psi_{V_i}(s)$$

However, recall that the restriction maps on $f_*\mathscr{G}$ are given by $\theta^{V_i}_{V_j} = \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}$, hence:

$$\begin{aligned} \tilde{F}_{V_i}(s)|_{V_j} =& \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)} \circ F_{f^{-1}(V_i)} \circ \mathrm{sh}_{f^{-1}(V_i)} \circ \psi_{V_i}(s) \\ =& F_{f^{-1}(V_j)} \circ \mathrm{sh}_{f^{-1}(V_j)} \circ \theta^{f^{-1}(V_i)}_{f^{-1}(V_j)} \circ \psi_{V_i}(s) \end{aligned}$$

where the final $\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}$ is the restriction map $(f_p^{-1}\mathscr{F})(f^{-1}(V_i)) \to (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$. We see that:

$$\psi_{V_i}(s) = [V_i, s]_i \in (f_p^{-1}\mathscr{F})(f^{-1}(V_i))$$

then:

$$\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}([V_i, s]_i) = [V_i, s]_j \in (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$$

However, note that we clearly have that $f(f^{-1}(V_j)) \subset V_j$, so:

$$\theta^{f^{-1}(V_i)}_{f^{-1}(V_j)}([V_i, s]_i) = [V_j, s|_{V_j}]_j = \psi_{V_j}(s|_{V_j})$$

where ψ_{V_j} is the map $\mathscr{F}(V_j) \to (f_p^{-1}\mathscr{F})(f^{-1}(V_j))$, hence:

$$\begin{aligned}\tilde{F}_{V_i}(s)|_{V_j} =& F_{f^{-1}(V_j)} \circ \mathrm{sh}_{f^{-1}(V_j)} \circ \psi_{V_j}(s|_{V_j}) \\ =& \tilde{F}_{V_j}(s|_{V_j})\end{aligned}$$

We have thus obtained a set/group/ring homomorphism:

$$\begin{aligned}\Phi : \mathrm{Hom}(f^{-1}\mathscr{F}, \mathscr{G}) &\longrightarrow \mathrm{Hom}(\mathscr{F}, f_*\mathscr{G}) \\ F &\longmapsto \tilde{F}\end{aligned}$$

We will define a set/group/ring homomorphism in the other direction and show that they are inverses of one another. Let $G : \mathscr{F} \to f_*\mathscr{G}$ be a morphism, then we wish to find a $\hat{G} : f^{-1}\mathscr{F} \to \mathscr{G}$. By the universal property of sheafification, it suffices to define a map $\hat{G}_p : f_p^{-1}\mathscr{F} \to \mathscr{G}$. Let $U \subset X$ be open, then on open sets we want $\hat{G}_p$ to be a map:

$$(f_p^{-1}\mathscr{F})(U) \longrightarrow \mathscr{G}(U)$$

For all V_i such that $f(U) \subset V_i$, it thus suffices to define maps $\xi_{V_i} : \mathscr{F}(V_i) \to \mathscr{G}(U)$ which commute with restriction by the universal property of the colimit. Let $s \in \mathscr{F}(V_i)$, and note that we have a map $\mathscr{F}(V_i) \to \mathscr{G}(f^{-1}(V_i))$. Note that $U \subset f^{-1}(f(U)) \subset f^{-1}(V_i)$, hence $U \subset f^{-1}(V_i)$. We thus define ξ_{V_i} by:

$$\xi_{V_i}(s) = \theta_U^{f^{-1}(V_i)} \circ F_{V_i}(s)$$

Suppose that $V_j \subset V_i$, and $f(U) \subset V_j$, then we see that:

$$\begin{aligned}\xi_{V_j}(s|_{V_j}) =& \theta_U^{f^{-1}(V_j)} \circ F_{V_j}(s|_{V_j}) \\ =& \theta_U^{f^{-1}(V_j)} \circ \theta_{f^{-1}(V_j)}^{f^{-1}(V_i)} \circ F_{V_i}(s_i) \\ =& \theta_U^{f^{-1}(V_i)} \circ F_{V_i}(s) \\ =& \xi_{V_i}(s)\end{aligned}$$

We thus obtain a unique a map $(f_p^{-1}\mathscr{F})(U) \to \mathscr{G}(U)$ given by:

$$(\hat{G}_p)_U([V, s]) = \xi_V(s)$$

We check that this is actually a presheaf morphism. Let $U_j \subset U_i$, and suppose $[V, s]_i \in (f_p^{-1}\mathscr{F})(U_i)$. Then we have that:

$$(\hat{G}_p)_{U_i}([V, s]_i)|_{U_j} = \theta_{U_j}^{U_i} \circ \xi_V^i(s)$$

where ξ_V^i is the map $\mathscr{F}(V) \to \mathscr{G}(U_i)$. It follows that:

$$\begin{aligned}(\hat{G}_p)_{U_i}([V, s]_i)|_{U_j} =& \theta_{U_j}^{U_i} \circ \theta_{U_i}^{f^{-1}(V)} \circ F_V(s) \\ =& \theta_{U_j}^{f^{-1}(V)} \circ F_V(s) \\ =& \xi_V^j(s) \\ =& (\hat{G}_p)_{U_j}([V, s]_j) \\ =& (\hat{G}_p)_{U_j}([V, s]_i|_{U_j})\end{aligned}$$

so $\hat{G}_p$ is presheaf morphism, and it follows that there exists a unique morphism $\hat{G} : f^{-1}\mathscr{F} \to \mathscr{G}$. We now define the set/group/ring homomorphism:

$$\begin{aligned}\Psi : \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G}) &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \\ G &\longmapsto \hat{G}\end{aligned}$$

Let $F \in \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G})$, then we want to show that $\Psi \circ \Phi(F) = \hat{\tilde{F}} = F$. It suffices to check that the two agree on arbitrary open set. Let $U \subset X$ be open, and take $(s_x) \in (f^{-1}\mathscr{F})(U)$, where (s_x)

is a sequence of stalks such that for all x there exists an open neighborhood of x, U_x, and a section $[V_x, f^x] \in (f_p^{-1}\mathscr{F})(U_x)$ such that for all $y \in U_x$, we have:

$$[V_x, f^x]_y = [U_x, [V_x, f^x]] = s_y$$

Now, $\hat{\tilde{F}}_U((s_x))$ is the unique section in $\mathscr{G}(U)$ such that:

$$\hat{\tilde{F}}_U((s_x))|_{U_x} = (\hat{\tilde{F}}_p)_{U_x}([V_x, f^x])$$

We see that by our previous work:

$$\begin{aligned}(\hat{\tilde{F}}_p)_{U_x}([V_x, f^x]) =& \theta_{U_x}^{f^{-1}(V_x)} \circ \tilde{F}_{V_x}(f^x) \\ =& \theta_{U_x}^{f^{-1}(V_x)} \circ F_{f^{-1}(V_x)} \circ \mathrm{sh}_{f^{-1}(V_x)} \circ \psi_{V_x}(f^x) \\ =& F_{U_x} \circ \mathrm{sh}_{U_x} \circ \theta_{U_x}^{f^{-1}(V_x)} \circ \psi_{V_x}(f^x)\end{aligned}$$

Note that $\psi_{V_x}(f^x) = [V_x, f^x] \in (f_p^{-1}\mathscr{F})(f^{-1}(V_x))$, so the restriction to U_x, is equal to $[V_x, f^x] \in (f_p^{-1}\mathscr{F})(U_x)$, we thus have that:

$$\begin{aligned}\hat{\tilde{F}}_U((s_x))|_{U_x} =& F_{U_x} \circ \mathrm{sh}_{U_x}([V_x, f^x]) \\ =& F_{U_x}((s_y)_{y \in U_x}) \\ =& F_{U_x}((s_x)|_{U_x}) \\ =& F_U((s_x))|_{U_x}\end{aligned}$$

Since $\{U_x\}$ is an open cover for U, and we have that:

$$(\hat{\tilde{F}}_U((s_x)) - F_U((s_x)))|_{U_x} = 0$$

for all x, it follows by sheaf axiom one that the two are equal on U, hence:

$$\Psi \circ \Phi = \mathrm{Id}$$

To show the other direction, let $G \in \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$, then we want to show that $\Psi \circ \Phi(G) = \tilde{\hat{G}} = G$. As before it suffices to prove this on open sets. Let $V \subset Y$ be open, and take $s \in \mathscr{F}(V)$, then we that:

$$\tilde{\hat{G}}_V(s) = \hat{G}_{f^{-1}} \circ \mathrm{sh}_{f^{-1}(V)} \circ \psi_V(s)$$

Now note that $\hat{G}_{f^{-1}} \circ \mathrm{sh}_{f^{-1}}(V) = (\hat{G}_p)_{f^{-1}(V)}$, hence:

$$\tilde{G}_V(s) = (\hat{G}_p)_{f^{-1}(V)}([V, s])$$

where $[V, s] \in (f_p^{-1}\mathscr{F})(f^{-1}(V))$. Then by our work defining the map Φ, we have that:

$$\tilde{\hat{G}}_V(s) = \xi_V(s) = \theta_{f^{-1}(V)}^{f^{-1}(V)} \circ G_V(s) = G_V(s)$$

implying that $\tilde{\hat{G}} = G$, and that:

$$\Phi \circ \Psi = \mathrm{Id}$$

It follows that:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{F}, f_*\mathscr{G})$$

as desired. $\square$

We end this section with the following corollaries:

Corollary 1.3.2. *If $U \subset X$ is open, and $\imath : U \to X$ the inclusion map, then for every sheaf $\mathscr{F}$ on X, we have that $\imath^{-1}\mathscr{F}$ is naturally isomorphic to $\mathscr{F}|_U$.*

Proof. Note that $\imath : U \to X$ is a homeomorphism onto it's image, and it's image is open in X, hence $\imath$ is an open map. Let $W \subset U$ be open, then we claim that every element in $(\imath_p^{-1}\mathscr{F})(W)$ can be written as the equivalence class $[W, s]$ for some $s \in \mathscr{F}|_U(W) = \mathscr{F}(W)$. Let $[V, t] \in (\imath_p^{-1}\mathscr{F})(W)$, then $\imath(W) = W \subset V$, hence we have that:

$$[V, t] = [W, t|_W]$$

so without loss of generality we can work with equivalence classes of the form $[W, s]$. We now define a map:

$$\begin{aligned} \phi^p : (\imath_p^{-1}\mathscr{F})(W) &\longrightarrow \mathscr{F}|_U(W) \\ [W, s] &\longmapsto s \end{aligned}$$

Note that this well defined as if $[W, s] = [W, t]$, then there exists some $V \subset W$ such that $\imath(W) = W \subset V$ such that $t|_V = s|_V$. However $V \subset W$ and $W \subset V$ implies that $W = V$, hence $s = t$. This induces a map on stalks given by:

$$\begin{aligned} \phi^p_x : (\imath_p^{-1}\mathscr{F})_x &\longrightarrow (\mathscr{F}|_U)_x \\ [V_x, [V_x, s]] &\longmapsto [V_x, s] \end{aligned}$$

where V_x is some open neighborhood of x. Suppose that $[V_x, [V_x, s]] \mapsto 0$, then this implies that:

$$s|_{Z_x} = 0$$

where Z_x is some open neighborhood of x such that $Z_x \subset V_x$. We want to show that $[V_x, [V_x, s]] = 0$; we note that:

$$[V_x, s]|_{Z_x} = [V_x, s] \in (f_p^{-1}\mathscr{F})(Z_x)$$

which is equal to $[Z_x, s|_{Z_x}] = [Z_x, 0]$ which is the zero section. Hence $[V_x, [V_x, s]] = 0$. The map is clearly surjective, hence ϕ^p_x is an isomorphism for all x. It follows that induced map on sheaves induces a stalk isomorphism for all x, thus by Lemma 1.2.1 we have the claim. □

Corollary 1.3.3. *A morphism $f : X \to Y$ of locally ringed spaces is equivalent to the data of a continuous map $f : X \to Y$, and a morphism of sheaves $\hat{f} : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. In particular, there exists natural stalk maps $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ which agree with the direct image counter part, and vice versa.*

Proof. The first statement follows from Theorem 1.3.1. Note that we have map $\hat{f}_x : (f^{-1}\mathscr{O}_Y)_x \to (\mathscr{O}_X)_x$, and moreover that there exists an isomorphism $(f_p^{-1})_x : (\mathscr{O}_Y)_{f(x)} \to (f_p^{-1}\mathscr{O}_Y)_x$, given by:

$$(f_p^{-1})_x([V, s]_{f(x)}) = [U, [V, s]]_x$$

where $[V, s]$ is the equivalence class defined in Proposition 1.3.4, and U is any open set of X such that $f(U) \subset V$. We define the map f_x by:

$$f_x = \hat{f}_x \circ \mathrm{sh}_x \circ (f_p^{-1})_x$$

and note that if $[V, s]_{f(x)} \in (\mathscr{O}_Y)_{f(x)}$, then:

$$f_x([U, s]_{f(x)}) = \hat{f}_x \circ \mathrm{sh}_x([U, [V, s]]_x)$$

Now let $f^\sharp$ be the map induced by by $\hat{f}$ under the isomorphism Φ. It follows that:

$$(f_*)_x \circ f^\sharp_{f(x)}([V, s]_{f(x)}) = [f^{-1}(V), f^\sharp_V(s)]_x$$

However, $f^\sharp_V$ is given by:

$$f^\sharp_V(s) = \hat{f}_{f^{-1}(V)} \circ \mathrm{sh}_{f^{-1}(V)} \circ \psi_V(s)$$

We note that ψ_V is the map $\mathscr{O}_Y(V) \to f_p^{-1}\mathscr{O}_Y(f^{-1}(V))$, given by $s \mapsto [V, s]$. It follows that:

$$\begin{aligned}(f_*)_x \circ f^\sharp_{f(x)}([V,s]_{f(x)}) =& [f^{-1}(V), \hat{f}_{f^{-1}(V)} \circ \mathrm{sh}_{f^{-1}(V)}([f^{-1}(V), [V,s]]_x)] \\ =& \hat{f}_x \circ \mathrm{sh}_x([f^{-1}(V), [V,s]]_x)\end{aligned}$$

We note that $f^{-1}(V)$ is an open set of X such that $f(f^{-1}(V)) \subset V$, so since $(f_p^{-1})_x$ is independent of the choice of U, we can choose $U = f^{-1}(V)$, implying that:

$$f_x = \hat{f}_x \circ \mathrm{sh}_x \circ (f_x^{-1}) = (f_*)_x \circ f^\sharp_{f(x)}$$

Now suppose that we are given the map $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$, and that $\hat{f} = \Psi(f^\sharp)$. We want to show that:

$$f_x = (f_*)_x \circ f^\sharp_{f(x)} = \hat{f}_x \circ \mathrm{sh}_x \circ (f^{-1})_x$$

Note that:

$$\hat{f}_x \circ \mathrm{sh}_x = (\hat{f} \circ \mathrm{sh})_x = \hat{f}_p$$

where $\hat{f}_p$ is the presheaf morphism $f_p^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ given on open sets $U \subset X$:

$$(\hat{f}_p)_U([V,s]) = \theta_U^{f^{-1}(V)} \circ f^\sharp_V(s)$$

Let $[V,s]_{f(x)} \in (\mathscr{O}_Y)_{f(x)}$, then we have that:

$$f_x([V,s]_{f(x)}) = [f^{-1}(V), f^\sharp_V(s)]_x$$

while:

$$\begin{aligned}(\hat{f}_p)_x \circ (f^{-1})_x([V,s]_{f(x)}) =& (\hat{f}_p)_x([f^{-1}(V), [V,s]]_x) \\ =& [f^{-1}(V), \theta^{f^{-1}(V)}_{f^{-1}(V)} \circ f^\sharp_V(s)]_x \\ =& [f^{-1}(V), f^\sharp_V(s)]\end{aligned}$$

implying the claim. □

We can now prove the final statement in Lemma 1.3.1.

Proof. Let $f, g : X \to Y$ be morphisms of (locally) ringed spaces, such that $f = g$, and $f_x = g_x$ for all $x \in X$, then we have that $f^{-1}\mathscr{O}_Y = g^{-1}\mathscr{O}_X$, that $f_p^{-1}\mathscr{O}_Y = g^{-1}\mathscr{O}_X$, and that $(f^{-1})_x = (g^{-1})_x$. It follows by the above proposition that since $f_x = g_x$:

$$\hat{f}_x \circ \mathrm{sh}_x \circ (f^{-1})_x = \hat{g}_x \circ \mathrm{sh}_x \circ (g^{-1})_x$$

where $\hat{f}$ and $\hat{g}$ are the images of $f^\sharp$ and $g^\sharp$ under the isomorphism Ψ. Note that $\mathrm{sh}_x \circ (f^{-1})_x$ is an isomorphism, hence we can apply the inverse map to both sides on the right and obtain that for all $x \in X$:

$$\hat{f}_x = \hat{g}_x$$

for all $x \in X$. It follows that $\hat{f} = \hat{g}$ as maps $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, so under the isomorphism Φ we have that $f^\sharp = g^\sharp$, as desired. □

We end our section on locally ringed spaces with the following corollary of Theorem 1.3.1:

Corollary 1.3.4. *Let $f : X \to Y$ be a morphism of topological spaces with sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. Then there are canonical morphisms:*

$$G : \mathscr{G} \longrightarrow f_* f^{-1}\mathscr{G} \qquad \textit{and} \qquad F : \mathscr{F} \to f^{-1} f_* \mathscr{F}$$

If f is a closed immersion (in the topological sense) then G is surjective. If f is an open immersion (in the topological sense) then F is an isomorphism.

Proof. Note that by Theorem 1.3.1 if $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on Y, we have a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathscr{G}, f_*\mathscr{G})$$

Let $\mathscr{F} = f^{-1}\mathscr{G}$, then we have that the identity morphism $\mathrm{Id} \subset \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G}, f^{-1}\mathscr{G})$ corresponds to a unique morphism $\tilde{\mathrm{Id}} : \mathscr{G} \to f_* f^{-1}\mathscr{G}$. Recall from Theorem 1.3.1 that this map is given on open sets $V \subset Y$ by:

$$\mathscr{G}(V) \xrightarrow{\ \psi_V\ } (f_p^{-1}\mathscr{G})(f^{-1}(V)) \xrightarrow{\ \mathrm{sh}_{f^{-1}(V)}\ } (f^{-1}\mathscr{G})(f^{-1}(V)) \xrightarrow{\ \mathrm{Id}_{f^{-1}(V)}\ } (f^{-1}\mathscr{G})(f^{-1}(V))$$

where ψ_V takes a section $s \in \mathscr{G}(V)$ to $[V, s] \in f_p^{-1}\mathscr{G}(f^{-1}(V))$. Now suppose that f is a closed immersion, and let $y \in Y$, if $y \notin f(X)$ then stalk of $(f_* f^{-1}\mathscr{G})_y$ is automatically trivial so $\tilde{\mathrm{Id}}_y$ must be surjective. Now suppose that $y = f(x)$ for some $x \in X$, and $[V, s]_{f(x)} \in (f_* f^{-1}\mathscr{G})_{f(x)}$ where $s \in (f^{-1}\mathscr{G})(f^{-1}(V))$. Since $s \in (f^{-1}\mathscr{G})(f^{-1}(V))$, we have that:

$$s = (t_p) \in \prod_{p \in f^{-1}(V)} (f_p^{-1}\mathscr{G})_x$$

where for each $p \in f^{-1}(V)$ there is a $U_p \subset f^{-1}(V)$ and a section $h \in f_p^{-1}\mathscr{G}(U_p)$ such that $h_q = t_q$ for all $q \in U_p$. Let $p = x$ such that $f(x) = y$ as above, then there exists an open subset $U_x \subset f^{-1}(V)$ and a section $h \in (f_p^{-1}\mathscr{G})(U_x)$ such that $h_x = t_x$. In particular, we have that the isomorphism $\mathrm{sh}_{f(x)} : (f_p^{-1}\mathscr{G})_x \to (f^{-1}\mathscr{G})_x$ sends h_x to s_x. For some $Z \subset Y$ such that $f(U_x) \subset Z$, and some $g \in \mathscr{G}(Z)$, we have that $h = [Z, g]_{U_x}$. Note that there is a smallest subset Z such that $f(U_x) = Z \cap f(X)$, so without loss of generality we can assume that $f(U_x) = Z \cap f(X)$, and that $f^{-1}(Z) = U_x$. We claim that $g \in \mathscr{G}(Z)$ satisfies $\tilde{\mathrm{Id}}_{f(x)}(g_{f(x)}) = [V, s]_{f(x)} \in (f_* f^{-1}\mathscr{G})_{f(x)}$. Indeed, we write that $g_{f(x)} = [Z, g]_{f(x)}$ then:

$$\mathrm{Id}_{f(x)}(g_{f(x)}) = [Z, \tilde{\mathrm{Id}}_Z(g)]_{f(x)}$$

we have that:

$$\begin{aligned}
\mathrm{Id}_Z(g) =& \mathrm{Id}_{f^{-1}(Z)} \circ \mathrm{sh}_{f^{-1}(Z)} \circ \psi_Z(g) \\
=& \, \mathrm{sh}_{f^{-1}(Z)}([Z, g]_{U_x}) \\
=& \, ([Z, g]_{U_x, p}) \\
=& (h_p) \\
=& (t_p)
\end{aligned}$$

where $(t_p) = s|_{U_x}$, but $U_x = f^{-1}(Z)$, so we have that:

$$[V, s]_{f(x)} = [Z, s|_{U_x}]_{f(x)} = \mathrm{Id}_{f(x)}(g_{f(x)})$$

It follows that $\tilde{\mathrm{Id}}_{f(x)}$ is surjective implying the first claim.

Now, suppose that f is an open embedding, and note that we again have an identity morphism of sheaves on Y given by:

$$\mathrm{Id} : f_*\mathscr{F} \to f_*\mathscr{F}$$

which induces a unique morphism $\hat{\mathrm{Id}} : f^{-1}f_*\mathscr{F} \to \mathscr{F}$. This map is the one induced by the sheafification of the map $\hat{\mathrm{Id}}_p : f_p^{-1}(f_*\mathscr{F}) \to \mathscr{F}$ given on open subsets of $U \subset X$ by:

$$(\hat{\mathrm{Id}}_p)_U([V, s]) = \xi_V(s) = \theta_U^{f^{-1}(V)} \circ \mathrm{Id}_V(s)$$

Note that if $[V, s] \in f_p^{-1}(f_*\mathscr{F})(U)$, then we have that $s \in f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V))$, and $f(U) \subset V$. We first claim that $[V, s] = [f(U), s|_{f(U)}] \in f_p^{-1}(f_*\mathscr{F})(U)$. Note that $f(U)$ is open, and that $f(U) \subset f(U) \cap V$, so essentially by definition we have that $[V, s] = [f(U), s|_{f(U)}]$. It follows for any $[V, s] \in f_p^{-1}(f_*\mathscr{F})(U)$ we can write $[V, s]$ as $[f(U), s|_{f(U)}]$. Now we see that:

$$(\hat{\mathrm{Id}}_p)_U([f(U), s]) = \theta_U^U \circ \mathrm{Id}_{f(U)}(s) = s \in \mathscr{F}(U) = f_*\mathscr{F}(f(U))$$

This is then trivially an isomorphism, so we have that $f_p^{-1}(f_*\mathscr{F})$ is actually a sheaf, and that $\mathrm{sh} : f_p^{-1}(f_*\mathscr{F}) \to f^{-1}f_*\mathscr{F}$ is an isomorphism. Since $\hat{\mathrm{Id}} \circ \mathrm{sh} = \hat{\mathrm{Id}}_p$, and both sh and $\hat{\mathrm{Id}}_p$ are isomorphisms, we have that $\hat{\mathrm{Id}}$ is an isomorphism as desired. □

1.4 The Structure Sheaf of Spec

Let A be a commutative ring; in this section we wish to equip the topological space $\operatorname{Spec} A$ with a sheaf of rings such that $\operatorname{Spec} A$ is a locally ringed space. Note that $A_{\mathfrak{p}}$ is a local ring by Example 1.3.1, so it would make sense to construct a sheaf on $\operatorname{Spec} A$ such that the stalk at $\mathfrak{p} \in \operatorname{Spec} A$ is $A_{\mathfrak{p}}$ (our choice of notation for stalks and the localization of a ring is intentionally suggestive). We begin with the following definition:

Definition 1.4.1. Let X be a topological space, and $\mathcal{B}$ be a basis for the topology of X. A **presheaf on a base** is the data of a set/group/ring, $\mathcal{F}(U)$ associated to each open set $U \in \mathcal{B}$, and restriction maps $\theta_V^U : \mathcal{F}(U) \to \mathcal{F}(U)$ whenever $U \subset V$, such that $\theta_W^V \circ \theta_V^U = \theta_V^U$. A **sheaf on a base** is a presheaf on a basis satisfying analogues of sheaf axioms one and two from Definition 1.2.1. Explicitly:

i) Let $\{U_i\} \subset \mathcal{B}$ be an open cover for $U \in \mathcal{B}$, then if $s, t \in \mathcal{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i then $s = t$.

ii) Let $\{U_i\} \subset \mathcal{B}$ be an open cover for $U \in \mathcal{B}$, and $s_i \in \mathcal{F}(U_i)$ sections such that for all basic opens $U_{ij} \subset U_i \cap U_j$:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

for all i and j, then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

We now show that sheaves on a base induce a sheaves on the total space which are unique up to unique isomorphism.

Theorem 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for the topology on X, and $\mathcal{F}$ a sheaf on the basis $\mathcal{B}$. Then, there exists a sheaf $\mathscr{F}$ on X induced by $\mathcal{F}$ satisfying the following universal property: for any sheaf $\mathscr{G}$, and any collection of set/group/ring morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ satisfying $\theta_V^U \circ \phi_U = \phi_V \circ \theta_V^U$ for all $V \subset U \in \mathcal{B}$, there exists a unique sheaf morphism $F : \mathscr{F} \to \mathscr{G}$, such that for all $U \in \mathcal{B}$ the following diagram commutes:*

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "F_U"] \arrow[d, "\psi_U"] & \mathscr{G}(U) \\
\mathcal{F}(U) \arrow[ur, "\phi_U"] &
\end{tikzcd}$$

where ψ_U is an isomorphism.

Proof. For all $x \in X$, we define the stalk $\mathcal{F}_x$ as:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$$

where we are clearly taking the colimit over $\mathcal{B}$, partially ordered by $U < V$ if $V \subset U$. The stalk is then the set of equivalence classes satisfying the same equivalence relation as the usual case, just restricted to basic sets. For each $W \subset X$ open, we define the set/group/ring by $\mathscr{F}(W)$:

$$\mathscr{F}(W) = \left\{ (s_x) \in \prod_{x \in W} \mathcal{F}_x : \forall y \in W, \exists U \in \mathcal{B}, y \in U, \text{ and } \exists f \in \mathcal{F}(U), \forall x \in U, f_x = s_x \right\}$$

In other words $(s_x) \in \prod_{x \in W} \mathcal{F}_x$ is an element of $\mathscr{F}(W)$ if for each $y \in W$, we can find an a basic open set U containing y, and a section $f \in \mathcal{F}(U)$ such that the sequence $(f_x) \in \prod_{x \in U} \mathcal{F}_x$ agrees with (s_x) on U. The restriction map θ_Z^W is then given by the restriction of the projection:

$$\prod_{x \in W} \mathcal{F}_x \longrightarrow \prod_{x \in Z} \mathcal{F}_x$$

to the sets/groups/rings $\mathscr{F}(W)$. The same argument as in Proposition 1.2.3 demonstrates that the restriction of the projection to $\mathscr{F}(W)$ has image in $\mathscr{F}(Z)$ when $Z \subset W$, and moreover that $\theta_Y^Z \circ \theta_Z^W = \theta_Z^W$. It follows that the assignment $Z \mapsto \mathscr{F}(Z)$ defines a presheaf on X.

We show that $\mathscr{F}$ is a sheaf. Suppose that $\{W_i\}$ is an open cover of $W \subset X$, and that $(s_x) \in \mathscr{F}(W)$ satisfies $(s_x)|_{W_i} = 0$ for all W_i. It follows that:

$$(s_x)|_{W_i} = (s_{x \in W_i}) = 0$$

implying that s_x is zero for all $x \in W_i$. Since $\{W_i\}$ covers W, it follows that for all $x \in W$ we have $s_x = 0$, hence $(s_x) = 0$.

Now suppose that we have $(s_x^i) \in \mathscr{F}(W_i)$ such that that:

$$(s_x^i)|_{W_i \cap W_j} = (s_x^j)|_{W_i \cap W_j}$$

hence we define an element $(s_x) \in \prod_x \mathcal{F}_x$ by:

$$(s_x) = (s_x^i)$$

whenever $x \in W^i$. This is well defined since $s_x^i = s_x^j$ whenever $x \in U_i \cap U_j$, so we need only show that $(s_x) \in \mathscr{F}(W)$, but this is clear. Indeed, for all $x \in W$, there exists an open set W_i such that $s_x = s_x^i$, however since $(s_x^i) \in \mathscr{F}(W_i)$, there exists a basic open U and a section $f \in \mathcal{F}(U)$ such that $f_y = s_y^i = s_y$ for al $y \in U$. We can do this for all x, hence $(s_x) \in \mathscr{F}(W)$, and clearly restricts to (s_x^i) for all i. It follows that $\mathscr{F}$ is indeed a sheaf.

We define $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ as follows: let $(s_x) \in \mathscr{F}(U)$, and let $U \in \mathcal{B}$, and $\{U_x\}$ be any open cover of U by basic opens such that for each $x \in U$ there is an $f^x \in \mathcal{F}(U_x)$ such that $s_y = f_y^x$ for all $y \in U_x$. We set $\psi_U((s_x))$ to be the unique element in $\mathcal{F}(U)$ satisfying $\psi_U((s_x))|_{U_x} = f^x$ for all U_x. Note that if such an element exists, it is independent of the cover and sections chosen. Indeed if $\{V_x\}$ is an other cover with sections e^x, then we denote the corresponding section $\psi_U^e((s_x))$. It follows that since $\psi_U^e((s_x))|_{V_x} = e^x$, we have for all $y \in U$:

$$\psi_U^e((s_x))_y = e_y^x = s_y = f_y^x = \psi_U((s_x))_y$$

Since the sections agree on stalks, we need only show that the natural map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is an injection, but this is clear by the same argument in Lemma 1.2.2. We now show such a section exists. Clearly, we need only show that $f^x|_{U_{xy}} = f^y|_{U_{xy}}$ for all $U_{xy} \subset U_x \cap U_y$, but this is vacuously true, as for any such U_{xy} we have that $f_z^x = s_z = f_z^y$ for all $z \in U_{xy} \subset U_x \cap U_y$, so by the preceding remark it must follows that $f^x|_{U_{xy}} = f^y|_{U_{xy}}$.

We now show that ψ_U is an isomorphism. Note that:

$$\psi_U((s_x))_y = s_y$$

hence if $\psi_U((s_x)) = 0$, we have that $\psi_U((s_x))_y = 0$ for all $y \in U$. It follows that $s_y = 0$ for all $y \in U$, hence $(s_x) = 0$. Moreover, if $s \in \mathcal{F}(U)$, then sequence $(s_x) \in \prod_{x \in U} \mathcal{F}_x$ clearly lies in $\mathscr{F}(U)$, and by definition we have that for all $y \in U$:

$$\psi_U((s_x))_y = s_y$$

for all $y \in U$, hence $\psi_U((s_x)) = s$ implying the claim.

Now suppose that $\mathscr{G}$ is a sheaf, equipped with morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ for all $U \in \mathcal{B}$, such that $\theta_V^U \circ \phi_U = \phi_V \circ \theta_V^U$. We define a morphism $F : \mathscr{F} \to \mathscr{G}$ on generic open sets $W \subset X$ as follows: let $(s_x) \in \mathscr{F}(W)$, then there exists an open cover $\{W_x\}$ of W by basic opens, along with sections $f^x \in \mathcal{F}(W_x)$ such that $f^x_y = s_y$ for all $y \in W_x$. Then we set $F_W((s_x))$ to be the unique section of $\mathscr{G}(W)$ such that $F_W((s_x))|_{W_x} = \phi_{W_x}(f^x)$. If this section exists, then it is well defined by the same argument as in the ψ_U case. We now show such a section exists; we need only $\phi_{W_x}(f^x)|_{W_x \cap W_y} = \phi_{W_y}(f^y)|_{W_x \cap W_y}$. Cover $W_x \cap W_y$ by basic opens V_i, then we know that $V_i \subset W_x$ and $V_i \subset W_y$ for all i. Since f^x and f^y agree on all open subsets of $W_x \cap W_y$, it follows that for all V_i:

$$(\phi_{W_x}(f^x)|_{W_x \cap W_y} - \phi_{W_y}(f^y)|_{W_x \cap W_y})|_{V_i} = 0$$

hence by sheaf axiom one the two agree on $W_x \cap W_y$. Now let U be a basic open, we want to show that:

$$F_U = \phi_U \circ \psi_U$$

Take $(s_x) \in \mathscr{F}(U)$, then there is a unique section $f \in \mathcal{F}(U)$ such that $\psi_U((s_x)) = f$. In particular, $f_x = s_x$ for all $x \in U$. Since our definition of F_U is independent of our cover and choice of sections, choose the trivial cover $\{U\}$ and the section $s \in \mathcal{F}(U)$. Since there is nothing to glue over, it follows that:

$$F_U((s_x)) = \phi_U(f) = \phi_U \circ \psi_U((s_x))$$

implying the claim. □

Corollary 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for its topology, and $\mathcal{F}$ a sheaf on $\mathcal{B}$. Then the induced sheaf $\mathscr{F}$ is unique up to unique isomorphism.*

Proof. Suppose that $\mathscr{G}$ is any other sheaf that satisfies the universal property, i.e. $\mathscr{G}$ is a sheaf on X equipped with isomorphisms $\psi_U^{\mathscr{G}} : \mathscr{G}(U) \to F(U)$ for all $U \in \mathcal{B}$, such that for any other sheaf $\mathscr{H}$ with morphisms $\phi_U : \mathcal{F}(U) \to \mathscr{H}(U)$ which commute with restrictions on basic opens, there is a unique morphism $\phi : \mathscr{G} \to \mathscr{H}$. Now note that $\mathscr{F}$ as constructed in Theorem 1.4.1 comes equipped with isomorphisms $\psi_U^{-1} : \mathcal{F}(U) \to \mathscr{F}(U)$ which trivially commute with restrictions on basic opens. It follows that there exists a unique morphism $F : \mathscr{G} \to \mathscr{F}$; we show that this is an isomorphism.

Let $g_x \in \mathscr{G}_x$, and note that g_x can be written as an equivalence class $[U, g]$ where U is a basic open set. Indeed, if $g_x = [V, g']$, then V is the union of basis open sets, hence there must be some basic open set U containing x. It follows that:

$$[V, g'] = [U, g'|_U]$$

as desired. We see that:

$$F_x(g_x) = [U, F_U(g)] = [U, \psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g)]$$

Suppose this equals zero, then there is an open set $V \subset U$ such that $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g)|_V$ is zero. Without loss of generality we can take V to be a basic open by our previous remark. It follows that:

$$\psi_V^{-1} \circ \psi_V^{\mathscr{G}}(g|_V) = 0 \Rightarrow g|_V = 0$$

as $\psi_V^{-1} \circ \psi_V^{\mathscr{G}}$ is an isomorphism. However, we have that:

$$[U, g] = [V, g|_V] = 0$$

hence $g_x = 0$. Now let $s_x \in \mathscr{F}_x$, we can represent s_x as an equivalence class $[U, s]$, where U is a basic open. Since $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}$ is an isomorphism, it follows that there exists a unique $g \in \mathscr{G}(U)$, such that $\psi_U^{-1} \circ \psi_U^{\mathscr{G}}(g) = s$. We thus have that F_x is an isomorphism for all x, hence $\mathscr{G}$ is uniquely isomorphic to $\mathscr{F}$ as desired. □

We now prove two results regarding sheafs on a base as a sanity check that things work as assumed. In particular, it should stand to reason that the stalks of a sheaf on a base are isomorphic, and that restricting a sheaf on to a sheaf on a base yields the same sheaf.

Proposition 1.4.1. *Let X be a topological space, $\mathcal{B}$ a basis for its topology, and $\mathcal{F}$ a sheaf on $\mathcal{B}$. Then the induced sheaf $\mathscr{F}$ satisfies $\mathscr{F}_x \cong \mathcal{F}_x$ for all $x \in X$.*

Proof. Note that we have morphisms $\phi_U : \mathscr{F}(U) \to \mathcal{F}_x$ for all $x \in U \subset X$ given by:

$$\phi_U((s_y)) = s_x$$

These maps trivially commute with restriction. It follows by the universal property of the colimit that there exists a unique morphism $\phi_x : \mathscr{F}_x \to \mathcal{F}_x$ given by:

$$\phi_x([U, (s_y)]) = s_x$$

Suppose that $s_x = 0$, then note that since $(s_y) \in \mathscr{F}(U)$, there exists an open neighborhood V_x of x, and a section $f^x \in \mathcal{F}(U)$ such that $f^x_y = s_y$ for all $y \in V_x$. We can thus write $s_x = [V_x, f^x]$, however this is zero, so there exists another open set such that $x \in Z_x \subset V_x$ such that $f^x|_{Z_x} = 0$. Since stalks commute with restriction, it follows that $(f^x|_{Z_x})_y = s_y$ for all $y \in Z_x$. However, this means that $s_y = 0$ for all $y \in Z_x$, hence:

$$(s_y)|_{Z_x} = 0 \Rightarrow [U, (s_y)] = [Z_x, (s_y)|_{Z_X}] = 0$$

so ϕ_x is injective. Moreover, suppose that $s_x = [U, s] \in \mathcal{F}_x$, then we see that $\psi_U^{-1}(s)$ is a sequence (t_y) in $\mathscr{F}(U)$ which satisfies $t_y = s_y$ for all $y \in U$. It follows that:

$$\phi_x([U, \psi_U^{-1}(s)]) = t_x = s_x$$

hence ϕ_x is surjective and thus an isomorphism as desired. □

Proposition 1.4.2. *Let $\mathscr{F}$ be a sheaf on X, and let $\mathcal{B}$ be a basis for the topology on X. Then for all $U \in \mathcal{B}$, the assignment $U \mapsto \mathcal{F}(U) = \mathscr{F}(U)$ defines a sheaf on a base such that the induced sheaf is uniquely isomorphic to $\mathscr{F}$.*

Proof. First note that since $\mathscr{F}$ is a sheaf, sheaf on a base axiom one is trivially fulfilled. Now let U be a basic open, and $\{U_i\}$ an open cover of U with sections $s_i \in \mathcal{F}(U_i)$ such that for all basic opens $U_{ij} \subset U_i \cap U_j$ we have:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

Note that we can cover $U_i \cap U_j$ by all such U_{ij}^k indexed by k, and that since $s_i \in \mathcal{F}(U_i) = \mathscr{F}(U_i)$, we can restrict each s_i to $U_i \cap U_j$. It suffices to show that:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

However, we have that for all $U_{ij}^k \subset U_i \cap U_j$:

$$(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})|_{U_{ij}^k} = 0$$

hence by sheaf axiom one s_i and s_j agree on $U_i \cap U_j$. It follows that since $\mathscr{F}$ is a sheaf there is a unique section $s \in \mathcal{F}(U) = \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

We show that $\mathscr{F}$ satisfies the universal property in Theorem 1.4.1, and thus by Corollary 1.4.1 is uniquely isomorphic to the induced sheaf. For all $U \in \mathcal{B}$, let $\phi_U : \mathcal{F}(U) \to \mathscr{G}(U)$ be a collection of morphisms which commute with restriction on a basic open sets, and note that $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ is the identity morphism for all U. We thus need to construct a map $F : \mathscr{F} \to \mathscr{G}$ such that $F_U = \phi_U$. Let W be an arbitrary open set, and $\{W_i\}$ an open cover by basic opens. If $s \in \mathscr{F}(W)$, then we define $F_W(s)$ as the unique element in $\mathscr{G}(W)$ such that $F_W(s)|_{W_i} = \phi_{W_i}(s|_{W_i})$.

Suppose such an element exists, and let $\{V_i\}$ be a different open cover of W by basic opens. Consider $V_i \cap W_j$, and let $Z_{ij} \subset V_i \cap W_j$ be any basic open set, then we have that:

$$\phi_{W_i}(s|_{W_i})|_{Z_{ij}} = \phi_{Z_{ij}}(s|_{Z_{ij}}) = \phi_{V_i}(s|_{V_i})|_{Z_{ij}}$$

It follows that since $\mathscr{G}$ is a sheaf, $\phi_{W_i}(s|_{W_i})$ and $\phi_{V_j}(s|_{V_j})$ agree on overlaps $W_i \cap V_j$. If $F_W(s) = g$ is the element such that $g|_{W_i} = \phi_{W_i}(s|_{W_i})$, and $F_W(s) = h$ is the element such that $h|_{V_i} = \phi_{W_i}(s|_{V_i})$, then we have that for all $V_i \cap W_j$:

$$(h - g)|_{V_i \cap W_j} = 0$$

Since all such intersections form a cover for W, and $\mathscr{G}$ is a sheaf it follows that $g = h$, so F_W is independent of the chosen cover.

Now we show that $F_W(s)$ exists. We need only show that $\phi_{W_i}(s|_{W_i})|_{W_i \cap W_j} = \phi_{W_j}(s_j)|_{W_i \cap W_j}$, however for all basic open sets $W_{ij} \subset W_i \cap W_j$ we have that:

$$\phi_{W_i}(s|_{W_i})|_{W_{ij}} = \phi_{W_{ij}}(s|_{W_{ij}}) = \phi_{W_j}(s|_{W_j})|_{W_{ij}}$$

so since $\mathscr{G}$ is a sheaf we must have that $\phi_{W_i}(s|_{W_i})|_{W_i \cap W_j} = \phi_{W_j}(s_j)|_{W_i \cap W_j}$, so $F_W(s)$ exists.

We need to check that F_W commutes with restrictions. Let $Z \subset W$, then we have an open cover of Z given by $\{Z \cap W_i\}$. For each i, we can cover $Z \cap W_i$ by basic opens Z_{ij} such that $Z_{ij} \subset W_i$ for all j. It follows that:

$$F_Z(s|_Z)|_{Z_{ij}} = \phi_{Z_{ij}}((s|_Z)|_{Z_{ij}}) = \phi_{Z_{ij}}(s|_{Z_{ij}}) = \phi_{W_i}(s|_{W_i})|_{Z_{ij}} = (F_W(s)|_{W_i})|_{Z_{ij}} = F_W(s)|_{Z_{ij}} = (F_W(s)|_Z)|_{Z_{ij}}$$

Since the set of all Z_{ij} cover Z, we must have that $F_Z \circ \theta_Z^W = \theta_Z^W \circ F_W$, hence F defines a sheaf morphism. It is then clear that:

$$F_U = \phi_U$$

whenever U is a basic open, implying the claim. □

Corollary 1.4.2. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheafs on X, and $\mathcal{B}$ a basis for the topology on X. Then, any morphism $F : \mathscr{F} \to \mathscr{G}$ is determined by the morphisms $F_U : \mathscr{F}(U) \to \mathscr{G}(U)$ where U is a basic open. In particular, F is an isomorphism if and only if F_U is an isomorphism for all $U \in \mathcal{B}$.*

Proof. By the preceding proposition, we know that $\mathscr{F}$ satisfies the universal property of the sheaf on a base defined by $\mathcal{F} : U \mapsto \mathscr{F}(U)$ for all $U \in \mathcal{B}$, hence if $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is some collection of morphisms which commute with restrictions on basis opens, then we have a unique map $F : \mathscr{F} \to \mathscr{G}$. Now suppose we are given a map $F : \mathscr{F} \to \mathscr{G}$, and define $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ by $\phi_U = F_U$. Since $\psi_U : \mathscr{F}(U) \to \mathcal{F}(U)$ is the identity, it follows that F trivially satisfies the diagram in Theorem 1.4.1, and is thus the unique morphism determined by $\phi_U = F_U$, implying that F is determined by the morphisms F_U as desired.

Now suppose that F is an isomorphism, then clearly for all $U \in \mathcal{B}$ we have that F_U is an isomorphism. Now suppose that for all $U \in \mathcal{B}$ we have that F_U is an isomorphism. If we can show that $F_x : \mathscr{F}_x \to \mathscr{G}_x$ is an isomorphism for all $x \in X$ then we are done. Let $[U, s] \in \mathscr{F}_x$, and suppose that $F_x([U, s]) = [U, F_U(s)] = 0$. Then this implies that $F_U(s)|_V = 0$ for some $V \subset U$. Since V is the union of basic opens, we can further restrict to a basis open W to obtain that $F_U(s)|_W = 0$, which implies that $s|_W = 0$, as F_W is an isomorphism. It follows that $[U, s] = [W, s|_W] = [W, 0] = 0$, so F_x is injective. Now let $[U, t] \in \mathscr{G}_x$, and let $W \subset U$ be any basic open set, then $[U, t] = [W, t|_W]$, and there is a unique element $s \in \mathscr{F}(W)$ such that $F_W(s) = t|_W$. It follows that $[W, s] \in \mathscr{F}_x$ satisfies $[W, F_W(s)] = [W, t|_W] = [U, t]$ hence F_x is surjective and thus an isomorphism, implying the claim. □

Now let A be a commutative ring; consider Spec A with the Zariski topology, then the set $\mathcal{B} = \{U_f\}_{f \in A}$ of distinguished opens forms a basis for the topology on Spec A by Lemma 1.1.2. We define a sheaf on $\mathcal{B}$ via the assignment:

$$U_f \longmapsto A_f \tag{1.4.1}$$

where A_f is the localization of A at f. By Lemma 1.1.3 we also have that $U_f = U_g$ if and only if $\sqrt{\langle f\rangle} = \sqrt{\langle g\rangle}$, and by Lemma 1.1.4 we then have that $A_f \cong A_g$, so the assignment is well defined. We need the following lemma:

Lemma 1.4.1. *Let A be a commutative ring, then every open cover of* $\operatorname{Spec} A$ *has a finite subcover. In particular, every distinguished open can be written as the finite union of distinguished opens.*

Proof. Let $\{V_i\}$ be an open cover of A, then we have that:

$$\operatorname{Spec} A = \bigcup_i V_i$$

For each i we have that:

$$V_i = \bigcup_j U_{f_{j_i}}$$

hence:

$$\begin{aligned}
\operatorname{Spec} A =& \bigcup_i \left(\bigcup_{j_i} U_{f_{j_i}} \right) \\
=& \bigcup_{i,j_i} U_{f_{j_i}} \\
=& \bigcup_{i,j_i} \mathbb{V}(\langle f_{j_i}\rangle)^c \\
=& \left(\bigcap_{i,j_i} \mathbb{V}(\langle f_{j_i}\rangle) \right)^c \\
=& \left(\mathbb{V}\left(\sum_{i,j_i} \langle f_{j_i}\rangle \right)\right)^c
\end{aligned}$$

It follows that since $\mathbb{V}(\langle 1\rangle) = \operatorname{Spec} A$, we have that 1 can be written as a finite linear combination:

$$1 = \sum_{k=1}^n a_k f_k$$

where $f_k = f_{j_i}$ for some j_i. We thus have that:

$$\operatorname{Spec} A = \bigcup_{k=1}^n U_{f_k}$$

By hypothesis we have that each U_{f_k} is contained in a V_k, hence:

$$\operatorname{Spec} A = \bigcup_{k=1}^n V_k$$

so the cover $\{V_i\}$, admits a finite subcover $\{V_k\}_{k=1}^n$.

Let U_f be a distinguished open, and $\{U_{g_i}\}$ an open covering of U_f. We see that:

$$U_f = (\mathbb{V}(\langle f\rangle))^c = \left(\mathbb{V}\left(\sum_i \langle g_i\rangle \right)\right)^c$$

It follows by Lemma 1.1.1 that:

$$\sqrt{\langle f\rangle} = \sqrt{\sum_i \langle g_i\rangle}$$

It follows that there exists an m such that $f^m \in \sum_i \langle g_i \rangle$ hence

$$f^m = \sum_{j=1}^{p} a_j g_j$$

for some $a_j \in A$ and g_i. We want to show that:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_{j=1}^{p} \langle g_j \rangle}$$

Let $a \in \sqrt{\langle f \rangle}$, then $a^n = f^k \cdot b$ for some $b \in A$, for some $n \in \mathbb{Z}^+$; we see that:

$$\begin{aligned} a^{nm} &= f^{mk} \cdot b^m \\ &= \left(\sum_{j=1}^{p} a_j g_j \right)^k b^m \in \sqrt{\sum_{j=1}^{n} \langle g_j \rangle} \end{aligned}$$

so $\sqrt{\langle f \rangle} \subset \sqrt{\sum_{j=1}^{n} \langle g_j \rangle}$. Now let $b \in \sqrt{\sum_{j=1}^{p} \langle g_j \rangle}$, then there exists an $n \in \mathbb{Z}^+$ such that:

$$b^n = \sum_{j=1}^{p} c_j g_j$$

for some $c_j \in A$. Now we note that $b^n \in \sqrt{\sum_i \langle g_i \rangle}$, hence $b^n \in \sqrt{\langle f \rangle}$ implying the claim. □

We also have the following:

Lemma 1.4.2. *Let X be a topological space and $\mathcal{F}$ a presheaf on a basis for its topology $\mathcal{B}$, such that every cover of a basic set by basic sets admits a finite subcover. If the sheaf on a base axioms hold for all such finite covers, then they hold in generality.*

Proof. We begin with sheaf axiom one; let U be a basic open, $\{U_i\}$ an open covering of U, and $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$ for all i. Let $\{U_j\}_{j=1}^k$ be a finite subcover, then we have $s|_{U_j} = 0$ for $1 \leq j \leq k$, so by the hypothesis it follows that $s = 0$ and sheaf axiom one is satisfied.

Now let $\{U_i\}$ be an cover of U, and $s_i \in \mathcal{F}(U_i)$ such that for all basic sets $U_{ij} \subset U_i \cap U_j$ we have:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

Then there exists a finite subcover $\{U_j\}_{j=1}^k$ such that for all basic open sets $U_{jl} \subset U_j \cap U_l$:

$$s_j|_{U_{jl}} = s_l|_{U_{jl}}$$

It follows by the hypothesis that there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_j} = s_j$ for all $1 \leq j \leq k$. We need only show that for $U_i \notin \{U_j\}_{j=1}^k$ we have $s|_{U_i} = s_i$. We have a finite cover of U_i by $\{U_i \cap U_j\}_{j=1}^k$, each $U_i \cap U_j$ has a cover of basic opens by $\{U_{ijm}\}_m$, hence we obtain a cover of U_i by basic opens $\{U_{ijm}\}$ such that $U_{ijm} \subset U_i \cap U_j \subset U_j$. We see that:

$$\theta^{U_i}_{U_{ijm}}(s|_{U_i}) = (s|_{U_i})|_{U_{ijm}} = s|_{U_{ijm}} = \theta^{U_j}_{U_{ijm}} \circ \theta^{U}_{U_j}(s) = s_j|_{U_{ijm}} = s_i|_{U_jm}$$

It follows that:

$$\theta^{U_i}_{U_{ijm}}(s|_{U_i} - s_i) = 0$$

for all j and m, hence $s|_{U_i} = s_i$, implying the claim. □

Proposition 1.4.3. *Let A be commutative ring, and $\mathcal{B}$ be the basis of distinguished opens for the Zariski topology on* Spec A*. Then the assignment* (1.4.1) *defines a sheaf $\mathcal{F}$ on $\mathcal{B}$.*

Proof. We first define restriction maps; by Lemma 1.1.3 we have that if $U_f \subset U_g$, then there exists an $m \in \mathbb{Z}^+$ and $a \in A$ such that $f^m = a \cdot g$. Note that we have maps $\pi_f : A \to A_f$ and $\pi_g : A \to A_g$, and that the image of g is a unit in A_f. Indeed, we have that:

$$\frac{g}{1} \cdot \frac{r}{f^m} = \frac{g \cdot r}{f^m} = \frac{f^m}{f^m} = 1$$

It follows that there exists a unique map $\theta^g_f : A_g \to A_f$ given by:

$$\theta^g_f\left(\frac{b}{g^k}\right) = \frac{b \cdot a^k}{f^{mk}}$$

Now suppose that $U_g \subset U_h$, then we have that there exists a $c \in A$, and an $n \in \mathbb{Z}^+$, such that $g^n = h \cdot c$. By the same argument we obtain a ring homomorphism:

$$\begin{aligned} \theta^h_g : A_h &\longrightarrow A_g \\ \frac{b}{h^k} &\longmapsto \frac{b \cdot c^k}{g^{nk}} \end{aligned}$$

We want to show that $\theta^g_f \circ \theta^h_g = \theta^h_f$. First note that we have:

$$f^m = a \cdot g \Rightarrow f^{mn} = a^n \cdot g^n = a^n \cdot c \cdot h$$

so the map θ^h_f is given by:

$$\frac{b}{h^k} \longmapsto \frac{b \cdot a^{nk} \cdot c^k}{f^{mnk}}$$

Now note that:

$$\begin{aligned} \theta^g_f \circ \theta^h_g\left(\frac{b}{h^k}\right) &= \theta^g_f\left(\frac{b \cdot c^k}{g^{nk}}\right) \\ &= \frac{b \cdot c^k \cdot a^{nk}}{f^{mnk}} \\ &= \theta^h_g\left(\frac{b}{h^k}\right) \end{aligned}$$

It is clear that $\theta^h_h = \mathrm{Id}_{A_h}$, hence $\mathcal{F}(U_f) = A_f$ defines a presheaf on $\mathcal{B}$.

We now check sheaf axiom one. Suppose that U_f is a distinguished open set, $\{U_{g_i}\}$ an open covering of U_f, and $s \in A_f$ such that $s|_{g_i} = 0$ for all i. By Lemma 1.4.1, and Lemma 1.4.2 it suffices to check this on all finite subcoverings of U_f, so without loss of generality we suppose that $\{U_{g_i}\}$ is finite. Since $U_{g_i} \subset U_f$, we have that for each i there exists an $m_i \in \mathbb{Z}^+$, and a $c_i \in A$ such that:

$$g_i^{m_i} = c_i \cdot f$$

Now we note that since f is a unit in A_f, so $s = 0$ if and only if $f^k s = 0$. Indeed, if $s = 0$ then clearly $f^k s = 0$, while if $f^k s = 0$, we have that $(f^k)^{-1} f^k s = s = 0$. If $s = a/f^k$, it thus suffices to show that $f^k s = a/1 = 0$. We have that $f^k s \in \ker \theta^f_{g_i}$, hence there exists an $l_i \in \mathbb{Z}^+$ such that:

$$g^{l_i} \cdot a = 0$$

Now note that:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_i \langle g_i \rangle} = \sqrt{\sum_i \left\langle g_i^{l_i} \right\rangle}$$

hence there exists a $k \in \mathbb{Z}^+$ such that:

$$f^k = \sum_i g_i^{l_i} c_i$$

for some $c_i \in A$. Since each $g^{l_i} \cdot a = 0$, we have that:

$$0 = \sum_i g^{l_i} a = \sum_i g^{l_i} c_i a = a \cdot f^k$$

hence $a/1$ is zero in A_f.

To check sheaf axiom two, it again suffices to assume that $\{U_{g_i}\}$ is a finite open cover of U_f. Let $s_i \in A_{g_i}$ be sections such that:

$$s_i|_{U_{ij}} = s_j|_{U_{ij}}$$

for all $U_{ij} \subset U_{g_i} \cap U_{g_j}$. Then since $U_{g_i} \cap U_{g_j} = U_{g_i g_j}$, we have that:

$$s_i|_{U_{g_i} \cap U_{g_j}} = s_j|_{U_{g_i} \cap U_{g_j}}$$

Since $U_{g_i g_j} \subset U_{g_i}, U_{g_j}$, we have that there exists $k_i \in \mathbb{Z}^+$ and $c_i \in A$ such that:

$$(g_i g_j)^{k_i} = g_i \cdot c_i \qquad \text{and} \qquad (g_i g_j)^{k_j} = g_j c_j$$

Clearly $k_i = 1$ with $c_i = g_j$ fit the bill, hence our restriction maps are given by:

$$\theta^{g_i}_{g_i g_j}\left(\frac{a_i}{g_i^{k_i}}\right) = \frac{a_i \cdot g_j^{k_i}}{g_i^{k_i} g_j^{k_i}}$$

hence on overlaps we have that:

$$\frac{a_i \cdot g_j^{k_i}}{g_i^{k_i} g_j^{k_i}} = \frac{a_j \cdot g_i^{k_j}}{g_i^{k_j} g_j^{k_j}}$$

Since $\{U_{g_i}\}$ is finite, there exists some K such that for all i and j:

$$(g_i g_j)^K \left((g_i g_j)^{k_j} \cdot a_i \cdot g_j^{k_i} - (g_i g_j)^{k_i} \cdot a_j \cdot g_i^{k_j}\right) = 0$$

We multiply by $g_i^{k_i} g_j^{k_j}$ to obtain:

$$\begin{aligned} 0 =& (g_i g_j)^K \left(g_i^{k_i} g_j^{k_j} (g_i g_j)^{k_j} a_i g_j^{k_i} - g_i^{k_i} g_j^{k_j} (g_i g_j)^{k_i} a_j g_i^{k_j}\right) \\ =& (g_i g_j)^{K + k_i + k_j} \left(a_i g_j^{k_j} - a_j g_i^{k_i}\right) \end{aligned} \tag{1.4.2}$$

Set K' to be large enough such that expression above holds for all i, j, and define:

$$h_i = a_i g_i^{K'}$$

Now since:

$$\sqrt{\langle f \rangle} = \sqrt{\sum_i \langle g_i \rangle} = \sqrt{\sum_i \left\langle g_i^{K' + k_i} \right\rangle}$$

we have that for some M, there exist c_i such that:

$$f^M = \sum_i c_i g_i^{K' + k_i}$$

We define s to be:

$$s = \sum_i \frac{c_i h_i}{f^M}$$

Now note that for each j, we have that $g_j^{n_j} = f \cdot b$, then the restriction is given by:

$$s|_{U_{g_j}} = \sum_i \frac{c_i h_i b^M}{g_j^{n_j \cdot M}}$$

We claim this equal to $s_j = a_j/g_j^{k_j}$; examine the expression:

$$g_j^{K'}\left(\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} - a_j \cdot g_j^{n_j M}\right)$$

Examine the first term,

$$\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'} = \sum_i c_i \cdot a_i \cdot g_i^{K'} \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'}$$

for each i we have that by (1.4.2):

$$g_i^{K'} \cdot g_j^{K'} \cdot g_j^{k_j} a_i = g_i^{K'} \cdot g_j^{K'} \cdot g_i^{k_i} a_j$$

hence we have that:

$$\begin{aligned}\sum_i c_i \cdot h_i \cdot b^M \cdot g_j^{k_j} \cdot g_j^{K'} &= \left(g_j^{K'} a_j b^M\right) \sum_i g_i^{K'} \cdot c_i \cdot g_i^{k_i} \\ &= \left(g_j^{K'} a_j b^M\right) f^M \\ &= g_j^{K'} a_j g_j^{n_j M}\end{aligned}$$

implying the claim. □

Definition 1.4.2. Let A be a commutative ring, the the **structure sheaf of** $\operatorname{Spec} \mathbf{A}$, denoted $\mathscr{O}_A$, is the sheaf induced by the sheaf on the base of distinguished opens given by the assignment:

$$U_f \longmapsto A_f$$

The pair $(\operatorname{Spec} A, \mathscr{O}_A)$ is called an **affine scheme**[14].

Proposition 1.4.4. *Let A be a commutative ring, then $(\operatorname{Spec} A, \mathscr{O}_A)$ is a locally ringed space. In particular, the stalk $(\mathscr{O}_A)_\mathfrak{p}$ is uniquely isomorphic to $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. By Proposition 1.4.1, it is sufficient to show that $\mathcal{F}_\mathfrak{p}$ is a local ring for all $\mathfrak{p} \in \operatorname{Spec} A$, where $\mathcal{F}$ is the sheaf on a base discussed defined by $U_f \mapsto A_f$. Let $\mathfrak{p} \in \operatorname{Spec} A$, then note that if $\mathfrak{p} \in U_f$, we have that $f \notin \mathfrak{p}$, hence $f \in A - \mathfrak{p}$. It follows that f is a unit in $A_\mathfrak{p}$, thus there exists a unique map $\phi_f : A_f \to A_\mathfrak{p}$ given by:

$$\phi_f : \frac{a}{f^k} \longmapsto \frac{a}{f^k}$$

Now suppose that $U_f \subset U_g$, then we have that $f^m = b \cdot g$, so the restriction map θ_f^g is given by:

$$\theta_f^g : \frac{a}{g^k} \longmapsto \frac{a \cdot b^k}{f^{mk}}$$

We want to show that $\phi_g = \phi_f \circ \theta_f^g$, i.e. that:

$$\frac{a \cdot b^k}{f^{mk}} = \frac{a}{g^k}$$

in $A_\mathfrak{p}$. We want to show that there exists a $u \in A - \mathfrak{p}$ such that:

$$u \cdot \left(a \cdot b^k \cdot g^k - a \cdot f^{mk}\right) = 0$$

However, $b^k \cdot g^k = f^{mk}$, so this statement is vacuously true. By the universal property of the colimit there thus exists a unique map $\phi : \mathcal{F}_\mathfrak{p} \to A_\mathfrak{p}$. Suppose we have $[U_f, s] \in \mathcal{F}_\mathfrak{p}$ such that:

$$\phi([U_f, s]) = 0$$

[14]This is a tentative definition of an affine scheme, but will be easily seen to be compatible with our future one.

Since $s \in A_f$, we have that s is of the form a/f^k, so we must have that:

$$\frac{a}{f^k} = 0$$

in $A_{\mathfrak{p}}$, thus there exists a $u \in A - \mathfrak{p}$ such that:

$$u \cdot a = 0$$

We claim that $[U_f, a/f^k] = 0$; well since $u \in A - \mathfrak{p}$, we have that $u \notin \mathfrak{p}$, hence $\mathfrak{p} \in U_u$. Note that:

$$[U_f, a/f^k] = [U_{fu}, a/f^k|_{U_u}]$$

Since $U_{fu} \subset U_f$, we have that there exists some $n \in \mathbb{Z}^+$ and some $c \in A$ such that:

$$(u \cdot f)^n = c \cdot f$$

however, $n = 1$, and $c = u$ fits the bill, hence our restriction map is given by:

$$a/f^k \longmapsto \frac{a \cdot u^k}{(u \cdot f)^k}$$

however $a \cdot u = 0$, hence we have that the above expression is 0, implying ϕ is injective. Now suppose that $a/r \in A_{\mathfrak{p}}$, then $r \in A - \mathfrak{p}$, hence $\mathfrak{p} \in U_r$. It follows that:

$$\phi([U_r, a/r]) = a/r$$

implying that ϕ is surjective and thus an isomorphism as desired. □

We also have the following facts:

Lemma 1.4.3. *Let* $(\operatorname{Spec} A, \mathscr{O}_A)$ *be an affine scheme, then there are unique isomorphisms* $\mathscr{O}_A(U_f) \cong A_f$*, and* $\mathscr{O}_A(\operatorname{Spec} A) \cong A$*.*

Proof. By Theorem 1.4.1 we have that $\mathscr{O}_A(U_f) \cong A_f$ as $\mathcal{F}(U_f) = A_f$. Moreover, note that:

$$U_1 = \{\mathfrak{p} \in \operatorname{Spec} A : 1 \notin \mathfrak{p}\}$$

which is equal to all of $\operatorname{Spec} A$, because no prime ideal contains 1. We easily see that $A_1 \cong A$, implying the claim. □

We now determine some topological properties of affine schemes.

Definition 1.4.3. A topological space X is **irreducible** if it is non empty, and cannot be written as the union of two proper closed subsets. A subspace $Z \subset X$ of a topological space is called an **irreducible subspace** if it is irreducible in the subspace topology. An **irreducible component** of a topological space is a maximal irreducible subspace.

Lemma 1.4.4. *Let* X *be a topological space which is irreducible. Then* X *is connected and every open subset of* X *is dense.*

Proof. Suppose that X is disconnected, then $X = U \cup V$ for some disjoint open sets U and V. It follows that since taking the closure over binary unions distributes that:

$$X = \bar{U} \cup \bar{V}$$

so X is reducible. The claim follows by the contrapositive.

Now let $U \subset X$ be any open set, and suppose that U is not dense. It follows that $\bar{U} \neq X$, and that the compliment U^c is closed. We claim that:

$$X = \bar{U} \cup U^c$$

However this is vacuously true, as:

$$X = U \cup U^c \Rightarrow X = \bar{U} \cup \bar{U^c} = \bar{U} \cup U^c$$

hence X is reducible and the claim again follows by the contrapositive. □

Lemma 1.4.5. *Let A be an integral domain, then* $\operatorname{Spec} A$ *is irreducible. In particular, every open set is dense, and* $\operatorname{Spec} A$ *is connected.*

Proof. Recall that if A is an integral domain then we have that $a \cdot b = 0$ if and only if a or b is zero. This then implies that that $\langle 0 \rangle$ is a prime ideal of A, and is thus a point in $\operatorname{Spec} A$. Now suppose that $\operatorname{Spec} A$ is reducible, then we have that by Proposition 1.1.1:

$$\operatorname{Spec} A = \mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J) = \mathbb{V}(\langle 0 \rangle)$$

for some ideals I and J. Lemma 1.1.1 then implies that:

$$\sqrt{I \cap J} = \sqrt{\langle 0 \rangle}$$

Since A is an integral domain, we must have that $\sqrt{\langle 0 \rangle} = \langle 0 \rangle$, so

$$\sqrt{I \cap J} = \langle 0 \rangle$$

However, we note that $\sqrt{I \cap J} = \sqrt{IJ}$, then we have that:

$$\sqrt{IJ} = \langle 0 \rangle$$

However, this implies that $IJ \subset \sqrt{IJ} = \langle 0 \rangle$, hence $IJ = \langle 0 \rangle$. It follows that every finite sum of the form:

$$\sum_k i_k j_k$$

where $i_k \in I$ and $j_k \in J$ is zero, hence either I or J is the zero ideal. The claim then follows from the contrapositive, and Lemma 1.4.4. $\square$

Note that when A is an integral domain, we have that every nonempty open set contains $\langle 0 \rangle$. Indeed, note that U_0 is the empty set, and that if $f \neq 0$, then $\langle 0 \rangle \in U_f$ as $f \notin \langle 0 \rangle$. In particular this implies that $\operatorname{Spec} A$ is not Hausdorff, as if $\mathfrak{p}$ and $\langle 0 \rangle$ are both contained in some open set U, then any open set containing $\mathfrak{p}$ will also contain $\langle 0 \rangle$. In general the Zariski topology will be non Hausdorff.

Example 1.4.1. Let k be a field, then we set $\mathbb{A}^n_k$ to be affine scheme:

$$\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$$

Note that in this case $k[x_1, \ldots, x_n]$ is an integral domain, so in particular $\mathbb{A}^n_k$ is irreducible, and connected. The singleton set consisting of the zero ideal is then clearly dense, and so not closed, nor is it open.

The remainder of this section will be dedicated to demonstrating that there is a category of affine schemes, denoted by Aff, which is (anti) equivalent to the category of commutative rings. The objects in Aff are affine schemes $(\operatorname{Spec} A, \mathscr{O}_A)$, and a morphism of affine schemes $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is simply a morphism of locally ringed spaces. Let $\phi : B \to A$ be a homomorphism, then we have an induced topological map $f : \operatorname{Spec} A \to \operatorname{Spec} B$ given by $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$. We want to define a morphism $f^\sharp : \mathscr{O}_B \to f_* \mathscr{O}_A$. By Theorem 1.4.1 it suffices to define morphisms $\psi_g : \mathcal{F}_B(U_g) \to (f_* \mathscr{O}_A)(U_g)$ for each $g \in B$ which commute with the restriction maps on distinguished opens. This means we need a morphism:

$$\psi_g : B_f \longrightarrow \mathscr{O}_A(f^{-1}(U_f))$$

We see that:

$$f^{-1}(U_g) = U_{\phi(g)}$$

so it suffices to define a map:

$$\phi_g : B_g \to A_{\phi(g)}$$

and compose it with the isomorphism $A_{\phi(f)} \to \mathscr{O}_A(U_{\phi(f)})$. We define a morphism $B \to A_{\phi(g)}$ by $b \mapsto \phi(b)/1$. Note that the image of g is a unit under this morphism, hence there exists a unique morphism $\phi_g : B_g \to A_{\phi(g)}$ by

$$\frac{b}{g^k} \longmapsto \frac{\phi(b)}{\phi(g^k)}$$

Note that the isomorphism $A_{\phi(g)} \to \mathscr{O}_A(U_{\phi(f)})$ is given by:

$$\frac{a}{\phi(g)^k} \longmapsto \left(\frac{a}{\phi(g^k)}_{\mathfrak{p}}\right) \in \prod_{\mathfrak{p}:\phi(g)\notin\mathfrak{p}} A_{\mathfrak{p}}$$

so ψ_g is the map:

$$\begin{aligned} \psi_g : \mathcal{F}_B(U_g) &\longrightarrow (f_*\mathscr{O}_A)(U_g) \\ b/g^k &\longmapsto ((\phi(b)/\phi(g^k))_{\mathfrak{p}}) \end{aligned} \tag{1.4.3}$$

It is clear that this map commutes with restrictions on a base, so we have morphism of sheaves:

$$f^\sharp : \mathscr{O}_B \longrightarrow f_*\mathscr{O}_A$$

We now need to check that for all $\mathfrak{p} \in \operatorname{Spec} A$ we have that:

$$f_{\mathfrak{p}} : (\mathscr{O}_B)_{f(\mathfrak{p})} \longrightarrow (\mathscr{O}_A)_{\mathfrak{p}}$$

is a local ring homomorphism. Let $s_{f(\mathfrak{p})} \in (\mathscr{O}_B)_{f(\mathfrak{p})}$, then without loss of generality, we can take $s_{f(\mathfrak{p})} = [U_g, (s_{\mathfrak{q}})]$ where $g \in B$ satisfies $g \notin f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, and $(s_{\mathfrak{q}}) \in \mathscr{O}_B(U_g) \cong B_g$. We than thus write $(s_{\mathfrak{q}}) = \psi_{U_g}^{-1}(b/g^k)$ for some $b/g^k \in B_g$. We thus have that:

$$\begin{aligned} f_{\mathfrak{p}}([U_g, \psi_{U_g}^{-1}(b/g^k)]_{f(\mathfrak{p})}) =& (f_{\mathfrak{p}}^*)([U_g, f_{U_g}^\sharp \circ \psi_{U_g}^{-1}(s_{\mathfrak{q}})]_{f(\mathfrak{p})}) \\ =& (f_{\mathfrak{p}}^*)([U_g, \psi_g(b/g^k)]_{f(\mathfrak{p})}) \\ =& [U_{\phi(g)}, (((\phi(b))/\phi(g^k))_{\mathfrak{p}})]_{\mathfrak{p}} \end{aligned}$$

We then have the following chain of isomorphisms:

$$\begin{aligned} (\mathscr{O}_A)_{\mathfrak{p}} &\longrightarrow \mathcal{F}_{\mathfrak{p}}^A \longrightarrow A_{\mathfrak{p}} \\ \left[U_{\phi(g)}, \left(\frac{\phi(b)}{\phi(g^k)}_{\mathfrak{p}}\right)\right]_{\mathfrak{p}} &\longmapsto \frac{\phi(b)}{\phi(g^k)}_{\mathfrak{p}} \longmapsto \frac{\phi(b)}{\phi(g^k)} \end{aligned}$$

and the same chain of isomorphisms in the opposite direction maps $b/g^k \in B_{f(\mathfrak{p})}$ to $[U_g, \psi_{U_g}^{-1}(b/g^k)]_{f(\mathfrak{p})}$. It thus suffices to check that if $b/g^k \in \mathfrak{m}_{f(\mathfrak{p})} \subset B_{f(\mathfrak{p})}$, then $\phi(b)/\phi(g^k) \in m_{\mathfrak{p}} \subset A_{\mathfrak{p}}$. However, this is clear, as if $b/g^k \in m_{f(\mathfrak{p})}$, we have that $b \in f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, so $\phi(b) \in \mathfrak{p}$, implying that $\phi(b)/\phi(g^k) \in m_{\mathfrak{p}}$. It follows that $f^\sharp$ is a morphism of local rings as desired.

Note that if $\phi : B \to A$ is a ring homomorphism, then both the morphism of sheaves and the maps on stalks are fully determined by the induced maps $B_{\phi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}}$, and $B_g \to A_{\phi(g)}$. Moreover note that the induced morphism $f^\sharp$ satisfies $\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B} = \phi$, where $\psi_{\operatorname{Spec} A/B}$ is the isomorphism $\mathscr{O}_{A/B}(\operatorname{Spec} A/B) \to A/B$. We now wish to prove the following:

Proposition 1.4.5. *If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes, then f and $f^\sharp$ are induced by a unique ring homomorphism $\phi : B \to A$.*

Proof. Let $\phi = \psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}$; we first want to show that the topological map f satisfies:

$$f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} A$. Consider the stalk $(\mathscr{O}_A)_\mathfrak{p} \cong A_\mathfrak{p}$, and the unique maximal ideal of $A_\mathfrak{p}$, $\mathfrak{m}_\mathfrak{p}$. We obtain a field $k'_\mathfrak{p}$ by taking the quotient $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$, where $\mathfrak{m}_\mathfrak{p}$ is the unique maximal ideal of $A_\mathfrak{p}$. Note that we now have a unique map:

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} A}(A) \cong A \longrightarrow A_\mathfrak{p} &\longrightarrow k'_\mathfrak{p} \\ a \longmapsto a/1 &\longmapsto [a/1] \end{aligned}$$

Denote this map by $\operatorname{ev}'_\mathfrak{p}$, then it is clear that $\psi^{-1}_{\operatorname{Spec} A}(\mathfrak{p}) \subset \ker \operatorname{ev}'_\mathfrak{p}$. Moreover, if $a \in \psi_{\operatorname{Spec} A}(\ker \operatorname{ev}'_\mathfrak{p})$, then we have that $a/1 \in \mathfrak{m}_\mathfrak{p}$, hence there must exist some $p \in \mathfrak{p}$, and some $c \in A - \mathfrak{p}$ such that:

$$\frac{a}{1} = \frac{p}{c} \Rightarrow a \cdot c - p = 0$$

This implies that either $a \in \mathfrak{p}$, or $c \in \mathfrak{p}$, however c can't lie in $\mathfrak{p}$ by construction, hence $a \in \mathfrak{p}$. It follows that $\psi^{-1}_{\operatorname{Spec} A}(\mathfrak{p}) = \ker \operatorname{ev}'_\mathfrak{p}$. Similarly, we have that $f(\mathfrak{p})$ is a prime ideal of B, so $\psi^{-1}_{\operatorname{Spec} B}(f(\mathfrak{p})) = \ker \operatorname{ev}'_{f(\mathfrak{p})}$. Via the unique isomorphism of stalks with localizations, and the isomorphism of global sections with the rings A and B, $\mathfrak{p}$ (and $f(\mathfrak{p})$) can be identified with global sections which vanish at $\mathfrak{p}$ (and $f(\mathfrak{p})$).

Now note that:

$$\begin{aligned} \phi^{-1}(\mathfrak{p}) &= (\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1}(\mathfrak{p}) \\ &= (f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1}\left(\psi^{-1}_A(\mathfrak{p})\right) \\ &= (f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})^{-1}\left(\ker \operatorname{ev}'_\mathfrak{p}\right) \end{aligned}$$

It thus suffices to show that:

$$(f^\sharp_{\operatorname{Spec} B})^{-1}(\ker \operatorname{ev}'_\mathfrak{p}) = \ker \operatorname{ev}'_{f(\mathfrak{p})}$$

as then we will have that:

$$\phi^{-1}(\mathfrak{p}) = \psi_{\operatorname{Spec} B}(\ker \operatorname{ev}'_{f(\mathfrak{p})}) = f(\mathfrak{p})$$

Let $s \in \ker \operatorname{ev}'_{f(\mathfrak{p})}$, then we want to show that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}'_\mathfrak{p}$. Let $b \in B$ be the unique element satisfying $\psi_{\operatorname{Spec} B}(s) = b$, then, under the isomorphism $B_{f(\mathfrak{p})} \cong (\mathscr{O}_{\operatorname{Spec} B})_{f(\mathfrak{p})}$, we see that the stalk $s_{f(\mathfrak{p})}$ gets mapped to $b/1 \in B_{f(\mathfrak{p})}$. Since $s \in \ker \operatorname{ev}_{f(\mathfrak{p})}$, it follows that $b/1 \in \mathfrak{m}_{f(\mathfrak{p})}$, so $s_{f(\mathfrak{p})} \in \mathfrak{m}'_{f(\mathfrak{p})} \subset (\mathscr{O}_{\operatorname{Spec} B})_{f(\mathfrak{p})}$. We thus have that $f_\mathfrak{p}(s_{f(\mathfrak{p})}) \in \mathfrak{m}'_\mathfrak{p} \subset (\mathscr{O}_{\operatorname{Spec} A})_\mathfrak{p}$[15], since f is a morphism of local rings. Hence:

$$f_\mathfrak{p}(s_{f_\mathfrak{p}}) = [\operatorname{Spec} A, f^\sharp_{\operatorname{Spec} B}(s)]_\mathfrak{p} \in \mathfrak{m}'_\mathfrak{p}$$

and under the isomorphism $(\mathscr{O}_{\operatorname{Spec} A})_\mathfrak{p} \cong A_\mathfrak{p}$, this gets mapped to $(\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B}(s))/1 = \phi(b)/1$, which must lie in $\mathfrak{m}_\mathfrak{p}$. It follows from our previous argument that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}'_\mathfrak{p}$, implying one inclusion.

Now let $s \in (f^\sharp_{\operatorname{Spec} B})^{-1}(\ker \operatorname{ev}_\mathfrak{p})$, then we have that $f^\sharp_{\operatorname{Spec} B}(s) \in \ker \operatorname{ev}_\mathfrak{p}$, so the stalk $f^\sharp_{\operatorname{Spec} B}(s)_\mathfrak{p}$ lies in $\mathfrak{m}'_\mathfrak{p}$, hence $f_\mathfrak{p}(s_{f(\mathfrak{p})}) \in \mathfrak{m}'_\mathfrak{p}$. Now suppose that $s_{f(\mathfrak{p})} \notin \mathfrak{m}'_{f(\mathfrak{p})}$, then we have that the corresponding element $b/1 \in B_{f(\mathfrak{p})}$ does not lie in $\mathfrak{m}_{f(\mathfrak{p})}$, but the element $\phi(b)/1 \in A_\mathfrak{p}$ corresponding to $f_\mathfrak{p}(s_{f(\mathfrak{p})})$ lies in $\mathfrak{m}_\mathfrak{p}$. Since $b/1 \notin m_{f(\mathfrak{p})}$, we know that $b \notin f(\mathfrak{p})$, hence we have that $b/1$ is a unit in $B_{f(\mathfrak{p})}$, and so $\phi(b)/1$ must be a unit in $A_\mathfrak{p}$ as well. It follows that $\mathfrak{m}_\mathfrak{p}$ contains the identity, so it is not maximal yielding a contradiction. We thus have that $s_{f(\mathfrak{p})} \in \mathfrak{m}'_{f(\mathfrak{p})}$, hence $s \in \ker \operatorname{ev}'_{f(\mathfrak{p})}$ as desired, implying the claim.

We now need to check that ϕ induces the same map sheaves as $f^\sharp : \mathscr{O}_{\operatorname{Spec} B} \to f_*\mathscr{O}_{\operatorname{Spec} A}$. First note that we have morphisms:

$$f^\sharp_{U_g} \circ \psi^{-1}_{U_g} : \mathcal{F}_B(U_g) \longrightarrow f_*\mathscr{O}_{\operatorname{Spec} A}(U_g) = \mathscr{O}_{\operatorname{Spec} A}(U_{\phi(g)})$$

for each distinguished $U_g \subset \operatorname{Spec} B$, which trivially make the diagram in Theorem 1.4.1 commute. Note that the equality follows from the fact the topological map is equal to taking preimages by ϕ. It follows

[15]The primed maximal ideals are the unique maximal ideal in the stalk.

that $f^\sharp$ is the unique morphism induced by these morphisms on a base, hence we need only show that ϕ induces the same map on distinguished opens. Note that for each $g \in B$, we have a map $\phi_g : B_g \to A_{\phi(g)}$ by:

$$\frac{b}{g^k} \longmapsto \frac{\phi(b)}{\phi(g^k)} = \frac{\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}(b)}{\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}(g^k)}$$

Since $\psi_{U_{\phi(g)}} : \mathscr{O}_{\operatorname{Spec} A}(U_{\phi(g)}) \to A_{\phi(g)}$ is an isomorphism, it thus suffices to show that:

$$\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}(b/g^k) = \frac{\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}(b)}{\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}(g^k)}$$

Note that ϕ_g is the unique map which satisfies $\phi_g \circ \theta^B_g = \phi'$, where ϕ' is the map $b \mapsto \phi(b)/1$, and θ^B_g is the restriction map, which is simply localization. By the universal property of localization, it then suffices to check that

$$(\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}) \circ \theta^B_g = \phi'$$

Well, note that isomorphisms ψ_U, and their inverses trivially commute with restrictions of a sheaf on a base, hence we have that:

$$\begin{aligned}(\psi_{U_{\phi(g)}} \circ f^\sharp_{U_g} \circ \psi^{-1}_{U_g}) \circ \theta^B_g(b) =& \theta^A_{\phi(g)} \circ (\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B})(b)\\ =& \frac{\psi_{\operatorname{Spec} A} \circ f^\sharp_{\operatorname{Spec} B} \circ \psi^{-1}_{\operatorname{Spec} B}(b)}{1}\\ =& \phi'(b)\end{aligned}$$

implying the claim. □

We briefly mention the definition of an (anti)-equivalence of categories.

Definition 1.4.4. Let C and D be categories, then C is **(anti) equivalent** to D if there is a a (contravariant) covariant functor $\mathcal{F} : C \to D$, such that that for every objects X and Y of C there is a bijection induced by $\mathcal{F}$[16]:

$$\operatorname{Hom}_C(X, Y) \to \operatorname{Hom}_D(\mathcal{F}(X), \mathcal{F}(Y))$$

and for every object Z of D there exists an object X of C such that $F(X)$ is isomorphic to Z, i.e. $\mathcal{F}$ is **essentially surjective.**

We end with the following corollary:

Corollary 1.4.3. *The category of commutative rings is anti equivalent to the category of affine schemes.*

Proof. Note that we have a contravariant functor Spec : Ring $\to$ Aff given by $A \mapsto (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$. We see that this if $(X, \mathscr{O}_X)$ is an affine scheme, then $(X, \mathscr{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some commutative ring A, implying that Spec is essentially surjective.

We need to show that the induced map:

$$\begin{aligned}\operatorname{Hom}_{\text{Ring}}(A, B) &\longrightarrow \operatorname{Hom}_{\text{Aff}}(\operatorname{Spec} B, \operatorname{Spec} A)\\ \phi &\longmapsto (f, f^\sharp)\end{aligned}$$

where $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, and $f^\sharp$ is the morphism of sheaves induced by the sheaf on a base morphisms given by (1.9)[17], is a bijection. Note that it is clearly injective, as if $\phi = \psi$, then $\phi^{-1}(\mathfrak{p}) = \psi^{-1}(\mathfrak{p})$, and for all $a/g^k \in \mathcal{F}_A(U_g)$ we have that:

$$\frac{\phi(a)}{\phi(g^k)} = \frac{\psi(a)}{\psi(g^k)}$$

[16] If $\mathcal{F}$ is contravariant then clearly the order switches.

[17] The domains and codomains have switched, but this is just a result of our choice of ϕ.

so the induced morphisms are equivalent as well. It follows that the morphisms of affine schemes are then equal so the map is injective.

Suppose that $(f, f^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ is morphism of affine schemes. Then it follows from Proposition 1.4.5 that the $\phi = \psi_{\operatorname{Spec} B} \circ f^\sharp_{\operatorname{Spec} A} \circ \psi^{-1}_{\operatorname{Spec} A}$ is a ring homomorphism that maps to $(f, f^\sharp)$, so Spec is anti equivalence of categories as desired. $\square$

Note that clearly if $\phi : A \to B$ is an isomorphism then $(f, f^\sharp)$ is an isomorphism and vice versa.

Schemes

2.1 Definition and Examples

We are now in a position to define a scheme in full generality.

Definition 2.1.1. Let $(X, \mathscr{O}_X)$ be a locally ringed space, then $(X, \mathscr{O}_X)$ is an **affine scheme** if $(X, \mathscr{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some commutative ring A. $(X, \mathscr{O}_X)$ is a scheme if every point $x \in X$ has an open neighborhood U of x such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. We call such open sets **affine opens**, and the topology on X is called the **Zariski topology**.

Note that a morphism of schemes is simply a morphism of locally ringed spaces, and hence an isomorphism of schemes is an isomorphism of locally ringed spaces. We denote the category of schemes by Sch, and use $\operatorname{Hom}(X, Y)$ and $\operatorname{Hom}_{\mathrm{Sch}}(X, Y)$ interchangeably to denote the hom set of scheme morphisms.

Example 2.1.1. It is clear that affine schemes are schemes essentially by definition. Indeed, if $(X, \mathscr{O}_X)$ is an affine scheme, then every point has a neighborhood isomorphic to an affine scheme, namely X itself. In this example, we show explicitly that distinguished opens are also open neighborhoods isomorphic to affine schemes. Let A be a commutative ring, then for every element $\mathfrak{p} \in \operatorname{Spec} A$, we need to show that there is open neighborhood U of $\mathfrak{p}$ such that $(U, \mathscr{O}_{\operatorname{Spec} A}|_U)$ is an affine scheme. Let $g \notin \mathfrak{p}$, then U_g is an open set containing $\mathfrak{p}$; we claim that $(U_g, \mathscr{O}_{\operatorname{Spec} A}|_U)$ is isomorphic to $(\operatorname{Spec} A_g, \mathscr{O}_{\operatorname{Spec} A_g})$. We already have from Proposition 1.1.3 that there exists a homeomorphism $\eta : U_g \to \operatorname{Spec} A_g$ given by:

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{g^k} \in A_g : p \in \mathfrak{p}, k \geq 0 \right\}$$

so we want to describe an isomorphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec} A_g} \longrightarrow \eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})$$

First recall that every distinguished open $U_{f/g^k} \subset \operatorname{Spec} A_g$ is equal to $U_{f/1} \subset \operatorname{Spec} A_f$, so it suffices to define a morphism on the set of distinguished opens of the form $U_{f/1}$ for some $f/1 \in A_g$. We see that:

$$\eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})(U_{f/1}) = \mathscr{O}_{\operatorname{Spec} A}(\eta^{-1}(U_{f/1})) = \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$$

So it suffices to prescribe maps $\phi_{f/1} : (A_g)_{f/1} \to A_{fg}$ and compose with the isomorphism $A_{fg} \to \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$. We set $\phi_{f/1}$ to be the unique isomorphism $(A_g)_{f/1} \to A_{fg}$, so we then obtain a set of isomorphisms $\psi_{f/1} : (A_f)_{f/1} \to \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$. By Theorem 1.4.1 there thus exists a unique morphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec} A_g} \longrightarrow \eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g})$$

We need to show that $\eta^\sharp$ is an isomorphism (and thus a morphism of locally ringed spaces), and it suffices to check by Corollary 1.4.2 that $\eta^\sharp$ is an isomorphism on distinguished open sets of $\operatorname{Spec} A_g$. In particular, since $\eta_*(\mathscr{O}_{\operatorname{Spec} A}|_{U_g}(U_{f/1})) = \mathscr{O}_{\operatorname{Spec} A}(U_{fg})$, we have the following diagram:

$$\begin{tikzcd}
\mathscr{O}_{\operatorname{Spec} A_g}(U_{f/1}) \arrow[r, "\eta^\sharp_{U_{f/1}}"] \arrow[d, "\psi_{U_{f/1}}"] & \mathscr{O}_{\operatorname{Spec} A}(U_{fg}) \\
(A_g)_{f/1} \arrow[r, "\phi_{f/1}"] & A_{fg} \arrow[u, "\psi^{-1}_{U_{fg}}"]
\end{tikzcd}$$

so $\eta^\sharp_{U_{f/1}}$ is the composition of isomorphisms, and is thus an isomorphism, implying that $\eta^\sharp$ is indeed a natural isomorphism.

The following lemmas will prove useful in the future:

Lemma 2.1.1. *Let $(X, \mathscr{O}_X)$ be a scheme, then the following hold:*

a) The set of open affines form a basis for the topology on X.

b) The sheaf on a base defined by $\mathcal{F}^X(U) = \mathscr{O}_X(U)$ induces a structure sheaf on X which is isomorphic to $\mathscr{O}_X$.

c) For every two open affines U, V, we $U \cap V$ can be covered by open affines which are simultaneously distinguished opens in both U and V.

Proof. We begin with *a*). It is clear that the set of affine opens cover X, so we need only check that any open set U can be written as the union of affine opens. For each $x \in U$ we have an affine open V_x, the collection $\{V_x\}$ then defines a cover of U by:

$$\{V_x \cap U\}$$

Equipped with the subspace topology, we have that $V_x \cap U$ is open in $V_x \cong \operatorname{Spec} A_x$ for some ring A_x. It follows that since $\operatorname{Spec} A_x$ can be covered by distinguished opens $U_{g^i_x}$, and each $U_{g^i_x}$ is itself an affine open of $\operatorname{Spec} A_x$, that $V_x \cap U$ can be covered by affine opens V_{x^i} isomorphic to $(U_{g^i_x}, \operatorname{Spec} A_x|_{U_{g^i_x}})$. We thus have that:

$$U = \bigcup_{x \in U} V_x \cap U = \bigcup_{x \in U} \bigcup_i V^i_x$$

so U can be covered by affine opens, as desired.

We see that *b*) follows from *a*) by Proposition 1.4.2.

For *c*), Let $U \cong \operatorname{Spec} A$, $V \cong \operatorname{Spec} B$, and $x \in U \cap V$. The isomorphisms $f : U \to \operatorname{Spec} A$ and $g : V \to \operatorname{Spec} B$ induce an isomorphism $h : f(U \cap V) \subset \operatorname{Spec} A \to g(U \cap V) \subset \operatorname{Spec} B$ such that $h \circ f|_{U\cap V} = g|_{U\cap V}$. So when we say that $U \cap V$ can be covered by open affines which are simultaneously distinguished opens in $\operatorname{Spec} A$ and $\operatorname{Spec} B$, we mean that that there exists an open cover $\{W_i\}$ of $U \cap V$, such that $f(W_i)$ are distinguished opens in $\operatorname{Spec} A$, and $g(W_i)$ is a distinguished open in $\operatorname{Spec} B$.[18]

It suffices to show that every $x \in U \cap V$ has such a neighborhood. Since $x \in U \cap V$, we have that $f(x) = \mathfrak{p} \in \operatorname{Spec} A$, $g(x) = \mathfrak{q} \in \operatorname{Spec} B$, and $h(\mathfrak{p}) = \mathfrak{q}$. Since $f(U \cap V)$ is an open set in $\operatorname{Spec} A$, there is a distinguished open $U_a = \operatorname{Spec} A_a$ such that $U_a \subset f(U \cap V)$, and $\mathfrak{p} \in U_f$. We see that $h(U_a)$ is an affine open subscheme of $g(U \cap V) \subset \operatorname{Spec} B$, hence there is an open embedding $\imath_a : \operatorname{Spec} A_a \hookrightarrow \operatorname{Spec} B$, such that $\imath_a = h|_{U_a}$. In particular, there is a distinguished open set $U_b \subset \imath_a(\operatorname{Spec} A_a)$ determined by some $b \in B$. Let

$$\phi : \mathscr{O}_{\operatorname{Spec} B}(\imath_a(\operatorname{Spec} A_a)) \to \mathscr{O}_{\operatorname{Spec} A_a}(\operatorname{Spec} A_a) \cong A_a$$

Let $U_{\phi(b)} \subset \operatorname{Spec} A_a$ be the distinguished open associated to $\phi(b)$, then we claim that:

$$\imath_a(U_{\phi(b)}) = U_b \tag{2.1.1}$$

Indeed for any $b \in B$ we have:

$$\begin{aligned}
\imath_a(U_{\phi(b)}) =& \{\imath_a(\mathfrak{p}) : \mathfrak{p} \in U_{\phi(b)}\} \\
=& \{\imath_a(\mathfrak{p}) : \phi(b) \notin \mathfrak{p}, \mathfrak{p} \in \operatorname{Spec} A_a\} \\
=& \{\imath_a(\mathfrak{p}) : b \notin \phi^{-1}(p), \mathfrak{p} \in \operatorname{Spec} A_a\} \\
=& U_b \cap \imath_a(\operatorname{Spec} A_a)
\end{aligned}$$

however since $U_b \subset \imath_a(\operatorname{Spec} A_a)$ we have obtained the equality (2.1.1) as desired.

[18] In the future, as we get more comfortable with the local nature of schemes, we will gradually suppress these isomorphisms for ease of notation, and simply work with affine opens as $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$.

Let $\phi(b) = c/a^n$, then by Lemma 1.1.4, $U_{\phi(b)}$ identified as a subset of $\operatorname{Spec} A$ is the distinguished open set $U_{c\cdot a} \subset \operatorname{Spec} A$. Since $\imath_a = h|_{h(U_a)}$, it follows that:

$$h(U_{c\cdot a}) = \imath_a(U_{\phi(b)}) = U_b$$

hence setting $W = f^{-1}(U_{c\cdot a}) \subset U \cap V$ is an affine which can simultaneously be identified with distinguished opens in both $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ as desired.

□

Lemma 2.1.2. *Let $(X, \mathscr{O}_X)$ be a scheme, then $(U, \mathscr{O}_X|_U)$ is scheme equipped with an open embedding $\iota : U \to X$.*

Proof. If we can show that the locally ringed space $(U, \mathscr{O}_X|_U)$ is a scheme, then the claim follows from Lemma 1.3.4. We need only show that for each $x \in U$ there is an open neighborhood V_x of x such that:

$$(V_x, \mathscr{O}_X|_U|_{V_x}) = (V_x, \mathscr{O}_X|_{V_x}) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$$

However, we have that by Lemma 2.1.1 U can be written as the union of affine opens, all of which will be open in the subspace topology of U. It follows that every $x \in U$ must lie in one of these affine opens, so by the definition of an affine open the claim follows. □

Note that by Lemma 2.1.1 and Corollary 1.4.2, it suffices to define morphisms between schemes on the basis affine opens.

Definition 2.1.2. Let X be a scheme, and U an open subset of X. The induced scheme $(U, \mathscr{O}_X|_U)$ is then called a **open subscheme**.

Lemma 2.1.3. *Let $f : X \to Y$ be a morphism of locally ringed spaces, then $(f, f^\sharp)$ is an isomorphism if and only if f is a homeomorphism and the stalk map $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an isomorphism for all $x \in X$.*

Proof. Suppose that $(f, f^\sharp)$ is an isomorphism of locally ringed spaces, then by definition f is a homeomorphism, and $f^\sharp$ is an isomorphism of sheaves. It follows that:

$$f^\sharp_{f(x)} : (\mathscr{O}_Y)_y \to (f_*\mathscr{O}_X)_y$$

is an isomorphism for all $y \in Y$. Since f is a homeomorphism, it suffices to check that:

$$(f_*)_x : (f_*\mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$$

is an isomorphism for all $x \in X$. We first show that $(f_*)_x$ is injective, suppose that $[U, s]_{f(x)}$ satisfies $[f^{-1}(U), s]_x = 0$, then there exists some open neighborhood of x $V \subset f^{-1}(U)$ such that $s|_V = 0$. Since f is a homeomorphism, we have that $f(V)$ is an open subset of U, and satisfies:

$$s|_{f(V)} = s|_{f^{-1}(f(V))} = s|_V = 0$$

so $[U, s]_{f(x)} = 0$. Now suppose that $[V, s]_x \in (\mathscr{O}_X)_x$, then we see that $f(V)$ is an open subset of Y, and thus $[f(V), s]_{f(x)} \in (f_*\mathscr{O}_X)_x$. It is then clear that $(f_*)_x([f(V), s]_{f(x)}) = [V, s]_x$ so $(f_*)_x$ is an isomorphism as desired. Since $f_x = (f_*)_x \circ f^\sharp_{f(x)}$, we have that f_x must be an isomorphism for all $x \in X$.

Now suppose that f is a homeomorphism, and $f_x : (\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is an isomorphism for all $x \in X$. It suffices to check that $f^\sharp_y$ is an isomorphism for all $y \in Y$. Let $y \in Y$, then since f is a homeomorphism, there is a unique element $x \in X$ such that $x = f^{-1}(y)$. We have that $f_{f^{-1}(y)}$ is an isomorphism, and is equal to $(f_*)_{f^{-1}(y)} \circ f^\sharp_y$, however by the proceeding paragraph, $(f_*)_{f^{-1}(y)}$ is an isomorphism, so we see that $f^\sharp_y = (f_*)^{-1}_{f^{-1}(y)} \circ f_{f^{-1}(y)}$, hence $f^\sharp_y$ is an isomorphism for al $y \in Y$. It follows that $f^\sharp$ is an isomorphism of sheaves so $(f, f^\sharp)$ is an isomorphism of locally ringed spaces. □

With the lemma above, we now have the following result, which is important for sanity reasons.

Corollary 2.1.1. *Let $f : X \to Y$ be a homeomorphism, $\mathscr{F}$ a sheaf on Y, and $\mathscr{G}$ a sheaf on X. Then a morphism $F : \mathscr{F} \to f_*\mathscr{G}$ is an isomorphism if and only if the unique map $\hat{F} : f^{-1}\mathscr{F} \to \mathscr{G}$ is an isomorphism. In other words, the isomorphism in Theorem 1.3.1 preserves isomorphisms.*

Proof. Suppose that $F : \mathscr{F} \to f_*\mathscr{G}$ is an isomorphism, then we have that the stalk map $F_y : \mathscr{F}_y \to (f_*\mathscr{G})_y$ is an isomorphism for all $y \in Y$. It suffices to check that the stalk map $\hat{F}_x : (f^{-1}\mathscr{F})_x \to \mathscr{G}_x$ is an isomorphism. By Corollary 1.3.3 we have that:

$$\hat{F}_x \circ \mathrm{sh}_x \circ (f_p^{-1})_x = (f_*)_x \circ F_{f(x)} \tag{2.1.2}$$

and by the preceding we have that $(f_*)_x$ is an isomorphism, and by hypothesis $F_{f(x)}$ is an isomorphism for all $f(x)$. It follows that $\hat{F}_x$ must be an isomorphism for all $x \in X$, as both sh_x and $(f_p^{-1})_x$ are isomorphisms for all $x \in X$, hence $\hat{F}$ is an isomorphism.

Conversely, suppose that $\hat{F}$ is an isomorphism, then $\hat{F}_x$ is an isomorphism for all $x \in X$. Then (2.1) implies that $F_{f(x)}$ is an isomorphism for all $f(x)$, as $(f_*)_x$ is an isomorphism and the composition on the left is a composition of isomorphisms. Since f is a homeomorphism and thus surjective, it follows that F_y is an isomorphism for all $y \in Y$, hence F is an isomorphism of sheaves as desired. □

As of this moment, we have two example of schemes, namely given a commutative ring A, we can construct an affine scheme, and given a scheme X we can take any open subset of X and obtain an open subscheme. We would like to be able to construct more examples, hence the following gluing proposition:

Theorem 2.1.1. *Let $\{X_i\}$ be a family of schemes, and suppose for each $i \neq j$ there exists an open subscheme $U_{ij} \subset X_i$. Suppose also that for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ satisfying $\phi_{ij}^{-1} = \phi_{ji}$, $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ for all i, j and k. Then, there exists a scheme X, together with morphisms $\psi_i : X_i \to X$ such that each ψ_i is an open embedding, $\psi_i(X_i)$ cover X, $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.*

Proof. We first begin by constructing the topological space X. As a set define X to be:

$$X = \left(\coprod_i X_i\right) / \sim$$

where the equivalence relation $\sim$ is given by $x_i \in X_i$ and $x_j \in X_j$ are equivalent if and and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $\phi_{ij}(x_i) = x_j$. We check that this is an equivalence relation. Note that we have $x_i \sim x_i$, as $x_i \in U_{ii} = X_i$, and $\phi_{ii} = \mathrm{Id}_{X_i}$. The relation is symmetric, as if $x_i \sim x_j$, then we have $\phi_{ij}(x_i) = x_j$, so $\phi_{ji}(x_j) = x_i$, hence $x_j \sim x_i$. Now suppose that $x_i \sim x_j$ and $x_j \sim x_k$, then $x_i \in U_{ij}$, $x_j \in U_{ji}$, $x_j \in U_{jk}$, and $x_k \in U_{kj}$. It follows that $\phi_{ij}(x_i) = x_j \in U_{ji} \cap U_{jk}$, so $x_i \in U_i j \cap U_{ik}$, and moreover that $x_k \in U_{kj} \cap U_{ki}$. We also see that $\phi_{jk} \circ \phi_{ij}(x_i) = \phi_{jk}(x_j) = x_k$, so we have that $\phi_{ik}(x_i) = x_k$, implying that $x_i \sim x_k$ as desired.

Note that since $\phi_{ii} = \mathrm{Id}$, no two elements $x_i, y_i \in X_i$ such that $x_i \neq y_i$ can be equivalent to one another. We thus have natural injections $\psi_i : X_i \to X$ given by $x_i \mapsto [x_i] \in X$, and thus ψ_i is a bijection onto it's image. We define a topology on X by $U \subset X$ is open if and only $\psi_i^{-1}(U) \subset X_i$ is open for all i. We check that this is a topology; note that the empty set is vacuously open, and that X is open as $\psi_i^{-1}(X) = X_i$. Moreover, arbitrary unions of open sets are open as:

$$\psi_i^{-1}\left(\bigcup_j U_j\right) = \bigcup_j \psi_i^{-1}(U_j)$$

which is the union of open sets in X_i by hypothesis, and so the original set is open in X. For finite intersections we have that:

$$\psi_i^{-1}(U \cap V) = \psi_i^{-1}(U) \cap \psi_i^{-1}(V)$$

which is open in X_i, so $U \cap V$ is open in X, so this assignment defines a topology on X.

Clearly, by the construction of the topology on X, we have that each $\psi_i : X_i \to X$ is a continuous map. We want to show that $\psi_i(X_i)$ cover X, and that each ψ_i satisfies $\psi_i(U_{ij}) = \psi(X_i) \cap \psi_j(X_j)$ and that $\psi_i = \psi_j \circ \phi_{ij}$. The first statement is clear, indeed let $x \in X$, then by the definition of X, x is an equivalence class with a class representative $x_i \in X_i$, so $\psi_i(x_i) = [x_i] = x$. Now let $U_{ij} \subset X_i$, and suppose that $x \in \psi_i(U_{ij})$, then x is an equivalence class with class representative $x_i \in X_i$. Since $x_i \in U_{ij}$, and $\phi_{ij} : U_{ij} \to U_{ji}$ is a homeomorphism, there must be a unique element $x_j \in U_{ji}$ such that $\phi_{ij}(x_i) = x_j$, hence $[x_i] = x = [x_j]$. It follows that $x \in \psi_j(X_j)$ as well, so $\psi_i(U_{ij}) \subset \psi_i(X_i) \cap \psi_j(X_j)$. Now suppose that $x \in \psi_i(X_i) \cap \psi_j(X_j)$, then $x = [x_i] = [x_j]$ for some $x_i \in X_i$ and $x_j \in X_j$. It follows that $x_i \sim x_j$, so $x_i \in U_{ij}$ and $x_j \in U_{ji}$ such that $\phi_{ij}(x_i) = x_j$, hence $[x_i] \in \psi_i(U_{ij})$, and we have that $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ as desired. Finally, let $x_i \in U_{ij}$, then $\psi_i(x_i) = [x_i]$, and $\psi_j \circ \phi_{ij}(x_i) = [\phi_{ij}(x_i)]$, however $\phi_{ij}(x_i) \in U_{ji}$, and we vacuously have that $\phi_{ij}(x_i) = \phi_{ij}(x_i)$, hence $\psi_i = \psi_j \circ \phi_{ij}$.

To show that $\psi_i : X_i \to \psi_i(X_i)$ is a homeomorphism, we first note that ψ_i is an injective open map. Indeed, let $x_i, y_i \in X_i$, such that $[x_i] = [y_i]$, implying that $x_i \sim y_i$, but x_i and y_i both lie in X_i, so we must have that $\phi_{ii}(x_i) = y_i$ implying that $x_i = y_i$. Now let $U \subset X_i$ be an open set, we want to show that $\psi_i(U)$ is open in X. It is clear that since ψ_i is injective we have that $\psi_i^{-1}(\psi_i(U)) = U$. Let $j \neq i$, then we want to show that $\psi_j^{-1}(\psi_i(U))$ is open in X_j. If $U \cap U_{ij}$ is empty then we see that there is no $x_j \in X_j$ such that $\psi_j(x_j) \in \psi_i(U)$, hence $\psi_j^{-1}(\psi_i(U)) = \emptyset$ and is thus open. Suppose that $U \cap U_{ij}$ is not empty, then we claim that:

$$\psi_j^{-1}(\psi_i(U)) = \phi_{ij}(U \cap U_{ij}) \tag{2.1.3}$$

which is an open subset of X_j as $U \cap U_{ij} \subset U_{ij}$ is open in the subspace topology, and ϕ_{ij} is a homeomorphism of open subspaces, so $\phi_{ij}(U \cap U_{ij}) \subset U_{ji}$ is open in the subspace topology, and thus open in X_j. Let $x_j \in \psi_j^{-1}(\psi_i(U))$, then we have that $\psi_j(x_j) = [x_j] \in \psi_i(U)$, hence there exists an $x_i \in U$ such that $[x_j] = [x_i]$ implying that $x_i \in U_{ij}$, $x_j \in U_{ji}$ and $\phi_{ij}(x_i) = x_j$. It follows that $x_i \in U \cap U_{ij}$, and that $x_j = \phi_{ij}(x_i)$ so $x_j \in \phi_{ij}(U \cap U_{ij})$. Now suppose that $x_j \in \phi_{ij}(U \cap U_{ij}) \subset U_{ji}$, then there exists a unique $x_i \in U \cap U_{ij}$ such that $\phi_{ij}(x_i) = x_j$. We see that $\psi_j(x_j) = [x_j]$, and that $[x_j] = [x_i]$ as $x_j \in U_{ji}$, $x_i \in U_{ij}$ and $\phi_{ij}(x_i) = x_j$. Since $x_i \in U \cap U_{ij} \subset U$, we have that $\psi_j(x_j) = \psi_i(x_i) \in \psi_i(U)$, hence $x_j \in \psi_j^{-1}(\psi_i(U))$, so (2.1.2) holds. It follows that $\psi_i(U)$ is thus open in X, and thus ψ_i is an open injective map, and is a bijection onto it's image, and thus a homeomorphism.

Now, denote $\psi_i(X_i)$ by $\mathscr{X}_i$, we want to put the structure of a scheme on $\mathscr{X}_i$. Note that each X_i is a scheme, hence comes equipped with a sheaf of local rings $\mathscr{O}_{X_i}$; we define the sheaf $\mathscr{O}_{\mathscr{X}_i}$ by:

$$\mathscr{O}_{\mathscr{X}_i} = \psi_{i*}\mathscr{O}_{X_i}$$

Since the $\psi_i : X_i \to \mathscr{X}_i$ is a homeomorphism, note that $(\psi_{i*})_x : (\mathscr{O}_{\mathscr{X}_i})_{\psi_i(x)} \to (\mathscr{O}_{\mathscr{X}_i})_x$ is an isomorphism for $x_i \in X_i$. It follows that the stalk of $\mathscr{O}_{\mathscr{X}_i}$ is a local ring, so $\mathscr{O}_{\mathscr{X}_i}$ is a locally ringed space. We now check that $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ is a scheme, let $x \in \mathscr{X}_i$, then there exists an open neighborhood U of $\psi_i^{-1}(x_i) \in X_i$ such that $(U, \mathscr{O}_{X_i}|_U) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A. It thus suffices to check that $(\psi_i(U), \mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})$ is isomorphism to $(U, \mathscr{O}_{X_i})$. We first note, that since ψ_i is a homeomorphism, that $\psi_i^{-1} : \psi_i(U) \to U$ is a homeomorphism. So we need only define a morphism $(\psi_i^{-1})^\sharp : \mathscr{O}_{X_i}|_U \to (\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})$. Let $V \subset U$ be an open set, then:

$$\mathscr{O}_{X_i}|_U(V) = \mathscr{O}_{X_i}(V)$$

while:

$$(\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(V) = (\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(\psi_i(V))$$

since $\psi_i(V) \subset \psi_i(U)$ we have that:

$$\begin{aligned}(\psi_i^{-1})_*(\mathscr{O}_{\mathscr{X}_i}|_{\psi_i(U)})(V) =& (\mathscr{O}_{\mathscr{X}_i})(\psi_i(V)) \\ =& (\psi_{i*}\mathscr{O}_{X_i})(\psi_i(V)) \\ =& \mathscr{O}_{X_i}(\psi_i^{-1}(\psi_i(V))) \\ =& \mathscr{O}_{X_i}(V)\end{aligned}$$

We thus define $(\psi_i^{-1})^\sharp_V$ to be the identity map; it is clear that this commutes with restrictions, hence this assignment defines a natural transformation, and since $(\psi_i^{-1})^\sharp_V$ is the identity for all $V \subset U$ we have that $(\psi_i^{-1})^\sharp$ is an isomorphism as desired. It follows that $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ is a scheme as desired.

Now we have that $\{\mathscr{X}_i\}$ is an open cover of of X, and moreover that $\psi_i : X_i \to \mathscr{X}_i$ is an isomorphism of schemes for each i, by applying the the same argument above to $U = X_i$. If we can show that there exist isomorphisms $\beta_{ij} : \mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j} \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}$, which satisfy the cocycle condition then we will obtain a sheaf on X such that $\mathscr{O}_X|_{\mathscr{X}_i} \cong \mathscr{O}_{\mathscr{X}_i}$ by Theorem 1.2.2. Note that we have:

$$\mathscr{X}_i \cap \mathscr{X}_j = \psi_i(X_i) \cap \psi_j(X_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$$

Furthermore, since $(X_i, \mathscr{O}_{X_i}) \cong (\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ via $(\psi_i^{-1}, (\psi_i^{-1})^\sharp)$, we have an inverse map given by $(\psi_i, \psi_i^\sharp)$, where $\psi_i^\sharp : \mathscr{O}_{\mathscr{X}_i} \to \psi_{i*}\mathscr{O}_{X_i}$, so we have map $(\psi_i^\sharp)_V : \mathscr{O}_{\mathscr{X}_i}(V) \to \mathscr{O}_{X_i}(\psi_i^{-1}(V))$. Now note that since $V \subset \psi_i(U_{ij})$, $\psi_i^{-1}(V) \subset U_{ij}$, so:

$$\mathscr{O}_{X_i}(\psi_i^{-1}(V)) = \mathscr{O}_{U_{ij}}(\psi_i^{-1}(V))$$

and we have an isomorphism $\phi_{ji}^\sharp : \mathscr{O}_{U_{ij}} \to \phi_{ji*}\mathscr{O}_{U_{ji}}$. We have that:

$$\phi_{ji}^{-1}(\psi_i^{-1}(V)) = (\psi_i \circ \phi_{ji})^{-1}(V) = \psi_j^{-1}(V)$$

so we have an isomorphism:

$$(\phi_{ji}^\sharp)_{\psi_i^{-1}(V)} : \mathscr{O}_{U_{ij}}(\psi_i^{-1}(V)) \longrightarrow \mathscr{O}_{U_{ji}}(\psi_j^{-1}(V)) = \mathscr{O}_{X_j}(\psi_j^{-1}(V))$$

Finally we have our isomorphism $(\psi_j^{-1})^\sharp : \mathscr{O}_{X_j} \to (\psi_j^{-1})_*\mathscr{O}_{\mathscr{X}_j}$, and note that:

$$(\psi_j^{-1})_*\mathscr{O}_{\mathscr{X}_j}(\psi_j^{-1}(V)) = \mathscr{O}_{\mathscr{X}_j}(V)$$

thus we have that the composition:

$$(\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V$$

is an isomorphism:

$$\mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j}(V) \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}(V)$$

We define β_{ij} on open sets $V \subset \mathscr{X}_i \cap \mathscr{X}_j$ as this composition. We check that this commutes with restriction maps, let $W \subset V$, then we see that:

$$(\psi_i^\sharp)_W \circ \theta^V_W = \theta^V_W \circ (\psi_i^\sharp)_W$$

On $\psi_*\mathscr{O}_{X_i}$, the restriction maps are given by $\theta^V_W = \theta^{\psi_i^{-1}(V)}_{\psi_i^{-1}(W)}$, so we see that:

$$(\phi_{ji}^\sharp)_{\psi_i^{-1}(W)} \circ \theta^V_W = \theta^{\psi_i^{-1}(V)}_{\psi_i^{-1}(W)} \circ (\phi_{ji}^\sharp)_{\psi_i^{-1}(V)}$$

Now on $\phi_{ji*}\mathscr{O}_{U_{ji}}$ the restriction maps are given by:

$$\theta^{\psi_i^{-1}(V)}_{\psi_i^{-1}(W)} = \theta^{\psi_j^{-1}(V)}_{\psi_j^{-1}(W)}$$

as $\psi_i \circ \phi_{ji} = \psi_j$ on U_{ji}, hence:

$$\begin{aligned}(\psi_j^{-1})^\sharp_{\psi_j^{-1}(W)} \circ \theta^{\psi_i^{-1}(V)}_{\psi_i^{-1}(W)} &= (\psi_j^{-1})^\sharp_{\psi_j^{-1}(W)} \circ \theta^{\psi_j^{-1}(V)}_{\psi_j^{-1}(W)} \\ &= \theta^{\psi_j^{-1}(V)}_{\psi_j^{-1}(W)} \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)}\end{aligned}$$

Finally, on $(\psi_j^{-1})_*\mathscr{O}_{\mathscr{X}_j}$, the restriction maps are given by:

$$\theta^{\psi_j^{-1}(V)}_{\psi_j^{-1}(W)} = \theta^V_W$$

so since $W \subset V \subset \mathscr{X}_i \cap \mathscr{X}_j$, we have that:

$$(\beta_{ij})_W \circ \theta^V_W = \theta^V_W \circ (\beta_{ij})_V$$

so $\beta_{ij} : \mathscr{O}_{\mathscr{X}_i}|_{\mathscr{X}_i \cap \mathscr{X}_j} \to \mathscr{O}_{\mathscr{X}_j}|_{\mathscr{X}_i \cap \mathscr{X}_j}$ is an isomorphism of sheaves as desired. It is clear that $\beta_{ii} = \mathrm{Id}$, so we want to check that $\beta_{ik} = \beta_{jk} \circ \beta ij$ on $\mathscr{X}_i \cap \mathscr{X}_j \cap \mathscr{X}_k$. However this is essentially a tautology, as on all open set $V \subset \mathscr{X}_i \cap \mathscr{X}_j \cap \mathscr{X}_k$:

$$\begin{aligned}
(\beta_{jk} \circ \beta_{ij})_V =& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi^\sharp_{kj})_{\psi_j^{-1}(V)} \circ (\psi_j^\sharp)_V \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\
=& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi^\sharp_{kj})_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\
=& (\psi_k^{-1})^\sharp_{\psi_k^{-1}(V)} \circ (\phi^\sharp_{ki})_{\psi_i^{-1}(V)} \circ (\psi_i^\sharp)_V \\
=& \beta_{ik}
\end{aligned}$$

Note that this chain of equality hinges on two statements. First the fact that:

$$(\psi_j^\sharp)_V \circ (\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} = \mathrm{Id}$$

However this is trivial, as:

$$(\psi_j^{-1})^\sharp_{\psi_j^{-1}(V)} : \mathscr{O}_{X_j}(\psi_j^{-1}(V)) \longrightarrow \mathscr{O}_{\mathscr{X}_j}(V)$$

is the identity map, and:

$$(\psi_j^\sharp)_V : \mathscr{O}_{\mathscr{X}_j}(V) \longrightarrow \mathscr{O}_{X_j}(\psi_j^{-1}(V))$$

is also the identity. The more challenging statement is the following:

$$(\phi_{kj})^\sharp_{\psi_j^{-1}(V)} \circ (\phi_{ji})^\sharp_{\psi_i^{-1}(V)} = (\phi^\sharp_{ki})_{\psi^{-1}(V)}$$

This follows from the fact that $\phi_{ki} = \phi_{ji} \circ \phi_{kj}$, so:

$$\phi^\sharp_{ki} = \phi_{ji*}\phi^\sharp_{kj} \circ \phi^\sharp_{ji}$$

so we have that:

$$\begin{aligned}
(\phi^\sharp_{ki})_{\psi_i^{-1}(V)} =& (\phi_{ji*}\phi^\sharp_{kj})_{\psi_i^{-1}(V)} \circ (\phi^\sharp_{ji})_{\psi_i^{-1}(V)} \\
=& (\phi^\sharp_{kj})_{\phi_{ji}^{-1}(\psi_i^{-1}(V))} \circ (\phi^\sharp_{ji})_{\psi_i^{-1}(V)} \\
=& (\phi^\sharp_{kj})_{\psi_j^{-1}(V)} \circ (\phi^\sharp_{ji})_{\psi_i^{-1}(V)}
\end{aligned}$$

implying the claim. It follows that the $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$ glue together to form a sheaf $X, \mathscr{O}_X$ such that $\mathscr{O}_X|_{\mathscr{X}_i} \cong \mathscr{O}_{\mathscr{X}_i}$, implying that each $\psi_i : X_i \to X$ is an open embedding. It is also clear that as morphisms of locally ringed space $\psi_i = \psi_j \circ \phi_{ij}$, essentially by the construction of our maps $\psi_i^\sharp$.

All that remains to show is that $(X, \mathscr{O}_X)$ is a scheme. Let $x \in X$, then $x \in \mathscr{X}_i$ for some i. There is then an isomorphism $(\mathscr{X}_i, \mathscr{O}_{X_i}|_{\mathscr{X}_i})$ to $(\mathscr{X}_i, \mathscr{O}_{\mathscr{X}_i})$, the latter of which is a scheme as it is isomorphic to $(X_i, \mathscr{O}_{X_i})$. Examine the image of $x \in X_i$ under this composition of isomorphisms, and denote it by x_i. Since X_i is a scheme, it follows that there is an open neighborhood V_{x_i} of X_i such that $(V_{x_i}, \mathscr{O}_{X_i}|_{V_i})$ is isomorphic to an affine scheme. Take the preimage of V_{x_i} under this composition of isomorphism, and we obtain an open neighborhood of x whose image under the composition of isomorphisms is isomorphic to an affine scheme. It follows that x has an open neighborhood W_x such that $(W_x, \mathscr{O}_X|_W)$ is isomorphic to an affine scheme implying the claim. $\square$

We have the obvious corollary:

Corollary 2.1.2. *Let $\{X_i\}$ be a family of schemes satisfying the criteria of Theorem 2.1.1, then the scheme X is unique up to unique isomorphism.*

Proof. This follows from Theorem 1.2.2, and the uniqueness of gluing topological spaces together, i.e. uniqueness of the quotient topology and the natural topology on the disjoint union of topological spaces. $\square$

We now show some easy examples of non affine schemes:

Example 2.1.2. Let $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, and note that for any $(z_1, z_2) \in \mathbb{C}$, the ideal $\langle x - z_1, y - z_2 \rangle$ is prime. It suffices to check that $\mathbb{C}[x,y]/\langle x - z_1, y - z_2\rangle$ is an integral domain; in fact, we claim that $\mathbb{C}[x,y]/\langle x - z_1, y - z_2\rangle \cong \mathbb{C}$.

We define a map $\phi : \mathbb{C}[x,y] \to \mathbb{C}$ by $p \mapsto p(z_1, z_2)$. This clearly a surjective morphism as the constant polynomial $p(x) = w$ maps to $w \in \mathbb{C}$. We thus see that $\mathbb{C} \cong \mathbb{C}[x,y]/\ker\phi$. Clearly $\langle x - z_1, y - z_2\rangle \subset \ker\phi$, suppose $p \in \ker\phi$, and write:

$$p = \sum_{ij} a_{ij} x^i y^j$$

Note that $x^n = (x - z_1 + z_1)^n$, and $y^n = (y - z_2 + z_2)^n$, hence there exists a P such that:

$$p(x,y) = P(x - z_1, y - z_2)$$

so $p(z_1, z_2) = P(0,0) = 0$, hence P as 0 constant term. Every term is then divisible by x or y, and we thus have that there exist polynomials Q and R such that:

$$P(x,y) = xQ(x,y) + yR(x,y)$$

so:

$$p(x,y) = (x - z_1)Q(x - z_1, y - z_2) + (y - z_2)R(x - z_1, y - z_2)$$

and so $p(x,y) \in \langle x - z_1, y - z_1\rangle$ implying that $\mathbb{C} \cong \mathbb{C}[x,y]/\langle x - z_1, y - z_2\rangle$.

We define a map $\psi : \mathbb{C}[x,y] \to \mathbb{C}$ by $p \longmapsto p(0,0)$, i.e. we evaluate the polynomial p in two variables at the point $(0,0$. Note that this is clearly a ring homomorphism, and that if $p \in \langle x, y\rangle$, that $p(0,0) = 0$, so $\langle x, y\rangle \subset \ker\psi$. We also see that if $p \in \ker\psi$, then the leading coefficient of p must be 0. It follows that:

$$p = \sum_{ij} w_{ij} x^i y^j$$

where if $i = j$, then $j \neq 0$, so:

$$\begin{aligned} p =& x \sum_{i>0,j} w_{ij} x^{i-1} y^j + \sum_{i=0,j} w_{ij} x^i y^j \\ =& x \sum \sum_{i>0,j} w_{ij} x^{i-1} y^j + y \sum_{i=0,j} w_{ij} x^i y^{j-1} \in \langle x, y\rangle \end{aligned}$$

so $\ker\psi = \langle x, y\rangle$. Moreover, this map is clearly surjective, as if $z \in \mathbb{C}$, the constant polynomial $z \in \mathbb{C}[x,y]$ maps to z as well. We thus get a unique isomorphism $\psi' : \mathbb{C}[x,y]/\langle x, y\rangle \to \mathbb{C}$ by the universal property of quotient rings. It follows that $\mathbb{C}[x,y]/\langle x - z_1, y - z_2\rangle \cong \mathbb{C}$, so every ideal of the form $\langle x - z_1, x - z_2\rangle$ is maximal[19], and thus prime.

It follows that we can identify $\mathbb{A}^2_{\mathbb{C}}$ with $\mathbb{C}^2$ along with some extra points (such as the zero ideal $\langle 0\rangle$). We thus denote the ideal $\langle x - z_1, x - z_2\rangle$ by (z_1, z_2), and claim that $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is an open subscheme of $\mathbb{A}^2_{\mathbb{C}}$ which is not affine. First note that:

$$\mathbb{A}^2_{\mathbb{C}} \setminus (0,0) = U_x \cup U_y$$

[19] In particular, it is a standard fact that any maximal ideal of a polynomial ring over an algebraically closed field $k = \bar{k}$ is of this form.

Indeed, if $\mathfrak{p} \in U_x \cup U_y$, then we have $x \notin \mathfrak{p}$ or $y \notin \mathfrak{p}$, hence $\mathfrak{p} \neq (0,0)$ implying that $p \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$. Now suppose that $\mathfrak{p} \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, then $\mathfrak{p} \neq (0,0)$, in particular, since $(0,0)$ is a maximal ideal we have that $(0,0) \not\subset \mathfrak{p}$. Now suppose that $x \in \mathfrak{p}$ and $y \in \mathfrak{p}$, then we clearly have that $(0,0) \subset \mathfrak{p}$, so either $x \notin \mathfrak{p}$, or $y \notin \mathfrak{p}$, implying that $\mathfrak{p} \in U_x \cup U_y$.

We know that there is a unique scheme structure on $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, and we can further deduce that this must be the one obtained by gluing the sheaf $\mathscr{O}_{U_x}$ to $\mathscr{O}_{U_y}$. Since there are only two sets which cover the space, we need only check that $\mathscr{O}_{U_x}|_{U_x \cap U_y} \cong \mathscr{O}_{U_y}|_{U_x \cap U_y}$. Let $V \subset U_x \cap U_y$, then we have that $V \subset U_x$ and $V \subset U_y$, hence:

$$\mathscr{O}_{U_x}|_{U_x \cap U_y}(V) = \mathscr{O}_{U_x}(V) = \mathscr{O}_X(V)$$

and similarly for $\mathscr{O}_{U_y}$, hence we get a sheaf of rings on $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$, and since every element $x \in \mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ lies in either U_x or U_y, and U_x and U_y are both affine schemes, it follows that with this structure sheaf, $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is the scheme isomorphic to the open subscheme $U_x \cup U_y$. We want to show that $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is not affine; denote by X the open subscheme $\mathbb{A}^1_{\mathbb{C}} \setminus (0,0)$, we want to calculate $\mathscr{O}_X(X)$. Note that by our work in Theorem 1.2.2 we have that:

$$\mathscr{O}_X(X) = \{(s_x, s_y) \in \mathscr{O}_{U_x}(U_x) \times \mathscr{O}_{U_y}(U_y) : s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}\}$$

as the morphism $\mathscr{O}_{U_x}|_{U_x \cap U_y} \to \mathscr{O}_{U_x}|_{U_x \cap U_y}$ is the identity morphism. Now note that since U_x and U_y are distinguished open sets, we have that:

$$\mathscr{O}_{U_x}(U_x) \cong (\mathbb{C}[x,y])_x = \mathbb{C}[x,y,1/x] \qquad \mathscr{O}_{U_y}(U_y) \cong (\mathbb{C}[x,y])_y = \mathbb{C}[x,y,1/y]$$

while we have that:

$$\mathscr{O}_{U_x}|_{U_x \cap U_y} = \mathscr{O}_{U_y}|_{U_x \cap U_y} \cong \mathbb{C}[x,y,1/x,1/y]$$

By our earlier work on affine schemes, we know that the restriction maps (up to isomorphism) here are just the obvious inclusions. It follows that if $s_x|_{U_x \cap U_y} = s_y|_{U_x \cap U_y}$, then s_x and s_y are in the image of the injections $\mathbb{C}[x,y] \to \mathbb{C}[x,y,1/x]$, and $\mathbb{C}[x,y] \to \mathbb{C}[x,y,1/y]$, as the must be polynomials with no $1/x$ or $1/y$ terms. It also follows that the preimages of s_x and s_y under these injections must be equal as well, hence:

$$\mathscr{O}_X(X) \cong \{(p,q) \in \mathbb{C}[x,y] \times \mathbb{C}[x,y] : p = q\} \cong \mathbb{C}[x,y]$$

Now suppose that X is affine, then we have that there is an isomorphism $(X, \mathscr{O}_X) \cong (\operatorname{Spec} A, \mathscr{O}_A)$ for some commutative ring A. We thus have that $\mathscr{O}_X(X) \cong A$, but we have just shown that $\mathscr{O}_X(X) \cong \mathbb{C}[x,y]$, implying that $X \cong \operatorname{Spec} \mathbb{C}[x,y]$ as topological spaces. Now in an affine scheme there is a bijection between the points of $\mathbb{A}^2_{\mathbb{C}}$ and the prime ideals of $\mathbb{C}[x,y]$, however $\mathbb{A}^2_{\mathbb{C}} \setminus (0,0)$ is missing the prime ideal $(0,0)$, so it cannot be affine.

Example 2.1.3. Let $\{X_i\}$ be a family of affine schemes, and then we claim that:

$$X = \coprod_i X_i$$

equipped with the natural disjoint union topology is a scheme which is affine if and only if the the family is finite. We can prove one direction immediately, suppose that X is an affine scheme, then we need to show that the family is finite. We prove this by the contrapositive, i.e. if the family is infinite then X is not affine, and we prove the contrapositive by contradiction. Assume that X is affine, then every open cover of X has a finite subcover by Lemma 1.4.1, however this is clearly not true as the infinite disjoint union of any family of topological spaces cannot be quasi-compact[20]. It follows that if the family is infinite then X is not affine, hence if X is affine then the family is finite.

[20] A topological space is quasi-compact if every open cover has a finite subcover. Note that this is often taken as the definition of compactness, but for some reason algebraic geometer's prefer this nomenclature.

Now suppose that the family is finite, by induction, and the associativity of the disjoint operation on topological spaces, it suffices to check that check that:

$$\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$$

is affine for any two rings A and B. Indeed, we claim that:

$$\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \cong \operatorname{Spec}(A_1 \times A_2)$$

In particular, we claim that $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$ is the coproduct in the category of affine schemes. Set $X = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$, then we want to first show that X is a scheme. Let $U \subset X$ be open, then we have that $\psi_1^{-1}(U) \subset \operatorname{Spec} A_1$ and $\psi_2^{-1}(U) \subset \operatorname{Spec} A_2$ are both open, where ψ_1 and ψ_2 are the canonical injections. We define:

$$\mathscr{O}_X(U) = \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(U)) \times \mathscr{O}_{\operatorname{Spec} A_2}(\psi_2^{-1}(U))$$

and restriction maps to be $\theta_V^U = (\theta_{\psi_1^{-1}(V)}^{\psi_1^{-1}(U)}, \theta_{\psi_2^{-1}(V)}^{\psi_2^{-1}(U)})$ We check that this is a sheaf, let $s \in \mathscr{O}_X(U)$, and U_i an open cover of U such that $s|_{U_i} = 0$ for all U_i, we want to show that $s = 0$. First note that we can write $s = (s_1, s_2) \in \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(U)) \times \mathscr{O}_{\operatorname{Spec} B}(\psi_2^{-1}(U))$, and that

$$\psi_1^{-1}(U) = \bigcup_i \psi_1^{-1}(U_i)$$

and similarly for A_2. It follows that

$$s|_{U_i} = \left(s_1|_{\psi_1^{-1}(U_i)}, s_2|_{\psi_2^{-1}(U_i)}\right)$$

implying that $s_1|_{\psi_1^{-1}(U_i)} = 0$ for all U_i, hence $s_1 = 0$, and similarly for A_2 implying sheaf axiom one. The same argument adapted to sheaf axiom two implies that this indeed a sheaf.

Now note that as a set:

$$X = \bigcup_i \{(\mathfrak{p}, i) : \mathfrak{p} \in A_i\}$$

so we define a map:

$$\eta : X \longrightarrow \operatorname{Spec}(A \times B)$$

by:

$$\eta((\mathfrak{p}, i)) = \begin{cases} \mathfrak{p} \times A_2 & \text{if } i = 1 \\ A_1 \times \mathfrak{p} & \text{if } i = 2 \end{cases}$$

We note that if $\mathfrak{p} \subset A_1$ is prime, then $\mathfrak{p} \times A_2$ is prime. Indeed, let (a, b) and (c, d) lie in $A_1 \times A_2$ such that $(ac, cd) \in \mathfrak{p} \times A_2$, then it follows that $ac \in \mathfrak{p}$, hence $a \in \mathfrak{p}$ or $c \in \mathfrak{p}$, implying that (a, b) or (c, d) in $\mathfrak{p} \times A_2$ so $\mathfrak{p} \times A_2$ is prime. It follows that this map is well defined. It is clearly injective, as we can't have $\mathfrak{p} \times A_2 = A_1 \times \mathfrak{q}$, so if $\mathfrak{p} \times A_2 = \mathfrak{q} \times A_2$, then this implies that $\mathfrak{p} = \mathfrak{q}$ hence $(\mathfrak{p}, 1) = (\mathfrak{q}, 1)$. To check that this map is surjective, first note that $\mathfrak{p} \times \mathfrak{q}$ is not prime for any ideals (not necessarily prime) $\mathfrak{p} \subset A_1$ and $\mathfrak{q} \subset A_2$ in $A_1 \times A_2$, where $\mathfrak{p}$ and $\mathfrak{q}$ are not the whole ring. Indeed, if $a \in \mathfrak{p}$, $b \in A_2$, $c \in A_1$, $d \in \mathfrak{q}$, then $(a, b), (c, d) \notin \mathfrak{p} \times \mathfrak{q}$, but $(ac, cd) \in \mathfrak{p} \times \mathfrak{q}$. Now let $\mathfrak{q} \subset A_1 \times A_2$ be a prime ideal, then it follows that $\mathfrak{q}_1 = \pi_2(\mathfrak{q})$ is an ideal and $\mathfrak{q}_2 = \pi_2(\mathfrak{p})$ are ideals of A and B respectively as the surjective image of an ideal is an ideal. We claim that:

$$\mathfrak{q} = \mathfrak{q}_1 \times \mathfrak{q}_2$$

It is clear that $\mathfrak{q} \subset \mathfrak{q}_1 \times \mathfrak{q}_2$, so now let $(a, b) \in \mathfrak{q}_1 \times \mathfrak{q}_2$. This implies that $(a, s_2) \in \mathfrak{q}$ and a $(s_1, b) \in \mathfrak{q}$ for some $s_i \in A_i$. Note that since $\mathfrak{q}$ is an ideal, we thus have that $(a, s_1) \cdot (1, 0) = (a, 0) \in \mathfrak{q}$, and similarly for $(0, b)$. Since $\mathfrak{q}$ is closed under addition it follows that $(a, b) \in \mathfrak{q}$. Since $\mathfrak{q}$ is prime however, we must have that $\mathfrak{q}_i = A_i$ for $i = 1$ or 2. Without loss of generality suppose that $\mathfrak{q}_2 = A_2$, then $\mathfrak{q}_1$ must be prime, as if $a \cdot c \in \mathfrak{q}_1$, then we must have that $(a, b) \cdot (c, d) \in \mathfrak{q} = \mathfrak{q}_1 \times A_2$, hence either $(a, b) \in \mathfrak{q}$ or $(c, d) \in \mathfrak{q}$, implying that either $a \in \mathfrak{q}_1$ or $c \in \mathfrak{q}_1$. Now let $\mathfrak{q} \subset A \times B$ be prime, then $\mathfrak{q} = \mathfrak{q}_1 \times A_2$ or $\mathfrak{q} = A_1 \times \mathfrak{q}_2$ where $\mathfrak{q}_i$ is prime, so it follows that $(\mathfrak{q}_i, i) \in X$, and satisfies $\eta((\mathfrak{q}_i, i)) = \mathfrak{q}$, so η is surjective.

We check that the map is continuos, by noting that η is continuous if and only if $\eta \circ \psi_i$ is continuous for each i. It suffices to check this on distinguished open sets. Let $U_{(a,b)} \subset \operatorname{Spec} A_1 \times A_2$ be a distinguished open, then:

$$\begin{aligned}\eta^{-1}(U_{(a,b)}) =& \{(\mathfrak{p}, i) \in X : \eta((\mathfrak{p}, i)) \in U_{(a,b)}\} \\ =& \{(\mathfrak{p}, i) : (a, b) \notin \eta((\mathfrak{p}, i))\}\end{aligned}$$

Then for $i = 1$:

$$\begin{aligned}\psi_1^{-1}(\eta^{-1}(U_{(a,b)})) =& \{\mathfrak{p} \in \operatorname{Spec} A_1 : \psi_1(\mathfrak{p}) \in \eta^{-1}(U_{(a,b)})\} \\ =& \{\mathfrak{p} \in \operatorname{Spec} A_1 : (\mathfrak{p}, 1) \in \eta^{-1}(U_{(a,b)})\} \\ =& \{\mathfrak{p} \in \operatorname{Spec} A_1 : \mathfrak{p} \times A_2 \in U_{(a,b)}\} \\ =& \{\mathfrak{p} \in \operatorname{Spec} A_1 : (a, b) \notin \mathfrak{p} \times \mathbb{A}_2\} \\ =& \{\mathfrak{p} \in \operatorname{Spec} A_1 : a \notin \mathfrak{p}\} \\ =& U_a\end{aligned}$$

similarly:

$$\psi_2^{-1}(\eta^{-1}(U_{(a,b)})) = U_b$$

so η is continuous. We want to show that η is an open map. Let $U \subset X$ be open, then:

$$U = \psi_1(\psi_1^{-1}(U)) \cup \psi_2(\psi_2^{-1}(U))$$

We can write $\psi_i^{-1}(U)$ as union of distinguished opens, hence:

$$U = \bigcup_j \psi_1(U_{a_j}) \cup \bigcup_k \psi_2(U_{b_k})$$

Taking the image of this under η we find that:

$$\eta(U) = \bigcup_j \eta(\psi_1(U_{a_j})) \cup \bigcup_k \eta(\psi_2(U_{b_k}))$$

so it suffices to check that $\eta \circ \psi_i$ is an open map. Let $U_a \subset \operatorname{Spec} A_1$, then:

$$\psi_1(U_a) = \{(\mathfrak{p}, 1) \in X : a \notin \mathfrak{p}\}$$

so:

$$\eta(\psi_1(U_a)) = \{\mathfrak{p} \times A_1 : a \notin \mathfrak{p}\}$$

We claim that:

$$\eta(\psi_1(U_a)) = U_{(a,0)} = \{\mathfrak{q} \in \operatorname{Spec}(A_1 \times A_2) : (a, 0) \notin \mathfrak{q}\}$$

Let $\mathfrak{q}$ in $U_{(a,0)}$, then $\mathfrak{q} \neq A_1 \times \mathfrak{p}$ for some $\mathfrak{p} \subset A_2$ as $a \in A_1$ and $0 \in \mathfrak{p} \subset A_2$. It follows that $\mathfrak{q} = \mathfrak{p} \times A_2$ for some $\mathfrak{p} \subset A_1$, and that $a \notin \mathfrak{p}$, hence $q \in \eta(\psi_1(U_a))$. Now suppose that $p \times A_1 \in \eta(\psi_1(U_a))$, then $a \notin \mathfrak{p}$, hence $(a, 0) \notin \mathfrak{p} \times A_1$, so $\mathfrak{p} \times A_1 \in U_{(a,0)}$. A similar proof follows for ψ_2, hence $\eta(U)$ is the union of open sets and is thus open. It follows that η is a homeomorphism as it is an open continuous bijection.

We now want to define sheaf isomorphism:

$$\eta^\sharp : \mathscr{O}_{\operatorname{Spec}(A \times B)} \longrightarrow \eta_* \mathscr{O}_X$$

It suffices to define the sheaf morphism on basic open sets of $\operatorname{Spec}(A \times B)$. Let $U_{(a,b)}$ be a basic open, and let $V = \eta^{-1}(U_{(a,b)}) \subset X$, then note that:

$$V = \psi_1(\psi_1^{-1}(V)) \cup \psi_2(\psi_2^{-1}(V))$$

hence:

$$\begin{aligned}(\eta_* \mathscr{O}_X)(U_{(a,b)}) =& \mathscr{O}_{\operatorname{Spec} A_1}(\psi_1^{-1}(V)) \times \mathscr{O}_{\operatorname{Spec} A_2}(\psi_2^{-1}(V)) \\ =& \mathscr{O}_{\operatorname{Spec} A_1}(U_a) \times \mathscr{O}_{\operatorname{Spec} A_1}(U_b) \\ =& (A_1)_a \times (A_2)_b\end{aligned}$$

It thus suffices to show by Corollary 1.4.2 that:

$$(A_1 \times A_2)_{(a,b)} \cong (A_1)_a \times (A_2)_b$$

and that the isomorphisms commute with restrictions on a base. We define a map $A_1 \times A_2 \to (A_1)_a \times (A_2)_b$ by:

$$(s,t) \longmapsto \left(\frac{s}{1}, \frac{t}{1}\right)$$

and note that the image (a,b) is a unit with inverse given by $(1/a, 1/b)$ so we obtain a unique map:

$$\begin{aligned}\phi : (A_1 \times A_2)_{(a,b)} &\longrightarrow (A_1)_a \times (A_2)_b \\ \frac{(s,t)}{(a,b)^k} &\longmapsto \left(\frac{s}{a^k}, \frac{t}{b^k}\right)\end{aligned}$$

It is clear that this map is surjective. If $\phi((s,t)/(a,b)^k) = 0$, then we have that:

$$\frac{s}{a^k} = 0 \qquad \text{and} \qquad \frac{t}{b^k} = 0$$

implying that there exists some m and some n such that:

$$a^m s = 0 \qquad \text{and} \qquad b^n t = 0$$

Let $K > \max\{m, n\}$ then:

$$a^K s = 0 \qquad \text{and} \qquad b^K t = 0$$

so:

$$(a,b)^K (s,t) = (a^K s, b^K t) = (0,0)$$

implying that $(s,t)/(a,b)^k = 0$ as well, so ϕ is injective and an isomorphism. It is clear (albeit a little messy to check explicitly) that the isomorphisms commute with restrictions on a base, hence $X \cong \operatorname{Spec}(A_1 \times A_2)$ and is thus an affine scheme.

To see that X is a coproduct, it suffices to check that $\operatorname{Spec}(A_1 \times A_2)$ satisfies the properties of the coproduct. Note that we have natural morphisms $\operatorname{Spec} A_i \to \operatorname{Spec}(A_1 \times A_2)$ given by the map $\pi_i^{-1} : \operatorname{Spec} A_i \to \operatorname{Spec}(A_1 \times A_2)$, and the induced map $\pi_i^\sharp : \mathscr{O}_{\operatorname{Spec}(A_1 \times A_2)} \to \operatorname{Spec}(A_i)$. Since there is an isomorphism:

$$\operatorname{Hom}(A, B) \cong \operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A)$$

and $A_1 \times A_2$ satisfies the universal property of the product in the category of rings, it follows that $\operatorname{Spec}(A_1 \times A_2)$ must satisfy the universal property of the coproduct, hence so must X. In particular, the isomorphism $X \cong \operatorname{Spec}(A_1 \times A_2)$ is unique.

As the preceding example states, the infinite disjoint unions of schemes (affine or not) is not affine. We will show later in this section that the disjoint union of schemes is the coproduct in the category of schemes. Funnily enough however, the product of schemes is not in general a scheme, so schemes are a category without products.

Example 2.1.4. Let $X = \mathbb{A}^1_{\mathbb{C}}$, i.e. the affine scheme $\operatorname{Spec}\mathbb{C}[x]$. Let Y be another copy of of $\mathbb{A}^1_{\mathbb{C}}$ but instead use the variable y for book keeping purposes (i.e. $Y = \operatorname{Spec}\mathbb{C}[y]$). Now examine $U_x \subset X$ and $U_y \subset Y$, both of these are affine schemes isomorphic to $\operatorname{Spec}\mathbb{C}[x, 1/x]$ and $\operatorname{Spec}\mathbb{C}[y, 1/y]$ respectively. Note that $\mathbb{C}$ is algebraically closed, so the only prime ideals of $\mathbb{C}[x]$ are of the form $x - z$ for some $z \in \mathbb{C}$, and of course the zero ideal $\langle 0 \rangle$. It follows that U_x and U_y contain every ideal but the ideal $\langle x \rangle$. Furthermore, with this identification we can truly vies $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$ with an extra point $\langle 0 \rangle$ which is 'close' to every other point. Obviously the usual topology on $\mathbb{C}$ differs from the one on $\mathbb{A}^1_{\mathbb{C}}$; in particular $\mathbb{A}^1_{\mathbb{C}}$ is clearly non Hausdorff.

We wish to glue these two schemes together along U_x and U_y. Since U_x and U_y are affine schemes, it suffices to give a ring isomorphism $\mathbb{C}[x, 1/x] \to \mathbb{C}[y, 1/y]$. We give the obvious one induced by the map $\mathbb{C}[x] \to \mathbb{C}[y]$ given by $x \mapsto y$. Clearly this isomorphism descends to an isomorphism $\mathbb{C}[x, 1/x] \mapsto \mathbb{C}[y, 1/y]$ which takes $x \mapsto y$ and $1/x \mapsto 1/y$. Since there are only two schemes to glue, there is only one subset of X and Y respectively to glue, and only one isomorphism $\phi_{xy} : U_x \to U_y$ so the conditions of Theorem 2.1.1 are trivially satisfied. Denote the induced scheme by Z, and note that the topological space:

$$Z = \left(X \coprod Y \right) / \sim$$

looks like $\mathbb{A}^1_{\mathbb{C}}$ with two origins $\langle x \rangle$ and $\langle y \rangle$. Indeed, the embeddings $\psi_x : X \to Z$ and $\psi_y : Y \to Z$ satisfies:

$$\psi_x(\langle x - z \rangle) = \psi_y(\langle y - z \rangle)$$

for all $z \neq 0$, and also agree on the zero ideal. We wish to show that this scheme is not affine, and we do so by calculating the ring of global sections. Now note that:

$$Z = \mathscr{X} \cup \mathscr{Y}$$

and so:

$$\mathscr{O}_Z(Z) = \{(s_x, s_y) \in \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) : s_x|_{\mathscr{X} \cap \mathscr{Y}} \cong \beta_{yx}(s_y|_{\mathscr{X} \cap \mathscr{Y}})\}$$

We have that $\mathscr{X} \cong \mathscr{Y} \cong \operatorname{Spec}\mathbb{C}[x]$, and that:

$$\mathscr{X} \cap \mathscr{Y} = \psi_x(X) \cap \psi_y(Y) = \psi_x(U_x) \cong \operatorname{Spec}\mathbb{C}[x, 1/x]$$

so under these identifications β_{yx} is equivalent to the map induced by $x \mapsto x$, hence:

$$\mathscr{O}_Z(Z) \cong \{(p, q) \in \mathbb{C}[x] \times \mathbb{C}[x] : \pi_x(p) = \pi_x(q)\}$$

where $\pi_x : \mathbb{C}[x] \to \mathbb{C}[x, 1/x]$ is the localization map. It follows that since the localization map of an integral domain is an injection that:

$$\mathscr{O}_Z(Z) \cong \mathbb{C}[x]$$

so if Z is affine then $Z \cong \operatorname{Spec}\mathbb{C}[x]$, but Z contains two copies of the zero ideal, hence cannot be affine by the same argument as in Example 2.1.3

This demonstrates an analogue of a failure of a scheme to be Hausdorff, in the sense that the $\mathbb{C}$ glued to itself everywhere except the origin is non Hausdorff. We will make this notion precise when we discuss separatedness. In our next example, we again glue two copies of an affine scheme together, just via a different isomorphism.

Example 2.1.5. Let X Y, $U_x \subset X$ and $U_y \subset Y$ be as previously defined in Example 2.1.4. Consider the map:

$$\mathbb{C}[x] \longrightarrow \mathbb{C}[y, 1/y]$$

induced by the assignment:

$$x \longmapsto 1/y$$

We note that $1/y$ is a unit in $\mathbb{C}[y, 1/y]$, hence this descend to a unique morphism:

$$\phi : \mathbb{C}[x, 1/x] \longrightarrow \mathbb{C}[y, 1/y]$$

We check that this is an isomorphism, $p \in \mathbb{C}[x, 1/x]$ be a polynomial such that $\phi(p) = 0$. We see that p can be written uniquely as:

$$p = \sum_{i=-n}^{m} z_i x^i$$

for some $n, m \in \mathbb{Z}^+$, and some $z_i \in \mathbb{C}$. It follows that:

$$\phi(p) = \sum_{i=-m}^{n} z_i y^i$$

and for this to be the zero polynomial, we must clearly have that $z_i = 0$ for all i, hence $p = 0$, so ϕ is injective. Clearly ϕ is surjective, as we can just invert any polynomial in $\mathbb{C}[y, 1/y]$ term by term and replace the variable y with x. It follows that ϕ is an isomorphism, with inverse induced by the assignment $y \mapsto 1/x$, so we obtain an isomorphism of schemes $U_x \mapsto U_y$ which trivially satisfy the criteria of Theorem 2.1.1.

As before, we first describe the topological space:

$$Z = \left(X \coprod Y\right) / \sim$$

and then calculate the ring of global sections. First note that the prime ideals $\langle x - z\rangle$ gets mapped to the prime ideal:

$$\eta(\langle x - z\rangle) = \left\{ \frac{p}{x^k} \in \mathbb{C}[x, 1/x] : p \in \langle x - z\rangle, k \geq 0 \right\}$$

so this is the ideal $\langle (x - z)/1\rangle \subset \mathbb{C}[x, 1/x]$. Under the isomorphism $\phi : \mathbb{C}[y, 1/y] \to \mathbb{C}[x, 1/x]$[21] we see that ϕ induces a homeomorphism f given by:

$$f(\langle x - z\rangle) = \langle 1/y - z\rangle \in \operatorname{Spec} \mathbb{C}[y, 1/y]$$

We claim that $\langle 1/y - z\rangle = \langle y - 1/z\rangle$ in $\mathbb{C}[y, 1/y]$. Let $p \in \langle 1/y - z\rangle$, then:

$$p = q \cdot (1/y - z)$$

for some $q \in \mathbb{C}[y, 1/y]$. Now note that element $(-y \cdot 1/z)/1$ is invertible in $\mathbb{C}[y, 1/y]$ hence we have that:

$$\begin{aligned} p =& q \cdot ((-y \cdot 1/z)/1) \cdot ((-y \cdot 1/z)/1)^{-1} \cdot (1/y - z) \\ =& q \cdot ((-y \cdot 1/z)/1)^{-1} \cdot (y - 1/z) \end{aligned}$$

so $p \in \langle y - 1/z\rangle$. The same argument in reverse demonstrates the other inclusion hence $\langle 1/y - z\rangle = \langle y - 1/z\rangle$. It follows that the ideal $\langle x - z\rangle$ is identified with ideal $\langle y - 1/z\rangle$ for all $z \neq 0$, and that $\langle 0\rangle$ is identified with $\langle 0\rangle$. As a set, we can make more this feel more familiar, identify X and Y with $\mathbb{C} \cup \{\langle 0\rangle\}$ and define the map:

$$F : X \coprod Y \longrightarrow \mathbb{P}^2 \cup \{\langle 0\rangle\}$$

where $\mathbb{P}^1 = \mathbb{C}^2 \setminus \{(0, 0)\}/\mathbb{C}^\times$, by:

$$F(z) = \begin{cases} \{0\} \text{ if } z \in X \text{ or } x \in Y \text{ and } z = \{0\} \\ [z, 1] \text{ if } z \in X \\ [1, z] \text{ if } z \in Y \end{cases}$$

[21] Abuse of notation alert! This is the technically the inverse of ϕ, but for notational reasons we redefined ϕ as it's inverse.

We see that $z \neq 0 \in X$ and $1/z \in Y$ then:

$$F(z) = [z, 1] = [1, 1/z] = F(1/z)$$

and similarly for $1/z \in X$ and $z \in Y$, hence there is a unique set map:

$$F' : Z \longrightarrow \mathbb{P}^1 \cup \{\langle 0 \rangle\}$$

This is surjective, as if $[w, z] \in \mathbb{P}^1$, both of which are non zero, then $[w, z] = [1, z/w]$ so $[z/w] \in Z$ maps to $[1, z/w]$. If either w or z is zero then $[w, z] = [0, 1]$ or $[1, 0]$ respectively, and the elements $[0_x]$ and $[0_y]$[22] map to $[0, 1]$ and $[1, 0]$ respectively. Moreover, the ideal $\langle 0 \rangle$ gets mapped to $\langle 0 \rangle$. The same argument in reverse essentially proves that F' is injection, and is thus a set isomorphism. For this reason, we see that Z is an algebraic geometry analogue of projective space, and thus we denote Z by $\mathbb{P}^1_{\mathbb{C}}$.

To see that $\mathbb{P}^1_{\mathbb{C}}$ is not affine, we calculate the ring of global sections. We see that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) = \{(s_x, s_y) \in \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) : s_x|_{\mathscr{X} \cap \mathscr{Y}} = \beta_{yx}(s_y|_{\mathscr{X} \cap \mathscr{Y}})\}$$

As before, we have that $\mathscr{O}_{\mathscr{X}}(\mathscr{X}) \cong \mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) \cong \mathbb{C}[x]$, and that:

$$\mathscr{X} \cap \mathscr{Y} = \psi_x(X) \cap \psi_y(Y) = \psi_x(U_x) \cong \operatorname{Spec} \mathbb{C}[x, 1/x]$$

so under these identifications, β_{yx} is equivalent to the map given by $x \mapsto 1/x$. It follows that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) \cong \{(p, q) \in \mathbb{C}[x] \times \mathbb{C}[y] : \pi(p) = \beta_{yx}(\pi(q))\}$$

Let $p = \sum_i z_i x^i$, and $q = \sum_i w_i x^i$, then we see that if $\pi(p) = \beta_{yx}(\pi(q))$ we must have that:

$$\sum_i z_i x^i = \sum_i w_i x^{-i}$$

hence $z_i = w_i = 0$ for $i > 0$, and $z_0 = w_0$. It follows that:

$$\mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{C}$$

so if $\mathbb{P}^1_{\mathbb{C}}$ was affine we would have that $\mathbb{P}^1_{\mathbb{C}} \cong \operatorname{Spec} k = \langle 0 \rangle$, which obviously cannot be the case.

This is our first example of what we will call a projective scheme. The entirety of the next section will be dedicated to the construction of the map (not a functor!)Proj : Ring $\to$ Scheme. In particular, we will have that $\mathbb{P}^n_k = \operatorname{Proj}(k[x_0, x_1, \ldots, x_n])$.

We continue with an extension of Example 2.1.3.

Proposition 2.1.1. *Let X and Y be schemes, then the topological space $X \coprod Y$ has the natural structure of a scheme, and is the coproduct in the category of schemes.*

Proof. Note that $\emptyset \subset X$ and $\emptyset \subset Y$, and since $\emptyset$ is an open subset of X and Y, it follows that $\emptyset$ is an open subscheme of X and Y, and there is an obvious isomorphism between the two. Since there are only two schemes to glue, it follows that this satisfies the criteria of Theorem 2.1.1, hence:

$$Z = \left(X \coprod Y\right) / \sim$$

has the natural structure of a scheme. However, this equivalence relation is the trivial one, hence:

$$Z = X \coprod Y$$

It follows that $X \coprod Y$ has the natural structure of a scheme, and we have have that the canonical injections $\psi_X : X \to Z$ and $\psi_Y : Y \to Z$ are scheme isomorphisms onto their images.

[22] We use this notation to denote the image of $0 \in X$ and $0 \in Y$ under the open embeddings ψ_x and ψ_y respectively.

Let $U \subset Z$ be open, then we have that since the gluing is trivial:

$$\begin{aligned}
\mathscr{O}_Z(U) = & \mathscr{O}_{\mathscr{X}}(U \cap \mathscr{X}) \times \mathscr{O}_{\mathscr{Y}}(U \cap \mathscr{Y}) \\
= & \mathscr{O}_{\mathscr{X}}(U \cap \psi_X(X)) \times \mathscr{O}_{\mathscr{Y}}(U \cap \psi_Y(Y)) \\
= & \mathscr{O}_X(\psi_X^{-1}(U) \cap X) \times \mathscr{O}_Y(\psi_Y^{-1}(U) \cap Y) \\
= & \mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U))
\end{aligned}$$

which is exactly the structure sheaf we put on $\operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$. Note that the Z already satisfies the universal property of the coproduct in the category of topological, spaces, i.e. for any topological space W and morphisms $\phi_X : X \to W$ and $\phi_Y : Y \to W$ there exists a unique morphism $\phi : Z \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\psi_X"] \arrow[ddr, bend right, "\phi_X"'] & & Y \arrow[dl, "\psi_Y"'] \arrow[ddl, bend left, "\phi_Y"] \\
& Z \arrow[d, "\phi"] & \\
& W &
\end{tikzcd}$$

We thus need to show that the sheaf morphisms commute 'in the opposite direction'. Suppose that W is actually a scheme, and the ϕ_X and ϕ_Y are morphisms of schemes, then note that we have:

$$\psi_X^\sharp : \mathscr{O}_Z \to \psi_{X*}\mathscr{O}_X \qquad \text{and} \qquad \phi_X^\sharp : \mathscr{O}_W \to \phi_{X*}\mathscr{O}_X^\sharp$$

and similarly for the Y morphisms. We thus need to construct a unique morphism $\phi^\sharp : \mathscr{O}_W \to \phi_*\mathscr{O}_Z$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathscr{O}_W \arrow[d, "\phi^\sharp"] \arrow[ddl, bend right, "\phi_X^\sharp"'] \arrow[ddr, bend left, "\phi_Y^\sharp"] & \\
& \phi_*\mathscr{O}_Z \arrow[dl, "\phi_*\psi_X^\sharp"'] \arrow[dr, "\phi_*\psi_Y^\sharp"] & \\
\phi_{X*}\mathscr{O}_X & & \phi_{Y*}\mathscr{O}_Y
\end{tikzcd}$$

First, as a sanity check, lets make sure this diagram makes sense. Note that:

$$\phi_*\psi_X^\sharp : \phi_*\mathscr{O}_Z \longrightarrow \phi_*(\psi_{X*}\mathscr{O}_X)$$

however, we have that $\phi \circ \psi_X = \phi_X$, so:

$$\phi_*(\psi_{X*}\mathscr{O}_X) = (\phi \circ \psi_X)_*\mathscr{O}_X = \phi_{X*}\mathscr{O}_X$$

so the diagram does indeed make sense. Now let U be an open subset of W, we define a map:

$$\phi_U^\sharp : \mathscr{O}_W(U) \longrightarrow \phi_*\mathscr{O}_Z(U)$$

by first noting that:

$$\begin{aligned}
\phi_*\mathscr{O}_Z(U) =&\mathscr{O}_Z(\phi^{-1}(U))\\
=&\mathscr{O}_X(\psi_X^{-1}(\phi^{-1}(U))) \times \mathscr{O}_Y(\psi_Y^{-1}(\phi^{-1}(U)))\\
=&\mathscr{O}_X((\phi\circ\psi_X)^{-1}(U)) \times \mathscr{O}_Y((\phi\circ\psi_Y)^{-1}(U))\\
=&\mathscr{O}_X(\phi_X^{-1}(U)) \times \mathscr{O}_Y(\phi_Y^{-1}(U))\\
=&\phi_{X*}\mathscr{O}_X(U) \times \phi_{Y*}\mathscr{O}_Y(U)
\end{aligned}$$

so the only reasonable definition of U is:

$$\phi_U^\sharp(s) = \Big((\phi_X^\sharp)_U(s), (\phi_Y^\sharp)_U(s)\Big)$$

We check that this commutes with restriction maps. Let $V \subset U$, and $s \in \mathscr{O}_W(U)$ then:

$$\begin{aligned}
\phi_V^\sharp \circ \theta_V^U(s) &= \Big((\phi_X^\sharp)_V, (\phi_Y^\sharp)_V\Big) \circ \theta_V^U(s)\\
&= \Big((\phi_X^\sharp)_V \circ \theta_V^U(s), (\phi_Y^\sharp)_V \circ \theta_V^U(s)\Big)
\end{aligned}$$

since $\phi_X^\sharp$ and $\phi_Y^\sharp$ are natural transformations, we have that:

$$\phi_V^\sharp \circ \theta_V^U(s) = \Big(\theta_V^U \circ (\phi_X^\sharp)_U(s), \theta_V^U \circ (\phi_Y^\sharp)_U(s)\Big)$$

However, note that restriction maps on $\phi_{X*}\mathscr{O}_X$ are given by $\theta_V^U = \theta_{\phi_X^{-1}(V)}^{\phi_X^{-1}(U)}$, so we have that:

$$\phi_V^\sharp \circ \theta_V^U = \left(\theta_{\phi_X^{-1}(V)}^{\phi_X^{-1}(U)} \times \theta_{\phi_Y^{-1}(V)}^{\phi_Y^{-1}(U)}\right) \circ \phi_U^\sharp$$

Now note that the restriction maps on $\phi_*\mathscr{O}_Z$ are given by:

$$\begin{aligned}
\theta_V^U =&\theta_{\phi^{-1}(V)}^{\phi^{-1}(U)}\\
=&\left(\theta_{\psi_X^{-1}(\phi^{-1}(V))}^{\psi_X^{-1}(\phi^{-1}(U))} \times \theta_{\psi_Y^{-1}(\phi^{-1}(V))}^{\psi_Y^{-1}(\phi^{-1}(U))}\right)\\
=&\left(\theta_{\phi_X^{-1}(V)}^{\phi_X^{-1}(U)} \times \theta_{\phi_Y^{-1}(V)}^{\phi_Y^{-1}(U)}\right)
\end{aligned}$$

hence:

$$\phi_V^\sharp \circ \theta_V^U = \theta_V^U \circ \phi_U^\sharp$$

so it follows that $\phi^\sharp : \mathscr{O}_W \to \phi_*\mathscr{O}_Z$ is indeed a natural transformation. We now need to check that $\phi_*\psi_X^\sharp \circ \phi^\sharp = \phi_X^\sharp$, and it suffices to check that they agree on all open sets of W. Recall that $\psi_X^\sharp$ is defined to be the identity on open sets when $U \subset Z$ is entirely contained in $\psi_X(X)$; since $\mathscr{O}_Z(U) = \mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U))$, it follows that $(\psi_X^\sharp)_U$ is the projection:

$$\mathscr{O}_X(\psi_X^{-1}(U)) \times \mathscr{O}_Y(\psi_Y^{-1}(U)) \longrightarrow \mathscr{O}_X(\psi_X^{-1}(U))$$

Now let $U \subset W$ be open, then:

$$\begin{aligned}
(\phi_*\psi_X^\sharp \circ \phi^\sharp)_U =&(\phi_*\psi_X^\sharp)_U \circ \phi_U^\sharp\\
=&(\psi_X^\sharp)_{\phi^{-1}(U)} \circ \Big((\phi_X^\sharp)_U \times (\phi_Y^\sharp)_U\Big)
\end{aligned}$$

Now note that $(\phi_X^\sharp)_U$ has image in $\mathscr{O}_X(\phi_X^{-1}(U)) = \mathscr{O}_X((\phi\circ\psi_X)^{-1}(U))$, and that $(\psi_X^\sharp)_{\phi^{-1}(U)}$ is the projection:

$$\mathscr{O}_X((\phi\circ\psi_X)^{-1}(U)) \times \mathscr{O}_Y((\phi\circ\psi_X)^{-1}(U)) \longrightarrow \mathscr{O}_X((\phi\circ\psi_X)^{-1}(U))$$

hence:

$$(\phi_*\psi_X^\sharp \circ \phi^\sharp)_U = (\phi_X^\sharp)_U$$

for all U. It follows that $\phi_*\psi_X^\sharp \circ \phi^\sharp = \phi_X^\sharp$, and similarly for Y, hence we have that $Z = X \coprod Y$ satisfies the universal property of the coproduct in the category of schemes as desired. $\square$

Before moving on to discuss closed subschemes, we prove the following result, which is an analogue of Corollary 1.4.3.

Proposition 2.1.2. *Let X be a scheme and $Y = \operatorname{Spec} A$ be an affine scheme. Then the set of morphisms $\operatorname{Hom}_{\mathrm{Sch}}(X, Y)$ is in natural bijection with the set of ring morphisms $\operatorname{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \cong \operatorname{Hom}(A, \mathscr{O}_X(X))$.*

Proof. Let $(f, f^\sharp)$ be a morphism $X \to Y$, then $f_Y^\sharp : \mathscr{O}_Y(Y) \to \mathscr{O}_X(f^{-1}(Y)) = \mathscr{O}_X(X)$ is a ring morphism. Define a set map:

$$\Phi : \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$$

by:

$$(f, f^\sharp) \longmapsto f_Y^\sharp$$

We want to define a map in the other direction, and show that these are inverses of another. Let $\psi : \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$ be a ring homomorphism, we want to define a map:

$$(f_\psi, f_\psi^\sharp) : X \longrightarrow Y$$

We first determine the topological map $f_\psi : X \to Y$; note that every point in $Y = \operatorname{Spec} A$ can be identified with prime ideal of $\mathscr{O}_Y(Y)$ via the isomorphism $\mathscr{O}_Y(Y) \cong A$. It thus suffices to assign a prime ideal of $\mathscr{O}_Y(Y)$ to each $x \in X$. Let $x \in X$, then we have that the ring $(\mathscr{O}_X)_x$ has a unique maximal (and thus prime) ideal $\mathfrak{m}_x$, and there is a unique stalk map map $\pi_x : \mathscr{O}_X(X) \to (\mathscr{O}_X)_x$. It follows that $\pi_x^{-1}(\mathfrak{m}_x)$ is a prime ideal of $\mathscr{O}_X(X)$, and $\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$ is a prime ideal of $\mathscr{O}_Y(Y)$. Let φ_A be the natural isomorphism $A \to \mathscr{O}_Y(Y)$, then we see that:

$$\varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))) \subset A$$

is a prime ideal of A, hence we define $f_\psi : X \to Y$ by:

$$f_\psi(x) = \varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$$

We check that this is continuous, and it suffices to check this on distinguished opens $U_a \subset \operatorname{Spec} A$. We see that:

$$\begin{aligned} f^{-1}(U_a) =& \{x \in X : f(x) \in U_a\} \\ =& \{x \in X : a \notin f(x)\} \\ =& \{x \in X : a \notin \varphi_A^{-1}(\psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))\} \\ =& \{x \in X : \varphi_A(a) \notin \psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))\} \\ =& \{x \in X : \pi_x(\psi(\varphi_A(a))) \notin \mathfrak{m}_x\} \end{aligned}$$

Let $\psi(\varphi_A(a)) = g \in \mathscr{O}_X(X)$, then we have that:

$$f_\psi^{-1}(U_a) = \{x \in X : g_x \notin \mathfrak{m}_x\}$$

It thus suffices to show that for every $g \in \mathscr{O}_X(X)$ the set:

$$X_g = \{x \in X : g_x \notin \mathfrak{m}_x\}$$

is an open set. Cover X with affine open sets $U_i = \operatorname{Spec} B_i$, then we see that:

$$X_g = \bigcup X_g \cap U_i$$

It thus suffices to check that $X_g \cap U_i$ is an open set for each i. Let $\beta_i : U_i \to \operatorname{Spec} B_i$ be an isomorphism of affine schemes, then we have an isomorphism $\mathscr{O}_X(U_i) \cong B_i$, which we denote by $\gamma_i : \mathscr{O}_X(U_i) \to B_i$. We claim that:

$$\beta_i(U_i \cap X_g) = U_{\gamma_i(g|_{U_i})}$$

which would imply that $U_i \cap X_g$ is an open subset of U_i, and thus an open subset of X. First note that since $(g|_{U_i})_x = g_x$:

$$U_i \cap X_g = \{x \in U_i : (g|_{U_i})_x \notin \mathfrak{m}_x\}$$

The homeomorphism β_i associates to each x a unique prime ideal $\mathfrak{p}_x \subset B_i$. Moreover, the unique maximal ideal $\mathfrak{m}_x$ is then isomorphic to $(B_i)_{\mathfrak{p}_x}$, hence:

$$\beta_i(U_i \cap X_g) = \{\mathfrak{p}_x \in \operatorname{Spec} B_i : \gamma(g|_{U_i})_{\mathfrak{p}_x} \notin (B_i)_{\mathfrak{p}_x}\}$$

Now $\gamma(g|_{U_i}) \in B_i$, so $\gamma(g|_{U_i})_{\mathfrak{p}_x}$ is given by:

$$\gamma(g|_{U_i})_{\mathfrak{p}_x} = \frac{\gamma(g|_{U_i})}{1} \notin (B_i)_{\mathfrak{p}_x}$$

We wish to show that if $b/1 \notin (B_i)_{\mathfrak{p}_x}$ then $b \notin \mathfrak{p}_x$, however, this clear by the contrapositive, i.e. if $b \in \mathfrak{p}_x$ then $b/1 \in (B_i)_{\mathfrak{p}_x}$ as:

$$(B_i)_{\mathfrak{p}_x} = \left\{\frac{p}{s} \in (B_i)_{\mathfrak{p}_x} : p \in \mathfrak{p}_x\right\}$$

hence if $b \in \mathfrak{p}_x$ then clearly $b/1 \in (B_i)_{\mathfrak{p}_x}$. It follows that:

$$\begin{aligned}\beta_i(U_i \cap X_g) =& \{\mathfrak{p}_x \in \operatorname{Spec} B_i : \gamma(g|_{U_i}) \notin \mathfrak{p}_x\} \\ =& U_{\gamma(g|_{U_i})}\end{aligned}$$

so $X_g \cap U_i$ is open for all i implying that X_g is open. It follows that $f_\psi^{-1}(U_a)$ is open, and thus f_ψ is continuous.

We now define the the map $f_\psi^\sharp : \mathscr{O}_Y \to f_{\psi*}\mathscr{O}_X$, and by Theorem 1.4.1 it suffices to define $f_\psi^\sharp$ on distinguished open set U_a. We thus need to define morphisms:

$$A_a \cong \mathscr{O}_Y(U_a) \longrightarrow f_{\psi*}\mathscr{O}_X(U_a) = \mathscr{O}_X(X_g)$$

where $g = \psi(\varphi_A(a))$. First consider the restriction map $\theta^X_{X_g} : \mathscr{O}_X(X) \to \mathscr{O}_X(X_g)$, then we want to show that image of g is a unit in $\mathscr{O}_X(X_g)$. Recall that for a local ring any element not in $\mathfrak{m}_x$ is a unit, hence $g_x \in (\mathscr{O}_X)_x$ is a unit for all $x \in X_g$. It follows that $(g_x) \in \prod_{x \in X_g}(\mathscr{O}_X)_x$ is a unit, in particular there is a sequence $(h_x) \in \prod_{x \in X_g}(\mathscr{O}_X)_x$ such that:

$$(h_x) \cdot (g_x) = (1_x)$$

Now for each $x \in X_g$, we write $h_x = [V_x, s^x]$ for some $V_x \subset X_g$, and some $s^x \in \mathscr{O}_X(V_x)$, it follows that:

$$h_x \cdot g_x = [V_x, s^x] \cdot [X, g] = [V_x \cap X, s^x|_{V_x \cap X} \cdot g|_{V_x \cap X}] = [V_x, s^x \cdot g|_{V_x}] = 1_x$$

There is an open subset of $W_x \subset V_x$ such that $(s^x \cdot g|_{V_x})|_{W_x} = 1$, implying that $s^x|_{W_x} = (g|_{W_x})^{-1}$. Repeating this for all $x \in X_g$ gives us an open cover of X_g by W_x, along with sections $s^x|_{W_x} := t^x \in \mathscr{O}_X(W_x)$. Now that since $t^x = (g|_{W_x})^{-1}$, we have that for all $y \in W_x$:

$$t^x_y = g_y^{-1} = h_y$$

hence $(h_x)/ \in \mathscr{O}^\sharp_X(X_g)$. Since $\mathscr{O}^\sharp_X \cong \mathscr{O}_X$ it follows that there is unique element $h \in \mathscr{O}_X(X_g)$ such that $h = (g|_{X_g})^{-1}$, hence $\theta^X_{X_g}(g)$ is a unit in $\mathscr{O}_X(X_g)$ so there exists a unique morphism:

$$\begin{aligned}\mathscr{O}_X(X)_g &\longrightarrow \mathscr{O}_X(X_g) \\ s/g^k &\longmapsto s|_{X_g} \cdot h^k\end{aligned}$$

We now show that there is a map $A_a \to \mathscr{O}_X(X)_g$, however this again follows from the universal property of localization, as we have that $\psi(\varphi_A(a)) = g$, so the image of a is a unit in $\mathscr{O}_X(X)_g$. We thus have have map:

$$\begin{aligned}A_a &\longrightarrow \mathscr{O}_X(X)_g \\ b/a^k &\longmapsto \psi(\varphi_A(b))/g^k\end{aligned}$$

and hence a morphism:

$$\begin{aligned} A_a &\longrightarrow \mathscr{O}_X(X_g) \\ b/a^k &\longmapsto \psi(\varphi_A(b))|_{X_g} \cdot h^k \end{aligned}$$

This clearly commutes with restrictions maps on the base, hence we get a sheaf morphism:

$$f_\psi^\sharp : \mathscr{O}_Y \to f_{\psi_*}\mathscr{O}_X$$

The assignment $\psi \mapsto (f_\psi, f_\psi^\sharp)$ then defines a set map $\Psi : \mathrm{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X)) \to \mathrm{Hom}(X, Y)$.

We check that Φ and Ψ are inverses of one another. Let $\psi \in \mathrm{Hom}(\mathscr{O}_Y(Y), \mathscr{O}_X(X))$, then we see that:

$$\Phi \circ \Psi(\psi) = (f_\psi^\sharp)_Y$$

It suffices to check that:

$$\psi \circ \varphi_A = (f_\psi^\sharp)_Y \circ \varphi_A$$

Well, note that by construction $(f_\psi^\sharp)_Y \circ \varphi_A$ is equivalent to the composition:

$$A \longrightarrow \mathscr{O}_X(X)_1 \longrightarrow \mathscr{O}_X(X)$$

which since there is nothing to invert is the map $b \mapsto \psi(\varphi_A(b))$, hence $(f_\psi^\sharp)_Y = \psi$, and $\Phi \circ \Psi = \mathrm{Id}$.

Now let $(f, f^\sharp) \in \mathrm{Hom}(X, Y)$, and set $\phi = f_Y^\sharp$, then we want to show that:

$$(f, f^\sharp) = (f_\phi, f_\phi^\sharp)$$

We first check that the topological maps are equal, in particular, we want to show that:

$$f(x) = \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$$

Let $a \in f(x)$, then since $f(x)$ is a prime ideal, we see that $\varphi_A(a)$ is a section which vanishes at $f(x)$, i.e $\varphi_A(a)_{f(x)}$ lies in the unique maximal ideal $\mathfrak{m}_{f(x)} \cong A_{f(x)}$. It follows that:

$$\pi_x(\phi(\varphi_A(a))) = (f_Y^\sharp(\varphi_A(a)))_x = [X, f_Y^\sharp(\varphi_a)] = [f^{-1}(Y), f_Y^\sharp(\varphi_a)] = f_x(\varphi_A(a)_{f(x)})$$

since f_x is a morphism of local rings, we must have that $\pi_x(\phi(\varphi_A(a))) \in \mathfrak{m}_x$, hence $a \in \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$. Now suppose that $a \in \varphi_A^{-1}(\phi^{-1}(\pi_x^{-1}(\mathfrak{m}_x)))$, then it follows that $f_x(\varphi_A(a)_{f(x)}) \in \mathfrak{m}_x$, and if $\varphi_A(a)_{f(x)} \notin \mathfrak{m}_{f(x)}$, then $\varphi_A(a)_{f(x)}$ is a unit in $(\mathscr{O}_Y)_{f(x)}$, implying that $\mathfrak{m}_x = (\mathscr{O}_X)_x$ contradicting the fact that $\mathfrak{m}_x$ is maximal, hence $a \in f(x)$ as well, so $f(x) = f_\phi(x)$ as desired.

To check that $f_\phi^\sharp = f^\sharp$, it suffices to check they agree on distinguished opens $U_a \subset \operatorname{Spec} A$ by Corollary 1.4.2. In particular, it suffices to check that the induced maps:

$$A_a \longrightarrow \mathscr{O}_X(X_g)$$

where $g = \phi(\varphi_A(a))$ agree. Let $b/a^k \in A_a$, then:

$$\begin{aligned} (f_\phi^\sharp)_{U_a}(\varphi_A(b)/\varphi_A(a^k)) =& \phi(\varphi_A(b))|X_g \cdot h^k \\ =& f_{U_a}^\sharp(\varphi_A(b)|_{U_a}) \cdot h^k \\ =& f_{U_A}^\sharp(\varphi_A(b)|_{U_a}) \cdot (g^k|_{X_g})^{-1} \end{aligned}$$

Now note that:

$$g|_{X_g} = f_Y^\sharp(\varphi_A(a))|_{X_g} = f_{U_a}^\sharp(\varphi_A(a)|_{U_a})$$

hence:

$$(g^k|_{X_g})^{-1} = f_{U_a}^\sharp(\varphi_A(a)|_{U_a})^{-k}$$

however $\varphi_A(a)|_{U_a}$ invertible in $\mathscr{O}_Y(U_a)$ hence:

$$(g^k|_{X_g})^{-1} = f^\sharp_{U_a}(\varphi_A(a)|_{U_a}^{-k})$$

so:

$$\begin{aligned}(f^\sharp_\phi)_{U_a}(\varphi_A(b)/\varphi_A(a^k)) =& f^\sharp_{U_A}(\varphi_A(b)|_{U_a}) \cdot f^\sharp_{U_a}(\varphi_A(a)|_{U_a}^{-k})\\ =& f^\sharp_{U_a}(\varphi_A(b)|_{U_a} \cdot \varphi_A(a)|_{U_a}^{-k})\\ =& f^\sharp_{U_a}\left(\frac{\varphi_A(b)}{1} \cdot \frac{1}{\varphi_A(a)^k}\right)\\ =& f^\sharp_{U_a}(\varphi_A(b)/\varphi_A(a^k))\end{aligned}$$

implying that $f^\sharp_\phi = f^\sharp$. We thus have that:

$$\Psi \circ \Phi(f, f^\sharp) = (f_\phi, f^\sharp_\phi) = (f, f^\sharp)$$

hence $\Psi \circ \Phi = \mathrm{Id}$ implying the claim. □

Note that since $\mathbb{Z}$ is the initial object in the category of rings, there exists a unique morphism $X \to \operatorname{Spec}\mathbb{Z}$ for every scheme X. As promised earlier, we now discuss how to put an induced subscheme structure on Zariski closed subsets of a scheme.

Definition 2.1.3. Let $(X, \mathscr{O}_X)$ be a scheme, and Y a Zariski closed subset of X, then the **sheaf of ideals** is given by the assignment $U \mapsto I(U)$, where:

$$I(U) = \{s \in \mathscr{O}_X(U) : \forall x \in Y \cap U, s_x \in \mathfrak{m}_x\}$$

That is $I(U)$ is the subgroup of sections on U which vanish on $Y \cap U$.

We quickly check that this is a sheaf:

Lemma 2.1.4. *The assignment $U \mapsto I(U)$ defines a sheaf on X.*

Proof. Clearly if $s \in I(U)$, and $V \cap U \neq \emptyset$, then $s|_V \in I(U)$ as $(s|_V)_x = s_x$ for all $x \in V$, so the restriction maps are precisely the same as the ones on X. Now let U_i be an open cover for U, and $s \in I(U)$ such that $s|_{U_i} = 0$ for all i. Then since $0 \in I(U)$, and $\mathscr{O}_X$ is a sheaf it follows that $s \in I(U)$ is equal to zero, implying sheaf axiom one. To prove sheaf axiom two, take U_i as before, and let $s_i \in I(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an $s \in \mathscr{O}_X(U)$ such that $s|_{U_i} = s_i$ for all U_i. For all $x \in U$ we have that $x \in U_i$ for some i, hence $s_x = (s_i)_x$, so if $x \in Y$ then $s_x \in \mathfrak{m}_x$ implying that $s \in I(U)$ so U is a sheaf. □

Given that we have just constructed a sheaf of ideals on X, it should be obvious that we are about construct a new 'quotient sheaf' of rings on X. Our plan of action is as follows: to construct this sheaf $\mathscr{O}_X/I$, then define the structure of sheaf on Y to be $\mathscr{O}_Y = \imath^{-1}(\mathscr{O}_X/I)$, and finally to show that this gives Y the structure of a scheme, when equipped with the subspace topology.

Let us first examine the affine case. Let $X = \operatorname{Spec} A$, and $Y = \mathbb{V}(I)$ for some radical ideal I of A. Then for a distinguished open $U_g \subset \operatorname{Spec} A$, we have that:

$$\mathscr{O}_X(U_g) \cong A_g$$

Now note that $\mathbb{V}(I) \cap U_g$ is a closed subset of U_g when equipped with the subspace topology, so $\mathbb{V}(I) \cap U_g$ corresponds to the vanishing set of an ideal $I_g \subset A_g$. We claim that:

$$I_g = \left\{a/g^k \in A_g : a \in I\right\}$$

As an abuse of notation, and confidence, denote the above set by I_g, then we need to show that:

$$\mathbb{V}(I_g) = \eta(\mathbb{V}(I) \cap U_g)$$

where η is the homeomorphism $\operatorname{Spec} A \to \operatorname{Spec} A_g$. We have that:

$$\mathbb{V}(I) \cap U_g = \{\mathfrak{p} \in \operatorname{Spec} A : I \subset \mathfrak{p} \text{ and } g \notin \mathfrak{p}\}$$

so:

$$\eta(\mathbb{V}(I) \cap U_g) = \{\eta(\mathfrak{p}) \in \operatorname{Spec} A_g : I \subset \mathfrak{p} \text{ and } g \notin \mathfrak{p}\}$$

However,

$$\eta(\mathfrak{p}) = \left\{ \frac{p}{g^k} : p \in \mathfrak{p}, k \geq 0 \right\}$$

Clearly if $g \notin \mathfrak{p}$, then $\eta(\mathfrak{p}) \in \operatorname{Spec} A_g$, as otherwise η is not defined. If $I \subset \mathfrak{p}$, then we also clearly have that $I_g \subset \mathfrak{p}$, as for $a/g^k \in I_g$ we have that $a \in I \subset \mathfrak{p}$. It follows that $\eta(\mathbb{V}(I) \cap U_g) \subset \mathbb{V}(I_g)$. Now let $\mathfrak{q} \in \mathbb{V}(I_g)$, then $I_g \subset \mathfrak{q}$, and $\mathfrak{q}$ is of the form $\eta(\mathfrak{p})$ for some $\mathfrak{p} \in U_g$. Moreover, we have that $\pi^{-1}(I_g) = I \subset \mathfrak{p}$, so $\mathfrak{p} \in \mathbb{V}(I) \cap U_g$, and thus $q = \eta(\mathfrak{p}) \in \eta(\mathbb{V}(I) \cap U_g)$. It follows that $\mathbb{V}(I_g) \subset U_g \cap \mathbb{V}(I_g)$, so we obtain the desired equality. We can now calculate $I(U_g)$ to be

$$\begin{aligned} I(U_g) &= \{s \in \mathscr{O}_X(U_g) : \forall \mathfrak{q} \in \mathbb{V}(I) \cap U_g, s \in \mathfrak{q}\} \\ &\cong \{a/g^k \in A_g : \forall \mathfrak{q} \in \mathbb{V}(I_g), a/g^k \in \mathfrak{q}\} \\ &\cong \bigcap_{\mathfrak{q} \in \mathbb{V}(I_\mathfrak{q})} \mathfrak{q} \\ &\cong \sqrt{I_q} \end{aligned}$$

Note that $I_q \subset \sqrt{I_q}$ automatically, and that if $a/g^k \in I_q$ we have that there is some r such that $a^r/g^{kr} \in I_q$, implying that $a^r \in I$, so $a \in \sqrt{I} = I$ as I is radical. It follows that:

$$I(U_g) \cong I_g$$

hence:

$$\mathscr{O}_X(U_g)/I(U_g) \cong A_g/I_g$$

We now have the following lemma:

Lemma 2.1.5. *Let $X = \operatorname{Spec} A$, and $Y = \mathbb{V}(I)$ for some radical ideal I, then the assignment $U_g \mapsto A_g/I_g$ defines a sheaf on the base of distinguished opens.*

Proof. Let $U_g \subset U_f$, then recall $\sqrt{\langle g \rangle} \subset \sqrt{\langle f \rangle}$, so there exists an $m > 0$, and $b \in A$ such that $g^m = f \cdot b$. It follows that the image of f is a unit in A_g, so we get a restriction map given by:

$$\frac{a}{f^k} \longmapsto \frac{a \cdot b^k}{g^{mk}}$$

If $a/f^k \in I_f$, then $a \in I$, and certainly $a \cdot b^k \in I$ hence $a/f^k|_{U_g} \in I_g$. It follows that we get well defined restriction maps given by $A_f/I_f \to A_g/I_g$:

$$[a/f^k] \longmapsto [a \cdot b^k/g^{mk}]$$

so we have a presheaf on the distinguished opens.

By Lemma 1.4.1 and Lemma 1.4.2 it suffices to take all covers to be finite. Now let U_{g_i} be an open cover U_f, and $[a/f^k] \in A_f/I_f$ such that $[a/f^k]|_{U_{g_i}} = 0$ for all i. Note, that $[a/f^k]$ induced a unique element in $\mathscr{O}_X(U_f)/I(U_f)$, and similarly for it's restrictions. Denote this element by s, if the restrictions are all 0, then $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in Y \cap U_f$, as $s_\mathfrak{p} = (s|_{U_{g_i}})_\mathfrak{p}$ and U_{g_i} cover U_f. It follows that $s \in I(U_f)$, hence $[a/f^k] \in I_g$, so $[a/f^k] = 0$.

Now let U_{g_i} be an open cover of U_f and $[a/g_i^{k_i}] \in A_{g_i}/I_{g_i}$ such that:

$$\left[\frac{a \cdot g_j^{k_i}}{(g_i g_j)^{k_i k_j}} \right] = \left[\frac{a \cdot g_i^{k_j}}{(g_i g_j)^{k_i k_j}} \right] \tag{2.1.4}$$

for all $U_{g_ig_j} = U_{g_i} \cap U_{g_j}$. We first show that $A_{g_i}/I_{g_i} \cong (A/I)_{[g_i]}$. Define the map:

$$\begin{aligned} A &\longrightarrow (A/I)_{[g_i]} \\ a &\longmapsto [a]/1 \end{aligned}$$

and note that g_i is clearly a unit in this map, with inverse given by $1/[g_i]$ so we have a unique homomorphism

$$\begin{aligned} A_{g_i} &\longrightarrow (A/I)_{[g_i]} \\ a/g_i^k &\longmapsto [a][g_i]^k \end{aligned}$$

This map is clearly surjective; now let $a/g_i^k \in I_{g_i}$, then $a \in I$, so $[a] = 0$, hence $a/g_i^k \mapsto 0/[g_i]^k = 0$. Suppose that $a/g_i^k \mapsto 0$, then we have that:

$$\frac{[a]}{[g_i]^k} = 0 \Rightarrow [g_i]^M \cdot [a] = 0$$

We see that this implies that $g_i^M \cdot a \in I$, so $g_i^M \cdot a/1 \in I_{g_i}$, implying that $a/1 \in I_{g_i}$, hence $a/g_i^k \in I_{g_i}$. It follows that the kernel of the map is equal to I_{g_i} hence the induced unique homomorphism:

$$\begin{aligned} A_{g_i}/I_{g_i} &\longrightarrow (A/I)_{[g_i]} \\ [a/g_i^k] &\longmapsto [a]/[g_i]^k \end{aligned}$$

is an isomorphism. The expression (2.3) is then equivalent to:

$$\frac{[a \cdot g_j^{k_i}]}{[(g_ig_j)^{k_ik_j}]} = \frac{[a \cdot g_i^{k_j}]}{[(g_ig_j)^{k_ik_j}]}$$

The same argument in Proposition 1.4.3 then proves the claim, as we are now just dealing with localizations of some ring A/I.

□

Take any $g \in I$, then note that $\mathbb{V}(I) \cap U_g = \emptyset$, indeed if $\mathfrak{p} \in \mathbb{V}(I)$, then $I \subset \mathfrak{p}$, hence $g \in \mathfrak{p}$, so $\mathfrak{p} \notin U_g$. Moreover, if $\mathfrak{p} \in U_g$, then $g \notin \mathfrak{p}$, so $\mathfrak{p} \notin \mathbb{V}(I)$. It follows that $\mathbb{V}(I_g) = \emptyset$, implying that $I_g = A_g$, hence:

$$\mathscr{O}_X(U_g)/I(U_g) = \{0\}$$

Lemma 2.1.6. *The assignment $U \mapsto \mathscr{O}_X(U)/I(U)$, where U is open and affine defines a sheaf on the basis of affine opens for X.*

Proof. Let U and be V be open affines in X such that $V \subset U$. Then, we define restriction maps by:

$$[s] \in \mathscr{O}_X(U)/I(U) \longmapsto [\theta_V^U(s)] \in \mathscr{O}_X(V)/I(U)$$

i.e. we choose a class representative $s \in [s]$, restrict to $\mathscr{O}_X(V)$ and the project again. Since I is a sheaf of ideals, it follows that this is independent of the class representative chosen, and thus well defined.

Now let U be open affine, U_i an open cover of U by open affines, and $[s] \in \mathscr{O}_X(U)/I(U)$ such that $[s]|_{U_i} = 0$ for all i. This implies that $s \in [s]$ restricts to an element in $I(U_i)$ for all i, and since $s_x = (s|_{U_i})_x$ for all $x \in U_i$, it follows that for all $x \in Y \cap U$ we have $s_x \in \mathfrak{m}_x$, implying that $s \in I(U)$, hence $[s] = 0$, so sheaf on a base axiom one is satisfied.

Now let U be open and affine, U_i be an open cover of U by open affines and $[s_i] \in \mathscr{O}_X(U_i)/I(U_i)$ be sections such that for all open affines $U_{ij} \subset U_i \cap U_j$ we have:

$$[s_i]|_{U_{ij}} = [s_j]|_{U_{ij}}$$

Now note that U is isomorphic as a scheme to $\operatorname{Spec} A$ for some ring A, so we can take $\{U_i\}$ to be a finite open cover by Lemma 1.4.1 and Lemma 1.4.2. We also have that $U \cap Y \cong \mathbb{V}(J)$ for some radical ideal $J \subset A$. Under this identification, each U_i can be written as a finite union of distinguished opens of $\operatorname{Spec} A$:

$$U_i = \bigcup_{a_i \in A} U_{a_i}$$

and see that:

$$\begin{aligned} U_i \cap U_j &= \left(\bigcup_{a_i} U_{a_i} \right) \cap \left(\bigcup_{a_j} U_{a_j} \right) \\ &= \bigcup_{a_i, a_j} U_{a_i} \cap U_{a_j} \\ &= \bigcup_{a_i, a_j} U_{a_i \cdot a_j} \end{aligned}$$

Now note that $U_{a_i \cdot a_j}$ is then an affine open subset of $U_i \cap U_j$, hence:

$$[s_i]|_{U_{a_i \cdot a_j}} = [s_j]|_{U_{a_i \cdot a_j}}$$

Moreover, we have that:

$$[s_i]|_{U_{a_i}}|_{U_{a_i \cdot a_j}} = [s_i]|_{U_{a_i \cdot a_j}}$$

and that for a_i and b_i we clearly have that $[s_i]|_{U_{a_i \cdot b_i}} = [s_i]|_{U_{a_i \cdot b_i}}$, so by reindexing to include all a_i, we obtain a finite open cover of $\operatorname{Spec} A$ by distinguished opens $\{U_{a_i}\}_{i \in I}$, and sections $[t_i] := [s_i]|_{U_{a_i}} \in \mathscr{O}_{\operatorname{Spec} A}(U_{a_i})/I(U_{a_i}) \cong (A/J)_{[a_i]}$ such that:

$$[t_i]|_{U_{a_i} \cap U_{a_j}} = [t_j]|_{U_{a_i} \cap U_{a_j}}$$

for all $U_{a_i} \cap U_{a_j}$. Lemma 2.1.5 then gives us an element $[s] \in \mathscr{O}_{\operatorname{Spec} A}(U)/I(U) \cong A/J$ such that $[s]|_{U_{a_i}} = [t_i]$ for all i. We show that $[s]|_{U_i} = [s_i]$. Recall that U_i is covered by distinguished opens U_{a_i}, and that for each a_i:

$$([s]|_{U_i} - [s_i])\,|_{U_{a_i}} = [t_i] - [t_i] = 0 \tag{2.1.5}$$

it follows by sheaf on a base axiom one that $[s]|_{U_i} = [s_i]$, implying the claim. □

Proposition 2.1.3. *Let $X = \operatorname{Spec} A$, $Y = \mathbb{V}(J)$ for some radical ideal J, I be the sheaf of ideals induced by Y, $\mathscr{O}_X/I$ the sheaf induced by Lemma 2.1.5, and $\iota : Y \to X$ the inclusion map. Then Y, equipped with subspace topology, and the structure sheaf $\mathscr{O}_Y = \iota^{-1}\mathscr{O}_X/I$ is an affine scheme isomorphic to* $\operatorname{Spec} A/J$.

Proof. We first define a homeomorphism $f : \mathbb{V}(J) \to \operatorname{Spec} A/J$. Let $\pi : A \to A/J$ be the projection map, and $\mathfrak{p} \subset \mathbb{V}(J)$, then we claim that $\pi(\mathfrak{p}) \subset A/J$ is a prime ideal in A/J. It is clear that $\pi(\mathfrak{p})$ is a group, we check that $\pi(\mathfrak{p})$ is an ideal. Suppose that $[a] \in \pi(\mathfrak{p})$ and $[b] \in A/J$, then there we see there is some $i \in J$ such that $a + i \in \mathfrak{p}$, and it follows that $(a + i) \cdot b \in \mathfrak{p}$. We thus must have that $[(a + i) \cdot b] \in \pi(\mathfrak{p})$, however:

$$[(a + i) \cdot b] = [ab + ib] = [ab] + i[b] = [ab] = [a] \cdot [b]$$

so $\pi(\mathfrak{p})$ swallows multiplication and is thus an ideal. We now show that $\pi(\mathfrak{p})$ is prime, let $[a]$ and $[b] \in A/J$, such that $[a] \cdot [b] \in \pi(\mathfrak{p})$. It follows that $[a \cdot b] \in \pi(\mathfrak{p})$, hence there is some $j_{ab} \in J$ such that $a \cdot b + j_{ab} \in \mathfrak{p}$. Since $\mathfrak{p}$ is closed under addition, and $-j_{ab} \in J \subset \mathfrak{p}$, it follows that $a \cdot b \in \mathfrak{p}$, hence either a or $b \in \mathfrak{p}$, implying that either $[a]$ or $[b]$ lies in $\pi(\mathfrak{p})$.

We thus define:

$$f : \mathbb{V}(J) \longrightarrow \operatorname{Spec} A/J$$

by $\mathfrak{p} \mapsto \pi(\mathfrak{p})$. This map is surjective, as if $\mathfrak{q} \in \operatorname{Spec} A/J$, we have that $\pi(\pi^{-1}(\mathfrak{q})) = \mathfrak{q}$, since π is surjective. Now suppose that $\pi(\mathfrak{p}) = \pi(\mathfrak{q})$, then we need to show that $\mathfrak{p} = \mathfrak{q}$. Let $a \in \mathfrak{p}$, then $[a] \in \pi(\mathfrak{p})$, and $[a] \in \pi(\mathfrak{q})$. Since $[a] \in \pi(\mathfrak{q})$, there is a $j \in J$ such that $a + j \in \mathfrak{q}$. However $J \subset \mathfrak{q}$, and again $\mathfrak{q}$ is closed under subtraction so $a + j - j = a \in \mathfrak{q}$, and $\mathfrak{p} \subset \mathfrak{q}$. The same argument shows that $\mathfrak{q} \subset \mathfrak{p}$, implying injectivity.

We claim that this map is continuous, and it suffices to check this on basic opens. Let $U_{[g]}$ be a distinguished open, then:

$$\begin{aligned} f^{-1}(U_{[g]}) =& \{\mathfrak{p} \in \mathbb{V}(J) : [g] \notin \pi(\mathfrak{p})\} \\ =& \{\mathfrak{p} \in \mathbb{V}(J) : \langle [g] \rangle \not\subset \pi(\mathfrak{p})\} \\ =& \{\mathfrak{p} \in \mathbb{V}(J) : \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}\} \end{aligned}$$

We claim that:

$$\{\mathfrak{p} \in \mathbb{V}(J) : \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}\} = \{\mathfrak{p} \in \mathbb{V}(J) : \langle g \rangle \not\subset \mathfrak{p}\}$$

Let $\mathfrak{p} \in \mathbb{V}(J)$ such that $\pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}$, then we want to show that $\langle g \rangle \not\subset \mathfrak{p}$. Well, we have that there exists an $a \in \pi^{-1}(\langle [g] \rangle) \not\subset \mathfrak{p}$, and $a = b \cdot g^k + j$ for some $j \in J$. Clearly, $b \cdot g^k + j \notin \mathfrak{p}$, but $j \in J \subset \mathfrak{p}$, so the only way this holds is if $b \cdot g^k \notin \mathfrak{p}$. We have that $b \cdot g^k \in \langle g \rangle$, so $\langle g \rangle \not\subset \mathfrak{p}$. Now suppose that $\langle g \rangle \not\subset \mathfrak{p}$, then there exists some $a \in \langle g \rangle$ such that $a \notin \mathfrak{p}$. However, $a = b \cdot g^k$, so $[a] = [b] \cdot [g]^k \in \langle [g] \rangle$, hence $a \in \pi^{-1}(\langle [g] \rangle)$ implying that $\langle [g] \rangle \not\subset \mathfrak{p}$. It follows that:

$$\begin{aligned} f^{-1}(U_{[g]}) =& \{\mathfrak{p} \in \mathbb{V}(J) : \langle g \rangle \not\subset \mathfrak{p}\} \\ =& \mathbb{V}(J) \cap U_g \end{aligned}$$

which is open the subspace topology.

We claim that this map is open and thus a homeomorphism. Note that $\{\mathbb{V}(J) \cap U_g\}_{g \in A}$ is a basis for $\mathbb{V}(J)$, and since f is a bijection, we have that:

$$f(\mathbb{V}(J) \cap U_g) = f(f^{-1}(U_{[g]}) = U_{[g]}$$

so f is open.

Now note that if $\imath : \mathbb{V}(J) \to \operatorname{Spec} A$ is the inclusion map, and $f^{-1} : \operatorname{Spec} A/J \to \mathbb{V}(J)$ is the homeomorphism, we have that $\imath \circ f^{-1} : \operatorname{Spec} A/J \to \operatorname{Spec} A$ comes from the ring homomorphism $\pi : A \to A/J$. We want to construct a sheaf isomorphism:

$$(f^{-1})^\sharp : \imath^{-1}\mathscr{O}_X/I \longrightarrow f_*^{-1}\mathscr{O}_{\operatorname{Spec} A/J}$$

and by Theorem 1.3.1 and Corollary 2.1.1 it suffices to define a sheaf isomorphism:

$$F : \mathscr{O}_X/I \longrightarrow \imath_* f_*^{-1}\mathscr{O}_{\operatorname{Spec} A/J} = (\imath \circ f^{-1})_*\mathscr{O}_{\operatorname{Spec} A/J}$$

We do so on a basis of distinguished opens U_g. Since $f^{-1} \circ \imath$ is topological map coming from the the projection $\pi : A \to A/J$, we have that if $g \in J$, then $(\imath \circ f^{-1})^{-1}(U_g) = U_{[g]} = U_0 = \emptyset$. By our earlier discussion we have that $\mathscr{O}_X/I(U_g) = \{0\}$ so our isomorphism on these open sets is trivial.

In the case where $g \notin J$, we have that $\mathscr{O}_X/I(U_g) = A_g/J_g \cong (A/J)_{[g]}$, while $\mathscr{O}_{\operatorname{Spec} A/J}(U_{[g]}) = (A/J)_{[g]}$. These isomorphisms clearly commute with restrictions on a distinguished base, so F is an isomorphism. We define $(f^{-1})^\sharp$ to be the sheaf isomorphism induced by the isomorphism in Theorem 1.3.1, hence $(f^{-1}, (f^{-1})^\sharp) : \operatorname{Spec} A/J \to \mathbb{V}(J)$ is an isomorphism as desired. □

We can now prove the desired claim:

Theorem 2.1.2. *Let X be a scheme, Y a Zariski closed subset of X, and I the sheaf of ideals on X induced by Y. Then there exists the natural structure of a scheme on Y, such that for all affine opens $U \subset X$, $\mathscr{O}_Y(U \cap Y) \cong \mathscr{O}_X(U)/I(U)$.*

Proof. Equip Y with the subspace topology, and the sheaf $\iota^{-1}\mathscr{O}_X/I$, where $\mathscr{O}_X/I$ is the sheaf induced by Lemma 2.1.6, and $\iota : Y \to X$ is the inclusion map. We need to show that every point in Y has an open neighborhood isomorphic to an affine scheme. Let $y \in Y$, then since $Y \subset X$, there is an open neighborhood U of y in X, such that $U \cong \operatorname{Spec} A$, and let $f : U \to \operatorname{Spec} A$ be the isomorphism. Now note that $U \cap Y$ is open in subspace topology on Y, and closed in the subspace topology on U. It follows that there is radical ideal $J \subset A$, such that $f(U \cap Y) = \mathbb{V}(J) \subset \operatorname{Spec} A$. Now $f^\sharp$ gives an isomorphism:

$$f^\sharp : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow f_*\mathscr{O}_U = f_*(\mathscr{O}_X|_U)$$

We claim that this induces an isomorphism:

$$f^\sharp : I_{\mathbb{V}(J)} \longrightarrow f_*(I|_U)$$

where $I_{\mathbb{V}(J)}$ is the sheaf of ideals on $\operatorname{Spec} A$ induced by $\mathbb{V}(J)$. Indeed, let $V \subset \operatorname{Spec} A$; if $s \in I_{\mathbb{V}(J)}(V)$, then $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in V \cap \mathbb{V}(J)$. For all all $\mathfrak{p} \in V \cap \mathbb{V}(J)$ let $\mathfrak{p} = f(x)$ for some unique $x \in f^{-1}(V) \cap U \cap Y$. Then since f is a morphism of locally ringed spaces we have that $f_x(s_\mathfrak{p}) \in \mathfrak{m}_x$ for all $x \in f^{-1}(V)$, hence:

$$f_x(s_\mathfrak{p}) = f_x([V, s]_\mathfrak{p}) = [f^{-1}(V), f^\sharp_V(s)]_x = (f^\sharp_V(s))_x \in \mathfrak{m}_x$$

so $f^\sharp_V(s) \in I(f^{-1}(V))$ for all $s \in I_{\mathbb{V}(J)}(V)$. Now let $t \in I(f^{-1}(V))$, then since $f^\sharp_V$ is an isomorphism there exists an $s \in \mathscr{O}_{\operatorname{Spec} A}(V)$ such that $f^\sharp_V(s) = t$, so $f_x(s_\mathfrak{p}) = t_x \in \mathfrak{m}_x$ for all $x \in f^{-1}(V) \cap Y$, where $\mathfrak{p} = f(x)$. However, f_x is an isomorphism, so since $\mathfrak{m}_x$ is the unique maximal ideal, and isomorphisms map maximal ideals to maximal ideals, we must have that $f_x(\mathfrak{m}_\mathfrak{p}) = \mathfrak{m}_x$, hence $s_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}$ for all $\mathfrak{p} \in \mathbb{V}(J) \cap V$. It follows that $f^\sharp$ induces an isomorphism of ideal sheafs, as desired.

We now claim that this induces an isomorphism:

$$\tilde{f}^\sharp : \mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)} \longrightarrow f_*(\mathscr{O}_X/I|_U)$$

Indeed, note that for any distinguished open set U_g, we clearly have that $f^{-1}(U_g) \subset U \subset X$ is then clearly an affine open, and we have that:

$$(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)})(U_g) \cong \mathscr{O}_{\operatorname{Spec} A}(U_g)/I_{\mathbb{V}(J)}(U_g)$$

while:

$$f_*(\mathscr{O}_X/I|_U)(U_g) = \mathscr{O}_X/I(f^{-1}(U_g)) \cong \mathscr{O}_X(f^{-1}(U_g))/I(f^{-1}(U_g))$$

By Corollary 1.4.2 it thus suffices to define morphisms:

$$\psi_{U_g} : \mathscr{O}_{\operatorname{Spec} A}(U_g)/I_{\mathbb{V}(J)}(U_g) \longrightarrow \mathscr{O}_X(f^{-1}(U_g))/I(f^{-1}(U_g))$$

which commute with restriction maps, but we clearly already have one induced by $f^\sharp$. Indeed, set:

$$\psi_{U_g}([s]) = [f^\sharp_V(s)]$$

which is clearly well defined, and obviously commute with said restrictions. The maps then induce the desired isomorphism of sheaves $\tilde{f}^\sharp : \mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)} \longrightarrow f_*(\mathscr{O}_X/I|_U)$.

We now switch to the topological picture and equip $U \cap Y$ with the subspace topology induced by Y, and $\mathbb{V}(J)$ equipped with the subspace topology on $\operatorname{Spec} A$, we want $f|_{U\cap Y} : U \cap Y \to \mathbb{V}(J)$ to be a homeomorphism. We first see that it is continuous, as if $W \subset \mathbb{V}(J)$ is open, then $W = V \cap \mathbb{V}(J)$ for some open subset $V \subset \operatorname{Spec} A$. It follows that:

$$f|_{U\cap Y}^{-1}(W) = f^{-1}(V \cap \mathbb{V}(J)) = f^{-1}(V) \cap f^{-1}(\mathbb{V}(J)) = f^{-1}(V) \cap (U \cap Y)$$

We see that $f^{-1}(V)$ is open in U, and thus open in X, so it follows that $f^{-1}(V) \cap Y$ is open in Y. Since:

$$f^{-1}(V) \cap (U \cap Y) = (f^{-1}(V) \cap Y) \cap (U \cap Y)$$

it follows that $f|_{U\cap Y}^{-1}(W)$ is open in $U\cap Y$, so f is continuous. Now let $W\subset U\cap Y$ be open, then $W=V\cap(U\cap Y)$ for some open subset $V\subset Y$, but for V to be open in Y we must have that $V=Z\cap Y$ for some Z open in X. We see that:

$$f|_{U\cap Y}(W)=f(Z\cap Y\cap(U\cap Y))=f(Z\cap(U\cap Y))=f((Z\cap U)\cap(U\cap Y))$$

and since f is a bijection:

$$f|_{U\cap Y}(W)=f(Z\cap U)\cap f(U\cap Y)=f(Z\cap U)\cap\mathbb{V}(J)$$

since $f:U\to\operatorname{Spec}A$ is a homeomorphism, it follows that $f(Z\cap U)$ is open in $\operatorname{Spec}A$, hence $f(Z\cap U)\cap\mathbb{V}(J)$ is open in $\mathbb{V}(J)$. We thus that have that $f|_{U\cap Y}$ is a homeomorphism $U\cap Y\to\mathbb{V}(J)$.

So now we have a homeomorphism $g=f|_{U\cap Y}:U\cap Y\longrightarrow\mathbb{V}(J)$, we claim that there then exists an isomorphism of sheaves:

$$\imath_{\mathbb{V}(J)}^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})\cong g_*(\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y})$$

We shall prove this by use of Theorem 1.3.1 and Corollary 2.1.1, and by noting that we have the following commutative square of topological maps:

$$\begin{tikzcd}
U \arrow[r, "f"] & \operatorname{Spec}A \\
U\cap Y \arrow[u, "\imath_{U\cap Y}"] \arrow[r, "g"] & \mathbb{V}(J) \arrow[u, "\imath_{\mathbb{V}(J)}"']
\end{tikzcd}$$

Now note that by Corollary 2.1.1 suffices to show that:

$$g^{-1}\left(\imath_{\mathbb{V}(J)}^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})\right)\cong\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Now note that since $(\imath_{\mathbb{V}(J)}\circ g)_*=g_*\circ\imath_{\mathbb{V}(J)*}$, so by Theorem 1.3.1, we have that $(\imath_{\mathbb{V}(J)}\circ g)^{-1}=g^{-1}\circ\imath_{\mathbb{V}(J)}^{-1}$, and by the diagram above we have that $\imath_{\mathbb{V}(J)}\circ g=f\circ\imath_{U\cap Y}$, so it suffices to show that:

$$(f\circ\imath_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})\cong\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Now we have that:

$$(f\circ\imath_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})=\imath_{U\cap Y}^{-1}\left(f^{-1}\left(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)}\right)\right)$$

Now by our earlier, we work we have that the image of $\tilde{f}^\sharp$ under the isomorphism in Theorem 1.3.1 gives an isomorphism $f^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})\cong\mathscr{O}_X/I|_U$, hence we have that:

$$(f\circ\imath_{U\cap Y})^{-1}(\mathscr{O}_{\operatorname{Spec}A}/I_{\mathbb{V}(J)})\cong\imath_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U)$$

so it suffices to show that:

$$\imath_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U)\cong\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}$$

Recall that $\mathscr{O}_X/I|_U\cong\imath_U^{-1}\mathscr{O}_X/I$ by Corollary 1.3.2, so we have that the left hand side satisfies:

$$\imath_{U\cap Y}^{-1}(\mathscr{O}_X/I|_U)\cong(\imath_U\circ\imath_{U\cap Y})^{-1}\mathscr{O}_X/I$$

while the right hand side satisfies:

$$\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y}\cong(\iota\circ\iota_{U\cap Y})^{-1}\mathscr{O}_X/I$$

Now the issue is that technically have two different inclusion maps $\iota_{U\cap Y}$. The first is $\imath_{U\cap Y}:U\cap Y\to U$, and the the second is $\iota_{U\cap Y}:U\cap Y\to Y$, however, clearly when composed with $\imath_U:U\to X$, and $\imath:Y\to X$, we find that $\iota\circ\iota_{U\cap Y}=\imath_U\circ\imath_{U\cap Y}$, as topological maps. It follows that:

$$(\imath_U\circ\imath_{U\cap Y})^{-1}\mathscr{O}_X/I=(\iota\circ\iota_{U\cap Y})^{-1}\mathscr{O}_X/I$$

So reversing this chain of isomorphisms gives the desired result:

$$g^\sharp : \iota^{-1}_{\mathbb{V}(J)}(\mathscr{O}_{\operatorname{Spec} A}/I_{\mathbb{V}(J)}) \longrightarrow g_*(\iota^{-1}(\mathscr{O}_X/I)|_{U\cap Y})$$

It follows that $(U\cap Y, \mathscr{O}_Y|_{U\cap Y}) \cong (\mathbb{V}(J), \mathscr{O}_{\mathbb{V}(J)})$, and hence by Proposition 2.1.3 that $(U\cap Y, \mathscr{O}_Y|_{U\cap Y}) \cong (\operatorname{Spec} A/J, \mathscr{O}_{\operatorname{Spec} A/J})$, so Y is indeed a scheme.

Moreover, we see that:

$$\mathscr{O}_Y(U\cap Y) = \mathscr{O}_Y|_{U\cap Y}(U\cap Y) \cong \mathscr{O}_{\operatorname{Spec} A/J}(\operatorname{Spec} A/J) \cong A/J$$

while:

$$\mathscr{O}_X(U) \cong A$$

and:

$$I(U) \cong J \subset A$$

hence:

$$\mathscr{O}_X(U)/I(U) \cong A/J$$

implying the claim. □

2.2 The Proj Construction

Our very first examples of schemes were affine ones, and now pretty much all the examples we have encountered are either open/closed subschemes of affine schemes, or a gluing of two affine schemes. In fact Example 2.1.5 is the motivating example for this section, it being the simplest example of what we will call a projective scheme. Indeed, our goal in this section is to discuss the analogue of projective space in differential geometry. We begin with the following example; reader be warned this is a mildly messy computation, and the checking of certain details are most likely best done on your own.

Example 2.2.1. Consider the variables $x_0, \ldots, x_n$, and $n+1$ rings:

$$A_i = \mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right] \cong \mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] / \langle x_i/x_i - 1\rangle$$

which gives us $n+1$ schemes $X_i = \operatorname{Spec} A_i$. We note that for each i, the x_j/x_i is just a dummy variable to remind us of how this object is related to the coordinate charts on $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus 0/\mathbb{C}^*$.

For all i, j and we set $U_{ij} \subset X_i$ to be U_{x_j/x_i}, i.e the distinguished open set corresponding to the localization of A_i at x_j/x_i. We need to write down isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$, and since all schemes are affine, it suffices to provide ring homomorphisms:

$$\phi^\sharp_{ij} : (A_j)_{x_i/x_j} \longrightarrow (A_i)_{x_j/x_i}$$

We suggestively denote $1/(x_j/x_i)$ by x_i/x_j, and consider the morphism:

$$\xi^\sharp_{ij} : \mathbb{C}\left[\frac{x_0}{x_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_n}{x_j}\right] \longrightarrow \mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]$$

induced by the following assignment on generators:

$$x_k/x_j \longmapsto \begin{cases} (x_k/x_i)\cdot(x_i/x_j)\cdot & \text{if } k \neq i \\ x_i/x_j & \text{if } k = i \end{cases}$$

Per our suggestive notation, we see that x_i/x_j is then a unit under the image of $\xi^\sharp_{ij}$ as:

$$\xi^\sharp_{ij}(x_i/x_j)\cdot x_j/x_i = (x_i/x_j)\cdot(x_j/x_i) = (1/(x_j/x_i))(x_j/x_i) = 1$$

We thus set $\phi_{ij}^\sharp$ to be the unique morphism induced by the universal property of localization, which is given on generators by:

$$\phi_{ij}^\sharp : \mathbb{C}\left[\frac{x_0}{x_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right] \longrightarrow \mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_i)\cdot(x_i/x_j) & \text{if } l \neq i, m = j \\ x_i/x_j & \text{if } l = i, m = j \\ x_j/x_i & \text{if } l = j, m = i \end{cases}$$

Note that this is an isomorphism, as the map in the other direction

$$\phi_{ji}^\sharp : \mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_j}{x_i}\right] \longrightarrow \mathbb{C}\left[\frac{x_0}{x_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_j)\cdot(x_j/x_i) & \text{if } l \neq j, m = i \\ x_j/x_i & \text{if } l = j, m = i \\ x_i/x_j & \text{if } l = i, m = j \end{cases}$$

satisfies $\phi_{ji}^\sharp = (\phi_{ij}^\sharp)^{-1}$, hence, we also have that the induced scheme morphisms must satisfy $\phi_{ij} = \phi_{ji}^{-1}$. Now we note that:

$$U_{ij} \cap U_{ik} = U_{x_j/x_i} \cap U_{x_k/x_i} = U_{(x_j/x_i)(x_k/x_i)} \cong \operatorname{Spec}\mathbb{C}\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}, \frac{x_i}{x_k}\right]$$

while:

$$U_{ji} \cap U_{jk} = U_{x_i/x_j} \cap U_{x_k/x_j} = U_{(x_i/x_j)(x_k/x_j)} \cong \operatorname{Spec}\mathbb{C}\left[\frac{x_0}{x_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}, \frac{x_j}{x_k}\right]$$

where again we have that $(x_i/x_k) := (x_k/x_i)^{-1}$ and $(x_j/x_k) := (x_k/x_j)^{-1}$. We thus want $\phi_{ij}(U_{(x_j/x_i)(x_k/x_i)}) = U_{(x_i/x_j)(x_k/x_j)}$, and consider $U_{(x_j/x_i)(x_k/x_i)}$ and $U_{(x_i/x_j)(x_k/x_j)}$ as distinguished open sets of the affine schemes $U_{x_j/x_i} \cong \operatorname{Spec}\mathbb{C}[\{x_k/x_i\}_{k\neq i}, x_i/x_j]$ and $U_{x_i/x_j} \cong \operatorname{Spec}\mathbb{C}[\{x_k, x_j\}_{k\neq j}, x_j/x_i]$. Note that if $k = i$, then the statement is trivial.

Now suppose that $\mathfrak{p} \in U_{(x_i/x_j)(x_k/x_j)}$, then we have that $\mathfrak{p} \in \operatorname{Spec}\mathbb{C}[\{x_k, x_j\}_{k\neq j}, x_j/x_i]$, and $x_k/x_j \notin \mathfrak{p}$, it follows that since $\phi_{ij}^\sharp : (A_j)_{x_i/x_j} \to (A_i)_{x_j/x_i}$ is an isomorphism, that $\phi_{ij}^\sharp(\mathfrak{p})$ is a prime ideal of $(A_i)_{x_j/x_i}$, which satisfies $(\phi_{ij}^\sharp)^{-1}(\phi_{ij}^\sharp(\mathfrak{p})) = \phi_{ij}(\phi_{ij}^\sharp(\mathfrak{p})) = \mathfrak{p}$. Moreover, since $x_k/x_j \notin \mathfrak{p}$, we have that $\phi_{ij}^\sharp(x_k/x_j) = x_k/x_i \cdot x_i/x_j \notin \phi_{ij}^\sharp(\mathfrak{p})$, hence $x_k/x_i \notin \phi_{ij}^\sharp(\mathfrak{p})$ as x_i/x_j is a unit in $(A_i)_{x_j/x_i}$. It follows that $\phi_{ij}^\sharp(\mathfrak{p}) \in U_{(x_j/x_i)(x_k/x_i)}$, so $\mathfrak{p} \in \phi_{ij}(U_{(x_j/x_i)(x_k/x_i)})$.

We now let $\mathfrak{p} \in \phi_{ij}(U_{(x_j/x_i)(x_k/x_i)})$, then $\mathfrak{p} = (\phi_{ij}^\sharp)^{-1}(\mathfrak{q})$, for some $\mathfrak{q} \in U_{(x_j/x_i)(x_k/x_i)}$, implying that $x_k/x_i \notin \mathfrak{q}$. Since $x_k/x_i \notin \mathfrak{q}$, we have that $(\phi_{ij}^\sharp)^{-1}(x_k/x_i) = x_k/x_j \cdot (x_j/x_i) \notin (\phi_{ij}^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}$, hence $x_k/x_j \notin \mathfrak{p}$. It follows by the same argument that $\mathfrak{p} \in U_{(x_i/x_j)(x_k/x_j)}$, so $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ as desired.

We now need to check that on $U_{ij} \cap U_{ik}$ we have:

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij}$$

Now note that $\phi_{ik}(U_{ij} \cap U_{ik}) = U_{kj} \cap U_{ki} = U_{(x_j/x_k)\cdot(x_k/x_i)} \cong \operatorname{Spec}\mathbb{C}\{\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i\}$, so the ring map which induces the morphism of schemes $\phi_{ik}|_{U_{ij}\cap U_{ik}} : U_{ij} \cap U_{ik} \longrightarrow U_{kj} \cap U_{kj}$ is given on generators by:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_k)\cdot(x_k/x_i)}} : \mathbb{C}[\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i] \longrightarrow \mathbb{C}[\{x_l/x_i\}_{l\neq i}, (x_i/x_j), (x_i/x_k)]$$
$$x_l/x_m \longmapsto \begin{cases} (x_l/x_i)\cdot(x_i/x_k) & \text{if } l \neq i, m = k \\ x_i/x_k & \text{if } l = i, m = k \\ x_k/x_i & \text{if } l = k, m = i \\ (x_i/x_j)\cdot(x_k/x_i) & \text{if } l = k, m = j \end{cases}$$

Now we essentially want to show that:

$$(\phi^\sharp_{ik})_{U_{(x_j/x_k)\cdot(x_k/x_i)}} = (\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)}$$

well similarly we have that $(\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)}$ is given on generators by:

$$(\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)} : \mathbb{C}[\{x_l/x_k\}_{l\neq k}, x_k/x_j, x_k/x_i] \longrightarrow \mathbb{C}[\{x_l/x_j\}_{l\neq j}, (x_j/x_i), (x_j/x_k)]$$

$$x_l/x_m \longmapsto \begin{cases} (x_l/x_j)\cdot(x_j/x_k) & \text{if } l\neq j, m=k \\ x_j/x_k & \text{if } l=j, m=k \\ x_k/x_j & \text{if } l=k, m=j \\ (x_j/x_i)\cdot(x_k/x_j) & \text{if } l=k, m=i \end{cases}$$

while:

$$(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}} : \mathbb{C}[\{x_l/x_j\}_{l\neq j}, (x_j/x_i), (x_j/x_k)] \longrightarrow \mathbb{C}[\{x_l/x_i\}_{l\neq i}, (x_i/x_j), (x_i/x_k)]$$

$$x_l/x_m \longmapsto \begin{cases} (x_l/x_i)\cdot(x_i/x_j) & \text{if } l\neq i, m=j \\ x_i/x_j & \text{if } l=i, m=j \\ x_j/x_i & \text{if } l=j, m=i \\ (x_i/x_k)\cdot(x_j/x_i) & \text{if } l=j, m=k \end{cases}$$

We now check that these agree on generators. Let $x_l/x_k \in \mathbb{C}[\{x_l, x_k\}_{l\neq k}, x_k/x_i, x_k/x_j]$, such that $l\neq j$, then:

$$(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)}(x_l/x_k) = (\phi^\sharp_{ij})_{U_{(x_i/x_j)}}((x_l/x_j)\cdot(x_j/x_k))$$

If $l\neq i$, we have that:

$$(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_l/x_j)\cdot(x_j/x_k)) = (x_l/x_i)\cdot(x_i/x_j)\cdot(x_i/x_k)\cdot(x_j/x_i) = (x_l/x_i)\cdot(x_i/x_k)$$

however:

$$(\phi^\sharp_{ik})_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_l/x_k) = (x_l/x_i)\cdot(x_i/x_k)$$

If $l=i$, then we have that:

$$(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_i/x_j)\cdot(x_j/x_k)) = (x_i/x_j)\cdot(x_i/x_k)\cdot(x_j/x_i) = x_i/x_k$$

but:

$$(\phi^\sharp_{ik})_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_i/x_k) = x_i/x_k$$

Now suppose that $l=j$, then:

$$\begin{aligned}(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)}(x_j/x_k) =& (\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}}(x_j/x_k) \\ =& (x_i/x_k)\cdot(x_j/x_i)\end{aligned}$$

while:

$$(\phi^\sharp_{ik})_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_j/x_k) = (x_j/x_i)\cdot(x_i/x_k)$$

Now for x_k/x_j, we have that:

$$\begin{aligned}(\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi^\sharp_{jk})_{(x_j/x_k)\cdot(x_k/x_i)}(x_k/x_j) =& (\phi^\sharp_{ij})_{U_{(x_i/x_j)\cdot(x_k/x_j)}}(x_k/x_j) \\ =& (x_k/x_i)\cdot(x_i/x_j)\end{aligned}$$

while:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_k/x_j) = (x_i/x_j)\cdot(x_k/x_i)$$

And finally for x_k/x_i:

$$\begin{aligned}(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}(x_k/x_i) =&(\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}}((x_j/x_i)\cdot(x_k/x_j))\\ =&(x_j/x_i)\cdot(x_k/x_i)\cdot(x_i/x_j)\\ =&x_k/x_j\end{aligned}$$

while:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_i)\cdot(x_k/x_i)}}(x_k/x_i) = x_k/x_i$$

so indeed we have that:

$$(\phi_{ik}^\sharp)_{U_{(x_j/x_k)\cdot(x_k/x_i)}} = (\phi_{ij}^\sharp)_{U_{(x_i/x_j)\cdot(x_k/x_j)}} \circ (\phi_{jk}^\sharp)_{(x_j/x_k)\cdot(x_k/x_i)}$$

implying that:

$$\phi_{ik} = \phi_{jk}\circ\phi_{ij}$$

as desired. It follows by Theorem 2.1.1 that the affine schemes $\operatorname{Spec} A_i$ glue together to form a scheme which we denote by $\mathbb{P}^n_{\mathbb{C}}$. We denote the open embeddings $\operatorname{Spec} A_i \to \mathbb{P}^n_{\mathbb{C}}$ by ψ_i, their topological images in $\mathbb{P}^n_{\mathbb{C}}$ by $\mathcal{A}_i$, and the sheaf isomorphisms $\mathscr{O}_{\mathcal{A}_i}|_{\mathcal{A}_i\cap\mathcal{A}_j} \to \mathscr{O}_{\mathcal{A}_j}|_{\mathcal{A}_i\cap\mathcal{A}_j}$ by β_{ij}.

We see that $\mathbb{P}^n_{\mathbb{C}}$ is not affine by calculating it's global ring of sections. We have that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) = \left\{(s_i)\in\prod_{i=0}^n \mathscr{O}_{\mathcal{A}_i}(\mathcal{A}_i) : \forall i,j, \beta_{ij}(s_i|_{\mathcal{A}_i\cap\mathcal{A}_j}) = s_j|_{\mathcal{A}_i\cap\mathcal{A}_j}\right\}$$

We first note that:

$$\mathcal{A}_i\cap\mathcal{A}_j = \psi_i(U_{ij}) = \psi_i((U_{x_j/x_i}))$$

and that:

$$\mathscr{O}_{\mathcal{A}_i}(\mathcal{A}_i)\cong A_i$$

Denote by π_{ij} the localization map $A_i\to(A_i)_{x_j/x_i}$, then it follows that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) \cong \left\{(s_i)\in\prod_{i=0}^n A_i : \forall i,j, \phi_{ji}^\sharp(\pi_{ij}(s_i)) = \pi_{ji}(s_j)\right\}$$

We know that any element in A_i and A_j can only be written as polynomials in the variables x_l/x_i and x_m/x_j, where $l\neq i$ and $m\neq j$, and the localization maps are the inclusions into the polynomial rings discussed above. We see that for $l\neq j$:

$$\phi_{ji}^\sharp(x_l/x_i) = (x_l/x_j)\cdot(x_j/x_i)\notin \operatorname{im}\pi_{ji}$$

and that if $l=j$ then:

$$\phi_{ji}^\sharp(x_j/x_i) = x_j/x_i \notin \operatorname{im}\pi_{ji}$$

hence the only polynomials $s_i\in A_i$ which can possibly satisfy $\phi_{ji}^\sharp(\pi_{ij}(s_i)) = \pi_{ji}(s_j)$ are the constant ones. However, $\phi_{ji}^\sharp\circ\pi_{ij}$ is the identity on constant polynomials, hence we must have that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}(\mathbb{P}^n_{\mathbb{C}}) \cong \left\{(s_i)\in\prod_{i=0}^n \mathbb{C} : \forall i,j, s_i = s_j\right\}\cong\mathbb{C}$$

so $\mathbb{P}^n_{\mathbb{C}}$ is certainly not affine.

We now discuss why we denote this by $\mathbb{P}^n_{\mathbb{C}}$, by showing that is an analogue of complex projective space $\mathbb{P}^n$ from differential geometry. First note that $\mathbb{C}$ is algebraically closed, so by the weak Nullstellensatz[23], the maximal ideals of A_i are of the form:

$$(z_0, \ldots, \hat{z}_i, \ldots, z_n) := \left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle$$

where $z_j \in \mathbb{C}$ for all j. It is easy that any maximal ideal corresponds to a closed point of $\operatorname{Spec} A_i$ and any closed point of $\operatorname{Spec} A_i$ must in turn be a maximal ideal. Since the embeddings ψ_i determine the topology on $\mathbb{P}^n_{\mathbb{C}}$, so a point $[x] \in \mathbb{P}^n_{\mathbb{C}}$ is closed, if only if $\psi_i^{-1}([x])$ is a maximal ideal of A_i for all i[24]. Let $[x] \in \mathcal{A}_i$ be closed, if $[x] \notin \mathcal{A}_j$ for any other $j \neq i$ then we claim that is the origin in $\operatorname{Spec} A_i$:

$$\psi^{-1}([x])_i = (0, \ldots, 0) := \left\langle \frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} \right\rangle$$

Indeed, if $[x] \notin \mathcal{A}_j$, then we must have that $\psi_i^{-1}([x]) \notin U_{ij}$ for all j, implying that $x_j/x_i \in \psi_i^{-1}([x])$ for all j, so clearly $\psi_i^{-1}([x])$ is the origin. Now if $[x] \in \mathcal{A}_i \cap \mathcal{A}_j$, we have that $[x] \in \psi_i(U_{ij}) = \psi_j(U_{ji})$, hence $\psi_i^{-1}([x])$, is equivalent to $\psi_j^{-1}([x])$. Indeed, we have that $\psi_i^{-1}([x]) \in U_{ij}$, and $\psi_j^{-1}([x]) \in U_{ji}$. We need to show that:

$$\phi_{ij}(\psi_i^{-1}([x])) = \psi_j^{-1}([x])$$

Well, apply ψ_j to the left hand side:

$$\psi_j(\phi_{ij}(\psi_i^{-1}([x]))) = \psi_i(\psi_i^{-1}([x])) = [x]$$

while clearly $\psi_j(\psi_j^{-1}([x])) = [x]$, so since ψ_j is injective we have the desired equality, implying $\psi_i^{-1}([x]) \sim \psi_j^{-1}([x])$. Now let:

$$\psi_i^{-1}([x]) = x_i = (z_0, \ldots, \hat{z}_i, \ldots, z_n) = \left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle$$

Now we see that under the isomorphism $U_{ij} \cong (\operatorname{Spec} A_i)_{x_j/x_i}$, x_i gets mapped to the ideal:

$$\left\langle \frac{x_0}{x_i} - z_0, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - z_n \right\rangle \subset \mathbb{C}[\{x_k/x_i\}_{k \neq i}, x_i/x_j]$$

and that under ϕ_{ij} we have that:

$$\phi_{ij}(x_i) = \left\langle \frac{x_0}{x_j} \cdot \frac{x_j}{x_i} - z_0, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_j}{x_i} - z_j, \ldots, \frac{x_n}{x_i}\frac{x_j}{x_i} - z_n \right\rangle$$

which by the same argument as in Example 2.1.5 can be rewritten as:

$$\phi_{ij}(x_i) = \left\langle \frac{x_0}{x_j} \cdot \frac{x_j}{x_i} - z_0, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_i}{x_j} - \frac{1}{z_j}, \ldots, \frac{x_n}{x_j}\frac{x_j}{x_i} - z_n \right\rangle$$

We claim that this ideal is equal to:

$$J = \left\langle \frac{x_0}{x_j} - \frac{z_0}{z_j}, \ldots, \frac{\hat{x}_j}{x_j}, \ldots, \frac{x_i}{x_j} - \frac{1}{z_j}, \ldots, \frac{x_n}{x_j} - \frac{z_n}{z_j} \right\rangle$$

Clearly, we need only show that any generator of J lies in $\phi_{ij}(x_i)$ and vice versa. Note that:

$$\frac{x_i}{x_j} \cdot \left(\frac{x_k}{x_j}\frac{x_j}{x_i} - z_k \right) + z_k \cdot \left(\frac{x_i}{x_j} - \frac{1}{z_j} \right) = \frac{x_k}{x_j} - z_k\frac{x_i}{x_j} + z_k\frac{x_i}{x_j} - \frac{z_k}{z_j} = \frac{x_k}{x_j} - \frac{z_k}{z_j}$$

[23]One could also potentially argue this fact using Zariski's lemma (Theorem 6.1.3), and the fact that $\mathbb{C}$ is algebraically closed.

[24]Note that if $x \notin \mathcal{A}_i$, then we have that $\psi_i^{-1}([x])$ is empty and thus closed

so $x_k/x_j - z_k/z_j \in \phi_{ij}(x_i)$. Meanwhile, we see that:

$$\frac{x_k}{x_j}\frac{x_j}{x_i} - z_k = \frac{x_j}{x_i} \cdot \left(\frac{x_k}{x_j} - \frac{z_k}{z_i}\right) - z_k \cdot \frac{x_j}{x_i}\left(\frac{x_i}{x_j} - \frac{1}{z_j}\right)$$

so $(x_k/x_J) \cdot (x_j/x_i) - z_k \in J$ as well. It follows that:

$$x_j = \left(\frac{z_0}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

We denote the set of closed points of $X_i = \operatorname{Spec} A_i$ by $|X_i|$, and construct a set map:

$$F : \coprod_{i=0}^{n} |X_i| \longrightarrow \mathbb{P}^n$$
$$x_i = (z_0, \ldots, \hat{z}_i, \ldots, z_n) \in X_i \longmapsto [z_0, \ldots, z_{i-1}, 1, z_i, \ldots, z_n]$$

and note that such a map is clearly surjective, as every equivalence class $[w_0, \ldots, w_n]$ must have at least one non zero entry w_k, so we can always rescale to obtain something of the above form. Moreover, we see that if $x_i \sim x_j$, then we have that by our previous work:

$$x_j = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \hat{z}_j, \ldots, \frac{z_n}{z_j}\right)$$

so:

$$\begin{aligned} F(x_j) &= \left[\frac{z_0}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_{i+1}}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, 1, \frac{z_{j+1}}{z_j} \ldots, \frac{z_n}{z_j}\right] \\ &= [z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n] \\ &= F(x_i) \end{aligned}$$

so F factors through the quotient, and we thus obtain a map:

$$\begin{aligned} \tilde{F} : |\mathbb{P}^n_{\mathbb{C}}| &\longrightarrow \mathbb{P}^n \\ [x] &\longmapsto F(\psi_i^{-1}(x)) \end{aligned}$$

for any i such that $[x] \in \mathcal{A}_i$. It is already surjective as F is surjective, so it suffices to check that if $F(x_i) = F(y_j)$ then $x_i \sim y_j$. First note that if $j = i$, then we clearly have that $x_i = y_j$, as the only way for:

$$(z_1, \ldots, z_{i-1}, 1, z_{i+1}, z_n) = \lambda(w_1, \ldots, z_{i-1}, 1, z_{i+1}, w_n))$$

is if $1 = \lambda$. Now suppose that:

$$[z_0, \ldots, z_{i-1}, 1, z_{i+1}, \cdots z_n] = [w_0, \ldots, w_{j-1}, 1, w_{j+1}, \ldots, w_n]$$

then we must clearly have that $w_i, z_j \neq 0$, hence $x_i \in U_{ij}$ and $y_j \in U_{ji}$. It follows that we can rewrite the right hand side as:

$$\left[\frac{w_0}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, 1, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, 1, \frac{w_{j+1}}{w_i}, \ldots, \frac{w_n}{w_i}\right]$$

Since the right hand side now has 1 in the ith spot, it follows that:

$$z_k = \frac{w_k}{w_i}$$

hence:

$$x_i = \phi_{ji}(y_j)$$

so the $x_i \sim y_j$ and $\tilde{F}$ injective. It follows that $\tilde{F}$ is a set isomorphism, so we can identify the closed points of $\mathbb{P}^n_{\mathbb{C}}$ with the classical projective space $\mathbb{P}^n$. We thus call $\mathbb{P}^n_{\mathbb{C}}$ the n-dimensional projective scheme over $\mathbb{C}$.

Note, that in the gluing process, we never used the fact that $\mathbb{C}$ was a field, or algebraically closed, so we could easily repeat this process with any ring or field A and obtain a projective scheme $\mathbb{P}^n_A$. We will however, lose the identification of closed points with classical projective space. Indeed, if we were to look at $\mathbb{P}^1_{\mathbb{R}}$, then $\langle (x/y)^2+1\rangle \in \operatorname{Spec}\mathbb{R}[x/y] \subset \mathbb{P}^1_{\mathbb{R}}$ is a closed point, but has no corresponding element in $\mathbb{RP}^1$.

Now before we move onto to discussing projective schemes in generality, we quickly show that $\mathbb{P}^n_{\mathbb{C}}$ satisfies another property which make it's remarkably similar to $\mathbb{P}^n$. Indeed, there exits a canonical map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ given by:

$$(z_0,\dots,z_n) \longmapsto [z_0,\dots,z_n]$$

We show now show a similar statement for the scheme $\mathbb{P}^n_{\mathbb{C}}$.

Lemma 2.2.1. *Let $\mathbb{P}^n_{\mathbb{C}}$ be the scheme constructed in Example 2.1.5, and $\mathbb{A}^{n+1} \setminus \{0\}$ the affine scheme* $\operatorname{Spec}\mathbb{C}[x_0,\dots,x_n]$ *minus the closed point $\langle x_0,\dots,x_n\rangle$. Then there exists a morphism:*

$$\mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

which one closed points satisfies:

$$(z_0,\dots,z_n) \longrightarrow [z_0,\dots,z_n]$$

under the identification of $|\mathbb{P}^n_{\mathbb{C}}|$ with $\mathbb{P}^n$.

Proof. Note that $\mathbb{A}^{n+1} \setminus \{0\}$ is indeed an open subscheme of $\mathbb{A}^{n+1}$, and admits an open cover of distinguished opens by:

$$\mathbb{A}^{n+1} \setminus 0 = \bigcup_{i=0}^{n} U_{x_i}$$

We also note by Corollary 1.2, that the structure sheaf on $\mathbb{A}^{n+1} \setminus \{0\}$, which we denote by Y going forward, is isomorphic to the one obtained by gluing the sheafs $\mathscr{O}_{U_{x_i}} = \mathscr{O}_{\mathbb{A}^{n+1}}|_{U_{x_i}}$ together, where the transition functions $\beta_{ij} : \mathscr{O}_{U_{x_i}}|_{U_{x_i}\cap U_{x_j}} \to \mathscr{O}_{U_{x_j}}|_{U_{x_i}\cap U_{x_j}}$ are the identity maps. It is then clear that the sheaf of global sections is satisfies $\mathscr{O}_Y(Y) \cong \mathbb{C}[x_0,\dots,x_n]$ by the same argument in Example 2.1.2.

We will now make use of of Proposition 1.3.3 to obtain a morphism of locally ringed spaces. We note that $X_i = \operatorname{Spec}\mathbb{C}[\{x_l/x_i\}_{l\neq i}]$ is an embedded open subscheme of $\mathbb{P}^n_{\mathbb{C}}$ for all $0 \le i \le n$, such that the closed points X_i correspond precisely to the closed points of $\mathcal{A}_i = \psi_i(X_i)$ which can be written in the form:

$$[z_0,\dots,z_{i-1},1,z_{i+1},\dots,z_n]$$

It thus makes sense to define morphisms $\xi_i : U_{x_i} \longrightarrow X_i$, and then compose with compose with the embedding $\psi_i : X_i \to \mathbb{P}^n_{\mathbb{C}}$. Since U_{x_i} and X_i are both affine schemes, we need only define a ring map:

$$\xi_i^\sharp : \mathbb{C}[\{x_l/x_i\}_{l\neq i}] \longrightarrow \mathbb{C}[x_0,\dots,x_n,x_i^{-1}]$$

Given our suggestive choice of notation, it should be no surprise that we define this map on generators by:

$$\frac{x_j}{x_i} \longrightarrow x_j \cdot x_i^{-1} \tag{2.2.1}$$

We see that any closed point of U_{x_i} is of the form $(z_0,\dots,z_i,\dots,z_n)$, where $z_i \neq 0$, and that this is the ideal:

$$I = \langle x_0 - z_0,\dots,x_i - z_i,\dots,x_n - z_n\rangle \subset \mathbb{C}[x_0,\dots,x_n,x_i^{-1}]$$

under the identification $U_{x_i} \cong \operatorname{Spec}\mathbb{C}[x_0,\dots,x_n,x_i^{-1}]$. We claim that:

$$(\xi_i^\sharp)^{-1}(I) = \left\langle \frac{x_0}{x_i} - \frac{z_0}{z_i},\dots,\frac{x_n}{x_i} - \frac{z_n}{z_i}\right\rangle$$

Since the right hand side is clearly a maximal ideal, we clearly need only show that each generator lies in $(\xi_i^\sharp)^{-1}(I)$. Now, note that:

$$\xi_i^\sharp(x_l/x_i - z_l/z_i) = x_l \cdot x_i^{-1} - z_l/z_i$$

however:

$$\begin{aligned} x_i^{-1}(x_l - z_l) + z_l \cdot (-x_i^{-1} \cdot z_i^{-1})(x_i - z_i) =& x_l x_i^{-1} - x_i^{-1} - z_l x_i^{-1} - z_l z_i^{-1} + z_l x_i^{-1} \\ =& x_l x_i^{-1} - z_l z_i^{-1} \end{aligned}$$

implying the claim. It follows that under the embedding ψ_i, we have that:

$$f_i((z_0, \ldots, z_n)) = \psi_i \circ (\xi_i^\sharp)^{-1}((z_0, \ldots, z_n)) = \psi_i((z_0/z_i, \ldots, \hat{z}_i, \ldots, z_n/z_i))$$

which is identified with $[z_0/z_i, \ldots, z_{i-1}/z_i, 1, z_{i+1}/z_i, \ldots, z_n] = [z_0, \ldots, z_n] \in \mathbb{P}^n$. So, on closed points f_i provides the correct map.

We now check that $f_i|_{U_{x_i} \cap U_{x_j}} = f_j|_{U_{x_i} \cap U_{x_j}}$. Note, that $f_i = \psi_i \circ \xi_i$, where ξ_i is the scheme morphism $U_{x_i} \to X_i$ induced by the ring map defined by (2.5). It thus suffices to check that *a*), $f_i|_{U_{x_i} \cap U_{x_j}}$ has image in U_{ij}, and *b*) that $\phi_{ij} \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \xi_j|_{U_{x_i} \cap U_{x_j}}$. Indeed, if *b*) holds, then we have that:

$$\psi_j \circ \phi_{ij} \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \psi_j \circ \xi_j|_{U_{x_i} \cap U_{x_j}} \Longrightarrow \psi_i \circ \xi_i|_{U_{x_i} \cap U_{x_j}} = \psi_j \circ \xi_j|_{U_{x_i} \cap U_{x_j}}$$

We first check that:

$$\xi_i(U_{x_i} \cap U_{x_j}) = U_{x_j/x_i} \subset X_i = \operatorname{Spec} \mathbb{C}[\{x_l/x_i\}_{l \neq i}]$$

Note that $U_{x_i} \cap U_{x_j} = U_{x_i x_j} \subset U_{x_i}$, is the distinguished open of U_{x_i} consisting of prime ideals $\mathfrak{p} \subset \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}]$ such that $x_j \notin \mathfrak{p}$. We need to show then that $x_j/x_i \notin (\xi_i^\sharp)^{-1}(\mathfrak{p})$. Well, if $x_j/x_i \in (\xi_i^\sharp)^{-1}(\mathfrak{p})$, then $\xi^\sharp(x_j/x_i) = x_j \cdot x_i^{-1} \in \mathfrak{p}$, implying that $x_j \in \mathfrak{p}$, as x_i^{-1} is a unit. It follows that ξ_i has image contained in $U_{x_i} \cap U_{x_j}$.

Now note that the ring map inducing the scheme morphism $U_{x_i x_j} \to U_{ij}$ is given by:

$$\begin{aligned} (\xi_i^\sharp)_{U_{x_i x_j}} : \mathbb{C}[\{x_l/x_i\}_{l \neq i}, x_i/x_j] &\longrightarrow \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}] \\ x_l/x_m &\longmapsto \begin{cases} x_l \cdot x_i^{-1} & \text{if } l \neq i, m = i \\ x_i \cdot x_j^{-1} & \text{if } l = i, m = j \end{cases} \end{aligned}$$

while for $U_{x_i x_j} \to U_{ji}$ it is given by:

$$\begin{aligned} (\xi_j^\sharp)_{U_{x_i x_j}} : \mathbb{C}[\{x_l/x_j\}_{l \neq j}, x_j/x_i] &\longrightarrow \mathbb{C}[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}] \\ x_l/x_m &\longmapsto \begin{cases} x_l \cdot x_j^{-1} & \text{if } l \neq j, m = j \\ x_j \cdot x_i^{-1} & \text{if } l = j, m = i \end{cases} \end{aligned}$$

and it now suffices to check that:

$$\xi_i^\sharp|_{U_{x_i x_j}} \circ \phi_{ij}^\sharp = \xi_j^\sharp|_{U_{x_i x_j}}$$

and we do so on generators. Let $x_l/x_j \in \mathbb{C}[\{x_l/x_j\}_{l \neq j}, x_j/x_i]$ such that $l \neq i$. Then:

$$\phi_{ij}^\sharp(x_l/x_j) = (x_l/x_i) \cdot (x_i/x_j)$$

and:

$$(\xi_i^\sharp)_{U_{x_i x_j}}((x_l/x_i) \cdot (x_i/x_j)) = x_l \cdot x_i^{-1} \cdot x_i \cdot x_j^{-1} = x_l \cdot x_j^{-1} = \xi_j^\sharp|_{U_{x_i x_j}}(x_l/x_j)$$

Now examine x_i/x_j, then:

$$\xi_i^\sharp|_{U_{x_i x_j}} \circ \phi_{ij}^\sharp(x_i/x_j) = \xi_i^\sharp|_{U_{x_i x_j}}(x_i/x_j) = x_i \cdot x_j^{-1}$$

while:

$$\xi_j^\sharp(x_i/x_j) = x_i \cdot x_j^{-1}$$

Finally, for x_j/x_i, we have that:

$$(\xi_i^\sharp)_{U_{x_ix_j}} \circ \phi_{ij}^\sharp(x_j/x_i) = (\xi_i^\sharp)_{U_{x_ix_j}}(x_j/x_i) = x_j \cdot x_i^{-1}$$

while:

$$(\xi_j^\sharp)_{U_{x_ix_j}}(x_j/x_i) = x_j \cdot x_i^{-1}$$

It thus follows that:

$$\xi_i^\sharp|_{U_{x_ix_j}} \circ \phi_{ij}^\sharp = \xi_j^\sharp|_{U_{x_ix_j}}$$

hence:

$$\phi_{ij} \circ \xi_i|_{U_{x_ix_j}} = \xi_j|_{U_{x_ix_j}}$$

and it follows that the scheme morphisms $f_i : \psi_i \circ \xi_i$ glue together to form a map:

$$f : \mathbb{A}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$

which clearly sends closed points:

$$(z_0, \ldots, z_n) \longmapsto [z_0, \ldots, z_n]$$

□

Now, as promised we move forward with the Proj construction. Much like Spec, we will see that Proj takes a commutative ring to a scheme, however, in this case, we will have that *a*) the ring must have a *grading*, *b*) the scheme will not in general be affine, and *c*) Proj is not a functor. We will in fact find that:

$$\mathbb{P}^n_{\mathbb{C}} \cong \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$$

We need the following definition:

Definition 2.2.1. A **$\mathbb{Z}$-graded ring** is a direct sum of abelian groups :

$$A = \bigoplus_{n\in\mathbb{Z}} A_n$$

equipped with a ring structure such that $A_i \cdot A_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$. We call elements of A_i **homogeneous elements of degree** i. A **homogeneous ideal** is an ideal generated by homogeneous elements, and a **graded ideal** is an ideal such that

$$I = \bigoplus_{n\in\mathbb{Z}} (I \cap A_n)$$

Clearly, we have that $A_0 \subset A$ is a subring, A_i is an A_0 module for all i, and A itself is and A_0 algebra. We often make the mild sin of referring to a $\mathbb{Z}$-graded ring as a graded ring, and so the reader should always assume we mean a ring with with $\mathbb{Z}$-graded structure unless state otherwise. Indeed there are other notions of a graded ring over other abelian groups, and so we will clarify should the need arise. We prove the following facts from commutative algebra:

Lemma 2.2.2. *Let A be a graded ring, and $I, J \subset A$ ideals of A. Then the following hold:*

a) *I is homogeneous if and only if it is graded*

b) *If I and J are homogeneous, then IJ, $I + J$, $I \cap J$, and $\sqrt{I}$ are homogeneous.*

c) *If I is homogeneous, then I is prime if $I \neq A$ and for any homogeneous elements $a, b \in A$, $a \cdot b \in I$ implies that $a \in I$ or $b \in I$.*

Proof. We start with a). Suppose that I is graded, then any $i \in I$ can be written as the finite sum:

$$i = \sum_i a_i$$

where each $a_i \in I \cap A_n$. Each a_i is homogeneous, so it follows that I is generated by homogeneous elements.

Suppose that I is generated by homogeneous elements, then any $i \in I$ can be written as the finite sum:

$$i = \sum_i a_i \cdot b_i$$

where $a_i \in A$, and each $b_i \in I \cap A_i$. Since A is graded, for each a_i we can write:

$$a_i = \sum_j a_{ij}$$

where each $a_{ij} \in A_j$. It follows that:

$$i = \sum_{i,j} a_{ij} b_i$$

It follows that $a_{ij} b_i \in A_{i+j}$ for all i and j, so we can rewrite i as the finite sum:

$$i = \sum_n \sum_{i+j=n} a_{ij} b_i$$

then for each n set:

$$d_n = \sum_{i+j=n} a_{ij} b_i$$

It is then clear that $d_n \in I \cap A_n$ hence:

$$i = \sum_n d_n$$

Since $(I \cap A_n) \cap (I \cap A_m) = I \cap (A_n \cap A_m) = I \cap \{0\} = \{0\}$ it thus follows that:

$$I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n)$$

Now let I and J be homogeneous ideals of A. We see that:

$$IJ = \langle ij : i \in I, j \in J \rangle$$

if S_I is the generating set of I, and S_J is the generating set of J, then we see that:

$$IJ = \langle S_I \cdot S_J \rangle$$

where:

$$S_I \cdot S_J = \{s \cdot t : s \in S_I, t \in S_J\}$$

Since all s and t are homogeneous, it follows that $s \cdot t$ is homogeneous, hence IJ is generated by homogeneous elements, implying that IJ is homogeneous.

The sum $I + J$ is the ideal:

$$I + J = \langle S_I \cup S_J \rangle$$

so $I + J$ is indeed generated by homogeneous elements, implying that $I + J$ is homogeneous.

Now consider $I \cap J$; we have that by a):

$$I = \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \qquad \text{and} \qquad J = \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$$

We claim that:

$$\bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n) = \bigoplus_{n \in \mathbb{Z}} (I \cap J \cap A_n)$$

Let $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap J \cap A_n)$, then:

$$i = \sum_n a_n$$

where $a_n \in I \cap J \cap A_n$. It follows that $a_n \in I \cap A_n$, and $a_n \in \cap J \cap A_n$ for all n, hence $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$. Now suppose that $i \in \bigoplus_{n \in \mathbb{Z}} (I \cap A_n) \cap \bigoplus_{n \in \mathbb{Z}} (J \cap A_n)$, then:

$$i = \sum_n a_n$$

where each $a_n \in I \cap A_n$ and:

$$i = \sum_n b_n$$

where $b_n \in J \cap A_n$. It follows that:

$$\sum_n a_n - b_n = 0$$

and since the intersection $A_n \cap A_m = \{0\}$ we must have that $a_n = b_n$ for all n. It follows that $a_n \in I \cap J \cap A_n$ for all n, implying the claim.

Now consider the radical of I:

$$\sqrt{I} = \{a \in A : a^n \in I\}$$

Let $a \in \sqrt{I}$, then since $a^n \in I$ we can write:

$$a^n = \sum_j b_j$$

where each $b_j \in I \cap A_n$. We can write a as:

$$a = \sum_i a_i$$

Then there exists a top degree element a_m, and it follows that $a_m^n = b_{nm} \in I \cap A_{nm}$, hence $a_m \in \sqrt{I} \cap A_m$. Now $a - a_m \in \sqrt{I}$, so we can apply the same argument to the next highest graded piece. It follows that $a_i \in \sqrt{I} \cap A_i$ for all i, so:

$$\sqrt{I} = \bigoplus_{n \in \mathbb{Z}} \sqrt{I} \cap A_n$$

and is thus homogeneous by a).

To prove c) suppose I homogeneous, not equal I, and suppose that for any homogeneous elements $a, b \in A$, $a \cdot b \in I$ implies that $a \in I$ or $b \in I$. Now let $a, b \in A$ be arbitrary, and write:

$$a = \sum_i a_i \qquad \text{and} \qquad b = \sum_i b_i$$

where $a_i, b_i \in A_i$. Suppose that $a \cdot b \in I$, but neither a nor $b \in I$; we will prove the contrapositive. Since $a, b \notin I$, and I is graded, there is lowest degree m and l such that $a_m, b_l \notin I$, and $a_i, b_j \in I$ for all $i < m$ and $j < l$[25]. Now note that the product is given by:

$$a \cdot b = \sum_{i,j} a_i b_j$$

and we have the $m + l$th component of $a \cdot b$ is given by:

$$(a \cdot b)_{m+l} = \sum_{i+j=m+l} a_i b_j \in I \cap A_{m+l}$$

For each such i and j not equal to m and l, we must either have that $i > m$ or $j > l$, but if $i > m$ or $j > l$ then $a_i \in I$ or $b_j \in I$, hence all such $a_i b_j \in I$. It follows that $a_m b_l \in I$, but neither $a_m \in I$ nor $b_l \in I$, so the claim follows by the contrapositive. □

Definition 2.2.2. Let A and B be rings, and $\phi : A \to B$ a ring homomorphism. Then, ϕ is a **graded ring homomorphism** if for all $n \in \mathbb{Z}$, $\phi(A_n) \subset B_n$. A graded ring homomorphism is a **graded ring isomorphism** if it is graded, and an isomorphism as a ring homomorphism.

We note that homogeneous ideals are precisely those that lead to graded quotients.

Lemma 2.2.3. *Let A be a graded ring, and I a homogeneous ideal, then:*

$$A/I = \bigoplus_{n \in \mathbb{Z}} \pi(A_n) \cong \bigoplus_{n \in \mathbb{Z}} A_n/I_n$$

where π is the quotient map, and $I_n = A_n \cap I_n$.

Proof. We note that since $\pi : A \to A/I$ is a surjective homomorphism that:

$$[a] = \left[\sum_n a_n\right] = \sum_n [a_n]$$

clearly each $[a_n] \in \pi(A_n)$, so any $[a]$ can be written as a finite sum, where each element lies in $\pi(A_n)$. To see that this admits a grading, we check that $\pi(A_n) \cap \pi(A_m) = \{0\}$. Let $[a] \in \pi(A_n) \cap \pi(A_m)$, then we have that there is $a_m \in A_m$ and $a_n \in A_n$ such that:

$$[a_m] = [a_n]$$

which implies that there exists an $i \in I$ such that:

$$a_m + i = a_n$$

Since i is graded we can write this as:

$$a_m + \sum_j b_j = a_n$$

where $b_j \in A_j \in I$. It follows that:

$$a_n - a_m = \sum_j b_J$$

but a_n and a_m are homogeneous, so $b_i = 0$ for $i \neq m, n$, and $b_m = -a_m$ and $b_n = a_n$. This then implies that both a_m and a_n lie in I, hence $[a] = 0$. It follows that:

$$A/I = \bigoplus_{n \in \mathbb{Z}} \pi(A_n)$$

Moreover, we see that $[a_m] \cdot [a_n] = [a_m \cdot a_n] \in \pi(A_{m+n})$, so A/I is a graded ring.

[25]If this is lowest degree is zero, then the elements are homogeneous and the contrapositive is immediate.

We now define the following homomorphism of abelian groups:

$$\begin{aligned} \phi_n : A_n &\longrightarrow A/I \\ a_n &\longmapsto [a_n] \end{aligned}$$

clearly this is a surjection onto $\pi(A_n)$, and clearly $\ker \phi_n = I_n$, hence ϕ_n descends to an isomorphism ψ_n:

$$\begin{aligned} \psi_n : A_n/I_n &\longrightarrow \pi(A_n) \\ [a_n]_n &\longmapsto [a_n] \end{aligned}$$

where the n subscript denotes taking the equivalence class in A_n/I_n. We take the direct sum of abelian modules:

$$\bigoplus_{n\in\mathbb{Z}} A_n/I_n$$

and equip with it the ring structure defined on homogeneous elements by:

$$[a_n]_n \cdot [a_m]_m = [a_m \cdot a_n]_{m+n}$$

and extend to linearly. This is clearly well defined as I is a graded ideal, hence we define the isomorphism:

$$\begin{aligned} \Psi : \bigoplus_{n\in\mathbb{Z}} A_n/I_n &\longrightarrow \bigoplus_{n\in\mathbb{Z}} \pi(A_n) \\ \sum_n [a_n]_n &\longmapsto \sum_n \psi_n([a_n]_n) = \sum_n [a_n] \end{aligned}$$

Which is clearly a ring homomorphism as:

$$\psi_{m+n}([a_m]_m \cdot [a_n]_n) = [a_m \cdot a_n]$$

while:

$$\psi_m([a_m]_m) \cdot \psi_n([a_n]_n) = [a_m \cdot a_n]$$

so it is clearly a graded isomorphism of rings. □

We have a similar result for localization:

Lemma 2.2.4. *Let A be a graded ring, and S be a multiplicatively closed subset of A containing only homogeneous elements. Then $S^{-1}A$ has the natural structure of a graded ring.*

Proof. We define a grading on $S^{-1}A$ by first defining the homogeneous elements of $S^{-1}A$ to be those of the form:

$$H = \left\{ \frac{a}{s} : s \in S, a \text{ is homogeneous} \right\}$$

Note that this indeed makes sense as S contains only homogeneous elements. We then define the degree of any $a/s \in H$ as $\deg a - \deg s$. We check that this well defined, suppose that:

$$\frac{a}{s} = \frac{b}{t}$$

then there is $u \in U$ such that:

$$u(at - bs) = 0$$

We note that since $a \cdot t$ and $b \cdot s$ are homogeneous, we must have that $\deg(a \cdot t) = \deg a + \deg t = \deg b + \deg s = \deg(d \cdot s)$, hence:

$$\deg a - \deg s = \deg b - \deg t$$

so the degree of an element is well defined. We define the set:

$$(S^{-1}A)_m = \left\{\frac{a}{s} \in H : \deg(a/s) = m \text{ or } \exists u \in S, u \cdot a = 0\right\}$$

We claim that this is a subgroup of $S^{-1}A$; indeed $0 \in (S^{-1}A)_m$ as $0/s$ satisfies $u \cdot 0 = 0$ for all $u \in S$. Now, suppose suppose that $a/s \in (S^{-1}A)_m$, then $\deg(-a/s) = m$, and $(a/s) + (-a/s) = 0$, so $(S^{-1}A)_m$ contains inverses. Now, let a/s and b/t in $(S^{-1}A)_m$, then:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

so:

$$\deg\left(\frac{at + bs}{st}\right) = \deg(at + bs) - \deg(st)$$

Note that:

$$\deg(st) = \deg(s) + \deg(t)$$

while:

$$\deg(at) = \deg(a) + \deg(t) \qquad \text{and} \qquad \deg(bs) = \deg(b) + \deg(s)$$

Since:

$$\deg a - \deg s = \deg b - \deg t$$

it follows that $\deg(at) = \deg(bs)$, hence:

$$\deg\left(\frac{at + bs}{st}\right) = \deg a + \deg t - \deg s - \deg t = \deg a - \deg s = m$$

so $(S^{-1}A)_m$ is closed under addition. Since the degree of an element is well defined, it follows by the construction of $(S^{-1}A)_m$ that for $m \neq n$, we have that $(S^{-1}A)_m \cap (S^{-1}A)_n = \{0\}$. Now finally, let $a/s \in S^{-1}A$, then a can be written as the sum:

$$a = \sum_i a_i$$

where $A_i \in a_i$ for each i. It follows that:

$$\frac{a}{s} = \sum_i \frac{a_i}{s}$$

each element is then homogeneous of degree $i - \deg s$. It follows that any element can be written as sum of homogeneous elements, hence:

$$S^{-1}A = \bigoplus_{n \in \mathbb{Z}} (S^{-1}A)_n$$

We now need only check that $(S^{-1}A)_n \cdot (S^{-1}A)_m \subset (S^{-1}A)_{m+n}$. Let $a/s \in (S^{-1}A)_n$, and $b/t \in (S^{-1}A)_m$, then we see that:

$$\deg\left(\frac{ab}{ts}\right) = \deg(ab) - \deg(st) = \deg a + \deg b - \deg s - \deg t = \deg(a/s) + \deg(b/t) = m + n$$

implying the claim. $\square$

We say that a $\mathbb{Z}$-graded ring A is $\mathbb{Z}^{\geq 0}$ graded if for all $n < 0$ we have $A_n = \{0\}$. Going forward, we assume that all rings are $\mathbb{Z}^{\geq 0}$ graded, unless explicitly stated otherwise. In particular, the localization of $\mathbb{Z}^{\geq 0}$ graded ring will obviously be $\mathbb{Z}$ graded.

Definition 2.2.3. We fix a **base ring** B, and say that a graded ring A is **graded over B** if $A_0 = B$. Moreover, the subset:

$$A_+ = \bigoplus_{i>0} A_i$$

is a prime ideal called the **irrelevant ideal**. If the irrelevant ideal is finitely generated, then we say that A is a **finitely graded ring over B**. Finally, if A is generated by A_1 as a B-algebra, we say that A is **generated in degree 1**.

We now begin the Proj construction:

Definition 2.2.4. Let A be a graded ring, then as a set $\mathbf{Proj}\,\mathbf{A}$ is defined by:

$$\operatorname{Proj} A = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \text{ is homogeneous and } A_+ \not\subset \mathfrak{p}\}$$

i.e. $\operatorname{Proj} A$ are the set of all homogeneous prime ideals which do not contain the irrelevant ideal.

If $f \in A$ is homogeneous, we denote by A_f the localization of A by the multiplicatively closed subset generated by f, equipped with the natural $\mathbb{Z}$ grading given by Lemma 2.2.4. We define $(A_f)_0$ to be the degree zero elements of A_f.

Proposition 2.2.1. *Let $f \in A_+$ be homogeneous, then there is a bijection between the prime ideals of $(A_f)_0$, the homogeneous prime ideals of A_f, and the homogeneous prime ideals of A which do not contain f.*

Proof. First note that there is a bijection between the prime ideals of A_f, and the prime ideals of A which does not contain f. Clearly, if $\mathfrak{p} \subset A$ is homogeneous, then the corresponding ideal $\mathfrak{p}_f$ is homogeneous in A_f, when equipped with the natural $\mathbb{Z}$ grading from Lemma 2.2.4. Now note that if $\mathfrak{p} \subset A_f$ is homogeneous, then $\mathfrak{p}$ is generated by homogeneous elements, and if $\pi : A \to A_f$, then $\pi^{-1}(\mathfrak{p})$ is the prime ideal of A corresponding to $\mathfrak{p}$. Now suppose that $\pi^{-1}(\mathfrak{p})$ is not generated by homogeneous elements, then if $\{a_i\}_i$ are the generators of $\pi^{-1}(\mathfrak{p})$, we have that $\{a_i/1\}_i$ are the generators of $\mathfrak{p}$, implying $\mathfrak{p}$ is not generated generated by homogeneous elements, so by the contrapositive, we have that $\pi^{-1}(\mathfrak{p})$ is homogeneous. It follows that the bijection between primes not containing f, and primes of A_f preserves homogeneous primes, implying the claim.

Now we have a natural inclusion homomorphism of rings $\imath : (A_f)_0 \hookrightarrow A_f$, so any homogeneous prime of A_f pulls back to a prime ideal of $(A_f)_0$. Given a prime $\mathfrak{p}_0 \in (A_f)_0$, then we set $\phi(\mathfrak{p}_0) = \sqrt{\mathfrak{p}_0 A_f}$, where $p_0 A_f$ is the the in A_f generated by $\mathfrak{p}_0$ as a subset of A_f. Since $\mathfrak{p}_0 A_f$ is generated by degree zero elements, it is homogeneous, so by Lemma 2.2.2 $\sqrt{\mathfrak{p}_0 A_f}$ is homogeneous. By Lemma 2.2.2 part *c)*, we need only prove this for homogeneous elements of $\sqrt{\mathfrak{p}_0 A_f}$. Let a and b be homogeneous elements of degree k and l, such that $a \cdot b \in \sqrt{\mathfrak{p}_0 A_f}$, so there must exist some $r \geq 0$ such that $(a \cdot b)^r \in \mathfrak{p}_0 A_f$ by the definition of the radical. Now, $(a \cdot b)^r$ has degree $(k+l)r$, so let $j = \deg f$, then:

$$\frac{(a \cdot b)^{jr}}{f^{(k+l)r}} \in (\mathfrak{p}_0 A_f)_0 = \mathfrak{p}_0$$

It follows that since $\mathfrak{p}_0$ is prime, either $a^{jr}/f^{kr} \in \mathfrak{p}_0$, or $b^{jr}/f^{kr} \in \mathfrak{p}_0$, hence either $a^{jr} \in \mathfrak{p}_0 A_f$, or $b^{jr} \in \mathfrak{p}_0 A_f$. Again by the definition of the radical we have that either $a \in \sqrt{\mathfrak{p}_0 A_f}$, or $b \in \sqrt{\mathfrak{p}_0 A_f}$, so $\sqrt{\mathfrak{p}_0 A_f}$ is indeed prime.

Now if we have $\mathfrak{p}_0 \in (A_f)_0$, then $\imath^{-1}(\sqrt{\mathfrak{p}_0 A_f}) = (\mathfrak{p}_0 A_f)_0 = \mathfrak{p}_0$, so one direction of the bijection is immediate. Now let $\mathfrak{p} \subset A_f$ be a homogeneous prime ideal, we want to show that:

$$\mathfrak{p} = \sqrt{\imath^{-1}(\mathfrak{p}) A_f}$$

Note that $\imath^{-1}(\mathfrak{p}) = (\mathfrak{p})_0$, i.e. the degree zero elements of $\mathfrak{p}$. Since both primes are homogeneous, it suffices to check equality on homogeneous elements. Let $a \in \mathfrak{p}$ have degree k, then $a^j/f^k \in (\mathfrak{p})_0$, so $a^j \in (\mathfrak{p})_0 A_f$, hence $a \in \sqrt{\imath^{-1}(\mathfrak{p}) A_f}$. Now suppose that $a \in \sqrt{\imath^{-1}(\mathfrak{p}) A_f}$, then there exists some r such that $a^r \in (\mathfrak{p})_0 A_f$, but this implies that $a^r \in \mathfrak{p}$, as $(\mathfrak{p})_0 \mathbb{A}_f \subset \mathfrak{p}$, Since $\mathfrak{p}$ is prime it follows that $a \in \mathfrak{p}$, implying the second direction of the bijection. $\square$

So to sum up the result of the last proposition, which is an analogue of Proposition 1.1.3 minus the topological information, we have that a homogeneous prime ideal which does not contain f induces a unique homogeneous prime ideal of A_f, which then induces induces a unique prime ideal of the subring $(A_f)_0$. Our next step is to put a topology on $\operatorname{Proj} A$.

Definition 2.2.5. Let T be a subset of homogeneous elements then the **projective vanishing set of T**, denoted $\mathbb{V}(T)$ is defined by:

$$\mathbb{V}(T) = \{\mathfrak{p} \in \operatorname{Proj} A : T \subset \mathfrak{p}\}$$

Similarly, if f is a homogeneous element of positive degree, and $I \subset A$ is a homogeneous ideal, we set:

$$\mathbb{V}(f) := \mathbb{V}(\langle f \rangle) = \{\mathfrak{p} \in \operatorname{Proj} A : f \in \mathfrak{p}\} \qquad \text{and} \qquad \mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} A : I \subset \mathfrak{p}\}$$

This leads us to our next lemma, which follows a very similar argument to Proposition 1.1.1:

Lemma 2.2.5. *Let A be a graded ring, then defining the closed sets of $\operatorname{Proj} A$ to be $\mathbb{V}(I)$ for all homogeneous ideals defines a topology on $\operatorname{Proj} A$*

Proof. We first see that zero element is contained in $(A)_d$ for every d, so 0 has any degree we wish. It follows that since $0 \subset \mathfrak{p}$ for all homogeneous primes, that:

$$\mathbb{V}(0) = \operatorname{Proj} A$$

so $\operatorname{Proj} A$ is closed. We also have that that:

$$\mathbb{V}(A_+) = \emptyset$$

as no $\mathfrak{p} \in \operatorname{Proj} A$ contains A_+, so the empty set is closed.

Now let I and J be homogeneous prime ideals, then we want to show that:

$$\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$$

Let $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, if $I \subset \mathfrak{p}$, then $I \cap J \subset I \subset \mathfrak{p}$, and similarly for J, hence $\mathfrak{p} \in \mathbb{V}(I \cap J)$. If $\mathfrak{p} \in \mathbb{V}(I \cap J)$, then $I \cap J \subset \mathfrak{p}$. Let $r \in I \cdot J$, then $r = i \cdot j$ for some $i \in I$ and some $j \in J$. It follows that $r \in I \cap J$, so $I \cdot J \subset I \cap J$. , hence $I \cdot J \subset \mathfrak{p}$. Now suppose that $I \not\subset \mathfrak{p}$, then there exists an $i \in I$ such that $i \notin \mathfrak{p}$, however since $I \cdot J \subset \mathfrak{p}$, we have that for all $j \in J$, $i \cdot j \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime it follows that $J \subset \mathfrak{p}$, and if $J \not\subset \mathfrak{p}$, the same argument demonstrates $I \subset \mathfrak{p}$. Note that if neither $I \subset \mathfrak{p}$, nor $J \subset \mathfrak{p}$, then $\mathfrak{p}$ can't be prime, as there exists $i \in I$ and $j \in J$ such that $i, j \notin \mathfrak{p}$, but $i \cdot j \in \mathfrak{p}$. It follows that $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(I) \cup \mathbb{V}(J)$, implying the second direction.

Now let $\{I_\alpha\}$ be an arbitrary family of homogeneous ideals. We claim that:

$$\bigcap_\alpha \mathbb{V}(I_\alpha) = \mathbb{V}\left(\sum_\alpha I_\alpha\right)$$

where $\sum_\alpha I_\alpha$ is the smallest ideal containing all I_α. Suppose that $\mathfrak{p} \in \bigcap_\alpha \mathbb{V}(I_\alpha)$, then we have that $I_\alpha \subset \mathfrak{p}$ for all α. Now since any $i \in \sum_\alpha I_\alpha$ can be written as the finite sum $\sum_{j=1}^n r_j$ where each $r_j \in I_\alpha \subset \mathfrak{p}$, we have that $i \in \mathfrak{p}$, hence $\sum_\alpha I_\alpha \subset \mathfrak{p}$, so $\mathfrak{p} \in \mathbb{V}(\sum_\alpha I_\alpha)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(\sum_\alpha I_\alpha)$, then since $I_\alpha \subset \sum_\alpha I_\alpha$, we have that $I_\alpha \subset \mathfrak{p}$ for all α, hence $\mathfrak{p} \in \bigcap_\alpha \mathbb{V}(I_\alpha)$. ☐

As before call this topology on $\operatorname{Proj} A$ the Zariski topology. Note that Lemma 1.1.1 holds in the sense that for any homogeneous ideals I and J, the following hold:

a) $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$

b) $J \subset I \implies \mathbb{V}(J) \supset \mathbb{V}(I)$

c) $\mathbb{V}(I) \subset \mathbb{V}(J) \iff \sqrt{I} \supset \sqrt{J}$

We define a basis of open sets similarly, though impose more restrictions on what our basic opens can be:

Definition 2.2.6. Let A be a graded ring, and f a homogeneous element of positive degree, then we define the **(projective) distinguished open** to be:

$$U_f = \mathbb{V}(f)^c$$

Lemma 2.2.6. *The set of (projective) distinguished opens form a basis for the Zariski topology on* $\operatorname{Proj} A$.

Proof. Let $U \subset \operatorname{Proj} A$ be open, then we have that for some homogeneous ideal $I \subset A$:

$$U = \mathbb{V}(I)^c$$

Note that:

$$I = \sum_{i \in I} \langle i \rangle$$

hence:

$$\begin{aligned} U =& \mathbb{V}\left(\sum_{i \in I} \langle i \rangle\right)^c \\ =& \left(\bigcap_{i \in I} \mathbb{V}(i)\right)^c \\ =& \bigcup_{i \in I} \mathbb{V}(i)^c \end{aligned}$$

Now we can split this into the following union:

$$U = \bigcup_{i \in I_+} U_i \cup \bigcup_{j \in I_0} \mathbb{V}(j)^c$$

where I_+ denotes the elements of I with positive degree, and I_0 are the degree zero elements of positive degree. Let $\{f_k\}$ be the generators of the irrelevant ideal A_+, then:

$$\emptyset = \mathbb{V}(A_+) = \bigcap_k \mathbb{V}(f_k) \Rightarrow \operatorname{Proj} A = \bigcup_k U_{f_k}$$

We claim that if $j \in I_0$, then:

$$U_j = \bigcup_k U_{jf_k}$$

Let $\mathfrak{p} \in U_j$, then $j \notin \mathfrak{p}$; since $A_+ \not\subset \mathfrak{p}$, we must have that there exists some k such that $f_k \notin \mathfrak{p}$. It follows that $jf_k \notin \mathfrak{p}$, hence $\mathfrak{p} \in \bigcup_k U_{jf_k}$. Now suppose that $\mathfrak{p} \in \bigcup U_{jf_k}$, then for some k we have that $jf_k \notin \mathfrak{p}$, hence $j \notin \mathfrak{p}$, and $f_k \notin \mathfrak{p}$, so $\mathfrak{p} \in U_j$. It follows that:

$$U = \bigcup_{i \in I_+} U_i \cup \bigcup_{j \in I_0} \bigcup_k U_{jf_k}$$

so the distinguished opens generate the Zariski topology on $\operatorname{Proj} A$. □

It should be no surprise that we are about to prove a similar result to Proposition 1.1.3, and from there we will use the projective distinguished opens to put the structure of a scheme on $\operatorname{Proj} A$ for any graded ring A.

Proposition 2.2.2. *Let A be a graded ring, and f a homogeneous element of positive degree. Then $U_f \subset \operatorname{Proj} A$ is homeomorphic to* $\operatorname{Spec}(A_f)_0$.

Proof. Recall that $U_f \subset \operatorname{Proj} A$ is defined by $\mathbb{V}(f)^c$, hence:

$$U_f = \{\mathfrak{p} \in \operatorname{Proj} A : f \notin \mathfrak{p}\}$$

From Proposition 2.2.1 we have a bijection $U_f \leftrightarrow$ homogeneous primes of $A_f \leftrightarrow \operatorname{Spec}(A_f)_0$, given by $F : \mathfrak{p} \mapsto \mathfrak{p}_f \mapsto \imath^{-1}(\mathfrak{p}_f)$, where $\imath : (A_f)_0 \to A_f$ is the inclusion map, and $\mathfrak{p}_f$ is the prime ideal generated by the image of $\mathfrak{p}$ under the localization map. In other words, $\mathfrak{p}_f = \eta(\mathfrak{p})$, where η is as defined in Proposition 1.1.3. Let $\mathbb{V}(I) \subset \operatorname{Spec}(A_f)_0$ be a closed subset, for some radical ideal $I \subset (A_f)_0$, then we have that:

$$\begin{aligned} F^{-1}(\mathbb{V}(I)) =& \{\mathfrak{p} \in U_f : \imath^{-1}(\mathfrak{p}_f) \in \mathbb{V}(I)\} \\ =& \{\mathfrak{p} \in U_f : I \subset \imath^{-1}(\mathfrak{p}_f)\} \end{aligned}$$

We first claim that $I \subset \imath^{-1}(\mathfrak{p}_f)$ if and only if $\imath(I) \subset \mathfrak{p}_f$. Suppose that $I \subset \imath^{-1}(\mathfrak{p}_f)$, then $i \in I$ implies that $i \in \iota^{-1}(\mathfrak{p}_f)$. By definition, it follows that $\iota(i) \in \mathfrak{p}_f$, hence $\imath(I) \subset \mathfrak{p}_f$. Now suppose that $\imath(I) \subset \mathfrak{p}_f$, since ι is injective, we thus have that $\iota^{-1}(I) = I$, hence $I \subset \iota^{-1}(\mathfrak{p}_f)$. It follows that:

$$F^{-1}(\mathbb{V}(I)) = \{\mathfrak{p} \in U_f : \iota(I) \subset \mathfrak{p}_f\}$$

Note that since $I \subset (A_f)_0$, we have that $\iota(I)$ consists of degree zero elements of A_f, and is thus homogeneous. We see that if $\pi : A \to A_f$ is the localization map, then $\pi^{-1}(\mathfrak{p}_f) = \mathfrak{p}$, hence:

$$F^{-1}(\mathbb{V}(I)) = \{\mathfrak{p} \in U_f : \pi^{-1}(\iota(I)) \subset \mathfrak{p}\} = U_f \cap \mathbb{V}(\pi^{-1}(\iota(I)))$$

which is a closed in the subspace topology on U_f hence F is continuous. We note that $f \notin \pi^{-1}(\iota(I))$, as this would imply that $f/1 \in I$ which can't be true as $I \subset (A_f)_0$. It follows that:

$$F^{-1}(\mathbb{V}(I)) = \mathbb{V}(\pi^{-1}(\iota(I))) \subset U_f$$

Now take $V \subset U_f$ be a closed subset. We must have that $V = \mathbb{V}(I) \cap U_f$ for some homogeneous ideal I. Moreover, if $f \in I$, then $\mathbb{V}(I) \cap U_f = \emptyset$, hence we actually have that $V = \mathbb{V}(I) \subset U_f$, and $f \notin I$. Now note that by Proposition 2.2.1:

$$\begin{aligned} F(\mathbb{V}(I)) =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : I \subset \pi_f^{-1}(\sqrt{\mathfrak{q}A_f})\} \\ =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : \pi_f(I) \subset \sqrt{\mathfrak{q}A_f}\} \\ =& \{\mathfrak{q} \in \operatorname{Spec}(A_f)_0 : \iota^{-1}(\pi_f(I)) \subset \mathfrak{q}\} \\ =& \mathbb{V}(\iota^{-1}(\pi_f(I))) \subset \operatorname{Spec}(A_f)_0 \end{aligned}$$

which is closed. It follows that F is a continuous closed bijection, and hence a homeomorphism as desired. $\square$

Our goal is to now equip $\operatorname{Proj} A$ with the structure of a scheme via Proposition 1.2.11. Note that we could also glue the affine schemes $\operatorname{Spec}(A_f)_0$ together via Theorem 2.1.1 and get the same result, but this would 'overkill', given that we have already in a sense glued the topological spaces $\operatorname{Spec}(A_f)_0$ together by our construction of the topology on $\operatorname{Proj} A$. We could also define the structure sheaf to be the sheaf on a base given by $U_f \mapsto \mathscr{O}_{\operatorname{Spec}(A_f)_0}$, and show that this truly defines a sheaf on the base of distinguished opens, but as we are about to see this equivalent description would be much more involved. We need the following lemma:

Lemma 2.2.7. *Let A be a graded ring, and $f, g \in A$ homogeneous elements of positive degree. Then:*

$$(A_{fg})_0 \cong ((A_f)_0)_h$$

where $h = g^{\deg f}/f^{\deg g}$. In particular, with $h^{-1} = f^{\deg g}/g^{\deg f}$ we have that:

$$((A_f)_0)_h \cong ((A_g)_0)_{h^{-1}}$$

Proof. We first examine the map:

$$\begin{aligned} \psi : A_f &\longrightarrow A_{fg} \\ \frac{a}{f^k} &\longmapsto \frac{a \cdot g^k}{(fg)^k} \end{aligned}$$

and note that if $a/f^k \in (A_f)_0$, then clearly $\psi(a/f^k) \in (A_{fg})_0$, so this descends to a morphism $\psi_0 : (A_f)_0 \to (A_{fg})_0$. We now see that

$$\psi_0 : h \longmapsto \frac{g^{\deg f} \cdot g^{\deg g}}{(fg)^{\deg g}}$$

which has an inverse in $(A_{fg})_0$ given by:

$$\frac{f^{\deg g + \deg f}}{(fg)^{\deg f}}$$

so there exists a unique map:

$$\begin{aligned} \theta_0 : ((A_f)_0)_h &\longrightarrow (A_{fg})_0 \\ \frac{a}{f^l} \cdot h^{-k} &\longmapsto \frac{a \cdot g^l}{(fg)^l} \cdot \left(\frac{f^{\deg g + \deg f}}{(fg)^{\deg f}} \right)^k \end{aligned}$$

We first claim this map injective. Suppose that $(a/f^l) \cdot h^{-k} \mapsto 0$, then we have that:

$$\frac{a \cdot g^l \cdot f^{k \deg g + k \deg f}}{(fg)^{l + k \deg f}} = 0$$

implying there exists a K such that:

$$(fg)^K (a \cdot g^l \cdot f^{k \deg g + k \deg f}) = 0$$

We want to then show that there exists an L such that:

$$\frac{a \cdot g^{L \deg f}}{f^{l \cdot L \deg f}} = 0$$

meaning that we really want to show there exists an L' such that:

$$f^{L'} \cdot (a g^{L \deg f}) = 0$$

Well set $L = K + l$, and $L' = K + k \deg g + k \deg f$, then:

$$f^{K + k \deg g + k \deg f} (a \cdot g^{K+l}) = (fg)^K (a \cdot g^l \cdot f^{k \deg g + k \deg f}) = 0$$

so $a/f^l \cdot h^{-k} = 0$ as desired, and the map is injective. Now let $a/(fg)^k \in (A_{fg})_0$, then we see that:

$$\begin{aligned} \theta_0 \left(\frac{g^{k \deg f - k} a}{f^{k \deg g + k}} \cdot h^{-k} \right) &= \frac{a \cdot g^{k \deg f - k + k + k \deg g}}{(fg)^{k \deg g + k}} \cdot \frac{f^{k \deg f + k \deg g}}{(fg)^{k \deg f}} \\ &= \frac{a g^{k \deg f + k \deg g} \cdot f^{k \deg f + k \deg g}}{(fg)^{k \deg g + k \deg f + k}} \\ &= \frac{a}{(fg)^k} \end{aligned}$$

so the map is also surjective, and thus an isomorphism. Clearly the second claim follows from the first. $\square$

Theorem 2.2.1. *There exists a unique (up to unique isomorphism) sheaf of rings $\mathscr{O}_{\operatorname{Proj} A}$ on $\operatorname{Proj} A$ which makes $\operatorname{Proj} A$ into a scheme, such that $(U_f, \mathscr{O}_{\operatorname{Proj} A}|_{U_f}) \cong (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec}(A_f)_0})$ for any homogeneous element f of positive degree.*

Proof. Let A_+^{hom} be the set of homogeneous elements of A of positive degree, and consider the cover of $\operatorname{Proj} A$ by $\{U_f\}_{f \in A_+^{\text{hom}}}$. For each U_f, let $\psi_f : \operatorname{Spec}(A_f)_0 \to U_f$ be the aforementioned homeomorphism, and set $\mathscr{F}_f := \psi_{f*} \mathscr{O}_{\operatorname{Spec}(A_f)_0}$. We need to define sheaf isomorphisms $\phi_{fg} : \mathscr{F}_f|_{U_f \cap U_g} \to \mathscr{F}_g|_{U_f \cap U_g}$ which satisfy $\phi_{fg} = \phi_{lg} \circ \phi_{fl}$ on triple overlaps $U_f \cap U_l \cap U_g$. First note that that:

$$U_f \cap U_g = \{\mathfrak{p} \in \operatorname{Proj} A : f, g \notin \mathfrak{p}\}$$

Since $\mathfrak{p}$ is prime, we have that $f, g \notin \mathfrak{p} \Leftrightarrow f \cdot g \notin \mathfrak{p}$, hence:

$$U_f \cap U_g = U_{fg} \cong \operatorname{Spec}(A_{fg})_0$$

Now by Lemma 2.2.7 and Corollary 1.4.3, we have that as affine schemes:

$$U_h = \operatorname{Spec}((A_f)_0)_h \cong \operatorname{Spec}(A_{fg})_0 \cong \operatorname{Spec}((A_g)_0)_{h^{-1}} = U_{h^{-1}}$$

where $h = g^{\deg f}/f^{\deg g}$, $h^{-1} = f^{\deg g}/g^{\deg f}$, and $U_h \subset \operatorname{Spec}(A_f)_0$, $U_{h^{-1}} \subset \operatorname{Spec}(A_g)_0$ are the distinguished open sets.

Moreover, we have $U_{fg} \subset U_f$, so we can examine the open set $\psi_f^{-1}(U_{fg}) \subset \operatorname{Spec}(A_f)_0$. We claim that this is equal to $U_h \subset \operatorname{Spec}(A_f)_0$; indeed, we have that if $\mathfrak{q} \in \psi_f^{-1}(U_{fg})$, then $\mathfrak{q} = \iota^{-1}(\mathfrak{p}_f)$ for some $\mathfrak{p} \in U_{fg}$. Since $f \cdot g \notin \mathfrak{p}$, we have that $g \notin \mathfrak{p}_f$, hence $h \notin \mathfrak{p}_f$, but $h \in (A_f)_0$, hence $h \notin \iota^{-1}(\mathfrak{p}_f)$ so $\mathfrak{q} \in U_h$. Now suppose that $\mathfrak{q} \in U_h$, we want to show that $\psi_f(\mathfrak{q}) \in U_{fg}$; well clearly $\psi_f(\mathfrak{q}) \in U_f$, and $g^{\deg f}/1 \notin \sqrt{\mathfrak{q}_0 A_f}$, hence $g^{\deg f} \notin \pi^{-1}(\sqrt{(}\mathfrak{q}_0)A_f) = \psi_f(\mathfrak{q})$, so $g \notin \psi_f(\mathfrak{q})$, implying that $\psi_f(\mathfrak{q}) \in U_{fg}$. Respectively, we have that $\psi_g^{-1}(U_{fg}) = U_{h^{-1}} \subset \operatorname{Spec}(A_g)_0$.

Now note since the isomorphism

$$(U_h, \mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h}) \cong (U_{h^{-1}}, \mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}})$$

is induced by the by the unique ring isomorphisms from Lemma 2.2.7, that the homeomorphism $U_h \to U_{h^{-1}}$ must be given by the restriction $\psi_g^{-1} \circ \psi_f|_{U_h}$. In particular, we have the following sheaf isomorphism:

$$\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}} \longrightarrow (\psi_g^{-1} \circ \psi_f|_{U_h})_* \mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h}$$

Since ψ_g is a homeomorphism, we thus obtain the following isomorphism of sheaves:

$$(\psi_g|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}}) \longrightarrow (\psi_f|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_f)_0}|_{U_h})$$

By noting that

$$(\psi_g|_{U_{fg}})_*(\mathscr{O}_{\operatorname{Spec}(A_g)_0}|_{U_{h^{-1}}}) = (\psi_{g*}\mathscr{O}_{\operatorname{Spec}(A_g)_0})|_{U_{fg}} = \mathscr{F}_g|_{U_{fg}}$$

and similarly for the ψ_f, we have the desired isomorphisms ϕ_{gf}.

Now let $f, g, l \in A_+^{\text{hom}}$, then we have the following unique ring isomorphisms:

$$((A_f)_0)_{g^{\deg f}/f^{\deg g}} \cong ((A_g)_0)_{f^{\deg g}} \qquad ((A_l)_0)_{g^{\deg l}/l^{\deg g}} \cong ((A_g)_0)_{l^{\deg g}/g^{\deg l}}$$
$$((A_f)_0)_{l^{\deg f}/l^{\deg g}} \cong ((A_l)_0)_{l^{\deg f}/f^{\deg l}}$$

which we denote by β_{fg}, β_{lg} and β_{fl} respectively. Note that these are the ring isomorphisms which induces the sheaf isomorphisms ϕ_{fg}. The triple overlap then satisfies:

$$U_{fgl} \cong U_{(gl)^{\deg f}/f^{\deg gl}} \cong U_{(fg)^{\deg l}/l^{\deg fg}} \cong U_{(fl)^{\deg g}/g^{\deg fl}}$$

which are the distinguished opens in $\operatorname{Spec}(A_f)_0$, $\operatorname{Spec}(A_l)_0$ and $\operatorname{Spec}(A_g)_0$. These isomorphisms of affine schemes come from the following unique ring isomorphisms induced by localization:

$$((A_f)_0)_{(gl)^{\deg f}/f^{\deg gl}} \cong ((A_l)_0)_{(fg)^{\deg l}/l^{\deg fg}} \cong ((A_g)_0)_{(fl)^{\deg g}/g^{\deg fl}}$$

In particular, these isomorphisms, which we denote by $\beta_{fl,g}$ and $\beta_{lg,f}$, are induced by β_{fl} and β_{lg}, composing with the relevant localization map, and the universal property of localization. These are also precisely the ring isomorphisms inducing the sheaf isomorphisms $\phi_{fl}|_{U_{fgl}}$ and $\phi_{lg}|_{U_{fgl}}$. The sheaf isomorphism $\phi_{fg}|_{U_{fgl}}$ comes from the ring isomorphism:

$$((A_f)_0)_{(gl)^{\deg f}/f^{\deg gl}} \cong ((A_g)_0)_{(fl)^{\deg g}/g^{\deg l}}$$

induced by β_{fg} in the same way. We denote this isomorphism by $\beta_{fg,l}$. Now, since $\beta_{fg,l}$ and $\beta_{lg,f} \circ \beta_{fl,g}$ both make the same localization diagram commute, we know they must be equal. It follows that:

$$\phi_{gf}|_{U_{fgl}} = \phi_{lg}|_{U_{fgl}} \circ \phi_{fl}|_{U_{fgl}}$$

so the sheaves glue to yield unique (up to unique isomorphism) sheaf of rings on Proj A.

All that remains to show is that Proj A is a scheme however this is now clear, as for any homogeneous element of positive degree f, we have a homeomorphism $\psi_f : \operatorname{Spec}(A_f)_0 \to U_f$, and a sheaf morphism:

$$\mathscr{O}_{\operatorname{Proj} A}|_{U_f} = \psi_{f*}\mathscr{O}_{\operatorname{Spec}(A_f)_0} \longrightarrow \psi_{f^*}\mathscr{O}_{\operatorname{Spec}(A_f)_0}$$

given by the identity map, so every point in x has an open neighborhood isomorphic to an affine scheme.

□

We now recall that the construction in Example 2.2.1 is valid for any commutative ring A, hence we have the following proposition:

Proposition 2.2.3. *Let A be a commutative ring, and consider the polynomial ring $A[x_0, \dots, x_n]$ with the standard grading induced by $\deg x_i = 1$, then:*

$$\mathbb{P}^n_A \cong \operatorname{Proj} A[x_0, \dots, x_n]$$

where $\mathbb{P}^n_A$ is the scheme constructed as in Example 2.2.1.

Proof. We first claim the distinguished opens $U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$ cover $\operatorname{Proj} A[x_0, \dots, x_n]$. Let $\mathfrak{p} \in \operatorname{Proj} A$, then we have that $\mathfrak{p}$ is a homogeneous prime ideal which does not contain the trivial idea, then $\mathfrak{p}$ can not be of the form:

$$\mathfrak{p} = \langle x_0, \dots, x_n \rangle$$

or contain such an ideal. It follows that at least one $x_i \in A[x_0, \dots, x_n]$ does not lie in $\mathfrak{p}$, hence we have that:

$$\operatorname{Proj} A[x_0, \dots, x_n] = \bigcup_{i=0}^{n} U_{x_i}$$

We now note that for each i we have that as schemes:

$$U_{x_i} \cong \operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0$$

and that the ring homomorphism:

$$\begin{aligned} \phi_i : (A[x_0, \cdots, x_n]_{x_i})_0 &\longrightarrow A[\{x_k/x_i\}_{k \neq i}] \\ x_m/x_i &\mapsto x_m/x_i \end{aligned}$$

is an isomorphism. We thus have scheme isomorphisms:

$$\operatorname{Spec} A[\{x_k/x_i\}_{k \neq i}] \longrightarrow U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$$

Now by noting we have that $\mathbb{P}^n_A$ is given by gluing the schemes $X_i = \operatorname{Spec} A[\{x_k/x_i\}_{k \neq i}]$ together as in Example 2.2.1, and via the open embeddings $\psi_i : X_i \to \mathbb{P}^n_A$, we have scheme isomorphisms:

$$f_i : \psi(X)_i \subset \mathbb{P}^n_A \longrightarrow U_{x_i} \subset \operatorname{Proj} A[x_0, \dots, x_n]$$

which trivially agree on overlaps, so we have a scheme isomorphism:

$$f : \mathbb{P}^n_A \longrightarrow \operatorname{Proj} A[x_0, \dots, x_n]$$

as desired. □

It is common in the literature to refer to a graded ring A over $A_0 = B$ to be *finitely generated in degree one over B* when the irrelevant ideal is finitely generated by degree one elements.[26]. We now define employ a more general definition of a projective scheme:

[26]Equivalently A is generated as a B-algebra by degree one elements.

Definition 2.2.7. A scheme X is a **projective scheme over B** if it is of the form $\operatorname{Proj} A$ for some graded ring A with $A_0 = B$, and A finitely generated in degree one over B. In particular, if A is any commutative ring, then the projective scheme $\mathbb{P}^n_A$ is defined by:

$$\mathbb{P}^n_A = \operatorname{Proj}(A[x_0, \ldots, x_n])$$

Now let k be any algebraically closed field, and recall the argument that the closed points of $\mathbb{P}^n_k$ are in bijection with standard projective space over k. We wish to identify the closed points $[z_0, \ldots, z_n]$ with homogeneous prime ideals of $k[x_0, \ldots, x_n]$. Note that at least one of these z_i must not be zero, so we can rewrite this point as:

$$[z_0/z_i, \ldots, 1, \ldots, z_n/z_i] \in \psi(X_i)$$

which then corresponds to the maximal ideal:

$$\mathfrak{p}_0 = \left\langle \frac{x_0}{x_i} - \frac{z_0}{z_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i} - \frac{z_n}{z_i} \right\rangle \in \operatorname{Spec} k[\{x_l/x_i\}_{l \neq i}]$$

which is the the same ideal in $\operatorname{Spec}(k[x_0, \ldots, x_n]_{x_i})_0$. Under the bijection between prime ideals of $(k[x_0, \ldots, x_n]_{x_i})_0$ and homogeneous prime ideals missing the trivial ideal of $k[x_0, \ldots, x_n]$, we then have that this corresponds to $\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}})$ where $A = k[x_0, \ldots, x_n]$. Since x_i is invertible in A_{x_i}, we see that:

$$\mathfrak{p}_0 A_{x_i} = \langle x_0 z_i - x_i z_0, \ldots, \hat{x}_i, \ldots, x_n z_i - x_i z_n \rangle \subset A_{x_i}$$

which is prime and thus radical. It follows that:

$$\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}}) = \langle x_0 z_i - x_i z_0, \ldots, \hat{x}_i, \ldots, x_n z_i - x_i z_n \rangle \subset k[x_0, \ldots, x_n]$$

Now note that for any k and l we have that:

$$x_k z_l - x_l z_k = (x_k z_i - x_i z_k) \cdot (z_l/z_i) - (x_l z_i - x_i z_l) \cdot (z_k/z_i)$$

so we have that:

$$\pi^{-1}(\sqrt{\mathfrak{p}_0 A_{x_i}}) = \langle x_i z_j - x_j z_i | 0 \leq i, j \leq n \rangle$$

Therefore the correspondence between closed points and homogeneous prime ideals of $k[x_0, \ldots, x_n]$ is given by:

$$[z_0, \ldots, z_n] \longleftrightarrow \langle x_i z_j - x_j z_i | 0 \leq i, j \leq n \rangle$$

Example 2.2.2. Let A be any commutative ring, we will examine $\operatorname{Proj} A[x]$ with two different gradings on $A[x]$. First, let $A[x]$ have the standard grading, and then note that $U_x = \operatorname{Proj} A[x]$. However, $U_x \cong \operatorname{Spec}(A[x]_x)_0$, and we see that:

$$(A[x]_x)_0 \cong (A[x, x^{-1}])_0 \cong A$$

so $\operatorname{Proj} A[x] \cong \operatorname{Spec} A$. Note that when $A = \mathbb{C}$, then we have that this implies that $\mathbb{P}^0_{\mathbb{C}} \cong \{\langle 0 \rangle\}$, i.e. the singleton set. This matches with the fact that $\mathbb{C} \smallsetminus \{0\}/\mathbb{C}^*$ is just a point.

Now let $A[x]$ have the trivial grading so that every element is homogeneous and of degree 0, we wish to describe $\operatorname{Proj} A[x]$, however this is easy. We see that

$$\operatorname{Proj} A[x] = \{\mathfrak{p} \in \operatorname{Spec} A[x] : \mathfrak{p} \text{ is homogeneous and } (A[x])_+ \notin \mathfrak{p}\}$$

is empty as $(A[x])_+ = \langle 0 \rangle$ and every ideal contains 0. It follows that $\operatorname{Proj} A[x]$ is the empty scheme.

Example 2.2.3. Let $X = \operatorname{Proj} \mathbb{C}[x, y, z]$, where $\mathbb{C}[x, y, z]$ is equipped with the grading $\deg x = 0$, $\deg y = \deg z = 1$, and all elements of $\mathbb{C}$ are degree zero. We know that in the standard grading case the closed points $\mathbb{P}^2_{\mathbb{C}}$ are precisely the points of $\mathbb{CP}^2$, we now wish to see how this changes with this new grading. We claim that $X = U_y \cup U_z$, where U_y and U_z are the projective distinguished open sets. Let $\mathfrak{p} \in X$, then $\mathfrak{p}$ is a homogeneous prime ideal which does not contain the trivial ideal. In particular, either y or x can't lie in $\mathfrak{p}$, so $\mathfrak{p} \in U_x \cup U_z$. We have that:

$$U_y \cong \operatorname{Spec}(\mathbb{C}[x,y,z]_y)_0 \cong \operatorname{Spec}\mathbb{C}[x,z/y] \qquad \text{and} \qquad U_z = \operatorname{Spec}(\mathbb{C}[x,y,z]_z)_0 \cong \operatorname{Spec}\mathbb{C}[x,y/z]$$

The closed points of each are of the form $\langle x - w_1, z/y - w_2\rangle$ and $\langle x - w_1, y/z - w_2\rangle$, and for a prime $\mathfrak{p}$ to be closed we necessitate that $\imath^{-1}(\mathfrak{p})$ in both U_x and U_y. We have that the gluing isomorphism along $U_z \cap U_y = U_{xy} \cong \operatorname{Spec}[x, y/z/, z/y]$ takes the closed point $\langle x - w_1, z/y - w_2\rangle$ to $\langle x - w_1, y/z - 1/w_2\rangle$. We define a set map

$$F : |U_x| \coprod |U_y| \longrightarrow \mathbb{C} \times \mathbb{P}^1$$

via the disjoint union set map induced by the maps:

$$\langle x - w_1, z/y - w_2\rangle \longmapsto (w_1, [w_2, 1]) \qquad \text{and} \qquad \langle x - w_1, y/z - w_1\rangle \longmapsto (w_1, [1, w_2])$$

Now this map is clearly surjective, and factors through the quotient condition, as if $\langle x - w_1, z/y - w_2\rangle \sim \langle x - w_1, y/z - v\rangle$, then we have that $v = 1/w_2$, so $\langle x - w - 1, y/z - v\rangle$ maps to $(w_1, [1, 1/w_2]) = (w_1, [w_2, 1])$. It follows there is induced map $\tilde{F} : |X| \to \mathbb{C}\times\mathbb{P}^1$, which is then also clearly injective. Via the identification of $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$ we see that the closed points of X are in bijection with $\mathbb{C} \times (\mathbb{C} \cup \{\infty\})$

Example 2.2.4. Let V be a vector space over a field k[27], then we define:

$$\mathbb{P}(V) := \operatorname{Proj}(\operatorname{Sym} V^*)$$

where $\operatorname{Sym} V^*$ is the symmetric algebra of the dual space to V. In particular:

$$\operatorname{Sym} V^* = T(V^*)/I = (k \oplus V^* \oplus V^* \otimes_k V^* \oplus \cdots)/I$$

where I is the homogeneous ideal:

$$I = \langle \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1 : \omega_i \in V^*\rangle$$

Note that with V finite dimensional, after fixing a basis $\{e_i\}_{i=1}^n$, and a corresponding dual basis $\{e^i\}_{i=1}^n$, we obtain an isomorphism

$$\operatorname{Sym} V^* \cong k[e^1, \dots, e^n] \cong k[x_1, \dots, x_n]$$

hence:

$$\mathbb{P}(V) \cong \mathbb{P}_k^{n-1}$$

We now suppose $k = \bar{k}$, and claim that any closed point of $\mathbb{P}(V)$ corresponds to a one dimensional linear subspace of $\ell \subset V$. Indeed, let $\ell \subset V$ be a one dimensional linear subspace, and define:

$$\mathfrak{p}_\ell = \langle \omega \in V^* : \omega(\ell) = \{0\}\rangle$$

i.e. we take the homogeneous ideal generated by degree 1 elements which vanish on all of ℓ. We claim that $\mathfrak{p}_\ell$ is prime; Fix a $v \in \ell$, and then let $u_1, \dots, u_{n-1}$ be a set of vectors such that $\{u_1, \dots, u_{n-1}, v\}$ is a basis. If we let $\{\mu_1, \dots, \mu_{n-1}, \nu\}$ be a dual basis basis such that $\mu_i(u_j) = \delta_{ij}$, $\mu_i(v) = 0$, then $\mathfrak{p}_l = \langle \mu_1, \dots, \mu_{n-1}\rangle$ which is manifestly prime.

We show that $\mathbb{V}(\mathfrak{p}_\ell) = \{\mathfrak{p}_\ell\}$. Indeed, suppose that there was a $\mathfrak{q} \in \operatorname{Proj}(\operatorname{Sym} V^*)$ such that $\mathfrak{p}_\ell \subset \mathfrak{q}$. In particular, we have that $V^* \cap \mathfrak{p}_\ell \subset \mathfrak{q} \cap V^*$, but in the process of showing that $\mathfrak{p}_\ell$ was prime, we showed that $\mathfrak{p}_\ell \cap V^*$ is an $n-1$ dimensional vector space, hence $\mathfrak{q} \cap V^* = \mathfrak{p}_\ell \cap V^*$ or V^*. In the latter case $\mathfrak{q} \notin \operatorname{Proj}(\operatorname{Sym} V^*)$, and in the former, we claim this implies that $\mathfrak{q} = \mathfrak{p}_\ell$. Suppose there was some

[27] Or more generally a free module over a ring A.

homogeneous $\omega \in \mathfrak{q}$ of degree n, that was not in $\mathfrak{p}_\ell$. Then, $\langle \mu_1, \ldots, \mu_{n-1}, \omega \rangle \subset \mathfrak{q}$; since $\omega \notin \mathfrak{p}_\ell$, we can write:

$$\omega = \mu + \alpha \cdot \nu$$

where $\mu \in \mathfrak{p}_\ell$. It follows that $\alpha \cdot \nu \in \mathfrak{q}$, so $\mathfrak{q}$ is not prime, hence we must have $\mathfrak{q} = \mathfrak{p}_\ell$. It follows that $\mathfrak{p}_\ell$ is maximal amongst prime ideals in $\operatorname{Proj}(\operatorname{Sym} V^*)$, and is thus a closed point.

Now suppose that $\mathfrak{p}$ is a closed point of $\mathbb{P}(V)$. We first note that since k is algebraically closed, $\mathfrak{p} \cap V^* \neq \{0\}$. Indeed, after choosing a basis, we can identify $\operatorname{Sym} V^*$ with $k[x_1, \ldots, x_n]$, and so $\mathfrak{p} \cap V^* = \{0\}$ implies that $\mathfrak{p} = \langle f_1, \ldots, f_m \rangle$ where each f_i is homogeneous of degree greater than 1. Since $\mathfrak{p}$ is closed, and thus maximal amongst homogeneous prime ideals not containing the irrelevant ideal, we have that by Proposition 2.2.1, the corresponding ideal in $k[x_1/x_i, \ldots, x_n/x_i]$ for some i is maximal.[28] The generating set of this ideal must contain a polynomial with leading term of degree greater than 1 as other wise x_i divides each f_i. However, this then implies the existence of a maximal ideal of $k[x_1/x_i, \ldots, x_n/x_i]$ which is not generated by linear factors, which contradicts Hilbert's Nullstellensatz. It follows that $\mathfrak{p} \cap V^* \neq \{0\}$, and thus must have dimension $n-1$ as otherwise it is not maximal. We send $\mathfrak{p} \cap V^*$ to the linear subspace:

$$\ell_{\mathfrak{p}} = \{v \in V : \omega(v) = 0, \forall \omega \in \mathfrak{p} \cap V^*\}$$

This is clearly one dimensional, and in particular the maps $\ell \mapsto \mathfrak{p}_\ell$, and $\mathfrak{p} \mapsto \ell_{\mathfrak{p}}$ are clear inverse of each other. We thus have the following obvious bijections:

$$|\mathbb{P}(V)| \longleftrightarrow \{\text{one dimensional linear subspaces of } V\}$$

We at times denote one dimensional linear subspaces by equivalence classes $[v]$, such that $[v] = [w]$ if and only if there exists a scalar $\lambda \in k^\times$ satisfying $v = \lambda w$. Note that if $k \neq \bar{k}$, then not every maximal homogeneous prime ideal corresponds to a linear subspace; in particular, we have that if $V = \mathbb{R}^2$, then $\left\langle x^2 + y^2 \right\rangle$ is such an ideal.

Note that we can do the same thing for free modules over say $\mathbb{Z}$, and obtain a projective space isomorphic to $\mathbb{P}^n_{\mathbb{Z}}$, however we lose the nice identification of the closed points. Moreover, one can also use the convention that $\mathbb{P}(V) = \operatorname{Proj} \operatorname{Sym} V$, however when V is a vector space over $k = \bar{k}$, we now canonically identify closed points with one dimensional quotients of V i.e. morphisms $V \to U$ with $\dim U = 1$. These two conventions are called the Fulton convention (lines in V) and the Grothendieck convention (one dimensional quotients of V).

Example 2.2.5. Fix $k = \bar{k}$; we wish to construct a closed subscheme of $G_k(d, n) \subset \mathbb{P}(W)$ for some k-linear vector space W, such that the closed points $|G_k(d, n)|$, can be identified with d dimensional linear subspaces of $V = k^n$. In other words, we wish to define a scheme which is the algebraic geometry analogue of the Grassmannian from differential geometry.

We claim that the correct W is given by $W = \Lambda^d V$, then:

$$\mathbb{P}(W) = \operatorname{Proj} \operatorname{Sym}(\Lambda^d V^*)$$

We define $D \subset \Lambda^d V$ as:

$$D = \{v_1 \wedge \cdots \wedge v_d \in \Lambda^d V : v_i \in V\}$$

Note that we are not taking this as a linear subspace or span, we are simply considering all elements in $\Lambda^d V$ which can be written in this form, i.e. alternating tensors which are simple or pure. We define an ideal via:

$$I = \{\omega \in \operatorname{Sym}(\Lambda^d V^*) : \omega(D) = \{0\}\}$$

and immediately note that $I \cap \Lambda^d V^* = 0$. We need to check that this ideal is homogeneous; let $\omega \in I$, and write:

$$\omega = \sum_i \omega_i$$

[28]The value i is clearly dependent on which open set U_{x_i} $\mathfrak{p}$ lives in.

where each ω_i has degree i. It suffices to check that if $\omega \in I$, then $\omega_i \in I$ for each i. Since $\omega(D) = 0$, we see that for any $\lambda \in k^\times$ that $\omega(\lambda \cdot D) = 0$. It follows that for all $v_1 \wedge \cdots \wedge v_d$, and all λ we have that:

$$\omega(\lambda v_1 \wedge \cdots \wedge v_d) = \sum_i \lambda^i \omega_i(v_1 \wedge \cdots \wedge v_d) = 0$$

Fixing $v_1 \wedge \cdots \wedge v_d$, and writing $a_i = \omega_i(v_1 \wedge \cdots \wedge v_d)$, we thus have a sequence of elements $(a_1, \ldots, a_m)$ for some m, such that:

$$\sum_i \lambda^i a_i = 0$$

for all non zero $\lambda \in k$. In particular, this means that the polynomial $p(x) \in k[x]$ given by:

$$p(x) = \sum_i x^i a_i$$

is the zero polynomial, hence each $a_i = 0$.[29] Since this hold for all $v_1 \wedge \cdots v_d$, it follows that each ω_i is identically zero on D, and thus I is generated by homogeneous elements.

We claim that $G_k(d, n) = \mathbb{V}(I)$ is the desired subscheme. Given a d dimensional linear subspace $W \subset V$, we choose a basis $\{v_1, \ldots, v_d\}$ and send it to $[v_1 \wedge \cdots \wedge v_d] \in \mathbb{P}(\Lambda^d V)$. Note that $[v_1 \wedge \cdots \wedge v_d] \in \mathbb{V}(I)$, as every element in I vanishes on $l = \text{span}\{v_1 \wedge \cdots \wedge v_d\}$, hence $I \subset \mathfrak{p}_l$. Note that this independent of the chosen basis, as another basis $\{w_1, \ldots, w_n\}$, yields an automorphism $g : W \to W$, such that

$$v_1 \wedge \cdots \wedge v_d = \det(g) \cdot w_1 \wedge \cdots \wedge w_d$$

which both determine the same $l \in |\mathbb{P}(\Lambda^d V)|$. Now let $\mathfrak{p}$ be a closed point of $\mathbb{P}(\Lambda^d V)$, and suppose that $\mathfrak{p} \in \mathbb{V}(I)$. Then we can uniquely identify $\mathfrak{p}$ with a linear subspace of $\Lambda^d V$, and since $\mathfrak{p} \in \mathbb{V}(I)$, this linear subspace must be spanned by some $v_1 \wedge \cdots \wedge v_d$ for some $v_i \in V$. We then send $\mathfrak{p}$ to the vector subspace spanned by $v_1, \ldots, v_d$. These operations are inverses of one another and thus we have obtained a bijection:

$$|G_k(d, n)| \longleftrightarrow \{d \text{ dimensional linear subspaces of } V\}$$

2.3 Fibre Products

Just as the coproduct does not generally exist in the category of rings, and is replaced with the more general notion of the tensor product of rings (which becomes the coproduct in the category of A algebras), we have a similar situation regarding direct products in the category of schemes. In particular, the direct product does not generally exist in the category of schemes, but is instead replaced with the more general notion of a fibre product.

Definition 2.3.1. Let X, Y and Z be objects in an arbitrary category, with morphisms $f : X \to Z$ and $g : Y \to Z$. The **fibre product of X and Y over Z** is the triplet $(X \times_Z Y, \pi_X, \pi_Y)$ such that the following hold:

i) $X \times_Z Y$ is an object in the aforementioned category.

ii) π_X and π_Y are morphisms $X \times_Z Y$ to X and Y respectively.

iii) If Q is any other object with morphisms $p_X : Q \to X$ and $p_Y : Q \to Y$ such that $f \circ p_X = g \circ \pi_Y$

[29] Note, that this argument only works if k has infinitely many elements, as then the ideal $\bigcap_{\lambda \in k} \langle x - \lambda \rangle = \langle 0 \rangle$. We will fix this later, when we give a better definition of the Grassmanian.

then there exists a unique morphism $\psi : Q \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\exists!\psi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We call $X \times_Z Y$ the **fibre products** and the morphisms π_X and π_Y **projection maps**.=

Note that this is the diagram defining a tensor product in the category of rings with the arrows reversed. Before we prove that fiber products of schemes exist, we will first prove some very general properties of fibre products. We will state most of our results in terms of schemes, but we alert the reader to the fact that the following results will hold in any category where fibre products exist. For now suppose we have already proven that fibre products exist in the category of schemes. First we employ the following definition:

Definition 2.3.2. Let Z be a scheme; a pair (X, f) where X is a scheme, and $f : X \to Z$ is a morphism of schemes is called **scheme over Z** or a **Z-scheme**. If (X, f) and (Y, g) are Z schemes, then a **morphism of Z schemes** $F : X \to Y$ is a morphism of schemes such that $f = g \circ F$.

One easily verifies the that collection of all Z schemes and their morphisms is a category which contains fibered products. In particular, we denote the category over all Z-schemes by Sch/Z.

Lemma 2.3.1. *Let (X, f), (Y, g), (W, h) be Z schemes, then there are canonical isomorphisms:*

$$X \times_Z Y \cong Y \times_Z X \qquad \textit{and} \qquad (X \times_Z Y) \times_Z W \cong X \times_Z (Y \times_Z W)$$

Proof. For notation purposes, we will denote projection maps on the left hand side of the first isomorphism with a superscript 1, and those on the right hand side with a superscript 2. Now note that we trivially have that the projection maps satisfy $f \circ \pi_X^i = g \circ \pi_Y^i$, so there exists unique morphisms $\psi : X \times_Z Y \to Y \times_Z X$ and $\phi : Y \times_Z X \to X \times_Z Y$. We thus have a morphism $\phi \circ \psi : X \times_Z Y \to X \times_Z Y$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "\phi\circ\psi"] \arrow[drr, bend left, "\pi_Y^1"] \arrow[ddr, bend right, "\pi_X^1"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y^1"] \arrow[d, "\pi_X^1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

However, the identity map also clearly satisfies this, so by uniqueness $\phi \circ \psi = \text{Id}$. A similar argument shows that $\psi \circ \phi = \text{Id}$, hence ψ (and ϕ) is a unique isomorphism.

Now note that $(X \times_Z Y) \times_Z W$ comes equipped with morphisms to $(X \times_Z Y)$ and W, given by $\pi^1_{X \times_Z Y}$ and π_W. We thus have a morphism from $(X \times_Z Y) \times_Z W$ to X and Y given by $\pi_X \circ \pi_{X \times_Z Y}$ and $\pi_Y \circ \pi_{X \times_Z Y}$. We see that $X \times_Z Y$ is a Z scheme when equipped with the morphism $f \circ \pi_X$ (equivalently $g \circ \pi_Y$), so we have morphisms:

$$\pi_Y \circ \pi_{X \times_Z Y} : (X \times_Z Y) \times_Z W \longrightarrow Y$$

and:

$$\pi_W : (X \times_Z Y) \times_Z W \longrightarrow W$$

which satisfy:

$$\begin{aligned} g \circ (\pi_Y \circ \pi_{X \times_Z Y}) =& (f \circ \pi_X) \circ \pi_{X \times_Z Y} \\ =& h \circ \pi_W \end{aligned}$$

so we have a unique morphism $\xi : (X \times_Z Y) \times_Z W \to Y \times_Z W$. Now we $Y \times_Z W$ is a Z scheme when equipped with the morphism $g \circ \pi_Y$ (or equivalently $h \circ \pi_W$). We see that:

$$\begin{aligned} (g \circ \pi_Y) \circ \xi =& g \circ \pi_Y \circ \pi_{X \times_Z Y} \\ =& f \circ \pi_X \circ \pi_{X \times_Z Y} \end{aligned}$$

so there is a unique morphism:

$$\psi : (X \times_Z Y) \times_Z W \longrightarrow X \times_Z (Y \times_Z W)$$

and the same argument gives a unique morphism:

$$\phi : X \times_Z (Y \times_Z W) \longrightarrow (X \times_Z Y) \times_Z W$$

which make similar diagrams commute. We see that the composition $\phi \circ \psi$ makes the following diagram commute:

$$\begin{tikzcd}
(X \times_Z Y) \times_Z W \arrow[dr, "\phi\circ\psi"] \arrow[drr, bend left, "\pi_W"] \arrow[ddr, bend right, "\pi_{X \times_Z Y}"'] & & \\
& (X \times_Z Y) \times_Z W \arrow[r, "\pi_W"] \arrow[d, "\pi_{X \times_Z Y}"] & W \arrow[d, "h"] \\
& X \times_Z Y \arrow[r, "f\circ\pi_X"] & Z
\end{tikzcd}$$

so $\phi \circ \psi = \mathrm{Id}$. The same argument then shows that $\psi \circ \phi = \mathrm{Id}$, so ϕ and ψ are isomorphisms as desired. $\square$

We also have the following analogue of the fact that for commutative rings $A \otimes_B B \cong A$:

Lemma 2.3.2. *Let X be a Z-scheme, then there is a natural isomorphism $X \times_Z Z \cong X$.*

Proof. We will show that (X, Id_X, f) satisfies the universal property of $X \times_Z Z$. Indeed, note that Z is naturally a Z-scheme when equipped with the identity morphism $\mathrm{Id}_Z : Z \to Z$. Trivially, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{Id}_X"] & Z \arrow[d, "\mathrm{Id}_Z"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Suppose Q is another scheme with morphisms $p_X : Q \to X$ and $p_Z : Q \to Z$, such that $f \circ p_X = \mathrm{Id}_Z \circ p_Z$, then the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_Z"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \arrow[r, "f"] \arrow[d, "\mathrm{Id}_X"] & Z \arrow[d, "\mathrm{Id}_Z"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We see that putting $p_X : Q \to X$ in the empty diagonal makes the diagram commute, and that any other morphism $\phi : Q \to X$ must satisfy $\mathrm{Id}_X \circ \phi = p_X$, so $\phi = p_X$ and the morphism is unique. It follows that X satisfies the universal property of the fibre product and is thus naturally isomorphic to $X \times_Z Z$. ☐

We have the following extension of the previous results:

Lemma 2.3.3. *Let X and Y be Z-schemes, and S a Y schemes viewed as an Z scheme via the composition $S \to Y \to Z$, then there is a canonical isomorphism of Z schemes:*

$$(X \times_Z Y) \times_Y S \cong X \times_Z S$$

where $(X \times_Z Y)$ is viewed as Y scheme via the second projection π_Y.

Proof. Let $f : X \to Z$, $g : Y \to Z$, and $h : S \to Y$ be the various morphisms that make X and Y Z-schemes, and S an Y scheme. We first know that by Lemma 2.3.1 and Lemma 2.3.2, as Y schemes $S \cong Y \times_Y S$; we claim these are also isomorphic as Z-schemes. There is then a unique isomorphism which makes the following diagram commute:

$$\begin{tikzcd}
S \arrow[dr, "\phi"] \arrow[drr, bend left, "\mathrm{Id}_S"] \arrow[ddr, bend right, "h"'] & & \\
& Y \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_Y"] & S \arrow[d, "h"] \\
& Y \arrow[r, "\mathrm{Id}_Y"] & Y
\end{tikzcd} \qquad \star$$

Now $Y \times_Y S$ is a Z scheme in one of two ways, via $g \circ \pi_Y$, or via $g \circ h \circ \pi_S$, however, $h \circ \pi_S = \mathrm{Id}_Y \circ \pi_Y = \pi_Y$, so these are actually equivalent Z-scheme structures and $Y \times_Y Z$ has a natural Z-scheme structure independent of choice. We thus see that:

$$g \circ \pi_Y \circ \phi = g \circ h$$

so ϕ is a Z scheme isomorphism as well. We now claim that:

$$X \times_Z S \cong X \times_Z (Y \times_Y S)$$

We have a morphism $\phi \circ \pi_S : X \times_Z S \to Y \times_Y S$ and a morphism $\pi_X : X \times_Z S \to X$ which satisfy:

$$g \circ \pi_Y \circ \phi \circ \pi_S = g \circ \pi_Y = f \circ \pi_X$$

so there is a unique morphism ψ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z S \arrow[dr, "\psi"] \arrow[drr, bend left, "\phi \circ \pi_S"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y \times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g \circ \pi_Y"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

In a similar vein, we have a Z scheme isomorphism $\phi^{-1} : Y \times_Y S \to Y$, which by the same argument induces a unique Z-scheme morphism $\xi : X \times_Z (Y \times_Y S) \to X \times_Z S$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\xi"] \arrow[drr, bend left, "\pi_S \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z S \arrow[r, "\pi_S"] \arrow[d, "\pi_X"] & S \arrow[d, "g \circ h"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The composition $\xi \circ \psi : X \times_Z S \to X \times_Z S$ then satisfies:

$$\pi_X \circ \xi \circ \psi = \pi_X \circ \psi = \pi_X$$

and:

$$\begin{aligned}
\pi_S \circ \xi \circ \psi =& \pi_S \circ \pi_{Y \times_Y S} \circ \psi \\
=& \pi_S \circ \phi \circ \pi_S \\
=& \pi_S \circ \mathrm{Id}_Y \\
=& \pi_S
\end{aligned}$$

So $\xi \circ \psi$ is the unique map making the following diagram commute:

$$\begin{tikzcd}
X \times_Z S \arrow[dr, "\xi \circ \psi"] \arrow[drr, bend left, "\pi_S"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z S \arrow[r, "\pi_S"] \arrow[d, "\pi_X"] & S \arrow[d, "g \circ h"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

however, as before the identity map satisfies this as well so by uniqueness $\xi \circ \psi$ is the identity. Similarly, $\psi \circ \xi$ is the identity map as well, so $X \times_Z S \cong X \times_Z (Y \times_Y S)$ as desired.

It now suffices to show that as Z-schemes:

$$(X \times_Z Y) \times_Y S \cong X \times_Z (Y \times_Y S)$$

We first note that as a Y-scheme we have the following commutative diagram:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_{X \times_Z Y}"] & S \arrow[d, "h"] \\
X \times_Z Y \arrow[r, "\pi_Y"] & Y
\end{tikzcd}$$

Now note that that $f \circ \pi_X = g \circ \pi_Y$, so we obtain that:

$$\begin{aligned}
f \circ \pi_X \circ \pi_{X \times_Z Y} =& g \circ \pi_Y \circ \pi_{X \times_Z Y} \\
=& g \circ h \circ \pi_S
\end{aligned}$$

Hence we have the following commutative diagram:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_X \circ \pi_{X \times_Z Y}"] & S \arrow[d, "g \circ h"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

We have a morphism $\pi_X \circ \pi_{X \times_Z Y} : (X \times_Z Y) \times_Y S \to X$, and a morphism $\phi \circ \pi_S : (X \times_Z Y) \times_Y S \to Y \times_Y S$ such that:

$$\begin{aligned}(g \circ h \circ \pi_S) \circ \phi \circ \pi_S =& g \circ h \circ \pi_S \\ =& f \circ \pi_X \circ \pi_{X \times_Z Y}\end{aligned}$$

hence there exists a unique morphism $\psi : (X \times_Z Y) \times_Y S \to X \times_Z (Y \times_Y S)$ such that the following diagram commutes:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[dr, "\psi"] \arrow[drr, bend left, "\phi \circ \pi_S"] \arrow[ddr, bend right, "\pi_X \circ \pi_{X \times_Z Y}"'] & & \\
& X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y \times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g \circ \pi_Y"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Now we go the other direction; we already have a morphism $\pi_S \circ \pi_{Y \times_Y S} : X \times_Z (Y \times_Y S) \to Y$, so we need to construct a morphism $\alpha : X \times_Z (Y \times_S Y) \to X \times_Z Y$. We have a morphism to X, and we have a morphism to Y given by $\pi_Y \circ \pi_{Y \times_Y S}$. We have that:

$$g \circ \pi_Y \circ \pi_{Y \times_Y S} = f \circ \pi_X$$

by the Z scheme structure on $X \times_Z (Y \times_S Y)$ so α is then the unique map that makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\alpha"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We now see that:

$$f \circ \pi_X \circ \alpha = f \circ \pi_X = g \circ \pi_Y \circ \pi_{Y \times_Y S}$$

so we have a unique map $\xi : X \times_Z (Y \times_Y S) \to (X \times_Z Y) \times_Y S$ that makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\xi"] \arrow[drr, bend left, "\pi_Y \circ \pi_{Y \times_Y S}"] \arrow[ddr, bend right, "\alpha"'] & & \\
& (X \times_Z Y) \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_{X \times_Z Y}"] & S \arrow[d, "g \circ h"] \\
& X \times_Z Y \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}$$

Now note that $\xi \circ \psi$ satisfies:

$$\pi_{X \times_Z Y} \circ \xi \circ \psi = \alpha \circ \psi$$

And moreover, see that:

$$\pi_X \circ \alpha \circ \psi = \pi_X \circ \psi = \pi_X \circ \pi_{X\times_Z Y}$$

as well as:

$$\begin{aligned}\pi_Y \circ \alpha \circ \psi =& \pi_Y \circ \pi_{Y\times_Y S} \circ \psi \\ =& \pi_Y \circ \phi \circ \pi_S \\ =& h \circ \pi_S \\ =& \mathrm{Id}_Y \circ \pi_Y \\ =& \pi_Y\end{aligned}$$

So $\alpha \circ \psi$ makes the following diagram commute:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[dr, "\alpha\circ\psi"] \arrow[drr, bend left, "\pi_Y\circ\pi_{Y\times_Y S}"] \arrow[ddr, bend right, "\pi_X\circ\pi_{X\times_Z Y}"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

However, replacing $\pi_{X\times_Z Y}$ also makes this diagram commute, so $\pi_{X\times_Z Y} = \alpha \circ \psi$, and we have that:

$$\pi_{X\times_Z Y} \circ \xi \circ \psi = \pi_{X\times_Z Y}$$

We also see that:

$$\begin{aligned}\pi_S \circ \xi \circ \psi =& \psi_S \circ \pi_{Y\times_Y S} \circ \psi \\ =& \pi_S \circ \phi \circ \pi_S \\ =& \pi_S\end{aligned}$$

It follows that $\xi \circ \psi$ makes the following diagram commute:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y S \arrow[dr, "\alpha\circ\psi"] \arrow[drr, bend left, "\pi_Y\circ\pi_{Y\times_Y S}"] \arrow[ddr, bend right, "\pi_X\circ\pi_{X\times_Z Y}"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

since the identity map makes this diagram commute as well we have that $\xi \circ \psi = \mathrm{Id}$. We now see that:

$$\begin{aligned}\pi_X \circ \psi \circ \xi =& \pi_X \circ \pi_{X\times_Z Y} \circ \xi \\ =& \pi_X \circ \alpha \\ =& \alpha\end{aligned}$$

while:

$$\begin{aligned}\pi_{Y\times_Y S} \circ \psi \circ \xi =& \phi \circ \pi_S \circ \xi \\ =& \phi \circ \pi_S \circ \pi_{Y\times_Y S}\end{aligned}$$

We claim that $\phi \circ \pi_S$ is the identity map; indeed note that we have:

$$\pi_S \circ \phi \circ \pi_S = \mathrm{Id}_S \circ \pi_S = \pi_S$$

while:

$$\pi_Y \circ \phi \circ \pi_S = h \circ \pi_S = \mathrm{Id}_Y \circ \pi_Y = \pi_Y$$

so $\phi \circ \pi_S$ makes the following diagram commute:

$$\begin{tikzcd}
Y \times_Y S \arrow[dr, "\phi\circ\pi_S"] \arrow[drr, bend left, "\pi_S"] \arrow[ddr, bend right, "\pi_Y"'] & & \\
& Y \times_Y S \arrow[r, "\pi_S"] \arrow[d, "\pi_Y"] & S \arrow[d, "g\circ h"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

However, so does the identity map, hence $\phi \circ \pi_S = \mathrm{Id}_{Y\times_Y S}$, and we have that:

$$\begin{aligned}
\pi_{Y\times_Y S} \circ \psi \circ \xi =& \phi \circ \pi_S \circ \pi_{Y\times_Y S} \\
=& \mathrm{Id}_{Y\times_Y S} \circ \pi_{Y\times_Y S} \\
=& \pi_{Y\times_Y S}
\end{aligned}$$

So it follows that $\psi \circ \xi$ makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z (Y \times_Y S) \arrow[dr, "\psi\circ\xi"] \arrow[drr, bend left, "\pi_{Y\times_Y S}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z (Y \times_Y S) \arrow[r, "\pi_{Y\times_Y S}"] \arrow[d, "\pi_X"] & Y \times_Y S \arrow[d, "g\circ\pi_Y"] \\
& X \arrow[r, "f\circ\pi_X"] & Z
\end{tikzcd}$$

but again so does the identity so $\psi \circ \xi = \mathrm{Id}$. It follows that:

$$(X \times_Z Y) \times_Y S \cong X \times_Z (Y \times_Y S) \cong X \times_Z S$$

implying the claim. □

The following lemmas are extremely helpful in identifying schemes as fibre products, as well as morphisms between them. They will be crucial in our existence proof of the fibre product.

Definition 2.3.3. Let Q, X, Y and Z, be schemes which fit into the following commutative square:

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}$$

If the induced map $Q \to X \times_Z Y$ is an isomorphism, then we call the above diagram a **cartesian square**.

Lemma 2.3.4. *Consider the following commutative diagram of schemes:*

$$
\begin{tikzcd}
X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}
$$

If the left and right squares are cartesian then the outer square is cartesian. Moreover, if the outer square and the right square are cartesian, then the left is as well.

Proof. We need to show that the following square:

$$
\begin{tikzcd}
X'' \arrow[r, "\pi_X \circ \pi_{X'}"] \arrow[d, "\pi_{S''}"] & X \arrow[d, "\pi_S"] \\
S'' \arrow[r, "f_S \circ f_{S'}"] & S
\end{tikzcd}
$$

is cartesian. We do so by showing that $(X'', \pi_{S''}, \pi_X \circ \pi_{X'})$ satisfies the universal property of the fibre product. Suppose that Q is a scheme equipped with morphisms $p_{S''}$ and p_X such that $\pi_S \circ p_X = f_S \circ f_{S'} \circ p_{S''}$, then we have the following commutative diagram:

$$
\begin{tikzcd}
Q \arrow[drrr, bend left, "p_X"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}
$$

In particular, since the outer square is cartesian we have a unique morphism $p_{X'}$ such that the following diagram commutes:

$$
\begin{tikzcd}
Q \arrow[drrr, bend left, "p_X"] \arrow[drr, "p_{X'}"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}
$$

So now Q comes equipped with maps $p_{X'} : Q \to X'$ and $p_{S'} : Q \to S''$ such that $f_{S'} \circ p_{S''} = \pi_{S'} \circ p_{X'}$. By hypothesis there is then a unique map $\phi : Q \to X''$ such that:

$$\pi_{S''} \circ \phi = p_{S''} \qquad \text{and} \qquad \pi_{X'} \circ \phi = p_{X'} \tag{2.3.1}$$

We thus need only show that:

$$\pi_X \circ \pi_{X'} \circ \phi = p_X$$

However, we know that $p_X = \pi_X \circ p_{X'}$ so by (2.6) we have that:

$$\pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'} = p_X$$

hence X'' satisfies the universal property of the fibre product and thus the outer square is a cartesian.

Now suppose the outer square and the right square are cartesian, and let Q be scheme equipped with morphisms $p_{S''} : Q \to S''$ and $p_{X'} : Q \to X'$ such that $f_{S'} \circ p_{S''} = \pi_{S'} \circ p_{X'}$. We thus have the following diagram:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_{X'}"] \arrow[dr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

Now note that the map $\pi_X \circ p_{X'} : Q \to X$ makes the following diagram a commute:

$$\begin{tikzcd}
Q \arrow[dr, "p_{X'}"] \arrow[drr, bend left, "\pi_X \circ p_{X'}"] \arrow[ddr, bend right, "f_S \circ p_{S''}"'] & & \\
& X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_{S'}"] \\
& S' \arrow[r, "f_S"] & S
\end{tikzcd} \qquad (\star)$$

and since X' is a fibre product, we have that $p_{X'}$ and $\pi_{X'} \circ p_{X'}$ are the unique maps that make this diagram commute. We then obtain the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "p_{X'}"] \arrow[drrr, bend left, "\pi_X \circ p_{X'}"] \arrow[ddr, bend right, "p_{S''}"'] & & & \\
& X'' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_{S''}"] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_{S'}"] & X \arrow[d, "\pi_S"] \\
& S'' \arrow[r, "f_{S'}"] & S' \arrow[r, "f_S"] & S
\end{tikzcd}$$

Clearly we have that $f_S \circ f_{S'} \circ p_{S''} = \pi_S \circ \pi_X \circ p_{X'}$, so since the outer square is cartesian we have a unique map $\phi : Q \to X''$ such that:

$$\pi_{S''} \circ \phi = p_{S''} \qquad \text{and} \qquad \pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'}$$

So we need only show that:

$$\pi_{X'} \circ \phi = p_{X'}$$

However, this is clear as $\pi_{X'} \circ \phi$ satisfies:

$$\begin{aligned} f_{S'} \circ p_{S''} =& f_{S'} \circ \pi_{S''} \circ \phi \\ =& \pi_{S'} \circ \pi_{X'} \circ \phi \end{aligned}$$

and trivially:

$$\pi_X \circ \pi_{X'} \circ \phi = \pi_X \circ p_{X'}$$

It follows that replacing $p_{X'}$ with $\pi_{X'} \circ \phi$ in $(\star)$ makes the diagram commute, so by the uniqueness of $p_{X'}$ we have that $\pi_{X'} \circ \phi = p_{X'}$. Therefore, X'' satisfies the universal property of the fibre product $S'' \times_{S'} X'$, and the left square is cartesian. □

We continue with our litany of lemmas regarding fibre products:

Lemma 2.3.5. *Let $F : X \to X'$ and $G : Y \to Y'$ be morphisms of Z-schemes. Then there is a morphism $F \times G : X \times_Z X \to Y \times_Z Y'$ which makes the following diagram commute:*

$$\begin{tikzcd}
X \arrow[r, "F"] & Y \\
X \times_Z Y \arrow[u, "\pi_X"] \arrow[r, "F\times G"] \arrow[d, "\pi_Y"] & X' \times_Z Y' \arrow[u, "\pi_{X'}"] \arrow[d, "\pi_{Y'}"] \\
Y \arrow[r, "G"] & Y'
\end{tikzcd}$$

Proof. Note that since F and G are Z scheme morphisms, we have morphisms $F \circ \pi_X : X \times_Z Y \to X'$ and $G \circ \pi_Y : X \times_Z Y \to Y'$ which satisfy:

$$f' \circ F \circ \pi_X = f \circ \pi_X = g \circ \pi_Y = g' \circ G \circ \pi_Y$$

so we have a unique map $F \times G$ which makes the following diagram commute:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "F\times G"] \arrow[drr, bend left, "G\circ\pi_Y"] \arrow[ddr, bend right, "F\circ\pi_X"] & & \\
& X' \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_{X'}"] & Y' \arrow[d, "g'"] \\
& X' \arrow[r, "f'"] & Z
\end{tikzcd}$$

We then see that:

$$G \circ \pi_Y = \pi_{Y'} \circ F \times G \qquad \text{and} \qquad F \circ \pi_X = \pi_{X'} \circ F \times G$$

so the diagram commutes as desired. □

We now come upon, and end our category theoretic results with, the first statement worthy of being called a theorem. We adopt Vakil's terminology and call this the magic square theorem, or the diagonal base change theorem.

Theorem 2.3.1. *Let X and X' be Y-schemes, and Y a Z-scheme; then the following square is cartesian:*

$$\begin{tikzcd}
X \times_Y X' \arrow[r] \arrow[d] & X \times_Z X' \arrow[d] \\
Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

Before we prove this theorem, let us actually check that the above square is commutative, and construct the maps. First, let f, f' and g be morphisms making X and X' Y-schemes, and Y a Z-scheme. The left vertical map is then given by $f \circ \pi_X$ (or equivalently $f' \circ \pi_{X'}$), and the right vertical map is the map $f \times f'$ constructed as in Lemma 2.3.5. Now, note that in the top right corner X and X' are Z-schemes with the morphisms $g \circ f$ and $g \circ f'$. Clearly, since $f \circ \pi_X = f' \circ \pi_{X'}$, we have that $(g \circ f) \circ \pi_X = (g \circ f') \circ \pi_{X'}$. It follows that there is then a map $\psi : X \times_Y X' \to X \times_Z X'$ such that the following diagram commutes[30]

$$\begin{tikzcd}
X \times_Y X' \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_{X'}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "g\circ f'"] \\
& X \arrow[r, "g\circ f"] & Z
\end{tikzcd}$$

Finally, the bottom map is what we call the diagonal map $\Delta : Y \to Y \times_Z Y$, and is the unique map which makes the following diagram commute:

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_Y"] \arrow[ddr, bend right, "\mathrm{Id}_Y"'] & & \\
& Y \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_Y"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Now, we want to show that $\Delta \circ f \circ \pi_X = (f \times f') \circ \psi$, and we do so by showing that the both make the following diagram commute:

$$\begin{tikzcd}
X \times_Y X' \arrow[dr] \arrow[drr, bend left, "f'\circ\pi_{X'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_Y"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

We see that:

$$\pi_Y \circ \Delta \circ f \circ \pi_X = f \circ \pi_X = f \circ \pi_{X'}$$

so $\Delta \circ f \circ \pi_X$ makes the diagram commute. Moreover:

$$\pi_Y \circ (f \times f') \circ \psi = f \circ \pi_X \circ \psi = f \circ \pi_X = f' \circ \pi_X$$

so the two are equal by the uniqueness of the morphism which makes the diagram commute. We thus have that the square in Theorem 2.3.1 commutes and is:

$$\begin{tikzcd}
X \times_Y X' \arrow[r, "\psi"] \arrow[d, "f\circ\pi_X"] & X \times_Z X' \arrow[d, "f\times f'"] \\
Y \arrow[r, "\Delta"] & Y \times_Z Y
\end{tikzcd}$$

We now begin with actually proving the statement:

[30]Abuse of notation alert! We are again denoting different projection maps in the same way. We hope our judicious inclusion of diagrams helps the reader parse through this poor choice.

Proof. We will show that $X \times_Y X'$ satisfies the universal property of the fibre product. Let Q be another scheme with morphisms $\alpha : Q \to Y$ and $\beta : Q \to X \times_Z X'$ such that:

$$\Delta \circ \alpha = (f \times f') \circ \beta \tag{2.3.2}$$

Now first note that β is the unique map the makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\beta"] \arrow[drr, bend left, "p_{X'}"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "g\circ f'"] \\
& X \arrow[r, "g\circ f"] & Z
\end{tikzcd}$$

where $p_X = \pi_X \circ \beta$ and $p_{X'} = \pi_{X'} \circ \beta$. The maps satisfy $g \circ f \circ p_X = g \circ f' \circ p_{X'}$, however we want to show that the maps satisfy $f \circ p_X = f' \circ p_{X'}$. Applying π_Y to both sides of (2.3.2) yields:

$$\begin{aligned}
\alpha =& \pi_Y \circ (f \times f') \circ \beta \\
=& f \circ \pi_X \circ \beta \\
=& f \circ p_X
\end{aligned}$$

However, $f \circ \pi_X = f \circ \pi_{X'}$ so we also have that:

$$\begin{aligned}
\alpha =& f' \circ \pi_X \circ \beta \\
=& f \circ p_{X'}
\end{aligned}$$

so $f \circ p_X = f \circ p_{X'}$. There is then a unique morphism $\phi : Q \to X \times_Y X'$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{X'}"] \arrow[ddr, bend right, "p_X"'] & & \\
& X \times_Y X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_X"] & X' \arrow[d, "f'"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Now note that:

$$\begin{aligned}
f \circ \pi_X \circ \phi =& f \circ p_X \\
=& \alpha
\end{aligned}$$

so we need to that:

$$\psi \circ \phi = \beta$$

and it suffices to show that:

$$\pi_X \circ \psi \circ \phi = p_X \qquad \text{and} \qquad \pi_{X'} \circ \psi \circ \phi = p_{X'}$$

We have that:

$$\pi_X \circ \psi \circ \phi = \pi_X \circ \phi = p_X$$

and that:

$$\pi_{X'} \circ \psi \circ \phi = \pi_{X'} \circ \phi = p_{X'}$$

so $\psi \circ \phi = \beta$ by the uniqueness of β. We thus have that ϕ is the unique map which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"] & & \\
& X \times_Y X' \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"] & X \times_Z X' \arrow[d, "f \times f'"] \\
& Y \arrow[r, "\Delta"] & Y \times_Z Y
\end{tikzcd}$$

Therefore, $X \times_Y X'$ is isomorphic to the fibre product $Y \times_Z X \times_Z X'$ and the square is cartesian as desired. $\square$

Now that we have sufficiently established our results regarding fibre products that have nothing to do with algebraic geometry, it is time to actually prove that fibre products of schemes indeed exist. We will prove this in sequential steps, slowly building up to the general case. We begin with the case where all schemes are affine:

Lemma 2.3.6. *Let X and Y be Z-schemes, and let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$. The fibre product $X \times_Z Y$ is then the affine scheme* $\operatorname{Spec}(A \otimes_C B)$.

Proof. Since X and Y are Z-schemes, there are ring morphisms $f_X^\sharp : C \to A$ and $g_Y^\sharp : C \to B$ which turn A and B into C algebras so we can construct the tensor product $A \otimes_C B$. The tensor product comes equipped with maps $\imath_A : A \to A \otimes_C B$, and $\imath_B : B \to A \otimes_C B$ given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$, and satisfies the universal property that for any two maps $\phi_A : A \to R$ and $\phi_B : B \to R$ such that:

$$\phi_A \circ f_X^\sharp = \phi_B \circ g_Y^\sharp$$

then there is a unique ring homomorphism $\alpha : A \otimes_B C \to R$ such that the following diagram commutes:

$$\begin{tikzcd}
R & & \\
& A \otimes_C B \arrow[ul, "\alpha"'] & B \arrow[l, "\imath_B"'] \arrow[ull, bend right, "\phi_B"'] \\
& A \arrow[u, "\imath_A"] \arrow[uul, bend left, "\phi_A"] & C \arrow[l, "f_X^\sharp"'] \arrow[u, "g_Y^\sharp"']
\end{tikzcd}$$

Via the anti equivalence between the category of commutative rings and affine schemes, have that $\operatorname{Spec}(A \otimes_C B)$ comes equipped with projection maps $\pi_X : \operatorname{Spec}(A \otimes_C B) \to X$, $\pi_Y : \operatorname{Spec}(A \otimes_C B) \to Y$ which make the obvious square commute. If Q is any scheme with maps $p_X : Q \to X$ and $p_Y : Q \to Y$ satisfying $f \circ p_X = g \circ p_Y$, then the induced ring homomorphisms satisfy the conditions of the universal property of the tensor product of commutative rings. It follows there is a unique ring homomorphism $A \otimes_B C \to \mathscr{O}_Q(Q)$ which by Proposition 2.1.2 induces a unique scheme morphism $\psi : Q \to \operatorname{Spec}(A \otimes_C B)$

which makes the following diagram commute:

$$\begin{tikzcd}
Q \arrow[dr, "\psi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& \operatorname{Spec}(A \otimes_C B) \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The affine scheme $\operatorname{Spec}(A \otimes_C B)$ then satisfies the universal property of the fibre product, implying the claim. ☐

When the base scheme is affine $Z = \operatorname{Spec} C$, we often denote the fibre product $X \times_Z Y$ by $X \times_C Y$. We thus immediately have that:

$$\mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^m_{\mathbb{C}} \cong \mathbb{A}^{m+n}_{\mathbb{C}}$$

via the isomorphism:

$$\mathbb{C}[x_1, \dots, x_n] \otimes_{\mathbb{C}} \mathbb{C}[y_1, \dots, y_m] \cong \mathbb{C}[x_1, \dots, x_{n+m}]$$

Clearly the same statement holds for any commutative ring. Before we continue with our construction, we need the following result, where we note that we make no assumption on U, Z, or Y being affine:

Lemma 2.3.7. *Let $f : U \to Z$ be an open embedding, and $g : Y \to Z$ be any morphism. Then $U \times_Z Y$ exists, and there is an induced open embedding $U \times_Z Y \to Y$.*

Proof. Let $V = f(U)$, then we claim that the open subscheme $(g^{-1}(V), \mathscr{O}_Y|_{g^{-1}(V)})$ is the fibre product $U \times_Z Y$. We first note that we have an inclusion map $\imath : g^{-1}(V) \hookrightarrow Y$, as well as an isomorphism $f^{-1} : V \to U$, so since $g|_{g^{-1}(V)}$ is a morphism $g^{-1}(V) \to V$, we have that $f^{-1} \circ g|_{g^{-1}(V)}$ is a morphism $g^{-1}(V) \to U$. Now note that:

$$f \circ f^{-1} \circ g|_{g^{-1}(V)} = g|_{g^{-1}(V)} = g \circ \imath$$

so we have the following commutative square which we wish to show is cartesian:

$$\begin{tikzcd}
g^{-1}(V) \arrow[r, "\imath"] \arrow[d, "f^{-1} \circ g|_{g^{-1}(V)}"'] & Y \arrow[d, "g"] \\
U \arrow[r, "f"] & Z
\end{tikzcd}$$

Let Q be a scheme and $p_U : Q \to U$, $p_Y : Q \to Y$ be morphisms such that $f \circ p_U = g \circ p_Y$, then we want to find a morphism $\phi : Q \to g^{-1}(V)$ such that $f^{-1} \circ g|_{g^{-1}(V)} \circ \phi = p_U$ and $\imath \circ \phi = p_Y$. Since $f \circ p_U = g \circ p_Y$, we must have p_Y maps into $g^{-1}(V)$, so there is a unique morphism $\phi : Q \to g^{-1}(V)$ such that $\imath \circ \phi = p_Y$. Now note that:

$$\begin{aligned}
f^{-1} \circ g|_{g^{-1}(V)} \circ \phi =& f^{-1} \circ g \circ \imath \circ \phi \\
=& f^{-1} \circ g \circ p_Y \\
=& f^{-1} \circ f \circ p_U \\
=& p_U
\end{aligned}$$

It follows that $g^{-1}(V)$ satisfies the universal property of the fibre product $U \times_Z Y$, so there is a unique isomorphism $\psi : U \times_Z Y \to g^{-1}(V)$, and thus an open embedding $U \times_Z Y \to Y$ given by $\imath \circ \psi$ as desired. ☐

Note that the morphism $\imath \circ \psi$ is equal to the canonical projection $\pi_Y : U \times_Z Y \to Y$. In particular, if $U \to Z$ and $V \to Z$ are two inclusion maps, then we have that by the lemma above $U \times_Z V \cong U \cap V$. This matches up with the fact $A_f \otimes_A A_g \cong A_{fg}$. We now have the following result:

Lemma 2.3.8. *Let X and Y be Z-schemes, with $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$. Moreover, let the morphism $\alpha : Y' \to Y$ be an open embedding. Then the fibre product $X \times_Z Y'$ exists, and the induced map $X \times_Z Y' \to X \times_Z Y$ is an open embedding.*

Proof. Note that $X \times_Z Y$ is a fibre product, and so by the previous lemma $(X \times_Z Y) \times_Y Y'$ is a fibre product as $Y' \to Y$ is an open embedding and we have a morphism $\pi_Y : X \times_Z Y \to Y$. It follows that the following diagram is commutative:

$$\begin{tikzcd}
(X \times_Z Y) \times_Y Y' \arrow[r, "\pi_{X \times_Z Y}"] \arrow[d, "\pi_{Y'}"] & X \times_Z Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[d, "f"] \\
Y' \arrow[r, "\alpha"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Since the right square and left square are both cartesian the outer square is cartesian, and we have by Lemma 2.3.5 that $(X \times_Z Y) \times_Y Y' \cong X \times_Z Y'$. By the preceding lemma we have that $(X \times_Z Y) \times_Y Y' \to X \times_Z Y$ is an open embedding, so the induced map $X \times_Z Y' \to X \times_Z Y$ is an open embedding as well. $\square$

Lemma 2.3.9. *Let X and Y be Z-schemes, with $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} C$, and Y arbitrary. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an open affine cover of Y. For each U_i, we have scheme morphisms $g|_{U_i} : U_i \to Z$ making each U_i a Z-scheme, hence the fibre product $X \times_Z U_i$ exists by Lemma 2.3.6. Let $U_{ij} = U_i \cap U_j$[31], then the scheme obtained by gluing each affine open along U_{ij} via the identity map is trivially isomorphic to Y. For each i let $V_i = X \times_Z U_i$, and moreover we have a morphism $g|_{U_{ij}} : U_{ij} \to Z$ which factors as $\imath \circ g : U_{ij} \to U_i \to Z$ where $\imath : U_{ij} \to U_i$. By Lemma 2.3.8 we have that the fibre product $X \times_Z U_{ij}$ exists and that there is an open embedding $\alpha_{ij} : X \times_Z U_{ij} \to V_i$.

We define $V_{ij} \subset V_i$ to then be the open subscheme $\alpha_{ij}(X \times_Z U_{ij})$. Now note that $U_{ij} = U_{ji} \subset Y$, so we have an equality $X \times_Z U_{ij} = X \times_Z U_{ji}$. Denoting by α_{ij}^{-1} the isomorphism $V_{ij} \to X \times_Z U_{ij}$, we obtain scheme isomorphisms $\phi_{ij} : V_{ij} \to V_{ji}$ given by $\alpha_{ji} \circ \alpha_{ij}^{-1}$.

We want to glue the schemes V_i together along the open subschemes V_{ij} via these scheme isomorphisms. Clearly we have that $\phi_{ij} = \phi_{ji}^{-1}$, so we need to check that $\phi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk}$ and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $V_{ij} \cap V_{ik}$. Note that $V_{ij} \cap V_{ik}$ is the fibre product $V_{ij} \times_{V_i} V_{ik}$[32], and similarly we have that $V_{ji} \cap V_{jk}$ is the fibre product $V_{ji} \times_{V_j} V_{jk}$. Now note that:

$$V_{ji} \cong X \times_Z U_{ij} \cong X \times_Z U_i \times_Y U_j \cong V_{ij}$$

while:

$$V_{ik} \cong X \times_Z U_i \times_Y U_k \qquad \text{and} \qquad V_{jk} \cong X \times_Z U_j \times_Y U_k$$

so:

$$\begin{aligned} V_{ij} \cap V_{ik} \cong V_{ij} \times_{V_i} V_{ik} &\cong (X \times_Z U_i \times_Y U_j) \times_{V_i} (X \times_Z U_i \times_Y U_k) \\ &\cong X \times_Z U_i \times_Y U_j \times_Y U_k \\ &\cong X \times_Z U_{ijk} \end{aligned}$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. Similarly, we have that:

$$\begin{aligned} V_{ji} \cap V_{jk} \cong V_{ji} \times_{V_j} V_{jk} &\cong (X \times_Z U_j \times_Y U_i) \times_{V_j} (X \times_Z U_j \times_Y U_k) \\ &\cong X \times_Z U_j \times_Y U_i \times_Y U_k \\ &\cong X \times_Z U_{ijk} \end{aligned}$$

[31] Note that the intersection of two affine opens is not necessarily affine.

[32] Which exists by Lemma 2.3.7 as both V_{ij} and V_{ik} are open subschemes of V_i

It follows that $V_{ij} \cap V_{ik}$ is uniquely isomorphic to $V_{ji} \cap V_{jk}$. We need to show that this isomorphism is precisely ϕ_{ij} restricted to $V_{ij} \cap V_{ik}$. We note that the embedding $\alpha_{ij} : X \times_Z U_{ij} \to V_i$ comes from the cartesian square:

$$\begin{tikzcd} X \times_Z U_{ij} \arrow[r, "\alpha_{ij}"] \arrow[d, "\pi_{U_{ij}}"] & V_i \arrow[d, "\pi_{U_i}"] \\ U_{ij} \arrow[r, "\imath"] & U_i \end{tikzcd}$$

and since $U_{ijk} \hookrightarrow U_{ij}$ we have an open embedding $\beta_{ijk} : X \times_Z U_{ijk} \to X \times_Z U_{ij}$. Let ψ_{ijk} be the isomorphism $V_{ij} \cap V_{ik} \to X \times_Z U_{ijk}$, then we obtain the following diagram of cartesian squares:

$$\begin{tikzcd} V_{ij} \cap V_{ik} \arrow[r, "\psi_{ijk}"] \arrow[d, "\pi_{U_{ijk}}"] & X \times_Z U_{ijk} \arrow[r, "\beta_{ijk}"] \arrow[d, "\pi_{U_{ijk}}"] & X \times_Z U_{ij} \arrow[r, "\alpha_{ij}"] \arrow[d, "\pi_{U_{ij}}"] & V_i \arrow[d, "\pi_{U_i}"] \\ U_{ijk} \arrow[r, "\mathrm{Id}"] & U_{ijk} \arrow[r, "\imath"] & U_{ij} \arrow[r, "\imath"] & U_i \end{tikzcd}$$

However, the inclusion map $\imath : V_{ij} \cap V_{ik} \to V_i$ also makes this diagram commute, so we have that:

$$\alpha_{ij} \circ \beta_{ijk} \circ \psi_{ijk} = \imath$$

similarly we have that:

$$\alpha_{ji} \circ \beta_{jik} \circ \psi_{jik} : V_{ji} \cap V_{jk} \to V_j$$

is the inclusion map. It follows that these maps are isomorphisms onto their images hence we have that:

$$\alpha_{ij}|_{\beta_{ijk}(X \times_Z U_{ijk})} \circ \beta_{ijk} \circ \psi_{ijk} = \mathrm{Id}_{V_{ij} \cap V_{ik}}$$

and:

$$\alpha_{ij}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik} \circ \psi_{jik} = \mathrm{Id}_{V_{ji} \cap V_{jk}}$$

so in particular,

$$\alpha_{ij}^{-1}|_{V_{ij} \cap V_{ik}} = \beta_{ijk} \circ \psi_{ijk}$$

Moreover, note that $\beta_{ijk} = \beta_{jik}$, and that the unique isomorphism $V_{ij} \cap V_{ik} \to V_{ji} \cap V_{jk}$ is given by $\psi_{jik}^{-1} \circ \psi_{ijk}$. We see that:

$$\psi_{jik}^{-1} = \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik}$$

hence:

$$\begin{aligned} \psi_{jik}^{-1} \circ \psi_{ijk} =& \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{jik})} \circ \beta_{jik} \circ \psi_{ijk} \\ =& \alpha_{ji}|_{\beta_{jik}(X \times_Z U_{ijk})} \circ \alpha_{ij}^{-1}|_{V_{ij} \cap V_{ik}} \\ =& (\alpha_{ji} \circ \alpha_{ij}^{-1})|_{V_{ij} \cap V_{ik}} \\ =& \phi_{ij}|_{V_{ij} \cap V_{ik}} \end{aligned}$$

implying that $\phi_{ij}(V_{ij} \cap V_{ik}) = V_{ji} \cap V_{jk}$ as desired. It follows that $\phi_{ik}(V_{ij} \cap V_{ik}) = V_{kj} \cap V_{ki}$ while:

$$\phi_{jk} \circ \phi_{ij}(V_{ij} \cap V_{ik}) = \phi_{jk}(V_{ji} \cap V_{jk}) = V_{ki} \cap V_{kj}$$

so $\phi_{jk} \circ \phi_{ij}$ is the unique isomorphism $V_{ij} \cap V_{ik} \to V_{ki} \cap V_{kj}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. We thus have that the schemes V_i and V_j glue together along V_{ij} for all i and j and are locally isomorphic to $X \times_Z U_i$.

We denote this scheme by S and show that it satisfies the universal property of the fibre product. We first construct projection maps $\pi_X : S \to X$ and $\pi_Y : S \to Y$. We see that the isomorphisms ϕ_{ij} fit into the diagram:

$$\begin{tikzcd}
V_{ij} \arrow[dr, "\phi_{ij}"] \arrow[drr, bend left, "\pi_{U_{ij}}"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& V_{ji} \arrow[r, "\pi_{U_{ji}}"] \arrow[d, "\pi_X"] & U_{ij} \subset Y \arrow[d, "g|_{U_{ij}}"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Note that here both π_X and $\pi_{U_{ij}}$ are the restrictions of the projection maps $\pi_X : V_i \to X$ and $\pi_{U_i} : V_i \to U_{ij} \subset Y$ to V_{ij} and similarly for V_j and V_{ji}. We thus have induced morphisms $\pi_X : S \to X$ and $\pi_Y : S \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

We want to show that this square is cartesian, so let Q be any other scheme, with projection maps $p_X : Q \to X$ and $p_Y : Q \to Y$ which make the relevant diagram commute. We have an open covering of Q by $\{\pi_Y^{-1}(U_i)\}$, and for each open we have a unique map ξ_i such that the following diagram commutes:

$$\begin{tikzcd}
\pi_Y^{-1}(U_i) \arrow[dr, "\xi_i"] \arrow[drr, bend left, "p_Y|_{\pi_Y^{-1}(U_i)}"] \arrow[ddr, bend right, "p_X"'] & & \\
& V_i \arrow[r, "\pi_{U_i}"] \arrow[d, "\pi_X"] & U_i \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We need to show that:

$$\xi_j|_{\pi_Y^{-1}(U_{ij})} = \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})}$$

We need only check that $\phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})}$ makes the relevant diagram commute. Note that:

$$\pi_X \circ \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = \pi_X \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = p_X$$

and:

$$\pi_{U_{ji}} \circ \phi_{ij} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = \pi_{U_{ij}} \circ \xi_i|_{\pi_Y^{-1}(U_{ij})} = p_Y|_{\pi_Y^{-1}(U_{ji})}$$

so the two are equal, and we thus have a unique morphism $\xi : Q \to S$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\xi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& S \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & U_i \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

so S satisfies the universal property of $X \times_Z Y$ implying the claim. $\square$

We now move to the next case:

Lemma 2.3.10. *Let X and Y be Z-schemes, with $Z = \operatorname{Spec} C$. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an open affine covering of X, then by Lemma 2.3.9 the fibre products $U_i \times_Z Y$ exist. We have open embeddings $U_{ij} = U_i \cap U_j \to U_i$ given by the inclusion map. We have that the scheme $U_{ij} \times_{U_i} (U_i \times_Z Y)$ exists, so we have the following commutative diagram:

$$\begin{tikzcd} U_{ij} \times_{U_i} (U_i \times_Z Y) \arrow[r, "\pi_{U_i \times_Z Y}"] \arrow[d, "\pi_{U_{ij}}"] & U_i \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{U_i}"] & Y \arrow[d] \\ U_{ij} \arrow[r, "\imath"] & U_i \arrow[r, "f|_{U_i}"] & Z \end{tikzcd}$$

where the left and right squares are cartesian, so the outer square is cartesian. It follows that the fibre product $U_{ij} \times_Z Y$ exists and comes with open embeddings $\alpha_{ij} : U_{ij} \times_Z Y \to U_i \times_Z Y$. These open embeddings satisfy the same properties as the ones in Lemma 2.3.9, so if $V_{ij} = \alpha_{ij}(U_{ij} \times_Z Y)$, we have isomorphisms $\alpha_{ji}^{-1} \circ \alpha_{ij} : V_{ij} \to V_{ji}$ which agree on triple overlaps. It follows that the V_i's glue together along V_{ij} for all i and j, hence we obtain a scheme S which is locally isomorphic to $U_i \times_Z Y$. The same argument as in Lemma 2.3.9 shows that S satisfies universal property of $X \times_Z Y$, implying the claim. $\square$

We now repeat the same result as in Lemma 2.3.8

Lemma 2.3.11. *Let X and Y be Z schemes, and suppose that there is an open embedding $Z \to Z'$, with Z' affine. Then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\alpha : Z \to Z'$ be the open embedding, and $f : X \to Z$, $g : Y \to Z$ the morphisms making X and Y Z-schemes. Then we have by Lemma 2.3.10 that the following square is cartesian:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "\alpha \circ g"] \\ X \arrow[r, "\alpha \circ f"] & Z' \end{tikzcd}$$

In particular, since α is a monomorphism, we have that $\pi_X \circ f = \pi_Y \circ g$, so the following square is commutative:

$$\begin{tikzcd} X \times_{Z'} Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Let Q be any scheme, with morphisms $p_X : Q \to X$ and $p_Y : Q \to Y$ such that the relevant diagram commutes. Then we have that $\alpha \circ f \circ p_X = \alpha \circ g \circ p_Y$ so there is a unique map $Q \to X \times_Z Y$ such that the fibre product diagram commutes. However, note that this same morphism makes the following diagram commute:

$$\begin{tikzcd} Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\ & X \times_{Z'} Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

so $X \times_{Z'} Y$ satisfies the universal property of $X \times_Z Y$, implying the claim. $\square$

We now prove the statement in generality:

Theorem 2.3.2. *Let X, and Y be Z-schemes, then the fibre product $X \times_Z Y$ exists.*

Proof. Let $\{U_i\}$ be an affine open cover of Z, then for all i, set $X_i = f^{-1}(U_i)$ and $Y_i = g^{-1}(U_i)$, then by Lemma 2.3.10 the fibre product $W_i = X_i \times_{U_i} Y_i$ exists for all i. Set $U_{ij} = U_i \cap U_j$, $X_{ij} = f^{-1}(U_{ij})$, and $Y_{ij} = g^{-1}(U_{ij})$, then by the preceding lemma $W_{ij} = X_{ij} \times_{U_{ij}} Y_{ij}$ exists for all i and j, and is isomorphic to $X_{ij} \times_{U_i} Y_{ij}$. There are then open embeddings W_{ij} into W_i and W_j by Lemma 2.3.8.

We now show that W_i satisfies the universal property of $X \times_Z Y_i$. Indeed, we have the following cartesian square:

$$\begin{tikzcd}
W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\pi_{X_i}"'] & Y_i \arrow[d, "g|_{Y_i}"] \\
X_i \arrow[r, "f|_{X_i}"] & U_i
\end{tikzcd}$$

Since $f|_{X_i} = f \circ \imath$, where $\imath$ is the inclusion map $X_i \hookrightarrow X$, and since we have inclusion maps $\imath : U_i \to Z$, we have the following commutative square:

$$\begin{tikzcd}
W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\imath \circ \pi_{X_i}"'] & Y_i \arrow[d, "\imath \circ g|_{Y_i}"] \\
X \arrow[r, "\imath \circ f"] & Z
\end{tikzcd}$$

Now suppose we are given a scheme Q, and morphisms $p_X : Q \to X$, $p_{Y_i} : Q \to Y_i$ such that:

$$\imath \circ f \circ p_X = \imath \circ g|_{Y_i} \circ p_{Y_i}$$

implying that $f \circ p_X = g|_{Y_i} \circ p_{Y_i}$. We see that p_X has image contained in X_i, and thus factors uniquely as $p_X = \imath \circ \alpha_{X_i}$ where $\imath : X_i \hookrightarrow X$ is the inclusion map. We hus have that $f \circ \imath \circ \alpha_{X_i} = f|_{X_i} \circ \alpha_{X_i} = g|_{Y_i} \circ p_{Y_i}$, so there is a unique morphism $\phi : Q \to W_i$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{Y_i}"] \arrow[ddr, bend right, "\alpha_{X_i}"'] & & \\
& W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\pi_{X_i}"'] & Y_i \arrow[d, "g|_{Y_i}"] \\
& X_i \arrow[r, "f|_{X_i}"] & U_i
\end{tikzcd}$$

However, since $\pi_{X_i} \circ \phi = \alpha_{X_i}$, we have that:

$$\imath \circ \pi_{X_i} = \imath \circ \alpha_{X_i} = p_X$$

so the following diagram commutes as well:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "p_{Y_i}"] \arrow[ddr, bend right, "p_X"'] & & \\
& W_i \arrow[r, "\pi_{Y_i}"] \arrow[d, "\imath \circ \pi_{X_i}"'] & Y_i \arrow[d, "\imath \circ g|_{Y_i}"] \\
& X \arrow[r, "\imath \circ f"] & Z
\end{tikzcd}$$

So $W_i \cong X \times_Z Y_i$ as desired, and similarly that $W_{ij} \cong X \times_Z Y_{ij}$. However, we are now in the same situation as Lemma 2.3.9 as the only point where we used that X and Z were affine was for the existence of the schemes $X \times_Z Y_i$ and $X \times_Z Y_{ij}$. We can thus glue the schemes W_i along W_{ij} as before, and the same argument shows that this scheme satisfies the universal property of $X \times_Z Y$, implying the claim. □

We now point out the following fact: fibre products in general have more points than naive cartesian products. Indeed, consider the scheme $X = \operatorname{Spec} \mathbb{C}[t]$, then the $X \times_{\mathbb{C}} X$, is the spectrum of the ring $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \cong \mathbb{C}[u,t]$. The prime ideals of this ring are then certainly not of the form $(\mathfrak{p}, \mathfrak{q})$ for primes $\mathfrak{p}, \mathfrak{q} \subset \mathbb{C}[t]$. However, note that we that the closed points of $\operatorname{Spec} \mathbb{C}[t,u]$ are in bijection with $\mathbb{C}^2$, which is the naive set product of the closed points of $\operatorname{Spec} \mathbb{C}[t]$ with itself (all points save the zero ideal are closed in $\operatorname{Spec} \mathbb{C}[t]$ though). We wish to extend this discussion to arbitrary situations, but first we need the following definition, which we will explore more in the subsequent chapter:

Definition 2.3.4. Let k be a field, and X a scheme over $\operatorname{Spec} k$. Then X is **locally of finite type over k** if there exists an affine open cover $\{U_i\}$ such that $\mathscr{O}_X(U_i)$ is a finitely generated k algebra.

We will need the following lemma:

Lemma 2.3.12. *Let X be a scheme locally of finite type over k, then $x \in X$ is a closed point if and only if there exists an affine open $U = \operatorname{Spec} A$ containing x such that x corresponds to a maximal ideal of A. In particular:*

$$|X| = \bigcup_i |U_i|$$

for any affine open cover $\{U_i\}$

Proof. Let $x \in X$, $U = \operatorname{Spec} A$ an affine open contain x, and $\mathfrak{p} \subset A$ a prime ideal corresponding to x. We first claim that $\overline{\{\mathfrak{p}\}} = \mathbb{V}(\mathfrak{p})$. Indeed, suppose $\mathbb{V}(I)$ is any closed set containing $\{\mathfrak{p}\}$, then we have that $I \subset \mathfrak{p}$, so $\mathbb{V}(\mathfrak{p}) \subset \mathbb{V}(I)$ implying that $\overline{\{\mathfrak{p}\}} = \mathbb{V}(\mathfrak{p})$. Suppose that x is closed in the subspace topology, so we have that $\{\mathfrak{p}\} = \mathbb{V}(\mathfrak{p})$; if $\mathfrak{p} \subset I$ for some ideal $I \subset A$, we have that $\mathbb{V}(I) \subset \mathbb{V}(\mathfrak{p})$ so $\mathbb{V}(I) = \{\mathfrak{p}\}$ or $\mathbb{V}(I) = \emptyset$. If $\mathbb{V}(I) = \emptyset$, then $I = A$, and if $\mathbb{V}(I) = \{\mathfrak{p}\}$ then $\mathbb{V}(I) = \mathbb{V}(\mathfrak{p})$, so we have that $\sqrt{I} = \mathfrak{p}$, but $I \subset \sqrt{I}$ so $I \subset \mathfrak{p}$, implying that $I = \mathfrak{p}$. We have thus shown that points in U which are closed in the subspace topology correspond precisely to the maximal ideals of A. In other words, the maximal ideals of $\operatorname{Spec} A$ are the closed points of $\operatorname{Spec} A$.

Now the stalk at x is the localization $A_{\mathfrak{p}}$, and the residue field k_x is given by:

$$k_x = A_{\mathfrak{p}}/\mathfrak{m}_x$$

where:

$$\mathfrak{m}_x = \left\{ \frac{p}{a} : p \in \mathfrak{p} \right\}$$

We claim that[33]:

$$A_{\mathfrak{p}}/\mathfrak{m}_x \cong A/\mathfrak{p}$$

Note that we have map $A \to A_{\mathfrak{p}}/\mathfrak{m}_x$ by combining the localization morphism with the projection morphism to the quotient. If $p \in \mathfrak{p}$, then $p/1 \in \mathfrak{m}_x$ so this map factors through the quotient hence we have a unique homomorphism:

$$\begin{aligned} \psi : A/\mathfrak{p} &\longrightarrow A_{\mathfrak{p}}/\mathfrak{m}_x \\ [a] &\longmapsto [a/1] \end{aligned}$$

We claim this map is an isomorphism; indeed $A/\mathfrak{p}$ is a field, so if $[a] \mapsto 0$ then $[a]$ is not invertible and thus must be the zero element. Now suppose that $[a/b] \in A_{\mathfrak{p}}/\mathfrak{m}_x$, then since $A/\mathfrak{p}$ is a field, there must be an element $h \in A$ such that $b \cdot h - 1 \in \mathfrak{p}$. We claim that $\psi([ah]) = [a/b]$; indeed note that:

$$\frac{ah}{1} - \frac{a}{b} = \frac{a(hb-1)}{b}$$

[33]This is only true as we are supposing that $\mathfrak{p}$ is a maximal ideal!

but $hb-1 \in \mathfrak{p}$, so this element lies in $\mathfrak{m}_x$ and thus $[ah/1]-[a/b]=0$ and ψ is an isomorphism.

It follows that k_x is a field extension of k, and is a finitely generated k algebra, so by Zariski's lemma [34] is a finite field extension of k. Now let $V = \operatorname{Spec} B$ be another open affine containing x, and suppose that $\mathfrak{q} \subset B$ is the prime ideal associated to x. We have that $B_{\mathfrak{q}}/\mathfrak{m}'_x \cong k_x$, and we now want to show that $B/\mathfrak{q}$ is a field. First note that there is a morphism $B \to B_{\mathfrak{q}}/\mathfrak{m}'_x$ which again sends any element in $\mathfrak{q}$ to zero, so we have a unique morphism $B/\mathfrak{q} \to B_{\mathfrak{q}}/\mathfrak{m}'_x$. This morphism is injective as if $[a] \mapsto 0$, then this implies that $a/1 \in \mathfrak{m}'_x$, but for this to be true a must lie in $\mathfrak{q}$. It follows that we can identify $B/\mathfrak{q}$ as (a priori) a sub k algebra of $B_{\mathfrak{q}}/\mathfrak{m}_x$, which is a finite dimensional k-vector space, so $B/\mathfrak{q}$ must also be a finite dimensional k-vector space. However, $B/\mathfrak{q}$ is prime so $B/\mathfrak{q}$ is an integral domain and the linear map of k vector spaces:

$$\begin{aligned} M_{[b]} : B/\mathfrak{q} &\longrightarrow B/\mathfrak{q} \\ [a] &\longmapsto [a]\cdot[b] \end{aligned}$$

is thus injective for all nonzero $[b] \in B$. Indeed, if $[a]\cdot[b]=0$, then $[a]=0$ so the map is injective. By rank nullity the map is an isomorphism, so there must exist an $[a]$ such that $[b]\cdot[a]=1$ implying that $B/\mathfrak{q}$ is a field, so $\mathfrak{q}$ is a maximal ideal.

We have thus shown that if $x \in U$ is closed in the subspace topology, then x corresponds to a maximal ideal in every affine open containing x, and is thus closed in every such open affine. Now let $\{U_i\}$ be an open affine cover of X, then:

$$X \smallsetminus \{x\} = \bigcup_i (U_i \smallsetminus \{x\})$$

We see that $U_i \smallsetminus \{x\}$ is open in X for all i, as either $U_i \smallsetminus \{x\}$ is U_i since $x \notin U_i$, or $U_i \smallsetminus \{x\}$ is open in U_i as $\{x\}$ is closed in U_i, so it is open in X. It follows that $X \smallsetminus \{x\}$ is open, so $\{x\}$ is a closed point. Therefore, we have proven that if $x \in X$ corresponds to a maximal ideal $\mathfrak{p} \in U = \operatorname{Spec} A$, then x is closed in X. Conversely, if x is closed, and $U = \operatorname{Spec} A$. is an open affine of X containing x, then we have that:

$$U \smallsetminus \{x\} = U \cap (X \smallsetminus \{x\})$$

so $U \smallsetminus \{x\}$ is open in the subspace topology, implying that $\{x\}$ is closed in the subspace topology and thus corresponds to a maximal ideal of A, as desired.

The second claim is now clear, because every closed point of X is a closed point of every affine open, and vice versa. □

We now turn to the main result:

Theorem 2.3.3. *Let X and Y be schemes locally of finite type over k with k algebraically closed. Then there exists a bijection:*

$$\begin{aligned} \phi : |X \times_k Y| &\longrightarrow |X| \times |Y| \\ z &\longmapsto (\pi_X(z), \pi_Y(z)) \end{aligned} \tag{2.3.3}$$

Proof. Let $\{U_i = \operatorname{Spec} A_i\}$ and $\{V_i = \operatorname{Spec} B_i\}$ be affine open covers of X and Y respectively. We then have that $\{U_i \times_k V_j = \operatorname{Spec} A_i \otimes_k B_i\}$ is an affine open cover of $X \times_k Y$. We see that each $A_i \otimes_k B_j$ is a a finite generated k-algebra. We first determine a bijection

$$|U_i \times_k V_j| \longleftrightarrow |U_i| \times |V_j|$$

for all i and j. We suppress the the indices going forward. The projection map π_X is locally induced by the inclusion $\imath_A : A \to A \otimes_k B$. Let $\mathfrak{m} \subset A \otimes_k B$ be a maximal ideal, then we have a morphism $\psi : A \to A \otimes_k B/\mathfrak{m}$ by composing with the projection onto the quotient. We have that $A \otimes_k B/\mathfrak{m}$ is a field, and a finitely generated k algebra so $A \otimes_k B/\mathfrak{m}$ is a finite field extension of k by Zariski's lemma. Note that if $a \in \ker \psi$, then we have that $\imath_A(a) \in \mathfrak{m}$, so $a \in \imath_A^{-1}(\mathfrak{m})$, and if $a \in \imath_A^{-1}(\mathfrak{m})$ then

[34] See Theorem 6.1.3

$\psi(a) = 0$, so $\ker\psi = \imath_A^{-1}(\mathfrak{m})$. We thus have an injective morphism $\psi' : A/\imath_A^{-1}(\mathfrak{m}) \to A \otimes_k B/\mathfrak{m}$, which is an isomorphism onto its image. We want to show that $\psi'(A/\imath_A^{-1}(\mathfrak{m}))$ is a subfield of $A \otimes_k B$. However, since $\mathfrak{m}$ is maximal, we have that $\imath_A^{-1}(\mathfrak{m})$ is prime so $\psi'(A/\imath_A^{-1}(\mathfrak{m}))$ is an integral domain. It follows that $A/\imath_A^{-1}(\mathfrak{m})$ is an integral domain. The argument in Lemma 2.3.12 then demonstrates that since $A \otimes_k B/\mathfrak{m}$ is a finite field extension of k, and $A/\imath_A^{-1}(\mathfrak{m})$ is a finite k-algebra as well as an integral domain, that $A/\imath_A^{-1}(\mathfrak{m})$ must be a field. Therefore, the morphisms π_X and π_Y take closed points to closed points.

We thus define our morphism $\phi : |U \times_k V| \longrightarrow |U| \times |V|$ by (2.8) restricted to $U \times_k V$. We define an inverse map by taking the pair $(\mathfrak{m}, \mathfrak{n}) \in |U| \times |V|$ and mapping it to the ideal $I = \langle \imath_A(\mathfrak{m}), \imath_B(\mathfrak{n})\rangle$. Now we claim that $A \otimes_k B/I$ is a field; indeed we have the following canonical isomorphism:

$$A \otimes_k B/I \cong A/\mathfrak{m} \otimes_k B/\mathfrak{n}$$

which is a finitely generated k algebra, and is finite as both $A/\mathfrak{m}$ and $B/\mathfrak{n}$ are finite field extensions of k. Since k is algebraically closed both fields are isomorphic to k as the only finite field extension of an algebraically closed field is k. We check that this is indeed an inverse, let $(\mathfrak{m}, \mathfrak{n}) \in |U| \times |V|$, then we have that $\phi \circ \phi^{-1}(\mathfrak{m}, \mathfrak{n}) = (\imath_A^{-1}(I), \imath_B^{-1}(I))$. We see that by definition $\imath_A(\mathfrak{m}) \subset I$, hence $\imath_A^{-1}(\imath_A(\mathfrak{m})) \subset \imath_A^{-1}(I)$, but $\mathfrak{m} \subset \imath_A^{-1}(\imath_A(\mathfrak{m}))$, so $\mathfrak{m} \subset \imath_A^{-1}(I)$ implying that $\mathfrak{m} = \imath_A^{-1}(I)$ as $\mathfrak{m}$ is maximal. The same argument holds for $\mathfrak{n}$, so we have that $\phi \circ \phi^{-1} = \mathrm{Id}$. Now suppose that $\mathfrak{m} \subset A \otimes_k B$, then $\mathfrak{m}$ is the kernel of a morphism $\psi : A \otimes_k B \to k$, and such a morphism induces morphisms $\psi_A : A \to k$ and $\psi_B : B \to k$ such that the following diagram commutes:

$$\begin{tikzcd}
k & & \\
& A \otimes_k B \arrow[ul, "\psi"'] & B \arrow[l, "\imath_B"'] \arrow[ull, bend right, "\psi_B"'] \\
& A \arrow[u, "\imath_A"] \arrow[uul, bend left, "\psi_A"] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

If $a \in \imath_A^{-1}(\mathfrak{m})$, then $a \in \ker(\imath_A \circ \psi) = \ker\psi_A$, so we have that $\imath_A^{-1}(\mathfrak{m}) = \ker\psi_A$, and similarly that $\imath_B^{-1}(\mathfrak{m}) = \ker\psi_B$. It therefore suffices to show that $\ker\psi = \langle \imath_A(\ker\psi_A), \imath_B(\ker\psi_B)\rangle$. Note that by the same argument we know that $\langle \imath_A(\ker\psi_A), \imath_B \ker\psi_B\rangle$ is maximal, so let $\omega \in \langle \imath_A(\ker\psi_A), \imath_B \ker\psi_B\rangle$, then we see that $\omega = \beta \cdot \imath_A(a) + \xi \cdot \imath_B(b)$ for some $a \in \ker\psi_A$, $b \in \ker\psi_B$, and some $\beta, \xi \in A \otimes_k B$. Clearly $\psi(\omega) = 0$, so we have that $\langle \imath_A(\ker\psi_A), \imath_B(\ker\psi_B)\rangle \subset \ker\psi$. We thus have that

$$\mathfrak{m} = \ker\psi = \langle \imath_A(\ker\psi_A), \imath_B(\ker\psi_B)\rangle = \left\langle \imath_A(\imath_A^{-1}(\mathfrak{m})), \imath_B(\imath_B^{-1}(\mathfrak{m}))\right\rangle$$

so $\phi^{-1} \circ \phi = \mathrm{Id}$.

Now by the preceding lemma we have that:

$$|X \times_k Y| = \bigcup_{ij} |U_i \times_k V_j| \qquad \text{and} \qquad |X| \times |Y| = \bigcup_{ij} |U_i| \times |V_j|$$

Since our projection maps agree on all overlapping open sets, they must agree on overlapping closed points, hence the local bijection induced by the inclusion homomorphisms described above also agrees on overlapping closed points. It follows that the bijections $|U_i \times_k V_j| \to |U_i| \times |V_j|$ glue together to yield the desired set bijection, implying the claim. □

We will use fibre products in the following section when we further discuss the topological and algebraic properties of schemes and their morphisms. For now, we end with the following examples:

Example 2.3.1. We claim that $\mathbb{P}^n_{\mathbb{C}} \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}$, where here the fibre product is taken over $\operatorname{Spec}\mathbb{Z}$. Note that we have a morphism $g : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{C}$, and a morphism $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ induced locally by the inclusion map:

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[\{x_l/x_i\}_{l \neq i}]$$

Indeed, for each i, the above morphism of rings induces morphisms of affine schemes $f_i : U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. We have that

$$U_{x_i} \cap U_{x_j} = U_{x_i x_j} \cong \operatorname{Spec} \mathbb{Z}[\{x_l/x_i\}_{l \neq i}, x_i/x_j]$$

It follows that the morphisms $f_i|_{U_{x_i x_j}} : U_{x_i x_j} \to \operatorname{Spec} \mathbb{Z}$ and $f_j|_{U_{x_i x_j}} : U_{x_i x_j} \to \operatorname{Spec} \mathbb{Z}$ are induced by the inclusion map:

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[\{x_l/x_i\}_{l \neq i}, x_i/x_j]$$

so they trivially agree. It follows that we have a morphism $f : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. Now we wish to define morphisms $p_Y : \mathbb{P}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$, and $p_X : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$. We define the first morphisms as we did in the case of $\mathbb{P}^n_{\mathbb{Z}}$, and we define the second morphism by first defining ring morphisms:

$$\mathbb{Z}[\{x_l/x_i\}_{l \neq i}] \hookrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}]$$

induced by the map $\mathbb{Z} \hookrightarrow \mathbb{C}$, and then noting that these give scheme morphisms $U_{x_i} \subset \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$ which have image contained in $U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}}$. These scheme morphisms then trivially agree on overlaps so we have a morphism $p_X : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}}$. We claim that:

$$f \circ p_X = g \circ p_Y$$

and it suffices to check this on affine opens. Indeed, if we restrict to $U_{x_i} \subset \mathbb{P}^n_{\mathbb{C}}$, then these are morphisms of affine schemes, so it suffices to check that the corresponding ring morphisms agree. We see that the first ring homomorphism is given by:

$$\begin{aligned} (f \circ p_X)|^{\sharp}_{U_{x_i}} : \mathbb{Z} &\longrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}] \\ z &\longmapsto z \end{aligned}$$

while the second is given by:

$$\begin{aligned} (g \circ p_X)|^{\sharp}_{U_{x_i}} : \mathbb{Z} &\longrightarrow \mathbb{C}[\{x_l/x_i\}_{l \neq i}] \\ z &\longmapsto z \end{aligned}$$

so the two agree. There is thus a unique morphism $\phi : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C}$, which we wish to check is an isomorphism. We have the following diagram:

$$\begin{tikzcd}
\mathbb{P}^n_{\mathbb{C}} \arrow[dr, "\phi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X"'] & & \\
& \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C} \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & \operatorname{Spec} \mathbb{C} \arrow[d, "g"] \\
& \mathbb{P}^n_{\mathbb{Z}} \arrow[r, "f"] & \mathbb{Z}
\end{tikzcd}$$

We see that $\pi_X \circ \phi|_{U_{x_i}} = p_X|_{U_{x_i}}$, and the $p_X|_{U_{x_i}}$ has image in $U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}}$, so $\phi|_{U_{x_i}}$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
U_{x_i} \arrow[dr, "\phi"] \arrow[drr, bend left, "p_Y"] \arrow[ddr, bend right, "p_X|_{U_{x_i}}"'] & & \\
& U_{x_i} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{C} \arrow[r, "\pi_Y"] \arrow[d, "\pi_{U_{x_i}}"] & \operatorname{Spec} \mathbb{C} \arrow[d, "g"] \\
& U_{x_i} \arrow[r, "f"] & \mathbb{Z}
\end{tikzcd}$$

where we have identified $U_{x_i} \times_Z \operatorname{Spec}\mathbb{C}$ with the open subset of $\mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec}\mathbb{C}$ which satisfies the same universal property[35]. Since all these schemes are affine, we now go to the ring picture, and see that we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}[\{x_l/x_i\}_{l\neq i}] & & \\
& \mathbb{Z}[\{x_l/x_i\}_{l\neq j}] \otimes_{\mathbb{Z}} \mathbb{C} \arrow[ul, "\phi|^\sharp_{U_{x_i}}"'] & \mathbb{C} \arrow[l, "\pi_Y^\sharp"'] \arrow[ull, bend right, "p_Y^\sharp"'] \\
& \mathbb{Z}[\{x_l/x_i\}_{l\neq j}] \arrow[u, "\pi_X|^\sharp_{U_{x_i}}"] \arrow[uul, bend left, "p_X|^\sharp_{U_{x_i}}"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

We note that these projections must be given by $p \mapsto p \otimes 1$, and $w \mapsto 1 \otimes m$, so the map $h : p \otimes w \mapsto p \cdot w$ makes the diagram commute. It follows that $\phi|^\sharp_{U_{x_i}} = h$, so $\phi|^\sharp_{U_{x_i}}$ is an isomorphism, implying that $\phi|_{U_{x_i}}$ is an isomorphism. Since $\phi|_{U_{x_i}}$ is an isomorphism for all x_i, we have that:

$$\phi(\mathbb{P}^n_{\mathbb{C}}) = \bigcup_{i=0}^{n} \phi(U_{x_i}) = \bigcup_{i=0}^{n} U_{x_i} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}$$

so ϕ is surjective, and is clearly injective. Moreover, we see that if $U \subset \mathbb{P}^n_{\mathbb{C}}$ is any open set, then we can write:

$$\phi(U) = \bigcup_{i=0}^{n} \phi(U \cap U_{x_i})$$

which is a finite union of open sets, so ϕ is a bijective open continuos map implying that ϕ is a homeomorphism. Moreover, the map $\phi^\sharp : \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}} \to \phi_* \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}}$ restricts to isomorphisms $\phi^\sharp|_{U_{x_i}} : \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}}|_{U_{x_i}} \to (\phi_* \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}})|_{U_{x_i}}$ as:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{C}}(U_{x_i}) = \mathbb{Z}[\{x_l/x_i\}_{l\neq i}] \otimes_{\mathbb{Z}} \mathbb{C} \qquad \text{and} \qquad (\phi_* \mathscr{O}_{\mathbb{P}^n_{\mathbb{C}}})(U_{x_i}) = \mathbb{C}[\{x_l/x_i\}]$$

and $\phi^\sharp|_{U_{x_i}}$ is then given by the isomorphism h. By Corollary 1.2.4, it follows that $\phi^\sharp$ is indeed an isomorphism of sheaves, so $(\phi, \phi^\sharp)$ is an isomorphism of schemes as desired, implying the claim.

Though we have proved this in the case of $\mathbb{C}$ and $\mathbb{Z}$, the same proof shows that $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_A \operatorname{Spec} B$, whenever B is an A algebra.

Example 2.3.2. Let A be any commutative ring, and I and J be ideals of A. We then claim that:

$$\mathbb{V}(I) \times_A \mathbb{V}(J) \cong \operatorname{Spec}(A/\langle I + J\rangle)$$

where $\mathbb{V}(I)$ and $\mathbb{V}(J)$ have the natural induced reduced subscheme structure. However, this follows from the easily verifiable fact that:

$$A/I \otimes_A A/J \cong A/\langle I + J\rangle$$

Moreover, since the scheme $\operatorname{Spec}(A/\langle I + j\rangle)$ is isomorphic to $\mathbb{V}(I + J)$, we have that:

$$\mathbb{V}(I) \times_A \mathbb{V}(J) \cong \mathbb{V}(I + J) = \mathbb{V}(I) \cap \mathbb{V}(J)$$

In particular, if X and Y are closed subsets of Z equipped with induced reduced subscheme structure, we have that:

$$X \times_Z Y \cong X \cap Y$$

where $X \cap Y$ is equipped with the induced reduced subscheme structure.

[35] All is well because this how we explicitly constructed the fibre product!

Example 2.3.3. Recall from Example 2.2.3 where we showed that $|X = \operatorname{Proj}\mathbb{C}[x,y,z]| \cong |\mathbb{A}^1_{\mathbb{C}}| \times |\mathbb{P}^1_{\mathbb{C}}|$ when $\mathbb{C}[x,y,z]$ is equipped the grading induced by $\deg x = 0$, and $\deg y = \deg z = 1$. We now claim that as schemes $X \cong \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$. Let U_y and U_z be the distinguished open sets of X, and ξ_{zy} the ring isomorphism $\mathbb{C}[x, y/z, z/y] \to \mathbb{C}[x, z/y, y/z]$ sending $x \mapsto x$, $y/z \mapsto z/y$, which induces the gluing isomorphism along $U_x \cap U_y$. Then we have ring homomorphisms:

$$\begin{aligned} \imath_{yx} : \mathbb{C}[x] &\longrightarrow (\mathbb{C}[x,y,z]_y)_0 \cong \mathbb{C}[x, z/y] \\ x &\longmapsto x \end{aligned}$$

and similarly a ring homomorphism $\imath_{zx}$ for $\mathbb{C}[x, y/z]$ which clearly satisfies $\xi_{zy} \circ \imath'_{yx} = \imath'_{zx}$, where the primed morphisms are the ones composed with the inclusions $\mathbb{C}[x, z/y], \mathbb{C}[x, y/z] \to \mathbb{C}[x, y/z, z/y]$. It follows that the induced scheme morphisms agree on U_{xy} so we get a unique morphism $p_{\mathbb{A}^1_{\mathbb{C}}} : X \to \mathbb{A}^1_{\mathbb{C}}$. Now we set $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj}\mathbb{C}[u,v]$ with the standard grading, and note that the ring homomorphism:

$$\begin{aligned} \imath_{yu} : \mathbb{C}[v/u] &\longrightarrow \mathbb{C}[x, z/y] \\ v/u &\longmapsto z/y \end{aligned}$$

induces a morphism of affine schemes:

$$p_{yu} : U_y \longrightarrow U_u \subset \mathbb{P}^1_{\mathbb{C}}$$

Similarly, the morphism $\imath_{zv} : \mathbb{C}[u/v] \to \mathbb{C}[x, y/z]$ given by $u/v \mapsto y/z$ gives a morphism of affine schemes $p_{zv} : U_z \to U_v \subset \mathbb{P}^1_{\mathbb{C}}$. We see that on the overlap $p_{yu}|_{U_{zy}}$ and $p_{zv}|_{U_{zy}}$ are induced by the ring homomorphisms:

$$v/u \in \mathbb{C}[v/u, u/v] \longmapsto z/y \in \mathbb{C}[x, z/y, y/z] \text{ and } u/v \in \mathbb{C}[u/v, v/u] \longmapsto y/z \in \mathbb{C}[x, z/y, y/z]$$

Clearly we have that $\xi_{zy}(z/y) = y/z$ so we have that $\xi_{zy} \circ \imath_{yu} = \imath_{vu}$, so $p_{yu}|_{U_{zy}} = p_{zv}|_{U_{zy}}$ implying that the morphisms glue together to yield our second map $p_{\mathbb{P}^1_{\mathbb{C}}} : X \to \mathbb{P}^1_{\mathbb{C}}$. If f and g are the morphisms making $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$ $\mathbb{C}$-schemes[36], then we clearly have that $f \circ p_{\mathbb{A}^1_{\mathbb{C}}} = g \circ p_{\mathbb{P}^1_{\mathbb{C}}}$, hence there is a unique morphism of schemes making the following diagram commute:

$$\begin{tikzcd}
X \arrow[drr, bend left, "p_{\mathbb{P}^1_{\mathbb{C}}}"] \arrow[dr, "\psi"] \arrow[ddr, bend right, "p_{\mathbb{A}^1_{\mathbb{C}}}"'] & & \\
& \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\pi_{\mathbb{P}^1_{\mathbb{C}}}"] \arrow[d, "\pi_{\mathbb{A}^1_{\mathbb{C}}}"] & \mathbb{P}^1_{\mathbb{C}} \arrow[d, "g"] \\
& \mathbb{A}^1_{\mathbb{C}} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

We claim this morphism is an isomorphism. Indeed, we have that $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u$ and $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_v$, cover $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$, and that U_z and U_y cover X. We see that by the constructions of the maps $p_{\mathbb{P}^1_{\mathbb{C}}}$ and $p_{\mathbb{A}^1_{\mathbb{C}}}$ that $\psi|_{U_y}$ must make the following the diagram commute:

$$\begin{tikzcd}
U_y \arrow[drr, bend left, "p_{\mathbb{P}^1_{\mathbb{C}}}|_{U_u}"] \arrow[dr, "\psi|_{U_u}"] \arrow[ddr, bend right, "p_{\mathbb{A}^1_{\mathbb{C}}}"'] & & \\
& \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u \arrow[r, "\pi_{U_u}"] \arrow[d, "\pi_{\mathbb{A}^1_{\mathbb{C}}}"] & U_u \arrow[d, "g"] \\
& \mathbb{A}^1_{\mathbb{C}} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

[36]The constructions are essentially the same as in Example 2.3.2

so we have the following commutative diagram in the category of rings:

$$\begin{tikzcd}
\mathbb{C}[x, y/z] & & \\
& \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[u/v] \arrow[ul, "\psi|^{\sharp}_{U_u}"] & \mathbb{C}[u/v] \arrow[l] \arrow[ull, bend right, "\imath_{yu}"'] \\
& \mathbb{C}[x] \arrow[u] \arrow[uul, bend left, "\imath_{yx}"] & \mathbb{C} \arrow[l] \arrow[u]
\end{tikzcd}$$

But the isomorphism $x \otimes (u/v) \mapsto x \cdot (u/v)$ makes this diagram commute so $\psi|_{U_y} : U_y \to \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u \subset \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is an isomorphism, and similarly for $\psi|_{U_z}$. Since $\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_C = (\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_u) \cup (\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} U_v)$ it follows that ψ itself is an isomorphism, implying the claim.

Example 2.3.4. We claim that there exists a morphism:

$$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$$

which on closed points satisfies:

$$([w_0, w_1], [z_0, z_1]) \longmapsto [w_0 z_0, w_1 z_0, w_0 z_1, w_1 z_1]$$

Set the first copy of $\mathbb{P}^1_{\mathbb{C}}$ to $\operatorname{Proj} \mathbb{C}[x_0, x_1]$ and the second to be $\operatorname{Proj} \mathbb{C}[y_0, y_1]$, also set $\mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[v_0, v_1, v_2, v_3]$. Now we have have an affine open cover of $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ given by $\{U_{x_i} \times_{\mathbb{C}} U_{y_j}\}_{ij}$, meanwhile $\mathbb{P}^3_{\mathbb{C}}$ is covered by $\{U_{v_k}\}_k$. We can write $[w_0, w_1]$ as the homogeneous prime ideal:

$$[w_0, w_1] = \langle x_0 w_1 - x_1 w_0 \rangle \subset \mathbb{C}[x_0, x_1]$$

If $[w_0, w_1] \in U_{x_0}$, then this corresponds to the prime ideal:

$$[w_0, w_1] = \langle x_1/x_0 - w_1/w_0 \rangle \subset \mathbb{C}[x_1/x_0]$$

and similarly if $[z_0, z_1] \in U_{y_0}$, then:

$$[z_0, z_1] = \langle y_1/y_0 - z_1/z_0 \rangle \subset \mathbb{C}[y_1/y_0]$$

We can thus rewrite $[w_0, w_1]$ and $[z_0, z_1]$ as $[1, w_1/w_0]$ and $[1, z_1/z_0]$. Our desired morphism will then send these two pairs of points to $[1, w_1/z_0, z_1/z_0, w_1 z_1/w_0 z_0]$ which lies in U_{v_0}. We thus need a morphism of affine schemes $U_{x_0} \times_{\mathbb{C}} U_{y_0} \to U_{v_0}$ which satisfies:

$$I = \left\langle \frac{x_1}{x_0} - \frac{w_1}{w_0}, \frac{y_1}{y_0} - \frac{z_1}{z_0} \right\rangle \longmapsto \left\langle \frac{v_1}{v_0} - \frac{w_1}{w_0}, \frac{v_2}{v_0} - \frac{z_1}{z_0}, \frac{v_3}{v_0} - \frac{w_1 z_1}{w_0 w_1} \right\rangle$$

and we claim this is given by the ring homomorphism:

$$\begin{aligned} \phi_0 : \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0] &\longrightarrow \mathbb{C}[x_1/x_0, y_1/y_0] \\ v_i/v_0 &\longmapsto \begin{cases} x_1/x_0 & \text{if } i = 1 \\ y_1/y_0 & \text{if } i = 2 \\ (x_1/x_0) \cdot (y_1/y_0) & \text{if } i = 3 \end{cases} \end{aligned}$$

It is then clear that:

$$\left\langle \frac{v_1}{v_0} - \frac{w_1}{w_0}, \frac{v_2}{v_0} - \frac{z_1}{z_0}, \frac{v_3}{v_0} - \frac{w_1 z_1}{w_0 w_1} \right\rangle \subset \phi_0^{-1} \left(\left\langle \frac{x_1}{x_0} - \frac{w_1}{w_0}, \frac{y_1}{y_0} - \frac{z_1}{z_0} \right\rangle \right)$$

as the first two generators of the left hand side trivially map into I, and the third generator satisfies :

$$\begin{aligned} \phi_0(v_3/v_0 - w_1 z_1/z_0 z_1) =& (x_1/x_0) \cdot (y_1/y_0) - w_1 z_1/z_0 z_1 \\ =& y_1/y_0 \,(x_1/x_0 - w_1/w_0) + w_1/w_0 \,(y_1/y_0 - z_1/z_0) \end{aligned}$$

Since the left hand ideal is maximal, we have equality, and thus our ring homomorphism ϕ_{00} induce scheme morphisms which satisfy the desired property on closed points. By the same logic we define $\phi_i : \mathbb{C}[\{v_j/v_i\}_{j\neq i}] \to \mathbb{C}[x_k/x_l, y_m/y_n]$ where we have that:

$$(k,l,m,n) = \begin{cases} (1,0,1,0) & \text{if } i=0 \\ (0,1,1,0) & \text{if } i=1 \\ (1,0,0,1) & \text{if } i=2 \\ (0,1,0,1) & \text{if } i=3 \end{cases}$$

by:

$$\begin{aligned} \phi_1 : \mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1] &\longrightarrow \mathbb{C}[x_0/x_1, y_1/y_0] \\ v_i/v_1 &\longmapsto \begin{cases} x_0/x_1 & \text{if } i=0 \\ (x_0/x_1)\cdot(y_1/y_0) & \text{if } i=2 \\ y_1/y_0 & \text{if } i=3 \end{cases} \\ \phi_2 : \mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2] &\longrightarrow \mathbb{C}[x_1/x_0, y_0/y_1] \\ v_i/v_2 &\longmapsto \begin{cases} y_0/y_1 & \text{if } i=0 \\ (x_1/x_0)\cdot(y_0/y_1) & \text{if } i=1 \\ x_1/x_0 & \text{if } i=3 \end{cases} \\ \phi_3 : \mathbb{C}[v_0/v_3, v_1/v_3, v_2/v_3] &\longrightarrow \mathbb{C}[x_0/x_1, y_0/y_1] \\ v_i/v_3 &\longmapsto \begin{cases} (x_0/x_1)\cdot(y_0/y_1) & \text{if } i=0 \\ y_0/y_1 & \text{if } i=1 \\ x_0/x_1 & \text{if } i=2 \end{cases} \end{aligned}$$

which then induce the morphisms:

$$\psi_i : U_{x_l} \times_{\mathbb{C}} U_{y_n} \longrightarrow U_{v_i}$$

We will show that these maps glue together in the specific case of $U_{x_0} \times_{\mathbb{C}} U_{y_1} \cap U_{x_1} \times_{\mathbb{C}} U_{y_0} \cong U_{x_0x_1} \times_{\mathbb{C}} U_{y_0y_1}$ which is isomorphic to the affine scheme $X = \operatorname{Spec}\mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1]$. When identifying these affine open subsets with this affine scheme, we see that the isomorphism gluing $U_{x_0} \times_{\mathbb{C}} U_{y_1}$ with $U_{x_1} \times_{\mathbb{C}} U_{y_0}$ along $U_{x_0x_1} \times_{\mathbb{C}} U_{y_0y_1}$ is given by the tensors product morphism induced by the gluing U_{x_0} and U_{x_1} along $U_{x_0x_1}$ and similarly for U_{y_0} and U_{y_1}. It follows that the gluing isomorphism $\xi : X \subset U_{x_1} \times_{\mathbb{C}} U_{y_0} \longrightarrow X \subset U_{x_0} \times_{\mathbb{C}} U_{y_1}$ is induced by the ring automorphism:

$$\xi^\sharp : \mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1] \longrightarrow \mathbb{C}[x_1/x_0, x_0/x_1, y_1/y_0, y_0/y_1]$$

which sends each generator to itself. The morphisms we wish to glue are clearly ψ_1 and ψ_2, and we see that $\psi_1|_X$ and $\psi_2|_X$ now clearly have image in $U_{v_1v_2}$ which as a subset of U_{v_1} we identify with $\operatorname{Spec}\mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1, v_1/v_2]$, and as a subset of U_{v_2} we identify of $\operatorname{Spec}\mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2, v_2/v_1]$. Let $\eta : U_{v_1v_2} \subset U_{v_1} \to U_{v_1v_2} \subset U_{v_2}$ be the gluing isomorphism, then to show that these agree, we have to show that:

$$\eta \circ \psi_1|_X = \psi_2|_X$$

so it suffices to show that:

$$(\psi_1|_X)^\sharp \circ \eta^\sharp = (\psi_2|_X)^\sharp$$

Recall that $\eta^\sharp$ is given by:

$$\begin{aligned} \eta^\sharp : \mathbb{C}[v_0/v_2, v_1/v_2, v_3/v_2, v_2/v_1] &\longrightarrow \mathbb{C}[v_0/v_1, v_2/v_1, v_3/v_1, v_1/v_2] \\ v_i/v_j &\longmapsto \begin{cases} (v_i/v_1)\cdot(v_1/v_2) & \text{if } i\neq 2 \text{ and } j=2 \\ v_1/v_2 & \text{if } i=1 \text{ and } j=2 \\ v_2/v_1 & \text{if } i=2 \text{ and } j=1 \end{cases} \end{aligned}$$

while the maps $(\psi_1|_X)^\sharp$ and $(\psi_2|_X)^\sharp$ are the maps induced by localization. We now calculate the image of each generator beginning with v_0/v_2:

$$\begin{aligned}(\psi_1|_X)^\sharp \circ \eta^\sharp(v_0/v_2) =&(\psi_1|_X)^\sharp(v_0/v_1 \cdot v_1/v_2)\\ =&x_0/x_1 \cdot x_1/x_0 \cdot y_0/y_1\\ =&y_0/y_1\end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_0/v_2) = y_0/y_1$$

For the next generator we have that:

$$\begin{aligned}(\psi_1|_X)^\sharp \circ \eta^\sharp(v_1/v_2) =&(\psi_1|_X)^\sharp(v_1/v_2)\\ =&x_1/x_0 \cdot y_0/y_1\end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_1/v_2) = x_1/x_0 \cdot y_0/y_1$$

For v_3/v_2 we have that:

$$\begin{aligned}(\psi_1|_X)^\sharp \circ \eta^\sharp(v_3/v_2) =&\psi_1^\sharp(v_3/v_1 \cdot v_1/v_2)\\ =&y_1/y_0 \cdot x_1/x_0 \cdot y_0/y_1\\ =&x_1/x_0\end{aligned}$$

while:

$$(\psi_2|_X)^\sharp(v_3/v_2) = x_2/x_0$$

Finally, we have that:

$$(\psi_1|_X)^\sharp \circ \eta^\sharp(v_2/v_1) = (\psi_1|_X)^\sharp(v_2/v_1) = x_0/x_1 \cdot y_1/y_0$$

while:

$$(\psi_2|_X)^\sharp(v_2/v_1) = x_0/x_1 \cdot y_1/y_0$$

2.4 Some Category Theory: Representable Functors

Over the next two sections we wish to develop an alternative but equivalent view of schemes, which will at times prove more convenient to work with. To do so, we first must take a detour through some abstract nonsense. Recall that a category $\mathscr{C}$ is locally small[37] if the Hom 'sets' are actually sets. We begin with the following lemma/notation:

Lemma 2.4.1. *Let $\mathscr{C}$ be a locally small category, and Y an object. Then there exists a contravariant functor $h_Y : \mathscr{C} \to \mathrm{Set}$ which sends an object X to* Set *via:*

$$X \longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

and sends $f \in \mathrm{Hom}_{\mathscr{C}}(X, Z)$ to the morphism:

$$\begin{aligned}h_Y(f) : \mathrm{Hom}_{\mathscr{C}}(Z, Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Y)\\ \alpha &\longmapsto f^*\alpha = \alpha \circ f\end{aligned}$$

[37] This is a borderline technicality that we honor here for the sake of being precise. In reality, we will almost never deal with a category which is not locally small. Moreover, its not exactly important that categories are locally small, the most vital results of this section, such as Yoneda's lemma, will still hold, the functor h_A will just have a different target category, namely the category of all classes, though doing such a thing may take us outside the realm of ZFC.

Proof. This is all essentially obvious, but we spell it out to fix our notation. Clearly if h_Y defines a functor then it is contravariant. Moreover, for $\mathrm{Id} \in \mathrm{Hom}_{\mathscr{C}}(X,X)$, we have that $h_Y(\mathrm{Id})$ is clearly the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(X,X)$. Now let $f \in \mathrm{Hom}_{\mathscr{C}}(X,Z)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Z,W)$, then $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(X,W)$. Let $\alpha \in \mathrm{Hom}_{\mathscr{C}}(W,Y)$, then:

$$h_Y(g \circ f)(\alpha) = (g \circ f)^*\alpha = \alpha \circ (g \circ f) = (g^*\alpha) \circ f = f^*(g^*\alpha) = (h_Y(f) \circ h_Y(g))(\alpha)$$

Since α was arbitrary we have that:

$$h_Y(g \circ f) = h_Y(f) \circ h_Y(g)$$

implying the claim. □

Definition 2.4.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The **product category** is the category where objects are pairs $(X_{\mathscr{C}}, X_{\mathscr{D}})$, and morphisms are pairs of morphisms $(f_{\mathscr{C}}, f_{\mathscr{D}})$, where $f_C : X_{\mathscr{C}} \to Y_{\mathscr{C}} \in \mathrm{Hom}_{\mathscr{C}}(X_{\mathscr{C}}, Y_{\mathscr{C}})$ and $g : X_{\mathscr{D}} \to Y_{\mathscr{D}} \in \mathrm{Hom}_{\mathscr{D}}(X_{\mathscr{D}}, Y_{\mathscr{D}})$.

One easily checks that the above is a category.

Example 2.4.1. Let $\mathscr{C}$ be a locally small category, and $\mathscr{D} = \mathscr{C}^{\mathrm{op}}$, i.e. the object of $\mathscr{D}$ are the objects of $\mathscr{C}$ but 'morphisms go the other way', so a morphism $X \to Y$ in $\mathscr{D}$ is given by $f \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$. The product category $\mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ is then of interest as we have a contravariant $\mathrm{Hom}(\cdot,\cdot)$ functor given by $(X,Y) \mapsto \mathrm{Hom}(X,Y)$, which sends a morphism $(f,g) : (X,Y) \to (W,Z)$ to the morphism:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(W,Z) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,Y) \\ \alpha &\longmapsto g \circ \alpha \circ f \end{aligned}$$

as $g : Y \to Z$ is an element of $\mathrm{Hom}_{\mathscr{C}}(Z,Y)$. One can make this covariant by considering $\mathrm{Hom}(\cdot,\cdot)$ as a functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, then if $(f,g) : (X,Y) \to (W,Z)$, we have that the natural set map is given by:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(W,Z) \\ \alpha &\longmapsto g \circ \alpha \circ f \end{aligned}$$

since in this case $f : X \to W$ is an element of of $\mathrm{Hom}_{\mathscr{C}}(W,X)$. The above is also an example of the fact that any contravariant functor can be viewed as a covariant functor from the opposite category.

Let $[\mathscr{C},\mathscr{D}]$ denote the category[38] of covariant functors from $\mathscr{C}$ to $\mathscr{D}$, and $[\mathscr{C}^{\mathrm{op}},\mathscr{D}]$ the category of contravariant functors from $\mathscr{C}$ to $\mathscr{D}$, where the objects in both are covariant/contravariant functors, and the morphisms are natural transformations. We denote the class[39] of natural formations between covariant/contravariant functors $\mathscr{F}$ and $\mathscr{G}$ by $\mathrm{Nat}(\mathscr{F},\mathscr{G})$, and note that $\mathrm{Nat}(\cdot,\cdot)$ can be viewed as a covariant, or contravariant functor from a suitable product category to the category of classes.

We also have the notion of an evaluation functor. That is given categories $\mathscr{C}$ and $\mathscr{D}$, we have a contravariant functor $\mathrm{ev} : \mathscr{C} \times [\mathscr{C}^{\mathrm{op}},\mathscr{D}]^{\mathrm{op}} \to \mathscr{D}$ given on objects by $(Y,\mathscr{F}) \mapsto \mathscr{F}(Y)$. Letting $(f,F) : (Y,\mathscr{F}) \to (X,\mathscr{G})$, we obtain the following the following commutative diagram:

$$\begin{array}{ccc} \mathscr{F}(Y) & \xleftarrow{\mathscr{F}(f)} & \mathscr{F}(X) \\ \uparrow{\scriptstyle F_Y} & & \uparrow{\scriptstyle F_X} \\ \mathscr{G}(Y) & \xleftarrow{\mathscr{G}(f)} & \mathscr{G}(X) \end{array}$$

so we send (f,F) to $\mathscr{F}(f) \circ F_X$, or equivalently $F_Y \circ \mathscr{G}(f)$. It is then clear that ev is a contravariant functor $\mathscr{C} \times [\mathscr{C}^{\mathrm{op}},\mathscr{D}]^{\mathrm{op}}$ to $\mathscr{D}$.

[38]Caution: if $\mathscr{C}$ and $\mathscr{D}$ are not both small categories one cannot talk of such a category in ZFC. Ignore this for now, we will speak more about set theoretic size issues in the next section.

[39]Generally the collection of all natural transformations do not form a set, but a class.

We also term the following contravariant functor $\mathscr{Y}$ from $\mathscr{C} \times [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]^{\mathrm{op}}$ to Class[40], the category of classes, given on objects by:

$$(Y, \mathscr{F}) \longmapsto \mathrm{Nat}(h_Y, \mathscr{F})$$

If $(f, F) : (Y, \mathscr{F}) \longrightarrow (X, \mathscr{G})$ is a morphism, then note that we have a natural transformation $\tilde{f} : h_Y \to h_X$ defined by:

$$\begin{aligned} \tilde{f}_Z : h_Y(Z) &\longrightarrow h_X(Z) \\ \alpha &\longmapsto f \circ \alpha \end{aligned}$$

It follows that $G \circ \tilde{f}$ is a natural transformation $h_y \to \mathscr{G}$, while F is a natural transformation $\mathscr{G} \to \mathscr{F}$. Hence, we send (f, F) to the morphism:

$$G \in \mathrm{Nat}(h_X, \mathscr{G}) \longmapsto F \circ G \circ \tilde{f} \in \mathrm{Nat}(h_Y, \mathscr{F})$$

We call $\mathscr{Y}$ the *Yoneda functor*[41], and note that by reversing arrows, this can be entirely formulated covariantly. The following famous result is known as Yoneda's lemma:

Lemma 2.4.2. *Let $\mathscr{C}$ be a locally small category, and consider the evaluation functor* $\mathrm{ev} : \mathscr{C} \times [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]^{\mathrm{op}} \longrightarrow \mathrm{Set}$. *There is a natural isomorphism:*

$$\mathscr{Y} \cong \mathrm{ev}$$

In particular, for all $Y \in \mathscr{C}$, and $\mathscr{F} \in [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$, we have that:

$$\mathrm{Nat}(h_Y, \mathscr{F}) \cong \mathscr{F}(Y)$$

Proof. Fixing an object $(Y, \mathscr{F})$, we first determine a morphism:

$$T_{Y,\mathscr{F}} : \mathrm{Nat}(h_Y, \mathscr{F}) \longrightarrow \mathscr{F}(Y)$$

Let G be a natural transformation, then this is the data of a morphism $G_Z : h_Y(Z) \to \mathscr{F}(Z)$ for all objects Z of $\mathscr{C}$ such that if $f : Z \to W$ is a morphism in $\mathscr{C}$ the following diagram commutes:

$$\begin{tikzcd} h_Y(Z) \arrow[r, "G_Z"] & \mathscr{F}(Z) \\ h_Y(W) \arrow[u, "h_Y(f)"] \arrow[r, "G_W"] & \mathscr{F}(W) \arrow[u, "\mathscr{F}(f)"'] \end{tikzcd}$$

In particular, G_Y is a map:

$$\mathrm{Hom}_{\mathscr{C}}(Y, Y) \longrightarrow \mathscr{F}(Y)$$

so we send $G \mapsto G_Y(\mathrm{Id}_A)$ for all F. We need to check that this is actually a natural transformation, i.e that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Nat}(Y, \mathscr{F}) \arrow[r, "T_{Y,\mathscr{F}}"] & \mathscr{F}(Y) \\ \mathrm{Nat}(X, \mathscr{G}) \arrow[u, "\mathscr{Y}(f,F)"] \arrow[r, "T_{X,\mathscr{G}}"] & \mathscr{G}(X) \arrow[u, "\mathrm{ev}(f,F)"'] \end{tikzcd}$$

Let G be a natural transformation $h_X \to \mathscr{G}$, then we have that:

$$\begin{aligned} \mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) =& \mathrm{ev}(f, F)(G_X(\mathrm{Id}_X)) \\ =& F_Y \circ \mathscr{G}(f) \circ G_X(\mathrm{Id}_X) \end{aligned}$$

[40] As we are about to see, this functor will actually have target in Set. We stress again that we do not really care that much about classes, and are simply paying heed for the moment out of necessity.

[41] This is not standard terminology.

However, G is a natural transformation $h_X \to \mathscr{G}$, so the following diagram commutes:

$$\begin{tikzcd} h_X(X) \arrow[r, "G_X"] \arrow[d, "h_X(f)"] & \mathscr{G}(X) \arrow[d, "\mathscr{G}(f)"] \\ h_X(Y) \arrow[r, "G_Y"] & \mathscr{G}(Y) \end{tikzcd}$$

hence:

$$\mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) = F_Y \circ G_Y \circ h_X(f)(\mathrm{Id}_X)$$

Now, $h_X(f)$ is the morphism:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, X) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y, X) \\ \alpha &\longmapsto \alpha \circ f \end{aligned}$$

hence $h_X(f) = f \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ so:

$$\mathrm{ev}(f, F) \circ T_{X,\mathscr{G}}(G) = F_Y \circ G_Y(f)$$

Similarly, we have that:

$$\begin{aligned} T_{Y,\mathscr{F}} \circ \mathscr{Y}(f, F)(G) =& (F \circ G \circ \tilde{f})_Y(\mathrm{Id}_Y) \\ =& F_Y \circ G_Y \circ \tilde{f}_Y(\mathrm{Id}_Y) \\ =& F_Y \circ G_Y(f) \end{aligned}$$

so the diagram is commutative and T defines a natural transformation $Y \to \mathrm{ev}$.

Now $\mathscr{F}(Y)$ is a set by assumption; let $x \in \mathscr{F}(Y)$, then we want to define a natural transformation $G_x \in \mathrm{Nat}(h_Y, \mathscr{F})$. Let Z be any object in $\mathscr{C}$, and define a morphism:

$$\begin{aligned} (G_x)_Z : h_Y(Z) &\longrightarrow \mathscr{F}(Z) \\ f &\longmapsto \mathscr{F}(f)(x) \end{aligned}$$

as $\mathscr{F}(f) : \mathscr{F}(Y) \to \mathscr{F}(Z)$. We need to show the following diagram commutes:

$$\begin{tikzcd} h_Y(Z) \arrow[r, "(G_x)_Z"] & \mathscr{F}(Z) \\ h_Y(W) \arrow[r, "(G_x)_W"] \arrow[u, "h_Y(g)"] & \mathscr{F}(W) \arrow[u, "\mathscr{F}(g)"] \end{tikzcd}$$

for any $g : Z \to W$. Let $f \in h_Y(W)$, then $h_Y(g)(f) = f \circ g$, and $(G_x)_Z(f \circ g) = \mathscr{F}(f \circ g)(x)$. Meanwhile, $(G_x)_W(f) = \mathscr{F}(f)(x)$, and $\mathscr{F}(g) \circ \mathscr{F}(f)(x) = \mathscr{F}(f \circ g)(x)$ as $\mathscr{F}$ is contravariant. It follows that G_x determines a natural transformation $h_Y \to \mathscr{F}$. Define $S_{Y,\mathscr{F}}$ by:

$$\begin{aligned} S_{Y,\mathscr{F}} : \mathscr{F}(Y) &\longrightarrow \mathrm{Nat}(h_Y, \mathscr{F}) \\ x &\longmapsto G_x \end{aligned}$$

then we need to show that this determines a natural transformation as well, so once again consider the diagram:

$$\begin{tikzcd} \mathscr{F}(Y) \arrow[r, "S_{Y,\mathscr{F}}"] & \mathrm{Nat}(h_Y, \mathscr{F}) \\ \mathscr{G}(X) \arrow[r, "S_{X,\mathscr{G}}"] \arrow[u, "\mathrm{ev}(f,F)"] & \mathrm{Nat}(h_X, \mathscr{G}) \arrow[u, "\mathscr{Y}(f,F)"] \end{tikzcd}$$

for all morphisms $(f, F) : (Y, \mathscr{F}) \to (X, \mathscr{G})$. Let $x \in \mathscr{G}(X)$, and set:

$$z = F_Y \circ \mathscr{G}(f)(x) = \mathscr{F}(f) \circ F_X(x)$$

then:

$$S_{Y,\mathscr{F}} \circ \mathrm{ev}(f, F)(x)$$

is the natural transformation G_z. We then need to show the following equality of natural transformations:

$$F \circ G_x \circ \tilde{f} = G_z$$

Let W be any object in $\mathscr{C}$, then $(G_z)_W$ send $g \in h_Y(W)$ to $\mathscr{F}(g)(z)$ so:

$$(G_z)_W(g) = \mathscr{F}(g)(\mathscr{F}(f) \circ F_X(x)) = \mathscr{F}(f \circ g) \circ F_X(x)$$

Meanwhile,

$$\begin{aligned}(F \circ G_x \circ \tilde{f})_W(g) =& F_W \circ G_x(f \circ g)\\ =& F_W \circ \mathscr{G}(f \circ g)(x)\end{aligned}$$

Note that $f \circ g : W \to X$, so by the naturality of F, we have that:

$$(F \circ G_x \circ \tilde{f})_W(g) = \mathscr{F}(f \circ g) \circ F_X(x)$$

Therefore S determines a natural transformation $\mathrm{ev} \to \mathscr{Y}$ as desired.

It remains to show that $S \circ T = \mathrm{Id}_{\mathscr{Y}}$ and $T \circ S = \mathrm{Id}_{\mathrm{ev}}$. We can do this object wise, let $(Y, \mathscr{F})$ be a pair, then:

$$S_{Y,\mathscr{F}} \circ T_{Y,\mathscr{F}} : \mathrm{Nat}(h_Y, \mathscr{F}) \longrightarrow \mathrm{Nat}(h_Y, \mathscr{F})$$

Let $G \in \mathrm{Nat}(h_Y, \mathscr{F})$, then $T_{Y,\mathscr{F}}(G) = G_Y(\mathrm{Id}_Y) \in \mathscr{F}(Y)$. We need to show that the natural transformation corresponding to $x = G_Y(\mathrm{Id}_Y)$, G_x is equal to G. Let W be an object of $\mathscr{C}$, and consider $g \in h_Y(W)$, then:

$$(G_x)_W(g) = \mathscr{F}(g)(x) = \mathscr{F}(g)(G_Y(\mathrm{Id}_Y)) = G_W \circ h_Y(g)(\mathrm{Id}_Y) = G_W(g)$$

It follows that $G_x = G$, hence $S_{Y,\mathscr{F}} \circ T_{Y,\mathscr{F}} = \mathrm{Id}_{\mathscr{Y}}$.

For the other direction we have:

$$T_{Y,\mathscr{F}} \circ S_{Y,\mathscr{F}} : \mathscr{F}(Y) \longrightarrow \mathscr{F}(Y)$$

Taking a point $x \in \mathscr{F}(Y)$, we need to show that $(G_x)_Y(\mathrm{Id}_Y) = x$. However, $(G_x)_Y(\mathrm{Id}_Y) = \mathscr{F}(\mathrm{Id}_Y)(x) = \mathrm{Id}_{\mathscr{F}(Y)}(x) = x$, implying the claim. □

The following corollary, known as the Yoneda embedding, is immediate:

Corollary 2.4.1. *Let X and Y be objects in a locally small category $\mathscr{C}$, then there is a natural bijection:*

$$\mathrm{Nat}(h_Y, h_X) \cong h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

Note that if we denote by h^Y and h^X the covariant analogues of h_X and h_Y, then almost verbatim the same proof shows that:

$$\mathrm{Nat}(h^Y, \mathscr{F}) \cong \mathscr{F}(Y)$$

where $\mathscr{F}$ is now a covariant functor. In particular,

$$\mathrm{Nat}(h^Y, h^X) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

Example 2.4.2. We explore the implications of the Yoneda lemma in a concrete algebraic category. Let $\mathscr{C} = \mathrm{Mod}_A$ be the category of A modules, and M and N modules. Then in particular, every natural transformation from $\mathrm{Hom}_{\mathrm{Mod}_A}(\cdot, M)$ to $\mathrm{Hom}_{\mathrm{Mod}_A}(\cdot, N)$ is uniquely determined by an A-module homomorphism $M \to N$.

Definition 2.4.2. Let $\mathscr{F}$ be contravariant functor $\mathscr{C} \to \mathrm{Set}$, then $\mathscr{F}$ is **representable**, if there exists a natural isomorphism $F \cong h_Y$ for some object Y.

Lemma 2.4.3. *Let $\mathscr{F}$ be a representable functor, represented by Y. Then the pair $(Y, F : h_Y \to \mathscr{F})$ is unique up to unique isomorphism.*

Proof. Let $F : h_Y \to \mathscr{F}$ be a natural isomorphism, and suppose that $G : h_X \to \mathscr{F}$ is another natural isomorphism. It follows that $G^{-1} \circ F : h_Y \to h_X$ is a natural isomorphism, and thus corresponds to a unique morphism in $\mathrm{Hom}_{\mathscr{C}}(Y, X)$. This morphism is given by $\alpha = G_Y^{-1} \circ F_Y(\mathrm{Id}_Y)$, and similarly we have a morphism $\beta = F_X^{-1} \circ G_X(\mathrm{Id}_X) \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. We see that we have the following diagrams:

$$\begin{tikzcd}
h_Y(Y) \arrow[r, "F_Y"] & \mathscr{F}(Y) \\
h_Y(X) \arrow[r, "F_X"] \arrow[u, "h_Y(\alpha)"] & \mathscr{F}(X) \arrow[u, "\mathscr{F}(\alpha)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
h_X(Y) \arrow[r, "G_Y"] \arrow[d, "h_X(\beta)"'] & \mathscr{F}(Y) \arrow[d, "\mathscr{F}(\beta)"] \\
h_X(X) \arrow[r, "G_X"] & \mathscr{F}(X)
\end{tikzcd}$$

It follows that for $\beta \in h_Y(X)$, we have that $h_Y(\alpha)(\beta) = \beta \circ \alpha$, and that $F_Y(\beta \circ \alpha) = \mathscr{F}(\alpha) \circ F_X(\beta)$. Since $\beta = F_X^{-1} \circ G_X(\mathrm{Id}_X)$:

$$\begin{aligned}
\mathscr{F}(\alpha) \circ F_X(\beta) = \mathscr{F}(\alpha) \circ G_X(\mathrm{Id}_X) = G_Y \circ h_X(\alpha)(\mathrm{Id}_X) =& G_Y(\alpha) \\
=& G_Y(G_Y^{-1} \circ F_Y(\mathrm{Id}_Y)) \\
=& F_Y(\mathrm{Id}_Y)
\end{aligned}$$

Since F_Y is an isomorphism it follows that $\beta \circ \alpha = \mathrm{Id}_Y$. Similarly, for $\alpha \in h_X(Y)$, we have that $G_X(\alpha \circ \beta) = \mathscr{F}(\beta) \circ G_Y(\alpha)$. The same argument shows that:

$$\begin{aligned}
\mathscr{F}(\beta) \circ G_Y(\alpha) =& \mathscr{F}(\beta) \circ F_Y(\mathrm{Id}_Y) \\
=& F_X \circ h_Y(\beta)(\mathrm{Id}_Y) \\
=& F_X(\beta) \\
=& F_X(F_X^{-1} \circ G_X(\mathrm{Id}_X)) \\
=& G_X(\mathrm{Id}_X)
\end{aligned}$$

Since G is an isomorphism, it follows that $\alpha \circ \beta = \mathrm{Id}_X$, so α and β are unique isomorphisms as desired. □

Example 2.4.3. Let Vec be the category of vector spaces over some field k. Consider the functor $D : \mathrm{Vec} \to \mathrm{Vec}$ given by $V \mapsto V^*$, and $A : V \to W$ maps to $A^* : W^* \to V^*$. This is a contravariant functor, is easily seen to be represented by k, essentially by definition.

Consider again the category of A modules Mod_A, and fix an object N. Define a functor by $\mathscr{F} : \mathrm{Mod}_A \to \mathrm{Set}$ by:

$$\mathscr{F}(M) = \{A - \text{bilinear forms on } M \oplus N\}$$

If $\phi : M \to M'$ is a morphism of A-modules, then we define a morphism:

$$\begin{aligned}
\mathscr{F}(\phi) : \mathscr{F}(M') &\longrightarrow \mathscr{F}(M) \\
\omega &\longmapsto \phi^* \omega
\end{aligned}$$

where $\phi^*\omega$ is the form on $M \oplus N$ given by $(\phi^*\omega)(m, n) = \omega(\phi(m), n)$. This clearly defines a contravariant functor, and in particular we claim is represented by $N^* := \mathrm{Hom}_{\mathrm{Mod}_A}(N, A)$. Indeed, define:

$$\begin{aligned}
F_M : \mathscr{F}(M) &\longrightarrow h_{N^*}(M) \\
\beta &\longmapsto f_\beta
\end{aligned}$$

where $f_\beta : M \to N^*$ is the morphism given by $m \mapsto \beta(m, \cdot) \in N^*$. This is clearly A-linear for each M, and given $\phi : M \to M'$ makes the relevant diagram commute, hence the assignment $M \mapsto F_M$ defines a natural transformation.

We define:

$$\begin{aligned} G_M : h_{N^*}(M) &\longrightarrow \mathscr{F}(M) \\ f &\longmapsto \beta_f \end{aligned}$$

by $\beta_f(m, n) = f(m)(n)$, as $f(m) \in N^*$. This is also clearly A-linear, an defines a natural transformation. We see that $F_M \circ G_M$ sends f to f_{β_f}, which is the morphism given by $f_{\beta_f}(m)(n) = \beta_f(m, n) = f(m)(n)$, hence $f_{\beta_f} = f$, so $F \circ G = \mathrm{Id}_{h_{N^*}}$. Similarly $G_M \circ F_M$ sends β to β_{f_β}, which is the bilinear form given by $\beta(m, n) = f_\beta(m)(n) = \beta(m, n)$, so $G \circ F = \mathrm{Id}_{\mathscr{F}}$, implying the claim.

We also have an example of a similar phenomenon happening in the covariant case:

Example 2.4.4. The forgetful functor $\mathscr{F}$: Ring $\to$ Set is represented by $\mathbb{Z}[x]$, by which we mean $h^{\mathbb{Z}[x]} \cong \mathscr{F}$. For any $A \in$ Ring we construct the following map:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], A) &\longmapsto \mathscr{F}(A) \\ f &\longmapsto f(x) \end{aligned}$$

which lies in the set A. This is a bijection because $\mathbb{Z}[x]$ is the free object on one generator in Ring, hence each element in $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], A)$ is determined precisely by where x is sent. The relevant diagram then obviously commutes implying the natural isomorphism.

We end this section by briefly exploring the notion of universal objects and how this is related to the representability of a functor. We provide no examples of this phenomenon, but this is extremely relevant in the study of moduli spaces.

Definition 2.4.3. Let $\mathscr{F} : \mathscr{C} \to$ Set be a contravariant functor, then (X, ξ) is a **universal object of** $\mathscr{F}$ if $X \in \mathscr{C}$, $\xi \in \mathscr{F}(X)$, and for all $Y \in \mathscr{C}$, $\alpha \in \mathscr{F}(Y)$ there is a unique morphism $f : Y \to X$ such that $\mathscr{F}(f)(\xi) = \alpha$.

We now prove the following:

Lemma 2.4.4. *Let $\mathscr{F} : \mathscr{C} \to$ Set be a contravariant functor, then $\mathscr{F}$ is representable if and only if there exists a universal object (X, ξ) of $\mathscr{F}$.*

Proof. Suppose that (X, ξ) is a universal object of $\mathscr{F}$, then we construct a natural isomorphism:

$$\Psi : h_X \longrightarrow \mathscr{F}$$

on objects via:

$$\begin{aligned} \Psi_Y : \mathrm{Hom}_{\mathscr{C}}(Y, X) &\longrightarrow \mathscr{F}(Y) \\ f &\longmapsto \mathscr{F}(f)(\xi) \end{aligned}$$

By definition this is injective and surjective. Take $g : Y \to Z$, and consider the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(Y, X) \arrow[r, "\Psi_Y"] & \mathscr{F}(Y) \\ \mathrm{Hom}_{\mathscr{C}}(Z, X) \arrow[r, "\Psi_Z"] \arrow[u, "h_X(g)"] & \mathscr{F}(Z) \arrow[u, "\mathscr{F}(g)"'] \end{tikzcd}$$

For any $f \in \mathrm{Hom}_{\mathscr{C}}(Z, X)$, we have that going up and to the right gives:

$$\mathscr{F}(h_X(f))(\xi) = \mathscr{F}(f \circ g)(\Xi) = (\mathscr{F}(g) \circ \mathscr{F}(f))(\xi)$$

which is precisely going right and then up. It follows that $h_X \cong \mathscr{F}$ as desired.

Now suppose that $\mathscr{F} \cong h_X$, and let $\Psi : h_X \to \mathscr{F}$ be the isomorphism. We set $\Psi_X(\mathrm{Id}) = \xi$, and claim that (X, ξ) is the universal object. Let $Y \in \mathscr{C}$, and $\alpha \in \mathscr{F}(Y)$, then $\Psi_Y^{-1}(\alpha) \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(Y,X) \arrow[r, "\Psi_Y"] & \mathscr{F}(Y) \\
\mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[u, "h_X(\Psi_Y^{-1}(\alpha))"] & \mathscr{F}(X) \arrow[l, "\Psi_X^{-1}"'] \arrow[u, "\mathscr{F}(\Psi_Y^{-1}(\alpha))"']
\end{tikzcd}$$

implying that:

$$\mathscr{F}(\Psi_Y^{-1}(\alpha)) = \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha)) \circ \Psi_X^{-1}$$

hence:

$$\begin{aligned}
\mathscr{F}(\Psi_Y^{-1}(\alpha))(\xi) &= \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha)) \circ \Psi_X^{-1}(\xi) \\
&= \Psi_Y \circ h_X(\Psi_Y^{-1}(\alpha))(\mathrm{Id}) \\
&= \Psi_Y(\Psi_Y^{-1}(\alpha)) \\
&= \alpha
\end{aligned}$$

Clearly our choice of $\Psi_Y^{-1}(\alpha)$ is unique, so (X, ξ) is a universal object. $\square$

Note that universal objects (X, ξ) are also clearly unique up to unique isomorphism.

2.5 Schemes are Functors and (Some) Functors are Schemes

In this section we note the boring fact that all schemes are functors, and then explore how some functors are schemes. In particular, we will introduce Grothendieck topologies in order to provide a refinement of the Yoneda lemma in the category of S-schemes, and prove a criterion for a functor to be representable by an S scheme. After this, we investigate examples of representable functors in the category of S-schemes with a particular emphasis on the representability of the fibre product, and some affine schemes. We will also argue that in many cases of interest, when given an arbitrary functor from the category of S schemes, it will suffices to prove representability in the category of schemes, and then base change to the category of S schemes, denoted Sch/S.

As previously noted, all schemes are uniquely schemes over $\operatorname{Spec} \mathbb{Z}$, hence the category Sch is the exact same as the category $\mathrm{Sch}/\mathbb{Z}$. In particular, every scheme defines a contravariant functor:

$$\begin{aligned}
h_X : \mathrm{Sch} &\longrightarrow \mathrm{Set} \\
Y &\longmapsto \mathrm{Hom}_{\mathrm{Sch}}(Y, X)
\end{aligned}$$

Given an S scheme, $f : X \to S$, we obtain another contravariant functor which we also denote by h_X:

$$\begin{aligned}
h_X : \mathrm{Sch}/S &\longrightarrow \mathrm{Set} \\
Y &\longmapsto \mathrm{Hom}_{\mathrm{Sch}/S}(Y, X)
\end{aligned}$$

where as always the morphisms in the category of S schemes are commutative diagrams in the category of schemes:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] & & X \arrow[dl] \\
& S &
\end{tikzcd}$$

Moreover, any scheme X viewed as the functor $h_X : \mathrm{Sch} \to \mathrm{Set}$, can be turned into a functor $\mathrm{Sch}/S \to \mathrm{Set}$ as follows: take the fibre product $X_S := X \times_{\mathbb{Z}} S$, and consider the contravariant functor $h_{X_S} : \mathrm{Sch}/S \to \mathrm{Set}$. Note that if X is already an S scheme and we forget this structure, then taking X_S obviously does not give us the back the original S scheme.

We will often refer to the hom set $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, X)$ as the 'Y-points of X', and when $Y = \operatorname{Spec} A$ we will refer to $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, X)$ as the 'A-points of X'. This choice of nomenclature has the following justification: if we let X be $\operatorname{Spec} B$, then for any field k a morphism $\operatorname{Spec} k \to \operatorname{Spec} B$ is a choice of a morphism $\phi : B \to k$. In particular, this morphism determines a point in $\operatorname{Spec} B$ given by $\ker \phi$ which in turn induces a field extension $B_{\ker \phi}/\mathfrak{m}_{\ker \phi} \hookrightarrow k$. Conversely, any such field extension $k/k_{\mathfrak{p}}$ induces a morphism $B \to k$, hence $\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} k, \operatorname{Spec} B)$ can be canonically identified with points $\mathfrak{p} \in \operatorname{Spec} B$ whose residue field $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ admits a field extension into k. Historically one calls such points k-points, and so via generalization we obtain the nomenclature Y-points of X as a way of describing $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, X)$.

We have our first example of a representable functor in the category of schemes:

Example 2.5.1. From Proposition 2.1.2 we have that the functor:

$$\begin{aligned} \mathrm{Sch} &\longrightarrow \mathrm{Set} \\ Y &\longrightarrow \mathrm{Hom}(A, \mathscr{O}_Y(Y)) \end{aligned}$$

is represented by $\operatorname{Spec} A$.

Recall from the previous section that the Yoneda embedding implies that in the category of S-schemes:

$$\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathrm{Sch}/S}(X, Y)$$

We denote by Aff/S the category of affine schemes over S, and note that any X scheme yields a functor $h_X^A : \mathrm{Aff}/S \to \mathrm{Set}$ by precomposing with the forgetful functor $\mathrm{Aff}/S \to \mathrm{Sch}/S$. Our first goal is to show that in some nice cases we have a refinement of the Yoneda lemma; that is if if $\mathscr{F}$ is a nice enough contravariant functor $\mathrm{Sch}/S \to \mathrm{Set}$, then there is a natural isomorphism:

$$\mathrm{Nat}(h_X^A, \mathscr{F}^A) \cong \mathscr{F}(X)$$

where $\mathscr{F}^A$ is the contravariant functor $\mathrm{Aff}/S \to \mathrm{Set}$ induced by $\mathscr{F}$. We now have to decide what it means for a functor to be nice enough; since schemes are covered by open affines, the above statement is essentially saying that if we look at how a functor behaves on a bunch open sets, then we can glue this data back together in order to see how the functor behaves on all of Y. This is very similar to statement that a presheaf is a sheaf, however there is one rather large subtlety: the category Sch/S is not a topological space, so how should one define a 'sheaf' on this category? We answer this question with the following discussion.

Recall from Section 1.2 that $\mathscr{C}_X$ was the category of of all open sets on X with morphisms given by inclusion. We want to construct something similar for the category Sch/S, and we do so as follows:

Definition 2.5.1. Let $\mathscr{C}$ be a category with fibre products. A **Grothendieck Topology on** $\mathscr{C}$ is a choice for every object X of a collection $C(X)$ who's elements are sets of morphisms $\{U_i \to X\}_{i \in I}$ satisfying the following:

a) If $Y \to X$ is an isomorphism, then the singleton set $\{Y \to X\}$ belongs to the collection $C(X)$.

b) If $\{U_i \to X\}_{i \in I} \in C(X)$, and $Y \to X$ is a morphism, then the base change $\{U_i \times_X Y \to Y\}_{i \in I}$ belongs to the collection $C(Y)$.

c) If $\{U_i \to X\}_{i \in I}$ is in $C(X)$, and $\{U_{ij} \to U_i\}_{j \in J_i}$ is in $C(U_i)$ for all $i \in I$, then $\{U_{ij} \to X\}_{i \in I, j \in J_i}$ belongs to the collection $C(X)$.

We call $C(X)$ the **coverings of X**. A category equipped with a Grothendieck topology is called a **site**.

There are a few things to note here. We follow the conventions of Vistoli, and Alper in such that we take a Grothendieck topology to be what SGA would call a pre-Grothendieck topology. Moreover, the stacks project definition of a site agrees with ours[42] but defines a Grothendieck topology in terms of sieves,[43] agreeing with the conventions of SGA. We now address the class sized elephant in the room: we have allowed $C(X)$ to be potentially larger than a set and in fact, our axiomatic malpractice goes back further to the discussion of the functor categories $[\mathscr{C}^{\mathrm{op}}, \mathscr{D}]$ and $[\mathscr{C}, \mathscr{D}]$. The problem is that we

[42]The stacks projects does take $C(X)$ to be a set in contradiction to our situation; more on this momentarily.
[43]To be defined later.

would ideally like to index over all objects in a category, and there is nothing in the ZFC axioms which allow us to do such a thing when the category is not small. There are in general two types of fixes to this problem: change the foundations of mathematics, or double and triple check that everything you want to be done can actually be done in ZFC. For an example of the former, the most common fix is to work inside a fixed Grothendieck universe, which is just fancy terminology for a set large enough to do everything one would like to do. In particular, if $\mathfrak{U}$ is a Grothendieck universe, we would define Sch to be the category of schemes in $\mathfrak{U}$, and similarly for other categories. Now every category is small, so every covering of interest as well as the functor categories all become sets. This is the approach taken in Vistoli's notes, as well as SGA. It has the minor drawback that the use of Grothendieck universes means we have to add an axiom to ZFC which says that if X is a set then there is a Grothendieck universe containing X. This ends up being equivalent to the existence of inaccessible cardinals; for some this is an unacceptable thing to assume largely because such an extension of ZFC *strengthens* our set theory, meaning statements which can be made in ZFC but are not provable all of a sudden become provable when one makes this assumption. There are however more sophisticated ways to add more axioms to ZFC which fix the set theoretic problems abound in category theory. Such methods yield axiomatic systems which are *conservative* extensions of ZFC, meaning that any statement that can be made in ZFC is true if and only if it is true in the extension. These fixes are usually spiritually similar to Grothendieck universes, but are more palatable as they do not strengthen our set theory. For an extremely readable, and comprehensive discussion on how one might reformulate the foundations of mathematics to fix these problems, including some more exotic approaches than those outlined above, we recommend Schulman's paper 'Set Theory for Category Theory'.

The second type of fix is taken by the Stacks Project. In this reference, the authors don't allow for categories or sites to be big and instead force everything to be a set. To get around the fact that Sch forms a proper class, they painstakingly show that given a set Sch_0 of schemes, one can always construct a larger set Sch_α containing Sch_0, which is closed under essentially any reasonable property one could wish for. In essence, using the axioms of ZFC, they construct by hand what is essentially a suitable Grothendieck universe[44] for schemes, and then take Sch_α to be the the category of schemes. They then show that for many of the sites of interest, what would be a proper class of coverings $C(X)$ can always be taken to be a set of coverings given by objects in Sch_α. This approach has the immense benefit that algebraic geometry can be done entirely in ZFC, but the obvious drawback of requiring quite a decent amount of effort.

At the end of the day, these notes are focused on geometry, not foundations. As such, with all due respect to the huge efforts of the Stacks Project, and those working on suitable foundations for category theory, we will reproduce none of the above arguments. The reader can assume that we are using whichever fix they find more spiritually comfortable, and look to the aforementioned references for the details. We will thus mention set theoretic size issues sparingly, only ever as a warning to the reader, and proceed as God intended: as if they did not exist. We have the following examples:

Example 2.5.2. Let X be a topological space, and $\mathscr{C}(X)$ the category of open sets with morphism given by inclusion as in Section 1.2. We put a Grothendieck topology on $\mathscr{C}(X)$ by declaring $C(V)$ to be the set of open covers of V. This obviously satisfies $a) - c)$ of Definition 2.5.1, and in this situation the Grothendieck topology on $\mathscr{C}(X)$ is essentially just the topology on X. When X is a scheme we call this site the *small Zariski site.*

Example 2.5.3. We consider the category $\mathrm{Ring}^{\mathrm{op}}$. For each ring A declare we $C(A)$ to consist of sets of the form $\{B_i \to A\}_{i\in I}$ where each morphism $A \to B_i$ satisfies the universal property of localization for some $a_i \in A$, and the set $\{a_i \in A : i \in I\}$ generates the unit ideal. The fact that this declaration satisfies properties $a)$ and $c)$ of Definition 2.5.1 is obvious from Lemma 1.1.4. For property $b)$, we need to show that if $\psi : A \to C$ is a ring homomorphism, and $\{\phi_i : B_i \to A\}_{i\in I}$ is an element of $C(A)$ that $\{\imath_C : B_i \otimes_A C \to C\}_{i\in I}$ is also in $C(C)$. We claim that $\imath_C : C \to B_i \otimes_A C$ satisfies the universal property of localizing at $c_i = \phi(a_i)$. Indeed, suppose that $\alpha : C \to D$ is a morphism of rings such that $\alpha(c_i)$ is invertible. It follows by the universal property of localization that there exists a morphism $B_i \to D$

[44]Technically what they construct is slightly weaker than a Grothendieck universe.

induced by by the morphism $\psi : A \to C$, and hence the universal property of the tensor product gives us a unique morphism $B_i \otimes_A C \to D$ which makes the relevant diagram commute. If the a_i generate the unit ideal, then so do the c_i, implying that our assignment is indeed a Grothendieck topology for $\mathrm{Ring}^{\mathrm{op}}$. We call this site the *Big affine site*, and denote it by $\mathrm{Aff}_{\mathrm{Zar}}$.[45]

Example 2.5.4. We consider the category Sch/S. For each S-scheme we declare $C(X)$ to be the collection of sets $\{f_i : U_i \to X\}_{i \in I}$ such that the $f_i(U_i)$ cover X and each f_i is an open embedding in the category of S schemes. Properties *a*) and *c*) are again obvious, as isomorphisms are open embedding, and the composition of open embedding is an open embedding. The covering property for *c*) is also clear as if $f_{ij}(U_{ij})$ cover U_i for fixed i, then $f_i(f_{ij}(U_{ij}))$ cover X when we index over all i and j. Given a cover $\{f_i : U_i \to X\}_{i \in I}$ and a morphism $g : Y \to X$ of S-schemes, Lemma 2.3.7 implies that the induced map $\pi_{Y_i} : U_i \times_X Y \to Y$ is also an open embedding in Sch/S. The only question is do the sets $\pi_i(U_i \times_X Y)$ cover Y? Well we have that $g^{-1}(X) = Y$, hence if $y \in Y$, then there is some U_i such that $y \in g^{-1}(f_i(U_i))$. However, $g^{-1}(f_i(U_i))$ satisfies the universal property of $U_i \times_X Y$, hence we have a unique isomorphism $\alpha : U_i \times_X Y \to g^{-1}(f_i(U_i))$ making the following diagram commute:

$$\begin{tikzcd}
U_i \times_X Y \arrow[dr, "\alpha"] \arrow[drr, bend left, "\pi_{Y_i}"] \arrow[ddr, bend right] & & \\
& g^{-1}(f_i(U_i)) \arrow[r, "\imath"] \arrow[d] & Y \arrow[d] \\
& U_i \arrow[r, "f_i"] & X
\end{tikzcd}$$

There is thus a unique $x \in U_i \times_X Y$ such that $\alpha(x) = y$, and so $\pi_{Y_i}(x) = y$ as well, implying that the $\pi_{Y_i}(U_i \times_X Y)$ cover Y. Therefore, our assignment description of $C(X)$ satisfies the requirements of Definition 2.5.1 and is thus a Grothendieck topology for Sch/S. We call this site the *Big Zariski site*, denote it by $\mathrm{Sch}/S_{\mathrm{Zar}}$, and call covers in this site *Zariski covers.*

Note that if $\mathscr{C}$ is a site, $\{U_i \to X\}_{i \in I}$ a cover, and $\mathscr{F}$ a contravariant functor $\mathscr{C} \to \mathrm{Set}$ then we get a direct product of maps:

$$(f_i^*) : \mathscr{F}(X) \longrightarrow \prod_i \mathscr{F}(U_i)$$

where $f_i^* = \mathscr{F}(f_i)$. For each $i, j \in I$ we also have two morphisms $\pi_0^{(i_0,i_j)} : U_{i_0} \times_X U_{i_1} \to U_{i_0}$ and $\pi_1^{(i_0,i_1)} : U_{i_0} \times_X U_{i_1} \to U_{i_1}$. We thus get direct product maps:

$$\pi_0^*, \pi_1^* : \prod_{i \in I} \mathscr{F}(U_i) \longrightarrow \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_X U_{i_1})$$

where for $k = 0, 1$:

$$\pi_k^* : (s_i)_{i \in I} \longmapsto \left(\pi_k^{(i_0,i_1)*}(s_{i_k})\right)_{i_k \in I}$$

Putting these maps together we get a diagram of the form:

$$\mathscr{F}(X) \xrightarrow{(f_i^*)} \prod_{i \in I} \mathscr{F}(U_i) \overset{\pi_0^*}{\underset{\pi_1^*}{\rightrightarrows}} \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_X U_{i_1})$$

Note that the fibre product diagram guarantees that $\pi_0^* \circ (f_i^*) = \pi_1^* \circ (f_i^*)$. We say that such a diagram is 'exact' if $(f_i)^*$ is injective, and if (s_i) satisfies $\pi_0^*((s_i)) = \pi_1^*((s_i))$, then there is some $s \in \mathscr{F}(X)$ such that $(f_i^*)(s) = (s_i)$. Note that if $\mathscr{F}$ takes values in Ab then this is equivalent to saying that:

$$\mathscr{F}(X) \xrightarrow{(f_i^*)} \prod_{i \in I} \mathscr{F}(U_i) \overset{\pi_0^*}{\underset{\pi_1^*}{\rightrightarrows}} \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_X U_{i_1})$$

[45]This is not standard terminology as far as we can tell.

is exact in the traditional sense.

Definition 2.5.2. Let $\mathscr{C}$ be a site, then a **presheaf** is a contravariant functor $\mathscr{C} \to \text{Set}$. A **sheaf** is a presheaf $\mathscr{F}$ such that for any covering $\{f_i : U_i \to X\}_{i \in I}$ we have an 'exact' sequence:

$$\mathscr{F}(X) \xrightarrow{(f_i^*)} \prod_{i \in I} \mathscr{F}(U_i) \underset{\pi_1^*}{\overset{\pi_0^*}{\rightrightarrows}} \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_X U_{i_1})$$

Example 2.5.5. Let X be a topological space, and give $\mathscr{C}(X)$ the Grothendieck topology from Example 2.5.2. We note that $U_i \times_V U_j = U_i \cap U_j$ for any open subsets U_i, U_j of $V \subset X$, and moreover that the maps f_i^* are simply restriction maps $\theta^V_{U_i}$. The maps $\pi_k^{(i_0,i_1)*}$ are the restriction maps $\theta^{U_{i_k}}_{U_{i_0} \cap U_{i_1}}$. Unraveling the definitions, one finds that the relevant sequence being exact is equivalent to the conditions laid out in Definition 1.2.2. In other words, sheaves on topological spaces are exactly sheaves on the site $\mathscr{C}(X)$.

Example 2.5.6. We consider $\text{Sch}/S_{\text{Zar}}$, and claim that the presheaf h_X is a sheaf. Indeed, let $\{f_i : U_i \to Y\}_{i \in I}$ be a Zariski cover of an S scheme, then we need to show that:

$$\text{Hom}_{\text{Sch}/S}(Y, X) \xrightarrow{(f_i^*)} \prod_{i \in I} \text{Hom}_{\text{Sch}/S}(U_i, X) \underset{\pi_1^*}{\overset{\pi_0^*}{\rightrightarrows}} \prod_{i_0, i_1 \in I} \text{Hom}_{\text{Sch}/S}(U_{i_0} \times_Y U_{i_1}, X)$$

is exact. Let $g, h : X \to Y$ be morphisms of schemes, then injectivity of the first map is equivalent to the statement: if $g \circ f_i = h \circ f_i$ for all i then $g = h$. If $g \circ f_i = h \circ f_i$ then we have that $g|_{f(U_i)} = h|_{f(U_i)}$; since the $f(U_i)$ are open sets which cover Y, we thus have that g must be equal to h and so (f_i^*) is injective. Now let $(g_i) \in \prod_{i \in I} \text{Hom}_{\text{Sch}/S}(U_i, X)$ satisfy $\pi_0^*((g_i)) = \pi_1^*((g_i))$. This implies that:

$$\left(\pi_0^{(i_0,i_j)*}(g_{i_0})\right) = \left(\pi_1^{(i_0,i_j)*}(g_{i_1})\right)$$

This implies that for all $i_0, i_1 \in I$, we have:

$$g_{i_0} \circ \pi_0^{(i_0,i_1)} = g_{i_1} \circ \pi_1^{(i_0,i_1)}$$

Let α_{i_k} be the isomorphism $f_{i_k}(U_{i_k}) \to U_{i_k}$, then we have that:

$$g_{i_0} \circ \alpha_{i_0} \circ \alpha_{i_0}^{-1} \circ \pi_0^{(i_0 i_1)} = g_{i_1} \circ \alpha_{i_1} \circ \alpha_{i_1}^{-1} \circ \pi_1^{(i_0 i_1)}$$

We have the following commutative diagram:

$$\begin{tikzcd}
U_{i_0} \times_Y U_{i_1} \arrow[dr, "\beta_{i_0,i_1}"] \arrow[drr, bend left, "\alpha_{i_1}^{-1} \circ \pi_1^{(i_0,i_1)}"] \arrow[ddr, bend right, "\alpha_{i_1}^{-1} \circ \pi_1^{(i_0,i_1)}"'] & & \\
& f_{i_0}(U_{i_0}) \cap f_{i_1}(U_{i_1}) \arrow[r, "\imath_{i_1}"] \arrow[d, "\imath_{i_0}"] & f_{i_1}(U_{i_1}) \arrow[d] \\
& f_{i_0}(U_{i_0}) \arrow[r] & Y
\end{tikzcd}$$

hence:

$$g_{i_0} \circ \alpha_{i_0} \circ \imath_{i_0} \circ \beta_{i_0,i_1} = g_{i_1} \circ \alpha_{i_1} \circ \imath_{i_1} \circ \beta_{i_0,i_j}$$

Since β_{i_0,i_1} is an isomorphism, we have that:

$$g_{i_0} \circ \alpha_{i_0} \circ \imath_{i_0} = g_{i_1} \circ \alpha_{i_1} \circ \imath_{i_1}$$

hence:

$$g_{i_0} \circ \alpha_{i_0}|_{f_{i_0}(U_{i_0}) \cap f_{i_1}(U_{i_1})} = g_{i_1} \circ \alpha_{i_1}|_{f_{i_0}(U_{i_0}) \cap f_{i_1}(U_{i_1})}$$

It follows that the $g_i \circ \alpha_i$ glue together to give a morphism $g : Y \to X$. This morphism is obviously an S scheme morphism by construction, and satisfies:

$$g \circ f_i = g|_{f_i(U_i)} \circ f_i = g_i \circ \alpha_i \circ f_i = g_i$$

hence $(f_i^*)(g) = (g_i)$ as desired. It follows that for any Zariski cover $\{U_i \to Y\}_{i \in I}$ the corresponding sequence induced by h_X is exact, and so h_X is a sheaf as desired. In particular, an representable functor is itself a sheaf.

As it turns out the sheaf condition on $\mathscr{F}$ is exactly what we need in order to obtain the desired refinement of the Yoneda lemma. Recall that if X is an S scheme, and $\mathscr{F}$ a contravariant functor, now called a presheaf, then h_X^A and $\mathscr{F}^A$ were contravariant functors $\mathrm{Aff}/S \to \mathrm{Set}$ given by precomposition with the forgetful functor. We now have the following:

Lemma 2.5.1. *Let $\mathscr{F}$ be a sheaf on the big Zariski site $\mathrm{Sch}/S_{\mathrm{Zar}}$, and X an S-scheme, then:*

$$\mathrm{Nat}(h_X^A, \mathscr{F}^A) \cong \mathscr{F}(X)$$

Proof. By Lemma 2.4.2, it suffices to show an isomorphism:

$$\mathrm{Nat}(h_X^A, \mathscr{F}^A) \cong \mathrm{Nat}(h_X, \mathscr{F})$$

Clearly, any any natural transformation $F : h_X \to \mathscr{F}$ induces a natural transformation $G : h_X^A \to \mathscr{F}^A$ by only considering affine schemes. Now let $G : h_X^A \to \mathscr{F}^A$ be a natural transformation, then we need to define a natural transformation $F : h_X \to \mathscr{F}$. This is equivalent to constructing the following diagram for any S scheme morphism $f : Y \to Z$:

$$\begin{tikzcd} h_X(Z) \arrow[r, "F_Z"] \arrow[d, "f^*"] & \mathscr{F}(Z) \arrow[d, "\mathscr{F}(f)"] \\ h_X(Y) \arrow[r, "F_Y"] & \mathscr{F}(Y) \end{tikzcd}$$

Let $\{U_i\}_{i \in I}$ be an open affine cover of Z, and $\{W_{ij}\}$ be an open affine cover of Y such that $f(W_{ij}) \subset U_i$. We have the following diagram:

$$\begin{tikzcd} h_X(Z) \arrow[r, "(\imath_i^*)"] & \prod_{i \in I} h_X(U_i) \arrow[r, shift left, "\pi_0^*"] \arrow[r, shift right, "\pi_1^*"'] \arrow[d, "(G_{U_i})"] & \prod_{i_0, i_1 \in I} h_Y(U_{i_0} \cap U_{i_1}) \\ \mathscr{F}(Z) \arrow[r, "(\imath_i^*)"] & \prod_{i \in I} \mathscr{F}(U_i) \arrow[r, shift left, "\pi_0^*"] \arrow[r, shift right, "\pi_1^*"'] & \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \cap U_{i_1}) \end{tikzcd}$$

Let $g \in h_X(Z)$, and consider the element $(g_i) = (\imath_i^*)(g) = (g|_{U_i})$. By definition we have that $\pi_0^*((g_i)) = \pi_1^*((g_i))$, and we want to show that $\pi_0^*((G_{U_i}(g_i))) = \pi_1^*((G_{U_i}(g_i)))$. For all $i_0, i_1 \in I$ it thus suffices to show that:

$$\pi_0^{(i_0,i_1)*} \circ G_{U_{i_0}}(g_{i_0}) = \pi_1^{(i_0,i_1)*} \circ G_{U_{i_1}}(g_{i_1})$$

We can cover $U_{i_0} \cap U_{i_1}$ with open affine schemes $\{V_j\}_{j \in J}$, and obtain inclusion maps $\imath_j : V_j \to U_{i_0} \cap U_{i_1}$. Note that $\imath_j^* \circ \pi_k^{(i_0,i_1)*}(g_{i_k}) = g|_{V_j}$; since G is a natural transformation we have the following diagram:

$$\begin{tikzcd} h_X(U_{i_k}) \arrow[r, "G_{U_{i_k}}"] \arrow[d, "\imath_j^* \circ \pi_k^{(i_0,i_1)*}"] & \mathscr{F}(U_{i_k}) \arrow[d, "\imath_j^* \circ \pi_k^{(i_0,i_1)*}"] \\ h_X(V_j) \arrow[r, "G_{V_j}"] & \mathscr{F}(V_j) \end{tikzcd}$$

so we obtain the following equality:

$$G_{V_j}\left(\imath_j^*\pi_0^{(i_0,i_1)*}(g_{i_0})\right) = G_{V_j}\left(\imath_j^*\pi_1^{(i_0,i_1)*}(g_{i_1})\right)$$

which by the commutativity of the diagram yields:

$$\imath_j^* \circ \pi_0^{(i_0,i_1)*} \circ G_{U_{i_0}}(g_{i_0}) = \imath_j^* \circ \pi_0^{(i_0,i_1)*} \circ G_{U_{i_1}}(g_{i_1})$$

for all $j \in J$. Since the map $(\imath_j^*)$ is injective, we thus have that:

$$\pi_0^{(i_0,i_1)*} \circ G_{U_{i_0}}(g_{i_0}) = \pi_1^{(i_0,i_1)*} \circ G_{U_{i_1}}(g_{i_1})$$

as desired. Since $\mathscr{F}$ is a sheaf this yields an element $x \in \mathscr{F}(Z)$ such that $\imath_i^*(x) = G_{U_i}(g_i)$. We thus define a set map $h_X(Z) \to \mathscr{F}(Z)$ by $g \mapsto x$, where x is obtained via the procedure outlined above. We need to check that the following diagram commutes:

$$\begin{tikzcd}
h_X(Z) \arrow[r, "F_Z"] \arrow[d, "f^*"] & \mathscr{F}(Z) \arrow[d, "f^*"] \\
h_X(Y) \arrow[r, "F_Y"] & \mathscr{F}(Y)
\end{tikzcd}$$

Let $g \in h_X(Z)$ then we need to show that $F_Y(g \circ f) = \mathscr{F}(f)(F_Z(g))$. If we restrict to an affine scheme $W_{ij} \subset Y$, then by construction we know that $\imath_{ij}^*(F_Y(g \circ f)) = G_{W_{ij}}(g \circ f|_{W_{ij}})$. Meanwhile, we have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{F}(Z) \arrow[r, "\imath_i^*"] \arrow[d, "f^*"] & \mathscr{F}(U_i) \arrow[d, "f|_{W_{ij}}^*"] \\
\mathscr{F}(Y) \arrow[r, "\imath_{ij}^*"] & \mathscr{F}(W_{ij})
\end{tikzcd}$$

hence:

$$\imath_{ij}^*(\mathscr{F}(f)(F_Z(g))) = \mathscr{F}(f|_{W_{ij}})(G_{U_i}(g|_{U_i}))$$

The commutative diagram from G:

$$\begin{tikzcd}
h_X(U_i) \arrow[r, "G_{U_i}"] \arrow[d, "f|_{W_{ij}}^*"] & \mathscr{F}(U_i) \arrow[d, "f|_{W_{ij}}^*"] \\
h_X(W_{ij}) \arrow[r, "G_{W_{ij}}"] & \mathscr{F}(W_{ij})
\end{tikzcd}$$

then tells us that:

$$\imath_{ij}^*(F_Y(g \circ f)) = G_{W_{ij}}(g \circ f|_{W_{ij}}) = \mathscr{F}(f|_{W_{ij}})(G_{U_i}(g|_{U_i})) = \imath_{ij}^*(\mathscr{F}(f)(F_Z(g)))$$

Since this holds for all W_{ij}, and since $\mathscr{F}$ is a sheaf, we have that the relevant diagram commutes, and so the assignment $Z \mapsto F_Z$ defines a natural transformation.

By construction if we restrict F to affine schemes we get G. If we start with F and restrict to affine schemes in order to get a functor G, and then apply the process above to get a new functor F' we will have that F and F' will have to agree on affines. Since $\mathscr{F}$ is a sheaf they will then have to agree, and so we obtain a natural isomorphism:

$$\mathrm{Nat}(h_X^A, \mathscr{F}^A) \cong \mathrm{Nat}(h_X, \mathscr{F})$$

The desired result then follows by Yoneda's lemma (Lemma 2.4.2). □

An immediate corollary is a refinement of the Yoneda embedding:

Corollary 2.5.1. *Let X and Y be S schemes, then there is a natural isomorphism:*

$$\mathrm{Nat}(h_X^A, h_Y^A) \cong \mathrm{Hom}_{\mathrm{Sch}/S}(X, Y)$$

In particular, to give an S-scheme morphism $X \to Y$, it suffices to define a natural map from the A points of X to the A points of Y for all affine S-schemes $\operatorname{Spec} A$.

Our next goal is to derive a representability criterion for a presheaf $\mathscr{F}$. As mentioned earlier, any representable functor would have to be sheaf on the big Zariski by virtue of being naturally isomorphic to h_X for some X, however not every sheaf need be representable. We will need the following definition:

Definition 2.5.3. Let $\mathscr{F}$ and $\mathscr{G}$ be contravariant functors $\mathrm{Sch}/S \to \mathrm{Set}$, then we employ the following terminology:

a) We call $\mathscr{G}$ a **subfunctor** of $\mathscr{F}$ if there exists a natural transformation $\imath : \mathscr{G} \to \mathscr{F}$ such that for all objects X, $\imath_X : \mathscr{G}(X) \to \mathscr{F}(X)$ is an inclusion map. In particular, $\mathscr{G}(X) \subset \mathscr{F}(X)$ for all objects X.

b) We say that $\mathscr{G}$ is an **open subfunctor** of $\mathscr{F}$, if for all pairs (X, s) where $s \in \mathscr{F}(X)$, there exists an open subscheme $U_s \subset X$ such that a morphism $f : Y \to X$ is in $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, U_s)$ if and only if $\mathscr{F}(f)(s) \in \mathscr{G}(Y)$.

c) A set $\{\mathscr{G}_i\}_{i \in I}$ is a **covering** of $\mathscr{F}$ if for all pairs (X, s) where $s \in \mathscr{F}(X)$ there exists an open covering of $\{U_{s_i}\}_{i \in I}$ of X such that $s|_{U_{s_i}} \in \mathscr{G}_i(U_{s_i})$.[46]

Note that *a*) makes sense in any category, and it is only *b*) and *c*) which depends on the domain category of $\mathscr{F}$ and $\mathscr{G}$ to be Sch/S.

Example 2.5.7. We consider Sch/S; if Z is an S scheme and $U \subset Z$ is an open subscheme we claim that h_U is an open subfunctor of h_Z. Clearly h_U is a subfunctor of h_Z; let X be another S scheme, $g \in h_Z(X)$, and $U_g = g^{-1}(U)$. Let $f : Y \to X$, and suppose that f is actually in $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, U_g)$, i.e. f has image contained in the open subscheme U_g. We see that $f^*g = g \circ f$ then has image contained in U hence $g \circ f \in h_U(Y)$. Conversely, suppose that $g \circ f \in h_U(Y)$, then $g \circ f(Y) \subset U$ hence $f(Y) \subset g^{-1}(U)$. It follows that $f \in \mathrm{Hom}_{\mathrm{Sch}/S}(Y, U_g)$ so h_U is an open subfunctor.

Suppose that $\{U_i\}$ is an open cover of Z, and let $g \in h_Z(X)$. Then $g^{-1}(U_i)$ cover X, and $g|_{g^{-1}(U_i)} \in \mathrm{Hom}_{\mathrm{Sch}/S}(g^{-1}(U_i), U_i)$ so the open subfunctors h_{U_i} form a cover of h_Z.

The following easy lemma cements the above connection between open subschemes of Z, and open subfunctors of h_Z.

Lemma 2.5.2. *Let Z be an S-scheme, and $\mathscr{G}$ a subfunctor of h_Z. Then $\mathscr{G}$ is an open subfunctor if and only if $\mathscr{G} = h_U$ for some open subscheme $U \subset Z$.*

Proof. If $\mathscr{G} = h_U$ for some open subscheme, then $\mathscr{G}$ is an open subfunctor by Example 2.5.7.

Suppose that $\mathscr{G}$ is an open subfunctor of h_Z, and set:

$$U = U_{\mathrm{Id}_Z}$$

i.e. we let U be the open subscheme of Z such that a morphism $f : Y \to Z$ is in $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, U)$ if and only if $f \in \mathscr{G}(Y)$. We need to show that $\mathscr{G} = h_U$, however this is now true by definition. $\square$

[46]By $s|_{U_i}$ we mean $\mathscr{F}(\imath_{U_i})(s)$ where $\imath_{U_i}$ is the open embedding.

With the above notion of an open subfunctor we can now state the following representability criterion:

Proposition 2.5.1. *Let $\mathscr{F}$ be a contravariant functor* $\mathrm{Sch}/S \to \mathrm{Set}$ *such that:*

a) *$\mathscr{F}$ is a sheaf in the Zariski topology on* Sch/S.

b) *There exists a cover $\{\mathscr{F}_i\}$ of $\mathscr{F}$ by open subfunctors of $\mathscr{F}$.*

c) *Each $\mathscr{F}_i$ is representable*

then $\mathscr{F}$ is representable.

Proof. As each $\mathscr{F}_i$ is representable, Lemma 2.4.4 implies that for each i there exists a universal object (X_i, ξ_i), and moreover that $\mathscr{F}_i \cong h_{X_i}$. The goal is to glue these X_i's together to obtain an S-scheme X which represents $\mathscr{F}$. We need to first determine open subschemes $U_{ij} \subset X_i$ and isomorphisms $\phi_{ij} : U_{ij} \to U_{ji}$ satisfying the cocycle condition. Consider $\xi_i \in \mathscr{F}_i(X_i) \subset \mathscr{F}(X_i)$, then since $\mathscr{F}_j \subset \mathscr{F}$ is open, there exists and open subscheme $U_{ij} \subset X_i$ such that a morphism $f : Y \to X_i$ is in $\mathrm{Hom}_{\mathrm{Sch}/S}(Y, U_{ij})$ if and only if $\mathscr{F}(f)(\xi_i) \in \mathscr{F}_j(Y)$. In particular, if $\imath_{ij}$ is the open embedding $U_{ij} \hookrightarrow X_i$ then we obtain that $\mathscr{F}(\imath_{ij})(\xi_i) = \xi_i|_{U_{ij}} \in \mathscr{F}_j(U_{ij})$. Since (X_j, ξ_j) is a universal object, there exists a unique morphism $\tilde{\phi}_{ij} : U_{ij} \to X_j$ such that $\mathscr{F}_j(\phi_{ij})(\xi_j) = \xi_i|_{U_{ij}}$. Since $\mathscr{F}_j$ is a subfunctor, it follows that $\mathscr{F}(\phi_{ij})(\xi_j) = \xi_i|_{U_{ij}} \in \mathscr{F}_i(U_{ij})$ hence by the defining property of U_{ji}, we must have that $\tilde{\phi}_{ij} \in \mathrm{Hom}_{\mathrm{Sch}/S}(U_{ij}, U_{ji})$. In order to be careful about codomains, we will denote the morphism $\tilde{\phi}_{ij}$ to be the original morphism $U_{ij} \to X_j$, and have ϕ_{ij} be the unique morphism $U_{ij} \to U_{ji}$ such that $\tilde{\phi}_{ij} = \imath_{ji} \circ \phi_{ij}$.

We now whish to show that $\phi_{ji} \circ \phi_{ij} = \mathrm{Id}_{U_{ij}}$. Note that $\xi_i|_{U_{ij}} = \mathscr{F}(f)(\xi_i)$ where $\imath_{ij} : U_{ij} \to X_i$ is the open embedding; similarly let $\imath_{ji} : U_{ji} \to X_j$ be the open embedding. Viewing ϕ_{ij} and ϕ_{ji} strictly as morphisms $U_{ij} \to U_{ji}$ and vice versa, we have that $\imath_{ij} \circ \phi_{ji}$ is $\tilde{\phi}_{ji}$, and $\imath_{ji} \circ \phi_{ij}$ is $\tilde{\phi}_{ij}$. We have that:

$$\begin{aligned}
\mathscr{F}(\phi_{ji} \circ \phi_{ij})(\xi_i|_{U_{ij}}) =& \mathscr{F}(\imath_{ij} \circ \phi_{ji} \circ \phi_{ij})(\xi_i) \\
=& \mathscr{F}(\phi_{ij}) \circ \mathscr{F}(\imath_{ij} \circ \phi_{ji})(\xi_i) \\
=& \mathscr{F}(\phi_{ij})(\xi_j|_{U_{ji}}) \\
=& \mathscr{F}(\imath_{ji} \circ \phi_{ij})(\xi_j) \\
=& \xi_i|_{U_{ij}}
\end{aligned}$$

Since the identity map $U_{ij} \to U_{ij}$ also satisfies $\mathscr{F}(\imath_{ij} \circ \mathrm{Id})(\xi) = \xi_i|_{U_{ij}}$, and ξ_i is a universal object we must have that $\imath_{ij} \circ \mathrm{Id} = \imath_{ij} \circ (\phi_{ji} \circ \phi_{ij})$. Since $\imath_{ij}$ is a monomorphism it follows that $\mathrm{Id} = \phi_{ji} \circ \phi_{ij}$ as desired. The same argument shows that $\phi_{ij} \circ \phi_{ji} = \mathrm{Id}$ hence the ϕ_{ij} are isomorphisms of schemes satisfying $\phi_{ij}^{-1} = \phi_{ji}$.

We need to show that $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$. Let $f : Y \to U_{ji}$ be any morphism, then $g = \phi_{ji} \circ f$ is a morphism $Y \to U_{ij}$. Suppose that f has image in $\phi_{ij}(U_{ij} \cap U_{ik})$, then g necessarily has image in U_{ik}. Conversely, suppose that g has image in U_{ik}, then $\phi_{ji}(f(Y)) \subset U_{ik}$, hence $f(Y) \subset \phi_{ji}^{-1}(U_{ik}) = \phi_{ij}(U_{ij} \cap U_{ik})$. Now, $\imath_{ij} \circ g$ is a morphism $Y \to X_i$, and by the definition of U_{ik}, we know that $\imath_{ij} \circ g$, has image in U_{ik}, if and only if $\mathscr{F}(\imath_{ij} \circ g)(\xi_i) \in \mathscr{F}_k(Y)$. Note this means that g has image U_{ik} if and only if $\mathscr{F}(\imath_{ij} \circ g)(\xi_i) \in \mathscr{F}_k(Y)$. Now we have that:

$$\mathscr{F}(\imath_{ij} \circ g)(\xi_i) = \mathscr{F}(\imath_{ij} \circ \phi_{ji} \circ g)(\xi_i) = \mathscr{F}(f)(\xi_j|_{U_{ji}}) = \mathscr{F}(\imath_{ji} \circ f)(\xi_j)$$

thus g has image in U_{ik} if and only if $\mathscr{F}(\imath_{ji} \circ f)(\xi_j) \in \mathscr{F}_k(Y)$. This implies that g has image in U_{ik} if and only if $\imath_{ji} \circ f$ has image in U_{jk}, i.e if and only if f has image in U_{jk}. By tracing the chain of if and only if statements, we have shown that:

$$f(Y) \subset \phi_{ij}(U_{ij} \cap U_{ik}) \Leftrightarrow f(Y) \subset U_{ji} \cap U_{jk}$$

By taking f to be the inclusion of a point, we have thus shown that $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ as desired.

We demonstrate the cocycle condition, i.e. that $\phi_{jk} \circ \phi_{ij}|_{U_{ijk}} = \phi_{ik}|_{U_{ijk}}$, where $U_{ijk} = U_{ij} \cap U_{ik}$. Restricting to U_{ijk} is the same as precomposing with the embedding $\imath_{ijk} : U_{ijk} \to U_{ij}$; consider the following:

$$\begin{aligned}
\mathscr{F}(\imath_{kj} \circ \phi_{jk} \circ \phi_{ij} \circ \imath_{ijk})(\xi_k) =& \mathscr{F}(\phi_{ij} \circ \imath_{ijk})(\xi_j|_{U_{jk}}) \\
=& \mathscr{F}(\imath_{jk} \circ \phi_{ij} \circ \imath_{ijk})(\xi_j)
\end{aligned}$$

We claim that $\imath_{jk}\circ\phi_{ij}\circ\imath_{ijk} = \imath_{ji}\circ\phi_{ij}\circ\imath_{ijk}$. Indeed, $\phi_{ij}(U_{ij}\cap U_{ik}) = U_{ji}\cap U_{jk}$, while $\imath_{jk}|_{U_{ji}\cap U_{jk}} = i_{ji}|_{U_{ji}\cap U_{jk}}$ as they are both inclusion maps, hence the claimed equality. It follows that:

$$\begin{aligned}\mathscr{F}(\imath_{kj}\circ\phi_{jk}\circ\phi_{ij}\circ\imath_{ijk})(\xi_k) =&\mathscr{F}(\imath_{ji}\circ\phi_{ij}\circ\imath_{ijk})(\xi_j)\\ =&\mathscr{F}(\imath_{ijk})(\xi_i|_{U_{ij}})\\ =&\xi_i|_{U_{ijk}}\end{aligned}$$

Meanwhile:

$$\mathscr{F}(\imath_{ki}\circ\phi_{ik}\circ\imath_{ijk})(\xi_k) = \mathscr{F}(\imath_{ijk})(\xi_i|_{U_{ik}}) = \xi_i|_{U_{ijk}}$$

It follows that:

$$\imath_{ki}\circ\phi_{ik}\circ\imath_{ijk} = i_{kj}\circ\phi_{jk}\circ\phi_{ij}\circ\imath_{ijk}$$

We show that $\imath_{ki}\circ\phi_{ik}\circ\phi_{ijk} = \imath_{kj}\circ\phi_{ik}\circ\phi_{ikj}$; from our work above, $\phi_{ik}(U_{ij}\cap U_{ik}) = U_{kj}\cap U_{ki}$, and $\imath_{ik}|_{U_{kj}\cap U_{ki}} = \imath_{kj}|_{U_{kj}\cap U_{ki}}$ as they are both inclusion maps, hence the claimed equality. It follows that:

$$\imath_{kj}\circ\phi_{ik}\circ\imath_{ijk} = i_{kj}\circ\phi_{jk}\circ\phi_{ij}\circ\imath_{ijk}$$

Since $\imath_{kj}$ is a monomorphism, the cocycle condition is satisfied.

Now Theorem 2.2.1 implies that the X_i glue together to form a scheme X together with open embeddings $\psi_i : X_i \to X$ such that the $\psi_i(X_i)$ cover X, $\psi_i(U_{ij}) = \psi_i(X_i)\cap\psi_j(X_j)$, and $\psi_i|_{U_{ij}} = \psi_j|_{U_{ji}}\circ\phi_{ij}$. We wish to define a universal object $\xi\in\mathscr{F}(X)$; denote by ψ_i^{-1} the isomorphism $\psi(X_i)\to X_i$, and set $\xi_i' = \mathscr{F}(\psi_i^{-1})(\xi_i)$. Then we claim that:

$$\xi_i'|_{\psi_i(X_i)\cap\psi_j(X_j)} = \xi_j'|_{\psi_i(X_i)\cap\psi_j(X_j)}$$

We see that:

$$\begin{aligned}\xi_i'|_{\psi_i(X_i)\cap\psi_j(X_j)} =&\mathscr{F}(\imath_{\psi_i(X_i)\cap\psi_j(X_j)})(\xi_i')\\ =&\mathscr{F}(\psi_i^{-1}\circ\imath_{\psi_i(U_{ij})})(\xi_i)\end{aligned}$$

We denote by $\psi_i^{-1}|_{\psi_i(U_{ij})}$ the unique morphism $\psi_i(U_ij)\to U_{ij}$ satisfying $\imath_{ij}\circ\psi_i^{-1}|_{\psi_i(U_{ij})} = \psi_i^{-1}\circ\imath_{\psi_i(U_{ij})}$. We then have that that $\psi_i|_{U_{ij}}$ has image $\psi_i(U_{ij})$, hence:

$$\psi_i^{-1}|_{\psi_i(U_[ij])}\circ\psi_i|_{U_{ij}} = \mathrm{Id}_{U_{ij}}$$

it follows that:

$$\mathrm{Id}_{U_{ij}} = \psi_i^{-1}|_{\psi_i(U_{ij})}\circ\psi_j|_{U_{ji}}\circ\phi_{ij}$$

By judiciously applying inverses, we obtain:

$$\phi_{ji} = \psi_i^{-1}|_{\psi_i(U_{ij})}\circ\psi_j|_{U_{ji}} \Rightarrow \phi_{ji}\circ\psi_j^{-1}|_{\psi_j(U_{ji})} = \psi_i^{-1}|_{\psi_i(U_{ij})}$$

We thus see:

$$\begin{aligned}\xi_i'|_{\psi_i(X_i)\cap\psi_j(X_j)} =&\mathscr{F}(\imath_{ij}\circ\phi_{ji}\circ\psi_j^{-1}|_{\psi_j(U_{ji})})(\xi_i)\\ =&\mathscr{F}(\psi_j^{-1}|_{\psi_j(U_{ji})})(\xi_j|_{U_{ji}})\\ =&\mathscr{F}(\imath_{ji}\circ\psi_j^{-1}|_{\psi_j(U_{ji})})(\xi_j)\\ =&\xi_j'|_{\psi_i(X_i)\cap\psi_j(X_j)}\end{aligned}$$

Since $\mathscr{F}$ is a sheaf, and the $\psi_i(X_i)$ cover X we obtain a unique element $\xi\in\mathscr{F}(X)$ such that $\xi|_{\psi_i(X_i)} = \xi_i'$. In particular, since the ψ_i are isomorphisms $X_i\to\psi_i(X_i)$, we have that $(\psi_i(X_i),\xi_i')$ is a universal object for $\mathscr{F}_i$.

We want to show that (X, ξ) is a universal object for $\mathscr{F}$, as then Lemma 2.4.4 will imply that $\mathscr{F}$ is representable. Let Y be any scheme, and $\alpha \in \mathscr{F}(Y)$. Since that $\mathscr{F}_i$ cover $\mathscr{F}$, there exists an open covering $\{V_i\}$ of Y such that $\alpha|_{V_i} \in \mathscr{F}_i(V_i)$. Since $(\psi_i(X_i), \xi_i')$ is a universal object for $\mathscr{F}_i$ there is a unique $f_i : V_i \to \psi_i(X_i)$ such that $\mathscr{F}_i(f_i)(\xi_i') = \alpha|_{V_i}$. We would like to show that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, where here we denote by $f_i|_{V_i \cap V_j}$ the morphism $\imath_{\psi_i(X_i)} \circ f_i \circ \imath_{V_i \cap V_j} : V_i \cap V_j \to X$, with $\imath_{\psi_i(X_i)}$ and $\imath_{V_i \cap V_j}$ the obvious open embeddings. First, note that obviously $\psi_i(U_{ij})$ and ξ_i' satisfy the same property as U_{ij} and ξ_i:

$$g : Y \to \psi_i(X_i) \in \mathrm{Hom}_{\mathrm{Sch}/S}(Y, \psi_i(U_{ij})) \Longleftrightarrow \mathscr{F}(g)(\xi_i') \in \mathscr{F}_j(Y)$$

Now since $\mathscr{F}(f_i \circ \imath_{V_i \cap V_j})(\xi_i') = \alpha|_{V_i \cap V_j}$, and $\alpha|_{V_i \cap V_j} \in \mathscr{F}_j(V_i \cap V_j)$, we must have that $f_i \circ \imath_{V_i \cap V_j}$ has image contained in $\psi_i(U_{ij})$. In particular, we have that $f_i \circ \imath_{V_i \cap V_j}$ then factors uniquely as:

$$f_i \circ \imath_{V_i \cap V_j} = \imath_{\psi_i(U_{ij})} \circ g_{ij}$$

where $g_{ij} : V_i \cap V_j \to \psi_i(U_{ij})$, and $\imath_{\psi_i(U_{ij})} : \psi_i(U_{ij}) \to \psi_i(X_i)$. There is a similar statement for g_{ji}, where we note that $\psi_{ji}(U_{ji}) = \psi_{ij}(U_{ij})$, but the embedding $\imath_{\psi_j(U_{ji})}$ is technically different as it has image a priori in $\psi_j(X_j)$. Of course, we have the equality:

$$\imath_{\psi_i(X_i)} \circ \imath_{\psi_i(U_{ij})} = \imath_{\psi_j(X_j)} \circ \imath_{\psi_j(U_{ji})}$$

hence it suffices to show that $g_{ij} = g_{ji}$. We have that:

$$\mathscr{F}(g_{ij})(\xi_i'|_{\psi_i(U_i j)}) = \mathscr{F}(\imath_{\psi_i(U_{ij})} \circ g_{ij})(\xi_i') = \alpha|_{V_i \cap V_j}$$

Meanwhile, we have that since $\xi_i'|_{\psi_i(U_{ij})} = \xi_j'|_{\psi_j(U_{ji})}$:

$$\mathscr{F}(g_{ji})(\xi_i'|_{\psi_i(U_{ij})}) = \mathscr{F}(g_{ji})(\xi_j'|_{\psi_j(U_{ji})}) = \mathscr{F}(i_{\psi_j(U_{ji})} \circ g_{ij})(\xi_j') = \alpha|_{V_i \cap V_j}$$

Since ξ_i' is a universal object, we must have that $\imath_{\psi_i(U_{ij})} \circ g_{ij} = \imath_{\psi_i(U_{ij})} \circ g_{ji}$, implying that $g_{ij} = g_{ji}$ as open embeddings are monomorphisms. It follows that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ and so the morphisms glue to yield a morphism $f : Y \to X$ which obviously satisfies $\mathscr{F}(f)(\xi) = \alpha$. This morphism is unique, as any other such morphism has to agree with f when restricted to V_i. The pair (X, ξ) is therefore a universal object, and $\mathscr{F}$ is representable as desired. $\square$

This proposition can be best understood as follows: if you want to show a sheaf is a scheme, give it a Zariski atlas, preferably an affine one. Almost every functor we shall encounter in the section will be representable by an affine scheme, hence we will not have much need for this result yet. However, we do have one example already available to us:

Example 2.5.8. Let $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ be functors from some category $\mathscr{C} \to \mathrm{Set}$. Let F and G be natural transformations $\mathscr{F} \to \mathscr{H}$ and $\mathscr{G} \to \mathscr{H}$, then we define the *fibre product* of functors to be:

$$(\mathscr{F} \times_{\mathscr{H}} \mathscr{G})(X) = \{(a, b) \in \mathscr{F}(X) \times \mathscr{G}(X) : F_X(a) = G_X(b)\}$$

Note that if $\mathscr{C} = \mathrm{Sch}$, $\mathscr{F} = h_X$, $\mathscr{G} = h_Y$ and $\mathscr{H} = h_Z$, then the Yoneda lemma tells us that the natural transformations $h_X \to h_Z$ and $h_Y \to h_Z$ come from morphisms $f : X \to Z$, and $g : Y \to Z$ given by post composition. It follows that:

$$(h_X \times_{h_Z} h_Y)(Q) = \{(\alpha, \beta) \in \mathrm{Hom}_{\mathrm{Sch}}(Q, X) \times \mathrm{Hom}_{\mathrm{Sch}}(Q, Y) : f \circ \alpha = g \circ \beta\}$$

In other words, the functor[47] $h_X \times_{h_Z} h_Y$ parameterizes pairs of morphisms $Q \to X$ and $Q \to Y$ which agree when post composed with f and g. Now we spent the entirety of Section 2.3 proving the existence of a scheme $X \times_Z Y$ such that there was a natural bijection:

$$\mathrm{Hom}_{\mathrm{Sch}}(Q, X \times_Z Y) \leftrightarrow \{(\alpha, \beta) \in \mathrm{Hom}_{\mathrm{Sch}}(Q, X) \times \mathrm{Hom}_{\mathrm{Sch}}(Q, Y) : f \circ \alpha = g \circ \beta\}$$

[47] If $\psi : P \to Q$ is a morphism of schemes, then $(h_X \times_{h_Z} h_Y)(\psi)$ is the morphism $(\alpha, \beta) \mapsto (\alpha \circ \psi, \beta \circ \psi)$.

In other words, our work on the fibre product was equivalent to showing that the functor $h_X \times_{h_Z} h_Y$ was represented by the scheme $X \times_Y Z$. In particular, there is a natural isomorphism $h_X \times_{h_Z} h_Y \cong h_{X \times_Y Z}$.

We now provide an alternative proof of this result. For ease of notation, fix $\mathscr{F} = h_X \times_{h_Z} h_Y$. We show that $\mathscr{F}$ is a sheaf, let $\{h_i : U_i \to Q\}_{i \in I}$ be a Zariski cover Q. We want to show that the following diagram is exact:

$$\mathscr{F}(Q) \xrightarrow{(h_i^*)} \prod_{i \in I} \mathscr{F}(U_i) \overset{\pi_0^*}{\underset{\pi_1^*}{\rightrightarrows}} \prod_{i_0, i_1 \in I} \mathscr{F}(U_{i_0} \times_Q U_{i_1})$$

By Example 2.5.6 we know that h_X, h_Y are sheaves. In particular, the condition that $(h_i^*)(\alpha_1, \beta_1) = (h_i^*)(\alpha_2, \beta_2)$ just means that[48] $(h_i^*)(\alpha_1) = (h_i^*)(\alpha_2)$ and $(h_i^*)(\beta_1) = (h_i^*)(\beta_2)$, and so since h_X and h_Y are sheaves, we have that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. The same argument shows that if $(\alpha_i, \beta_i) \in \mathscr{F}(U_i)$ agree on the overlaps $U_i \times_Q U_j$, then $\alpha_i \in h_X(U_i)$ and $\beta_i \in h_Y(U_i)$ agree on the overlaps as well. The sheaf condition then gives us morphisms $\alpha \in h_X(Q)$ and $\beta \in h_Y(Q)$ hence we obtain the morphism $(\alpha, \beta) \in h_X(Q) \times h_Y(Q)$. It is clear that $(\alpha, \beta) \in \mathscr{F}(Q)$ as on an affine cover of Q, $\{h_i(U_i)\}_{i \in I}$, we have that $f \circ \alpha|_{h_i(U_i)} = g \circ \beta|_{h_i(U_i)}$, hence $f \circ \alpha = g \circ \beta$. It follows that $\mathscr{F}$ is a sheaf.

We define open subfunctors $\mathscr{F}_{ijk}$ as follows; let $\{W_i\}$ be an open affine cover of Z, and let $\{U_{ij}\}_{ij}$ and $\{V_{ik}\}_{ik}$ be open affine covers of X and Y respectively such that that the covers $\{U_{ij}\}_j$ and $\{V_{ik}\}_k$ cover $\alpha^{-1}(W_i)$ and $\beta^{-1}(W_i)$ respectively. Define:

$$\mathscr{F}_{ijk}(Q) = \{(\alpha, \beta) \in \mathscr{F}(Q) : \alpha(Q) \subset U_{ij}, \beta(Q) \subset V_{ik}\}$$

Note that the $\mathscr{F}_{ijk}$ are clearly subfunctors. Let $(\alpha, \beta) \in \mathscr{F}(Q)$, and set $U_{(\alpha,\beta)} = \alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik})$. Suppose that morphism $\phi : S \to Q$ has image in $\alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik})$, then we claim that $\mathscr{F}(\phi)(\alpha, \beta) \in \mathscr{F}_{ijk}(S)$. Indeed, $\mathscr{F}(\phi)(\alpha, \beta) = (\alpha \circ \phi, \beta \circ \phi)$. Since $\phi(Y) \subset \alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik})$, we have that $\alpha \circ \phi(Y) \subset U_{ij}$, and $\beta \circ \phi(S) \subset V_{ik}$, hence $\mathscr{F}(\phi)(\alpha, \beta) \in \mathscr{F}_{ijk}(S)$. Now suppose that $\mathscr{F}(\phi)(\alpha, \beta) \in \mathscr{F}_{ijk}(S)$, then $\alpha \circ \phi(S) \subset U_{ij}$ and $\beta \circ \phi(S) \subset V_{ik}$. In particular, $\phi(S) \subset \beta^{-1}(V_{ik})$, and $\phi(S) \subset \alpha^{-1}(U_{ij})$. It follows that $\phi(S) \subset U_{(\alpha,\beta)}$, hence ϕ can be viewed as an element of $\mathrm{Hom}_{\mathrm{Sch}}(S, U_{(\alpha,\beta)})$.

To show that these subfunctors cover, let $(\alpha, \beta) \in \mathscr{F}(Q)$, and take our open covering to be $\{\alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik})\}_{ijk}$. To see that these cover, let $x \in Q$, then $f \circ \alpha(x) = g \circ \beta(x) \in W_i$ for some i. In particular, it follows that $\alpha(x) \in U_{ij}$ for some j, and $\beta(x) \in V_{ik}$ for some j, hence $x \in \alpha^{-1} \cap \beta^{-1}(V_{ik})$. Essentially by construction restricting α and β to these open subset buts the element $(\alpha, \beta)|_{\alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik})}$ in the set $\mathscr{F}_{ijk}(\alpha^{-1}(U_{ij}) \cap \beta^{-1}(V_{ik}))$, hence the subfunctors cover.

Finally, we show that $\mathscr{F}_{ijk}$ are representable. Let $U_{ij} = \operatorname{Spec} A_{ij}$, $V_{ik} = \operatorname{Spec} B_{B_i k}$, and $W_i = \operatorname{Spec} C_i$. Then we claim that $\mathscr{F}_{ijk}$ is represented by the affine scheme $\operatorname{Spec} A_{ij} \otimes_{W_i} B_{ik}$. In particular, if α is a morphism landing U_{ij}, then we can view α as a morphism to the affine scheme $\operatorname{Spec} A_{ij}$, and similarly with β. We can also view $f|_{U_{ij}}$ and $g|_{V_{ik}}$ as morphisms between the affine schemes $\operatorname{Spec} A_{ij} \to \operatorname{Spec} C_i$ and $\operatorname{Spec} B_{ik} \to \operatorname{Spec} C_i$. Lemma 2.3.6 then implies that there is a unique morphism $\psi : Q \to \operatorname{Spec}(A_{ij} \otimes_{C_i} B_{ik})$. This defines a bijection between $\mathscr{F}_{ijk}(Q)$ and $\mathrm{Hom}_{\mathrm{Sch}}(Q, \operatorname{Spec}(A_{ij} \otimes_{C_i} B_{ik}))$, which is natural as given a morphism $S \to Q$, the relevant diagram necessarily commutes. It follows that $\mathscr{F}_{ijk} \cong h_{\operatorname{Spec}(A_{ij} \otimes_{C_i} B_{ik})}$, and so $\mathscr{F}_{ijk}$ is representable. Our representability criterion, Proposition 2.5.1, then implies that $h_X \times_{h_Z} h_Y$ is representable as desired.

Note that this is not really an easier treatment of the fibre product, we have just split the work up differently. In particular, in Section 2.3, we proved the fibre product exists in the affine case, and then slowly glued schemes together piece by piece in successively more general situations. Here, we glued abstract objects together in vast generality, broke our specific functor down into tractable pieces, and then used the fact that the fibre product exists in the affine case to show our functor could be represented.

Before moving onto more examples, we want to show that in most cases of interest, it suffices to prove representability over $\operatorname{Spec} \mathbb{Z}$, i.e. in Sch. For each S denote by $U_S : \mathrm{Sch}/S \to \mathrm{Sch}$ the forgetful functor.

[48] Abuse of notation alert! We are using (h_i^*) to refer to three different maps. In particular, we have (h_i^*) with respect to the functor $\mathscr{F}$, (h_i^*) with respect to the functor h_X, and (h_i^*) with respect to the functor h_Y. We trust the astute reader to pick up which (h_i^*) comes from which functor from context clues.

Lemma 2.5.3. *Suppose we have functors $\mathscr{F}_S : \mathrm{Sch}/S \to \mathrm{Set}$, and $\mathscr{F} : \mathrm{Sch} \to \mathrm{Set}$. For any S-scheme X denote by h'_X the functor $\mathrm{Hom}_{\mathrm{Sch}/S}(-, X)$. If $\mathscr{F}_S \cong \mathscr{F} \circ U_S$ and $\mathscr{F} \cong h_X$, then $\mathscr{F}_S \cong h'_{X \times_{\mathbb{Z}} S}$. In particular, if (X, η) a universal object for $\mathscr{F}$, then $(X \times_{\mathbb{Z}} S, \mathscr{F}(\pi_X)(\eta))$ is a universal object for $\mathscr{F}_S$.*

Proof. This is all essentially obvious but we spell it out; it suffices to prove that $\mathscr{F} \circ U_S \cong h'_{X \times_{\mathbb{Z}} S}$. Let $G : \mathscr{F} \to h_X$ be the natural isomorphism, and Y be an S-scheme, and $f_Y : Y \to S$ be the structural morphism. We define a map $F_Y : \mathscr{F} \circ U_S(Y) \to h_{X \times_{\mathbb{Z}} S}(Y)'$, by sending an element $s \in \mathscr{F} \circ U_s(Y)$ to the unique morphism $Y \to X \times_{\mathbb{Z}} S$ induced by by $G_Y(s)$ and f_Y. This is a bijection of sets essentially by the universal property of the fibre product. If $F_Y(s) = F_Y(t)$, then in particular, we have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[dr, "F_Y(s)=F_Y(t)"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_{\mathbb{Z}} S \arrow[r, "\pi_S"] \arrow[d, "\pi_X"] & S \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

where the left most arrow can be either $G_Y(s)$ or $G_Y(t)$. This means $G_Y(s) = \pi_X \circ F_Y(s) = G_Y(t)$ hence $s = t$ and F_Y is injective. Suppose that $g : Y \to X \times_{\mathbb{Z}} Y$ is an S-scheme morphism; then since $\pi_Y \circ g = f_Y$, we have that $G_Y^{-1}(\pi_X \circ g)$ maps to g under F_Y, as both $F_Y(G_Y^{-1}(\pi_X \circ g))$ and g make the same diagram commute.

To show that this natural let $h : Z \to Y$ be a morphism of S-schemes, then we need to show that the following square commutes:

$$\begin{tikzcd}
\mathscr{F} \circ U_S(Y) \arrow[r, "F_Y"] \arrow[d, "\mathscr{F} \circ U_S(h)"] & \mathrm{Hom}_{\mathrm{Sch}/S}(Y, X \times_{\mathbb{Z}} S) \arrow[d, "h^*"] \\
\mathscr{F} \circ U_S(Z) \arrow[r, "F_Z"] & \mathrm{Hom}_{\mathrm{Sch}/S}(Z, X \times_{\mathbb{Z}} S)
\end{tikzcd}$$

Let $s \in \mathscr{F} \circ U_S(Y)$, then $F_Y(s)$ is the unique morphism g corresponding to $(G_Y(s), f_Y)$, so $h^* \circ F_Y(s) = g \circ h$. We have that $\mathscr{F} \circ U_S(h)(s) = \mathscr{F}(h)(s)$ as the forgetful functor just tells us to treat h as a scheme morphism with no extra structure. By naturality of G, we thus have that $F_Z(\mathscr{F}(h)(s))$ is the unique morphism g' corresponding to the pair $(G_Y(s) \circ h, f_Z)$, where f_Z is the morphism making Z an S-scheme. Now note that $g \circ h$ corresponds to $(G_Y(s) \circ h, f_Y \circ h)$, so since $f_Y \circ h = f_Z$, we have that $g \circ h = g'$, implying the claim.

For the second claim, it again suffices to prove this for $\mathscr{F}_S \circ U_S$. Our work in Lemma 2.4.4 implies that the pair $(X, G_X^{-1}(\mathrm{Id}_X))$ is our universal object for $\mathscr{F}$, and that $(X \times_{\mathbb{Z}} S, F^{-1}_{X \times_{\mathbb{Z}} S}(\mathrm{Id}_{X \times_{\mathbb{Z}} S}))$ is our universal object for $\mathscr{F} \circ U_S$. If $\pi_X : X \times_Z S \to X$ is the projection, we need to show that $\mathscr{F}(\pi_X) \circ G_X^{-1}(\mathrm{Id}_X) = F^{-1}_{X \times_{\mathbb{Z}} S}(\mathrm{Id}_{X \times_{\mathbb{Z}} S})$. Since G_X is natural, we have that

$$\mathscr{F}(\pi_X) \circ G_X^{-1}(\mathrm{Id}) = G^{-1}_{X \times_{\mathbb{Z}} S}(\pi_X) \in \mathscr{F}(X \times_Z S) = \mathscr{F} \circ U_S(X \times_Z S)$$

Now, note that $\pi_X = \pi_X \circ \mathrm{Id}_{X \times_Z S}$, hence our work above demonstrates that $F_{X \times_{\mathbb{Z}} S}$ sends π_X to $\mathrm{Id}_{X \times_Z S}$. Since $F_{X \times_{\mathbb{Z}} S}$ is injective it follows that $G^{-1}_{X \times_{\mathbb{Z}} S}(\pi_X) = F^{-1}_{X \times_{\mathbb{Z}} S}(\mathrm{Id}_{X \times_{\mathbb{Z}} S})$ as desired.

□

We now begin with examples of schemes that are functors.

Example 2.5.9. Proposition 2.1.2 demonstrates that any affine scheme Spec A represents the contravariant functor:

$$X \longmapsto \mathrm{Hom}(A, \mathscr{O}_X(X))$$

Example 2.5.10. For any base scheme S, we define a functor:

$$\mathscr{F}_S : \mathrm{Sch}/S \longrightarrow \mathrm{Set}$$

taking an S-scheme X to $\mathscr{O}_X(X)^n$. We send morphisms $f : X \to Y$ to the n fold direct product of the morphism $f^\sharp_Y : \mathscr{O}_Y(Y) \to (f_*\mathscr{O}_X)(Y) = \mathscr{O}_X(X)$. It is clear that this defines a functor. Note that if $\mathscr{F}$ is the same functor defined with $S = \operatorname{Spec} Z$ then we clearly have that $\mathscr{F}_S = \mathscr{F} \circ U_S$ where U_S is the forgetful functor. It follows from Lemma 2.5.3 that we can demonstrate representability of $\mathscr{F}_S$ by first showing that $\mathscr{F}$ is representable and base changing to Sch/S.

One can easily see from Example 2.5.9, the universal property of the direct product, and the fact that $\mathbb{Z}$ is the initial object in Ring, that $\mathscr{F}$ is represented by $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. We will take a more hands on approach. Let $f \in \mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^n_{\mathbb{Z}})$, then we obtain an element $F_X(f) \in \mathscr{O}_X(X)^n$ by sending f to the n tuple $(f^\sharp_X(x_1), \ldots, f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_n))$. We first claim this is injective; suppose that $F_X(f) = F_X(g)$, and let $U = \operatorname{Spec} A$ be any open affine of X. Since $f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i) = g^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)$ for all i, we have that $f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)|_U = g^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)|_U$. Note that $f^\sharp_X(x_i)|_{\mathbb{A}^n_{\mathbb{Z}}} = (f \circ \imath_U)^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)$ where $\imath_U$ is the open embedding, and similarly for $g^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)|_U$. In other words, the restricted maps $f|_U, g|_U : U \to \mathbb{A}^n_{\mathbb{Z}}$ satisfy $(f|_U)^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i) = (g|_U)^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i)$. However, since $f|_U$ and $g|_U$ are morphisms of affine scheme, they are fully determined by the induced ring homomorphisms on global sections. Since a ring homomorphism $\mathbb{Z}[x_1, ;x_n] \to A$ is fully determined by where x_i is sent for each i, it follows that $(f|_U)^\sharp_U = (g|_U)^\sharp_U$ and so $f|_U = g|_U$. Since U was arbitrary, we know that $f = g$ as they agree on every open affine, hence F_X is injective.

Now let $(a_1, \ldots, a_n) \in \mathscr{O}_X(X)^n$, and $\{U_i = \operatorname{Spec} A_i\}$ an affine open cover. We define morphisms $f_i : U_i \to \mathbb{A}^n_{\mathbb{Z}}$ to be the ones induced by the ring maps $\mathbb{Z}[x_1, \ldots, x_n] \to A_i$ given by $x_i \mapsto a_i|_{U_i}$. We claim that the f_i agree on overlaps; indeed let $V = \operatorname{Spec} B \subset U_i \cap U_j$ be any affine open, then $f_i|_V = f_j|_V$ because $f_i|_V$ is induced by the ring homomorphism $\mathbb{Z}[x_1, \ldots, x_n] \to B$ given by $x_k \mapsto a_k|_V$. It follows that $f_i|_V$ and $f_j|_V$ are induced by the same ring homomorphism, are thus equal. Since V was arbitrary we must have that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so the f_i glue together to give a morphism f, which by construction satisfies $f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_i) = a_i$. It follows that F_X is an isomorphism for each X.

Let $g : Y \to X$, then we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^n_{\mathbb{Z}}) & \xrightarrow{F_X} & \mathscr{O}_X(X)^n \\
\Big\downarrow{\scriptstyle g^*} & & \Big\downarrow{\scriptstyle (g^\sharp_X)^n} \\
\mathrm{Hom}_{\mathrm{Sch}}(Y, \mathbb{A}^n_{\mathbb{Z}}) & \xrightarrow{F_Y} & \mathscr{O}_Y(Y^n)
\end{array}$$

Let $f \in \mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^n_{\mathbb{Z}})$, then $(g^\sharp_X)^n \circ F_Z(f)$ is equal to

$$\Big(g^\sharp_X \circ f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_1), \ldots, g^\sharp_X \circ f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}(x_n)\Big)$$

We know that $g^\sharp_X \circ f^\sharp_{\mathbb{A}^n_{\mathbb{Z}}} = (f \circ g)^\sharp_{\mathbb{A}^n_{\mathbb{Z}}}$, hence we have that the above is equal to $F_Y(g^* f)$ as desired. It follows that F is a natural isomorphism.

2.6 Group Schemes

In this section we study a particular type of functor representable by schemes: "group schemes". In the category of smooth manifolds, a group manifold, i.e. a Lie group, is a smooth manifold equipped with a group structure such that multiplication and inversion are smooth. This is an acceptable definition because a direct product of manifolds $M \times N$ is the direct product in the category of sets. However, in the category Sch/S, our direct product is given by the fibre product over S, which is almost never equal to the cartesian product of the underlying sets. We therefore require a slightly more sophisticated definition?

Definition 2.6.1. Let $\mathscr{C}$ be a category with finite direct products and a terminal object $\star$. An object $G \in \mathscr{C}$ is a **group object** if there exists a contravariant functor $\mathscr{F} : \mathscr{C} \to \mathrm{Grp}$ such that $h_G \cong U \circ \mathscr{F}$, where U is the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$. In other words, $h_G(A)$ has a natural group structure for all objects $A \in \mathscr{C}$.

There is a more down to earth definition of a group object, i.e. let $f : G \to \star$ be the unique morphism, then G is a group object if there exists morphisms $m : G \times G \to G$, $e : \star \to G$, and $i : G \to G$ such that the following diagrams group diagrams commute:

$$\begin{tikzcd}
G \arrow[r, "{(\mathrm{Id},i)}"] \arrow[d, "{(i,\mathrm{Id})}"'] \arrow[dr, "e\circ f"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times (G \times G) \arrow[r, "\mathrm{Id}\times m"] \arrow[dd, "\cong"] & G \times G \arrow[dr, "m"] & \\
 & & G \\
(G \times G) \times G \arrow[r, "m\times\mathrm{Id}"] & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
G \times \star \arrow[r, "\mathrm{Id}\times e"] \arrow[dr, "\pi_G"'] & G \times G \arrow[d, "m"] \\
 & G
\end{tikzcd}$$

where (f, g) denotes the map obtained from the universal property of the direct product with f and g.[49] Clearly, if these diagrams occurred in Set they would endow G with the structure of a group, hence this is in a sense the more obvious extension of a group. We will shortly show that these two definitions are equivalent, but first note that the if these diagrams occurred in Set, then our diagram corresponding to the identity element only encodes the axiom:[50]

$$g \cdot e = g$$

and not that $e \cdot g = g$. This is actually all that is necessary; let $g \in G$ then the inverse and associativity axioms tell us that

$$e \cdot g = (g \cdot g^{-1})g = g \cdot (g^{-1} \cdot g) = g \cdot e = g$$

We now prove a similar statement for general group objects:

Lemma 2.6.1. *Let G be a group object in a category $\mathscr{C}$. Then the following diagram commutes:*

$$\begin{tikzcd}
\star \times G \arrow[r, "e\times Id"] \arrow[dr, "\pi_G"'] & G \times G \arrow[d, "m"] \\
 & G
\end{tikzcd}$$

Proof. Note that $f \circ \pi_G = \pi_\star$ as $\star$ is the terminal object. Moreover, we have that $e \times \mathrm{Id}$ is the morphism $(e \circ \pi_\star, \pi_G)$. We want to show that:

$$m \circ (e \circ \pi_\star, \pi_G) = \pi_G$$

The inversion diagram tells that $m \circ (\mathrm{Id}, i) = e \circ f$. Since $f \circ \pi_G = \pi_\star$ and $(\mathrm{Id}, i) \circ \pi_G = (\pi_G, i \circ \pi_G)$, we then have that:

$$m \circ (\pi_G, i \circ \pi_G) = e \circ \pi_\star$$

[49] Note that this map is equal to $f \times g \circ \Delta$, with Δ the diagonal morphism as always.
[50] Here, since we are in the category Set, we view e not as morphism from the singleton set, but as the identity element in G.

We can thus write that:

$$m \circ (e \circ \pi_\star, \pi_G) = m \circ (m \circ (\pi_G, i \circ \pi_G), \pi_G)$$

Let F be the isomorphism $(G \times G) \times G \to G$, then associativity tells us that:

$$m \circ (e \circ \pi_\star, \pi_G) = m \circ (\mathrm{Id} \times m) \circ F \circ ((\pi_G, i \circ \pi_G), \pi_G)$$

By the definition of F, for any triple of morphisms $((f,g),h)$, we have that $F \circ ((f,g),h) = (f,(g,h))$, hence:

$$\begin{aligned} m \circ (e \circ \pi_\star, \pi_G) =& m \circ (\mathrm{Id} \times m) \circ (\pi_G, (i \circ \pi_G, \pi_G)) \\ =& m \circ (\pi_G, m \circ (i \circ \pi_G, \pi_G)) \end{aligned}$$

Now the same argument with inversion tells us that $m \circ (i \circ \pi_G, \pi_G) = e \circ \pi_\star$, hence:

$$m \circ (e \circ \pi_\star, \pi_G) = m \circ (\pi_G, e \circ \pi_\star)$$

Now $(\pi_G, e \circ \pi_\star)$ comes from the diagram:

$$\begin{tikzcd}
& \star \times G \arrow[dl, bend right, "\pi_G"'] \arrow[d] \arrow[dr, bend left, "e\circ\pi_\star"] & \\
G & G \times G \arrow[l, "\pi_G^L"] \arrow[r, "\pi_G^R"'] & G
\end{tikzcd}$$

where π_G^L and π_G^R denote the left and right projections. Denote by s denote the swap isomorphism $\star \times G \to G \times \star$, respectively. Note that s come from the diagram:

$$\begin{tikzcd}
& \star \times G \arrow[dl, bend right, "\pi_G"'] \arrow[d] \arrow[dr, bend left, "\pi_\star"] & \\
G & G \times \star \arrow[l, "\pi_G"] \arrow[r, "\pi_\star"'] & \star
\end{tikzcd}$$

We first claim that $(\pi_G, e \circ \pi_\star) = (\pi_G, e \circ \pi_\star)' \circ s$, where $(\pi_G, e \circ \pi_\star)'$ comes from the diagram:

$$\begin{tikzcd}
& G \times \star \arrow[dl, bend right, "\pi_G"'] \arrow[d] \arrow[dr, bend left, "e\circ\pi_\star"] & \\
G & G \times G \arrow[l, "\pi_G^L"] \arrow[r, "\pi_G^R"'] & G
\end{tikzcd}$$

It suffices to check that $(\pi_G, e \circ \pi_\star)' \circ s$ makes the defining diagram defining $(\pi_G, e \circ \pi_\star)$ commute. We see that $\pi_G^L \circ (\pi_G, e \circ \pi_\star)' = \pi_G$, and that $\pi_G \circ s = \pi_G$. Similarly, $\pi_G^R \circ (\pi_G, e \circ \pi_\star)' = e \circ \pi_\star \circ s$, and $\pi_\star \circ s = \pi_\star$, hence $e \circ \pi_\star \circ s = e \circ \pi_\star$, and our claimed equality follows. By the diagram corresponding the right identity element we therefore obtain:

$$m \circ (e \circ \pi_\star, \pi_G) = m \circ (\pi_G, e \circ \pi_\star)' \circ s = \pi_G \circ s = \pi_G$$

which is the desired equality. □

As promised, we now show that the two definitions of a group a object agree. For the sake of clarity, we reaffirm that we take a group object to a priori be an object which represents a functor to Grp post composed with the forgetful functor.

Proposition 2.6.1. *Let $\mathscr{C}$ be a category with finite products, and terminal object $\star$. An object G is a group object if and only if there exist maps $m : G \times G \to G$, $e : \star \to G$, and $i : G \to G$ making the following diagrams commute:*

$$\begin{tikzcd}
G \arrow[r, "(Id,i)"] \arrow[d, "(i,Id)"'] \arrow[dr, "e\circ f"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times (G \times G) \arrow[r, "Id\times m"] \arrow[dd, "\cong"] & G \times G \arrow[dr, "m"] & \\
& & G \\
(G \times G) \times G \arrow[r, "m\times Id"] & G \times G \arrow[ur, "m"] &
\end{tikzcd}$$

$$\begin{tikzcd}
G \times \star \arrow[r, "Id\times e"] \arrow[dr, "\pi_G"'] & G \times G \arrow[d, "m"] \\
& G
\end{tikzcd}$$

Proof. Suppose that G is a group object, then via the functor $\mathscr{F}$ we can naturally endow $h_G(A)$ with a group structure. Indeed, let $F : h_G \to U \circ \mathscr{F}$, then we define multiplication in $h_G(A)$ via:

$$a_1 \cdot a_2 = F_A^{-1}(F_A(a_1) \cdot F_A(a_2))$$

Note that the even though we are post composing with the forgetful functor as to avoid formulating a notion of what it means for h_G to be naturally isomorphic to a functor with a different codomain, we still have access to hte multiplication and inversion maps, and the identity element, all satisfying the usual axioms. We define the identity and inversion similarly:

$$e_A = F_A^{-1}(e) \qquad \text{and} \qquad a^{-1} = F_A^{-1}()$$

One easily sees this gives $h_G(A)$ the structure of a group for every object A. Importantly, we claim that these actually define natural transformations $m_G : h_G \times h_G \to h_G$, $i_G h_G \to h_G$, and $e_G : h_\star \to h_G$. WE need to check naturality, i.e. if $f : A \to B$ then:

$$m_{G,A} \circ (f^*, f^*) = f^* \circ m_{G,B}$$

Let $(b_1, b_2) \in h_G(B)$ then since $\mathscr{F}(f)$ is a group homomorphism:

$$\begin{aligned}
f^* \circ m_{G,B}(b_1, b_2) =& f^*(F_B^{-1}(F_B(b_1) \cdot F_B(b_2))) \\
=& F_A^{-1}((U \circ \mathscr{F})(f)(F_B(b_1) \cdot F_B(b_2))) \\
=& F_A^{-1}((U \circ \mathscr{F})(f)(F_B(b_1)) \cdot (U \circ \mathscr{F})(f)(F_B(b_2))) \\
=& F_A^{-1}(F_A(f^*b_1) \cdot F_A(f^*b_2)) \\
=& m_{G,A} \circ (f^*, f^*)(b_1, b_2)
\end{aligned}$$

and so m_G is natural. Similarly, if $b \in h_G(B)$ then:

$$\begin{aligned}
f^* \circ i_{G,B}(b) =& f^* \circ F_B^{-1}(F_B(b)^{-1}) \\
=& F_A^{-1}((U \circ \mathscr{F})(f)(F_B(b)^{-1})) \\
=& F_A^{-1}((U \circ \mathscr{F})(f)(F_B(b))^{-1}) \\
=& F_A^{-1}(F_A(f^*b)^{-1}) \\
=& i_{G,A} \circ f^*(b)
\end{aligned}$$

and so i_G is natural as well. Note that $e_{G,A}$ is given by sending the unique element in $h_\star(A)$ to identity element in $h_G(A)$. Since $(U \circ \mathscr{F})(f)$ fulfills the constraints of a group homomorphism it is clear that e_G is natural as well.

Since these natural transformations endow each $h_G(A)$ with the structure of a group, it follows that we have commutative diagrams of the form:

$$\begin{tikzcd}
h_G \arrow[r, "(\mathrm{Id}_G, i_G)"] \arrow[d, "(i_G, \mathrm{Id}_G)"'] \arrow[dr, "e_G \circ f_G"] & h_G \times h_G \arrow[d, "m"] \\
h_G \times h_G \arrow[r, "m_G"] & h_G
\end{tikzcd}
\qquad
\begin{tikzcd}
h_G \times (h_G \times h_G) \arrow[r, "\mathrm{Id}_G \times m_G"] \arrow[d, "\cong"] & h_G \times h_G \arrow[dr, "m_G"] & \\
 & & G \\
(h_G \times h_G) \times h_G \arrow[r, "m_G \times \mathrm{Id}_G"] & h_G \times h_G \arrow[ur, "m_G"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
h_G \times h_\star \arrow[r, "\mathrm{Id}_G \times e_G"] \arrow[dr, "\pi_G"'] & h_G \times h_G \arrow[d, "m_G"] \\
 & h_G
\end{tikzcd}$$

where f_G is the unique natural transformation $h_G \to h_\star$ coming from Corollary 2.4.1, and π_G is the obvious natural transformation $h_G \times h_\star \to h_G$. It follows by Corollary 2.4.1 that there are morphisms $m : G \times G \to G$, $i : G \to G$, and $e : \star \to G$ which make the desired diagrams commute.

Now suppose that we have morphisms m, i, and e which make the desired diagrams commute. Furthermore, let $f_G : G \to \star$ be the unique morphism. We first endow $\mathrm{Hom}(A, G)$ with a group structure. Define multiplication by:

$$a_1 \cdot a_2 = m \circ (a_1, a_2)$$

where (a_1, a_2) is the unique morphism $A \to G \times G$ given by the universal property of the direct product. We immediately check that this is associative; denote by F the isomorphism $(G \times G) \times G \to G \times (G \times G)$, then :

$$\begin{aligned}
(a_1 \cdot a_2) \cdot a_3 =& (m \circ (a_1, a_2)) \cdot a_3 \\
=& m \circ (m \circ (a_1, a_2), a_3) \\
=& m \circ m \times \mathrm{Id} \circ ((a_1, a_2), a_3) \\
=& m \circ \mathrm{Id} \times m \circ F \circ ((a_1, a_2), a_3) \\
=& m \circ \mathrm{Id} \times m \circ (a_1, (a_2, a_3)) \\
=& m \circ (a_1, m \circ (a_2, a_3)) \\
=& a_1 \cdot (a_2 \cdot a_3)
\end{aligned}$$

We define the identity element e_A in $\mathrm{Hom}(A, G)$ to be $e \circ f_A$ where f_A is the unique morphism $A \to \star$. We check that this is a right identity:

$$a \cdot e_A = m \circ (a, e \circ f_A) = m \circ \mathrm{Id} \times e \circ (a, f_A) = \pi_G \circ (a, f_A) = a$$

We have the left identity diagram from Lemma 2.6.1, and so the same argument implies that $e \circ f_A$ is the left identity as well. Finally, we define a^{-1} to be $i \circ a$; then our inversion diagram tells us that:

$$a \cdot a^{-1} = m \circ (a, i \circ a) = m \circ (\mathrm{Id}, i) \circ a = e \circ f \circ a = e \circ f_A = e_A$$

so $i \circ a$ is a right inverse. The other half of the diagram tell us that $i \circ a$ is a left inverse as well. It follows that $\mathrm{Hom}(A, G)$ has the structure of a group.

We claim that we can now obtain a functor $\mathscr{F} : \mathscr{C} \to \text{Grp}$ which satisfies $U \circ \mathscr{F} = h_G$. We define $\mathscr{F}$ by sending objects to the $\text{Hom}(A, G)$ equipped with the group object above, and send morphism $f : B \to A$ to f^*. We need only show that f^* is a group homomorphism. Let $a_1, a_2 \in \text{Hom}(A, B)$ then:

$$f^*(a_1 \cdot a_2) = f^*(m \circ (a_1, a_2)) = m \circ (a_1, a_2) \circ f = m \circ (a_1 \circ f, a_2 \circ) = (f^* a_1) \cdot (f^* a_2)$$

hence f^* is a group homomorphism as desired. It follows that G is trivially a group object as desired. $\square$

We employ the following definition:

Definition 2.6.2. A **group scheme over S** is a group object in the category S.

We immediately note that $\mathbb{A}^n_S$ is a group scheme in the category Sch/S. Indeed, $\mathbb{A}^n_S$ represents the functors taking a scheme X to $\mathscr{O}_X(X)^n$ which has the natural structure of an abelian group given by element wise addition. We at times denote the group scheme corresponding to the affine line as $\mathbb{G}_{a,S}$, the additive group. Obviously, over S, this implies that $\mathbb{G}^n_m = \mathbb{A}^n_S$. We will often forgo the precision of Definition 2.6.1 and refer to a group scheme as representing the functor $\text{Sch}/S \to \text{Grp}$, and instead of the post composition with the forgetful functor. Moreover we frequently denote by $\mathbb{G}_{a,S}$ both the functor $\text{Sch}/S \to \text{Grp}$, and the scheme representing said functor. If $S = \operatorname{Spec} \mathbb{Z}$, or if the base scheme is implicitly understood, we will simply write $\mathscr{G}_a$. We will employ this convention with many group schemes.

Example 2.6.1. We define the functor $\mathbb{G}_{m,S} : \text{Sch}/S \to \text{Grp}$ by sending X to $\mathscr{O}_X(X)^\times$, i.e. we send an S scheme to the multiplicative group of invertible elements in its ring of global sections. Note that since this satisfies the criteria of Lemma 2.5.3, it suffices to show that $\mathscr{G}_m$ is a group scheme over $\operatorname{Spec} \mathbb{Z}$. We claim that $\mathscr{G}_m = U_x \subset \mathbb{A}^1_{\mathbb{Z}}$. This time, we will use Example 2.5.9, i.e. that h_{U_x} is naturally isomorphic to the functor $\mathscr{F} : \text{Sch} \to \text{Set}$ given by sending X to $\text{Hom}(\mathbb{Z}[x]_x, \mathscr{O}_X(X))$. We define a natural transformation $F : \mathbb{G}_m \to \mathscr{F}$. Let $a \in \mathscr{O}_X(X)^\times$, then define a ring homomorphism $\mathbb{Z}[x] \to \mathscr{O}_X(X)$ by sending x to a. The universal property of localization then yields a unique morphism $\phi_a : \mathbb{Z}[x]_x \to \mathscr{O}_X(X)$, and so define F_X by $a \mapsto \phi_a$. This is obviously a bijection so F_X is an isomorphism.

We need to show that F_X is natural, i.e. that if $f : X \to Y$ is a morphism of schemes, then:

$$\mathscr{F}(f) \circ F_Y = F_X \circ \mathscr{G}_m(f)$$

Note that $\mathbb{G}_m(f)$ is simply $f^\sharp_Y$, and that $\mathscr{F}(f)$ is given by post composing with $f^\sharp_Y$. If $b \in \mathscr{O}_Y(Y)^\times$, it then suffices to show that:

$$f^\sharp_Y \circ \phi_b = \phi_{f^\sharp_Y(b)}$$

however this is clear, as $f^\sharp_Y \circ \phi_b$ and $\phi_{f^\sharp_Y(b)}$ send x to the same element. It follows that $\mathbb{G}_m$

Note that since group schemes represent functors to Grp, the hom functors can naturally be viewed as functors to groups as well. In particular, we have that the T points of any group scheme G, $G(T) := \text{Hom}_{\text{Sch}/S}(T, G)$, form a bona fide group which can be naturally identified with the functorial description of G. In the two examples above we see that $\mathbb{G}_{a,S}(T)$ and $\mathbb{G}_{m,S}(T)$ are abelian groups for every S-scheme T. It follows from Corollary 2.4.1 that since the multiplication maps correspond to the multiplication natural transformation on the level of the group functors that the $m : \mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ must commute with the isomorphism $\mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S} \to \mathbb{G}_{m,S} \times_S \mathbb{G}_{m,S}$ swapping the two factors. Similarly, if one has group scheme G such that multiplication commutes the isomorphism swapping the two factors of G in the fibre product, then $G(T)$ must be commutative. We thus have the following definition:

Definition 2.6.3. A group scheme G over S is a **commutative or abelian group scheme over S** if for every S scheme T, the T points of G are an abelian group. Equivalently, if $s : G \times_S G \to G \times_S G$ is the isomorphism swapping the two factors of G, and m is the multiplication map, we have that $m \circ s = m$.

Here is an example of a non abelian group scheme:

Example 2.6.2. For any commutative ring, denote by $GL_n(A)$ the group of invertible matrices with coefficients in A, or equivalently $\text{Aut}_A(A^n)$. Now note that in any commutative ring A, if $B : A^n \to A^n$ is represented by a matrix, then we can form the adjugate of B, $\text{adj}(B)$ which satisfies:

$$\text{adj}(B) \cdot B = B \cdot \text{adj}(B) = \det(B) \cdot I_n$$

where I_n is the identity matrix. It follows that B is invertible if and only if $\det B$ is invertible in the ring A. With this in mind, we define a functor $\mathbb{GL}_{n,S} : \mathrm{Sch}/S \to \mathrm{Sch}$ which sends X to $GL_n(\mathscr{O}_X(X))$, and a morphism $f : X \to Y$ to the morphism:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} f_Y^\sharp(a_{11}) & \cdots & f_Y^\sharp(a_{1n}) \\ \vdots & \ddots & \vdots \\ f_Y^\sharp(a_{n1}) & \cdots & f_Y^\sharp(a_{nn}) \end{pmatrix}$$

Note that $\mathbb{GL}_{n,S}(f)$ is clearly a group homomorphism. By Lemma 2.5.3 it suffices to show representability when $S = \operatorname{Spec}\mathbb{Z}$; consider affine n^2 space, $\mathbb{A}_{\mathbb{Z}}^{n^2}$. For any commutative ring A, the A points of this scheme are elements in A^{n^2}, which we can view as square matrices with coefficients in A. Now, suppose we have matrix B with coefficients $b_{ij} \in A$, then for B to invertible as a matrix we want:

$$\det(B) = \det((b_{ij})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

to be invertible in A. When we wanted to detect invertible elements in A, our representing object was $\mathscr{G}_m = \operatorname{Spec}\mathbb{Z}[x, x^{-1}]$, i.e. we inverted the polynomial x. Heuristically it follows that if we want to detect invertible, then we should invert the polynomial in n^2 variables associated to the determinant. In other, words if label our variables x_{ij}, and let (x_{ij}) denote the matrix with coefficients in $\mathbb{Z}[x_{ij} : 1 \le i, j \le n]$, such that the ijth coefficient is x_{ij}, then our desired ring is $\mathbb{Z}[x_{ij} : 1 \le i, j \le n]_{\det((x_{ij}))}$, where:

$$\det((x_{ij})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)}$$

An extremely similar argument as in Example 2.6.1 then demonstrates

$$\mathbb{GL}_n = \operatorname{Spec}\mathbb{Z}[x_{ij} : 1 \le i, j \le n]_{\det((x_{ij}))}$$

and so $\mathbb{GL}_{n,S} = \mathbb{GL}_n \times_{\mathbb{Z}} S$. In particular, if $S = \operatorname{Spec} k$ for k a field then

$$\mathbb{GL}_{n,k} = \operatorname{Spec} k[x_{ij} : 1 \le i, j \le n]_{\det((x_{ij}))}$$

and the k-points are in bijection with $GL_n(k)$, our standard general linear group.

Properties of Schemes and their Morphisms

3.1 Closed Embeddings

In this chapter we will broadly discuss some topological, and algebraic properties of schemes and subschemes, along with their morphisms. Reader be warned: this chapter may feel like whiplash. Recall that in Definition 1.3.7 we defined what an open embedding is; we now define a similar class of morphisms:

Definition 3.1.1. Let $f : X \to Y$ be a morphism of schemes, then f is a **closed embedding**[51] if $f(X) \subset Y$ is closed, f is a homeomorphism onto it's image, and $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is surjective.

Example 3.1.1. Let A be a ring and $I \subset A$ be an ideal. We claim that the natural map $g : \operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection map $\pi : A \to A/I$ is a closed embedding. First note that if $\mathfrak{p} \subset A/I$ is a prime ideal, then we have that $I \subset \pi^{-1}(\mathfrak{p})$. Indeed, we have that $\ker \pi = I$, so $\pi^{-1}(0) = I$, and $\pi^{-1}(0) \subset \pi^{-1}(\mathfrak{p})$. It follows that we get a induced continuous map $g : \operatorname{Spec} A/I \to \mathbb{V}(I)$. However, we have already shown in Proposition 2.1.3 that there is a homeomorphism $f : \mathbb{V}(I) \to \operatorname{Spec} A/I$ given by $\mathfrak{p} \mapsto \pi(\mathfrak{p})$. We see that $f \circ g(\mathfrak{p}) = \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$, so $f \circ g = \operatorname{Id}$. We want to show that $\pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ as well. Note that:

$$\pi^{-1}(\pi(\mathfrak{p})) = \{a \in A : [a] \in \pi(\mathfrak{p})\}$$

while:

$$\pi(\mathfrak{p}) = \{[a] \in A/I : a \in \mathfrak{p}\}$$

If $a \in \mathfrak{p}$, then clearly we have that $[a] \in \pi(\mathfrak{p})$ so $a \in \pi^{-1}(\pi(\mathfrak{p}))$ implying that $\mathfrak{p} \subset \pi^{-1}(\pi(\mathfrak{p}))$. If $a \in \pi^{-1}(\pi(\mathfrak{p}))$ then $[a] \in \pi(\mathfrak{p})$, so $a + i \in \mathfrak{p}$ for some $i \in I$. We have that $I \subset \mathfrak{p}$, so $i \in \mathfrak{p}$, hence $a + i - i = a \in \mathfrak{p}$, implying that $\pi^{-1}(\pi(\mathfrak{p})) \subset \mathfrak{p}$. It follows that $g \circ f(\mathfrak{p}) = \pi^{-1}(\pi(\mathfrak{p})) = \mathfrak{p}$ so $g \circ f = \operatorname{Id}$ as well. We thus have that g is a homeomorphism onto the closed subspace A/I.

We now check that the morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow g_*\mathscr{O}_{\operatorname{Spec} A/I}$ is surjective, and it suffices to check that $g^\sharp_{U_h}$ is surjective for every distinguished open U_h, as then the induced morphism on stalks will always be surjective. Note that:

$$g_*\mathscr{O}_{\operatorname{Spec} A/I}(U_h) = \mathscr{O}_{\operatorname{Spec} A/I}(U_{[h]}) \cong (A/I)_{[h]}$$

Note that that $g^\sharp_{U_h}$ is given by:

$$\begin{aligned} g^\sharp_{U_h} : A_h &\longrightarrow (A/I)_{[h]} \\ a/h^k &\longmapsto [a]/[h]^k \end{aligned}$$

which is clearly surjective so $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a closed embedding as desired.

With this example in mind, we wish to show that every closed embedding is locally of this form.

[51] This is sometimes referred to in the literature as a closed immersion.

Lemma 3.1.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is a closed embedding if and only if for every open affine $U = \operatorname{Spec} A \subset Y$ there exists an ideal $I \subset A$ such that $f^{-1}(U) = \operatorname{Spec} A/I \subset X$, and $f|_{f^{-1}(U)}$ comes from the projection (up to isomorphism).*

Proof. Let $f : X \to Y$ be a closed immersion, and let $I_{X/Y}$ be the sheaf of ideals on Y given by $\ker f^\sharp$. If $U = \operatorname{Spec} A \subset Y$ is an affine open then $I = I_{X/Y}(U)$ is an ideal of A and thus determines a closed subset $\mathbb{V}(I) \subset U$. Let $V = f^{-1}(U)$ then we have an induced morphism of schemes $f|_V : V \to U$ which must be a homeomorphism onto its image, so we simply need to show that $f(V) = \mathbb{V}(I)$. By Proposition 2.1.2, we have this morphism of schemes is uniquely determined by the morphism $(f|_V)^\sharp_U : \mathscr{O}_U(U) = A \to \mathscr{O}_V(V)$, which we denote by ψ going forward. If $x \in V$, then we have that:

$$f|_V(x) = \psi^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$$

where π_x is the morphism $\mathscr{O}_V(V) \to (\mathscr{O}_V)_x$. We have that I is the kernel of ψ, and so $I \subset f(x)$ as $0 \in \pi_x^{-1}(\mathfrak{m}_x)$ hence $\psi^{-1}(0) \subset \psi^{-1}(\pi_x^{-1}(x))$. It follows that $f|_V : V \to U$ has image in $\mathbb{V}(I)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(I)$, we want to show that $\mathfrak{p} \in f(V)$; since $f|_V$ is a closed embedding, we have that the stalk map:

$$(f|_V)^\sharp_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow ((f|_V)_*\mathscr{O}_V)_{\mathfrak{p}}$$

is surjective with kernel $I_{\mathfrak{p}}$. If $\mathfrak{p} \notin f(V)$ then we clearly have that $((f|_V)_*\mathscr{O}_V)_{\mathfrak{p}}$ is zero, implying that $I_{\mathfrak{p}} = A_{\mathfrak{p}}$. However, $I \subset \mathfrak{p}$, so this means that $\mathfrak{m}_{\mathfrak{p}} = A_{\mathfrak{p}}$ as $I_{\mathfrak{p}} \subset \mathfrak{m}_{\mathfrak{p}}$. This is clearly a contradiction, so we have that if $I \subset \mathfrak{p}$ then $\mathfrak{p} \in f(V)$ as desired. It follows that $f|_V : V \to U$ is a homeomorphism onto $\mathbb{V}(I)$.

Note that $\mathbb{V}(I) \cong \operatorname{Spec} A/I$, so we can freely identify the two. Let $g : V \to \operatorname{Spec} A/I$ be the homeomorphism induced by $f|_V : V \to U$. We note that for all $x \in V$, we have that $f|_V(x) = g(x)$. If $W \subset U$ is open, we have that $W \cap \mathbb{V}(I)$ is open in $\operatorname{Spec} A/I$, and we thus have that:

$$(f|_V)^{-1}(W) = (f|_V^{-1})(W) \cap (f|_V)^{-1}(\mathbb{V}(I)) = f|_V^{-1}(W \cap \mathbb{V}(I)) = g^{-1}(W \cap \mathbb{V}(I))$$

It follows that for any open set $Z = W \cap \mathbb{V}(I) \subset \mathbb{V}(I)$:

$$g_*\mathscr{O}_V(Z) = (f|_V)_*\mathscr{O}_V(W)$$

In particular, if U_g is an affine open of $\operatorname{Spec} A$, then:

$$g_*\mathscr{O}_V(U_{[g]}) = (f|_V)_*\mathscr{O}_V(U_g)$$

We thus define a morphism $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ on a basis of affine opens by noting that for each U_g we have a morphism:

$$(f|_V)^\sharp_{U_g} : A_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

whose kernel is precisely I_g. It follows that we get a unique morphism:

$$g^\sharp_{U_{[g]}} : \mathscr{O}_{\operatorname{Spec} A/I}(U_{[g]}) = A_g/I_g \longrightarrow g_*\mathscr{O}_V(U_{[g]})$$

which is trivially injective on each distinguished open. Moreover, these maps then clearly commute with the restriction maps, since localization commutes with taking quotients, as we have shown earlier. It follows that $g^\sharp : \mathscr{O}_{\operatorname{Spec} A/I} \to g_*\mathscr{O}_V$ is an injective morphism of sheaves, and is surjective on stalks because $(f|_V)^\sharp$ is. Since it is injective and surjective on stalks, we have that $g^\sharp$ is an isomorphism, implying that $f^{-1}(U) \cong \operatorname{Spec} A/I$ as schemes as desired. It follows that $f|_V : V \to U$ is now a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$, such that the kernel of $\psi : A \to A/I$ is precisely I, hence up to isomorphism ψ is the projection map as desired.

Now suppose that for every affine open $U = \operatorname{Spec} A \subset Y$ we have that $f^{-1}(U) \cong \operatorname{Spec} A/I$, for some ideal I. Then with $V = f^{-1}(U)$, we have that $f|_V : V \to U$ is a morphism of affine schemes $\operatorname{Spec} A/I \to \operatorname{Spec} A$ induced by the projection. By Example 3.1.1, we have that $f|_V$ is a a closed immersion for all V. Since locally we have that $f^\sharp$ comes from the projection, we have that the stalk map $(f^\sharp)_y : (\mathscr{O}_Y)_y \to (f_*\mathscr{O}_X)_y$, is surjective. It follows that $f^\sharp$ is surjective by Proposition 1.2.8. Moreover, since each $f|_V$ is a homeomorphism onto it's image for all U, we have that $f : X \to Y$ must also be a homeomorphism onto it's image. Let $\{U_i\}$ be an open cover of Y, and $V_i = f^{-1}(U_i)$ then $f(X) \cap U_i = f|_{V_i}(V_i)$ which is closed in U_i. It follows that $U_i \setminus f|_{V_i}(V_i)$ is open in Y. We claim that:

$$Y \setminus f(X) = \bigcup_i U_i \setminus f|_{V_i}(V_i)$$

Indeed, suppose that $y \in Y \setminus f(X)$, then for all i, we have that there is no $x \in V_i$ such that $f|_{V_i}(x) = y$. It follows that $y \in U_i \setminus f|_{V_i}(V_i)$ for all i, hence $Y \setminus f(X) \subset \bigcup_i U_i \setminus f|_{V_i}(V_i)$. Now suppose that:

$$y \in \bigcup_i U_i \setminus f|_{V_i}(V_i)$$

then for all i we have that there so no x such that $f|_{V_i}(x) = y$, hence there is no $x \in X$ such that $f(x) = y$ so $y \in Y \setminus f(X)$ giving us the other inclusion. Since $Y \setminus f(X)$ is the union of open sets, it is open, implying that $f(X)$ is closed, f is a homeomorphism onto its image, and $f^\sharp$ is surjective, hence f is a closed embedding implying the claim. □

We have the following obvious corollaries:

Corollary 3.1.1. *If $X \to \operatorname{Spec} A$ is a closed embedding then $X \cong A/I$ for some I.*

Corollary 3.1.2. *A morphism $f : X \to Y$ is a closed embedding if and only if there exists an affine cover $\{U_i\}$ of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a closed embedding.*

We can properly define closed subschemes now:

Definition 3.1.2. Let X be a scheme, then a **closed subscheme** of X is an equivalence class of closed immersions $f : Z \to X$, where two closed immersions f and g are equivalent if and only if there is an isomorphism $F : Z \to Z$ such that $f \circ F = g$.

The clunky nature of the definition of above can be best explained by noting that for $X = \operatorname{Spec} \mathbb{C}[x]$, we have that $\mathbb{V}(x) = \mathbb{V}(x^2)$ as $\sqrt{\langle x^2 \rangle} = \sqrt{x}$, but $\operatorname{Spec} \mathbb{C}[x]/\langle x \rangle \ncong \operatorname{Spec} \mathbb{C}[x]/\langle x^2 \rangle$. So even though the two topological spaces agree, and both are the same from a topological embedding point of view, the two closed subschemes are not isomorphic. In particular, there are a multitude of scheme structures one can put on a closed subspace of any scheme X, with the induced reduced subscheme structure being just one of many.

Example 3.1.2. Let $X = \operatorname{Proj} A$ for a graded ring A, and Z a closed subscheme of X. Furthermore, suppose that the irrelevant ideal satisfies[52]:

$$A_+ = \sqrt{\langle g_1, \ldots, g_n \rangle} \tag{3.1.1}$$

for some $g_i \in A_+^{\text{hom}}$. Note that this condition is equivalent to $\operatorname{Proj} A$ being quasi-compact; indeed, suppose that $\operatorname{Proj} A$ is quasi-compact then there clearly exists a finite covering of X by projective distinguished opens $\{U_{g_i}\}$. Since $\mathbb{V}(A_+) = \emptyset$, we have that:

$$\mathbb{V}(A_+) = \left(\bigcup_{i=1}^n U_{g_i} \right)^c = \bigcap_{i=1}^n \mathbb{V}(\langle g_i \rangle) = \mathbb{V}(\langle g_1, \ldots, g_n \rangle)$$

so (3.1.1) follows immediately. Now suppose that (3.1.1) holds, then X is equal to the union of U_{g_i}, which is finite, hence X is a finite union of quasi-compact schemes and is thus quasi-compact[53].

With the quasi-compactness assumption on X, we wish to show that Z is of the form $\operatorname{Proj} A/I$ for some homogeneous ideal $I \subset A$. Supposing (2.4.1), we have an open cover of X given by $\{U_{g_i}\}$, and

[52]Note that A_+ is radical, as if $f \in \sqrt{A_+}$, then for some n, $f^n \in A_+$. If f has a degree zero part then f^n has a degree zero part hence $f^n \notin A_+$. It follows that f is a sum of positively graded elements, and thus $f \in A_+$.

[53]In general topology this is the same as say if X is a finite union of compact spaces then X is compact. This setting just feels weird as for Hausdorff spaces compact sets are closed.

thus we obtain a finite open cover of Z by $\{V_i = f^{-1}(U_{g_i})\}$. Since f is a closed embedding, each $V_i = \operatorname{Spec}(A_{g_i})_0/I_i$; our goal is to construct I out of these I_i. Let $m_i = \deg g_i$, for each i, and define:

$$J_{i,d} = \begin{cases} \{0\} & \text{if } m_i \nmid d \\ \{a \in A_d : a/g_i^{d/m_i} \in I_i\} & \text{if } m_i \mid d \end{cases}$$

Note that $\deg(a/g_i^{d/m_i}) = d - d/m_i \cdot m_i = 0$, so $a/g_i^{d/m_i} \in (A_{g_i})_0$. We set:

$$J_i = \bigoplus_d J_i$$

It is clear that J_i is a homogeneous ideal for each i, hence we set:

$$I = \bigcap_{i=1}^n J_i$$

We want to show that $f(Z) = \mathbb{V}(I)$, and it suffices to show that $f|_{V_i}(V_i) = \mathbb{V}(I) \cap U_{g_i}$ for all i. If $\pi_i : A \to A_{g_i}$ is the localization map, and $\imath_i : (A_{g_i})_0 \to A_{g_i}$ is the inclusion, then we set:

$$(I_{g_i})_0 = \imath_i^{-1}(\langle \pi_i(I) \rangle)$$

Let $\phi : U_{g_i} \to (\operatorname{Spec} A_{g_i})_0$ be the homeomorphism from Proposition 2.2.2 given by $\mathfrak{p} \mapsto \mathfrak{p}_{g_i} \mapsto (\mathfrak{p}_{g_i})_0$; we first claim that:

$$\mathbb{V}(I) \cap U_{g_i} = \phi^{-1}(\mathbb{V}((I_{g_i})_0)) \subset U_{g_i}$$

Let $\mathfrak{p} \in \mathbb{V}(I) \cap U_{g_i}$, then $\mathfrak{p}$ is a homogeneous prime ideal such that $I \subset \mathfrak{p}$, and $g_i \notin \mathfrak{p}$. Since $\mathfrak{p} \in U_{g_i}$, we have that $\phi(\mathfrak{p}) = (\mathfrak{p}_{g_i})_0 \subset (A_{g_i})_0$. Since $I \subset \mathfrak{p}$, we have that $I_{g_i} \subset \mathfrak{p}_{g_i}$, hence $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$, so $\mathfrak{p} \in \phi^{-1}(\mathbb{V}((I_{g_i})_0))$.

Now suppose that $\mathfrak{p} \in \phi^{-1}(\mathbb{V}(I_{g_i})_0) \subset U_{g_i}$, then $\mathfrak{p} \in U_{g_i}$ vacuously, so we need to show that $\mathfrak{p} \in \mathbb{V}(I)$. By definition, $(I_{g_i})_0 \subset (\mathfrak{p}_{g_i})_0$; in A_{g_i}, we have that $(\mathfrak{p}_{g_i})_0$ corresponds to $\sqrt{(\mathfrak{p}_{g_i})_0 A_f}$, so we have that $\sqrt{(I_{g_i})_0 A_f} \subset \sqrt{(\mathfrak{p}_{g_i})_0 A_f}$ as well. It thus suffices to show that $I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$, as then:

$$I \subset \pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right) \subset \pi_i^{-1}\left(\sqrt{(\mathfrak{p}_{g_i})_0 A_f}\right) = \mathfrak{p}$$

Furthermore, as I is homogeneous, we need only check that every homogeneous element of I lies in $\pi_i^{-1}\left(\sqrt{(I_{g_i})_0 A_f}\right)$. Let $a \in I$ be homogeneous of degree d; if $a \in \ker \pi_i$ then we are done, otherwise, we have that $a^{m_i}/g_i^d \in (I_{g_i})_0$. It follows that $a^{m_i}/1 \in (I_{g_i})_0$, hence $a^{m_i}/1 \in (I_{g_i})_0 A_f$, so $a/1 \in \sqrt{(I_{g_i})_0 A_f}$ by definition[54].

It now suffices to show that $f|_{V_i}(V_i) = \phi^{-1}(\mathbb{V}(I_{g_i}))$. Since $f|_{V_i}(V_i) \subset U_{g_i}$, we have that $f|_{V_i}$ is a homeomorphism onto the closed subset $\mathbb{V}(I_i) \subset \operatorname{Spec}(A_{g_i})_0$. Therefore, it suffices to check that $\mathbb{V}(I_i) = \mathbb{V}((I_{g_i})_0)$, and in particular that $I_i = (I_{g_i})_0$ for all i. Now note that the only elements in A_{g_i} which have degree zero are those of the form a/g_i^n where a is homogeneous and satisfying $\deg a = n \cdot m_i$. Let $a/g_i^n \in (I_{g_i})_0$, then $a/g_i^n \in I_{g_i}$, so $a/1 \in I_{g_i}$ as well. It follows that $a \in I \cap A_{n \cdot m_i}$, hence $a/g_i^n \in I_i$ for all i, so $(I_{g_i})_0 \subset I_i$ as desired.

Now let $a/g_i^n \in I_i$, and $l = \operatorname{lcm}(m_1, \ldots, m_n)$. We have that there exists a $k \leq r \in \mathbb{N}$ such that:

$$n = k \cdot l + r \Rightarrow n + (k - r) = (k+1)l$$

so by taking $a/g_i^n = ag^{k-r}/g^{k-r+n}$, we may assume that l divides n. Since $\ker f^\sharp$ is a sheaf of ideals, if $I_{ij} = \ker f^\sharp_{U_{g_i} \cap U_{g_j}}$, we have that $a|_{U_{g_i} \cap U_{g_j}} \in I_{ij}$. Recall that $U_{g_i} \cap U_{g_j} = U_{g_i g_j} = \operatorname{Spec}(A_{g_i g_j})_0$, hence we have that:

$$a/g^n|_{U_{g_i} \cap U_{g_j}} = ag_j^n/(g_i g_j)^n \in I_{ij} \subset (A_{g_i g_j})_0$$

[54]Note that we have now shown that for any homogeneous ideal I, $\mathbb{V}(I) \cap U_h = \mathbb{V}((I_h)_0) \subset U_h$

Moreover, we also have that

$$U_{g_ig_j} \cong \mathrm{Spec}((A_{g_i})_0)_h$$

where $h = g_j^{m_i}/g_i^{m_j}$. The ring homomorphism

$$f^\sharp_{U_{g_i}} : (A_{g_i})_0 \to (A_{g_i})_0/I_i$$

determines a morphism of affine schemes which on all distinguished opens of $\mathrm{Spec}(A_{g_i})_0$ of the form U_b, has kernel given by $(I_i)_b$. The morphism determined by $f^\sharp_{U_i}$ must agree with f on all open subsets of U_{g_i}, hence we have that I_{ij} is naturally isomorphic to the ideal $(I_i)_h$, via the unique isomorphism $((A_{g_i})_0)_h \cong (A_{g_ig_j})_0$ from Lemma 2.2.7. Similarly, with $h^{-1} = g_i^{m_j}/g_j^{m_i}$, we must have that $(I_j)_{h^{-1}}$ is naturally isomorphic to I_{ij} via the same isomorphism. Any element in $(I_j)_{h^{-1}}$ can be written as:

$$\frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}}\right)^{-e} \tag{3.1.2}$$

where $b/g_j^k \in I_j$. Recall that we took n to be divisible by l, so $n = m_i \cdot p$ and $n = m_j \cdot q$ for some p and q. Hence, under the isomorphism $(I_i)_h \cong (I_j)_{h^{-1}}$ we have that:

$$\frac{a}{g_i^{m_j\cdot q}} \longmapsto \frac{a}{g_j^{m_i\cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i\cdot q}}\right)^{-1}$$

So for an element of the form (3.1.2) we must have that:

$$\frac{a}{g_i^{m_j\cdot q}} \longmapsto \frac{a}{g_j^{m_i\cdot q}} \cdot \left(\frac{g_i^n}{g_j^{m_i\cdot q}}\right)^{-1} = \frac{b}{g_j^k} \cdot \left(\frac{g_i^{m_j}}{g_j^{m_i}}\right)^{-e}$$

We thus have that by the definition of localization we have that:

$$\frac{g_i^{m_j\cdot e} a}{g_j^{m_i\cdot e + m_i\cdot q}} \in I_j$$

We can take e large enough so that $e'_j = m_j \cdot e$ is divisible by l, hence we can write that:

$$\frac{g_i^{e'_j} a}{g_j^{(e'+n)\cdot(m_i/m_j)}} \in I_j$$

Do this for all j, and let $e' = \max(e'_1, \ldots, e'_n)$, then $g_i^{e'} a \in J_j$ for all j. It follows that $g_i^{e'} a \in I$, hence:

$$\frac{g_i^{e'} a}{1} \in I_{g_i}$$

so $a/1 \in I_{g_i}$, giving us that $a/g^n \in (I_{g_i})_0$. It follows that $I_i = (I_{g_i})_0$ so $f(Z) = \mathbb{V}(I)$ as desired.

We now show that $\mathbb{V}(I)$ is homeomorphic to $\mathrm{Proj}\, A/I$. Let $\pi : A \to A/I$ be the projection map, where A/I has the induced grading, and $\mathfrak{p} \in \mathrm{Proj}\, A/I$. The prime ideal $\pi^{-1}(\mathfrak{p})$ is homogeneous, as if $a \in \pi^{-1}(\mathfrak{p})$ then we write a as:

$$a = \sum_d a_d \tag{3.1.3}$$

where $a_d \in A_d$. It follows that $\pi(a) \in \mathfrak{p}$, and since $\mathfrak{p}$ is homogeneous each $\pi(a_d)$ is in $\mathfrak{p}$ so each $a_d \in \pi^{-1}(\mathfrak{p})$. Each $\pi^{-1}(\mathfrak{p})$ contains I so this defines a map $F : \mathrm{Proj}\, A/I \to \mathbb{V}(I)$. Via the bijection between prime ideals of A/I and prime ideals of A containing I it follows that this map is a bijection, so it suffices to check that this is continuous and open.

We can do this on the distinguished basis for $\mathrm{Proj}\, A/I$ and the basis $\{\mathbb{V}(I) \cap U_g\}_{g \in A_+^{\mathrm{hom}}}$ for $\mathbb{V}(I)$. Let U_g be the projective distinguished open in $\mathrm{Proj}\, A$, then

$$F^{-1}(V(I) \cap U_g) = F^{-1}(V(I)) \cap F^{-1}(U_g) = F^{-1}(U_g)$$

I claim that this is equal to $U_{[g]}$. Suppose $[g] \notin \mathfrak{p} \subset A/I$, then for all $i \in I$ we must have that $g+i \notin \pi^{-1}(\mathfrak{p})$ hence $g \notin \pi^{-1}(\mathfrak{p})$. It follows that $\mathfrak{p} \in U_g$ so $U_{[g]} \subset U_g$. Now let $\mathfrak{p} \in f^{-1}(U_g)$, then $g \notin \pi^{-1}(\mathfrak{p})$, but this implies that $[g] \notin \pi(\pi^{-1}(\mathfrak{p})) = \mathfrak{p}$ so $\mathfrak{p} \in U_{[g]}$. Therefore $f^{-1}(U_g) = U_{[g]}$ and f is continuous.

To show that F is open we claim that $F(U_{[g]}) = V(I) \cap U_g$, but this is now clear as $F : \operatorname{Proj} A/I \to V(I)$ is bijective, so since $F^{-1}(V(I) \cap U_g) = U_{[g]}$ we get that $F(F^{-1}(V(I) \cap U_g) = V(I) \cap U_g = U_{[g]}$. It follows that f is a continuous open bijective map and thus a homeomorphism.

Now note that the structure sheaf $\mathscr{O}_{\operatorname{Proj} A/I}$ satisfies:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = ((A/I)_{[g]})_0$$

However, recall that there is a unique surjective homomorphism

$$\begin{aligned} A_g &\longrightarrow (A/I)_{[g]} \\ a/g^k &\longmapsto [a]/[g]^k \end{aligned}$$

which commutes with localization maps, and clearly preserves grading. It follows, that we have a unique surjective homomorphism commuting with the isomorphisms from Lemma 2.2.7:

$$\begin{aligned} (A_g)_0 &\longrightarrow ((A/I)_{[g]})_0 \\ a/g^k &\longrightarrow [a]/[g]^k \end{aligned}$$

where $\deg a = k \cdot \deg g$. Note that clearly $(I_g)_0$ maps to zero under this map, so we have unique surjective homomorphism:

$$\begin{aligned} \phi : (A_g)_0/(I_g)_0 &\longrightarrow ((A/I)_{[g]})_0 \\ [a/g^k] &\longrightarrow [a]/[g]^k \end{aligned}$$

Now suppose that $\phi([a/g^k]) = 0$, then we have that $[a]/[g]^k = 0 \in ((A/I)_{[g]})_0 \subset (A/I)_{[g]}$. It follows that there an M such that $[g^M \cdot a] = 0 \in A/I$, hence $g^M a \in I$. We thus have that $g^M a/1 \in I_g$, so $g^M a/g^{M+k} = a/g^k \in (I_g)_0$. By the naturality[55] of these isomorphisms it follows that up to a unique sheaf isomorphism:

$$\mathscr{O}_{\operatorname{Proj} A/I}(U_{[g]}) = (A_g)_0/(I_g)_0$$

Now equip $\mathbb{V}(I)$ with the sheaf $\mathscr{O}_{\mathbb{V}(I)} = F_* \mathscr{O}_{\operatorname{Proj} A/I}$, and note that this endows $\mathbb{V}(I)$ with the structure of a scheme isomorphic to $\operatorname{Proj} A/I$[56].

Let $\tilde{f}$ be restriction of the codomain to $\mathbb{V}(I)$. In particular, we have that:

$$\tilde{f} : Z \longrightarrow \mathbb{V}(I)$$

Since $I_i = (I_{g_i})_0$, we define a sheaf morphism on the open cover $\{\mathbb{V}(I) \cap U_{g_i}\}$ as the identity map:

$$\tilde{f}^\sharp_{\mathbb{V}(I) \cap U_{g_i}} : \mathscr{O}_{\mathbb{V}(I)}(\mathbb{V}(I) \cap U_{g_i}) = (A_{g_i})_0/(I_{g_i})_0 \longrightarrow \mathscr{O}_Z(V_i) = (A_{g_i})_0/I_i$$

These then agrees on overlaps $U_{g_i} \cap U_{g_j}$ as $((I_{g_i})_0)_h \cong I_{ij} \cong ((I_{g_j})_0)_{h^{-1}}$ via the natural isomorphisms which glue $\operatorname{Proj} A$ together. It follows that this defines a sheaf isomorphism:

$$\tilde{f}^\sharp : \mathscr{O}_{\mathbb{V}(I)} \longrightarrow \mathscr{O}_Z$$

hence $(\tilde{f}, \tilde{f}^\sharp)$ determines a scheme isomorphism $Z \to \mathbb{V}(I)$. Since $\mathbb{V}(I) \cong \operatorname{Proj} A/I$ as schemes, we thus have that $Z \cong \operatorname{Proj} A/I$ as desired.

[55] Note that $(A_g)_0/(I_g)_0$ does not depend on the class representative g, as for any homogeneous i of degree equal to g, $[a/(g+i)^k] = [a/g^k]$.

[56] This is not the reduced scheme structure, rather one induced by the sheaf of ideals determined by I itself. If $\mathbb{V}(I)$ was equipped with the reduced structure, then as schemes $\mathbb{V}(I) \cong \operatorname{Proj} A/\sqrt{I}$.

We now briefly show that the condition that $\operatorname{Proj} A$ be quasi-compact is extremely necessary. Indeed take:

$$X = \operatorname{Proj} k[x_1, x_2, \dots]$$

for any field k. Let:

$$Z = \coprod_{i=1}^{\infty} X_i = \operatorname{Spec} k[x_1/x_i, x_2/x_i, \dots, \hat{x}_i/x_i, \dots]/\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

where the ideal:

$$\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

is generated by all ith fold products of elements in $\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$. Note that each X_i is is a singleton set as

$$\sqrt{\langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i} = \langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle$$

Denote each point by $0^i \in X_i \subset Z$, and define a closed embedding by:

$$\begin{aligned} f : Z &\longrightarrow X \\ 0^i &\longmapsto [0, \dots, 0, 1, 0 \dots, 0, \dots] \end{aligned}$$

where the 1 is in the ith position. If we take the homogeneous ideal $I = \langle x_i x_j : i \neq j\rangle$, then clearly for all k:

$$(I_{x_k})_0 = \langle x_1/x_k, x_2, x_k, \dots \hat{x}_k/x_k, \dots\rangle$$

So under the identification $\mathbb{V}(I) \cap U_{x_i} = \mathbb{V}((I_{x_i})) \subset U_{x_i}$, we discern that $\mathbb{V}(I) \cap U_{x_i}$ contains only the point $[0, \dots, 0, 1, 0 \dots, 0, \dots]$, where the 1 is again in the ith position. Clearly we then have that for all U_{x_i}, $f(Z) \cap U_{x_i} = \mathbb{V}(I) \cap U_{x_i}$, hence $f(Z) = \mathbb{V}(I)$, and f has closed image.

We set:

$$I_i = \langle x_1/x_i, x_2/x_i, \dots, \hat{x}^i/x_i, \dots\rangle^i$$

and define a sheaf morphism on the affine open cover $\{U_{x_i}\}_{i=1}^{\infty}$ via the canonical projections:

$$\begin{aligned} f^\sharp_{U_{x_i}} : k[\{x_j/x_i\}_{j=1, j\neq i}^{\infty}] &\longrightarrow k[\{x_j/x_i\}_{j=1, j\neq i}^{\infty}]/I_i \\ g &\longmapsto [g] \end{aligned}$$

and note that there is nothing to glue as $f^{-1}(U_{x_i} \cap U_{x+j})$ is the empty set. This sheaf homomorphism is clearly surjective on stalks so $Z \hookrightarrow \mathbb{P}^{\infty}_k$ is a closed embedding.

We claim that there is no homogeneous ideal I such that $Z \cong \operatorname{Proj} A/I$. Indeed, suppose there was. Then by the work above we would have that for all x_i,

$$(I_{x_i})_0 = I_i$$

Let $f \in I$ be a nonzero homogeneous element of degree d, then $f/1 \in I_{x_i}$, and $f/x_i^d \in (I_{x_i})_0$ for all i.[57] For the above to be true, we must then have that $f/x_i^d \in I_i$ for all i as well. However, if $k > d$, for f/x_k^d to lie in I_k, we must have that f/x_k^d is a sum of k fold products of elements of the form x_i/x_k, an obvious contradiction if f is nonzero, hence $f = 0$. Since this can be done for arbitrary degree d, as there is no upper bound on the ideals I_i, we have that Z cannot possibly be isomorphic to $\operatorname{Proj} A/I$, as no homogeneous ideal can agree with I_i on the affine open cover.

With the above example in mind, we can classify all projective schemes over some fixed ring B:

[57]Note that f/x_i^d cannot be zero as k is a field, so localization maps are injective.

Theorem 3.1.1. *X is a projective scheme over B if and only if it is a closed subscheme of $\mathbb{P}^n_B$ for some n.*

Proof. Suppose that X is a projective scheme over B, then by Definition 2.2.7, we have that:

$$X = \operatorname{Proj} A$$

where A is a graded ring, satisfying $A_0 = B$, and is finitely generated in degree one as a B algebra. Since A is finitely generated in degree one, let $a_1, \ldots, a_n$ be a generating set of degree one elements; this defines a surjection

$$\phi : B[x_0, \ldots, x_n] \to A$$

which preserves grading. It follows that $\ker \phi$ is a homogeneous ideal, and that $A \cong B[x_0, \ldots, x_n]/\ker \phi$, hence:

$$X = \operatorname{Proj}(B[x_0, \ldots x_n]/\ker \phi)$$

As a scheme, X is canonically isomorphic to $\mathbb{V}(\ker \phi) \subset \mathbb{P}^n_B$[58], hence X determines a closed subscheme of $\mathbb{P}^n_B$.

If X is a closed subscheme of $\mathbb{P}^n_B$, then since $\mathbb{P}^n_B$ is quasicompact, we have that by Example 3.1.2 $X \cong \operatorname{Proj} B[x_0, \ldots, x_n]/I$ for some homogeneous ideal I. If I contains the irrelevant ideal, then X is the empty scheme and thus isomorphic to $\operatorname{Proj} B$, where B has the trivial grading, so X is trivially a projective B scheme. If I does not contain the irrelevant ideal, then $B[x_0, \ldots, x_n]/I$ is a graded, finitely generated in degree one, B-algebra, hence X is projective B scheme as desired. □

Example 3.1.3. Recall from Example 2.3.4 that locally the morphism:

$$f : \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}$$

is given by scheme morphisms:

$$U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$$

The U_{v_i} cover $\mathbb{P}^3_{\mathbb{C}}$, and $f^{-1}(U_{v_i}) = U_{x_l} \times_{\mathbb{C}} U_{y_n}$. We claim that this morphism is a closed embedding, and by Corollary 3.1.2 it suffices to check that each $U_{x_l} \times_{\mathbb{C}} U_{y_n} \to U_{v_i}$ is a closed embedding. By Corollary 3.1.2, it suffices to check that $U_{x_l} \times_{\mathbb{C}} U_{y_n} \cong \operatorname{Spec} \mathbb{C}[\{v_k/v_i\}_{k \neq i}]/I$ for some ideal I. We check this in case of $i = 0$. Note that the morphism of affine schemes comes from the ring homomorphism:

$$\begin{aligned} \phi_0 : \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0] &\longrightarrow \mathbb{C}[x_1/x_0, y_1/y_0] \\ v_i/v_0 &\longmapsto \begin{cases} x_1/x_0 & \text{if } i = 1 \\ y_1/y_0 & \text{if } i = 2 \\ (x_1/x_0) \cdot (y_1/y_0) & \text{if } i = 3 \end{cases} \end{aligned}$$

This is clearly surjective, hence $\mathbb{C}[x_1/x_0, y_1/y_0] \cong \mathbb{C}[v_1/v_0, v_2/v_0, v_3/v_0]/\ker \phi_0$, and it follows that the induced morphism is a closed embedding. The kernel of this homomorphism is:

$$I = \langle v_1/v_0 \cdot v_2/v_0 - v_3/v_0 \rangle$$

and so the homogeneous ideal cutting out $f(\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}})$ is given by $J = \langle v_1 v_2 - v_3 v_0 \rangle$. It follows that as schemes,

$$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1 \cong \operatorname{Proj} \mathbb{C}[v_0, v_1, v_2, v_3]/\langle v_1 v_2 - v_3 v_0 \rangle$$

[58]Note that $\mathbb{V}(\ker \phi)$ is not necessarily equipped with the reduced subscheme structure, but instead equipped with scheme structure determined by the sheaf of ideals induced by $\ker \phi$. This only coincides with the reduced structure if $\ker \phi$ satisfies $\sqrt{\ker \phi} = \ker \phi$.

Example 3.1.4. Let $Z \subset X$ be a closed subset of a scheme X, and equip Z with the induced reduced closed subscheme structure, then we have that the inclusion map $\imath : Z \to X$ is a homeomorphism onto its image. We want to define a sheaf morphism $\imath^\sharp : \mathscr{O}_X \to \imath_* \mathscr{O}_Z$. Recall that if $I_{Z/X}$ is the sheaf of ideals associated to the closed subset Z then $\mathscr{O}_Z = \imath^{-1}\mathscr{O}_X/I_{Z/X}$. By Corollary 1.3.4, we have that there is a canonical morphism:

$$\mathscr{O}_X/I_{Z/X} \longrightarrow \imath_*\imath^{-1}\mathscr{O}_X/I_{Z/X}$$

which is surjective. There is a surjective morphism $\mathscr{O}_X \to \mathscr{O}_X/I_{Z/X}$, so we define $\imath^\sharp$ to the be composition of these sheaf morphisms. It follows that $(\imath, \imath^\sharp)$ is a closed embedding as desired.

We now go to our next result regarding closed embeddings which is an analogue of Lemma 2.3.7:

Lemma 3.1.2. *Let $f : X \to Z$ be a closed embedding, and let $g : Y \to Z$ be any morphism. Then the base change $X \times_Z Y \to Y$ is also a closed embedding.*

Proof. We have the following Cartesian square:

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

Let $\{U_i = \operatorname{Spec} A_i\}$ be an affine open cover of Z, and choose an affine open cover $\{V_{ij} = \operatorname{Spec} B_{ij}\}$ of Y such that $g(V_{ij}) \subset U_i$. Note that $f^{-1}(U_i) \cong \operatorname{Spec} A_i/I_i$ for some ideal I_i. We have that:

$$\pi_Y^{-1}(V_{ij}) \cong X \times_Z V_{ij}$$

We claim that this is isomorphic to $f^{-1}(U_i) \times_{U_i} V_i$. Indeed, we need to show that the following diagram is cartesian:

$$\begin{tikzcd} f^{-1}(U_i) \times_{U_i} V_i \arrow[r, "\pi_Y"] \arrow[d, "\pi_X \circ \imath"] & V_{ij} \arrow[d, "g|_{V_{ij}}"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

where $\imath : f^{-1}(U_i) \to X$ is the inclusion, is Cartesian. Let Q be any scheme with maps $p_X : Q \to X$ and $p_{V_{ij}} : Q \to V_{ij}$ which make the relevant diagram commute. Since $g(V_{ij}) \subset U_i$, we have that $g \circ p_{V_{ij}}(Q) \subset U_i$. Since $f \circ p_X = g \circ p_{V_{ij}}$, we have that $f \circ p_X(Q) \subset U_i$ as well, and thus $p_X(Q) \subset f^{-1}(U_i)$. Since $X \times_Z V_{ij}$ is a fibre product we have a unique morphism $\phi : Q \to X \times_Z V_{ij}$ such that $\pi_X \circ \imath \circ \phi = p_X$. We thus have that $\pi_X \circ \imath \circ \phi(Q) = p_X(Q) \subset f^{-1}(U_i)$. We see that $(\pi_X \circ \imath)^{-1}(f^{-1}(U_i)) \subset f^{-1}(U_i) \times_Z V_i$, hence $\phi(Q) \subset p_X(Q)$. Since both $f(f^{-1}(U_i)) \subset U_i$, and $g(V_{ij}) \subset U_i$, we have that this $f^{-1}(U_i) \times_Z V_{ij} = f^{-1}(U_i) \times_{U_i} V_{ij}$, and it follows that $\phi(Q) \subset f^{-1}(U_i) \times_{U_i} V_{ij}$, so ϕ factors uniquely through the open embedding $f^{-1}(U_i) \times_{U_i} V_{ij} \to X \times_Z V_{ij}$, and we have a unique morphism $Q \to f^{-1}(U_i) \times_{U_i} V_{ij}$. It follows that $f^{-1}(U_i) \times_{U_i} V_{ij} \cong f^{-1}(U_i) \times_{U_i} V_{ij}$ as desired. We thus have the following chain of isomorphisms:

$$\begin{aligned} \pi_Y^{-1}(V_{ij}) &\cong X \times_Z V_{ij} \\ &\cong f^{-1}(U_i) \otimes_{U_i} V_{ij} \\ &\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} \operatorname{Spec} B_{ij} \\ &\cong \operatorname{Spec} A_i/I_i \otimes_{A_i} B_{ij} \end{aligned}$$

Let $\phi : A_i \to B_{ij}$ be the ring homomorphism making B_{ij} an A_i algebra, and set $J = \langle \phi(I_i) \rangle$, then we have that:

$$A_i/I_i \otimes_{A_i} B_{ij} \cong B_{ij}/J$$

hence:

$$\pi_Y^{-1}(V_i) \cong \operatorname{Spec} B_{ij}/J$$

so $\pi_Y : X \times_Z Y \to Y$ is a closed embedding by Corollary 3.1.2. ◻

These two lemmas each provide an example of properties of morphisms we are about to study, namely being *local on target* and *stable under base change*. More precisely, let $f : X \to Z$ be a morphism of schemes and P a property morphisms of schemes, then P is local on target if for any affine cover of $\{U_i\}$ of Z such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ satisfies P for all i we have that f satisfies P, and if $f : X \to Y$ satisfies P, then for all affine opens U, $f|_{f^{-1}(U)}$ satisfies P as well. In other words, a property of a morphism of schemes is called local on target if it can be checked affine locally. Let $g : Y \to Z$ be any other morphism of schemes, and let $f : X \to Z$ be a morphism satisfying P, then P is stable under base change if $X \times_Z Y \to Y$ also satisfies the property.

There is a third property of morphisms important to study, and that is notion of begin *closed under composition*. In particular if P is a property of morphisms, then P is closed under composition if for all $f : X \to Y$ and $g : Y \to Z$ satisfying P, then $g \circ f$ satisfies P as well.

Lemma 3.1.3. *Closed embeddings are closed under composition.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be closed embeddings, and let $U = \operatorname{Spec} A$ be an open affine. Then since g is a closed embedding, there is some $I \subset A$ such that:

$$(g \circ f)^{-1}(U) = f^{-1}(\operatorname{Spec} A/I)$$

Since g is a closed embedding, there is some $J \subset A/I$ such that:

$$(g \circ f)^{-1}(U) = \operatorname{Spec}(A/I)/J$$

Let I' be the kernel of the morphism $A \to A/I \to (A/I)/J$, then $(A/I)/J \cong A/I'$, so $g \circ f$ is a closed embedding by Lemma 3.1.1. $\square$

We end the section with the following general result:

Theorem 3.1.2. *Let P be a property of a morphism of Z-schemes $f : X \to Y$ such that P is closed under composition and stable under base change. Then if $f : X \to Y$ and $g : X' \to Y'$ both satisfy P then the induced morphism $f \times g : X \times_Z X' \to Y \times_Z Y'$ satisfies property P.*

Proof. Let h and h' be the morphisms making Y and Y' Z-schemes, and q and q' the morphisms making X and X' Z-schemes. We have that $f \times g$ comes from the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[dr, "f\times g"] \arrow[drr, bend left, "g\circ\pi_{X'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h'"] \\
& Y \arrow[r, "h"] & Z
\end{tikzcd}$$

It is clear that $f \times g = \mathrm{Id} \times g \circ f \times \mathrm{Id}$, so it suffices to show that $f \times \mathrm{Id}$ and $\mathrm{Id} \times g$ both satisfy proper P. We have the following commutative diagram:

$$\begin{tikzcd}
X \times_Z X' \arrow[r, "f\times \mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_Y"] & X' \arrow[d, "q'"] \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

The right square is Cartesian, and since $h \circ f = q$, and $\pi_{X'} \circ f \times \mathrm{Id} = \pi_{X'}$ the outer diagram is Cartesian, so the left square is also Cartesian. Since the left square is Cartesian, it follows that $f \times \mathrm{Id}$ is the base change of f, and thus satisfies property P. Now note that we also have the following commutative diagram:

$$\begin{tikzcd} Y \times_Z X' \arrow[r, "\mathrm{Id}\times g"] \arrow[d, "\pi_{X'}"] & Y \times_Z Y' \arrow[r, "\pi_Y"] \arrow[d, "\pi_{Y'}"] & Y \arrow[d, "h"] \\ X' \arrow[r, "g"] & Y' \arrow[r, "h'"] & Z \end{tikzcd}$$

The right square is Cartesian, and the outer square satisfies $h' \circ g = q'$, and $\pi_Y \circ \mathrm{Id} \times g = \pi_Y$, so it is Cartesian as well. It follows that the left square is Cartesian, and that $\mathrm{Id} \times g$ is the base change of g, so $\mathrm{Id} \times g$ satisfies property P as well. Since P is closed under composition, we have that $f \times g$ satisfies property P as well. □

3.2 Reduced, Irreducible, and Integral Schemes

In the following sections, we will study some algebraic and topological properties schemes may have, and the interplay between them. We begin with the following definition:

Definition 3.2.1. Let X be a scheme, then X is **irreducible** if it is irreducible as a topological space as in Definition 1.4.3. We also have that X is **reduced** if $\mathscr{O}_X(U)$ has no nilpotents for all $U \subset X$, and is **integral** if $\mathscr{O}_X(U)$ is an integral domain for all $U \subset X$.

We first check that being reduced is an inherently local property.

Lemma 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is reduced

b) There exists an affine open cover $\{U_i\}$ such that each U_i is reduced

c) Every stalk $(\mathscr{O}_X)_x$ is reduced

Proof. Clearly $a \Rightarrow b$, so we first show that $b \Rightarrow c$. Let $x \in U_i = \operatorname{Spec} A$, then $(\mathscr{O}_X)_x \cong A_\mathfrak{p}$ so it suffices to check that $A_\mathfrak{p}$ has no nilpotents. Let $a/g \in A_\mathfrak{p}$ where $g, h \notin \mathfrak{p}$. Then if $(a/g)^k = 0$ for some k there exists a $c \in \mathbb{A} - \mathfrak{p}$ such that:

$$c \cdot a^k = 0$$

We see that $c \neq 0$, so since A has not nilpotents we have that either $a = 0$ hence $a/g = 0$, implying the claim.

Now we show that $c \Rightarrow a$. Let U be an open set of X, and $s \in \mathscr{O}_X(U)$ such that $s^k = 0$ for some k. Then for every $x \in U$ we have that $(s^k)_x = s_x^k = 0$ implying that $s_x = 0$ for all s. However, the map:

$$\mathscr{O}_X(U) \longrightarrow \prod_{x \in U} (\mathscr{O}_X)_x$$

is injective so $s = 0$, hence $\mathscr{O}_X(U)$ has no nilpotents. □

Example 3.2.1. Let X be a scheme, and Y a closed subset of X, then Y equipped with the induced reduced closed subscheme structure is reduced. Indeed, let $\{U_i = \operatorname{Spec} A\}$ be an affine open cover of X, then $U_i \cap Y \cong \operatorname{Spec} A_i/I_i$ determines an affine open cover of Y. Each I_i is radical, hence we have that if $[a] \in A_i/I_i$ satisfies $[a]^k = 0$ then $a^k \in I_i$, implying that $a \in I_i$. It follows that $[a] = 0$, so A_i/I_i is reduced. We thus have an affine open cover of Y such that each affine scheme is reduced, so by Lemma 3.2.1 we have that Y is reduced as well. In particular, X is a closed subset of X, and thus there is a reduced scheme X_{red}, such that the underlying topological space is X, and it's structure sheaf is $\mathscr{O}_{X_{\text{red}}} = \mathscr{O}_X/I$, where I is the sheaf of ideals corresponding to X. In particular, the stalks $\mathscr{O}_{X_{\text{red}},x}$ are isomorphic to $\mathscr{O}_{X,x}/\sqrt{\langle 0 \rangle}$, and on any affine open, $\mathscr{O}_{X_{\text{red}}}(U) \cong \mathscr{O}_X(U)/\sqrt{\langle 0 \rangle}$.

We now show some properties of X being irreducible. We need the following definition:

Definition 3.2.2. Let X be a topological space, then a **generic point** is a point $\eta \in X$ which is dense, i.e. $\overline{\{\eta\}} = X$.

Lemma 3.2.2. *Let X be an irreducible topological space, then every non empty open subset of X is irreducible when equipped with the subspace topology. Moreover, a topological space is irreducible if and only if the intersection of every two non empty open sets is non empty.*

Proof. Suppose that X is irreducible, then by Lemma 1.4.4 we have that X is connected and every open subset of X is dense. Let U be a non empty open subset of X, then we claim that U is irreducible when equipped with the subspace topology. Indeed, suppose that $U = Y_1 \cup Y_2$ for two proper closed subsets of U. Then $Y_1 = Z_1 \cap U$ and $Y_2 = Z_2 \cap U$, then we have that $U = (Z_1 \cup Z_2) \cap U$, implying that $U \subset Z_1 \cup Z_2$ hence U is contained in the closed subset $Z_1 \cup Z_2$. However, $\bar{U} = X$, so $X = Z_1 \cup Z_2$ implying that X is reducible. The claim follows from the contrapositive.

Now let U and V be two nonempty open subsets of X such that $U \cap V = \emptyset$. Then $U^c \cup V^c = X$, so X is reducible. By the contrapositive we have that if X is irreducible then $U \cap V \neq \emptyset$.

Suppose that $U \cap V \neq \emptyset$ for every open set, and let $Z_1, Z_2 \subset X$ be two proper closed subsets. We see that since $Z_1^c \cap Z_2^c \neq \emptyset$ that $Z_1 \cup Z_2 \neq X$, so X is irreducible □

Lemma 3.2.3. *Let X be a scheme, then X is irreducible if and only if X has unique generic point η.*

Proof. Suppose that X is reducible, then $X = Z_1 \cup Z_2$ for two closed proper subsets of X. It follows that every $x \in X$ lies in Z_1 or Z_2 so the closure of every point is contained in Z_1 or Z_2. We thus have that X has no generic points, let alone a unique one. By the contrapositive, we have that if X has a unique generic point, then X is irreducible.

Now let X be irreducible, by Lemma 3.2.2 we have that $U = \operatorname{Spec} A$ is a irreducible topological space as well. We claim that the nilradical:

$$I = \{a \in A : \exists k \in \mathbb{N}, a^k = 0\}$$

is prime. Let $f, g \in A$ then if $f, g \in I$ we have that $U_f = U_{f^k} = U_0 = \emptyset$ and similarly for g. Similarly, if $U_f = U_0$ then there is some k such that $f^k = 0$ so we have that a distinguished open is empty if and only if the element lies in I. Now suppose that $U_f \cap U_g$ is not empty, the fact that $U_f \cap U_g$ is not empty implies that $fg \notin I$. It follows by the contrapositive that if $fg \in I$ then either f or g are in I so I is prime. The closure of the singleton set $\{I\} \in \operatorname{Spec} A$ is given by $\mathbb{V}(I)$ and we claim that this is equal to $\operatorname{Spec} A$. We need only show that $I \subset \mathfrak{p}$ for any prime in A, however this is clear as $0 \in \mathfrak{p}$, and for any $f \in I$ we have that $f^k = 0 \in \mathfrak{p}$, hence $f \in \mathfrak{p}$, so $I \subset \mathfrak{p}$. We show that I is unique, suppose that $\mathfrak{q}$ is prime, and satisfies $\mathfrak{q} \subset \mathfrak{p}$ for every prime. Then $\mathbb{V}(\mathfrak{q}) = \operatorname{Spec} A$, so $\mathfrak{q} = \sqrt{\mathfrak{q}} = \sqrt{I} = I$ implying uniqueness.

We now claim that the point $x \in X$ corresponding to $I \in \operatorname{Spec} A$ is actually a generic point of X. Indeed, suppose that $\overline{\{x\}} = V$ for some closed subset of X, then we have that:

$$V = \bigcap_{Z \ni x} Z$$

where $Z \subset X$ is closed. In the subspace topology, since x is a generic point, we have that:

$$U = \bigcap_{Y \ni x} Y$$

where $Y \subset U$ is closed. The subsets of U which are closed are of the form $Z \cap U$ where Z is closed in X, hence we have that:

$$U = \bigcap_{Z \ni x} Z \cap U = V \cap U$$

hence $U \subset V$. However, the only closed set of X which contains U is X itself, so $V = X$ and $\{x\}$ is generic.

To show uniqueness, note that x lies in every open set of $U \subset X$, as other wise, $x \in U^c$, which is closed and thus contradicts the fact that $\{x\}$ is dense. Now suppose that U is any open affine, and $y \in X$ is a generic point not equal to x. Then y is clearly a generic point of every open affine, so $y, x \in U$ are both generic points. But then $x = y$ as every irreducible affine scheme only has one generic point, implying the claim. □

Note that if the nilradical of a ring A is prime then its vanishing locus is the whole of $\operatorname{Spec} A$, so $\operatorname{Spec} A$ contains a generic point, and is thus irreducible. In particular, $\operatorname{Spec} A$ is irreducible if and only if the nilradical is prime.

Lemma 3.2.4. *Let X be a scheme, which is not the empty scheme. Then the following are equivalent:*

a) X is irreducible

b) There exists an affine open covering $\{U_i\}$ of X such that U_i is irreducible for all i, and $U_i \cap U_j \neq \emptyset$ for all i and j.

c) Every nonempty open affine $U \subset X$ is irreducible.

Proof. Note that Lemma 3.2.2 implies that $a \Rightarrow b, c$. We show that $b \Rightarrow a$. Suppose that $X = Z_1 \cup Z_2$, then we have that since each U_i is irreducible $U_i \subset Z_1$ or Z_2. Indeed suppose otherwise, then $U_i \cap (Z_1 \cup Z_2) = (U_i \cap Z_1) \cup (U_i \cap Z_2)$ which are both closed in the subspace topology, thus $U_i \cap Z_j$ must equal Z_j for at least one j. Without loss of generality suppose that $U_i \subset Z_1$ and take any other U_j. Then $U_i \cap U_j$ is non empty and in dense in U_j. Since $U_i \subset Z_1$, we have that $U_i \cap U_j \subset Z_1 \cap U_j$, which is closed in U_j. It follows that the closure of $U_i \cap U_j$ is contained in Z_1, thus $U_j \subset Z_1$. We thus have that $\bigcup U_i = X \subset Z_1$, so $X = Z_1$, implying that X is irreducible.

For $c \Rightarrow a$, let $U \cap V$ be empty for some open affines, then $U \cup V$ is affine as it is trivially a disjoint union, and thus the coproduct in the category of schemes, and finite coproducts of affine schemes are affine by Example 2.1.3. However, irreducible spaces are connected, and $U \cup V$ is an affine open so is not irreducible contradicting c. It follows that the intersection of every open affine is non trivial, and since the open affines generate the topology on X we must have that the intersection of every open set is non empty, thus by Lemma 3.2.2 we have that X is irreducible. □

Example 3.2.2. Note that any disconnected scheme is not irreducible, we now give an example of a connected but reducible scheme. We first note that an affine scheme $\operatorname{Spec} A$ is connected if and only if it only has no nontrivial idempotents. Indeed, suppose that A has a nontrivial idempotent a, then $a \cdot a = a$. Note that $\langle a \rangle + \langle 1 - a \rangle = A$, implying that

$$\mathbb{V}(a) \cap \mathbb{V}(1-a) = \mathbb{V}(1) = \emptyset$$

Since $\langle a \rangle$ and $\langle 1 - a \rangle$ are coprime, we have that $\langle a \rangle \cap \langle 1 - a \rangle = \langle a \rangle \cdot \langle 1 - a \rangle = \langle 0 \rangle$. We thus have that:

$$\mathbb{V}(a) \cup \mathbb{V}(1-a) = \mathbb{V}(0) = \operatorname{Spec} A$$

But this then implies that $\mathbb{V}(a)^c = \mathbb{V}(1-a)$, so we have that $\operatorname{Spec} A$ is the union of two open disjoint sets, and thus disconnected. It follows by the contrapositive that if $\operatorname{Spec} A$ is connected, then there are no nontrivial idempotents.

Now suppose $\operatorname{Spec} A$ is disconnected, then there exist open sets such that $U \cap V = \emptyset$, and $U \cup V = \operatorname{Spec} A$. It follows that U and V are both also closed so $U = \mathbb{V}(I)$ and $V = \mathbb{V}(J)$ for two radical ideals I and J. Now we have that $I + J = A$, and $I \cap J = \{0\}$ so by the Chinese remainder theorem there is an isomorphism:

$$A \to A/I \times A/J$$

It follows that A is a product of two rings A/I and A/J so $\operatorname{Spec} A$ is the disjoint union of two affine schemes. It follows that $(1, 0)$ is a nontrivial idempotent of A, hence disconnected and affine implies the existence of an idempotent, and the claim follows from contradiction.

We thus wish to find a ring with no nontrivial idempotents and a nilradical which is not prime[59]. Consider $\mathbb{Z}[x]/\langle 2x\rangle$, then the nilradical contains $[0]$ but $[2]\cdot[x]=0$ so the nilradical is not prime. It follows that $\operatorname{Spec}\mathbb{Z}[x]/\langle 2x\rangle$ is reducible, but there are no non trivial idempotents. Indeed, if $[p]\in\mathbb{Z}[x]/\langle 2x\rangle$ satisfies $[p]^2=[p]$ then we have that $p^2-p\in\langle 2x\rangle$, but the only way this can be true if p^2-p is divisible by $2x$ or is just actually equal to zero. This is only satisfied if $[p]=0$ or if $p=1$, hence $[p]=1$. It follows that $\mathbb{Z}[x]/\langle 2x\rangle$ has no non trivial idempotents, and is thus connected but not irreducible.

We now turn to proving results regarding integral schemes. We have our first theorem of the section:

Theorem 3.2.1. *Let X be a scheme, then X is integral if and only if it is reduced and irreducible.*

Proof. Suppose that X is integral, then X is automatically reduced. Moreover, every open affine of X corresponds to $\operatorname{Spec} A$ where A is an integral domain, so every open affine is irreducible by Lemma 1.4.5. It follows by Lemma 3.2.4 that X is irreducible as well.

Now suppose that X is irreducible and reduced, then every open affine is irreducible and reduced, so we have that for each affine open $\operatorname{Spec} A$, A is an integral domain. Indeed, this implies that the generic point of A is the zero ideal, hence $\{0\}$ is a prime ideal implying that A is an integral domain. We need to show that for any open set U, $\mathscr{O}_X(U)$ is an integral domain. For any affine open, $V\subset U$ we have that $\mathscr{O}_X(V)$ is integral domain as V is an affine open in X, so it suffices to show that $\mathscr{O}_X(U)\to\mathscr{O}_X(V)$ is injective, as then $\mathscr{O}_X(U)$ is isomorphic to a subring of an integral domain.

Let $f\in\mathscr{O}_X(U)$ lie in the kernel of the restriction map $\mathscr{O}_X(U)\to\mathscr{O}_X(V)$. Let $W\subset V$ be any open set, then we claim that $\mathscr{O}_X(V)\to\mathscr{O}_X(W)$ is injective. Note that since V is affine, W can be covered with distinguished U_{g_i} for $g_i\in\mathscr{O}_X(V)$ so that the restriction maps $\mathscr{O}_X(V)\to\mathscr{O}_X(U_{g_i})$ come from localizations of an integral domain and are thus injections. It follows that if $f\in\mathscr{O}_X(V)$ satisfies $f|_W=0$, then $f|_{U_{g_i}}=0$ and so $f=0$. Now cover U with affine opens $\{W_i\}$, and note that by the preceding argument applied to W_i and $W_i\cap V$ we have that $\mathscr{O}_X(W_i)\to\mathscr{O}_X(W_i\cap V)$ is injective. Therefore $f|_{W_i}=0$ because $f|_{W_i\cap V}=f|_{W_i}|_{W_i\cap V}$ which is equal to zero as $f|_V=0$. It follows that f vanishes on an open cover of U and thus must be equal to zero, implying the claim. □

We now have the obvious corollary:

Corollary 3.2.1. *Let X be a scheme, then the following are equivalent:*

a) X is integral

b) There exists an affine cover $\{U_i\}$ of X such that $U_i\cap U_j\neq\emptyset$ and U_i is integral for all i.

c) Every open affine $U\subset X$ is integral

Proof. We have that $a\Rightarrow b$ as if X is integral then X is irreducible by Theorem 3.2.1, so there exists an affine open cover of X such that each U_i is irreducible and $U_i\cap U_j\neq\emptyset$. Since every affine open is reduced we have the claim by Theorem 3.2.1 as well.

For $a\Rightarrow c$, we see that every open set $\mathscr{O}_X(U)$ is an integral domain, so if $U=\operatorname{Spec} A$ is integral, we have that $\mathscr{O}_X(U)=A$ is an integral domain implying that $\operatorname{Spec} A$ irreducible by Lemma 1.4.5. Every affine open of U is reduced, so U is reduced, and irreducible implying that U is integral again by Theorem 3.2.1.

For $b\Rightarrow a$, note that each U_i is reduced and irreducible by Theorem 3.2.1, so by Lemma 3.2.4 we have that X irreducible, and by Lemma 3.2.1 we have that X is reduced. By Theorem 3.2.1, X is integral.

For $c\Rightarrow a$,the same argument holds. □

Example 3.2.3. We claim that $\mathbb{P}^n_A$ is integral if A is integral domain. Indeed, we have an affine open cover by:

$$U_{x_i}=A[\{x_j/x_i\}_{j\neq i}]$$

such that $U_{x_i}\cap U_{x_j}=U_{x_ix_j}\neq\emptyset$. Each of these is integral, so we have that $\mathbb{P}^n_A$ is integral as well.

[59] As by the affine case in Lemma 3.2.3, if the nilradical is not prime then X is not irreducible.

Proposition 3.2.1. *Let X and Y be integral schemes over an algebraically closed field k. If X is locally of finite type then $X \times_k Y$ is an integral scheme.*

Proof. It suffices to prove that for any affine opens $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} B \subset Y$, that $A \otimes_k B$ is an integral domain. We first claim that the natural map:

$$A \longrightarrow \prod_{\mathfrak{m} \in |\operatorname{Spec} A|} A/\mathfrak{m}$$

is injective. Indeed, we can write $A \cong k[x_1, \dots, x_n]/I$ for some prime ideal I, and some $n \in \mathbb{N}$. The maximal ideals of A are then precisely the maximal ideals of $k[x_1, \dots, x_n]$ such that $I \subset \mathfrak{m}$. Suppose that $[f] \in A \mapsto (0_{\mathfrak{m}}) \in \prod_{\mathfrak{m} \in |\operatorname{Spec} A|} A/\mathfrak{m}$. Then we have that $f \in \mathfrak{m}$ for every $I \subset \mathfrak{m}$. By Hilbert's strong Nullstellensatz we have that there exists a k such that $f^k \in I$, but since I is prime we have that $f \in I$ hence $[f] = 0$.

Now note that for every $\mathfrak{m} \in |\operatorname{Spec} A|$, we have that $A/\mathfrak{m} \cong k$ as k is algebraically closed. For each $\mathfrak{m}$, let $\phi_{\mathfrak{m}}$ be the unique isomorphism $A \to k$ with kernel $\mathfrak{m}$, then we have the following chain of maps:

$$A \otimes_k B \longrightarrow A/\mathfrak{m} \otimes_k B \longrightarrow B$$

given on simple tensors by:

$$a \otimes b \longmapsto \phi_{\mathfrak{m}}([a]) \cdot b$$

Let:

$$x = \sum_i a_i \otimes b_i \qquad \text{and} \qquad y = \sum_i c_i \otimes d_i$$

be such that $x \cdot y = 0$. By the bilinearity of the tensor product, and the fact that A and B are both vector spaces, we can take $\{b_i\}$ and $\{d_i\}$ to be linearly independent sets over k. We see that for every $\mathfrak{m} \in |\operatorname{Spec} A|$:

$$x \cdot y \longmapsto (\phi_{\mathfrak{m}}([a_i])b_i) \cdot (\phi_{\mathfrak{m}}([c_i)]d_i) = 0$$

Since B is an integral domain, we have that it follows that either:

$$(\phi_{\mathfrak{m}}([a_i])b_i) = 0 \qquad \text{or} \qquad (\phi_{\mathfrak{m}}([c_i])d_i) = 0$$

Suppose the first summation is zero, then since $\{b_i\}$ is linearly independent, we have that $\phi_{\mathfrak{m}}([a_i]) = 0$ for all a_i. This implies that each $a_i \in \mathfrak{m}$ for all $\mathfrak{m}$. By the injectivity of the map $A_i \to \prod A_{\mathfrak{m}}$, it follows that each $a_i = 0 \in A$, hence:

$$x = \sum_i 0 \otimes b_i = 0$$

The same argument demonstrates that if the second sum is equal to zero, then $y = 0$, thus if $x \cdot y = 0$, we have that either $x = 0$ or $y = 0$ so $A \otimes_k B$ is an integral domain. □

3.3 Normal Schemes

Recall that if A is an integral domain, and $\eta = \langle 0 \rangle$ is the zero ideal, then $A_\eta = \operatorname{Frac}(A)$, that is the localization at the zero prime ideal is the field of fractions. This can be seen easily by noting that a), A_η is easily seen to be a field, and b), that the constructions of $\operatorname{Frac}(A)$ is identical to A_η. Further recall that if $A \subset B$, then B is an A algebra, and we say that $b \in B$ is *integral over* A, if there exists a monic polynomial $p \in A[x]$ such that $p(b) = 0$. We set the integral closure of A to be:

$$\bar{A} = \{b \in B : b \text{ is integral over } A\}$$

We now have the following definition:

Definition 3.3.1. Let A be an integral domain, then A is an **integrally closed domain** if $\bar{A} = A$, where A is being viewed as a subring of $\mathrm{Frac}(A)$[60].

We have the following example:

Example 3.3.1. The integers are an integrally closed domain. Indeed, note that $\mathrm{Frac}\,\mathbb{Z} = \mathbb{Q}$, clearly $\mathbb{Z} \subset \bar{\mathbb{Z}}$ as for any element in $a \in \mathbb{Z}$ we have that $x - a$ has a a root. Now let $a/b \in \bar{\mathbb{Z}}$, such that a and b have greatest common divisor equal to 1. Then their must exist some monic polynomial:

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with $a_i \in \mathbb{Z}$, such that $p(a/b) = 0$. It follows that:

$$a^n/b^n + a_{n-1}a^{n-1}/b^{n-1} + \cdots a_1 a/b + a_0 = 0$$

Multiplying throughout by b^n we obtain that:

$$a^n + a_{n-1}a^{n-1}b + \cdots a_1 ab^{n-1} + a_0 b^n = 0$$

however, since $a_{n-1}a^{n-1}b + \cdots a_1 ab^{n-1} + a_0 b^n$ is divisible by b, we must have that a^n is divisible by b. Since a and b both have unique factorizations into primes, it follows that a is divisible by b, a clear contradiction, implying the claim.

As a counter example, take $\mathbb{C}[x,y]/\left\langle x^2 - y^3 \right\rangle$. We first claim that this ring is isomorphic to $\mathbb{C}[t^2,t^3]$. Indeed, consider the ring homomorphism $\mathbb{C}[x,y] \to \mathbb{C}[t^2,t^3]$ given by $x \mapsto t^3$ and $y \mapsto t^2$, then we see that $x^2 - y^3 \mapsto t^6 - t^6 = 0$, so there is a unique ring homomorphism given by $[x] \mapsto t^3$ and $[y] \mapsto t^2$. We define an inverse by sending $t^3 \mapsto x$ and $t^2 \mapsto y$, and composing with the projection. This is easily seen to be an isomorphism, and $\mathbb{C}[t^2,t^3]$ is obviously an integral domain. It's field of fractions is the localization at the zero ideal, which contains $\mathbb{C}[t,t^{-1}]$, as $t^2 \cdot t^{-3} = t^{-1}$ and $t^3 \cdot t^{-2} = t$. However, t is integral over $\mathbb{C}[t^2,t^3]$ as it is the root of the polynomial $(\mathbb{C}[t^2,t^3])[\alpha]$ given by $\alpha^2 - t^2$.

We now develop a scheme theoretic analogue of the above construction:

Definition 3.3.2. Let X be a scheme, then X is **normal** if for all $x \in X$, the stalk $(\mathscr{O}_X)_x$ is an integrally closed domain.

We have the following (non)examples:

Example 3.3.2. We claim that $\mathbb{P}^n_{\mathbb{C}}$ is a normal scheme. Indeed, the $U_i = \mathrm{Spec}(\mathbb{C}[x_0,\ldots,x_n]_{x_i})_0$ cover $\mathbb{P}^n_{\mathbb{C}}$, so suppose $x \in U_i$. Then x corresponds to a prime ideal $\mathfrak{p}$ of the ring $\mathbb{C}[\{x_j/x_i\}_{j\neq i}]$. Any polynomial ring is a unique factorization domain, and so is it's localization at $\mathfrak{p}$, so the argument that $\mathbb{Z}$ is an integrally closed domain holds pretty much verbatim for $\mathbb{C}[\{x_j/x_i\}_{j\neq i}]_{\mathfrak{p}}$, hence $\mathbb{P}^n_{\mathbb{C}}$ is normal.

As a counter example take $X = \mathrm{Spec}\,\mathbb{C}[t^2,t^3]$, and consider the maximal ideal $\mathfrak{m} = \left\langle t^2, t^3 \right\rangle$. Then the stalk at $\mathfrak{m}$ does not invert t^2 or t^3, hence the same argument as in Example 3.3.1 demonstrates that X is not a normal scheme.

We now wish to describe a process in which we take an integral scheme X and normalize it. We first need the following definition:

Definition 3.3.3. Let $f : X \to Y$ be a morphism of schemes. Then f is **dominant** if $f(X)$ is a dense subset of Y.

We need the following lemma:

Lemma 3.3.1. *Let $f : X \to Y$ be a morphism of integral schemes, then the following are equivalent:*

a) f is dominant.

b) f takes the generic point of X to the generic point of Y.

c) For every open affines $U \subset X$, $V \subset Y$, such that $f(U) \subset V$, the ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective.

d) For all $x \in X$ the map of local rings $(\mathscr{O}_Y)_{f(x)} \to (\mathscr{O}_X)_x$ is injective.

[60] Recall that the localization map for an integral domain is injective, so A is indeed a subring.

Proof. Let $f : X \to Y$ be dominant and by Lemma 3.2.3 let $\eta_X \in X$ and $\eta_Y \in Y$ be the unique generic points. It follows that since f is dominant that $f(X) \subset Y$ is a dense subset. We first note that $f(X)$ is an irreducible subspace, as if $Z_1, Z_2 \subset f(X)$ are closed such that $Z_1 \cup Z_2 = f(X)$, then we can write $Z_1 = W_1 \cap f(X)$, and $Z_2 = W_2 \cap f(X)$, hence $f(X) = (W_1 \cup W_2) \cap f(X)$, but then $f(X) \subset W_1 \cup W_2$, so $W_1 \cup W_2 = X$ as $f(X)$ is dense. Since Y is irreducible we must have that $W_1 = Y$ or $W_2 = Y$, either way it follows that $Z_1 = f(X)$ or $Z_2 = f(X)$. It follows that $f(X)$ must contain a unique generic point point η, and this point must also be a generic point for Y, so $\eta = \eta_Y$.

We now claim that $f(\eta_X) = \eta_Y$. Note that for any subset U we have that $f(\bar{U}) \subseteq \overline{f(U)}$. Indeed, if f is continuous, then $f^{-1}(\overline{f}(U))$ is closed, and since $f(U) \subset \overline{f(U)}$, we have that $f^{-1}(f(U)) \subset f^{-1}(\overline{f(U)})$, so $U \subset \overline{f}^{-1}(\overline{f(U)})$ implying that $\bar{U} \subset f^{-1}(\overline{f(U)})$, and finally that $f(\bar{U}) \subset \overline{f(U)}$. It follows that $f(X) = f(\bar{\eta_X}) \subset \overline{f(\eta_X)}$ which must be equal to Y as $f(X)$ is dense. It follows that $f(\eta_X)$ is dense, hence $f(\eta_X)$ must be η_Y. We thus have that $a \Rightarrow b$. Clearly if f takes the generic point of X to the generic point of Y then $f(X)$ is dense in Y so $b \Rightarrow a$ as well.

To see that $b \Rightarrow c$, let $U \subset X$, and $V \subset Y$ be affine opens such that $f(U) \subset V$. Then we have an induced morphism of affine schemes $f|_U : U \to V$. Since $\mathscr{O}_X(U)$ and $\mathscr{O}_Y(V)$ are integral domains, and $\eta_X \in U$ and $\eta_Y \in V$ both correspond to the zero ideal, we have that by b), $f|_U$ must come from a ring homomorphism ϕ satisfying $\phi^{-1}(\langle 0 \rangle) = \langle 0 \rangle$, hence $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective. If this holds for all such open affine, then f must take the generic point to the generic point so $c \Rightarrow b$ as well.

For $c \Rightarrow d$, let $x \in U$ and $f(x) \in V$. Then writing $U = \operatorname{Spec} A$, and $V = \operatorname{Spec} B$, we let $x = \mathfrak{p}$, and $f(x) = \phi^{-1}(\mathfrak{p})$, where $\phi : B \to A$ is the ring homomorphism inducing $f|_U$. The map $(\mathscr{O}_Y)_{f(x)} \to (f_* \mathscr{O}_X)_{f(x)}$ is clearly injective, so it suffices to check that $(f_* \mathscr{O}_X)_{f(x)} \to (\mathscr{O}_X)_x$ is injective. Let $U_g \subset \operatorname{Spec} B$, and take $[U_g, s]_{\phi^{-1}(\mathfrak{p})} \in (f_* \mathscr{O}_X)_{f(x)} \cong ((f|_U)_* \mathscr{O}_{\operatorname{Spec} A})_{\phi^{-1}(p)}$, then we have that this maps to $[f|_U^{-1}(U_g), s]_{\mathfrak{p}} = [U_{\phi(g)}, s]_{\mathfrak{p}}$, where $\phi(g) \neq 0$. If this is zero, then there exists some distinguished open $U_h \subset U_{\phi(g)}$ such that $s|_{U_h} = 0$, but the restriction maps on an integral affine scheme are injective, so this implies $s = 0$, hence $[U_g, s] = 0$, hence $c \Rightarrow d$ as desired. To see that $d \Rightarrow c$, it suffices to reduce to the case of affine schemes, let $\phi : B \to A$ be the ring homomorphism inducing $\operatorname{Spec} A \to \operatorname{Spec} B$. The stalk map $(\mathscr{O}_{\operatorname{Spec} B})_{\phi^{-1}(\mathfrak{p})} \to (\mathscr{O}_{\operatorname{Spec} A})_{\mathfrak{p}}$ is then the localization of the map $B \to A_{\mathfrak{p}}$ at $\phi^{-1}(\mathfrak{p})$, which exists as $\phi(\phi^{-1}(\mathfrak{p})) \subset \mathfrak{p}$. We have the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A \arrow[d] \\
B_{\phi^{-1}(\mathfrak{p})} \arrow[r] & A_{\mathfrak{p}}
\end{tikzcd}$$

Since A and B are integral domains the vertical arrows are injective, and by hypothesis the bottom arrow is injective. It follows that if $\phi(b) = 0$, then $\phi(b)/1 \in A_{\mathfrak{p}}$ is zero, implying that $b/1 \in B_{\phi^{-1}(\mathfrak{p})}$ is zero hence $b \in B$ is zero. Therefore ϕ is injective as desired, so $c \Rightarrow d$. $\square$

We have the following definition:

Definition 3.3.4. Let X be an integral scheme, then the **normalization of X** is the scheme $\tilde{X}$, equipped with a morphism $N : \tilde{X} \to X$, such that for every normal integral scheme Z, and every dominant $f : Z \to X$ the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "\exists! \tilde{f}"'] & X \\
\tilde{X} \arrow[ur, "N"'] &
\end{tikzcd}$$

where

As with every object defined this way we must show that such an object exists and is unique up to unique isomorphism. We do so now:

Theorem 3.3.1. *Let X be an integral scheme, then it's normalization, $\tilde{X}$ exists, and is unique up to unique isomorphism.*

Proof. If such an object exists it is obviously unique to up to unique isomorphism, as the morphism N we construct will be dominant so we need only check the universal property.

First consider the case where $X = \operatorname{Spec} A$ is affine, then we take $\tilde{X} = \operatorname{Spec} \tilde{A}$, where $\tilde{A}$ is the integral closure of A in $\operatorname{Frac}(A)$. This comes with a canonical injection map $A \to \tilde{A}$, so we get a dominant morphism $N : \operatorname{Spec} \tilde{A} \to \operatorname{Spec} A$. Now let Z be a normal integral scheme, and $f : Z \to \operatorname{Spec} A$ be a dominant morphism, then for every affine open $U \subset Z$, we have that $f|_U : U \to \operatorname{Spec} A$, is induced by an injective ring map. The homomorphism $A \to \mathscr{O}_Z(U)$ is given by the ring homomorphism $A \to \mathscr{O}_Z(Z)$ composed with restriction to $\mathscr{O}_Z(U)$. This second map is injective, and since the composition is injective, we must have that $A \to \mathscr{O}_Z(Z)$ is injective as well.

We want to show that the ring homomorphism $A \to \mathscr{O}_Z(Z)$ factors through the inclusion $A \to \tilde{A}$. We first show that $\mathscr{O}_Z(Z)$ is integrally closed. Let $a \in \operatorname{Frac}(\mathscr{O}_Z(Z))$ be integral over $\mathscr{O}_Z(Z)$, and let $\operatorname{Spec} B \subset Z$ be an affine open. Then since Z is integral, we have that $\mathscr{O}_Z(Z) \subset B$[61], so $\operatorname{Frac}(\mathscr{O}_Z(Z)) \subset \operatorname{Frac}(B)$. It follows that $a \in \operatorname{Frac}(B)$, and that a is integral over B. Let $I = \{b \in B : ab \in B\}$, if $I = B$, then $a \in B$ so we are done. If $I \neq B$, then $I \subset \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset B$. We see that a is integral over $B_{\mathfrak{p}}$, and thus $a \in B_{\mathfrak{p}}$ as Z is normal. However, there then exists an $s \in B \setminus \mathfrak{p}$ such that $s \cdot a \in B$, implying that $s \in I$, contradicting the fact that $s \in B \setminus \mathfrak{p}$, so $I = B$. It follows that $a \in B = \mathscr{O}_Z(V)$. Cover Z with affine opens V_i, and the same argument shows that $b \in \mathscr{O}_Z(V_i)$ for all i. For all affine opens $V_{ijk} \subset V_i \cap V_j$, we have that we can identify $\mathscr{O}_Z(V_i)$ and $\mathscr{O}_Z(V_j)$ as subrings of $\mathscr{O}_Z(V_{ijk})$, so $b \in \mathscr{O}_Z(V_i)$ and $b \in \mathscr{O}_Z(V_j)$ both map to the same element in $\mathscr{O}_Z(V_{ijk})$[62]. Since the affine opens form a basis for the topology on Z, and thus determine a sheaf on a base, it follows that $b \in \mathscr{O}_Z(Z)$ so $\mathscr{O}_Z(Z)$ is indeed integrally closed.

It follows that since A injects into $\mathscr{O}_Z(Z)$, and $\mathscr{O}_Z(Z)$ is integrally closed, that $\tilde{A}$ injects into $\mathscr{O}_Z(Z)$ as well, thus we have a morphism $\tilde{A} \to Z$. Since A injects into $\tilde{A}$ we clearly have the following commutative diagram in the category of rings:

$$\begin{tikzcd}
\mathscr{O}_Z(Z) & A \arrow[l] \arrow[dl] \\
\tilde{A} \arrow[u] &
\end{tikzcd}$$

which yields the following commutative diagram in the category of schemes:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \operatorname{Spec} A \\
\operatorname{Spec} \tilde{A} \arrow[ur] &
\end{tikzcd}$$

implying the result for affine integral schemes.

Now let X be an integral scheme, and $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover for X. Then we have isomorphisms $\beta_{ij} : U_{ij} \subset \operatorname{Spec} A_i \to \operatorname{Spec} A_j$ which agree on triple overlaps. For each i, set $\tilde{U}_i = \operatorname{Spec} \tilde{A}_i$, and let $N_i : \operatorname{Spec} \tilde{A}_i \to \operatorname{Spec} A_i$ be the normalization map. Finally set $\tilde{U}_{ij} \subset \operatorname{Spec} \tilde{A}_i$ to be $N_i^{-1}(U_{ij})$. We claim that $\tilde{U}_{ij}$ satisfies the universal property of the normalization of U_{ij}. Indeed, we have a morphism $N_i|_{\tilde{U}_{ij}} : \tilde{U}_{ij} \to U_{ij}$ which must be dominant as it sends the unique generic point of $\tilde{U}_{ij}$ to U_{ij}. Now let $f : Z \to U_{ij}$ be any dominant morphism from an integrally closed scheme Z, then the composition $\imath \circ f : Z \to \operatorname{Spec} A_i$ is dominant, and there is a unique morphism $g : Z \to \operatorname{Spec} \tilde{A}_i$ such that $N \circ g = \imath \circ f$. But this implies that $g(Z) \subset \tilde{U}_{ij}$, so g factors through the inclusion map$\tilde{U}_{ij} \to \operatorname{Spec} \tilde{A}_i$ implying that $\tilde{U}_{ij}$ is indeed the normalization of U_{ij}.

[61] Via the inclusion map.

[62] If one is unconvinced, then they can write out the restriction maps themselves, and find that this must be true by examining the induced injective maps $\operatorname{Frac}(\mathscr{O}_Z(Z)) \to \operatorname{Frac}(\mathscr{O}_Z(V_i))$.

We want to show that there exist scheme isomorphisms $\phi_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ji}$ which agree on triple overlaps. Fix the notation $N_i|_{\tilde{U}_{ij}} = N_{ij}$, and note that we have a dominant morphism $\beta_{ij} \circ N_{ij} : \tilde{U}_{ij} \to U_{ji}$. It follows that there is a unique morphism ϕ_{ij} such that the following diagram commutes:

$$\begin{tikzcd} \tilde{U}_{ij} \arrow[r, "\beta_{ij}\circ N_{ij}"] \arrow[d, "\phi_{ij}"'] & U_{ji} \\ \tilde{U}_{ji} \arrow[ur, "N_{ji}"'] & \end{tikzcd}$$

Similarly, we have a morphism $\phi_{ji} : \tilde{U}_{ji} \to \tilde{U}_{ji}$ such that a similar diagram commutes. We thus claim that the following diagram commutes:

$$\begin{tikzcd} \tilde{U}_{ij} \arrow[r, "N_{ij}"] \arrow[d, "\phi_{ji}\circ\phi_{ij}"'] & U_{ij} \\ \tilde{U}_{ij} \arrow[ur, "N_{ij}"'] & \end{tikzcd}$$

Indeed, note that $N_{ij} \circ \phi_{ji} = \beta_{ji} \circ N_{ji}$, so:

$$\begin{aligned} N_{ij} \circ \phi_{ji} \circ \phi_{ij} =& \beta_{ji} \circ N_{ji} \circ \phi_{ij} \\ =& \beta_{ji} \circ \beta_{ij} \circ N_{ij} \\ =& N_{ij} \end{aligned}$$

so the diagram commutes. But the identity map also makes this diagram commutes so $\phi_{ji} \circ \phi_{ij} = \mathrm{Id}$, and similarly $\phi_{ij} \circ \phi_{ji}$ is the identity, implying that they are isomorphisms. It is easily seen by a similar argument that these morphisms agree on triple overlaps, as the β_{ij} agree on triple overlaps so the $\tilde{U}_i$ glue together to form an integral normal scheme $\tilde{X}$.

It follows that the N_i then also glue together to form a dominant morphism $N : \tilde{X} \to X$, such that $N|_{\tilde{U}_i} = N_i$. Given $f : Z \to X$ with f dominant and Z integral normal, we obtain an open cover of Z by $V_i = f^{-1}(U_i)$. Each of these schemes is normal integral, and the restriction is clearly dominant, so we obtain gives unique morphisms $V_i \to \tilde{U}_i$ which which clearly agree on $V_i \cap V_j$. These maps then glue to yield a unique dominant morphism $Z \to \tilde{X}$ such that the relevant diagram commutes, so $\tilde{X}$ is indeed the normalization of X. ☐

3.4 Noetherian Schemes

We now turn to defining another important class of schemes, called Noetherian schemes, which again have an interesting interplay between the algebraic properties of their structure sheaf, and the topological properties of the total space. To begin, we review some commutative algebra:

Definition 3.4.1. Let A be a commutative ring, then A is **Noetherian** if every strictly increasing chain of ideals:

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

terminates. In other words, there exists some m such that $I_m = I_{m+k}$ for all $k \geq 0$.

Example 3.4.1. Any field is obviously Noetherian, any finite ring is also obviously Noetherian. Mildly more interestingly, $\mathbb{Z}$ is Noetherian. Indeed, every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$, so suppose we have the following infinite chain of ideals:

$$\langle n_1 \rangle \subset \langle n_2 \rangle \subset \cdots$$

We see that if $\langle n_1 \rangle \subset \langle n_2 \rangle$, then $n_1 \in \langle n_2 \rangle$, hence $n_1 = a \cdot n_2$ for some $a \in \mathbb{Z}$. It follows that n_2 divides n_1. If this is chain is infinite, then n_1 has infinitely many divisors, which is absurd implying the claim.

We have the following useful lemma which makes the example above a bit more immediate:

Lemma 3.4.1. *Let A be a ring, then A is Noetherian if and only if every ideal of A is finitely generated.*

Proof. Suppose that every ideal of A is finitely generated, and that:

$$I_1 \subset I_2 \subset \cdots$$

is a strictly increasing chain of ideals, and let:

$$I = \bigcup_i I_i$$

We claim that I is an ideal (no generating set needed!). Indeed, we see that if $0 \in I$, and that if $a, b \in I$ then $a \in I_i$ and $b \in I_j$ for some i and j. Without loss of generality suppose that $i \leq j$, then $a_i \in I_j$ so $a + b \in I_j$ hence $a + b \in I$. We see that I clearly contains all of it's inverses so I is a subgroup. Now let $a \in I$, and $b \in A$, then $a \in I_i$ for some i, and $a \cdot b \in I_i$ so $a \cdot b \in I$ as well implying that I is an ideal.

Since I is finitely generated, let $I = \langle a_1, \ldots, a_n \rangle$ for some $n \in \mathbb{Z}$. We have that each a_i lies in some I_{j_i} for some j_i, so let $j_k = \max(j_1, \cdots j_n)$, then since $I_{j_i} \subset I_{j_k}$ for all $i \in \{1, \ldots, n\}$ we must have that I_{j_k} contains each a_i. Let $j_k = m$, then it follows that $I \subset I_m$, so $I = I_m$ as $I_m \subset I$ by definition. For any $l \geq m$, we have that $I_m = I \subset I_l$ so the chain clearly terminates, and A is Noetherian.

Conversely, let $I \subset A$ be any ideal with minimal generating set $\{a_i\}_{i \in J}$ where J is a totally ordered set that is not finite. For any $j \in J$ we set $I_j = \{a_i\}_{i \leq j}$, and note that for any $j < k$, we have that $I_j \subset I_k$ and that this inclusion is strict. Indeed, if $I_j = I_k$ then for all $j < l \leq k$, we have that $a_l \in I_j$, implying that $a_l = \sum_{i \leq j} b_i a_i$ hence a_l is not a generating element of I, a contradiction, so $I_j \subset I_k$. We can label the initial segment of J with natural numbers regardless of it's cardinality, hence:

$$I_1 \subset I_2 \subset \cdots$$

is an infinite strictly increasing chain of ideals, so A is not Noetherian. The claim then follows by the contrapositive. □

We also have the following collection results:

Lemma 3.4.2. *Let A be a Noetherian ring then:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Noetherian.

b) If $I \subset A$ is an ideal then A/I is Noetherian.

Proof. Let $I_S \subset S^{-1}A$ be an ideal, then we first claim that

$$I_S = S^{-1}I := \left\{ \frac{a}{s} : a \in I, s \in S \right\}$$

for some $I \subset A$. In particular, let $I = \pi^{-1}(I_S)$ where $\pi : A \to S^{-1}A$ is the localization map. Indeed, we have that:

$$S^{-1}\pi^{-1}(I_S) = \left\{ \frac{a}{s} : a \in \pi^{-1}(I_S), s \in S \right\}$$

Suppose that $a/s \in I_S$, then we have that $a/1 \in I_S$, so $a \in \pi^{-1}(I_S)$. It follows that $a/s \in S^{-1}\pi^{-1}$ giving us one inclusion. Now suppose that $a/s \in S^{-1}\pi^{-1}$, then $a \in \pi^{-1}$, so $a/1 \in I_S$ by definition. It follows that $a/s \in I$, by $a/1 \cdot 1/s = a/s$, hence $I = S^{-1}\pi^{-1}(I_S)$ implying the claim.

Since A is Noetherian, it follows that $\pi^{-1}(I_S)$ is finitely generated. In particular, since any of ideal of $S^{-1}A$ is generated by elements of the form $a/1$ as $1/s$ is invertible, we clearly see that $S^{-1}I$ is finitely generated as well. By the above paragraph, it follows that I_S is finitely generated, hence $S^{-1}A$ is Noetherian by Lemma 3.4.1 implying b).

Now let $I \subset A$ be an ideal. We see that if J is an ideal of A/I, then J is of the form $\pi(\pi^{-1}(J))$ as the quotient map $\pi : A \to A/J$ is surjective. We see that $\pi^{-1}(J)$ is finitely generated as A is Noetherian, so J itself must be finitely generated as well. Indeed suppose that $\{a_1, \dots, a_n\}$ are generating elements of $\pi^{-1}(J)$, and let $[j] \in J$. We see that $j \in \pi^{-1}(J)$ can be written as $\sum_i b_i a_i$, hence $[j] = \sum_i [b_i][a_i]$, so $\{[a_1], \dots, [a_n]\}$ generates J. It follows that every ideal of A/I is finitely generated, hence A/I is Noetherian by Lemma 3.4.1 implying b). ☐

The following results are some of the most famous results in commutative algebra, the first of which is known as the Hilbert Basis theorem.

Theorem 3.4.1. *Let A be a ring, then $A[x_1, \dots, x_n]$ is Noetherian if and only if A is Noetherian.*

Proof. We see that if $A[x_1, \dots, x_n]$ is Noetherian, then $A[x_1, \dots, x_n]/\langle x_1, \dots, x_n\rangle \cong A$ must be Noetherian by Lemma 3.4.3.

Now suppose that A is Noetherian, since we trivially have that $A[x, y] \cong (A[x])[y]$, it suffices by an induction argument to show that $A[x]$ is Noetherian. Let $I \subset A[x]$ be an ideal, we will show that I is finitely generated. We have a partial order on I, by writing:

$$f = a_n x^n + \cdots + a_1 x^1 + a_0 \qquad g = b_k x^k + \cdots + b_1 x^1 + b_0$$

and saying that $f \leq g$ if and only if $n \leq k$, we call n and k the degree of f and g respectively, and write it as $\deg f$. Choose an element of least degree $f_0 \in I$, i.e. an element f_0 such that there is no g in I satisfying $\deg g < \deg f$. If $\langle f_0\rangle = I$ we are done, if not, then we choose an element f_2 in $I \setminus \langle f_0\rangle$ of least degree. We perform this recursively obtaining a sequence[63] $\langle f_0, f_1, \dots\rangle \subset I$. For each f_i, let $a_{\deg f_i}$ be the leading coefficient of f_i, and consider the ideal $J = \langle a_{\deg f_0}, \dots\rangle \subset A$. Then, since A is Noetherian, we know that the sequence:

$$\langle a_{\deg f_0}\rangle \subset \langle a_{\deg f_0}, a_{\deg f_1}\rangle \subset \cdots$$

terminates, so for some $m \geq 0$, we have that this chain must terminate with $\langle a_{\deg f_0}, \dots, a_{\deg f_m}\rangle$ implying that $J = \langle a_{\deg f_0}, \dots, a_{\deg f_m}\rangle$. We claim that $I = \langle f_0, \dots, f_m\rangle$. Suppose otherwise, then by construction $f_{m+1} \notin \langle f_0, \dots, f_m\rangle$, but $a_{\deg f_{m+1}} \in J$, so we can write:

$$a_{\deg f_{m+1}} = \sum_{i=0}^{m} a_{\deg f_i} b_i$$

for some $b_i \in A$. Define g by:

$$g = \sum_i b_i f_i x^{\deg f_{m+1} - \deg f_i}$$

Note that this clearly lies in $\langle f_0, \dots, f_m\rangle$, but this element as the same degree as f_{m+1} with $a_{\deg g} = a_{\deg f_{m+1}}$. We thus see that $f_{m+1} - g$ has degree strictly less than f_{m+1}, and that $f_{m+1} - g \notin \langle f_0, \dots, f_m\rangle$, so $f_{m+1} - g$ is the minimal element of $I \setminus \langle f_0, \dots, f_m\rangle$, a contradiction. It follows that $I = \langle f_0, \dots, f_m\rangle$, so every ideal of $A[x]$ is finitely generated and thus by Lemma 3.4.1 we have that $A[x]$ is Noetherian. ☐

We now have the following obvious corollary:

Corollary 3.4.1. *Let A be a Noetherian and B be any finitely generated A algebra, then B is Noetherian.*

To prove our second famous result, we need to extend the idea of a Noetherian ring to modules.

Definition 3.4.2. Let M be an A module, then M is Noetherian if for every strictly increasing chain of submodules:

$$N_1 \subset N_2 \subset \cdots$$

terminates.

We prove the following analogue Lemma 3.4.1

Lemma 3.4.3. *Let M be an A module, then the following hold:*

[63] This is equivalent to using the axiom of dependent choice.

a) *M is Noetherian if and only if every submodule is finitely generated.*

b) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then M_2 is Noetherian if and only if M_1 and M_3 are.*

c) *If A is Noetherian, and M is finitely generated then M is Noetherian.*

Proof. We begin with a). Suppose that every submodule of M finitely generated, and consider the following sequence of submodules:

$$N_1 \subset N_2 \subset \cdots$$

Let:

$$N' = \bigcup_i N_i$$

Then this has finitely many generators $(m_1, \dots, m_n)$ for some n, and each must lie in N_i for some i, so choose the largest such i, and call it k. We have that $(m_1, \dots, m_n) \subset N_k$ essentially by construction, $N' = N_k$. It follows that for any $l \geq k$, we have that $N_l \subset N' = N_k$, so for all $l \geq k$ we have that $N_l = N_k$, so M is Noetherian.

Now suppose that M is Noetherian, and let N be a submodule which is not finitely generated. Let $\{m_i\}_{i \in I}$ be the minimal generating set of N where I is a totally ordered set of any cardinality. For any $j \in I$ let N_j be the submodule generated by the elements $\{m_i\}_{i \leq j}$, then for any $k < j$, we have that $N_k \subseteq N_j$. If $N_k = N_j$ then for each $k < l \leq j$, we have that m_l can be written as a linear combination of $\{m_i\}_{i \leq k}$, hence m_l is not a generating element. It follows that N_k is a strict subset of N_j for each $k < j$. Since we can write the initial segment of any totally ordered set as the natural numbers, we have that:

$$N_1 \subset N_2 \subset N_3 \subset \cdots$$

is a strictly increasing chain of ideals which does not terminate, hence M is not Noetherian, a contradiction. It follows that every submodule of M (including M itself) must be finitely generated, thus we have a).

Now suppose that we have an exact sequence:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is an exact sequence of A modules. If M_2 is Noetherian, then we have that $M_3 \cong M_2/\ker g$ so M_3 is Noetherian, as every submodule of M_3 must be finitely generated. Moreover, every submodule of M_1 is a submodule of M_2, so we have that every submodule of M_1 is finitely generated hence M_1 is also Noetherian.

Let M_1 and M_3 be Noetherian modules, and consider the following chain of strictly increasing submodules of M_2:

$$N_1 \subset N_2 \subset \cdots$$

Then we have that:

$$f^{-1}(N_1) \subset f^1(N_2) \cdots \qquad \text{and} \qquad g(N_1) \subset g_2(N_2) \subset \cdots$$

are are strictly increasing chains of ideals in M_1 and M_3 respectively. Since M_1 and M_3 it follows that there exist n_1 and n_3 such that $f^{-1}(N_{n_1})$ and $g(N_{n_3})$ make the above chains terminate. Without loss of generality let $n_3 > n_2$, and denote n_3 by n. Then we claim that for all $k > n$, $N_k = N_n$. Indeed consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}(N_n) \arrow[r, "f"] \arrow[d] & N_n \arrow[r, "g"] \arrow[d] & g(N_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^{-1}(N_k) \arrow[r, "f"] & N_k \arrow[r, "g"] & g(N_k) \arrow[r] & 0
\end{tikzcd}$$

The vertical arrows are inclusion maps, so the leftmost and rightmost arrows are the identities. We want to show that the middle arrow is the identity as well, and it suffices to show that $N_k \subset N_n$. Let $m \in N_k$, and consider $g(m) \in g(N_k)$. Since the right most arrow is the identity, we have that $g(m) \in g(N_n)$, since g is surjective there exists an element $l \in N_n$ which maps to $g(m)$. Let $\imath : N_n \to N_k$ denote the inclusion map, then since:

$$g(\imath(l)) = g(m)$$

It follows that $\imath(l) - m \in \ker g$, but the kernel of g is the image of f, so we have that there exists an $\eta \in f^{-1}(N_k)$ such that $f(\eta) = \imath(l) - m$. Since the left most arrow is the identity, we have that $\eta \in f^{-1}(N_k)$, so $f(\eta) \in N_n$. It follows that $\imath(l) - m \in N_n \subset N_l$ hence $m \in N_n$ as well so $N_k = N_n$, and the middle arrow is the identity. We thus have that N_n makes the chain terminate implying b).

To prove c), let A be a Noetherian ring, and suppose that M is finitely generated. Then M is a quotient of the free module A^n for some n, and so it suffices to check that A^n is a Noetherian A-module. Note that every submodule of A is by definition of an ideal, as it is a subgroup and swallows multiplication, so A is Noetherian as a module over itself as well. We proceed by induction, suppose that A^n is Noetherian, then we have the following short exact sequence:

$$0 \longrightarrow A \xrightarrow{f} A^{n+1} \xrightarrow{g} A^n \longrightarrow 0$$

Since A^n is Noetherian and A are Noetherian, it follows by b) that A^{n+1} is Noetherian implying c) as desired.

□

We are now in position to prove the following result, known as the Artin-Tate lemma:

Theorem 3.4.2. *Let $A \subset B \subset C$ be rings where A is Noetherian, C is finitely generated over A, and C is a finite B module. Then B is finitely generated as an A algebra.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the generators of C as an A algebra, and let $\{y_1, \ldots, y_m\}$ be the generators of C as a B module. Then we have that for some $b_{ij}, b_{ijk} \in B$ that:

$$x_i = \sum_j b_{ij} y_j \qquad \text{and} \qquad y_i y_j = \sum_k b_{ijk} y_k \tag{3.4.1}$$

Let B_0 be the A algebra generated by $\{b_{ij}, b_{ijk}\}$. By Corollary 3.4.1, we have that B_0 is Noetherian, and we have that $A \subseteq B_0 \subseteq B$.

It is clear that C is a B_0-algebra, so we claim that C is finite over B_0, i.e. is a finitely generated B_0 module. Every element $c \in C$ can be written as:

$$c = \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

By making repeated use of the equations in (2.4.1) we can rewrite this in terms of a linear combination of y_k and elements of B_0, hence C is a finitely generated B_0 module. It follows from Lemma 3.4.2 part c) that C is a Noetherian B_0 module, so every submodule of C is finitely generated. We thus have that B is a finitely generated B_0 module, and thus a finitely generated A algebra as desired. □

After our brief detour into commutative algebra, we are now ready to dive back into scheme theory. It should be no surprise that the class of schemes we are about to study are intimately related to Noetherian rings. We begin with the following definition:

Definition 3.4.3. Let X be a topological space, then X is Noetherian if every decreasing sequence of closed subsets:

$$Y_1 \supset Y_2 \supset \cdots$$

terminates. In other words there exits an integer m such that for all $k \geq m$ we have $Y_m = Y_k$.

Example 3.4.2. Let A be a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space. Indeed, any descending sequence of closed subsets can be written uniquely as a sequence of the vanishing locus of radical ideals I_k:

$$\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$$

This then corresponds to an increasing sequence of ideals:

$$I_1 \subset I_2 \subset \cdots$$

which must terminate as A is Noetherian. It follows that the chain $\mathbb{V}(I_1) \supset \mathbb{V}(I_2) \supset \cdots$ must terminate as well.

Note that not every affine scheme which is a Noetherian topological space comes from a Noetherian ring. Indeed consider the infinite polynomial ring $A = k[x_1, x_2, \dots]/\left\langle x_1^2, x_2^2, \dots \right\rangle$ over a field k. Every prime ideal must contain the nilpotents $[x_i]$ for all i, so the only prime is given by $\mathfrak{p} = \langle [x_1], [x_2], \dots \rangle$ implying that $\operatorname{Spec} A$ is a single point and thus Noetherian. It clear that A is not Noetherian as $\mathfrak{p}$ is not finitely generated.

Lemma 3.4.4. *Let X be a Noetherian topological space, then every non empty closed subset $Z \subset X$ can be expressed uniquely as $Z = Z_1 \cup \cdots \cup Z_n$ where each Z_n is an irreducible closed subspace, and for all i, j we have that $Z_i \not\subset Z_j$.*

Proof. Suppose there exists a closed subset Y_1 that cannot be expressed as a finite union of irreducible closed subspaces. If Y_1 contains another such closed subset Y_2, then we have that $Y_1 \supset Y_2$. We can repeat this process ad infinitum, but since X is Noetherian, we must have that this chain eventually terminates for some Y_r. Now since this chain terminates, it follows that every proper closed subset of Y_r can be written as the finite union of irreducible closed subspaces. We see that Y_r is not irreducible as other wise it is trivially a finite of union of irreducible closed subspaces, hence $Y_r = W_1 \cup W_2$ for proper closed subsets of Y_r. However, W_1 and W_2 can be written as a finite union of irreducible closed subsets, a contradiction. It follows that every closed subset of X can thus be written as a finite union of irreducible closed subsets, and by discarding those that satisfy $W_i \subset W_j$, we have that every closed subset of X can be written as a finite union of irreducible closed subspaces none of which fully contain each other.

To show uniqueness, suppose that:

$$Z = Z_1 \cup \cdots \cup Z_n = Y_1 \cup \cdots \cup Y_m$$

where Z_i and Y_j are irreducible closed subspaces none of which contain the other. It follows that for any $Z_1 \subset Y_1 \cup \cdots \cup Y_m$, so $Z_1 = (Y_1 \cap Z_1) \cup \cdots \cup (Y_m \cap Z_1)$, but then for some i we have that $Z_1 = Y_i \cap Z_1$ as Z_1 is irreducible. It follows that $Z_1 \subset Y_i$, and similarly for some j we have that $Y_i \subset Z_j$, but then $j = 1$ as we have that $Z_1 \subset Y_i \subset Z_j$ and Z_1 is only contained in Z_1. It follows that $Y_i = Z_1$, so repeating this process for all $1 \leq i \leq n$ we have that the lists are the same, implying the claim. $\square$

Definition 3.4.4. Let X be a scheme, then X is **locally Noetherian** if there exists a cover $\{U_i\}$ of X by affine schemes such that each U_i is the spectrum of a Noetherian ring. Moreover, X is **Noetherian** if it can be covered by finitely many such affine schemes.

Example 3.4.3. Lemma 3.4.3 demonstrates that every affine scheme $\operatorname{Spec} A$ where A is Noetherian is Noetherian.

Lemma 3.4.5. *A topological space X is Noetherian if and only if every subspace of X is quasi-compact. In particular, X is quasi-compact, and every subspace of X is Noetherian.*

Proof. Suppose X is Noetherian. Let $Y \subset X$ be a subset equipped with subspace topology, and $\{U_i \cap Y\}_{i \in I}$ be an open cover of Y. Consider the set:

$$\mathscr{U} = \{\text{finite unions of elements in } \{U_i\}\}$$

and equip this set with the partial order given by $V \leq W$ if and only if $V \subset W$. Consider an ascending chain of elements in $\mathscr{U}$:

$$V_1 \subset V_2 \subset \cdots$$

then we obtain the descending chain of closed subsets of X:

$$V_1^c \supset V_2^c \supset \cdots$$

which must terminate for some m as X is Noetherian. By Zorn's lemma, there must then be a maximal element of $\mathscr{U}$, call it W. Then we have that for some $\{i_1, \ldots, i_n\}$

$$W = U_{i_1} \cup \cdots \cup U_{i_n}$$

and moreover that:

$$W \cap Y = (U_{i_1} \cap Y) \cup \cdots \cup (U_{i_n} \cap Y)$$

Suppose that $Y \not\subset W$, then there is a $y \in Y$ such that $y \notin W$. However, $\{U_i \cap\}_{i \in I}$ covers Y, so for some $k \in I$, we have that $y \in U_k$. It follows that $W \subset W \cup U_k$, contradicting the fact that W is maximal, hence $Y \subset W$. Therefore, $Y = W \cap Y$, and the set $\{U_{i_j} \cap Y\}_{j=1}^n$ is a finite subcover, so Y is quasi-compact. In particular, we have that X is quasi-compact.

Now suppose that every subspace of X is compact, and let:

$$V_1 \supset V_2 \supset \cdots$$

be a descending chain of closed subsets of X. Then we obtain an ascending chain of open sets:

$$U_1 \subset U_2 \subset \cdots$$

by letting $U_i = V_1^c$. Consider the open subspace:

$$U = \bigcup_{i=1}^{\infty} U_i$$

which has an open cover given by $\{U_i\}_{i \in \mathbb{N}}$. Since U is quasi-compact, this subspace has an open cover given $U_{i_1} \cup \cdots U_{i_n}$ for some $\{i_1, \cdots i_n\} \subset \mathbb{N}$. Via reordering we can assume that $U_{i_1} \subset \cdots \subset U_{i_n}$, so $U = U_{i_n}$. We claim that the ascending chain stabilize with U_{i_n}. Indeed suppose that $m > i_n$, then $U_{i_n} \subset U_m$, however, by construction, $U_m \in U$, so $U_m = U_{i_n}$. By taking compliments again we obtain that the descending chain of closed subsets:

$$V_1 \supset V_2 \supset \cdots$$

stabilizes so X is Noetherian.

Now finally let Y be a subspace of a Noetherian topological space X. Let $W \subset Y$, then the subspace topology on W induced from Y is the same as the one induced from X. That is, $U \subset W$ is open in the subspace topology if and only $U = Y \cap V$ for some open set $V \subset Y$. However, V is open in Y if and only if $V = X \cap Z$ for some Z open in X. It follows that U is open in W if and only if $U = Y \cap V = X \cap Y \cap Z = X \cap Z$, hence the topologies are equivalent. Since W is quasi-compact as a subspace of X, it follows that W is quasi-compact as subspace Y, hence Y must be Noetherian by argument above. $\square$

Proposition 3.4.1. *Let X be a Noetherian scheme, then X is a Noetherian topological space*

Proof. By Example 3.4.2 we have that X is the union of finitely many Noetherian topological spaces, so it suffices to prove that any such topological space is Noetherian. Let $\{U_i\}_{i \in I}$ be the finite cover of X by Noetherian affine schemes, and suppose that:

$$Y_1 \supset Y_2 \supset \cdots$$

is a descending chain of closed subsets. Then for each i we have that there exists an m_i such that the following chain terminates at m_i:

$$Y_1 \cap U_i \supset Y_2 \cap U_i \supset \cdots \supset Y_{m_i} \cap U_i$$

Take $\max\{m_i\}_{i \in I}$ which exists as I is a finite set, and let m be the maximum number. Then we claim that for any $k \geq m$ we have $Y_m = Y_k$. Indeed, we can write:

$$Y_m = \bigcup_i Y_m \cap U_i = \bigcup_i Y_k \cap U_i = Y_k \tag{3.4.2}$$

as the $\{U_i\}$ cover X, implying the claim. □

As Example 3.4.2 shows, the converse does not hold. We continue to prove topological properties of Noetherian schemes:

Lemma 3.4.6. *Let X be a Noetherian scheme, then X has finitely many irreducible components. In particular, X has a finite number of connected components, each of which is the finite union of irreducible components.*

Proof. Note that any irreducible component is closed. Indeed, if $Z \subset X$ is irreducible then clearly so is $\bar{Z}$, so since Z is maximal we must have that $Z = \bar{Z}$ as Z is by definition a subset of $\bar{Z}$. Since X is a Noetherian topological space by Proposition 3.4.1, and by Lemma 3.4.4 we have that every closed subset of X can be written as the finite union of irreducible closed subsets it follows that:

$$X = Z_1 \cup \cdots \cup Z_n$$

where each Z_i irreducible. Let $\{Y_i\}$ be the set of irreducible components, then since each Z_i must be contained in one of these irreducible components, it follows that:

$$X = \bigcup_i Y_i$$

However, this is a decomposition of X into irreducible closed subspaces, each of which is not contained in the other as they are all maximal. It follows that each Y_i must be equal to some Z_j for some i and j by the uniqueness part of Lemma 3.4.4, hence there can only be finitely many irreducible components.

Let $\{X_i\}$ be the set of connected components, then since each X_i is closed we have that by Lemma 3.4.4:

$$X_i = Z_{1_i} \cup \cdots \cup Z_{n_i}$$

for irreducible closed subsets of X_i. It follows that:

$$X = \bigcup_i Z_{1_i} \cup \cdots \cup Z_{n_i}$$

which must be a finite union as X is Noetherian, implying there are only finitely many X_i. It follows that each X_i must be a finite union of irreducible components Y_i of X by uniqueness of the decomposition of X into irreducible components, again by Lemma 3.4.4, implying the claim. □

It turns out we can check the locally Noetherian condition affine locally (hence the name):

Proposition 3.4.2. *Let X be a scheme, then X is locally Noetherian if and only if every open affine is Noetherian.*

Proof. If every open affine is Noetherian, then clearly X is locally Noetherian.

Suppose that $\{U_i = \operatorname{Spec} A_i\}$ is an affine open cover of X with A_i Noetherian for all i, and $V = \operatorname{Spec} B \subset X$ be any open affine. Then we obtain an open cover of V by $\{V \cap U_i\}$, and for each of these there is an open cover by distinguished opens $U_f \subset \operatorname{Spec} B$ and $U_g \subset \operatorname{Spec} A_i$. Since the schemes $U_i \cap V \subset \operatorname{Spec} A_i$ and $U_i \cap V \subset \operatorname{Spec} B$ are clearly isomorphic (just take the identity), it follows that V admits a cover of distinguished opens all of which are Noetherian schemes. In particular, we have that there exists a finite set of elements $\{f_i\}$ of B which generate the unit ideal $\langle 1\rangle$ such that for all i B_{f_i} is a Noetherian ring.

Now let $I \subset B$ be an ideal, and let $\pi_i : A \to A_{f_i}$ be the localization map. If I_{f_i} is the localized ideal in B_{f_i} then we claim that:

$$I = \bigcap_i \pi_i^{-1}(I_{f_i})$$

For each i we have that $I \subset \pi_i^{-1}(\pi_i(I))$ so it follows that $I \subset \bigcap_i \pi_i^{-1}(I_{f_i})$. Now let $b \in \bigcap_i \pi_i^{-1}(I_{f_i})$, then for each i we have that $\pi_i(b) \in I_{f_i}$, so we have that for some $a_i \in I$, and some integer m_i:

$$\frac{b}{1} = \frac{a_i}{f_i^{m_i}}$$

It follows that there exists an M_i such that $f_i^{M_i} b \in I$. Let M be the maximum of all such M, then since $\langle \{f_i\}\rangle = \langle 1\rangle$, we have that $\langle \{f_i^M\}\rangle = \langle 1\rangle$ so we there exist c_i in B such that:

$$1 = \sum_i c_i f_i^M$$

hence:

$$b = \sum_i c_i f_i^M b \in I$$

Now suppose that:

$$I_1 \subset I_2 \subset \cdots$$

is an increasing chain of ideals, then for each i we have that:

$$I_{1_{f_i}} \subset I_{2_{f_i}} \subset \cdots$$

terminates for some m_{f_i}. Let m be the maximum of all such m_{f_i}, then for all $k > m$ and all i, we have that $I_{k_{f_i}} = I_{m_{f_i}}$. It follows that for all $k > m$ we have that:

$$I_k = \bigcap_i \pi_i^{-1}(I_{k_{f_i}}) = \bigcap_i \pi_i^{-1}(I_{m_{k_{f_i}}}) = I_m$$

so the chain in B terminates with I_m, implying that B is Noetherian, and that V is Noetherian. □

We have the following corollary:

Corollary 3.4.2. *Let X be a scheme, then X is Noetherian if and only if it is quasi-compact, and for every affine open $\mathscr{O}_X(U)$ is a Noetherian ring.*

The condition that X is Noetherian is in a sense a finiteness condition that allows us to prove some striking results. Often times we will restrict to the case where we deal with Noetherian or locally Noetherian schemes, as they are easier to work with, and the condition is actually quite a reasonable one. As an example, note that we showed that X is a reduced scheme if and only if all of it's stalks have no nontrivial nilpotents. The astute reader will recognize that we did not have a similar equivalent condition for a scheme to be integral. As the following theorem shows, we can deduce such a result if we work with sufficiently nice schemes:

Theorem 3.4.3. *Let X be a connected and Noetherian scheme, then X is integral if and only if the stalk $(\mathscr{O}_X)_x$ is an integral domain.*

Proof. Note that if X is integral then stalks are integral domains.

Conversely, suppose that X is a connected Noetherian scheme, such that all the stalks are integral domains. Then all the stalks also contain no nontrivial nilpotents hence X is reduced by Lemma 3.2.1.

By Theorem 3.2.1 we need only show that X is irreducible. As X is connected we have only one connected component, and by Lemma 3.4.6 we have that X has finitely many irreducible components. Let X have a decomposition into:

$$Z_1 \cup \cdots \cup Z_n$$

where each Z_i is an irreducible component. We see that if $Z_1 \cap Z_j = \emptyset$ for all j then Z_1 is open as its compliment is the finite union of closed subset. Since Z_1 is irreducible and thus connected, it follows that either $n = 1$ and $Z_1 = X$ so we are done, or that Z_1 and $Z_2 \cup \cdots \cup Z_n$ are disjoint open sets that cover X so X is disconnected. It follows that if $n \neq 1$, every irreducible component of X must intersect with at least one other irreducible component.

Suppose that $n \neq 1$, then there exist irreducible components Z and Y such that $Z \cap Y \neq \emptyset$. Let $x \in Z \cap Y$ and let $U = \operatorname{Spec} A$ be a affine open containing x. Note that if for all $x \in Z \cap Y$ and all $U = \operatorname{Spec} A$ we have that $Z \cap \operatorname{Spec} A = Y \cap \operatorname{Spec} A$, we can conclude that $Y = Z$, so without loss of generality assume that $Z \cap \operatorname{Spec} A \neq Y \cap \operatorname{Spec} A$. By Lemma 3.2.2, we have that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible closed subsets of $\operatorname{Spec} A$. We claim that they are irreducible components, indeed, suppose that there was an irreducible closed subset $S \subset \operatorname{Spec} A$ such that $Z \cap \operatorname{Spec} A \subset S$, then the closure of S in X is an irreducible closed subset of X containing the closure of $Z \cap \operatorname{Spec} A$, however this is equal to $\bar{Z} = Z$ contradicting the fact that Z is irreducible. It follows that $Z \cap \operatorname{Spec} A$ and $Y \cap \operatorname{Spec} A$ are irreducible components of $\operatorname{Spec} A$.

Now let x correspond to the prime ideal $\mathfrak{p} \subset A$, $Z \cap \operatorname{Spec} A = \mathbb{V}(I)$, and $Y \cap \operatorname{Spec} A = \mathbb{V}(J)$ for radical ideals $I \neq J \subset A$. We claim that I and J are minimal prime ideals over $\langle 0 \rangle$, in the sense that a) they are prime ideals, and b) for every prime ideal we have that if $\mathfrak{q} \subset I$ then $I = \mathfrak{q}$. Let $a, b \in A$ such that $a \cdot b \in I$, then we have that:

$$U_{ab} \cap \mathbb{V}(I) = (U_a \cap \mathbb{V}(I)) \cap (U_b \cap \mathbb{V}(I)) = \emptyset$$

Since $\mathbb{V}(I)$ is irreducible, it follows that either $U_a \cap \mathbb{V}(I)$ or $U_b \cap \mathbb{V}(I)$ are empty, hence either $a \in I$ or $b \in I$ so I and J are both prime. To see that they are minimal, suppose that there exists a prime ideal $\mathfrak{q} \subset I$, then $\mathbb{V}(I) \subset \mathbb{V}(\mathfrak{q})$, but by reversing the argument above we have that $\mathbb{V}(\mathfrak{q})$ is an irreducible closed subset so it follows that $\mathbb{V}(I) = \mathbb{V}(\mathfrak{q})$ as $\mathbb{V}(I)$ is maximal. We thus have that $I = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$ so I and J are both minimal prime ideals over $\langle 0 \rangle$.

We see that A is not an integral domain. Indeed, if A were an integral domain, then $\langle 0 \rangle$ is the unique minimal prime ideal over $\langle 0 \rangle$. In particular, there is a bijection between prime ideals which are contained in $\mathfrak{p}$ and prime ideals of $A_\mathfrak{p}$, hence we must have that there $I_\mathfrak{p}$ and $I_\mathfrak{q}$ are minimal primes of $A_\mathfrak{p}$. It follows that $A_\mathfrak{p} \cong \mathscr{O}_{X,x}$ is not an integral domain, a contradiction, hence we must have that $n = 1$, implying that X is irreducible, so X is reduced and irreducible and thus by Theorem 3.2.1 an integral scheme as desired. $\square$

3.5 Morphisms of Finite Type

Recall that in Definition 2.3.4 we defined what it meant for a k-scheme to be locally of finite type. We now extend this definition to arbitrary schemes as follows:

Definition 3.5.1. Let $f : X \to Y$ be a morphism of schemes. Then f is **locally of finite type** if there exists an affine open cover $\{V_i = \operatorname{Spec} B_i\}$ of Y, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i algebra. The morphism is of **finite type** if the cover of $f^{-1}(V_i)$ is finite.

We have the following obvious examples:

Example 3.5.1. Let A be a finitely generated B algebra, then $\operatorname{Spec} A \to \operatorname{Spec} B$ is obviously of finite type. Let X be a k-scheme of locally finite type, and $f : X \to \operatorname{Spec} k$ the morphism making X a k-scheme, then f is also trivially locally of finite type. If we can take X to be Noetherian k-scheme of locally finite type, then we also have that f is of finite type.

We now show that being locally of finite type is local on target:

Proposition 3.5.1. *Morphisms of locally finite type are:*

a) Local on target.

b) Stable under base change.

c) Closed under composition.

Moreover, morphisms of finite type are closed under composition as well.

Proof. Clearly we have that if for every affine open $V \subset Y$ the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is locally of finite type then f is.

Now suppose that $f : X \to Y$ is a morphism of locally finite type. Let $\{V_i = \operatorname{Spec} B_i\}$ be an open cover for Y, and for each i, let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. Let $V = \operatorname{Spec} B$ be any affine open, then we can write:

$$V = \bigcup_i V_i \cap V$$

hence:

$$\begin{aligned} f^{-1}(V) =& \bigcup_i f^{-1}(V_i) \cap f^{-1}(V) \\ =& \bigcup_{i,j} U_{ij} \cap f^{-1}(V) \end{aligned}$$

Now note that $U_{ij} \cap f^{-1}(V) \subset U_{ij} \cong \operatorname{Spec} A_{ij}$, thus there exist elements $f_{ijk} \in A_{ij}$ such that:

$$U_{ij} \cap f^{-1}(V) = \bigcup_k U_{f_{ijk}}$$

We note that $U_{f_{ijk}} \cong \operatorname{Spec}(A_{ij})_{f_{ijk}}$, hence doing this for all i and j we have obtained an affine open cover:

$$f^{-1}(V) = \bigcup_{i,j,k} U_{f_{ijk}} = \bigcup_{i,j,k} \operatorname{Spec}(A_{ij})_{f_{ijk}}$$

It thus suffices to show that if A is a finitely generated B algebra, then A_f is also a finitely generated B algebra for all $f \in A$. Let $\pi : A \to A_f$ be the localization map, and $\phi : B \to A$ be the map making A a finitely generated B algebra. The map $\pi \circ \phi$ which takes $b \mapsto \phi(b)/1$ is then the map making A_f a B algebra. Let $\{a_1, \ldots, a_n\}$ be the generators of A as a B algebra, then any element $a \in A$ can be written as:

$$a = \sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}$$

We claim that $\{a_1/1, \ldots, a_n/1, 1/f\}$ is a generating set for A_f. Indeed, we see that any element in A_f, can be written as a/f^k, hence:

$$\begin{aligned} a/f^k =& (1/f^k) \cdot a/1 \\ =& (1/f^k) \cdot \frac{\sum_{i_1 \cdots i_n} \phi(b_{i_1 \cdots i_k}) a_1^{i_1} \cdots a_n^{i_n}}{1} \\ =& \sum_{i_1 \cdots i_n} \frac{1}{f^k} \cdot \frac{\phi(b_{i_1 \cdots i_n})}{1} \cdot \frac{a_1^{i_1}}{1} \cdots \frac{a_n^{i_n}}{1} \end{aligned}$$

implying a).

Let $\{V_i = \operatorname{Spec} B_i\}$ be a cover of Z by affine opens, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ a cover of X by affine opens such that $f(U_{ij}) \subset V_i$. Moreover, let $\{W_{ij} = \operatorname{Spec} C_{ij}\}$ be a cover of Y of affine opens such that $g(W_{ij}) \subset V_i$. It follows that $\pi_Y^{-1}(W_{ij}) \cong X \times_{V_i} W_{ij} \cong f^{-1}(V_i) \times_{V_i} W_{ij}$. Now $f^{-1}(V_i) \times_{V_i} W_{ij}$ admits an open affine cover of the form $U_{ik} \times_{V_i} W_{ij} = \operatorname{Spec}(A_{ik} \otimes_{B_i} C_{ij})$. We then need only show that $A_{ik} \otimes_{B_i} C_{ij}$ is a finitely generated C_{ij} algebra. However, this is then clear, as if $\{a_1, \ldots, a_n\}$ are the generators of A_{ik} as a B_i algebra, then $\{a_1 \otimes 1, \ldots, a_n \otimes 1\}$ are generators of $A_{ik} \otimes_{B_i} C_{ij}$ as C_{ij} algebra. Indeed, we can write any element ω in $A_{ik} \otimes B_i C_{ij}$ as a sum of trivial tensors:

$$\omega = \sum_i \alpha_i \otimes c_i = \sum_i (\alpha_i \otimes 1) \cdot (1 \otimes c_i)$$

Each α_i can be written as the finite sum:

$$\alpha_i = \sum_{j_1 \cdots j_n} b_{ij_1 \cdots j_n} a_1^{j_1} \cdots a_n^{j_n}$$

hence:

$$\begin{aligned}\omega =& \sum_i \sum_{j_1 \cdots j_n} (b_i j_1 \cdots j_n a_1^{j_1} \cdots a_n^{j_n} \otimes 1) \cdot (1 \otimes c_i) \\ =& \sum_i \sum_{j_1 \cdots j_n} (a_1 \otimes 1)^{j_1} \cdots (a_n \otimes 1)^{j_n} \cdot (1 \otimes b_{ij_1 \cdots j_n} c_i)\end{aligned}$$

By collecting terms, and relabeling we obtain that:

$$\omega = \sum_{i_1 \cdots i_n} (a_1 \otimes 1)^{i_1} \cdots (a_n \otimes 1)^{i_n} \cdot (1 \otimes c_{i_1 \cdots i_n})$$

implying that $A_{ik} \otimes_{B_i} C_{ij}$ is indeed a finitely generated C_{ij} algebra, and thus b).

Let $\{W_i = \operatorname{Spec} C_i\}$ be an open affine cover for Z. Since g is (locally) of finite type, there exists an open affine cover $g^{-1}(W_i)$, $\{V_{ij} = \operatorname{Spec} B_{ij}\}_j$, such that each B_{ij} is a a finitely generated C_i algebra. By the same logic, there exists an affine open cover of each $f^{-1}(V_{ij})$, $\{U_{ijk} = \operatorname{Spec} A_{ijk}\}_k$, such that each A_{ijk} is a finitely generated B_{ij} algebra. Now note that for each i:

$$\begin{aligned}\bigcup_{jk} U_{ijk} =& \bigcup_{ij} \left(\bigcup_k U_{ijk} \right) \\ =& \bigcup_j f^{-1}(V_{ij}) \\ =& f^{-1} \left(\bigcup_j V_{ij} \right) \\ =& f^{-1}(g^{-1}(W_i))\end{aligned}$$

hence for each i, the $\{U_{ijk}\}_{jk}$ form an affine open cover of $(g \circ f)^{-1}(W_i)$. It now suffices to show that each A_{ijk} is a finitely generated C_i algebra. Each A_{ijk} is a finitely generated B_{ij} algebra, so let $\{a_1, \ldots, a_n\}$ generate A_{ijk} as a B_{ijk} algebra. Moreover, we have that each B_{ij} is a finitely generated C_i algebra so let $\{b_1, \ldots, b_m\}$ generate B_{ij} as a C_i algebra. We claim that $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$[64] generates A_{ijk} as C_i algebra. Indeed, let $a \in A$, then:

$$a = \sum_{l_1 \cdots l_n} b_{l_1 \cdots l_n} a_1^{l_1} \cdots a_n^{l_n}$$

We can write:

$$b_{l_1 \cdots l_n} = \sum_{\lambda_1 \cdots \lambda_m} c_{l_1 \cdots l_n \lambda_1 \cdots \lambda_m} b_1^{\lambda_1} \cdots b_m^{\lambda_m}$$

hence:

$$a = \sum_{l_1 \cdots l_n \lambda_1 \cdots \lambda_m} c_{l_1 \cdots l_n \lambda_1 \cdots \lambda_m} b_1^{\lambda_1} \cdots b_m^{\lambda_m} a_1^{l_1} \cdots a_n^{l_n}$$

[64]Here it understood that by b_l we mean the image of b_l in A_{ijk} under the homomorphism making A_{ijk} a B_{ij} algebra.

implying c).

For the last claim, if g and f are of finite type, then every cover can be taken to be finite, hence $\{U_{ijk}\}_{jk}$ is a finite cover of $(g \circ f)^{-1}(W_i)$, so $g \circ f$ is of finite type as well. $\square$

Example 3.5.2. Let $f : X \to Y$ be a closed embedding, then f is of finite type. Indeed, for every affine open $U = \operatorname{Spec} A \subset Y$, we have that $f^{-1}(U) = \operatorname{Spec} A/I$, so admits a finite cover of affine opens of X. It remains to show that A/I is a finitely generated A algebra, however this clear as any $[a] \in A$ can be written as $a \cdot [1] = [a \cdot 1] = [a]$, hence A/I is finitely generated over A by $[1]$.

Let $\imath : U \to X$ be an open embedding, then $\imath$ is locally of finite type. Indeed, let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover of X, then $\imath^{-1}(V_i) = U \cap V_i$ and $\imath|_{U \cap V_i} : U \cap V_i \to V_i$ is an open embedding into an affine scheme. We can cover each $U \cap V_i$ with $U_{f_{ij}} \subset \operatorname{Spec} B_i$ for some $f_{ij} \in B$. It follows that $\{U_{f_{ij}}\}_j$ is a cover for $\imath^{-1}(V_i)$, and that $\imath|_{U_{f_{ij}}}$ is the given by the localization map $\pi_{ij} : B_i \to (B_i)_{f_{ij}}$. Consider the morphism:

$$\begin{aligned} \phi : B_i[x] &\longrightarrow (B_i)_{f_{ij}} \\ x &\longmapsto 1/f_{ij} \end{aligned}$$

Let $b/f_{ij}^n \in (B_i)_{f_{ij}}$, then $bx^n \mapsto b/f_{ij}^n$ so $(B_i)_{f_{ij}}$ is finitely generated by $\{1, 1/f_{ij}\}$ as a B_i algebra. If X is Noetherian, then we can take $\imath$ to be of finite type.

Locally finite type schemes over a field have the following useful property:

Proposition 3.5.2. *Let X and Y be schemes locally of finite type over k, and $f : X \to Y$ a morphism of k-schemes*[65]*. Then if $x \in |X|$ we have that $f(x) \in |Y|$, i.e. f takes closed points to closed points. In particular, if x is a point with residue field equal to k*[66]*, then $f(x)$ also has residue field k.*

Proof. Let $x \in |X|$, and choose affine open $V = \operatorname{Spec} B \subset Y$ such that $f(x) \in V$. There then exists an affine open $U = \operatorname{Spec} A \subset X$, containing x which maps into V. It follows that $f|_U : U \to V$ is a morphism of affine schemes; let it be induced by the ring map $\phi : B \to A$. Since x is closed, x corresponds to a maximal idea $\mathfrak{m}$ of A; by Lemma 2.3.12 it suffices to show that $\phi^{-1}(\mathfrak{m})$ is also maximal. Note that since f is a morphism of k schemes, that ϕ is a morphism of k algebras. Since A is finitely generated we have that $A/\mathfrak{m} \cong k_x$ is a finite field extension of k by Zariski's lemma[67]. There is an induced map:

$$\pi \circ \phi : B \to A/\mathfrak{m} \cong k_x$$

where $\pi : A \to A/\mathfrak{m}$ is the projection. The kernel of this morphism is:

$$(\pi \circ \phi)^{-1}(0) = \phi^{-1}(\pi^{-1}(0)) = \phi^{-1}(\mathfrak{m})$$

Moreover, the image of this morphism is the k algebra $B/\phi^{-1}(\mathfrak{m})$ which now obviously sits in the following inclusions:

$$k \subset B/\phi^{-1}(\mathfrak{m}) \subset k_x$$

Since k_x is a finite field extension, and thus a finite dimensional vector space over k, and $B/\phi^{-1}(\mathfrak{m})$ is an integral domain, it follows that $B/\phi^{-1}(\mathfrak{m})$ is a field,[68] implying that $\phi^{-1}(\mathfrak{m})$ is maximal as desired. The final statement follows by noting that if $k_x = k$ then $B/\phi^{-1}(\mathfrak{m})$ is clearly equal to k as well. $\square$

3.6 Separated Z-Schemes

In the category of topological spaces, direct products exist, and a space is Hausdorff if and only if the map $\Delta : X \to X \times X$ has closed image. In the category of schemes, the topological spaces we are dealing

[65] I.e the relevant diagram commutes.
[66] Also known as a k-rational point.
[67] See Theorem 6.1.3
[68] We used a similar argument in Lemma 2.3.12.

with are almost never dealing with Hausdorff spaces and we do not have product. Indeed, consider the affine plane $\mathbb{A}^n_{\mathbb{C}}$, then this space is modeled off of $\mathbb{C}^n$, but is certainly not Hausdorff, as the unique generic point is contained in every open set. Moreover, we have that $\mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^m_{\mathbb{C}} \cong \mathbb{A}^{n+m}_{\mathbb{C}}$, so fibre products mildly behave like direct products, but $\mathbb{A}^{n+m}_{\mathbb{C}}$ has many more points than the naive cartesian product[69].

However, if we restrict ourself to the category of Z-schemes, then fibre product, $X \times_Z Y$, does satisfy the universal property of the direct product. Indeed, this is true essentially by constriction, if $f_X : X \to Z$ and $f_Y : Y \to Z$ are Z-schemes, then their fibre product is a Z-scheme. If $f_Q : Q \to Z$ is a Z-scheme, and $p_X : Q \to X$ and $p_Y : Q \to Y$ are morphisms of Z-schemes, then we automatically have $f_X \circ p_X = Q_X$ and $f_Y \circ p_Y = Q_Y$, so there exists a unique morphism $Q \to X \times_Z Y$ of Z-schemes which satisfies the direct product diagram. With this in mind, we wish to develop an analogue to a scheme being Hausdorff, which mimics the definition of Hausdorff in the category of topological spaces, leading us to the next definition:

Definition 3.6.1. Let X be a Z-scheme, then X is **separated over Z**, or alternatively a separated **Z**-scheme, if the diagonal map $\Delta : X \to X \times_Z X$ has closed image. A morphism $f : X \to Z$ is **separated** if $\Delta : X \to X \times_Z X$ is a closed embedding.

The notion of separatedness is our analogue of Hausdorff in the category of schemes, and we will spend the next few pages discussing the implications of such a result.

Example 3.6.1. Let $X = \mathbb{A}^n_k$, then we claim that $\mathbb{A}^n_k$ is separated over k. Indeed, we have that $X \times_k X \cong \operatorname{Spec} k[x_1, \ldots, x_n] \otimes_k k[x_1, \ldots, x_n]$, and that the diagonal morphism is induced by the ring homomorphism given on simple tensors by $\phi : f \otimes g \mapsto fg$. This is a surjective ring homomorphism, so if $I = \ker \phi$, we have that $\Delta(X) = \mathbb{V}(I) \subset X \times_k X$. It follows that X is separated over k.

We would actually like to show that the notion of being separated over a scheme Z is the same as the morphism $f : X \to Z$ being a separated morphism. To do so we will need to show that the diagonal map is a closed embedding if it has closed image[70] We need the following definition:

Definition 3.6.2. A morphism $f : X \to Y$ is a **locally closed immersion**[71] if f factors as a closed embedding followed by an open embedding. In other words we have the following commutative diagram for some open subset $U \subset Y$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & U \arrow[ur, "\imath"'] & \end{tikzcd}$$

where g is a closed embedding, and $\imath$ is the inclusion.

We want to show every diagonal map is a locally closed immersion.

Lemma 3.6.1. *Let $f : X \to Z$ be a morphism, then $\Delta : X \to X \times_Z X$ is a locally closed immersion.*

Proof. Let $\{V_i\}$ be an affine open cover for Z, and for each i let: $\{U_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$. We have that $\{U_{ij} \times_{V_i} U_{ik}\}_{i,j,k}$ is an open affine cover for $X \times_Z X$, and claim that:

$$U = \bigcup_{ij} U_{ij} \times_{V_i} U_{ij}$$

contains the image of Δ. However, this is clear because $\Delta|_{U_{ij}}$ has image in $U_{ij} \times_{V_i} U_{ij}$, so Δ has image in U, and we have that Δ factors as:

$$X \longrightarrow U \longrightarrow X \times_Z X$$

The second morphism is clearly an open embedding, so we need only show that the morphism with restricted image, which we denote by g, is a closed embedding. This is also clear, as if $U_{ij} = \operatorname{Spec} A_{ij}$, and $V_i = \operatorname{Spec} B$, then $g|_{U_{ij}}$ is induced by the ring homomorphism $A_{ij} \otimes_{B_i} A_{ij} \to A_{ij}$ which is surjective, and is thus a quotient map. By Corollary 3.1.2 we have that g is a closed embedding, implying the claim. $\square$

[69] Which is not even a scheme!

[70] The other direction is immediate.

[71] In the literature this is sometimes called a locally closed embedding, or simply an immersion.

We now prove the following more general statement:

Proposition 3.6.1. *Let $f : X \to Y$ be a locally closed embedding, then f is a closed embedding if and only if $f(X)$ has closed image in Y.*

Proof. Suppose that f is a closed embedding, then f trivially has closed image. Moreover, every closed embedding is a locally closed immersion as Y is trivially an open subscheme of Y.

Now let f be a locally closed immersion, and factor as $\imath \circ g$ where $g : X \to U$ is a closed embedding, and $\imath$ is the inclusion map into Y. Suppose $f(X)$ has closed image in Y, then by Corollary 3.1.2 we need to find an open cover of Y such that f restricts to a closed embedding. Note that:

$$Y = U \cup f(X)^c$$

as $f(X) \subset U$. We have that $f|_{f^{-1}U} : f^{-1}(U) = X \to U$ is a closed embedding as it is equal to g, and that $f^{-1}(f(X)^c)$ is the empty scheme $\emptyset$, so $f|_\emptyset$ is the empty map which is also trivially a closed embedding, implying the claim. $\square$

We now have the following corollary:

Corollary 3.6.1. *Let $f : X \to Z$ be a morphism of schemes, then f is separated if and only if X is separated over Z.*

Before calculating some examples, the following lemma will prove useful:

Lemma 3.6.2. *Let $X \to Z$ be a Z scheme, and $U_i, U_j \subset X$ open subschemes mapping into $V \subset Z$. Then $\Delta^{-1}(U_i \times_V U_j) = U_i \cap U_j$.*

Proof. We have that $U_i \times_V U_j$ is an open subscheme of $X \times_Z X$, and that X is an $X \times_Z X$ scheme via the diagonal map. Lemma 2.3.7 then tells us that $(U_i \times_V U_j) \times_{X \times_Z X} X$ is canonically an open subscheme of X given by $\Delta^{-1}(U_i \times_V U_j)$.

We see that we have the following canonical isomorphism:

$$(U_i \times_V U_j) \times_{X \times_Z X} X \cong (U_i \times_{X \times_Z X} X) \times_X (X \times_{X \times_Z X} U_j)$$

Now $U_i \times_{X \times_Z X} X$ makes the following square cartesian:

$$\begin{tikzcd}
U_i \times_{X \times_Z X} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\
U_i \arrow[r, "\Delta|_{U_i}"] & X \times_Z X
\end{tikzcd}$$

We claim that U_i satisfies the universal property of $U_i \times_{X \times_Z X} X$ with projections given by the identity map $U_i \to U_i$, and the inclusion map $\imath : U_i \to X$. Let Q be a scheme such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_{U_i}"'] & & \\
& U_i \arrow[r, "\imath"] \arrow[d, "\mathrm{Id}"] & X \arrow[d, "\Delta"] \\
& U_i \arrow[r, "\Delta|_{U_i}"] & X \times_Z X
\end{tikzcd}$$

Then in particular, we have that since $\Delta|_{U_i} = \Delta \circ \imath$:

$$\Delta \circ \imath \circ f_{U_i} = \Delta \circ f_X$$

Let $\pi_1 : X \times_Z X \to X$ be the projection onto the first factor, then:

$$\pi_1 \circ \Delta \circ \imath \circ f_{U_i} = \pi_1 \circ \Delta \circ f_X$$

However, $\pi_1 \circ \Delta = \mathrm{Id}_X$, hence:

$$\imath \circ f_{U_i} = f_X$$

It follows that the choosing as f_{U_i} as the middle morphism makes the diagram commute, and it is unique because any other g morphism needs to satisfy $\mathrm{Id} \circ g = f_{U_i}$. Therefore there is a unique isomorphism $U_i \times_{X \times_Z X} X \cong U_i$.

It follows that:

$$(U_i \times_V U_j) \times_{X \times_Z X} X \cong U_i \times_X U_j \cong U_i \cap U_j$$

implying the claim.

□

We now list some examples (and non-examples) of separated schemes and morphisms:

Example 3.6.2. Every morphism of affine schemes is separated. Indeed, let $\operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine scheme, then $\operatorname{Spec} A \times_B \operatorname{Spec} A = \operatorname{Spec} A \otimes_B A$, and the diagonal morphism is given by $a_1 \otimes a_2 \mapsto a_1 \dot{a}_2$ which is surjective, so Δ is a closed embedding. In particular, $\operatorname{Spec} A$ is separated over $\operatorname{Spec} B$.

Example 3.6.3. We claim that $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$ is separated over A. We construct the map $\mathbb{P}^n_A \to \operatorname{Spec} A$ given on the open cover $\{U_{x_i} = \operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0\}$ by the morphism of affine schemes induced by the ring homomorphisms $A \hookrightarrow (A[x_0, \dots, x_n]_{x_i})_0$. We have an open cover of $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ by $\{U_{x_i} \times_A U_{x_j}\}_{i,j}$. Now note that $\Delta^{-1}(U_{x_i} \times_A U_{x_j})$ is equal to the intersection:

$$U_{x_i} \cap U_{x_j} = (\operatorname{Spec}(A[x_0, \dots, x_n]_{x_i})_0)_{x_j/x_i} \cong \operatorname{Spec} A[\{x_k/x_i\}_{k \neq i}, x_i/x_j]$$

We have that:

$$U_{x_i} \times_A U_{x_j} = \operatorname{Spec} A[\{z_k/z_i\}_{k \neq i}, \{y_k/y_j\}_{k \neq j}]$$

We have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is induced by the ring homomorphism which makes the following diagram of rings commute:

$$\begin{tikzcd}
A[\{x_k/x_i\}_{k\neq i}, x_i/x_j] & & \\
& A[\{z_k/z_i\}_{k\neq i}, \{y_k/y_j\}_{k\neq j}] \arrow[ul, "\Delta^\sharp"] & A[\{x_k/x_j\}_{k\neq j}] \arrow[l, "\imath_j"'] \arrow[ull, bend right, "\phi"'] \\
& A[\{x_k/x_i\}_{k\neq i}] \arrow[u, "\imath_i"] \arrow[uul, bend left, "\psi"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

where ψ is the inclusion, ϕ is the morphism $x_k/x_j \mapsto x_k/x_i \cdot x_i/x_j$ for $j \neq i$, and $x_i/x_j \mapsto x_i/x_j$. The maps $\imath_i$ and $\imath_j$ take x_k/x_i and x_k/x_j to z_k/z_i and y_k/y_j respectively. It follows that $\Delta^\sharp(z_k/z_i) = x_k/x_i$, and that $\Delta^\sharp(y_i/y_j) = x_i/x_j$ so $\Delta^\sharp$ is indeed surjective. Therefore, on the open cover $U_{x_i} \times U_{x_j}$ we have that $\Delta|_{U_{x_i} \cap U_{x_j}}$ is a closed embedding, so Δ is a closed embedding thus $\mathbb{P}^n_A$ is separated over $\operatorname{Spec} A$.

Example 3.6.4. As a generalization of the preceding example, we let A now be any graded ring, and claim that $\operatorname{Proj} A$ is separated over $\operatorname{Spec} A_0$. Note that for any $f \in A_+^{\mathrm{hom}}$, the morphism $\operatorname{Proj} A \to \operatorname{Spec} A_0$ is induced by the obvious morphism $A_0 \to (A_f)_0$. The maps obviously agree on overlaps as any element in the image of this map is over the form $a/1$.

Now, $\operatorname{Proj} A \times_{A_0} \operatorname{Proj} A$, has an open cover by $U_f \times_{A_0} U_g \cong \operatorname{Spec}(A_f)_0 \otimes_{A_0} (A_g)_0$ for all $f, g \in A_+^{\text{hom}}$. By Lemma 3.6.2, we have that $\Delta^{-1}(U_f \times_{A_0} U_g) = U_f \cap U_g = U_{fg}$, hence it suffices to check that the morphism of affine schemes $U_{fg} \to U_f \times_{A_0} U_g$ is a closed embedding. By replacing f with $f^{\deg g}$ and $g^{\deg f}$, we may assume that f and g are of the same degree. Now the diagonal morphism $U_{fg} \to U_f \times_{A_0} U_g$ must come from a ring homomorphism $\Delta^\sharp : (A_{fg})_0 \longrightarrow (A_f)_0 \otimes_{A_0} (A_g)_0$. In particular, this homomorphisms must make the following diagram commute:

$$\begin{tikzcd}
(A_{fg})_0 & & \\
& (A_f)_0 \otimes (A_g)_0 \arrow[ul, "\Delta^\sharp"'] & (A_g)_0 \arrow[l] \arrow[ull, bend right, "\theta_g"'] \\
& (A_f)_0 \arrow[u] \arrow[uul, bend left, "\theta_f"] & A_0 \arrow[l] \arrow[u]
\end{tikzcd}$$

where in this case the restriction maps $\theta_f : (A_f)_0 \to (A_{fg})_0$ are given by $a/f^k \mapsto a \cdot g^k/(fg)^k$. Note that this clearly lands in the degree 0 part of A_{fg}. It follows that $\Delta^\sharp$ must be given on simple tensors by:

$$a/f^k \otimes b/g^l \longmapsto \frac{abf^l g^k}{(fg)^{k+l}}$$

Now let $a/(fg)^k \in (A_{fg})_0$, then $\deg a = 2 \cdot k \cdot \deg f$, so consider the element:

$$\frac{a}{f^{2k}} \otimes \frac{f^k}{g^k} \in (A_f)_0 \otimes (A_g)_0$$

Under $\Delta^\sharp$, we have that this maps to:

$$\frac{ag^{2k} f^{2k}}{(fg)^{3k}} = \frac{a}{(fg)^k}$$

It follows that $\Delta^\sharp$ is surjective, and so $\Delta|_{U_{fg}}$ is a closed embedding for all all f and g, hence Δ is a closed embedding and $\operatorname{Proj} A$ is separated over $\operatorname{Spec} A_0$.

We have the following non example:

Example 3.6.5. Let Z be the scheme obtained by gluing $X = \operatorname{Spec} \mathbb{C}[x]$ and $Y = \operatorname{Spec} \mathbb{C}[y]$ along the affine open U_x and U_y via the isomorphism induced by $x \mapsto y$. We claim that Z is not separated over $\operatorname{Spec} \mathbb{C}$. If ψ_X and ψ_Y are the open embeddings $X \to Z$ and $Y \to Z$ respectively, we have that Z has an open cover given by $\psi_X(X)$ and $\psi_Y(Y)$. It follows that $Z \times_{\mathbb{C}} Z$ has an open cover given by:

$$\{\psi_X(X) \times_{\mathbb{C}} \psi_X(X), \psi_Y(Y) \times_{\mathbb{C}} \psi_Y(Y), \psi_X(X) \times_{\mathbb{C}} \psi_Y(Y), \psi_Y(Y) \times_{\mathbb{C}} \psi_X(X)\}$$

Each of these is isomorphic to the affine plane $\mathbb{A}^2_{\mathbb{C}}$, so we need to determine how these schemes glue together. We label these schemes by X_1, X_2, X_3 and X_4 in the order which they appear, and set:

$$X_i = \operatorname{Spec} \mathbb{C}[x_i] \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}[y_i] = \operatorname{Spec} \mathbb{C}[x_i, y_i]$$

Then X_1 and X_2 are glued on $U_{x_1} \cap U_{y_1}$ and $U_{x_2} \cap U_{y_2}$, X_1 and X_3 are glued along U_{y_1} and U_{y_3}, X_1 and X_4 are glued along U_{x_1} and U_{x_4}, X_2 and X_3 are glued along U_{x_2} and U_{x_3}, X_2 and X_4 are glued on U_{y_2} and U_{y_4} and X_3 and X_4 are glued along $U_{x_3} \cap U_{y_3}$ and $U_{x_4} \cap U_{y_4}$. All of these morphisms are induced by the by isomorphism $x_i, y_i \mapsto x_j, y_j$.

It follows that $Z \times_{\mathbb{C}} Z$ is the affine plane with four origins, and doubled axis. The diagonal $\Delta(Z)$ is equal to $\Delta(\psi_X(X)) \cup \Delta(\psi_Y(Y))$, which via the above identification is contained in $X_1 \cup X_2$[72]. In particular,

[72]Abuse of notation alert! Technically, each X_i is a copy of $\mathbb{A}^2_{\mathbb{C}}$ which we glue together to get $Z \times_{\mathbb{C}} Z$, so only their images under the canonical open embeddings are contained in $Z \times_{\mathbb{C}} Z$. We employ this abuse so as to not clutter the page with notation.

geometrically $\Delta(Z) \cap X_1$ and $\Delta(Z) \cap X_2$ is the diagonal in $\mathbb{A}^2_{\mathbb{C}}$, while $\Delta(Z) \cap X_3$ and $\Delta(Z) \cap X_4$ is the diagonal of $\mathbb{A}^2_{\mathbb{C}}$ minus the origin. Therefore, $\Delta(Z) \cap X_i$ is not closed for all i, hence by definition of the topology on $Z \times_{\mathbb{C}} Z$, we have that Z is not separated.

Note that this also shows that Z is not an affine scheme by Example 3.6.2.

We now show that every open and closed embedding is also a separated morphism:

Proposition 3.6.2. *Let $f : X \to Z$ be a closed or open embedding, then X is separated over Z.*

Proof. First suppose that $f : X \to Z$ is a closed embedding, then there exist an open affine cover $\{V_i = \operatorname{Spec} B_i\}$ of Z such that $U_i = f^{-1}(V_i) = \operatorname{Spec} B_i/I_i$ for some ideal I. It follows that $X \times_Z X$ admits an open affine cover of the form:

$$\{U_i \times_{V_i} U_i = \operatorname{Spec}(B_i/I_i \otimes_{B_i} B_i/I_i)\}$$

Since $B_i/I_i \otimes_{B_i} B_i/I_i \cong B_i/I_i$ we have that $U_i \times_{V_i} U_i \cong U_i$ so $X \times_Z X \cong X$ and the diagonal map is just the identity. In particular, one can also see this by noting the $f(X) \times_Z f(X) \cong f(X) \cap f(X) \cong f(X) \cong X$.

Now suppose that $f : X \to Z$ is an open embedding, then $X \cong U$ for some open subscheme of Z. We have that $X \times_Z X \cong U \times_Z U \cong U \times_U U = U$, so again the diagonal map is just the identity, implying the claim. ☐

Recall that morphisms/topological properties of schemes are generally considered 'nice' if they are either local on target or stable under base change. We want to see that separated morphisms fall into this category as well:

Proposition 3.6.3. *Let $f : X \to Z$ be a morphism of schemes, then the following hold:*

a) Separated morphisms are local on target.

b) Separated morphisms are stable under base change.

c) Separated morphisms are closed under composition.

Proof. To show a), we first assume that f is separated. It follows that $\Delta : X \to X \times_Z X$ is a closed embedding, so $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is also a closed embedding. It follows that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is a separated morphism as well.

Now suppose that we have affine open cover $\{V_i = \operatorname{Spec} B_i\}$ such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is separated. This then implies that $\Delta|_{f^{-1}(V_i)} : f^{-1}(V_i) \to f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is a closed embedding. Let $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover for $f^{-1}(V_i)$, then we have that $\{U_{ij} \times_{V_i} U_{ik}\}$ is an affine open cover for $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$, and that $\Delta|^{-1}_{f^{-1}(V_i)}(U_{ij} \times_{V_i} U_{ik}) = U_{ij} \cap U_{ik}$. Since this is a closed embedding, we thus have that $U_{ij} \cap U_{ik}$ is affine and of the form $\operatorname{Spec} A_{ij} \otimes_{B_i} A_{ik}/I$ for some ideal I. Doing this for all i, we obtain an open affine cover of $X \times_Z X$ such Δ restricts to a closed embedding on $\Delta^{-1}(U_{ij} \times_{V_i} U_{ik})$ so it follows that Δ itself is a closed embedding.

To show b), suppose that $f : X \to Z$ is separated, and let $g : Y \to Z$ be any morphism. We want to show that $X \times_Z Y$ is separated over Y. We have that:

$$\begin{aligned}
(X \times_Z Y) \times_Y (X \times_Z Y) &\cong (X \times_Z Y) \times_Y (Y \times_Z X) \\
&\cong ((X \times_Z Y) \times_Y Y) \times_Z X \\
&\cong (X \times_Z Y) \times_Z X \\
&\cong (X \times_Z X) \times_Z Y
\end{aligned}$$

The diagonal map $\Delta : X \times_Z Y \to (X \times_Z Y) \times_Y (X \times_Z Y)$ is then the map induced by $\Delta_X : X \to X \times_Z X$ and the identity on Y, composed with the above chain of isomorphisms. In other words we have the following diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "\pi_Y"] \arrow[ddr, bend right, "\Delta_X \circ \pi_X"'] & & \\
& (X \times_Z X) \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{X \times_Z X}"] & Y \arrow[d, "g"] \\
& X \times_Z X \arrow[r, "f \circ \pi_X"] & Z
\end{tikzcd}$$

The claim then follows from Theorem 3.1.2.

We write $\Delta : X \to X \times_Z X$ for the diagonal map we wish to prove is a closed embedding, and $\Delta_X : X \to X \times_Y X$, $\Delta_Y : Y \to Y \times_Z Y$ for the diagonal maps we know to be closed embeddings. From Theorem 2.3.1 we have the following Cartesian square:

$$\begin{tikzcd}
X \times_Y X \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"] & X \times_Z X \arrow[d, "f \times f"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_Z Y
\end{tikzcd}$$

where ψ is the map coming from the following diagram[73]:

$$\begin{tikzcd}
X \times_Y X \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\
& X \arrow[r, "g \circ f"] & Z
\end{tikzcd}$$

It follows that ψ is closed embedding as it is the base change of the closed embedding Δ_Y. We claim that $\Delta = \psi \circ \Delta_X$. Indeed, Δ comes from the following diagram:

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g \circ f"] \\
& X \arrow[r, "g \circ f"] & Z
\end{tikzcd}$$

Now $\pi_X \circ \psi \circ \Delta_X = \pi_X \circ \Delta_X = \mathrm{Id}_X$, so $\psi \circ \Delta_X$ makes the above diagram commute. It follows that Δ is the composition of a closed embeddings, and thus a closed embedding, hence $g \circ f$ is separated. $\square$

Note that the above implies that if X is a separated Z scheme, and Z is separated over $\operatorname{Spec} \mathbb{Z}$ then X is separated over $\operatorname{Spec} \mathbb{Z}$. In particular, every separated k scheme is separated over $\operatorname{Spec} \mathbb{Z}$, and every affine scheme, and scheme of the form $\operatorname{Proj} A$ is separated over $\operatorname{Spec} \mathbb{Z}$. In particular, the following lemma demonstrates that any scheme which is separated over $\operatorname{Spec} \mathbb{Z}$ is separated over any other scheme.

[73] Abuse of notation alert! We are once again using the notation π_X to refer to multiple maps.

Lemma 3.6.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated, then so is f.*

Proof. Let $\psi : X \times_Y X \to X \times_Z X$ be the morphism from Proposition 3.6.3, then by our work in that proposition we know that ψ is the base change of a closed embedding and is thus a closed embedding. Moreover, we know that the diagonal map $\Delta : X \to X \times_Z X$ factors as:

$$\begin{tikzcd} X \arrow[r, "\Delta_Y"] \arrow[rr, bend right, "\Delta"] & X \times_Y X \arrow[r, "\psi"] & X \times_Z X \end{tikzcd}$$

Now $\Delta(X) \subset X \times_Z X$ is closed by assumption, and $\psi(\Delta_Y(X)) = \Delta(X)$, hence since ψ is injective, we have that:

$$\psi^{-1}(\Delta(X)) = \psi^{-1}(\psi(\Delta_Y(X))) = \Delta_Y(X)$$

hence $\Delta_Y(X)$ is closed in $X \times_Y X$. The claim follows from Corollary 3.6.1. □

Now we have the following corollary:

Corollary 3.6.2. *Let X be separated over* $\operatorname{Spec}\mathbb{Z}$*, then any morphism $f : X \to Y$ is separated.*

Proof. Let $g : Y \to \operatorname{Spec}\mathbb{Z}$ be the unique morphism from Y to $\operatorname{Spec}\mathbb{Z}$. Then $g \circ f$ is the unique morphism from $X \to \operatorname{Spec}\mathbb{Z}$ which is separated by assumption. It follows from the proceeding lemma that f is separated. □

One less than ideal artifact of the topology on a scheme is that the intersection of two affine opens need not be affine. Indeed, let X be the affine plane over $\mathbb{C}$ with doubled origin, then their are two copies of $\mathbb{A}^2_{\mathbb{C}}$ contained in X, but their intersection is two copies of the zero ideal $\langle 0 \rangle$ which is manifestly not affine, i.e. no ring has two copies of the zero ideal as a prime spectrum. We now show that separated morphisms provide a solution to this problem:

Proposition 3.6.4. *Let $f : X \to Z$ be a separated morphism, and let $V =$ Spec $B \subset Z$ be an open affine. Then for every open affine $U_i =$ Spec $A_i \subset X$ which maps into V, we have that $U_i \cap U_j$ is an open affine.*

Proof. Let $\Delta : X \to X \times_Z X$, and then note that $\Delta(f^{-1}(V))$ is contained in $f^{-1}(V) \times_V f^{-1}(V)$. We see that if U_i and U_j are as above, we have that $\Delta^{-1}(U_i \times_V U_j) = U_i \cap U_j$, but Δ is a closed embedding so $U_i \cap U_j$ is of the form $\operatorname{Spec}(A_i \otimes_B A_j)/J$ hence $U_i \cap U_j$ is indeed an open affine. □

We have the following obvious corollary:

Corollary 3.6.3. *Let X be separated over an affine scheme* Spec A*. Then the intersection of every affine open in X is an affine open.*

Note that this text in algebraic geometry has never once mentioned the notion of a variety, largely because the author was first introduced to algebraic geometry through the language of schemes. However, we now have sufficient language to give the definition of a variety, which are often the most geometric feeling schemes. We note that the definition of a variety varies wildly throughout the literature, and will change in this text when we discuss Abelian varieties.

Definition 3.6.3. Let X be a scheme, then X is a **variety over k** if X is of finite type over a field k, reduced, and separated over Spec k.

Note that every variety is immediately quasi-compact as it is the finite union of affine schemes. Each of these affine schemes is Spec of a finitely generated k-algebra thus every variety is locally Noetherian. In particular, by Corollary 3.4.2 every variety is Noetherian.

Example 3.6.6. The n-dimensional affine plane $\mathbb{A}^n_{\mathbb{C}}$, and projective space $\mathbb{P}^n_{\mathbb{C}}$ are varieties. In general, the closed points of 'nice enough' varieties over $\mathbb{C}$, when equipped with the standard topology induced by that on $\mathbb{C}^n$ have the structure of smooth manifolds. We will make this notion precise later in the text.

Example 3.6.7. Let X be a variety, then every closed subset of $Z \subset X$ is a variety when equipped with the induced reduced subscheme structure. Every reduced closed subscheme of X is isomorphic to such a Z, so every reduced closed subscheme of X is a variety.

Let U an open subscheme of X, then U is a variety. Indeed, open embeddings are separated by Proposition 3.6.2, and are locally of finite type by Example 3.5.2. Since X is Noetherian the open embedding $\imath : U \to X$ is of finite type, thus U. Finally U is reduced as being reduced is a local property.

Suppose that Y is a reduced locally closed subscheme of X, i.e. there exists a morphism $\imath : Y \to X$ such that $\imath$ is a locally closed immersion. Then $\imath$ factors as an open embedding in to a reduced closed subscheme $Z \subset X$, followed by the closed embedding $Z \hookrightarrow X$. It follows that Y is a variety as it is an open subscheme of the variety Z.

We have the following result:

Theorem 3.6.1. *Let X be a reduced projective k-scheme, then X is a variety. In particular, any closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure is a variety.*

Proof. By Theorem 3.1.1 a projective k scheme is closed subscheme of $\mathbb{P}^n_k$ for some k, hence there exists some closed embedding $X \hookrightarrow \mathbb{P}^n_k$. Since closed embeddings are separated by Proposition 3.6.2, and separated morphisms are closed under composition by Proposition 3.6.3, we have that the natural morphism $X \hookrightarrow \mathbb{P}^n_k \to \operatorname{Spec} k$ making a X a k scheme is separated. Moreover, by Example 3.5.2, $X \hookrightarrow \mathbb{P}^n_k$ is separated, so Proposition 3.6.3 implies that X is separated over k. Since X is assumed to be reduced, we have that X naturally carries the structure of a scheme of variety over k.

Let $X \subset \mathbb{P}^n_k$ be any closed subset, then equipped with the induced reduced subscheme structure, we have that the above discussion applies to X as well, hence X is a variety. □

With Theorem 3.6.1, and Example 3.6.6 in mind we employ the following definitions:

Definition 3.6.4. A scheme X is a **projective variety** if it is a reduced closed subscheme of $\mathbb{P}^n_k$ for some n. In particular, every projective variety is isomorphic to a closed subset of $\mathbb{P}^n_k$ equipped with the induced reduced subscheme structure. A scheme X is a **quasi-projective variety** if it is an open subscheme of $\mathbb{P}^n_k$ for some n.

We note that $\mathbb{A}^n_k$ is a quasi-projective variety, and that every reduced closed subscheme of $\mathbb{A}^n$ is quasi-projective variety. Moreover, every projective variety is quasi-projective, as they $Z \to Z \hookrightarrow X$ is a locally closed immersion. We therefore end this discussion, by remarking that most varieties one comes across in nature are quasi-projective, and the construction of a variety that is not quasi-projective was a research area of great interest until Nagata provided such an example in 1950's.

3.7 Proper Z-Schemes

A compact topological space X is generally one where every open cover has a finite subcover. Throughout this text, we have called this property quasi-compactness, largely because this definition is not restrictive enough. Indeed, the analogue of the complex vector space $\mathbb{C}^n$ in algebraic geometry is $\mathbb{A}^n_{\mathbb{C}}$. Under the usual definition of compactness, $\mathbb{A}^n_{\mathbb{C}}$ is compact as every affine scheme is quasi-compact, but $\mathbb{C}^n$ is most definitely not. Given this, instead we follow the lead of our separatedness condition, and define our analogue of compactness relative to a base scheme.

In topology, a proper map $f : X \to Y$ is one in which the inverse image of a compact set is compact. This is the correct way of thinking of 'relative compactness' in the setting of topological spaces. However, in this sense, when working with schemes, almost every morphism is proper. Indeed, if we deal with Noetherian schemes, which are Noetherian topological spaces, every subset of a scheme is compact, so every map between Noetherian topological spaces is proper in the topological sense. This is not very a helpful condition, so, following our treatment of separatedness, we analyze an equivalent definition of proper maps.

Recall that if X and Y are locally compact Hausdorff spaces, then $f : X \to Y$ being proper is equivalent to f being universally closed. That is, in topology, if $g : Z \to Y$ is another continuous map, then there exists a fibre product:

$$X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}$$

equipped with the subspace topology. The map f is then universally closed if f is closed, and the projection $X \times_Y Z \to Z$ is also closed for every topological space Z. It is easy to check that these two descriptions of properness are equivalent in the setting of locally compact, Hausdorff spaces.

In the setting of schemes, the definition of universally closed is the same:

Definition 3.7.1. Let $f : X \to Z$ be a closed morphism of schemes, i.e. f takes closed subsets to closed subsets[74]. Then f is **universally closed** if for every Z-scheme Y the projection $X \times_Z Y \to Y$ is also closed. In other words a closed morphism is universally closed if it closed under base change.

Now, we know what the analogue of Hausdorff is in the category of Z-schemes, so we need a good analogue of what it means for a Z-scheme to be locally compact. However, we have already encountered such an analogue, indeed if X if of finite type over Z, i.e. if $f : X \to Z$ is of finite type, then this morally feels like X being locally compact in the usual sense. This motivates our definition of proper morphisms and 'compactness' in the category of schemes:

Definition 3.7.2. Let $f : X \to Z$ be a morphism of schemes. Then f is a **proper morphism** if f separated, of finite type, and universally closed. We call any Z-scheme $f : X \to Y$ a **proper Z-scheme**, or **proper over Z** if f is proper.

So our usual analogues of compactness, and proper maps in algebraic geometry are proper morphisms and proper Z-schemes respectively We wish to show that proper morphisms are local on target, stable under base change, and closed under composition. It clearly suffices to prove the following:

Lemma 3.7.1. *Universally closed morphism are:*

a) *Local on target.*

b) *Stable under base change.*

c) *Closed under composition.*

Proof. Let $f : X \to Z$ be a universally closed morphism, and $g : Y \to Z$ be any morphism of schemes. Let $\{V_i = \operatorname{Spec} C_i\}$ be an affine open cover of Z, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$, $\{W_{ik} = \operatorname{Spec} B_{ik}\}$ be affine open covers of X and Y such that U_{ij} and W_{ik} map into V_i. We want to show that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is universally closed. First note that $f|_{f^{-1}(V_i)}$ is indeed a closed map, as if $S \subset f^{-1}(V_i)$ is a closed subset then $S = T \cap f^{-1}(V_i)$ for some closed $T \subset X$. We have that:

$$f|_{f^{-1}(V_i)}(S) = f(T) \cap V_i$$

so since f is closed, it follows that the restriction is too. Now note that $f^{-1}(V_i) \times_{V_i} Y \cong f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$. We already know that $\pi_Y : X \to_Z Y \to Y$ is a closed map, so it's restriction to the open set $f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)$ must now also be a closed map, hence $f|_{f^{-1}(V_i)}$ is again universally closed.

Now suppose that $f|_{f^{-1}(V_i)}$ is a universally closed map for all i. We first claim that f is closed. We have that $f(T) \cap V_i$ is closed for all i, hence:

$$Y \setminus f(T) = \bigcup_i V_i \setminus (f(T) \cap V_i)$$

which is an infinite union of open sets and thus open. It follows that if $f|_{f^{-1}(V_i)}$ is universally closed for all i, then $\pi_Y|_{f^{-1}(V_i) \times_{V_i} g^{-1}(V_i)}$ is closed for all i, so the same argument above shows that π_Y is closed, implying a).

To show b), we need to show that $\pi_Y : X \times_Z Y \to Y$ is universally closed. Let $h : Y' \to Y$ be a Y scheme, and note that:

$$\begin{aligned}(X \times_Z Y) \times_Y Y' &\cong X \times_Z (Y \times_Y Y') \\ &\cong X \times_Z Y'\end{aligned}$$

[74] Note that this does not mean that f is a closed embedding!

The map $\pi_{Y'} : X \times_Z Y' \to Y'$ is closed, and is equal to the map $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ composed with the above isomorphisms, hence $\pi_{Y'} : (X \times_Z Y) \times_Y Y' \to Y'$ is closed as well, so π_Y is also universally closed.

To show c), let $f : X \to Y$, and $g : Y \to Z$ be universally closed maps. We see that $g \circ f$ is a closed map, so we need only show that it is universally closed. Let Y' be a Z-scheme, then we need to show that $\pi_{Y'} : X \times_Z Y' \to Y'$ is a closed map. We have the following commutative diagram:

$$\begin{tikzcd}
X \times_Z Y' \arrow[r, "f\times\mathrm{Id}"] \arrow[d, "\pi_X"] & Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where $f \times \mathrm{Id}$ comes from the following diagram:

$$\begin{tikzcd}
X \times_Z Y' \arrow[dr, "f\times\mathrm{Id}"] \arrow[drr, bend left, "\pi_{Y'}"] \arrow[ddr, bend right, "f\circ\pi_X"'] & & \\
& Y \times_Z Y' \arrow[r, "\pi_{Y'}"] \arrow[d, "\pi_Y"] & Y' \arrow[d, "h"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The right and outer squares are cartesian, so it follows as that the left square is cartesian as well. We have that f is universally closed, so $f \times \mathrm{Id}$ must be a closed map. It follows that $\pi_{Y'} = \pi_{Y'} \circ f \times \mathrm{Id}$ is the composition of closed maps and is thus closed. Therefore we have that $g \circ f$ is universally closed as desired. □

We now have the following corollary:

Corollary 3.7.1. *Proper morphisms are local on target, stable under base change, and closed under composition.*

Proof. This follows because separated maps, universally closed maps, and maps of finite type are all local on target, stable under base change, and closed under composition. □

Note that if a scheme is proper over a field, i.e. $X \to \operatorname{Spec} k$ is proper for a field k, then X is in a sense 'compact'. We now demonstrate that $\mathbb{A}^n_{\mathbb{C}}$ is not proper over $\mathbb{C}$:

Example 3.7.1. Clearly the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is closed, separated, and of finite type. We need to show that this morphism is not universally closed. Consider the scheme morphism $\pi : \mathbb{A}^n_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. This morphism comes from (up to isomorphism) the ring homomorphism $\mathbb{C}[y] \hookrightarrow \mathbb{C}[x_1, \dots, x_{n+1}]$. Consider the closed subset $\mathbb{V}(x_1 \cdots x_{n+1} - 1)$, we claim that:

$$\mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \cong \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

which is an integral domain. Indeed, note that there is a map:

$$\mathbb{C}[x_1, \dots, x_{n+1}] \longrightarrow \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

given by $x_i \mapsto x_i$ for $i \leq n$, and $x_{n+1} \mapsto 1/(x_1 \dots x_n)$. This map clearly factors through the quotient hence we have well defined map:

$$\phi : \mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle \to \mathbb{C}[x_1, \dots, x_n]_{x_1 \cdots x_n}$$

Now note that there is map:

$$\mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{C}[x_1, \dots, x_{n+1}] / \langle x_1 \cdots x_{n+1} - 1 \rangle$$

given by the composition of the inclusion map with the map with the projection map. We have that $[x_1 \cdots x_n]$ is invertible in $\mathbb{C}[x_1, \ldots, x_{n+1}]/\langle x_1 \cdots x_{n+1} - 1\rangle$ so the there is a well defined map:

$$\psi : \mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n} \longrightarrow \mathbb{C}[x_1, \ldots, x_{n+1}]/\langle x_1 \cdots x_{n+1} - 1\rangle$$

These are then clearly inverses of one another, so we have that the two rings are isomorphic. As the localization of an integral domain is an integral domain, it follows that $x_1 \cdots x_{n+1} - 1$ is irreducible.

The induced projection map then takes $\langle x_1 \cdots x_{n+1} - 1\rangle \subset \mathbb{V}(x_1 \cdots x_{n+1} - 1)$ to the zero ideal, which is the generic point in $\mathbb{A}^1_{\mathbb{C}}$. It follows that $\pi(\mathbb{V}(x_1 \cdots x_{n_1} - 1))$ cannot be closed, so the map $\mathbb{A}^n_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not universally closed.

We now see that all closed embedding's are proper:

Example 3.7.2. Let $f : X \to Z$ be a closed embedding, then f is separated, of finite type and closed. We need only show that f is universally closed, but closed embeddings are stable under base change, so $\pi : X \times_Z Y \to Y$ is a closed embedding as well. It follows that π must be a closed map, hence f universally closed, and thus proper.

For our first nontrivial example we show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, however, we need to be able to characterize the scheme-theoretic fibre of a scheme morphism. In other words, for $f : X \to Y$, we would like to know how to make sense of the preimage of $f^{-1}(y)$ for $y \in Y$ as a scheme.

First note that in the category of topological spaces, if $f : X \to Y$ is continuous map, then we can naturally identify $f^{-1}(p)$ with $\{y\} \times_Y X$, where $\{y\} \hookrightarrow Y$ is the inclusion map. In the category of schemes, we can naturally equip $\{y\}$ with a scheme structure given by $\operatorname{Spec} k_y$, where k_y is the residue field. We define a scheme morphism $g : \operatorname{Spec} k_y \to Y$ by first defining the topological map to be $\eta = \langle 0\rangle \mapsto y$, and the sheaf morphism $g^\sharp : \mathscr{O}_Y \to g_* \mathscr{O}_{\operatorname{Spec} k_y}$ by first noting that if $y \in U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(g^{-1}(U)) = k_y$, and if $y \notin U$ then $\mathscr{O}_{\operatorname{Spec} k_y}(\emptyset) = \{0\}$. We thus define $g^\sharp$ on open sets by:

$$g^\sharp_U(s) = \begin{cases} 0 \in \{0\} & \text{if } y \notin U \\ [s_y] \in k_y & \text{if } y \in U \end{cases}$$

which trivially commutes with restriction maps[75]. This then motivates our following definition:

Definition 3.7.3. Let $f : X \to Y$ be a scheme, then the for any $y \in Y$, the **scheme theoretic fibre** over y, denoted X_y is given by $\operatorname{Spec} k_y \times_Y X$.

Note that this naturally has the structure of a scheme, so it is mainly important to show that there is a natural identification with elements in the fibre over y and elements in $\operatorname{Spec} k_y \times_Y X$.

Lemma 3.7.2. *Let $f : X \to Y$ be a morphism of schemes. Then there is a natural identification between $\operatorname{Spec} k_y \times_Y X$ with the fibre $f^{-1}(y)$.*

Proof. We have the following diagram:

$$\begin{tikzcd} X_y \arrow[r, "\pi_X"] \arrow[d, "\pi_y"] & X \arrow[d, "f"] \\ \operatorname{Spec} k_y \arrow[r, "g"] & Y \end{tikzcd}$$

We first want to show that the image of π_X is the fibre $f^{-1}(y)$, and then demonstrate that π_X is a homeomorphism onto it's image. Note that if suffices to check this on an affine open cover of X_y, so let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover of Y, and $\{U_{ij} = \operatorname{Spec} A_{ij}\}$ an open cover of X such that $f(U_{ij}) \subset V_i$ for all i and j. It follows that:

$$X_y = \bigcup_{ij} \operatorname{Spec} k_y \times_{V_i} U_{ij}$$

[75]Note that by Corollary 1.3.1 we have that g is a monomorphism, as the stalk map $g_\eta : (\mathscr{O}_Y)_{g(\eta)} \to (\mathscr{O}_{\operatorname{Spec} k_y})_\eta$ is always the projection $\mathscr{O}_{Y,y} \to k_y$

We will show that $\pi_X|_{\operatorname{Spec} k_y \times_{V_i} U_{ij}} = \pi_{U_{ij}}$ is a homeomorphism onto $U_{ij} \cap f^{-1}(y) = f|_{U_{ij}}^{-1}(y)$. Moreover, supposing that $y \in V_i$ as otherwise $\operatorname{Spec} k_y \times_{V_i} U_{ij}$ is clearly empty, we can write y as a prime ideal $\mathfrak{p} \subset B_i$, so $k_y = k_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$. Suppressing the i and j notation for clarity, we have the following diagram:

$$\begin{tikzcd}
\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \arrow[r, "\pi_U"] \arrow[d, "\pi_y"] & \operatorname{Spec} A \arrow[d, "f|_U"] \\
\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \arrow[r, "g"] & \operatorname{Spec} B
\end{tikzcd}$$

where it is understood that g is now the morphism $\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \to \operatorname{Spec} B$ induced by the localization map followed by the projection to the residue field. Now note that:

$$\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \cong \operatorname{Spec}(B_{\mathfrak{p}} \otimes_B A)/\left\langle \mathfrak{m}_{\mathfrak{p}} \otimes 1 \right\rangle$$

and that A is a B algebra via the ring homomorphism $\phi : B \to A$ inducing $f|_U$. We define $A_{\mathfrak{p}}$ to be $\phi(B \smallsetminus \mathfrak{p})^{-1}A$, and claim that:

$$B_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}$$

We have a map:

$$\beta : B_{\mathfrak{p}} \otimes_B A \longrightarrow A_{\mathfrak{p}}$$

given on simple tensors by $b/s \otimes a \mapsto \phi(b) \cdot a/\phi(s)$. Moreover, we have a ring homomorphism $A \to B_{\mathfrak{p}} \otimes A$ given by $a \mapsto 1 \otimes a$. For all $\phi(s) \in \phi(B \smallsetminus \mathfrak{p})$, we have $1 \otimes \phi(s)$ is invertible, as $1 \otimes \phi(s) = s/1 \otimes 1$, which has inverse given by $1/s \otimes 1$. It follows that there is ring homomorphism:

$$\begin{aligned} \alpha : A_{\mathfrak{p}} &\to B_{\mathfrak{p}} \otimes A \\ a/\phi(s) &\longmapsto 1/s \otimes a \end{aligned}$$

These maps are clearly inverses of each other so we have that:

$$B_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}$$

Now note that under the map β we have that:

$$\beta(\langle \mathfrak{m}_{\mathfrak{p}} \otimes 1 \rangle) = \{a/\phi(s) \in A_{\mathfrak{p}} : a \in \langle \phi(\mathfrak{p}) \rangle\} = \langle \phi(\mathfrak{p})/1 \rangle \subset A_{\mathfrak{p}}$$

so it follows that we have the following isomorphism:

$$\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes A \cong \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$$

The projection π_U is now induced by the ring homomorphism $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle$, and the projection π_y is given by the composition $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \otimes_B A \cong A_{\mathfrak{p}}/\left\langle \phi(q)/1 \right\rangle$. In the category of commutative rings, we thus have the following commutative diagram:

$$\begin{tikzcd}
A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p})/1 \right\rangle & A \arrow[l, "\pi \circ \pi_l"'] \\
B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} \arrow[u, "\imath"] & B \arrow[l, "\nu"] \arrow[u, "\phi"']
\end{tikzcd}$$

where π_l is the localization map, and π is the quotient map. Let $\mathfrak{p} \in \operatorname{Spec} B$, then:

$$f|_U^{-1}(\mathfrak{p}) = \left\{ \mathfrak{q} \in \operatorname{Spec} A : \phi^{-1}(\mathfrak{q}) = \mathfrak{p} \right\}$$

However, clearly from the commutativity of the first diagram, we have that $\pi_U(\operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle) \subset f^{-1}|_U(\mathfrak{p})$, so we need to define an inverse map $\eta : f|_U^{-1}(\mathfrak{p}) \to \operatorname{Spec} A_{\mathfrak{p}}/\left\langle \phi(\mathfrak{p}) \right\rangle$.

Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, implying that $\phi(\mathfrak{p}) \subset \mathfrak{q}$. We first show that:

$$\langle \pi_l(\mathfrak{q}) \rangle = \{a/s \in A_\mathfrak{p} : a \in \mathfrak{q}\}$$

is a prime ideal. This is clearly an ideal by construction, so suppose that $a/s, c/t \in A_\mathfrak{q}$ such that $ac/st \in \langle \pi_l(\mathfrak{q}) \rangle$. It follows that $ac/st = d/r$ such that $d \in \mathfrak{q}$, hence there exists some $u \in \phi(B \smallsetminus \mathfrak{p})$ such that:

$$u(acr - dst) = 0$$

Note that $u, r, s, t \in \phi(B \smallsetminus \mathfrak{p})$, hence $u, r, s, t \notin \phi(\mathfrak{p}) \subset \mathfrak{q}$. We thus have that $acru \in \mathfrak{q}$, so $ac \in \mathfrak{q}$, so either a or c are in $\mathfrak{q}$. Note that $\langle \pi_l(\mathfrak{q}) \rangle$ is not all of $A_\mathfrak{q}$, as otherwise we have that $\mathfrak{q} \cap \phi(B \smallsetminus \mathfrak{p}) \neq \emptyset$, which would imply that $\phi^{-1}(\mathfrak{q}) \cap \phi^{-1}(\phi(B \smallsetminus \mathfrak{p})) \neq \emptyset$, so $\mathfrak{p} \cap B \smallsetminus \mathfrak{p} \neq \emptyset$ which is a clear contradiction.

Let $\psi = \pi \circ \pi_l$; since $\langle \pi_l(\mathfrak{q}) \rangle$ clearly contains $\langle \phi(\mathfrak{p})/1 \rangle$, we have that $\pi(\langle \pi_l(\mathfrak{q}) \rangle)$ is a prime ideal of $A_\mathfrak{p}$. We thus define $\eta(\mathfrak{q}) \in \operatorname{Spec} A_\mathfrak{p} / \langle \phi(\mathfrak{p})/1 \rangle$ by:

$$\eta(\mathfrak{q}) = \{[a/s] : a \in \mathfrak{q}\}$$

which clearly then satisfies $\eta(\mathfrak{q}) = \langle \psi(\mathfrak{q}) \rangle = \pi(\langle \pi_l(\mathfrak{q}) \rangle)$. Let $U_{[a/1]}$ be a distinguished open of $\operatorname{Spec} A_\mathfrak{p} / \langle \phi(\mathfrak{p})/1 \rangle$, then we see that:

$$\begin{aligned} \eta^{-1}(U_{[a/1]}) =& \{\mathfrak{q} \in f|_U^{-1}(\mathfrak{p}) : [a/1] \notin \langle \psi(\mathfrak{q}) \rangle\} \\ =& \{\mathfrak{q} \in f_U^{-1}(\mathfrak{p}) : a \notin \mathfrak{q}\} \\ =& U_a \cap f|_U^{-1}(\mathfrak{p}) \end{aligned}$$

which is open in $f_U^{-1}(\mathfrak{p})$. Since $U_a \cap f|_U^{-1}(\mathfrak{p})$ form a basis we have that η is indeed continuous.

We see that $\psi^{-1}(\eta(\mathfrak{q})) = \mathfrak{q}$, so $\pi_U \circ \eta = \mathrm{Id}$. Now let $\mathfrak{q} \in \operatorname{Spec} A_\mathfrak{p} / \langle \phi(\mathfrak{q}) \rangle$, then:

$$\eta(\psi^{-1}(\mathfrak{q})) = \{[a/s] : a \in \psi^{-1}(\mathfrak{q})\}$$

Suppose that $[a/s] \in \mathfrak{q}$, then $[a/1] \in \mathfrak{q}$, and $a \in \psi^{-1}(\mathfrak{q})$. Now suppose that $[a/s]$ satisfies $a \in \psi^{-1}(\mathfrak{q})$, then $[a/1] \in \mathfrak{q}$ so $[a/s] \in \mathfrak{q}$ as well. It follows that $\eta(\psi^{-1}(\mathfrak{q})) = \mathfrak{q}$ hence $\eta \circ \pi_U = \mathrm{Id}$, and π_U is a homeomorphism onto $f|_U^{-1}(\mathfrak{p})$.

Since the above argument holds for all affine opens $\operatorname{Spec} k_y \times_{V_i} U_{ij}$, it follows that $\pi_X : X_s \to X$ is a homeomorphism onto $f^{-1}(y)$ implying the claim.

□

We can now show that $\mathbb{P}^n_A$ is proper.

Example 3.7.3. Note that $\mathbb{P}^n_A \cong \mathbb{P}^n_\mathbb{Z} \times_\mathbb{Z} \operatorname{Spec} A$, so if $\mathbb{P}^n_\mathbb{Z} \to \operatorname{Spec} \mathbb{Z}$ is proper, we have that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, as proper morphisms are stable under base change.

We have already shown that $\mathbb{P}^n_\mathbb{Z}$ is separated, and it is clearly of finite type, so we need only show that $f : \mathbb{P}^n_\mathbb{Z} \to \operatorname{Spec} \mathbb{Z}$ is universally closed. Let $g : Y \to \operatorname{Spec} \mathbb{Z}$ be any $\mathbb{Z}$ scheme, then we want to show that $\mathbb{P}^n_\mathbb{Z} \times_\mathbb{Z} Y \to Y$ is closed. As we have shown, being closed is local on target, so it suffices to show that for any open affine $U = \operatorname{Spec} A \subset Y$ that $\pi : \mathbb{P}^n_\mathbb{Z} \times_\mathbb{Z} \operatorname{Spec} A \cong \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map.

Let $Z = \mathbb{V}(I) \subset \mathbb{P}^n_A$ where $I = \langle g_1, g_2, \dots \rangle$ is a homogeneous ideal. We need to determine the primes $\mathfrak{p} \in \operatorname{Spec} A$ which lie in $\pi(Z)$. In other words, by the preceding lemma, we want to know for which $\mathfrak{p}$, the fiber $\pi^{-1}(\mathfrak{p}) \cap Z \cong \operatorname{Spec} k_\mathfrak{p} \times_A Z = Z_\mathfrak{p}$ is non empty. We have that $k_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p})$ which is an A algebra, therefore, $Z_\mathfrak{p} \subset (\mathbb{P}^n_A)_\mathfrak{p}$, and $(\mathbb{P}^n_A)_\mathfrak{p} = \mathbb{P}^n_A \times_A \operatorname{Spec} k_\mathfrak{p} \cong \mathbb{P}^n_{k_\mathfrak{p}}$. It follows that $Z_\mathfrak{p}$ is a closed subset of $\mathbb{P}^n_{k_\mathfrak{p}}$, and that locally

$$Z_\mathfrak{p} \cap \operatorname{Spec} k_\mathfrak{p} \times_A U_{x_i} = \operatorname{Spec} k_\mathfrak{p} \otimes_A (A[x_0, \dots, x_n]_{x_i})/(I_{x_0})_0 \cong \operatorname{Spec}(k_\mathfrak{p}[x_0, \dots, x_n]_{x_i})_0 / J$$

where J is the ideal generated by the image of $(I_{x_0})_0$ under the map $(A[x_0, \dots, x_n]_{x_i})_0 \to (k_\mathfrak{p}[x_0, \dots, x_n]_{x_i})_0$. Hence, $Z_\mathfrak{p} = \mathbb{V}(I_\mathfrak{p})$, where $I_\mathfrak{p} = \langle [g_1], [g_2], \dots \rangle$, and $[g_i]$ is the image of the map:

$$A[x_0, \dots, x_n] \longrightarrow k_\mathfrak{p}[x_0, \dots, x_n]$$

induced by the projection $\pi : A \to A/\mathfrak{p}$, followed by the inclusion $A/\mathfrak{p} \hookrightarrow \operatorname{Frac}(A/\mathfrak{p})$. It follows that $Z_\mathfrak{p}$ is non empty if and only if $\mathbb{V}(I_\mathfrak{p}) \neq \mathbb{V}(\langle x_0, \dots, x_n\rangle)$, hence $\sqrt{I_\mathfrak{p}} \not\supset \langle x_0, \dots, x_n\rangle$. Equivalently for all $n > 0$, we have that:

$$\langle x_0, \dots, x_n\rangle^n \not\subset \langle [g_1], [g_2], \dots \rangle$$

If $S = k_\mathfrak{p}[x_0, \dots, x_n]$, then non containment is equivalent to the map:

$$\begin{aligned} \bigoplus_i (A[x_0, \dots, x_n])_{d - \deg g_i} &\longrightarrow S_d \\ f_i &\longmapsto [f_i g_i] \end{aligned}$$

not begin surjective for all d. Let $d_0 = \dim_{k_\mathfrak{p}} S_d$[76], then this gives us a matrix with coefficients in A, d_0 rows, and potentially infinite columns. All of the $d_0 \times d_0$ minors of this matrix must have determinant zero in $k_\mathfrak{p}$, so the determinants lie in $\mathfrak{p}$, and therefore the ideal generated by these determinants, $\tilde{J}$, is contained in $\mathfrak{p}$. It follows that the fibre $Z_\mathfrak{p} = \pi^{-1}(\mathfrak{p}) \cap Z$ is non empty if and only $\mathfrak{p}$ lies in $\mathbb{V}(\tilde{J})$.

Now if $\mathfrak{p} \in \pi(Z)$, then $\pi^{-1}(\mathfrak{p}) \subset Z$, hence $Z_\mathfrak{p}$ is nonempty so $\mathfrak{p} \in \mathbb{V}(\tilde{J})$, and if $\mathfrak{p} \in \mathbb{V}(\tilde{J})$ then the fibre $Z_\mathfrak{p}$ is non empty, so $\mathfrak{p} \in \pi(Z)$. Therefore, $\pi(Z) = \mathbb{V}(\tilde{J})$, hence π is closed map, and $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper as desired.

We have the following corollary:

Corollary 3.7.2. *Let $Z \subset \mathbb{P}^n_A$ be a closed subscheme, then Z is proper over* $\operatorname{Spec} A$*.*

Proof. The map $Z \to \operatorname{Spec} A$ is given by the closed embedding $\imath : Z \to \mathbb{P}^n_A$, followed by the canonical morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$ from Example 2.3.1, then by Example 3.7.3, we have that this map is proper. By Example 3.7.2, closed embeddings are proper, and by Corollary 3.7.1 proper morphisms are closed under composition. It follows that $Z \to \operatorname{Spec} A$ is proper. ☐

Recall that if $f : X \to Y$ is a continuous map between Hausdorff topological with X compact, then f is proper. We wish to prove the algebraic geometry analogue of this result; i.e. if X and Y are S-schemes, with X proper over S and Y separated over S, then any morphism $f : X \to Y$ is proper as well. This will follow from the following lemma:

Lemma 3.7.3. *Let X and X' be Y-schemes, and Y a separated Z-scheme. Then the map $X \times_Y X' \to X \times_Z X'$ is a closed embedding.*

Proof. This follows from Theorem 2.3.1 as the following diagram is Cartesian:

$$\begin{tikzcd} X \times_Y X' \arrow[r] \arrow[d] & X \times_Z X' \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

It follows that the morphism $X \times_Y X' \to X \times_Z X'$ is the base change of $\Delta_Y : Y \to Y \times_Z Y$, which is a closed embedding. Since closed embeddings are stable under base change the claim follows. ☐

We can now prove the desired result:

Theorem 3.7.1. *Let X and Y be Z-schemes, with Y separated over Z, and $f : X \to Y$ a Z-scheme morphism. Then the following hold:*

a) If X is universally closed over Z, then f is universally closed.

b) If X is proper over Z, then f is proper.

[76]Note this that this is finite, as $\dim_{k_\mathfrak{p}}$ is equal to the partitions of d.

Proof. Let g and h be the morphisms which make X and Y Z-Schemes respectively. Let α be the unique morphism making the following the diagram commute:

$$\begin{tikzcd} X \arrow[drr, bend left, "f"] \arrow[dr, "\alpha"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\ & X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "h"] \\ & X \arrow[r, "g"] & Z \end{tikzcd}$$

It follows that f factors as:

$$X \xrightarrow{\alpha} X \times_Z Y \xrightarrow{\pi_Y} Y$$

We see that π_Y is the base change of a universally closed morphism, and is thus universally closed. It thus suffices to show that α is a universally closed. With $X' = Y$, we claim that, up to isomorphism, α is the top horizontal map making the diagram in Lemma 3.7.3 commute. Indeed, if $X' = Y$ then $X \times_Y Y$ is uniquely isomorphic to X, with projections given by $\mathrm{Id} : X \to X$ and $f : X \to Y$. X then fits into the following Cartesian diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f \times \mathrm{Id}"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

By our work in Theorem 2.3.1, the horizontal map is then precisely the one defining α, so by Lemma 3.7.3 α is a closed embedding. Since closed embedding are universally closed by Example 3.7.2, we have proven a).

Now suppose that X is proper over Z, then π_Y is the base change of a proper map and is thus proper. In particular α is a closed embedding which is proper by Example 3.7.2, so the same argument guarantees that f is a proper map implying b). □

Example 3.7.4. Let X be a projective variety, then X is proper by Corollary 3.7.2 so any k morphism $X \to Y$ with Y separated over k is proper. In particular, every k morphism from X to a variety Y is proper.

3.8 Affine Morphisms

In this section we introduce affine morphisms, though it will more fruitful to study special types of affine morphisms as in the next section.

Definition 3.8.1. Let $f : X \to Y$ be a morphism of schemes, then f is affine if for every open affine $V \subset Y$, we have that $f^{-1}(V)$ is also affine.

Example 3.8.1. Any closed embedding is an affine morphism.

We prove the following structure result regarding schemes:

Lemma 3.8.1. *Let X be a scheme, and $\mathscr{O}_X(X) = A$. Suppose that $g_1, \ldots, g_n \in A$ generate the unit ideal, and that X_{g_i} is affine for each i, then $X \cong \operatorname{Spec} A$.*

Proof. Recall that:

$$X_{g_i} = \{x \in X : (g_i)_x \notin \mathfrak{m}_x\} \tag{3.8.1}$$

is an open set in X. Moreover, since the g_i generate the unit ideal in A, we have that for every affine open $U \subset X$, $g_i|_U$ generate the unit ideal of $\mathscr{O}_X(U)$. It follows that the distinguished open $U_{g_i|_U} \subset U$ cover U, however by our work in Proposition 2.1.2 we know that:

$$U_{g_i|_U} = X_{g_i} \cap U$$

It follows that X_{g_i} cover X as if $x \in X$, then there is an open affine U containing x, and thus an i such that $x \in U_{g_i|_U}$, hence $x \in X_{g_i}$ and $\bigcup_i X_{g_i} \subset X$.

Set $X_{g_i} = \operatorname{Spec} A_i$, and $X_{ij} = X_{g_i} \cap X_{g_j}$. Since each X_{g_i} is affine, by our work in Proposition 2.1.2, we have that each X_{ij} is a distinguished open in both $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$, thus:

$$\operatorname{Spec}(A_i)_{g_j|_{X_i}} \cong X_{ij} \cong \operatorname{Spec}(A_j)_{g_i|_{X_j}}$$

The rings $\mathscr{O}_X(X_{g_j})$ and $\mathscr{O}_X(X_{ij})$ have canonical $\mathscr{O}_X(X)$ module structures given by by the restriction maps $\theta^X_{X_{g_j}}$ and $\theta^X_{X_{ij}}$. There is a natural map

$$\begin{aligned} \alpha : \mathscr{O}_X(X) &\longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j}) \\ s &\longmapsto (s|_{X_{g_j}}) \end{aligned}$$

where by $(s|_{X_{g_j}})$ we mean $(s|_{X_{g_0}}, \ldots, s|_{X_{g_n}})$. This map is an injection as the X_{g_j} cover X. We define another map:

$$\begin{aligned} \beta : \bigoplus_j \mathscr{O}_X(X_{g_j}) &\longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj}) \\ (s_j) &\longmapsto (s_{kj}) \end{aligned}$$

where:

$$(s_{kj}) = \left(s_k|_{X_{kj}} - s_j|_{X_{kj}}\right)$$

Note that $\beta \circ \alpha = 0$, as:

$$\beta((s|_{X_{g_j}})) = ((s|_{X_{g_k}})|_{X_kj} - (s|_{X_{g_j}})|_{X_{kj}}) = (s|_{X_{kj}} - s|_{X_{kj}}) = 0$$

Similarly, if $\alpha((s_j)) = 0$, then we have sections $s_j \in \mathscr{O}_X(X_{g_j})$ such that for all k and j:

$$s_j|_{X_{kj}} = s_k|_{X_{kj}}$$

It follows by the sheaf axioms, that there exists an $s \in \mathscr{O}_X(X)$ such that $s|_{X_{g_j}} = s_j$. We have thus shown that $\ker \beta = \operatorname{im} \alpha$, and so we have the following sequence of $\mathscr{O}_X(X)$ modules:

$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \bigoplus_j \mathscr{O}_X(X_{g_j}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj})$$

hence the following exact sequence of A modules:

$$0 \longrightarrow A \longrightarrow \bigoplus_j A_j \longrightarrow \bigoplus_{k<j} A_{kj}$$

We can localize[77] the sequence at g_i, to obtain the exact sequence:

$$0 \longrightarrow A_{g_i} \longrightarrow \bigoplus_j (A_j)_{g_i|_{X_{g_j}}} \longrightarrow \bigoplus_{k<j} (A_{kj})_{g_i|_{X_{kj}}}$$

[77] We take this on a faith for the moment. A precise proof is given in greater generality in Lemma 5.3.1.

Note that first morphism, which we denote α_i, is induced by the unique ones which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\theta^X_{X_{g_j}}"] \arrow[d, "\pi_{g_i}"] & A_j \arrow[d, "\theta^{X_{g_j}}_{X_{ij}}"] \\ A_{g_i} \arrow[r, "(\alpha_i)_j"] & (A_j)_{g_i|_{X_j}} \end{tikzcd}$$

where $(\alpha_i)_j$ is the jth component of the map α_i. Moreover, the second morphism is given by:

$$\beta_i : \bigoplus_j \mathscr{O}_X(X_{ji}) \longrightarrow \bigoplus_{k<j} \mathscr{O}_X(X_{kj} \cap X_i)$$
$$(s_j) \longmapsto (s_k|_{X_{ki} \cap X_j} - s_j|_{X_{ji} \cap X_k})$$

Finally, note that $(A_i)_{g_i|_{X_i}}$ is A_i as $g_i|_{X_i}$ is invertible in A_i, so the map $(\alpha_i)_i$ is given by the localization of the restriction map $\theta^X_{X_i}$. We wish to show that $(\alpha_i)_i$ is an isomorphism.

Let $a/g_i^k \in A_{g_i}$ satisfy $(\alpha_i)_i(a/g_i^k) = 0$, then, since g_i maps to an invertible element, we have that $a/1$ also maps to zero. We claim that $\alpha_i(a/1) = 0$; indeed, we have that $(\alpha_i)_i(a/1) = 0$ by assumption, and that:

$$\begin{aligned} (\alpha_i)_j(a/1) =& (\alpha_i)_j(\pi_{g_i}(a)) \\ =& (a|_{X_{f_j}})|_{X_{ij}} \\ =& a|_{X_{ij}} \\ =& (a|_{X_{f_i}})|_{X_{ij}} \end{aligned}$$

Since $\theta^{X_{g_i}}_{X_{ii}} = \theta^{X_{g_i}}_{X_{g_i}} = \mathrm{Id}$, it follows that:

$$a|_{X_{f_i}} = (\alpha_i)_i(\pi_{g_i}(a)) = (\alpha_i)_i(a/1) = 0$$

hence $(\alpha_i)_j(a/1) = 0$ for all $j \neq i$ as well. By exactness, have that $a/1 = 0$, hence $(\alpha_i)_i$ is injective.

Now let $s \in A_i$; then $s|_{X_{ij}} \in (A_j)|_{f_i|_{X_j}}$ for all j, hence we have an element $(s_j) \in \bigoplus_j (A_j)|_{f_i|_{X_j}}$. It follows that:

$$\beta_i((s_j)) = (s_k|_{X_{ki} \cap X_j} - s_j|_{X_{ji} \cap X_k})$$

but:

$$s_k|_{X_{ki} \cap X_j} = s|_{X_{ik}}|_{X_{ij} \cap X_j} = s|_{X_{ij} \cap X_j}$$

and similarly for j, hence $\beta_i((s_j)) = 0$. It follows by exactness that there exists some $a/g_i^k \in A_{g_i}$ such that $\alpha_i(a/g_i^k) = (s_j)$, hence $(\alpha_i)_i$ is surjective. Therefore, we have $A_{g_i} \cong A_i$, and so $X_{g_i} \cong \operatorname{Spec} A_{g_i}$.

By Proposition 2.1.2, there is a natural map $f' : X \to \operatorname{Spec} A$ induced by the identity map $A \to \mathscr{O}_X(X)$. Furthermore, since the g_i generate the unit ideal in A, we know that U_{g_i} cover $\operatorname{Spec} A$. The morphism:

$$f'|_{X_{g_i}} : X_{g_i} \longrightarrow U_{g_i}$$

is the one induced by the ring homomorphism:

$$A_{g_i} \longrightarrow \mathscr{O}_X(X_{g_i})$$
$$a/g_i^k \longmapsto a|_{X_{g_i}} \cdot (g_i|_{X_{g_i}})^{-k}$$

however this is precisely $(\alpha_i)_i$, which we just showed was an isomorphism. Since f' restricts to an isomorphism on the inverse image of an open cover of $\operatorname{Spec} A$, we have that $X \cong \operatorname{Spec} A$ $\square$

Proposition 3.8.1. *Affine morphisms are local on target.*

Proof. Suppose that $f : X \to Y$ is an affine morphism, and let $V \subset Y$ be an affine open. We wish to show that the morphism $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as well. Well, let $W \subset V$ be an affine open, then, in particular, W is an affine open in Y, and $(f|_{f^{-1}(V)})^{-1}(W) = f^{-1}(W)$ which is affine by assumption. It follows that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an affine morphism as desired.

Let $f : X \to Y$ be a morphism, and let $\{V_i = \operatorname{Spec} B_i\}$ be an affine open cover such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism. By assumption, each $f^{-1}(V_i)$ is affine so set $f^{-1}(V_i) = \operatorname{Spec} A_i$, and let $V = \operatorname{Spec} B \subset Y$ be an arbitrary open affine of Y. We have that:

$$V = \bigcup_i V_i \cap V$$

By Lemma 2.1.1, each $V_i \cap V$ can be covered by open affines:

$$V_i \cap V = \bigcup_j U_{ij}$$

where U_{ij} is a distinguished open affine in V_i and V. Hence:

$$V = \bigcup_{ij} U_{ij}$$

and each U_{ij} satisfies:

$$(f|_{f^{-1}(V)})^{-1}(U_{ij}) = f^{-1}(U_{ij}) = (f|_{f^{-1}(V_i)})^{-1}(U_{ij})$$

But $f|_{f^{-1}(V_i)} : \operatorname{Spec} A_i \to \operatorname{Spec} B_i$ is a morphism of affine schemes, and U_{ij} is a distinguished open, hence $(f|_{f^{-1}(V_i)})^{-1}(U_{ij})$ is a distinguished open of $\operatorname{Spec} A_i$ and thus an affine open of $f^{-1}(V)$. It follows that $V = \operatorname{Spec} B$ admits a cover of distinguished opens U_{ij} such that $f^{-1}(U_{ij}) \subset f^{-1}(V)$ is an affine open.

We have thus reduced the result to the following problem: if $f : X \to \operatorname{Spec} B$ is a morphism of schemes such that there is a cover of $\operatorname{Spec} B$ by distinguished opens $\{U_{b_i}\}_{i=0}^n$ with $f^{-1}(U_{b_i})$ affine, then X is an affine scheme. Let $\phi : B \to \mathscr{O}_X(X)$ be the unique ring homomorphism inducing f; by our work in Proposition 2.1.2, we know that $f^{-1}(U_{b_i}) = X_{\phi(b_i)}$. Since b_i generate the unit ideal in in B, we have that $\phi(b_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. Therefore, by Lemma 3.8.1 we have that X is affine, hence if $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is an affine morphism, then f is an affine morphism. □

Corollary 3.8.1. *Morphisms between affine schemes are affine. In particular, affine morphisms are local on target.*

Proof. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes. Let $V \subset \operatorname{Spec} B$ be an open affine scheme, then we would like to show that $f^{-1}(V)$ is an affine scheme.

Set $V = \operatorname{Spec} C$, and set $X = f^{-1}(V)$. Then we have have morphism $g : X \to \operatorname{Spec} C$ given by $f|_{f^{-1}(V)}$. We can cover $\operatorname{Spec} C$ with distinguished opens U_{c_i} which are also distinguished opens of $\operatorname{Spec} B$, hence $g^{-1}(U_{c_i})$ are open affines, as f is a morphism of affine schemes. In particular, if $\phi : C \to \mathscr{O}_X(X)$ is the unique morphism inducing g, then $g^{-1}(U_{c_i}) = X_{\phi(c_i)}$. Since c_i generate the unit ideal in C, $\phi(c_i)$ generate the unit ideal in $\mathscr{O}_X(X)$. It follows by Lemma 3.8.1 that X is affine, hence morphisms between affine schemes are affine.

Now let $f : X \to Y$ be an affine morphism; then for any open affine cover $\{U_i\}$, we have that $f^{-1}(U_i)$ is open affine by definition. The restricted morphism is then a morphism of affine schemes, and thus an affine morphism by the discussion above. Conversely, if $f^{-1}(U_i)$ is an affine scheme, then $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a morphism of affine schemes, and thus an affine morphism. It follows by Proposition 3.8.1 that f is an affine morphism. □

We of course need to also check that affine morphisms are stable under base change, and that the composition of affine morphisms is affine:

Proposition 3.8.2.

a) Affine morphisms are stable under base change.

b) The composition of affine morphisms is again affine.

Proof. For a), let $f : X \to Z$ be an affine morphism, and $g : Y \to Z$ be any morphism. Let $\{V_i\}$ be an affine cover of Z, then $\{W_i = f^{-1}(V_i)\}$ is an affine cover for X, and we can obtain an affine open cover of $\{U_{ij}\}$ of Y such that $g(U_{ij}) \subset V_i$ for all j. We need only show that $\pi_Y^{-1}(U_{ij})$ is an affine scheme; indeed we claim that $\pi_Y^{-1}(U_{ij}) \cong W_i \times_{V_i} U_{ij}$, which is manifestly an affine scheme. For ease of notation, set $S = \pi_Y^{-1}(U_{ij})$, then $\pi_Y|_S(S) \subset U_{ij}$, and we have that:

$$f \circ \pi_X|_S(S) = g \circ \pi_Y|_S(S) \subset V_i$$

It follows that:

$$\pi_X|_S(S) \subset f^{-1}(V_i) = W_i$$

We thus have unique morphisms $\pi_{U_{ij}} : S \to U_{ij}$ and $\pi_{W_i} : S \to W_i$ such that $\imath_{U_{ij}} \circ \pi_{U_{ij}} = \pi_y|_S$ and $\imath_{W_i} \circ \pi_{W_i} = \pi_X|_S$. Moreover, these morphisms make the following diagram commute:

$$\begin{tikzcd}
S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"'] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\
W_i \arrow[r, "f|_{w_i}"] & V_i
\end{tikzcd}$$

Now suppose that we have morphisms $p_{W_i} : Q \to W_i$ and $p_{U_{ij}} : Q \to U_{ij}$ which make the relevant diagram commute. Then by composing with open embeddings, we obtain a unique morphism $\phi : Q \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\phi"] \arrow[drr, bend left, "\imath_{U_{ij}} \circ p_{U_{ij}}"] \arrow[ddr, bend right, "\imath_{W_i} \circ p_{W_i}"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

We first claim that $\phi(Q) \subset S$. Indeed, we have that

$$\pi_Y(\phi(Q)) = \imath_{U_{ij}} \circ p_{U_{ij}}(Q) \subset U_{ij}$$

hence:

$$\phi(Q) \subset \pi_Y^{-1}(U_{ij}) = S$$

Therefore, there exists a unique map $\psi : Q \to S$ such that $\imath_S \circ \psi = \phi$. We need to check that that ψ makes the relevant diagram commute. We see that:

$$\begin{aligned}
\imath_{U_{ij}} \circ (\pi_{U_{ij}} \circ \psi) =& \pi_Y|_S \circ \psi \\
=& \pi_Y \circ \imath_S \circ \psi \\
=& \pi_Y \circ \phi \\
=& \imath_{U_{ij}} \circ p_{U_{ij}}
\end{aligned}$$

and similarly that:

$$\imath_{W_i} \circ (\pi_{W_i} \circ \psi) = \imath_{W_i} \circ p_{W_i}$$

Since open embeddings are monomorphisms, it follows that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, "\psi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S \arrow[r, "\pi_{U_{ij}}"] \arrow[d, "\pi_{W_i}"] & U_{ij} \arrow[d, "g|_{U_{ij}}"] \\
& W_i \arrow[r, "f|_{W_i}"] & V_i
\end{tikzcd}$$

so S satisfies the universal property of $W_i \times_{V_i} U_{ij}$ and is thus affine. It follows by Corollary 3.8.1 that π_Y is an affine morphism, as desired.

For b), let $f : X \to Y$, and $g : Y \to Z$ be affine morphisms, then clearly we have that for any affine open $U \subset Z$, that $f^{-1}(g^{-1}(U))$ is affine; it follows that $g \circ f$ is an affine morphism implying the claim. $\square$

3.9 Finite and Integral Morphisms

In this section, we discuss finite and integral morphisms of schemes. Recall that a morphism of rings $\phi : B \to A$ is finite if it makes A a finitely generated B module. That is, there is a finite set $\{a_1, \ldots, a_n\}$ such that any a can be written as:

$$a = \sum_{i=1}^{n} \phi(b_i) a_i$$

for some $b_i \in B$. Often times the notation ϕ is suppressed and we write $b_i \cdot a_i$. Furthermore, a morphism $\phi : B \to A$, is integral if every element of A is integral over B. That is, every $a \in A$ is the root of some monic polynomial in $\phi(B)[x]$. If a finite morphism, or an integral morphisms is injective, i.e. an inclusion of rings, then they are called finite extensions, or integral extensions respectively. In either case, we will often suppress the notation $\phi(p)$ for a polynomial in $\phi(B)[x]$, and simply write $p \in B[x]$ with evaluation on A understood to be the one induced by ϕ.

Definition 3.9.1. Let $f : X \to Y$ be a morphism of schemes, then f is **finite** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from a finite morphism of rings. Similarly, f is **integral** if for every open affine $V \subset Y$ we have that $f^{-1}(V)$ is affine, and the induced morphism $f|_{f^{-1}(V)}$ of affine schemes comes from an integral morphism of rings.

Note that finite and integral morphisms are examples of affine morphisms. We need to show that finite morphisms are closed under composition before moving forward:

Lemma 3.9.1. *Let $f : X \to Y$ and $g : Y \to Z$ be finite morphisms. Then $g \circ f$ is a finite morphism.*

Proof. This statement clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are finite, then $\psi \circ \phi$ is finite. Suppose ψ and ϕ are finite, then there exists $\{b_1, \ldots, b_m\}$ and $\{a_1, \ldots, a_n\}$ which generate B as a C-module and A as a B-module. Let $a \in A$, then there exist β_i such that:

$$a = \sum_{i=1}^{n} \phi(\beta_i) \cdot a_i$$

There exist c_{ij} such that each β_i satisfies:

$$\beta_i = \sum_{j=1}^{m} \psi(c_i j) \cdot b_j$$

hence:

$$a = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(\psi(c_{ij})) \cdot \phi(b_j) a_i$$

hence the set $\{\phi(b_j) \cdot a_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ generates A as a C module, which is finite, so $\psi \circ \phi$ is finite. $\square$

We also demonstrate the following relationship between integral and finite morphisms:

Proposition 3.9.1. *Let $f : X \to Y$ be a finite morphism, then f is integral. If $f : X \to Y$ is integral and locally of finite type, then f is finite.*

Proof. For the first statement, it suffices to show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is finite then it is integral. This then reduces to the case that if $\phi : B \to A$ is a finite morphism then it is integral.

Suppose $\phi : B \to A$ is finite, then A is a finitely generated b module, hence there exists $a_1, \ldots, a_n \in A$ such that for all $a \in A$ there are $b_1, \ldots, b_n \in B$ satisfying:

$$a = b_1 a_1 + \cdots + b_n a_n$$

We want to show that any $a \in A$ is the root of a monic polynomial in $\phi(B)[x]$. First note that we have a surjective map of B-modules:

$$\begin{aligned} \pi : B^{\oplus n} &\longrightarrow A \\ (b_1, \ldots, b_n) &\longmapsto \sum_{i=1}^{n} a_i b_i \end{aligned}$$

and that for any $a \in A$ we have B-module endomorphism $\psi_a \in \operatorname{End}_B(A)$ given by $s \mapsto a \cdot s$. For each i we have:

$$a \cdot a_i = \sum_{ij} b_{ij} a_j$$

for some $b_{ij} \in B$. This gives us an $n \times n$ matrix T with coefficients in B given by:

$$T = \begin{pmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{nn} \end{pmatrix}$$

The following diagram then commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "T"] & A \arrow[d, "\psi_a"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

Let $p \in B[x]$, and consider $p(T)$ and $p(\psi_a)$, where the latter polynomial p is technically a polynomial in $\phi(B)[x]$ as p is acting on elements of a via the ring homomorphism ϕ. The following diagram also commutes:

$$\begin{tikzcd} B^{\oplus n} \arrow[r, "\pi"] \arrow[d, "p(T)"] & A \arrow[d, "p(\psi_a)"] \\ B^{\oplus n} \arrow[r, "\pi"] & A \end{tikzcd}$$

as it would commute for any endomorphism of A and it's induced matrix T. Suppose that $p(T)$ is the zero morphism, and let $a' \in A$. Then there exists $(b_1, \ldots, b_n) \in B^{\oplus n}$ such that $\pi(b_1, \ldots, b_n) = a'$ so:

$$p(\psi_a)(a') = p(\psi_a) \circ \pi(b_1, \ldots, b_n) = \pi \circ p(T)(b_1, \ldots, b_n) = 0$$

so $p(\psi_a)$ is also zero. If $p(\psi_a) = 0$, then $p(a)$ is also zero as the ring homomorphism $a \mapsto \psi_a$ is injective; it thus suffices to show that there exists a polynomial $p \in B[x]$ such that $p(T) = 0$.

Note that if B is a field then this holds by the Cayley-Hamilton theorem. Consider the surjection:

$$\begin{aligned} F : \mathbb{Z}[x_{ij}] &\longrightarrow B \\ x_{ij} &\longrightarrow b_{ij} \end{aligned}$$

and the inclusion:

$$G : \mathbb{Z}[x_{ij}] \longrightarrow \mathbb{Q}(x_{ij})$$

where $\mathbb{Q}(X_{ij})$ is the field of fractions $\mathrm{Frac}(\mathbb{Z}[x_{ij}])$. We have an induced ring homomorphism:

$$F' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_B(B^n)$$

which is given by[78]:

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & \ddots & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} F(p_{11}) & \cdots & F(p_{n1}) \\ \vdots & \ddots & \vdots \\ F(p_{1n}) & \cdots & F(p_{nn}) \end{pmatrix}$$

and a similar inclusion:

$$G' : \mathrm{End}_{\mathbb{Z}[x_{ij}]}(\mathbb{Z}[x_{ij}]^n) \longrightarrow \mathrm{End}_{\mathbb{Q}(x_{ij})}(\mathbb{Q}(x_{ij})^n)$$

Let:

$$T' = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix}$$

then $F'(T') = T$. Since $\mathbb{Q}(x_{ij})$ is a field, there is a monic polynomial $q \in \mathbb{Q}(x_{ij})[y]$ such that $q(G'(T')) = 0$. This polynomial is given by $\det(y \cdot I_n - G'(T'))$, where I_n is the $n \times n$ identity matrix. Since each component $G'(T')_{ij} \in \mathbb{Z}[x_{ij}] \subset \mathbb{Q}(x_{ij})$, it follows that $q \in (\mathbb{Z}[x_{ij}])[y] \subset \mathbb{Q}(x_{ij})[y]$, and $q(T') = 0$. We have an induced ring homomorphism $F'' : (\mathbb{Z}[x_{ij}])[y] \to B[y]$, and it follows that:

$$0 = F'(q(T')) = F''(q)(F'(T')) = F''(q)(T)$$

so $p = F''(q)$ is a monic polynomial in $B[y]$ which has T as a root. By our earlier discussion it follows that $p(a) = 0$, and since $a \in A$ was arbitrary the map $\phi : B \to A$ is integral, implying the claim.

For the second statement, it also suffices to show that if $\phi : B \to A$ is an integral morphism which makes A a finitely generated B-algebra then ϕ is finite. Let $\{a_1, \ldots, a_n\}$ generate A as a B algebra. Then the morphism:

$$\begin{aligned} B[x_1, \ldots, x_n] &\longrightarrow A \\ x_i &\longrightarrow a_i \end{aligned}$$

is surjective. Moreover, for all $a \in A$, there exists a monic $p \in B[y]$ such that $p(0) = a$. Let $p_i \in B[y]$ satisfy $p_i(0) = a_i$, and let $d_i = \deg(p_i)$, then we claim that the set:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

generate A as a B module. Let $a \in A$, then we have that:

$$a = \sum_{i_1 \cdots i_n} b_{i_1 \cdots i_n} a_1^{i_1} \cdots a_n^{i_n}$$

[78]Since both $\mathbb{Z}[x_{ij}]^n$ and B^n are free modules of rank n, their endomorphism rings are $n \times n$ matrices with coefficients in their respective rings.

for some $b_{i_1 \cdots i_n}$, then we need only show that each $i_j \leq d_i - 1$. We prove this by induction on n; if $n = 1$ then we have that a can be written as:

$$a = \sum_i b_i a_1^i$$

Now $\phi(p)(a_1) = 0$, so:

$$a_1^{d_1} = -(b_{d_1-1} a_1^{d_1-1} + \cdots + b_0) \tag{3.9.1}$$

We need only show that any $a_1^{d_1+m}$ for $m \geq 0$ is in the B-span of $\{a_1^i : 0 \leq i \leq d_i - 1\}$. The base case $m = 0$ is proven, now suppose $m-1$th case so that:

$$\begin{aligned} a_1^{d_1+m} =& (a_1^{d_1+m-1}) \cdot a_1 = a_1 \left(b'_{d_1-1} \cdot a_1^{d_1-1} + \cdots + b'_0 \right) \\ =& a_1^{d_1} b'_{d_1-1} + \cdots + a_1 \cdot b'_0 \end{aligned}$$

Since $a_1^{d_1}$ can be written as in (3.9.1), when $n = 1$ we have that A is a finitely generated B module. Now supposing the $n-1$th case, we have that the sub algebra $A' \subset A$ generated by $\{a_1, \ldots, a_{n-1}\}$ is a finite B module. Since $\phi(B) \subset A'$, we have that A is integral over A', and A is clearly finitely generated over A' by a_n, hence A by the $n = 1$ case we have that A is a finite A' module. By Lemma 3.9.1, it follows that A is a finitely generated module with generators given by:

$$\{a_1^{m_1} \cdots a_n^{m_n} : 0 \leq m_i \leq d_i - 1\}$$

as desired. □

Corollary 3.9.1. *Let $\phi : B \to A$ be a ring homomorphism, and $a_1, a_2 \in A$ be integral over B. Then, $a_1 + a_2$, $a_1 \cdot a_2$, and $b \cdot a_i$ are integral elements over B.*

Proof. Let $A' \subset B$ be the B algebra generated by a_1 and a_2. The same induction argument in the second part of Proposition 3.9.1 then shows that A' is a finite B module[79], and thus A' is integral over B implying the claim. □

Example 3.9.1. Consider the map $\mathbb{Q} \to \bar{\mathbb{Q}}$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. This is integral by construction, but is not finite as $\bar{\mathbb{Q}}$ is not a finite dimensional $\mathbb{Q}$-vector space. Indeed, suppose that $\bar{\mathbb{Q}}$ is n dimensional as a $\mathbb{Q}$ vector space, and consider the polynomial $x^{n+1} - 2$; this polynomial as $n+1$ roots over $\mathbb{C}$ all of which must lie in $\bar{\mathbb{Q}} \setminus \mathbb{Q}$. These roots are all linearly independent hence $\bar{\mathbb{Q}}$ contains an $n+1$ dimensional $\mathbb{Q}$-linear subspace, and $\mathbb{Q} \to \bar{\mathbb{Q}}$ can't be finite.

Note that when dealing with varieties over a fixed field k, then every morphism is of finite type[80]. Indeed, let A and B be finitely generated k algebras, with generating sets $\{b_1, \ldots, b_n\}$ and $\{a_1, \ldots, a_m\}$. if $\phi : B \to A$ is a morphism, then consider the induced morphism:

$$\begin{aligned} \phi' : B[x_1, \ldots, x_m] &\longrightarrow A \\ x_i &\longmapsto a_i \end{aligned}$$

which on B acts by ϕ. This map is surjective as $k \subset B$, hence if:

$$a = p(a_1, \ldots, a_m)$$

for some $p \in k[x_1, \ldots, x_m]$, then $p \in B[x_1, \ldots, x_n]$, and $\phi'(p) = a$. It follows that A is a finitely generated B algebra so any morphism of varieties must be of finite type. It follows that in this setting we have that integral morphisms and finite morphisms between varieties are the same.

We now proceed with the rest of our standard results:

Proposition 3.9.2. *The following hold:*

[79]This is because the argument only uses that the generators are integral.

[80]Not locally of finite type, because every variety is quasi-compact, hence we can take every open cover to be finite

a) Integral morphisms are stable under composition.

b) Finite and integral morphisms are stable under base change.

c) Finite and integral morphism are local on target.

Proof. As in Lemma 3.9.1, a) clearly reduces to the following: if $\phi : C \to B$, and $\psi : B \to A$ are integral, then $\psi \circ \phi$ is integral. Suppose that ϕ and ψ are integral; let $a \in A$, then there exists a monic polynomial $p \in B[x]$ such that $p(a) = 0$. Set:

$$p(x) = b_0 + b_1 x + \cdots + x^n$$

and let $B' \subset B$ be the C algebra generated by $\{b_0, \ldots, b_n\}$. Note that B' is integral over C as B is integral over C hence B' is a finite C module. Let $A' \subset A$ be the B' algebra generated by a, then A' is obviously finitely generated over B', and integral over B' by Corollary 3.9.1, so Proposition 3.9.1 show that A' is finite over B'. We thus have the composition:

$$C \to B' \to A'$$

is a composition of finite morphisms and is thus finite. It follows by Proposition 3.9.1 that $C \to A'$ is integral, thus there exists a monic polynomial $q \in C[y]$ such that $q(a) = 0$. Since $a \in A$ was arbitrary we have that $C \to A$ is integral as well.

We now have that b) reduces as: if $\phi : B \to A$ is finite/integral, and and $\psi : B \to C$ is any morphism, then the induced map $C \to A \otimes_B C$ is finite/integral. Suppose that ϕ is finite, and let $\{a_1, \ldots, a_n\}$ generate A as a finite B-module. We claim that $S = \{a_1 \otimes 1, \ldots, a_n \otimes 1\}$ generates $A \otimes_B C$ as a C module. Indeed, since $A \otimes_B C$ is generated as an abelian group by simple tensors, it suffices to show that any $a \otimes c$ lies in the C span of S. Well, for some $b_i \in B$:

$$\begin{aligned} a \otimes c &= \left(\sum_i a_i b_i \right) \otimes c \\ &= \sum_i (a_i b_i) \otimes c \\ &= \sum_i a_i \otimes (b_i c) \\ &= \sum_i (a_i \otimes 1) \cdot (1 \otimes b_i c) \end{aligned}$$

as desired[81].

Now suppose that ψ is an integral morphism, then by Corollary 3.9.1 we need only show that $a \otimes 1$ is integral over C. We know there exists a monic polynomial $p \in B[x]$, so consider it's image in $C[x]$, which we also denote by p. This polynomials image $A \otimes_B C[x]$ is given by:

$$(1 \otimes p)(x) = (1 \otimes b_0) + \cdots (1 \otimes b_n) x^n$$

then:

$$\begin{aligned} (1 \otimes p)(a \otimes 1) =& (1 \otimes b_0) + \cdots (1 \otimes 1)(a^n \otimes 1) \\ =& a \otimes b_0 + \cdots + a^n \otimes b_n \\ =& b_0 \otimes 1 + a^n \otimes 1 \\ =& (p(a)) \otimes 1 \\ =& 0 \end{aligned}$$

so $C \to A \otimes_B C$ is an integral, implying b).

For c), suppose that $f : X \to Y$ is an integral/finite morphism, and U is any affine open of Y, and set $V = f^{-1}(U)$. We need to show that $f|_V : V \to U$ is integral/finite. Note that by Proposition 3.8.1,

[81] Recall that the canonical C module structure on $A \otimes_B C$ is given by $c\dot{(}a \otimes c') = (1 \otimes c) \cdot (a \otimes c')$.

we have that $f|_V : V \to U$ is an affine morphism, and that any open affine ove U is an affine open of Y, hence $(f|_V)|_{(f|_V)^{-1}(U)} = f|_{f^{-1}(U)}$ must come from an integral/finite morphism of rings by the definition of integral/finite morphisms. It follows that $f|_V$ is integral/finite.

Let $\{U_i = \operatorname{Spec} A_i\}$ be an open affine cover of Y, and $\{V_i = f^{-1}(U_i) = \operatorname{Spec} B_i\}$ be the corresponding open cover of X. Suppose that each $f : X \to Y$ is a morphism with each $f|_{V_i}$ integral/finite, and let $U = \operatorname{Spec} A \subset Y$ be an affine open of Y. Since f is affine by Proposition 3.8.1 we know that $f^{-1}(U) = \operatorname{Spec} B$ is affine. By Lemma 2.1.1, we can cover $\operatorname{Spec} A$ with open sets which are simultaneously distinguished in $\operatorname{Spec} A$ and $\operatorname{Spec} A_i$ for some i, hence there exists a distinguished open cover $\{U_{a_j}\}$ of $\operatorname{Spec} A$ such that the induced morphism $f^{-1}(U_{a_j}) \to U_{a_j}$ is integral/finite. Let $\phi : A \to B$ be the ring homomorphism induces $f|_{f^{-1}(U)}$, then we have reduced the problem to the following situation: let $\{a_1, \dots, a_n\} \subset A$ generate the unit ideal, and the induced map $\phi_j : A_{a_j} \to B_{\phi(a_j)}$ be integral/finite, then ϕ is integral/finite.

First suppose that each ϕ_j is finite; then there exist $s_{1_j}, \dots, s_{n_j} \in B_{\phi(a_j)}$ which generate $B_{\phi(a_j)}$ as an A_{a_j} module. We can write each s_{i_j} as:

$$s_{i_j} = \frac{b_{i_j}}{\phi(a_j)^{k_{i_j}}}$$

for some $b_{i_j} \in B$, some $k_{i_j} \in \mathbb{N}$. Since $1/a_j \in A_{a_j}$, it follows that we can take our generators to be of the form:

$$s_{i_j} = \frac{b_{i_j}}{1}$$

for all i_j. This gives us a finite set $\{b_{i_j}\} \subset B$, which we claim generates B as an A module; let $N = |\{b_{i_j}\}$, and consider the morphism of A modules:

$$\begin{aligned} \psi : A^{\oplus N} &\longrightarrow B \\ (a_{i_j}) &\longmapsto \sum_i \sum_j a_{i_j} b_{i_j} \end{aligned}$$

Let $\pi : B \to C$ be cokernel of this map, then since cokernels commute with localization[82], we have that the induced map $\pi_{a_j} : B_{\phi(a_j)} \to C_{\pi(\phi(a_j))}$ is the cokernel of:

$$A_{a_j}^{\oplus N} \longrightarrow B_{\phi(a_j)}$$

which is surjective hence $C_{\pi(\phi(a_j))} = 0$ for all j. Now let $c \in C$, then $c/1 \in C_{\pi(\phi(a_j))} = 0$, hence there exists some m_j such that $\pi(\phi(a_j))^{m_j} c = 0$. The a_j generate the unit ideal, so $a_j^{m_j}$ generate the unit ideal as well, hence $1 = \sum_j a_j^{m_j} \alpha_j$, therefore:

$$c = 1 \cdot c = \sum_j \pi(\phi(a_j^{m_j} \alpha_j)) \cdot c = 0$$

hence $C = 0$ and so ψ is surjective.[83]

Now suppose that each ϕ_j is integral. Let $b \in B$, then for all j, $b/1 \in B_{\phi(a_j)}$ is the root of a monic polynomial $p_j \in A_{a_j}[x]$. Note that $A_{a_j}[x] = (A[x])_{a_j}$; let:

$$p_j = x^{n_j} + \frac{b_{n_j-1}}{a_j^{k_{n_j-1}}} x^{n_j-1} + \dots + \frac{b_0}{a_j^{k_0}}$$

There exists a M_j such that:

$$a_j^{M_j} p_j = a_j^{M_j} x^{n_j} + \frac{b'_{n_j-1}}{1} x^{n_j-1} + \dots + \frac{b'_0}{1}$$

[82]See Lemma 5.3.3 parti *iv*).

[83]If this feels like like there is some sheaf business going on here, that's because there is!

There thus exists a $p'_j \in A[x]$ such that $p'_j/1 \in (A[x])_{a_j}$ is equal to $a_j^{M_j} p_j$. Since $\phi(a_j)^{M_j} p_j(b) = 0$ it follows that there is an L_j such that $\phi(a_j)^{L_j+M_j} p_j(b) = 0$. Moreover, if we set $q_j = a_j^{L_j} \cdot p'_j$, then $q_j(b) = 0$, and $q_j/1 = a_j^{M_j+L_j} p_j$. Let N be the maximum degree of the q_j, and let $m_j = N - n_j$. Now again, we have that the set $\{a_1^{K_1}, \dots, a_n^{K_n}\}$ generates the unit ideal, hence there are h_j such that:

$$1 = \sum_j h_j a_j^{K_j}$$

so we define $q \in A[x]$ by:

$$q = \sum_j h_j x^{m_j} q_j$$

Note that each $x^{m_j} q_j$ has degree N, and that the degree N term of q is given by:

$$q = \sum_j h_j x^{m_j} a_j^{K_j} x^{n_j} = x^N \sum_j h_j a_j^{K_j} = x^N$$

hence q is a monic polynomial in $A[x]$. We claim that $q(b) = 0$, however this is clear as $q_j(b) = 0$ for all j. It follows that $\phi : A \to B$ is integral, implying the claim.

□

Our goal is to now further justify the the nomenclature 'finite morphism' in the sense that we wish to prove that these maps have finite fibres. Let $f : X \to Y$ be a finite morphism, and recall that the scheme theoretic fibre of $y \in Y$ is given by:

$$X_y = \operatorname{Spec} k_y \times_Y X$$

Note that if $U = \operatorname{Spec} A \subset Y$ is an affine scheme containing y then we have the following isomorphism:

$$X_y \cong \operatorname{Spec} k_y \times_U f^{-1}(U)$$

If f is finite then it is affine as well, and so with $f^{-1}(U) = \operatorname{Spec} B$, it suffices to show that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

is a finite topological space which ultimately amounts to showing that $k_y \otimes_A B$ has finitely many prime ideals. To do so we will need to develop the theory of Artinian rings, a class of rings which satisfy a condition dual to the Noetherian one.

Definition 3.9.2. Let A be a commutative ring, then A is commutative if every strictly decreasing chain of ideals:

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

terminates.

One quickly sees that being Artinian is a much less reasonable finiteness condition than being Noetherian. Indeed, let $A = \mathbb{Z}$, then the following chain never terminates:

$$2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots$$

so $\mathbb{Z}$ is not Artinian. Furthermore, in contrast to Theorem 3.4.1, we have that $A[x_1, \dots, x_n]$ is never Artinian as the following chain never terminates:

$$\langle x_i \rangle \supset \langle x_i^2 \rangle \supset \cdots \supset \langle x_i^n \rangle \supset \cdots$$

Example 3.9.2. Let $A = k^n$ with the ring structure given the canonical product ring structure. Then we have that every ideal is a vector subspace and the length of any chain of ideals is bounded above by $n+1$, hence must be finite. It follows that A is Artinian (and Noetherian). Moreover, any finite k-algebra is Artinian, and any ring that is finite as a set is also Artinian, i.e. $\mathbb{Z}/n\mathbb{Z}$.

The following is an analogue of Lemma 3.4.2:

Lemma 3.9.2. *Let A be a Artinian, then the following hold:*

a) If S is any multiplicatively closed subset then $S^{-1}A$ is Artinian.

b) If $I \subset A$ is an ideal then A/I is Artinian.

Proof. For *a*) let:

$$J_1 \supset J_2 \supset \cdots$$

be a strictly descending chain of ideals in $S^{-1}A$. If $\pi : A \to S^{-1}A$ is the localization map, then we have that:

$$\pi^{-1}(J_1) \supset \pi^{-1}(J_2) \supset \cdots$$

is chain of ideals in A. For some n this must terminate, hence for all $m \geq n$ we have that $\pi^{-1}(J_m) = \pi^{-1}(J_n)$. It now suffices to show that $\left\langle \pi(\pi^{-1}(J_m)) \right\rangle = J_m$ for any m. Clearly, we have the inclusion $\left\langle \pi(\pi^{-1}(J_m)) \right\rangle \subset J_m$; let $a/s \in J_m$, then $a/1 \in J_m$,and $a \in \pi^{-1}(J_m)$. It follows that $a/1 \in \pi(\pi^{-1}(J_m))$, hence $a/s \in \left\langle \pi(\pi^{-1}(J_m)) \right\rangle$ implying the equality.

For *b*), we employ the same argument; however since $\pi : A \to A/I$ is surjective we automatically have the equality $\left\langle \pi(\pi^{-1}(J_m)) \right\rangle = J_m$. □

The above gives us the following strange result:

Proposition 3.9.3. *Let A be Artinian, then every $\mathfrak{p} \in \operatorname{Spec} A$ is maximal. In particular, A is an integral domain if and only if it is a field.*

Proof. Let A be Artinian, and $\mathfrak{p} \in \operatorname{Spec} A$, then by Lemma 3.9.2 we have that $A/\mathfrak{p}$ is an Artinian integral domain. Let $[a] \in A/\mathfrak{p}$ be nonzero and consider the following chain:

$$\langle [a] \rangle \supset \left\langle [a]^2 \right\rangle \cdots$$

which must stabilize, hence for some n we have that $\langle [a]^n \rangle = \left\langle [a]^{n+1} \right\rangle$. This implies that $[a]^n \in \left\langle [a]^{n+1} \right\rangle$ so there exists $[b] \in A/\mathfrak{p}$ such that $[a]^{n+1}[b] = [a]^n$, thus:

$$[a]^n([a] \cdot [b] - [1]) = 0 \Rightarrow [a] \cdot [b] - 1 = 0$$

as $[a]$ is assumed nonzero. It follows that $[b] = [a]^{-1}$ hence every nonzero element of $A/\mathfrak{p}$ is invertible so $A/\mathfrak{p}$ is a field implying that $\mathfrak{p}$ is maximal. In particular, if A is an integral domain then $\langle 0 \rangle$ is prime and thus maximal so A is a field. □

We now need the following general lemma:

Lemma 3.9.3. *Let A be a commutative ring, and $\mathfrak{q}, \mathfrak{p}_i \in \operatorname{Spec} A$ for $1 \leq i \leq n$. Then $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$ if and only if for some i we have $\mathfrak{p}_i \subset \mathfrak{q}$.*

Proof. We proceed by induction, the base case $n = 1$ is trivial, and if $\mathfrak{p}_i \subset \mathfrak{q}$ for some i, then clearly we have that $\bigcap_i \mathfrak{p}_i \subset \mathfrak{q}$. Assuming the $n-1$th case, we have that:

$$\left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right) \cap \mathfrak{p}_n \subset \mathfrak{q}$$

If $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \subset \mathfrak{q}$, we are done by induction, so assume that $\bigcap_{i=1}^{n-1} \mathfrak{p}_i \not\subset \mathfrak{q}$. Let $a \in \mathfrak{p}_n$, then by assumption there exists some $b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right)$ such that $b \notin \mathfrak{q}$. It follows that $a \cdot b \in \left(\bigcap_{i=1}^{n-1} \mathfrak{p}_i \right) \cap \mathfrak{p}_n$ which lies in $\mathfrak{q}$, however $\mathfrak{q}$ is prime hence either $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$, thus again by assumption we have that $a \in \mathfrak{q}$. It follows that $\mathfrak{p}_n \subset \mathfrak{q}$.

□

Proposition 3.9.4. *Let A be Artinian, then $\operatorname{Spec} A$ is a finite topological space and carries the discrete topology*[84]*.*

[84] Recall that in the discrete topology every subset is open

Proof. Suppose that Spec A has infinitely many maximal ideals, then we can choose some infinite sequence $\{\mathfrak{m}_i\}_{i=1}^{\infty}$ of pairwise distinct maximal ideals. Consider the following chain:

$$\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \cdots$$

We claim that this chain is strictly decreasing and never stabilizes, implying A is not Artinian. Suppose:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1}$$

then we have that:

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1} \subset \mathfrak{m}_{n+1}$$

It follows that one of the $\mathfrak{m}_i$ is contained $\mathfrak{m}_{n+1}$ by Lemma 3.9.3, hence $\mathfrak{m}_i = \mathfrak{m}_{n+1}$ as these are all maximal ideals. However this is impossible as all maximal ideals are pairwise distinct by assumption, so A is not Artinian.

Supposing A is Artinian, we have by the above that Spec A has only finitely many maximal ideals. Since every prime ideal is maximal, by Proposition 3.9.3 we have that Spec A is a finite topological space equal to $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ where each $\mathfrak{m}_i$ is a maximal ideal. We see that $\mathbb{V}(\mathfrak{m}_i) = \{\mathfrak{m}_i\}$ so the singleton sets are closed, hence every subset of Spec A is closed, so every subset of Spec A is open implying that Spec A carries the discrete topology. ☐

We can now show that finite morphisms have finite fibres as initially discussed:

Corollary 3.9.2. *let $f : X \to Y$ be a finite morphism, then for all $y \in Y$, the fibre $X_y = \operatorname{Spec} k_y \times_Y X$ is a finite topological space.*

Proof. From our earlier discussion, if $U = \operatorname{Spec} A \subset Y$ contains y, and $\operatorname{Spec} B = f^{-1}(U)$, then we have that:

$$X_y \cong \operatorname{Spec}(k_y \otimes_A B)$$

Since f is finite, we have that B is a finite A algebra, hence by Proposition 3.9.2 we have that $k_y \otimes_A B$ is a finite k_y algebra. Example 3.9.2 then implies that $k_y \otimes_A B$ is Artinian, hence $\operatorname{Spec}(k_y \otimes_A B)$ is a finite topological space with the discrete topology by Proposition 3.9.4 as desired. ☐

3.10 Finite Morphisms are Proper

We now end our discussion on integral and finite morphisms by connecting them to the other classes of morphisms discusses in this chapter. In particular we wish to show that integral morphisms are precisely those morphisms which are affine and universally closed, and finite morphisms are precisely those morphisms which are affine and proper. To do so, as usual, we will need to prove a slew of results from commutative algebra. Namely, this section could just as easily be called Lying Over, Going Up, and Nakayama's Lemma as our desired results will be applications of these lemmas.

We begin with Nakayama's Lemma; it comes in many flavors, and we prove five of them:

Lemma 3.10.1. *Let A be a ring, $I \subset A$ an ideal, and M a finitely generated A module. The following then hold:*

a) If $IM = M$ then there exists and $a \in A$ such that $[a] = [1] \in A/I$, and $a \cdot M = 0$.

b) If $IM = M$, and

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then $M = 0$.

c) Let N' and N be A-modules with $M, N \subset N'$, and suppose that I is contained in all maximal ideals of A as in b). Then if $N' = N + IM$, $N' = N$.

d) *Let $f : N \to M$ be an A module morphism and suppose I is contained in all maximal ideals of A. Then if $\bar{f} : N/IN \to M/IM$ is surjective, f is surjective.*

e) *Suppose I is contained in all maximal ideals of A, and let $\pi : M \to M/IM$ be the natural surjection. If the image $\{f_1, \ldots, f_n\} \subset M$ generates M/IM then $\{f_1, \ldots, f_n\}$ generate M.*

Proof. We start with a); note that:

$$IM = \{i \cdot m : i \in I, m \in M\}$$

Choose generators $f_1, \ldots, f_n$ of M, then we claim that the map:

$$\begin{aligned} \alpha : I^n &\longrightarrow M \\ (b_1, \ldots, b_n) &\longmapsto \sum_i b_i f_i \end{aligned}$$

is surjective. Let $m \in M$, then since $IM = M$ we have that $m = i \cdot n$ for some $i \in I$ some $n \in N$. However, $n = \sum_i a_i f_i$ as the f_i generate M, hence $m = \sum_i (ia_i) f_i$, and each $ia_i \in I$ implying the initial claim. In particular, we can write each generator as:

$$f_i = \sum_j c_{ij} f_j$$

for some $c_{ij} \in I$. Consider the matrix with coefficients in A given by:

$$S = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

which determines a morphism $A^n \to A^n$. Let $\beta : A^n \to M$ be the natural surjection[85] and set:

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is in the ith position, and note that $\beta(e_i) = f_i$. Then:

$$\beta \circ S(e_i) = \beta \left(\sum_j c_{ij} e_j \right) = \sum_j c_{ij} f_j = f_i$$

Now observe that for all i:

$$\beta \circ (\mathrm{Id} - S)(e_i) = 0$$

hence $\beta \circ (\mathrm{Id} - S)$ is identically zero. Define $a \in A$ by:

$$a = \det(\mathrm{Id} - S) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(\delta_{1\sigma(1)} - c_{1\sigma(1)}) \cdots (\delta_{n\sigma(n)} - c_{n\sigma(n)})$$

where S_n is the symmetric group. Note that $[a] = [1] \in A/I$ as if σ is not the identity then under the projection $\pi : A \to A/I$, we have

$$\pi(\delta_{i\sigma(i)} - c_{i\sigma(i)}) = \pi(c_{i\sigma(i)}) = 0$$

[85] Defined the same as α, just on all of A^n.

and if σ is the identity, then:

$$\pi(\delta ii - c_{ii}) = \pi(1 - c_{ii}) = [1]$$

Moreover, recall that for any matrix T there exists an adjugate matrix $\mathrm{adj}(T)$ satisfying:

$$\mathrm{adj}(T) \cdot T = T \cdot \mathrm{adj}(T) = \det(T) \cdot \mathrm{Id}$$

hence for all i, we have that:

$$\begin{aligned} a \cdot f_i =& a \cdot \beta(e_i) \\ =& \beta(a \cdot e_i) \\ =& \beta \circ (\det(\mathrm{Id} - S)\mathrm{Id})(e_i) \end{aligned}$$

Now note that for any matrix T, we have that:

$$\beta \circ T(e_i) = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix}$$

hence:

$$(\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

and so:

$$\beta \circ (\det(\mathrm{Id} - S))(e_i) = \mathrm{adj}(\mathrm{Id} - S) \cdot (\mathrm{Id} - S) \cdot \begin{pmatrix} 0 \\ \vdots \\ f_i \\ \vdots \\ 0 \end{pmatrix} = 0$$

implying that $a \cdot f_i = 0$ as desired. In particular, since a annihilates each generator, we have that $a \cdot M = 0$, implying a).

For b) suppose in addition that:

$$I \subset \bigcap_{\mathfrak{m} \in |\operatorname{Spec} A|} \mathfrak{m}$$

then with a as defined in a), we claim that a is invertible. Indeed, there exists $i \in I$ such that $a = 1 + i$, and this $i \in \mathfrak{m}$ for all $\mathfrak{m} \in |\operatorname{Spec} A|$. Consider the ideal $\langle 1 + i \rangle$, then if this ideal is not all of A, there must be some $\mathfrak{m} \in |\operatorname{Spec} A|$ such that $\langle 1 + i \rangle \subset \mathfrak{m}$. However, $\imath \in \mathfrak{m}$ as well so $1 \in \mathfrak{m}$ which is a contradiction. It follows that $\langle i + 1 \rangle = A$ hence a invertible. Let $m \in M$, then $a \cdot m = 0$ by construction, but:

$$0 = a^{-1} \cdot (a \cdot m) = m$$

hence $M = 0$ implying b).

For c), suppose that $N' = N + IM$, then note this implies that $N' = N + M$ as if $n' \in N'$ then we have $n' = n + i \cdot m$ for $n \in N$, $i \in I$, and $m \in M$. However $i \cdot m \in M$ hence $N' \subset N + M$. Since N

and M are submodules of N' it follows that $N' = N + M = N + IM$. In particular, we have that N'/N is finitely generated, as if $\{f_1, \dots, f_k\}$ generate M, then we claim that $\{[f_1], \dots, [f_k]\}$ generate N'/N. Indeed, let $[n'] \in N'/N$, then any class representative n' can be written as $n + m$ for $n \in N$ and $m \in M$. Any $m \in M$ can be written as:

$$m = \sum_i a_i f_i$$

hence:

$$[n'] = \sum_i a_i [f_i] + [n] = \sum_i a_i [f_i]$$

Moreover we claim that $I(N'/N) = N'/N$; clearly we have $I(N'/N) \subset (N'/N)$, so let $[n'] \in N'/N$. Then any class representative n' can be written as $n + i \cdot m$, hence $[n'] = [i \cdot m] = i \cdot [m]$ so $[n'] \in I(N'/N)$. It follows by b) that $N'/N = 0$, implying the claim.

For d), let $f : N \to M$ be an A module homomorphism. Note that $\bar{f} : N/IN \to M/IM$ is induced by $\pi \circ f : N \to M/IM$ and factors uniquely through the quotient as $IN \subset \ker(\pi \circ f)$. Obviously, we have that $M = \operatorname{im}(f) + IM$, and M is finitely generated hence by c) we have that $\operatorname{im}(f) = M$ and f is surjective, as desired.

For e), I be as in b), and consider the natural projection $\pi : M \to M/IM$. We have that M/IM is finitely generated by $\{[f_1], \dots, [f_n]\}$. Let $N \subset M$ be the submodule generated by $f_1, \dots, f_n$, then we claim that $M = N + IM$. Let $m \in M$, and consider $[m]$. Then:

$$[m] = \sum_i a_i [f_i] = \left[\sum_i a_i f_i \right]$$

It follows that there exists $\beta \in IM$ such that:

$$m = \sum_i a_i f_i + \beta$$

hence $m \in N + \mathfrak{m}M$. Since M is finitely generated, it follows by c) that $M = N$ hence $\{f_1, \dots, f_n\}$ generate M. $\square$

We need the following lemma for both Lying Over and Going Up

Lemma 3.10.2. *Let $\phi : B \to A$ be an integral morphism, $I \subset A$, $J \subset B$ ideals, and $T \subset B$ a multiplicatively closed set. Then the following hold:*

a) The morphism $B \to A/I$ is integral.

b) The morphism $B/J \to A/\langle \phi(J) \rangle$ is integral.

c) The morphism $T^{-1}B \to \phi(T)^{-1}A$ is integral.

Proof. To show a), recall that the composition of integral morphisms is integral, so it suffices to show that $\pi : A \to A/I$ is integral. Let $[a] \in A/I$, then $p(x) = x - a \in A[x]$ is a monic polynomial with $[a]$ as a root, hence π is integral.

For b) let $J \subset B$ be an ideal, then the morphism $\psi : B/J \to A/\langle \phi(J) \rangle$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd}
B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\
B/J \arrow[r, "\psi"] & A/\langle \phi(J) \rangle
\end{tikzcd}$$

Let $[a] \in A/\langle \phi(J) \rangle$, then $a \in \pi_A^{-1}([a])$, and there is a polynomial $p \in B[x]$:

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

of which a is a root. There is then a polynomial $q \in B/J[x]$ given by:

$$q(x) = x^n + [b_{n-1}]x^{n-1} + \cdots + b_0$$

We see that:

$$\begin{aligned} q([a]) =& [a]^n + [b_{n-1}][a]^{n-1} + \cdots + b_0 \\ =& [a^n + b_{n-1}a^{n-1} + \cdots + b_0] \\ =& [p(a)] \\ =& 0 \end{aligned}$$

hence ψ is integral.

For c), the morphism $\psi : T^{-1}B \to \phi(T)^{-1}A$ is the unique one which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_B"] & A \arrow[d, "\pi_A"] \\ T^{-1}B \arrow[r, "\psi"] & \phi(T)^{-1}A \end{tikzcd}$$

It suffices to show that $a/1$ and $1/\phi(t)$ are the roots of monic polynomials in $T^{-1}B[x]$ by Corollary 3.9.1. Let $a/1 \in \phi(T)^{-1}A$, then there exists $a \in A$ which maps to $a/1$ under π_A. Let $p \in B[x]$ be given by:

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

and satisfy $p(a) = 0$. Define $q \in T^{-1}B[x]$ by:

$$q(x) = x^n + \frac{b_{n-1}}{1}x^{n-1} + \cdots + \frac{b_0}{1}$$

then:

$$q(a) = \frac{p(a)}{1} = 0$$

as desired. For $1/\phi(t)$ we claim that:

$$q(x) = x - \frac{1}{t} \in T^{-1}B[x]$$

satisfies $q(1/\phi(t)) = 0$. However, this clear as:

$$q(1/\phi(t)) = \frac{1}{\phi(t)} - \psi\left(\frac{1}{t}\right)$$

which by the definition of ψ reduces to:

$$\frac{1}{\phi(t)} - \frac{1}{\phi(t)} = 0$$

as desired. □

As the following example shows,

Example 3.10.1. If $S \subset A$ is multiplicatively closed, and $\phi : B \to A$ is a morphism of rings, then the natural map $\psi : B \to S^{-1}A$ given by $\psi = \pi \circ \phi$ is not in general integral even if ϕ is. Indeed, if this were true then every localization would be integral as the identity map is integral; as a counter example take the localization map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$, then since $\mathbb{C}[t]$ is an integrally closed domain it follows that if $\alpha \in \mathrm{Frac}(\mathbb{C}t)$ is integral then $\alpha \in \mathbb{C}[t]$. However $t^{-1} \notin \mathbb{C}[t]$ hence t^{-1} can't be integral over $\mathbb{C}[t]$ so the map $\mathbb{C}[t] \to \mathrm{Frac}(\mathbb{C}[t])$ is not integral.

With our many flavours of Nakayama's lemma at hand, as well as Lemma 3.10.2 we can now prove the Lying Over, and Going Up result, beginning with the former:

Lemma 3.10.3. *Let $\phi : B \to A$ be an integral extension of rings, then induced map on schemes $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is surjective.*

Note that this is called 'Lying Over' because it implies that for any $\mathfrak{p} \in \operatorname{Spec} B$ we can find a prime $\mathfrak{q} \in \operatorname{Spec} A$ which maps to it.

Proof. Given $\mathfrak{p} \in \operatorname{Spec} B$, we simply need to show that the fibre $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A \times_B \operatorname{Spec} k_{\mathfrak{p}}$ is non empty. By Lemma 3.7.2, we have that:

$$f^{-1}(\mathfrak{p}) \cong \operatorname{Spec} A_{\mathfrak{p}} / \langle \phi(\mathfrak{p})/1 \rangle$$

where $A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1} A$. It follows that $f^{-1}(\mathfrak{p})$ is empty if and only if $\langle \phi(\mathfrak{p})/1 \rangle = A_{\mathfrak{p}}$, as the only ring without a maximal ideal is the 0 ring.

The localization map $\phi_{\mathfrak{p}} : B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is an integral morphism by Lemma 3.10.2. In particular, if $b/s \in B_{\mathfrak{p}}$, and $\phi(b)/\phi(s) = 0 \in A_{\mathfrak{p}}$, then there exists some $\phi(t) \in \phi(B \setminus \mathfrak{p})$ such that

$$\phi(bt) = 0$$

This implies $b \cdot t = 0$, but then $b/s = 0 \in B_{\mathfrak{p}}$. It follows that $\phi_{\mathfrak{p}}$ is injective as well. Let $\mathfrak{m}_{\mathfrak{p}}$ be the unique maximal ideal in $B_{\mathfrak{p}}$, then by the commutativity of the diagram:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\pi_{\mathfrak{p}}"] & A \arrow[d, "\pi_{\mathfrak{p}}"] \\ B_{\mathfrak{p}} \arrow[r, "\phi_{\mathfrak{p}}"] & A_{\mathfrak{p}} \end{tikzcd}$$

we have that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = \langle \phi(\mathfrak{p})/1 \rangle$. Indeed, suppose that $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$, then by definition, we have that $a \in \phi(\mathfrak{p})$, and $s \in \phi(B \setminus \mathfrak{p})$. There is then a unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$ such that $\phi_{\mathfrak{p}}(b/t) = \phi(b)/\phi(t) = a/s$. Similarly, if $a/s \in \langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle$, then $a/s = \phi(b)/\phi(t)$ for some unique $b \in \mathfrak{p}$, and $t \in B \setminus \mathfrak{p}$, hence $a/s \in \langle \phi(\mathfrak{p})/1 \rangle$.

The condition that $\langle \phi_{\mathfrak{p}}(\mathfrak{m}_{\mathfrak{p}}) \rangle = A_{\mathfrak{p}}$ is now more aptly written as $\mathfrak{m}_{\mathfrak{p}} \cdot A_{\mathfrak{p}} = A_{\mathfrak{p}}$. For the sake of contradiction, suppose this holds, then we have that $1 \in A_{\mathfrak{p}}$ can be written as:

$$1 = \sum_{i=1}^{n} m_i \cdot g_i \tag{3.10.1}$$

with $m_i \in \mathfrak{m}_{\mathfrak{p}}$, and $g_i \in A_{\mathfrak{p}}$. Take the subalgebra $A' \subset A_{\mathfrak{p}}$ generated by $\{g_1, \ldots, g_n\}$, then A' is integral over $B_{\mathfrak{p}}$ and finitely generated, hence a a finite $B_{\mathfrak{p}}$ module by Proposition 3.9.1. We then have that (3.10.1) implies $1 \in \mathfrak{m}_{\mathfrak{p}} \cdot A'$, hence $\mathfrak{m}_{\mathfrak{p}} \cdot A' = A'$. However, $\mathfrak{m}_{\mathfrak{p}}$ is the only maximal ideal of $B_{\mathfrak{p}}$, so by Nakayama's lemma[86], we have that $A' = 0$, contradicting the injectivity of $\phi_{\mathfrak{p}}$. □

Going Up is now a borderline immediate consequence of Lying Over:

Lemma 3.10.4. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be an integral morphism, and $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$. Let $\mathfrak{q} \in \operatorname{Spec} A$ satisfy $f(\mathfrak{q}) = \mathfrak{p}$, then there exists $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.*

Note that this is called 'Going Up' as it implies that we can lift chains of prime ideals.

Proof. Let $\phi : B \to A$ be the ring homomorphism which induces f; in particular, ϕ makes A integral over B. With $\mathfrak{p}, \mathfrak{p}'$, and $\mathfrak{q}$ as stated, consider the induced map $\phi' : B \to A/\mathfrak{q}$. Note that by Lemma 3.10.2 this map is integral. We claim that $\ker \phi' = \mathfrak{p}$. Indeed, let $b \in \mathfrak{p}$, then $\phi'(p) = [\phi(b)]$, but $\phi(b) \in \mathfrak{q}$ as $\phi(\mathfrak{p}) \subset \mathfrak{q}$. Now suppose that $\phi'(b) = 0$, then $\phi(b) \in \mathfrak{q}$, so $b \in \phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It follows that the map $B/\mathfrak{p} \to A/\mathfrak{q}$ is injective; in particular it is integral by Lemma 3.10.2 as $(A/\mathfrak{q})/\phi'(\mathfrak{p}) = (A/\mathfrak{q})/\langle 0 \rangle = A/\mathfrak{q}$. By Lying

[86] Part b) of Lemma 3.10.1.

Over we have that that the induced map $\operatorname{Spec} A/\mathfrak{q} \to \operatorname{Spec} B/\mathfrak{p}$[87] is surjective, so there exists a prime $\mathfrak{q}'$ containing $\mathfrak{q}$ which maps to $\mathfrak{p}'$ as desired. □

Example 3.10.2. Let $N : \tilde{X} \to X$ be the normalization map of an integral scheme X. We claim that N is integral, and surjective. First note by the proof in Theorem 3.3.1, where we define N on an affine cover $\operatorname{Spec} A_i$, of X, that $N^{-1}(\operatorname{Spec} A_i) \cong \operatorname{Spec} \bar{A}_i$, so N is affine by Proposition 3.8.1. On this open cover, N is given by the $A \hookrightarrow \bar{A}$ which is integral extension by definition, hence N is integral by Proposition 3.9.2. In particular, by Lemma 3.10.3 we have that $\operatorname{Spec} \bar{A}_i \to \operatorname{Spec} A_i$ is surjective for all i, hence N is surjective.

If $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism satisfying:

> For any $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ with $f(\mathfrak{q}) = \mathfrak{p}$, there exists a $\mathfrak{q}' \in \operatorname{Spec} A$ containing $\mathfrak{q}$ such that $f(\mathfrak{q}') = \mathfrak{p}'$.

we say that Going Up holds for f. In particular, Going Up is equivalent to f being a closed map:

Proposition 3.10.1. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism, then f is closed if and only if Going Up holds for f.*

Proof. Suppose that $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is closed, and let $\phi : B \to A$ be the ring homomorphism inducing f. Let $\mathfrak{p} \subset \mathfrak{p}' \in \operatorname{Spec} B$, and $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $f(\mathfrak{q}) = \mathfrak{p}$. Consider $\mathbb{V}(\mathfrak{q})$, then $f(\mathbb{V}(\mathfrak{q}))$ is closed, and contains $\mathfrak{p}$, hence $f(\mathbb{V}(\mathfrak{q}))$ contains the closure of $\mathfrak{p}$, $\mathbb{V}(\mathfrak{p})$. Since $\mathfrak{p}'$ is contained in $\mathbb{V}(\mathfrak{p})$, we have that $\mathfrak{p}' \in f(\mathbb{V}(\mathfrak{q}))$ hence there exists some $\mathfrak{q}' \in \mathbb{V}(\mathfrak{q})$ such that $f(\mathfrak{q}') = \mathfrak{q}$. It follows that Going Up holds for f.

Now suppose that going up holds for f, and let $\mathbb{V}(I) \subset \operatorname{Spec} A$ be a closed subset. Note that since $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is integral, we have that Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} A$, thus clearly Going Up holds for $\operatorname{Spec} A/I \to \operatorname{Spec} B$. It thus suffices to show that if Going Up holds for $f : \operatorname{Spec} A \to \operatorname{Spec} B$ then $f(\operatorname{Spec} A)$ has closed image.

Let $Z = f(\operatorname{Spec} A)$, and let $\mathfrak{p} \in \bar{Z}$. Then for any open set containing $\mathfrak{p}$ we must have that $U \cap Z \neq \emptyset$, as other wise U^c is a closed set containing Z, and thus contains $\bar{Z}$. However, $\mathfrak{p} \notin U^c$ so $\mathfrak{p} \notin \bar{Z}$, a contradiction. Hence, for all $g \notin \mathfrak{p}$, we have that $U_g \cap Z \neq \emptyset$. In particular, since $U_g \cap Z = f(U_{\phi(g)})$, we have that $U_{\phi(g)}$ is not empty for $g \notin \mathfrak{p}$.

This implies that $A_\mathfrak{p} = \phi(B \setminus \mathfrak{p})^{-1}A$ is not the zero ring. Indeed, if $A_\mathfrak{p}$ is the zero ring that $1 = 0$, hence there would exist some $g \in B \setminus \mathfrak{p}$ such that $\phi(g) = 0$, but that would imply that $U_{\phi(g)}$ is empty, a contradiction. We now consider the composition:

$$\operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A \to \operatorname{Spec} B$$

where the first map is induced by the localization map $\pi : A \to A_\mathfrak{p}$. Now let $\tilde{\mathfrak{q}} \in \operatorname{Spec} A_\mathfrak{p}$, and consider $\mathfrak{p}' = f(\pi^{-1}(\tilde{\mathfrak{q}}))$; we claim that $\mathfrak{p}' \subset \mathfrak{p}$. Suppose the contrary, then there exists a $g \in \mathfrak{p}'$ such that $g \notin \mathfrak{p}$. It follows that $\phi(g)/1 \in \tilde{\mathfrak{q}}$, but if $g \notin \mathfrak{p}$, then $\phi(g) \in \phi(B \setminus \mathfrak{p})$, so $\tilde{\mathfrak{q}} = A_\mathfrak{p}$, a contradiction.

In particular, we have shown that there exists $\mathfrak{p}' \subset \mathfrak{p} \in \operatorname{Spec} B$, and $\mathfrak{q}' = \pi^{-1}(\tilde{\mathfrak{q}})$ satisfying $f(\mathfrak{q}') = \mathfrak{p}'$. Since Going Up holds for f, it follows that there exists a $\mathfrak{q} \in \operatorname{Spec} A$ satisfying $\mathfrak{q}' \subset \mathfrak{q}$ and $f(\mathfrak{q}) = \mathfrak{p}$. Therefore, if $\mathfrak{p} \in \bar{Z}$, we have $\mathfrak{p} \in Z$, so Z is closed, implying the claim. □

Lemma 3.10.5. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be induced by $\phi : B \to A$. Then, the closure of the image, $\operatorname{cl}(f(\operatorname{Spec} A))$ is equal to $\mathbb{V}(\ker \phi)$.*

Proof. Set $Z = \operatorname{cl}(f(\operatorname{Spec} A))$. First, let $\mathfrak{p} \in f(\operatorname{Spec} A)$, then $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Spec} A$. Since $0 \in \mathfrak{q}$, we have that $\phi^{-1}(0) = \ker \phi \subset \mathfrak{p}$, hence $\mathfrak{p} \in \mathbb{V}(\ker \phi)$. It follows that $f(\operatorname{Spec} A) \subset \mathbb{V}(\ker \phi)$ hence $Z \subset \mathbb{V}(\ker \phi)$. Now by definition:

$$Z = \bigcap_{f(\operatorname{Spec} A) \subset \mathbb{V}(I)} \mathbb{V}(I)$$

[87] Which is topologically equivalent to the map $f|_{\mathbb{V}(\mathfrak{q})}$.

If $f(\operatorname{Spec} A) \subset \mathbb{V}(I)$, then $I \subset \phi^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Let $b \in I$, then $\phi(b) \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, so $\phi(b) \in \sqrt{\langle 0 \rangle}$, i.e. there exists some n such that $\phi(b)^n = 0$. This however, implies that $b \in \sqrt{\ker \phi}$, hence $I \subset \sqrt{\ker \phi}$, and we have that $\mathbb{V}(I) \supset \mathbb{V}(\ker \phi)$. It follows that $\mathbb{V}(\ker \phi) = Z$ as desired. □

Lemma 3.10.6. *Let $f : X \to Z$ be a surjective*[88] *morphism of schemes, and $g : Y \to Z$ any other morphism. Then the base change $X \times_Z Y \to Y$ is surjective.*

Proof. Let $y \in Y$, then we need to show that the fibre:

$$\pi_Y^{-1}(y) = \operatorname{Spec} k_y \times_Y (Y \times_Z X)$$

is not empty. Note that:

$$\operatorname{Spec} k_y \times_Y (Y \times_Z X) \cong \operatorname{Spec} k_y \times_Z X$$

Let $z = g(y)$, then we also have that:

$$f^{-1}(z) \times_{k_z} \operatorname{Spec} k_y \cong (X \times_Z k_z) \times_{k_z} \operatorname{Spec} k_y \cong g^{-1}(y)$$

where the morphism making $\operatorname{Spec} k_y$ a k_z scheme comes from composing the stalk map $g_y : (\mathscr{O}_Z)_z \to (\mathscr{O}_Y)_y$ with the projection $\pi_z : (\mathscr{O}_Y)_y \to k_y$. Since g is a morphism of locally ringed spaces, this gives rise to a field morphism $k_z \to k_y$, which we take to induce the structural morphism of $\operatorname{Spec} k_y$ as a k_z scheme.

Now since $f^{-1}(z)$ is not empty, we have that there is a non empty affine open $U = \operatorname{Spec} A \subset f^{-1}(z)$. It thus suffices to show that $\operatorname{Spec} A \otimes_{k_z} k_y$ is nonempty. We claim that $A \otimes_{k_z} k_y$ is a nonzero ring, indeed since $A \neq 0$ we have that A is a non zero k_z vector space. Any k_z basis then extends to a k_y basis for $A \otimes_{k_z} k_y$ of the same cardinality, hence $A \otimes_{k_z} k_y$ cannot be the zero vector space. Since every ring has a maximal ideal, it follows that that $\pi_Y^{-1}(y)$ is non empty implying the claim. □

We now prove the first major result of the section:

Theorem 3.10.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is integral if and only if f is affine, and universally closed.*

Proof. Suppose f is integral, then f is automatically affine, so it suffices to show f is universally closed. Since f is integral, it's base change is integral by Proposition 3.9.2, so it suffices to show that an integral morphism is closed. Clearly, it then suffices to show this in the case $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} B$, but this follows from the fact Going Up holds for integral morphism, and Proposition 3.10.1.

Now suppose that f is affine and universally closed. It again clearly suffices to show that f is integral in the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so let $\phi : B \to A$ be the morphism inducing f. We want to show that for all $a \in A$, there exists a monic polynomial $p \in B[x]$ such that p lies in the kernel of the map $\operatorname{ev}_a : B[x] \to A$, given by sending x to a. Consider the composition:

$$\psi : B[x] \to A[x] \to A[x]/\langle ax - 1 \rangle$$

Let $\beta \in \ker \psi$, then with $\beta = \sum_i b_i x^i$, there exists some polynomial $q \in A[x]$ such that:

$$\sum_i \phi(b_i) x^i = (ax - 1)q$$

Let $q = \sum_i c_i x^i$, then in particular we must have that:

$$\phi(b_i) = a \cdot c_{i-1} - c_i$$

If $\deg q = d$, we claim that:

$$p = \sum_{i=0}^{d} b_i x^{d+1-i} \in \ker \operatorname{ev}_a$$

[88] Set theoretically.

We rewrite the sum as follows:

$$\begin{aligned}\sum_{i=0}^{d} \phi(b_i)x^{d+1-i} =& \sum_{i=0}^{d}(a \cdot c_{i-1} - c_i)x^{d+1-i} \\ =& a\sum_{i=0}^{d} c_{i-1}x^{d+1-i} - x\sum_{i=0} c_i x^{d-i}\end{aligned}$$

Now note that the $c_{-1} = 0$, so we we can rewrite the first sum as:

$$\begin{aligned}\sum_{i=0}^{d} \phi(b_i)x^{d+1-i} =& a\sum_{i=0}^{d} c_i x^{d-i} - x\sum_{i=0} c_i x^{d-i} \\ =& (a-x)\cdot \sum_{i=0} c_i x^{d-i}\end{aligned}$$

which certainly maps to zero under the morphism $B[x] \to A$ sending x to a, hence $p \in \ker \mathrm{ev}_a$. Moreover, if $b_0 = 1$ then p is monic, which would imply A is integral over B. It thus suffices to show that $\ker\psi$ contains a β satisfying $b_0 = 1$.

We claim this is equivalent to $\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$ being empty. Certainly, if $\beta \in \ker\psi$ with $b_0 = 1$ then $\ker\psi + \langle x\rangle = B[x]$. If $\ker\psi + \langle x\rangle = B[x]$, then that means $1 \in \ker\psi + \langle x\rangle$ hence:

$$1 = \beta + xg$$

for $\beta \in \ker\psi$, and $g \in B[x]$. However this clearly implies that $b_0 = 1$.

Note that the morphism $\varphi : B[x] \to A[x]$ is induced by the following diagram:

$$\begin{tikzcd}
A \otimes_{\mathbb{Z}} \mathbb{Z}[X] & & \\
& B \otimes_{\mathbb{Z}} \mathbb{Z}[X] \arrow[ul, "\exists!\varphi"'] & \mathbb{Z}[X] \arrow[l, "\imath_{\mathbb{Z}[x]}"'] \arrow[ull, bend right, "\mathrm{Id}"'] \\
& B \arrow[u, "\imath_B"] \arrow[uul, bend left, "\imath_A \circ \phi"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

By Theorem 3.1.1, we have that φ induces a unique morphism $f' : \operatorname{Spec} A[x] \to \operatorname{Spec} B[x]$ which is universally closed. We claim that $f'(\mathbb{V}(ax-1)) = \mathbb{V}(\ker\psi)$; note that $f'|_{\mathbb{V}(ax-1)}$ is induced by ψ, hence the closure of the image of $f'|_{\mathbb{V}(ax-1)}$ is equal to $\mathbb{V}(\ker\psi)$ by Lemma 3.10.5. However, f' is a closed map, so it's restriction to any closed set is a closed map, hence $f'(\mathbb{V}(ax-1)) = \mathbb{V}(\ker\psi)$ as desired.

We claim that:

$$(B[x]/\ker\psi) \otimes_{B[x]} B \cong B[x]/(\ker\psi + \langle x\rangle)$$

Indeed, the morphism $\mathrm{ev}_0 : B[x] \to B$ is what makes B a $B[x]$ algebra, hence by our work in Lemma 3.1.2:

$$(B[x]/\ker\psi) \otimes_{B[x]} B \cong B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

It thus suffices to show that:

$$B/\langle \mathrm{ev}_0(\ker\psi)\rangle \cong B[x]/(\ker\psi + \langle x\rangle)$$

Consider the composition:

$$B \hookrightarrow B[x] \to B[x]/(\ker\psi + \langle x\rangle)$$

and note that if $b \in \langle \mathrm{ev}_0(\ker\psi)\rangle$, then:

$$b = \sum_i b_i p_i(0)$$

where $p_i \in \ker\psi$. If we consider b as an element in $b[x]$, then b is in $\ker\psi + \langle x\rangle$ as it is given by:

$$\sum_i b_i p_i - \sum_i b_i(p_i - p_i(0))$$

where clearly each $p_i - p_i(0) \in \langle x\rangle$. It follows that this factors through the quotient to give us a well defined homomorphism:

$$F : B/\langle \mathrm{ev}_0(\ker\psi)\rangle \longrightarrow B[x]/(\ker\psi + \langle x\rangle)$$

Now consider the composition:

$$B[x] \to B \to B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

If $p \in \ker\psi + \langle x\rangle$, then p can be written as:

$$p = q + xp'$$

where $q \in \ker\psi$, and $p' \in B[x]$. It follows that $q(0) \in \langle \mathrm{ev}_0(\ker\psi)\rangle$ hence this map also factors through the quotient to yield a well defined homomorphism:

$$G : B[x]/(\ker\psi + \langle x\rangle) \longrightarrow B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

Now let $[p] \in B[x]/(\ker\psi + \langle x\rangle)$, then:

$$G([p]) = [p(0)] \in B/\langle \mathrm{ev}_0(\ker\psi)\rangle$$

while:

$$F([p(0)]) = [p(0)] \in B[x]/(\ker\psi + \langle x\rangle)$$

However:

$$[p] - [p(0)] \in \langle x\rangle$$

so $F \circ G = \mathrm{Id}$. Clearly $G \circ F = \mathrm{Id}$, so the two are isomorphic as desired. It follows that the following diagram is Cartesian:

$$\begin{tikzcd}
\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} B[x]/\ker\psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

Moreover, we claim that the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle \arrow[r] \arrow[d] & \operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A[x]/\langle ax-1\rangle \arrow[r] & \operatorname{Spec} B[x]/\ker\psi \arrow[r] & \operatorname{Spec} B[x]
\end{tikzcd}$$

The right square is Cartesian, so we need only show the left square commutes, but this is equivalent to the following diagram commuting:

$$\begin{tikzcd}
B \otimes_{B[x]} A[x]/\langle ax-1\rangle & B[x]/(\ker\psi + \langle x\rangle) \arrow[l, "\imath_B"'] \\
A[x]/\langle ax-1\rangle \arrow[u, "\imath_A"] & B[x]/\ker\psi \arrow[l] \arrow[u]
\end{tikzcd}$$

However, $0 \in B$ is equal to $\mathrm{ev}_0(x)$, hence in $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$:

$$0 \otimes [a] = \mathrm{ev}_0(x) \otimes 1 = 1 \otimes [ax] = 1 \otimes 1$$

so $0 = 1$, and $B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is the zero ring. It follows that the left square trivially commutes, so by Lemma 2.3.4 the left square is Cartesian. Now note, that the morphism:

$$\operatorname{Spec} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/\ker\psi$$

is surjective as it is given by $f'|_{\mathbb{V}(ax-1)}$ with restricted image, so by Lemma 3.10.6 we have that the morphism:

$$\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle \to \operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$$

is also surjective. However, $\operatorname{Spec} B \otimes_{B[x]} A[x]/\langle ax-1\rangle$ is empty, hence $\operatorname{Spec} B[x]/(\ker\psi + \langle x\rangle)$ is also empty, so by our earlier remarks $\operatorname{Spec} A \to \operatorname{Spec} B$ is integral as desired. □

We now proceed with showing that all finite morphisms are proper, though much of the leg work has already been covered. We first need the following immediate result:

Lemma 3.10.7. *Let $f : X \to Y$ be affine, then f is separated.*

Proof. Since the property of being separated is local on target, and f is affine, it suffices to show this in the case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. However this clear by Example 3.6.2, hence f is separated. □

The above borderline immediately implies the following:

Theorem 3.10.2. *Let $f : X \to Y$ be a morphism. Then f is finite if and only if it is affine and proper.*

Proof. Suppose f is finite, then f is automatically affine, and integral. It follows that f is separated by Lemma 3.10.7, and universally closed by Theorem 3.10.1. Moreover, f is of finite type as every finite morphism is automatically finite[89]. It follows that f is affine and proper.

Now suppose f is affine and proper, then f is affine and universally closed so it is integral by Theorem 3.10.1. Since f is of finite type, we then obtain that f is finite by Proposition 3.9.1, implying the claim. □

3.11 Quasicompact and Quasiseparated Morphisms

In this section we discuss quasicompact and quasiseparated morphisms. These can be thought of weaker versions of the finite type and separated hypothesis, but, as we will see, a quasicompact-quasiseparated hypothesis, often denoted qcqs, will be extremely fruitful. The definition of a quasicompact morphism is obvious:

Definition 3.11.1. Let $f : X \to Y$ be a morphism of schemes, then f is **quasicompact** if for every quasicompact open set $U \subset Y$, $f^{-1}(U)$ is quasicompact. If X is a Z-scheme, we say that X is a **quasicompact Z-scheme**, or is **quasicompact over Z** if the structural morphism $f : X \to Z$ is quasicompact.

Note that obviously a scheme is quasicompact as a topological if and only if it is a quasicompact $\mathbb{Z}$ scheme. Moreover, if Z is a quasicompact topological space, then every quasicompact Z scheme is quasicompact as topological space. We prove the standard results about morphisms:

Lemma 3.11.1. *Quasicompact morphisms are:*

a) Closed under composition.

b) Stable under base change.

c) Local on target.

[89] In particular if A is finitely generated as B module, then it is finitely generated as a B algebra by the same generating set.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasicompact morphism, and suppose that $U \subset Z$ is a quasicompact open subset of Z. Then, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ which is a quasicompact open subset of as g and f are both quasicompact. It follows that $g \circ f$ is a quasicompact morphism proving a).

Let $f : X \to Z$ be a quasicompact morphism, and $g : Y \to Z$ any morphism. We need to show that $\pi_Y : X \times_Z Y \to Y$ is a quasicompact morphism. Let $U \subset Y$ be quasicompact, then we want to show that $\pi_Y^{-1}(U) = X \times_Z U$ is also quasicompact. Since U is quasicompact, we know that $g(U)$ is quasicompact, and there thus exist finitely many open affines $V_1, \ldots, V_n \subset Z$ such that:

$$g(U) \subset \bigcup_{i=1}^{n} V_i = V$$

Note that V is quasicompact as it is a finite union of quasicompact spaces. It follows that:

$$\pi_Y^{-1}(U) = f^{-1}(V) \times_V U$$

where we know that $f^{-1}(V)$ is quasicompact because f is quasicompact. Let $\{W_{ij}\}$ and $\{U_{ik}\}$ be finite affine open covers of $f^{-1}(V)$ and U such that $f(W_{ij}) \subset V_i$ and $g(U_{ik}) \subset V_i$ respectively. It follows that $\pi_Y^{-1}(U)$ is a finite union of affine schemes, and thus a finite union of quasicompact spaces, and so must be quasicompact, proving b).

For iii), let $f : X \to Y$ be a quasicompact morphism, $U \subset Y$ be an open subset of Y. We need to check that $f|_{f^{-1}(U)}$ is quasicompact; let $V \subset U$ be an open quasicompact subset, then V is an open quasicompact subset of Y, and so $f|_{f^{-1}(U)}^{-1}(V) = f^{-1}(V)$ is a quasicompact open subset of $f^{-1}(U)$, and so $f|_{f^{-1}(U)}$ is quasicompact. Now suppose that U_i an open cover of Y such that $f|_{f^{-1}(U_i)}$ is a quasicompact morphism for each i. If V is a quasicompact open set, then finitely many of the U_i cover it, and without loss generality we can assume it is the first n of the U_i. Now cover each $V \cap U_i$ open affine schemes W_{ij}; the union of all such W_{ij} is equal to V, hence there is a finite subcover of the W_{ij}. Note that each W_{ij} is quasicompact because every affine scheme is quasicompact. It follows that:

$$f^{-1}(V) = f^{-1}\left(\bigcup_{ij} W_{ij}\right) = \bigcup_{ij} f^{-1}(W_{ij})$$

Now, $f^{-1}(W_{ij}) = (f|_{f^{-1}(U_i)})^{-1}(W_{ij})$ and is thus quasicompact by hypothesis. It follows that $f^{-1}(V)$ is a finite union of quasicompact spaces, and hence quasicompact implying c). ☐

The quasiseparated condition is a hair stranger than the quasicompact one, and best viewed as weakening of the separated separated condition.

Definition 3.11.2. Let $f : X \to Z$ be a morphism of schemes, then f is **quasiseparated** if the diagonal map $\Delta : X \to X \times_Z X$ is quasicompact. A Z-scheme X is a **quasiseparated Z scheme** if the structural morphism is quasiseparated. A scheme is **quasiseparated** if it is quasiseparated as $\mathbb{Z}$ scheme.

As with the quasicompact condition we prove the standard results about morphisms:

Lemma 3.11.2. *Quasiseparated morphisms are:*

a) Closed under composition.

b) Stable under base change.

c) Local on target.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasiseparated morphisms. We write $\Delta : X \to X \times_Z X$ for the diagonal map we wish to show is quasicompact, and $\Delta_X : X \to X \times_Y X$, $\Delta_Y : Y \to Y \times_Z Y$ for the morphisms we know to be quasicompact. We emulate the proof of Proposition 3.6.3; from Theorem 2.3.1 we have the following cartesian square:

$$\begin{tikzcd}
X \times_Y X \arrow[r, "\psi"] \arrow[d, "f \circ \pi_X"'] & X \times_Z X \arrow[d, "f \times f"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_Z Y
\end{tikzcd}$$

where ψ is the morphism coming from the following diagram:[90]

$$\begin{tikzcd}
X \times_Y X \arrow[dr, "\psi"] \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_X"'] & & \\
& X \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "g\times f"] \\
& X \arrow[r, "g\circ f"] & Z
\end{tikzcd}$$

We claim that $\Delta = \psi \circ \Delta_X$; however this is obvious as:

$$\pi_X \circ \psi \circ \Delta_X = \mathrm{Id}_X \circ \pi_X \circ \Delta_X = \mathrm{Id}_X$$

so $\psi \circ \Delta_X$ makes the diagram defining Δ commute. Since ψ is the base change of Δ_Y and thus quasicompact, and Δ_X is quasicompact by assumption, we have that Δ is quasicompact by Lemma 3.11.1. This proves a).

For b), let $f : X \to Z$ be quasiseparated, and $g : Y \to Z$ another morphism. We need to show that $\pi_Y : X \times_Z Y \to Y$, and so need to show that the diagonal morphism:

$$X \times_Z Y \longrightarrow (X \times_Z Y) \times_Y (X \times_Z Y)$$

is quasicompact. Note that:

$$\begin{aligned}(X \times_Z Y) \times_Y (X \times_Z Y) &\cong X \times_Z (Y \times_Y X) \times_Z X \\ &\cong X \times_Z X \times_Z Y\end{aligned}$$

It follows that the diagonal map is, up to isomorphism, equal to $\Delta_X \times \mathrm{Id}_Y$ which is quasicompact by Theorem 3.1.2, probing b).

For c), suppose $f : X \to Y$ is quasiseparated, and $U \subset Y$ is an open subset. The diagonal morphism $f^{-1}(U) \to f^{-1}(U) \times_U f^{-1}(U)$ is given by $\Delta|_{f^{-1}(U)}$, which is quasicompact by Lemma 3.11.1. If f is a morphism such that for an open cover $\{U_i\}$ of Y, we have that each $f|_{f^{-1}(U_i)}$ is quasiseparated, then

$$\Delta|_{f^{-1}(U_i)} : f^{-1}(U_i) \longrightarrow f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$$

Since the $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$ cover $X \times_Y X$ it follows by Lemma 3.11.1 that Δ is quasicompact, hence c). □

There is an equivalent, more topological formulation of a morphism of schemes being quasiseparated which mimics Proposition 3.6.4. We explore this with the following result:

Proposition 3.11.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is quasiseparated if and only if for every quasicompact open set $U \subset Y$, and quasicompact opens V_1 and V_2 of X which map into U, we have that $V_1 \cap V_2$ is quasicompact.*

Proof. Suppose that f is quasiseparated, and let U, V_1 and V_2 be as stated. Since $V_1 \times_U V_2$ is quasicompact, and f is quasiseparated, we have that $\Delta^{-1}(V_1 \times_U V_2)$ is quasicompact. Lemma 3.6.2 then implies that $V_1 \cap V_2$ is quasicompact.

Now suppose that for ever quasicompact open set $U \subset Y$, and every quasicompact opens V_1 and V_2 which map into U we have that $V_1 \cap V_2$ is quasicompact. We need to show that the diagonal morphism Δ is quasicompact. Let $W \subset X \times_Y X$ be a quasicompact open subset; then there is a finite open cover

[90] Abuse of notation alert! As always we are using π_X to refer to multiple maps.

of W by affine schemes of the form $V_{ij} \times_{U_i} V_{ik}$, where V_{ij} and V_{ik} map into U_i. Note that U_i, V_{ij} and V_{ik} are quasicompact for all i, j, and k. We have that:

$$\Delta^{-1}(W) = \bigcup_{ijk} \Delta^{-1}(V_{ij} \times_{U_i} V_{ik}) = \bigcup_{ijk} V_{ij} \cap V_{ik}$$

which is a finite union of quasicompact sets, and is thus quasicompact. □

Note that the above implies that a scheme is quasiseparated over $\mathbb{Z}$, or any other quasicompact space for that matter, if and only if the intersection of any two open quasicompact sets is quasicompact. We fix some nomenclature: by a scheme X qcqs over Z, we mean a scheme which is quasicompact and quasiseparated over Z. If we do not make the base scheme explicit, we will always mean $\operatorname{Spec}\mathbb{Z}$, which will in turn imply that X is a quasicompact scheme such that every intersection of quasicompact opens is again quasicompact.

With the above work on the quasiseparated condition, we may now shed light on why the qcqs hypothesis will be so fruitful: it will allow us to consider finite open covers whose intersections also admit finite open covers. Our first example of this fruitfulness will be in next section on the valuative criterion, but we will again employ this hypothesis in our chapter on $\mathscr{O}_X$ modules. For the moment, we explore some examples and non examples:

Example 3.11.1. A separated morphism is quasiseparated. In particular, since closed embeddings, open embeddings, proper morphisms, and affine morphisms[91] are separated, they are also quasiseparated.

Moreover, morphisms of finite type, closed embeddings, proper morphisms, and affine morphisms are all quasicompact.

Note that every Noetherian scheme is a Noetherian topological space, so every subspace is quasicompact. It follows that Noetherian schemes are qcqs.

Example 3.11.2. Any scheme that is of the form $\operatorname{Proj} A$ where the irrelevant ideal A_+ is not finitely generated up to radical is quasiseparated[92] but not quasicompact.

Example 3.11.3. Let Z be the scheme obtained by gluing $X = \operatorname{Spec}\mathbb{C}[x]$ nad $Y = \operatorname{Spec}\mathbb{C}[y]$ along the affine open U_x and U_y via the isomorphism induced by $x \mapsto y$. In particular, Z is the line with a double origin from Example 2.1.4. In Example 3.6.4 we demonstrated that Z is not separated, however, it is quasiseparated. Indeed, if $U_1, U_2 \subset Z$ are quasicompact, and do not contain any copy of either origin then the lie in an open subscheme isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ which is is obviously quasiseparated over $\operatorname{Spec}\mathbb{C}$, and thus over $\operatorname{Spec}\mathbb{Z}$. It follows that $U_1 \cap U_2$ is quasicompact. Now if U_1 contains a copy of one origin, and U_2 does not, then U_1 and U_2 are again contained in a single copy of $\mathbb{A}^1_{\mathbb{C}}$, and so their intersection is again quasicompact. The same is true if U_1 and U_2 contain the same copy of the origin. Now suppose that U_1 and U_2 contain different copies of the origin, then their intersection contains no copy of the origin, and so:

$$U_1 \cap U_2 = (U_1 \cap U_x) \cap (U_2 \cap U_y)$$

where by U_x and U_y we actually mean their image under the open embeddings. We have thus reduced this case to the original one where neither U_1 nor U_2 contain any copy of origin, hence $U_1 \cap U_2$ is quasicompact. It follows that every intersection of quasicompact opens is quasicompact and so Z is quasiseparated. Since Z is obviously quasicompact, Z is qcqs.

Example 3.11.4. Let $X = \operatorname{Spec} k[x_0, \dots]$ and $Y = \operatorname{Spec} k[y_0, \dots]$, and Z be the scheme glued along the open sets $\mathbb{V}(\langle x_0, \dots \rangle)^c$ and $\mathbb{V}(\langle y_0, \dots \rangle)^c$ induced by the map $x_i \mapsto y_i$. In particular, Z is quasicompact, but not quasiseparated because $X \cup Y = \mathbb{V}(\langle x_0, \dots \rangle)^c \subset Z$ is the infinite plane with the origin removed. This is equal to the infinite union of the distinguished opens U_{x_i} which has no finite subcover, and so cannot be quasicompact.

Example 3.11.5. Consider $X = \operatorname{Spec} k[x_0, \dots]$, and let $U = \mathbb{V}(\langle x_0, \dots \rangle)^c$. U is the infinite union of distinguished opens U_{x_i} and so is not quasicompact as above. It follows that $\imath : U \to X$ is not a quasicompact morphism, and so open embeddings are not necessarily quasicompact.

[91] And thus finite and integral morphisms

[92] As it is separated over $\operatorname{Spec} A_0$, and thus over $\operatorname{Spec}\mathbb{Z}$ by Example 3.6.4.

We end our discussion of qcqs schemes with the following result:

Lemma 3.11.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. The following hold:*

a) If $g \circ f$ is quasiseparated then so is f.

b) If $g \circ f$ is quasicompact, and g is quasiseparated, then f is quasicompact.

Proof. For a), assume that $g \circ f$ is quasiseparated, and let $V \subset Y$ be a quasicompact open subset, then in particular it's image in Z is quasicompact, hence by taking a finite open cover the image by affine opens, we can find a quasicompact open subset W which V maps ito. Now consider quasicompact opens $U_1, U_2 \subset X$ mapping into V. Since V maps into W, and $g \circ f$ is quasiseparated, we have that $U_1 \cap U_2$ is quasicompact, hence by Proposition 3.11.1 f must quasiseparated as well.

For b), note that f factors as:

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X \times f"] \arrow[rr, bend right, "f"'] & X \times_Z Y \arrow[r, "\pi_Y"] & Y \end{tikzcd}$$

The morphism π_Y is the base change of the quasicompact morphism $g \circ f$, and thus quasicompact. Note that the first morphism satisfies $\pi_X \circ 1 \times f = \mathrm{Id}$. in particular, π_X is quasiseparated as it is the base change of $g : Y \to Z$. Up to isomorphism, $1 \times f$ comes from the morphism $X \times_Y Y \to X \times_Z Y$ which fits into the following diagram:

$$\begin{tikzcd} X \times_Y Y \arrow[r, "1 \times f"] \arrow[d, "\pi_Y"] & X \times_Z Y \arrow[d, "f \times \mathrm{Id}_Y"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_Z Y \end{tikzcd}$$

By Theorem 2.3.1 this square is cartesian, and so $1 \times f$ is quasicompact, as desired. It follows that f is the composition of quasicompact morphisms, and thus quasicompact itself. ☐

Corollary 3.11.1. *Let X be a scheme. If X is quasiseparated then any morphism $f : X \to Y$ is quasiseparated. If X is quasicompact, and Y is quasiseparated, then $f : X \to Y$ is also quasicompact. In particular, any morphism of qcqs schemes is qcqs.*

Proof. Let X be a quasiseparated, and $g : Y \to \operatorname{Spec} \mathbb{Z}$ the unique morphism. Then $g \circ f$ is the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ so the claim follows from Lemma 3.11.3. The same argument demonstrates the other two claims. ☐

3.12 The Valuative Criterion for Being Universally Closed

In the next sections we discuss a group of results, colloquially known as the valuative criteria, which will allow us to test whether a morphism is universally closed, or separated by examining lifting properties of certain commutative diagrams. We begin with the definition of a valuation ring:

Definition 3.12.1. A ring A is a **valuation ring** if A is an integral domain, A is local, and for every $a \in \operatorname{Frac}(A)^\times$, at least one of a or a^{-1} lies in A.

Example 3.12.1. We localize the integers at a prime ideal $\mathfrak{p} = \langle p \rangle$ for p a prime, then $\mathbb{Z}_\mathfrak{p}$ is a valuation ring. Indeed, $\mathbb{Q}$ is it's field of fractions, and if $a/b \in \mathbb{Q}$, then we can take a and b to be integers such that have no common multiples. In particular, either a is a multiple of p, b is a multiple of p or neither are multiples of p. If a is a multiple of p, then $b \notin \mathfrak{p}$ hence $a/b \in \mathbb{Z}_\mathfrak{p}$, if b is a multiple of p then $a \notin \mathfrak{p}$ hence $b/a \in \mathbb{Z}_\mathfrak{p}$, and if neither are multiples of p both a/b and b/a are in $\mathbb{Z}_\mathfrak{p}$.

Now consider $k[x, y]$ localized at $\mathfrak{m} = \langle x, y \rangle$; it's field of fractions is the ring of rational functions in two variables $k(x, y)$. This is a local integral domain, but not a valuation ring as $k(x, y)$ is in a sense 'too big'. Indeed we have that $x/y \in k(x, y)$ but neither x/y nor y/x are in $k[x, y]_\mathfrak{m}$ as both x and y lie in $\mathfrak{m}$.

It is unimportant why this is called a valuation ring, but we discuss it briefly for historical reason. If we have a field k, then a valuation is a surjective map:

$$\nu : k^\times \longrightarrow \Gamma$$

where Γ is an ordered abelian group[93] satisfying:[94]

$$\nu(xy) = \nu(x) + \nu(y) \qquad \text{and} \qquad \nu(x + y) \geq \min\{\nu(x), \nu(y)\}$$

The ring associated to the valuation is the union:

$$A_\nu = \{0\} \cup \{a \in k^\times : \nu(a) \geq 0\}$$

It is easy to see from the definition of that any ring associated to a valuation is a valuation ring with field of fractions k. In particular, the unique maximal ideal of A_ν is given by:

$$\mathfrak{m}_\nu = \{0\} \cup \{a \in k^\times : \nu(a) > 0\}$$

If $\nu(a) = 0$, then we have that:

$$0 = \nu(a \cdot a^{-1}) = \nu(a) + \nu(a^{-1}) = \nu(a^{-1})$$

so every element out side of $\mathfrak{m}_\nu$ is a unit in A_ν. It follows that $\mathfrak{m}_\nu$ is maximal, and in fact unique because any other maximal ideal not contained in $\mathfrak{m}_\nu$, and thus equal to $\mathfrak{m}_\nu$, would contain units. The following proof that these are equivalent ways of thinking of valuation rings is due to Krull:

Lemma 3.12.1. *Let k be a field, then there is a bijection:*

$$\left\{\begin{matrix} \textit{valuations from } k \textit{ up to} \\ \textit{order preserving isomorphisms of } \Gamma \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{valuation rings with} \\ \textit{fraction field } k \end{matrix}\right\}$$

Proof. The assignment $\nu \mapsto A_\nu$ is one direction of this bijection. Let A be a valuation ring with field of fractions k, and set $\Gamma = k^\times / A^\times$. This is an abelian group, and we make the unfortunate notational choice of setting:

$$[x] + [y] = [x \cdot y]$$

We give it a total order via:

$$[x] \leq [y] \Leftrightarrow y/x \in A \smallsetminus \{0\}$$

with equality if $a/b = 1$. Note that this well defined, if we choose different class representatives $x \cdot a_1$ and $y \cdot a_2$, then $ya_1/xa_2 = (y/x) \cdot (a_1/a_2)$, but $a_i \in A^\times$ hence $(y/x) \cdot (a_1/a_2) \in A^\times$. We check that this is a total order; it is clearly reflexive, and if we have that $[x] \leq [y]$ and $[y] \leq [z]$, then $x/y, y/z \in A^\times$, hence $x/z \in A^\times$ so $[x] \leq [z]$. If $[x] \leq [y]$ and $[y] \leq [x]$, then x/y and y/x lie in A, and hence $x/y \in A^\times$, so $[x] - [y] = [1] = 0 \in \Gamma$, implying that $[x] = [y]$. Finally we check that for all $[x]$ and $[y]$ we have either $[x] \leq [y]$ or $[y] \leq [x]$, however this follows from Definition 3.12.1, as if $x/y \notin A$, then $y/x \in A$ and vice versa.

We define ν_A to be the quotient map $k^\times \to k^\times / A^\times$. This satisfies $\nu_A(xy) = \nu_A(x) + \nu_A(y)$ by construction, hence we need to show that $[x+y] \geq \min\{[x], [y]\}$. Without loss of generality, suppose that $[x] \leq [y]$, then we have that $(x + y)/x = 1 + y/x \in A$ as $[x] \leq [y]$, implying the claim.

All that remains to be shown is that the assignments $\phi : \nu \mapsto A_\nu$ and $\psi : A \mapsto \nu_A$ are inverses of each other. We first show that for a valuation $\nu : k^\times \to \Gamma$:

$$\ker \nu = A_\nu^\times$$

[93] I.e. the integers. In particular, any ordered abelian group has to be compatible with addition and thus cannot have torsion.

[94] The following axioms, along with the fact that Γ should be torsion free, imply that $\nu(1)$ and $\nu(-1)$ are zero.

for any valuation ν. We know that anything in the kernel of ν is a unit in $A_\nu^\times$. Suppose that $a \in A_\nu^\times$, then in particular, $a \notin \mathfrak{m}_\nu$ hence $a \in A_\nu^\times$. It follows that up to $\Gamma \cong k^\times/A_n^\times$, and that ν is up to isomorphism the quotient morphism. It is obvious that this isomorphism is order preserving, hence we have that $\psi \circ \phi$ is the identity.

Now suppose that A is a valuation ring, then we want to show that:

$$A = \{a \in k^\times : [a] \geq [1] \in k^\times/A^\times\}$$

However, this is obvious as if $a \in A$, then $a/1 \in A$ hence $[a] \geq [1]$, and if $[a] \geq [1]$ then $a/1 \in A$. This implies that $\phi \circ \psi = \mathrm{Id}$, and hence the claim. □

Example 3.12.2. Continuing with $\mathbb{Z}_\mathfrak{p}$, where $\mathfrak{p} = \langle p \rangle$ for some prime, we claim that $\mathbb{Q}^\times/\mathbb{Z}_\mathfrak{p}^\times \cong \mathbb{Z}$, where $\mathbb{Z}$ has its totally order additive group structure. . Every non zero rational number a/b can be written in lowest terms, and since we can factor out powers of p from both a/b, every element of $\mathbb{Q}$ can be written uniquely as $p^n a'/b'$, with $n \in \mathbb{Z}$. We therefore define a surjective group homomorphism $\mathbb{Q}^\times \to \mathbb{Z}$ by sending $p^n a'/b'$ to n. If $a/b \in \mathbb{Z}_\mathfrak{p}^\times$, then neither a nor b have a power of p hence we we see that the kernel of this map is obviously $\mathbb{Z}_\mathfrak{p}^\times$. It follows that $\mathbb{Q}^\times/\mathbb{Z}_\mathfrak{p}^\times \cong \mathbb{Z}$ as desired.

The main connection that valuation rings have to algebraic geometry comes from a different but further equivalent characterization. We need the following definition:

Definition 3.12.2. Let A and B be local subrings of a field k, then B **dominates** A if $A \subset B$, and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$. We say that a local integral domain is **maximally dominant** if for all local subrings $B \subset \mathrm{Frac}(A)$, if B dominates A or A dominates B then $B = A$.

We first show that any maximally dominant ring is integrally closed:

Lemma 3.12.2. *Let A be maximally dominant, then A is integrally closed.*

Proof. We need to show that:

$$A = \{x \in k : x \text{ is integral over } A\}$$

Suppose that $x \in k$ is integral over A, and let B be the subring of k generated by A and x. Then the inclusion $\imath : A \to B$ is integral, hence by Lemma 3.10.3 hence there is a prime ideal $\mathfrak{p} \in \operatorname{Spec} B$ lying over the unique maximal ideal $\mathfrak{m} \in \operatorname{Spec} A$. We have that $B_\mathfrak{p}$ is naturally a subring of k, and we claim that $B_\mathfrak{p}$ dominates A. Indeed, we need only show that:

$$\mathfrak{m} = A \cap \mathfrak{m}_\mathfrak{p}$$

In particular, if $\pi : B \to B_\mathfrak{p}$ is the localization map, then since π and $\imath$ are both naturally viewed as inclusions, we have that:

$$\mathfrak{m} = \imath^{-1}(\mathfrak{p}) = A \cap \mathfrak{p} = A \cap \pi^{-1}(\mathfrak{m}_\mathfrak{p}) = A \cap B \cap \mathfrak{m}_\mathfrak{p} = A \cap \mathfrak{m}_\mathfrak{p}$$

so $B_\mathfrak{p}$ dominates A. It follows that either $B_\mathfrak{p} = A$ or k. If $B_\mathfrak{p} = A$ then $x \in A$, hence every element integral over A lies in A, and A is integrally closed. □

We now show that a ring is a valuation ring if and only if it is maximally dominant:

Lemma 3.12.3. *A ring is maximally dominant if and only if it is a valuation ring.*

Proof. Suppose that A is a valuation ring with maximal ideal $\mathfrak{m}$, and set $k = \mathrm{Frac}(A)$. Let $B \subset k$ be a local ring with maximal ideal $\mathfrak{m}_B$, and suppose that B dominates A. Then we have that:

$$\mathfrak{m} = A \cap \mathfrak{m}_B$$

Let $b \in B$, if $b \notin A$ then $b^{-1} \in A$ by definition. However, if $b \notin A$ then $b^{-1} \in \mathfrak{m}$ as well as b^{-1} is not a unit in A. It follows that $b^{-1} \in \mathfrak{m}_B$ a contradiction, hence $b \in A$. It follows that $B \subset A$ and thus $A = B$, hence A is maximally dominant.

Suppose that A is maximally dominant, and let $x \in K$. We want to show that at least one of x or x^{-1} are in A. Assume without loss of generality that $x \notin A$, and let B be the subring of k generated by x and A. If there is a prime ideal $\mathfrak{p}$ of B lying over A then $B_{\mathfrak{p}}$ dominates A by our work in Lemma 3.12.2, implying that $B_{\mathfrak{p}} = A$. But then $B \subset B_{\mathfrak{p}} = A \subset B$ hence $A = B$ so no such $\mathfrak{p}$ can exist. Suppose that $\langle \mathfrak{m} \rangle \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} B$, then $\imath^{-1}(\mathfrak{p}) \in \operatorname{Spec} A$, and $\imath^{-1}(\langle \mathfrak{m} \rangle) = \mathfrak{m}$, hence $\mathfrak{m} \subset \imath^{-1}(\mathfrak{p})$, implying that $\imath^{-1}(\mathfrak{p}) = \mathfrak{m}$, which as we just mentioned is impossible. It follows that $\mathbb{V}(\langle \mathfrak{m} \rangle) \subset \operatorname{Spec} B$ is the emptyset, and thus $\langle \mathfrak{m} \rangle = B$, so we can write:

$$1 = \sum_i m_i b_i$$

for $m_i \in \mathfrak{m}$ and $b_i \in B$. Since any b_i can be written as a sum of polynomials in x with coefficients in A, and $m_i \cdot a \in \mathfrak{m}$ for all $a \in A$, we have that we can we replace the m_i and b_i to be of the form:

$$1 = \sum_{n=0}^{d} m_n x^n$$

Multiply both sides by x^{-d} to obtain that:

$$\begin{aligned} x^{-d} =& m_0 x^{-d} + m_1 x^{-d+1} + \cdots + m_{d-1} x^{-1} + m_d \\ =& m_0 x^{-d} + \sum_{n=1}^{d} m_n (x^{-1})^{d-n} \end{aligned}$$

We thus have that:

$$(1 - m_0) x^{-d} - \sum_{n=1}^{d} m_n (x^{-1})^{d-n} = 0$$

So the polynomial $p \in A[y]$ given by:

$$p(y) = (1 - m_0) y^d - \sum_{n=1}^{d} m_n (y)^{d-n}$$

satisfies $p(x^{-1}) = 0$. Moreover, we have that $1 - m_0 \notin \mathfrak{m}$ hence it is a unit, and so $p(y)/(1 - m_0)$ is a monic polynomial which has x^{-1} as a zero. Therefore x^{-1} is integral over A, and by Lemma 3.12.2 lies in A. □

We now begrudgingly introduce the following terminology to describe some very simple phenomena. Recall that on a scheme there are non-closed points, i.e. elements $x \in X$ such that $\overline{\{x\}} \neq \{x\}$. We say that y is a *specialization* of x, or x is *generalization* of y if $y \in \overline{\{x\}}$. We denote by $x \rightsquigarrow y$ y is a specialization of x, or x is a generalization . In essence, a specialization is a choice of point in side the closure another point. Now note that if A is a valuation ring, then there is unique specialization of the generic point to the unique maximal ideal, $\eta \rightsquigarrow \mathfrak{m}$. In fact, we will show that a choice of a specialization in a scheme X is equivalent to morphism $\operatorname{Spec} A \to X$ for some valuation ring A. Once we show this, we will demonstrate the connection between valuation rings, and a morphism being universally closed.

We need the following lemma:

Lemma 3.12.4. *Let k be a field, and $B \subset k$ a local subring, then there is a valuation ring with fraction field k dominating B.*

Proof. We partially order the set L_B of local subrings of k which dominate B by:

$$A_1 < A_2 \Leftrightarrow A_2 \text{ dominates } A_1$$

Let $\{A_i\}_{i \in I}$ be a totally ordered subset of L_B, then we set:

$$C = \bigcup_{i \in I} A_i$$

If $A_i < A_j$ then $A_i \subset A_j$ hence the above set is naturally a ring. Moreover, it the ideal:

$$\mathfrak{m}_C = \bigcup_i \mathfrak{m}_{A_i}$$

is an ideal of C. It is maximal because if $\mathfrak{m}_C \subset I$, then $I \cap A_i$ is an ideal containing $\mathfrak{m}_{A_i}$, and is thus equal to $\mathfrak{m}_{A_i}$ or A_i. If $I \cap A_i = A_i$, then $1 \in I$ and $I = C$, otherwise it follows that $\mathfrak{m}_C = C$ hence $\mathfrak{m}_C$ is maximal. Let $\mathfrak{n}$ be any other maximal ideal of C, then $\mathfrak{n} \cap A_i$ is a prime ideal of A_i and so contained in $\mathfrak{m}_{A_i}$ for all i. It follows that $\mathfrak{n} \subset \mathfrak{m}_A$ and thus $\mathfrak{n} = \mathfrak{m}_C$, so C is a local ring. It is obvious that C dominates every A_i by definition, and also dominates B.

By Zorn's lemma, it follows that there exists a maximal element of L_B, hence we need to show that if A is a maximal element of L_B then it has fraction field k. In particular, by the contrapositive, it suffices to show that if A does not have fraction field $\mathrm{Frac}(A) \subsetneq k$, then there exists a local ring $C \neq A$ which dominates A. Let $t \in k$ but not in $\mathrm{Frac}(A)$, then let $A[t]$ denote the A algebra generated by t.[95] This is still a subring of K, and the ideal $\langle t, \mathfrak{m} \rangle$ is obviously maximal; it is unique because every other element of $A[t]$ is invertible. It follows that $A[t]_{\langle t, \mathfrak{m} \rangle}$ is a local ring not equal to A contained in k which dominates A.

Now suppose that t is not transcendental, then there is a polynomial $p \in A[x]$ such that $p(t) = 0$. If:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Then if we instead take the polynomial:

$$q(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

we have that $a_n t$ satisfies:

$$q(a_n t) = a_n^{n-1} p(t) = 0$$

so there exists some a such that at is integral over A. Let A' be the subring generated by A and at, then $A \hookrightarrow A'$ is an integral extension so by Lemma 3.10.3 there exists a $\mathfrak{p} \subset A'$ lying over $\mathfrak{m}$. We take $C = A'_{\mathfrak{p}}$, then C dominates A by the same argument in Lemma 3.12.2, and is obviously not equal to A. $\square$

In Proposition 2.1.2, we were able to construct a bijection between ring homomorphisms $A \to \mathscr{O}_X(X)$ and scheme morphisms $X \to \operatorname{Spec} A$. In order to show that certain valuation rings map into schemes, we will need construct morphism $\operatorname{Spec} A \to X$ when A is a local ring.

Lemma 3.12.5. *Let X be a scheme, and A local ring. Then the following hold:*

a) Then there is bijection between $\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$ and pairs (x, ϕ) where $x \in X$ and $\phi : \mathscr{O}_{X,x} \to A$ is a local ring morphism.

b) $x \rightsquigarrow y$ if and only if x is in the image of $\operatorname{Spec} \mathscr{O}_{X,y} \to X$.

Proof. Let $f \in \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$, and let $x = f(\mathfrak{m})$. Then we have the induced local ring homomorphism $f_{\mathfrak{m}} : \mathscr{O}_{X,x} \to A_{\mathfrak{m}}$, but $A_{\mathfrak{m}} = A$ as every element outside of $\mathfrak{m}$ is already invertible. It follows that $(f(\mathfrak{m}), f_{\mathfrak{m}})$ a a pair of the desired form.

Now let (x, ϕ) be an aforementioned pair. Let $\operatorname{Spec} B$ be an affine open containing x, and identify x with $\mathfrak{p}$. Then ϕ is of the form $B_{\mathfrak{p}} \to A$, and so we get a morphism $B \to A$ by precomposing with the localization map. This yields a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$, which gives a morphism $f_{(x,\phi)} : \operatorname{Spec} A \to X$ by post composing with the open embedding.

By construction, the stalk map $f_{(x,\phi)_{\mathfrak{m}}} : \mathscr{O}_{X,x} \cong B_{\mathfrak{p}} \to A$ is equal to ϕ up to isomorphism. Moreover, we have that $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}_{\mathfrak{p}}$, and $\pi^{-1}(\mathfrak{m}_{\mathfrak{p}}) = \mathfrak{p} = x$, hence $f_{(x,\phi)}(\mathfrak{m}) = x$.

To complete the proof, we want to show that $f_{\mathfrak{m}}$ induces the same scheme morphism f. We first show that $\operatorname{im} f$ is contained in any affine open containing $f(\mathfrak{m})$. Suppose that $f(\mathfrak{m}) \in \operatorname{Spec} B \subset X$, then

[95] This is a free A algebra as there are no polynomial relations in t.

$f^{-1}(\operatorname{Spec} B)$ contains $\mathfrak{m}$. It follows that $f^{-1}(\operatorname{Spec} B)$ is a union of distinguished opens U_a, one of which must satisfy $a \notin \mathfrak{m}$, but then a is invertible and $U_a = \operatorname{Spec} A$. It follows that the construction of $f_{(f(\mathfrak{m}),f_\mathfrak{m})}$ is independent of the chosen affine open containing $f(\mathfrak{m})$. Choose such an open affine $\operatorname{Spec} B$, then f and $f_{(f(\mathfrak{m}),f_\mathfrak{m})}$ come from ring morphisms $\phi, \psi : B \to A$. The stalk maps are the unique ones coming from localization, and so satisfy:

$$\phi = f_\mathfrak{m} \circ \pi_\mathfrak{p} \qquad \text{and} \qquad \psi = f_{(f(\mathfrak{m}),f_\mathfrak{m})_\mathfrak{m}} \circ \pi_\mathfrak{p}$$

Since $f_\mathfrak{m} = f_{(f(\mathfrak{m}),f_\mathfrak{m})_\mathfrak{m}}$ by construction, we have that $\phi = \psi$, and thus $f = f_{(f(\mathfrak{m}),f_\mathfrak{m})}$ implying a).

For b), first note that if $x \rightsquigarrow y$ in any topological space X, and $f : X \to Y$ is continuous map, then $f(x) \rightsquigarrow f(y)$ as

$$y \in \overline{\{x\}} \Rightarrow f(y) \in f(\overline{\{x\}})$$

but a map is continuous if and only $(\bar{V}) \subset \overline{f(V)}$ for all subsets $V \subset X$. By the work above, if we let $\operatorname{Spec} A$ be any affine open containing y, and set $y = \mathfrak{p} \in \operatorname{Spec} A$, then the map $\operatorname{Spec} \mathscr{O}_{X,y} \to X$ comes from the morphism $\operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A$ post composed with an open embedding. Now suppose that x is in the image of $\operatorname{Spec} A_\mathfrak{p}$, then there is some $\mathfrak{q} \in \operatorname{Spec} A_\mathfrak{p}$ such that $\mathfrak{q} \rightsquigarrow \mathfrak{m}_\mathfrak{p}$. It follows that $x \rightsquigarrow y$.

If $x \rightsquigarrow y$, then let $U = \operatorname{Spec} A$ be an affine open containing y. Suppose that $x \notin U$, then $x \in X \smallsetminus U$, which is a closed subset. It follows that $\overline{\{x\}} \subset X \smallsetminus U$, hence $y \in X \smallsetminus U$ a contradiction. Therefore, we have tha $x \in U$ as well. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals of A associated to y and x respectively. We have that $\mathfrak{q} \subset \mathfrak{p}$, so if π is the localization map then the prime ideal $\langle \pi(\mathfrak{q}) \rangle \subset A_\mathfrak{p}$ is contained in $\mathfrak{m}_\mathfrak{p}$. The morphism $\operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A$ coming from π then sends $\mathfrak{m}_\mathfrak{p}$ to $\mathfrak{p}$, and $\langle \pi(\mathfrak{q}) \rangle$ to $\mathfrak{q}$, hence x is in the image of $\operatorname{Spec} \mathscr{O}_{X,y} \to X$ implying b). □

We now prove the aforementioned and desired result regarding valuation rings:

Proposition 3.12.1. *Let X be a scheme, and $x \rightsquigarrow y$ specialization of points. Then the following hold:*

a) There exists a valuation ring A, and a morphism $f : \operatorname{Spec} A \to X$ such that $f(\eta) = x$ and $f(\mathfrak{m}) = y$.

b) Given any field extension k/k_x, we can find an f such that the induced extension k_η/k_x is isomorphic to the given one.

Proof. Fix $x \rightsquigarrow y$ and a field extension k/k_x. By part a) of the previous lemma, we have a morphism:

$$\operatorname{Spec} \mathscr{O}_{X,y} \to X$$

which for any $\operatorname{Spec} C$ containing $y = \mathfrak{p}$, comes from the morphism $\operatorname{Spec} C_\mathfrak{p} \to \operatorname{Spec} A$ given by the localization map π. As mentioned, we have that $x = \mathfrak{q}$ is in the image of $\operatorname{Spec} C_\mathfrak{p} \to \operatorname{Spec} C$. In particular, $\mathfrak{q}_\mathfrak{p} = \langle \pi(\mathfrak{q}) \rangle$ maps to $\mathfrak{q}$, and there is a morphism $C_\mathfrak{p} \to (C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ given by localizing further. However, everything invertible in $(C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ is invertible in $C_\mathfrak{q}$ and vice versa, so there is a canonical isomorphism $(C_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}} \cong C_\mathfrak{q}$. It follows that there is a canonical morphism $C_\mathfrak{p} \to k_x = C_\mathfrak{q}/\mathfrak{m}_\mathfrak{q}$. In particular, the above implies that the morphism $\operatorname{Spec} k_x \to X$ taking η to x factors as:

$$\operatorname{Spec} k_x \to \operatorname{Spec} \mathscr{O}_{X,y} \to X$$

which now takes η to x, and $\mathfrak{m}_y$ to y. We thus have a morphism of local rings:

$$\mathscr{O}_{X,y} \to k_x \to k$$

Let B be the image of $\mathscr{O}_{X,y}$ in k,[96] and A be any valuation ring which dominates B and satisfies $\operatorname{Frac}(A) = k$. Such an A exists by Lemma 3.12.4.

Since A dominates B, we have that $B \subset A$, hence there is a morphism:

$$\phi : \mathscr{O}_{X,y} \to k_x \hookrightarrow A$$

[96] The image of a local ring is always a local ring.

We let $f : \operatorname{Spec} A \to X$ be the induced morphism. Since $k_\eta = A_{\langle 0\rangle} = k$, we have that k/k_x is isomorphic to k_η/k essentially by construction. We need to check that $f(\eta) = x$, and $f(\mathfrak{m}) = y$. In particular, we have that $\phi^{-1}(\eta)$ is the kernel of the morphism $\beta : C_\mathfrak{p} \to k_x$. We have the following commutative diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_\mathfrak{q} \arrow[r] & k_x \\ C_\mathfrak{p} \arrow[ur, dashed, "\alpha"] & & \end{tikzcd}$$

The kernel of $C_\mathfrak{q} \to k_x$ is:

$$\mathfrak{m}_\mathfrak{q} = \left\{ \frac{q}{a} \in C_\mathfrak{q} : q \in \mathfrak{q} \right\}$$

We obviously then have that:

$$\alpha^{-1}(\mathfrak{m}_\mathfrak{q}) = \mathfrak{q}_\mathfrak{p} = \left\{ \frac{q}{a} \in C_\mathfrak{q} : q \in \mathfrak{q} \right\}$$

It follows that η maps to $\mathfrak{q}_\mathfrak{p}$ which as discussed maps to $\mathfrak{q} = x$. Now let $\beta : C_\mathfrak{p} \to B$, and $\imath : B \to A$. We have that $\phi : C_\mathfrak{p} \to A$ factors as $\imath \circ \beta$, and that $\imath^{-1}(\mathfrak{m}) = \mathfrak{m}_B = \beta(\mathfrak{m}_\mathfrak{p})$. It follows that $\phi^{-1}(\mathfrak{m}) = \mathfrak{m}_\mathfrak{p}$ which maps to $\mathfrak{p} = y$. Given $x \rightsquigarrow y$, we have thus found a valuation ring A satisfying $a)$ and $b)$. $\square$

If $f : X \to Y$ is a morphism of schemes, and $y_1 \rightsquigarrow y_2$ is a specialization in Y, then we say that $y_1 \rightsquigarrow y_2$ *lifts along* f if for any x_1 such that $f(x_1) = y_1$ there exists a specialization $x_1 \rightsquigarrow x_2$ such that $f(x_2) = y_2$. The going up lemma can be rephrased as follows: let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be an integral morphism, then specializations lift along f. Moreover Proposition 3.10.1 can be rephrased as $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is closed if and only if specializations lifts along f. The connection between valuation rings, specializations, and a morphism being universally closed is hinted at with the following characterization of universally closed morphisms:

Proposition 3.12.2. *Let $f : X \to Y$ be a morphism of schemes then the following hold:*

a) If f is universally closed then specializations lift along any base change of f.

b) If f is quasicompact, and specializations lift along any base change of f then f is universally closed.

Proof. For $a)$ it suffices to prove that if f is closed then specializations lift along f. Indeed, if we prove this generic case, then since f is universally closed, any base change will be closed, and so specializations will lift along the base change as well. Let $y_1 \rightsquigarrow y_2$ be a specialization, and suppose that $x_1 \in X$ satisfies $f(x_1) = y_1$. Since f is continuous:

$$\overline{f(\overline{\{x_1\}})} = \overline{f(\{x_1\})}$$

But since f is closed, we have that $f(\overline{\{x\}})$ is closed, hence:

$$f(\overline{\{x_1\}}) = \overline{f(\{x_1\})} = \overline{\{y_1\}}$$

Since $y_2 \in \overline{\{y_1\}}$, there must exist an $x_2 \in \overline{\{x_1\}}$ such that $f(x_2) = y_2$. This proves $a)$.

For $b)$, by Lemma 3.11.1 any base change of f is quasicompact, so by the same argument as in $a)$ it suffices to show that f is quasicompact, and specializations lift along f, then f is closed. Let $Z \subset X$ be closed a set, then we want to show that $f(Z)$ is closed. In particular, we can equip Z with induced reduced subscheme structure, and consider the closed embedding $\imath : Z \to X$. Closed embeddings are quasicompact, hence the induced map $Z \to Y$ is quasicompact. Moreover, since closed embeddings are proper, we have that by part $a)$ specializations lift along $\imath : Z \to X$.

Note that if if $f : X \to Y$ and $g : Z \to X$ are scheme morphisms such that specializations lift along f and g then specializations lift along $f \circ g$. Indeed, suppose that $y_1 \rightsquigarrow s_2$, and z_1 satisfies $f \circ g(z_1) = y_1$. We see that $x_1 = g(z_1)$ maps to y_1 so since specializations lift along f there is an x_2 satisfying $g(z_1) \rightsquigarrow x_2$ with $f(x_2) = y_2$. Since $g(z_1) = x_1$, and specializations lift along g there is an element z_2 such that

$g(z_2) = x_2$ and $z_1 \rightsquigarrow z_2$. It follows that $f \circ g(z_2) = f(x_2) = y_2$, hence specializations lift along $g \circ f$. Moreover, if specializations lift along f, then have that $f(X)$ is closed under specializations in the sense that for all $y_1 \in f(X)$, if $y_1 \rightsquigarrow y_2$ then $y_2 \in f(X)$. Indeed, if $y_2 \notin f(X)$ but $y_1 \in f(X)$ then there would be no x_2 such that $f(x_2) = y_2$, contradicting the fact that specializations lift.

In our situation, we have that $f \circ \imath$ is a quasicompact morphism along which specializations lift, and so $f \circ \imath(Z) = f(Z)$ is also closed under specialization. We want to show that $f(Z)$ is closed, and it suffices to show that $f(Z) \cap U$ is closed for any affine open schemes $U = \operatorname{Spec} B \subset Y$. Since $f(Z)$ is closed under specialization, we claim that $f(Z) \cap U$ is closed under specialization in U. If $y_1 \rightsquigarrow y_2$ with $y_1 \in f(Z) \cap U$, then we have that $y_2 \in \overline{\{y_1\}}$ where the closure is taken with respect to the subspace topology on U. We have that $y_2 \in \overline{\{y_1\}}$ with respect to U, hence $y_2 \in \overline{\{y_1\}}$ with respect to Y, so $y_2 \in f(Z)$. It follows that $y_2 \in f(Z) \cap U$ so $f(Z) \cap U$ is closed under specialization.

By restricting to the preimage of U, we may therefore assume $f \circ \imath$ is a quasicompact morphism whose image is closed under specialization with $Y = \operatorname{Spec} B$. Since $f \circ \imath$ is quasicompact, we have that Z admits a finite affine open cover $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$. Note that:

$$f(Z) = \bigcup_i f \circ \imath(U_i) = \bigcup_i f \circ \imath|_{U_i}(U_i)$$

It follows that if we take the morphism:

$$g : \coprod_i U_i \longrightarrow Y$$

induced by the morphisms $f \circ \imath|_{U_i}$, we have that $\operatorname{im} g = f(Z)$. By Example 2.1.3, since each U_i is affine, and there are finitely many, we hae that:

$$\coprod_i U_i = \operatorname{Spec}(A_1 \times \cdots A_n)$$

so g is a morphism of affine schemes whose image is stable under specialization. We therefore need only show that if $g : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of affine schemes with $g(\operatorname{Spec} A)$ stable under specialization then $g(\operatorname{Spec} A)$ is closed. By Lemma 3.10.5 we know that $\operatorname{cl}(g(\operatorname{Spec} A)) = \mathbb{V}(\ker \phi)$, where $\phi : B \to A$ is the ring morphism inducing g. Obviously $g(\operatorname{Spec} A) \subset \mathbb{V}(\ker \phi)$ by definition. We want to show the reverse inclusion. Let $\mathfrak{p} \in \mathbb{V}(\ker \phi)$, then it suffices to a find a $\mathfrak{q} \in g(\operatorname{Spec} A)$ such that $\mathfrak{q} \subset \mathfrak{p}$ as then $\mathfrak{q} \rightsquigarrow \mathfrak{p}$. Suppose further that $b \notin \mathfrak{p}$, then $\mathfrak{p} \in U_b$; if $U_b \cap g(\operatorname{Spec} A) = \emptyset$, then $g(\operatorname{Spec} A) \subset \mathbb{V}(b)$ which is a contradiction as $\mathfrak{p} \notin \mathbb{V}(b)$. Hence we may assume that $U_b \cap g(\operatorname{Spec} A)$ is non empty for all $b \notin \mathfrak{p}$.

Since $g^{-1}(U_b) = U_{\phi(b)} \subset \operatorname{Spec} A$, the restricted morphism:

$$g|_{U_{\phi(b)}} : \operatorname{Spec} A_{\phi(b)} \longrightarrow \operatorname{Spec} B$$

has image equal to $U_b \cap g(\operatorname{Spec} A)$ which is as mentioned cannot be empty. It follows that $A_{\phi(b)}$ is not the zero ring for any $b \notin \mathfrak{p}$. Recall from Lemma 3.7.2 that:

$$A_{\mathfrak{p}} = \phi(B \setminus \mathfrak{p})^{-1} A$$

and so since $A_{\phi(b)}$ is not the zero ring for all $b \notin \mathfrak{p}$, we have that $A_{\mathfrak{p}}$ is not the zero ring as well. There is then a morphism:

$$\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A \to \operatorname{Spec} B$$

We claim that any $\mathfrak{q}' \in \operatorname{Spec} A_{\mathfrak{p}}$ must satisfy $\phi^{-1}(\pi^{-1}(\mathfrak{q}')) \subset \mathfrak{p}$. Indeed, let $b \in \phi^{-1}(\pi^{-1}(\mathfrak{q}'))$, then $\phi(b) \notin \phi(B \setminus \mathfrak{p})$, hence $b \notin B \setminus \mathfrak{p}$. Since $b \notin B \setminus \mathfrak{p}$, we have that $b \in \mathfrak{p}$, implying the claim. We take $\mathfrak{q} = \phi^{-1}(\pi^{-1}(\mathfrak{q}'))$, which then satisfies $\mathfrak{q} \in g(\operatorname{Spec} A)$, and $\mathfrak{q} \subset \mathfrak{p}$. Since $g(\operatorname{Spec} A)$ is closed under specialization, $\mathfrak{p} \in g(\operatorname{Spec} A)$, so $g(\operatorname{Spec} A) = \mathbb{V}(\ker \phi)$ and is thus closed.

This completes the proof of b), but we briefly recap. We have shown that if f is quasicompact and specializations lift along f, then for any closed set $Z \subset X$ the induced morphism $h = f \circ \imath : Z \to Y$ is a quasicompact morphism along which specializations lift. We then showed that $f(Z)$ is closed under specialization, and for any open affine $U \subset Y$, the intersection $f(Z) \cap U$ is also closed under specialization in U. Since it suffices to check $U \cap f(Z)$ is closed in U for all affine U, it suffices to consider a quasicompact morphism $h : Z \to \operatorname{Spec} B$ whose image is closed under specialization. By replacing Z with a finite union of affine schemes, we were able to write $h(Z)$ as the image of a morphism from an affine scheme equal to the disjoint union of the affine schemes covering Z. We finally showed that any morphism of affine schemes whose image is closed under specialization has closed image. It follows that if $Z \subset X$ is a closed subset, then $f \circ \imath(Z) = f(Z)$ is closed, hence f is a closed map implying b). □

At this point we have seen that both universally closed morphisms and valuation rings are connected in someway to the concept of specializations and how they lift. The question is then how can we leverage these connections to get a direct link between universally closed morphisms and valuation rings. The answer, lies in what we will call a valuative diagram.

Definition 3.12.3. Let $f : X \to Y$ be a morphism of a schemes, and A a valuation ring with field of fractions k. Then a **valuative diagram** is a commutative square:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where the dashed arrow may or may not exist, and may or may not be unique. If the dashed arrow exists for all such possible valuative diagrams then we say that f **satisfies the existence part of the valuative criteria**. If there is at most one dashed arrow for all possible valuative diagrams then we say that f **satisfies the uniqueness part of the valuative criteria.**

We will consider the consequences of the uniqueness part of the valuative criteria in the next section. For now we have the following:

Theorem 3.12.1. *Let $f : X \to Y$ be a morphism of schemes. Then the following are equivalent:*

a) Specializations lift along any base change of f.

b) f satisfies the existence part of the valuative criteria.

Proof. We first show that $a) \Rightarrow b)$. Suppose we have a valuative diagram:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "g"] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since specializations lift along any base change by assumption, we can instead consider a diagram of the form:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "h'"] \arrow[d] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "\mathrm{Id}"] \arrow[ur, dashed] & \operatorname{Spec} A \arrow[r, "g"] & Y \end{tikzcd}$$

where $X' = X \times_Y \operatorname{Spec} A$, and π is the canonical projection map, and h' comes from the relevant fibre product diagram, so that $\pi_X \circ h' = h$. In particular, if we can show that the dashed arrow l exists which makes the left square commute, then $\pi_X \circ l$ will make the original diagram commute.

Let $h'(\eta) = x_1 \in X'$, then $k_{x_1} \subset k$. If we consider the canonical specialization in $\operatorname{Spec} A$, $\eta \rightsquigarrow \mathfrak{m}$, we have that $\pi(x_1) = \eta$, hence since specializations lift along π there exists an $x_2 \in X$ such that $x_1 \rightsquigarrow x_2$ and $\pi(x_2) = \mathfrak{m}$. The stalk map $\pi_{x_2} : \mathscr{O}_{\operatorname{Spec} A, \mathfrak{m}} \to \mathscr{O}_{X,x_2}$ gives a morphism of local rings:

$$A \to \mathscr{O}_{X,x_2}$$

Note that morphism h' factors as:

$$\operatorname{Spec} k \to \operatorname{Spec} k_{x_1} \to X'$$

while the morphism $\operatorname{Spec} k_{x_1} \to X'$ factors as:

$$\operatorname{Spec} k_{x_1} \to \operatorname{Spec} \mathscr{O}_{X,x_2} \to X'$$

Putting this together, we have that h' is equal to the following composition:

$$\operatorname{Spec} k \to \operatorname{Spec} k_{x_1} \to \operatorname{Spec} \mathscr{O}_{X,x_2} \to X'$$

and so the morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ is given by:

$$\operatorname{Spec} k \to \operatorname{Spec} k_{x_1} \to \operatorname{Spec} \mathscr{O}_{X,x_2} \to X' \to \operatorname{Spec} A$$

We now have a morphism $\operatorname{Spec} \mathscr{O}_{X,x_2} \to \operatorname{Spec} A$, which by Lemma 3.12.5 is entirely determined by where $\mathfrak{m}_{x_2}$ is sent, and the stalk map $A \to \mathscr{O}_{X,x_2}$. However, this stalk map is canonically given by π_{x_2}, and $\mathfrak{m}_{x_2}$ is sent to $\mathfrak{m}$, as $\mathfrak{m}_{x_2}$ is sent to x_2. In this situation, as both A and $\mathscr{O}_{X,x_2}$ are local, the stalk is map is equal to the ring homomorphism inducing $\operatorname{Spec} \mathscr{O}_{X,x_2} \to \operatorname{Spec} A$, so by taking global sections we obtain the following chain of ring homomorphisms:

$$A \to \mathscr{O}_{X,x_2} \to k_{x_1} \to k$$

which is equal to the inclusion map $A \to k$. Denote by B the image of $\mathscr{O}_{X,x_2}$ in k which is local, we claim that B dominates A. Indeed, we have that $A \to \mathscr{O}_{X,x_2}$ must be injective as otherwise this composition can't be the inclusion, hence $A \subset B$. Moreover, since A is a local ring we have that $\mathfrak{m}$ embeds into $\mathfrak{m}_2$, hence $\mathfrak{m} \subset \mathfrak{m}_B$, so $\mathfrak{m} = A \cap \mathfrak{m}_B$ trivially. Since A is a valuation ring, we have that $A = B$ by Lemma 3.12.3, hence we have obtained a morphism:

$$\mathscr{O}_{X,x_2} \to A$$

by restricting the codomain. This yields the desired scheme morphism $l : \operatorname{Spec} A \to X'$, we just need to check that $\pi \circ l = \mathrm{Id}$. However, the stalk map on this morphism is given by:

$$A \to \mathscr{O}_{X,x_2} \to A$$

which as discussed is the identity map. Since A is a local ring, this must be equal to the ring map $A \to A$ inducing $\pi \circ l$, hence $\pi \circ l = \mathrm{Id}$. Therefore $\pi_X \circ l$ makes the original diagram commute, and the existence part of the valuative criteria is satisfied by f.

To show that $b) \to a)$, suppose that f satisfies the existence part of the valuative criteria. Let $g : Z \to Y$, then we claim that existence part of the valuative criteria holds for $\pi_Z : X \times_Y Z \to Z$. Indeed, if $X' = X \times_Y Z$ we have the following diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "h"] \arrow[d] & X' \arrow[r, "\pi_X"] \arrow[d, "\pi_Z"] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[urr, dashed, "l"] & Z \arrow[r, "g"] & Y
\end{tikzcd}$$

where l exists as f satisfies the existence part of the valuative criteria. THere is then a morphism l' induced by the fibre product diagram:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[dr, "l'"] \arrow[drr, bend left, "l"] \arrow[ddr, bend right, "j"] & & \\
& X' \arrow[r, "\pi_X"] \arrow[d, "\pi_Z"] & X \arrow[d, "f"] \\
& Z \arrow[r, "g"] & Y
\end{tikzcd}$$

which we claim makes the following diagram commute:

$$
\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "h"] \arrow[d, "\imath_\eta"'] & X' \arrow[d, "\pi_Z"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[ur, dashed, "l'"] & Z
\end{tikzcd}
$$

By construction $\pi_Z \circ l' = j$, so we need only show that $l' \circ \imath_\eta = h$. Note that $l \circ \imath_\eta = \pi_X \circ h$, and so $\pi_X \circ l' \circ \imath_\eta = \pi_X \circ h$. Since the square commutes we also have that $\pi_Z \circ l' \circ \imath_\eta = \pi_Z \circ h$. It follows that both $l' \circ \imath_\eta$ and h make the following diagram commute:

$$
\begin{tikzcd}
\operatorname{Spec} k \arrow[dr] \arrow[drr, bend left, "\pi_X \circ h"] \arrow[ddr, bend right, "\pi_Z \circ h"'] & & \\
& X' \arrow[r, "\pi_X"] \arrow[d, "\pi_Z"] & X \arrow[d, "f"] \\
& Z \arrow[r, "g"] & Y
\end{tikzcd}
$$

hence $\imath_\eta \circ l' = h$, and so π_Z satisfies the existence part of the valuative criteria.

It thus suffices to check that if f satisfies the existence part of the valuative criteria then specializations lift, as then the same will be true of any base change. Let $y_1 \rightsquigarrow y_2$ be a specialization, and let $x_1 \in X$ be such that $f(x_1) = y_1$. Let $k = k_{x_1}$, then note that the stalk map $\mathscr{O}_{Y,y_1} \to \mathscr{O}_{X,x_1}$ yields a field extension k_{x_1}/k_{y_1}. By Proposition 3.12.1 there is a valuation ring A with $\operatorname{Frac}(A) = k_{x_1}$, and a morphism $j : \operatorname{Spec} A \to Y$ such that $j(\eta) = y_1$ and $j(\mathfrak{m}) = y_2$. Since $\operatorname{Frac}(A) = k_{x_1}$, we thus have the following diagram:

$$
\begin{tikzcd}
\operatorname{Spec} k_{x_1} \arrow[r, "\imath_{x_1}"] \arrow[d, "\imath_\eta"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}
$$

where l exists by assumption. We have that $x_1 \rightsquigarrow l(\mathfrak{m})$ is then specialization as it is the image of the canonical specialization $\eta \rightsquigarrow \mathfrak{m}$. By the commutativity of the diagram we have that $f(l(\mathfrak{m})) = y_2$, so specializations lift. $\square$

The following theorem, know as the valuative criterion for being universally closed, is now immediate:

Theorem 3.12.2. *Let $f : X \to Y$ be a quasicompact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criteria.*

Proof. By Theorem 3.12.1, if f satisfies the existence part of the valuative criteria, then specializations lift along any base change of f. By Proposition 3.12.2 since f is quasicompact, and we thus have that f is universally closed. If f is universally closed, the by Proposition 3.12.2 we have that specializations lift along any base change, so by Theorem 3.12.1 we must have that the existence part of the valuative criteria is satisfied by f. $\square$

In Example 3.7.1 we showed that affine space over $\mathbb{C}$ was not proper. With the valuative criteria we can show that affine space over any ring is not proper.

Example 3.12.3. It suffices to show that $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is not universally closed as $\mathbb{A}^n_A \to \operatorname{Spec} A$ comes from the base change $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. In particular, since $\mathbb{A}^1_{\mathbb{Z}}$ is a closed subscheme of $\mathbb{A}^n_{\mathbb{Z}}$, it suffices to show that $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is not universally closed. Indeed, if $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed then $\mathbb{A}^1_{\mathbb{Z}} \hookrightarrow \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed as closed embeddings are universally closed, and universally closed morphisms are closed under composition. Since the morphism $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is unique, if $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is universally closed, so is $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$. So by the contrapositive, if $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is not universally closed, neither is $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$.

Consider the following valuative diagram:

$$\begin{tikzcd} \operatorname{Spec}\mathbb{Q} \arrow[r] \arrow[d] & \mathbb{A}^1_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec}\mathbb{Z}_{\langle 2\rangle} \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

coming from the following commutative square of rings:

$$\begin{tikzcd} \mathbb{Q} & \mathbb{Z}[x] \arrow[l, "\eta"'] \\ \mathbb{Z}_{\langle 2\rangle} \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] \end{tikzcd}$$

where every map is an inclusion map except for the morphism $\eta : \mathbb{Z}[x] \to \mathbb{Q}$ which is given by sending x to $1/2$. The existence of a map $l : \operatorname{Spec}\mathbb{Z}_{\langle 2\rangle} \to \mathbb{A}^1_{\mathbb{Z}}$ which makes the valuative diagram commute, is equivalent to a ring morphism $\phi : \mathbb{Z}[x] \to \mathbb{Z}_{\langle 2\rangle}$ such that $\phi \circ \imath = \imath$ and $\imath \circ \phi = \eta$. This would imply that $\phi(x)$ has to map to $1/2$ in $\mathbb{Q}$ under the inclusion map $\mathbb{Z}_{\langle 2\rangle} \to \mathbb{Q}$, but no such element exists, since we don't invert 2 in $\mathbb{Z}_{\langle 2\rangle}$. It follows by Theorem 3.12.2 that $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is not universally closed, and thus $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is not universally closed by the earlier discussion.

In Example 3.7.3 we showed the map $\mathbb{P}^n_A \to \operatorname{Spec} A$ was universally closed by explicitly calculating the image of a closed subset $Z \subset \mathbb{P}^n_A$ in $\operatorname{Spec} A$. We now wish to do so using the valuative criterion for being universally closed. We need the following lemma:

Lemma 3.12.6. *Let k be field, then there is a natural bijection:*

$$\operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} k, \mathbb{P}^n_{\mathbb{Z}}) \leftrightarrow \left(k^{n+1} \setminus 0\right)/k^\times$$

In other words k points of $\mathbb{P}^n_{\mathbb{Z}}$ are honest to god points in projective space.

Proof. Let $\imath : \operatorname{Spec} k \to \mathbb{P}^n_{\mathbb{Z}}$ be a k point, and let $\mathfrak{p} = \imath(\eta)$. Note that in any U_{x_i}, that:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}},\mathfrak{p}} = \mathbb{Z}[x_1/x_i, \ldots, x_n/x_i]_{(\mathfrak{p}_{x_i})_0}$$

where $(\mathfrak{p}_{x_i})_0$ is the corresponding prime ideal in $\operatorname{Spec}(\mathbb{Z}[x_0, \ldots, x_n]_{x_i})_0$. In the above ring, we have inverted everything away from $(\mathfrak{p}_{x_i})_0$, which is the same as inverting every homogeneous element away from $\mathfrak{p}$ and taking the degree zero part. In particular, if $\mathbb{Z}[x_0, \ldots, x_n] = A$ we have that:[97]

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}},\mathfrak{p}} = ((A^{\mathrm{hom}} \setminus \mathfrak{p})^{-1}A)_0 := A_{(\mathfrak{p})}$$

It follows that $\imath$ induces a morphism on stalks:

$$\imath_\eta : A_{(\mathfrak{p})} \longrightarrow k$$

At least one $x_i \notin \mathfrak{p}$ as otherwise $\mathfrak{p} \notin \mathbb{P}^n_{\mathbb{Z}}$, so define a tuple by:

$$[\imath_\eta(x_0/x_i), \ldots, 1, \ldots, \imath_\eta(x_0/x_i)] \in k^{n-1} \setminus 0/k^\times$$

This is independent of our choice of x_i, as if x_j is another generator not in $\mathfrak{p}$ we have that:

$$(x_k/x_i) \cdot (x_i/x_j) = x_k/x_j$$

[97]For a more formal and general proof of this fact, see Lemma 5.6.1 and Lemma 5.6.3.

hence the the two tuples will differ by multiplication with the constant $\imath_\eta(x_i/x_j)$.

If $(a_0,\ldots,a_n)\in k^{n+1}$ then we define a homogeneous prime ideal by:

$$\mathfrak{p}=\left\langle p\in A^{\mathrm{hom}}:p(a_0,\ldots,a_n)=0\right\rangle$$

where we evaluate p on k^{n+1} by writing the integer coefficients of the polynomial as coefficients in k via the unique map $\mathbb{Z}\to k$. This is homogeneous because it is generated by homogeneous elements, and it is prime because if $p(a_0,\ldots,a_n)\cdot q(a_0,\ldots,a_n)=0\in k$, then either $p(a_0,\ldots,a_n)=0$ or $q(a_0,\ldots,a_n)=0$. This is does not contain the irrelevant ideal as at least one of the $x_i\in A$ must not map to zero. Clearly everything homogeneous element not in $\mathfrak{p}$ is invertible in k, so we obtain a map $A_{(\mathfrak{p})}\to k$, and so since k is trivially a local ring we obtain by Lemma 3.12.5 a morphism morphism $\operatorname{Spec}k\to\mathbb{P}^n_{\mathbb{Z}}$ taking η to $\mathfrak{p}$.

The stalk map induced by the above morphism is equal to the map $A_{(\mathfrak{p})}\to k$, thus for some a_i the corresponding tuple in k^{n+1} is $(a_0/a_i,\ldots,a_n/a_i)$ which, up to rescaling, is the tuple we started with. In the other direction, if we start with a morphism $\imath:\operatorname{Spec}k\to\mathbb{P}^n_{\mathbb{Z}}$, and then take the homogeneous prime ideal induced by $(\imath_\eta(x_n/x_i),\ldots,\imath_\eta(x_n/x_i))$, then we claim that that the map $f:\operatorname{Spec}k\to\mathbb{P}^n_{\mathbb{Z}}$ induced by $(\imath_\eta(x_n/x_i),\ldots,\imath_\eta(x_n/x_i))$ is equal to $\imath$. First note that $f(\eta)\in U_{x_i}$ as x_i is not the homogeneous prime ideal induced $(\imath_\eta(x_n/x_i),\ldots,\imath_\eta(x_n/x_i))$. We can thus view both f and $\imath$ as coming from ring homomorphisms $(A_{x_i})_0\to k$;[98] by assumption $\imath^\sharp_{U_{x_i}}$ sends x_k/x_i to $\imath_\eta(x_k/x_i)$ as the stalk map comes from the universal property of localization applied to $\imath^\sharp_{U_{x_i}}$. By construction, we have that $f^\sharp_{U_{x_i}}$ sends x_k/x_i to $\imath_\eta(x_k/x_i)$ as $f^\sharp_{U_{x_i}}$ comes from the universal property of localization applied to the morphism:

$$\begin{aligned}\mathbb{Z}[x_0,\ldots,x_n]&\longrightarrow k\\ x_k&\longmapsto \imath_\eta(x_k/x_i)\end{aligned}$$

Since this map sends x_i to one, when we localize we find that $f^\sharp_{U_{x_i}}$ is precisely $\imath^\sharp_{U_{x_i}}$ hence $f=\imath$ as desired. □

We now use the valuative criterion for being universally closed to show that $\mathbb{P}^n_A\to\operatorname{Spec}A$ is universally closed.

Example 3.12.4. The same argument as in Example 3.7.3 demonstrates that it suffices to check this in the case of $A=\mathbb{Z}$. Note that the structure morphism is quasicompact as $\mathbb{P}^n_{\mathbb{Z}}$ is quasicompact. Suppose that we have a valuative diagram:

$$\begin{tikzcd} \operatorname{Spec}k \arrow[r, "\imath"] \arrow[d, "\imath_\eta"'] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "\pi"] \\ \operatorname{Spec}A \arrow[r, "j"'] \arrow[ur, dashed, "l"] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

where A is now a valuation ring. We need to show that the dashed line l exists. By Lemma 3.12.6 $\operatorname{Spec}k\to\mathbb{P}^n_{\mathbb{Z}}$ is equivalent to a choice an $n+1$ tuple up to rescaling $(a_0,\ldots,a_n)$ such that not all $a_i=0$. Since A is a valuation ring, we can order our our tuple in k so that:

$$a_{i_0}\le a_{i_1}\le\cdots\le a_{i_n}$$

with order given by $c<b$ if and only if $b/c\in A$. It follows that we can scale the tuple by $1/a_{i_0}$ to obtain an $n+1$ tuple of elements $(a_0,\ldots,a_n)$ in A, where at least one $a_i=1$. This morphism then induces a morphism:

$$\begin{aligned}\phi:\mathbb{Z}[x_0,\ldots,x_n]&\to A\subset k\\ x_i&\longmapsto a_i\end{aligned}$$

[98]We showed in Lemma 3.12.5 that a morphism from a local ring to a scheme is independent of what affine open the image of the morphism is in.

which has the same kernel as the morphism $\mathbb{Z}[x_0, \dots, x_n] \to k$ inducing the map $\operatorname{Spec} k \to \mathbb{P}^n_{\mathbb{Z}}$. Let $B = \mathbb{Z}[x_0, \dots, x_n]$, $\mathfrak{m}$ be the unique prime ideal in A, and $\mathfrak{q}$ be the ideal:

$$\mathfrak{q} = \{p \in B^{\text{hom}} : p \in \phi^{-1}(\mathfrak{m})\}$$

This is homogeneous by construction, and prime by Lemma 2.2.2 part c). Moreover, we know that $\mathfrak{q}$ does not contain the irrelevant ideal because at least one a_i can be taken to be equal to 1, so at least one x_i can't map into the unique maximal ideal of A. Trivially, anything not in $\mathfrak{q}$ maps to something invertible in A hence we obtain a ring homomorphism:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}},\mathfrak{q}} = B_{(\mathfrak{q})} \longrightarrow A$$

which induces a morphism $l : \operatorname{Spec} A \to \mathbb{P}^n_{\mathbb{Z}}$ by Lemma 3.12.5. In particular, we have that $\eta \in \operatorname{Spec} A$ maps to $\mathfrak{p}$, and $\mathfrak{m} \in \operatorname{Spec} A$ maps to $\mathfrak{q}$. The bottom triangle commutes trivially by the uniqueness of morphisms to $\operatorname{Spec} \mathbb{Z}$. For the top triangle, we have that $\ell \circ \imath_\eta(\eta) = \mathfrak{p}$, and $\imath(\eta) = \mathfrak{p}$. Moreover, we have that $x_i \notin \mathfrak{q}$ hence $\mathfrak{p} \subset \mathfrak{q} \in U_{x_i}$. By our work in Lemma 3.12.5 it suffices to check that $\ell \circ \imath_\eta$ and $\imath$ agree when viewed as morphism of affine schemes into U_{x_i}. By our work in Lemma 3.12.6, the morphism $\imath^\sharp_{U_{x_i}} : (B_{x_i})_0 \to k$ is given by sending x_j/x_i to a_j. The morphism $\imath_\eta \circ \ell : \operatorname{Spec} A \to U_{x_i}$ is then induced by the following composition:

$$(B_{x_i})_0 \to B_{(\mathfrak{q})} \to A \hookrightarrow k$$

where the first map is induced by the localization map $B_{x_i} \to (A^{\text{hom}} \setminus \mathfrak{q})^{-1}B$. It follows this map also sends x_j/x_i to $a_j \in A$ as $B_{(\mathfrak{q})} \to A$ is induced by ϕ. Upon including into k, we see that the above composition clearly agrees with $\imath^\sharp_{U_{x_i}}$ hence the upper triangle commutes and $\pi : \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{Z}$ is universally closed by Theorem 3.12.2.

3.13 The Valuative Criterion for Being Separated

In this section we develop the valuative criterion for being separated. In particular, we will show that a quasiseparated morphism of schemes is separated if and only if the the uniqueness part of the valuative criteria is satisfied. Putting these together we will obtain that valuative criteria for properness as a formal consequence of the results in this section, and the results in the previous section. Unlike the last section, demonstrating the connection to separatedness will be relatively easy. In a sense we have already done all of the 'hard work' by relating specializations to valuation rings to being universally closed.

Usually when showing a morphism of schemes is unique we invoke some form of universal property, however in this general setting we have no access to a universal property. We will thus have to devise a mechanism to compare two morphisms and decide when they are equal. We do this first in the setting of abstract nonsense:

Definition 3.13.1. Let $\mathscr{C}$ be a category, and $f, g \in \operatorname{Hom}_{\mathscr{C}}(A, B)$. Then the **equalizer of f and g** is an object $E_{f,g}$ of $\mathscr{C}$ satisfying the following criteria:

a) There is a morphism $\text{eq} : E_{f,g} \to A$ such that $f \circ \text{eq} = g \circ \text{eq}$.

b) If $h : D \to A$ is a morphism satisfying $f \circ h = g \circ h$ there exists a unique morphism $\mu : D \to A$ such that the following diagram commutes:

$$\begin{tikzcd} E_{f,g} \arrow[r, "\text{eq}"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ D \arrow[u, "\mu"] \arrow[ur, "h"'] & & \end{tikzcd}$$

In general, such objects tend to exist. We look to some examples:

Example 3.13.1. Let $\mathscr{C}$ be Top, the category of topological spaces, with morphisms given by continuous maps. If $f, g : X \to Y$ then define:

$$E_{f,g} = \{x \in X : f(x) = g(x)\}$$

and endow $E_{f,g}$ with the subspace topology. Then the inclusion map $\imath : E_{f,g} \to X$ is continuous, and any map $h : Z \to X$ satisfying $f \circ h = g \circ h$ clearly factors through $E_{f,g}$ hence $E_{f,g}$ as defined is indeed the equalizer of f and g in the category of topological spaces.

Example 3.13.2. In the category of abelian groups, the equalizer is the kernel of the difference morphism $f - g$.

Example 3.13.3. In the category of commutative rings, if $\phi, \psi : A \to B$ we take:

$$E_{\phi,\psi} = \{a \in A : \phi(a) = \psi(a)\}$$

which is obviously a subring of A. We take eq to again be the inclusion map, and this then obviously satisfies the universal property of the equalizer.

The following lemma demonstrates why we call this object the equalizer of f and g:

Lemma 3.13.1. *Let $\mathscr{C}$ be a category with equalizers, and let $f, g : A \to B$ be two morphisms. Then $f = g$ if and only if (A, Id_A) satisfies the universal property of the equalizer. In other words $f = g$ if and only if $E_{f,g} \cong A$.*

Proof. If (A, Id_A) satisfy the universal property of the equalizer then we obviously have that $f = g$.

If $f = g$ then $f \circ \mathrm{Id}_A = g \circ \mathrm{Id}_A$. If $h : D \to A$ satisfies $f \circ h = g \circ h$, then the following diagram trivially commutes:

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}_A"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ D \arrow[u, "h"] \arrow[ur, "h"'] & & \end{tikzcd}$$

and any other morphism which makes this diagram commute has to be equal to h hence A with the identity map satisfies the universal property of the equalizer. ☐

Our first goal is to construct an equalizer object in the category of schemes, however this will be shockingly easy with the language of fibre products.

Lemma 3.13.2. *Let X and Y be Z-schemes, and $f, g : X \to Y$ morphisms in the category of Z schemes. Then the equalizer of f and g exists in the category of Z schemes. In particular* eq *is a locally closed immersion, and a closed embedding if Y is separated over Z.*

Proof. We define $E_{f,g}$ to be the following fibre product diagram:

$$\begin{tikzcd} E_{f,g} \arrow[r, "\mathrm{eq}"] \arrow[d, "\pi_Y"] & X \arrow[d, "f \times g"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

We show that this satisfies the universal property of the equalizer of f and g. Suppose that $h : S \to X$

satisfies $f \circ h = g \circ h$, then $f \times g \circ h$ fits into the following diagram:

$$\begin{tikzcd}
S \arrow[dr, "f \times g \circ h"] \arrow[drr, bend left, "f \circ h"] \arrow[ddr, bend right, "g \circ h"'] & & \\
& Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

Since $f \circ h = g \circ h$, without loss of generality we use $f \circ h$. If we can show that $\Delta \circ f \circ h = f \times g \circ h$ also makes the above diagram commute, then we will get a unique morphism $\mu : S \to E_{f,g}$ satisfying $\mathrm{eq} \circ \mu = h$. Now note that if π_Y is the morphism $Y \times_Z Y \to Y$, then:

$$\pi_Y \circ \Delta \circ f \circ h = \mathrm{Id}_Y \circ f \circ h = f \circ h$$

so $\Delta \circ f \circ h$ makes the diagram commute hence $\Delta \circ f \circ h = f \times g \circ h$. It there is a unique μ making the following diagram commute:

$$\begin{tikzcd}
S \arrow[dr, "\mu"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "f \circ h"'] & & \\
& E_{f,g} \arrow[r, "\mathrm{eq}"] \arrow[d] & X \arrow[d, "f \times g"] \\
& Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

and so $(E_{f,g}, \mathrm{eq})$ satisfies the universal property of the equalizer.

In general eq is a locally closed immersion because it is the base change of a locally closed immersion $\Delta : Y \to Y \times_Z Y$. When Y is separated over Z, Δ is a closed embedding, so by the same base change argument eq is a closed embedding. $\square$

When f and g are morphisms of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ we have the following apt description of the equalizer:

Example 3.13.4. Let $f, g : \operatorname{Spec} A \to \operatorname{Spec} B$ be morphisms of affine schemes over $\operatorname{Spec} C$. Equivalently f and g come from C-algebra morphisms $\phi, \psi : B \to A$. The equalizer is given by:

$$E_{f,g} = \operatorname{Spec} A \times_{B \otimes_C B} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_{B \otimes_C B} B)$$

where B is made $B \otimes_C B$ algebra via multiplication, and A is a $B \otimes_C B$ algebra $\phi \otimes \psi : B \otimes_C B \to A$ by sending $b_1 \otimes b_2$ to $\phi(b_1) \cdot \psi(b_2)$. The equalizer map is induced by $\imath_A : A \to A \otimes_{B \otimes_C B} B$ sending a to $a \otimes 1$. By Lemma 3.13.2 we know that $\imath_A$ is a surjection, and claim it's kernel is given by:

$$I = \langle \phi(b) - \psi(b) : b \in B \rangle$$

This is however clear as $A \otimes_{B \otimes_C B} B$ satisfies the universal property of the co-equalizer[99] of ϕ and ψ by the contravariant equivalence of categories between commutative rings and affine schemes. Moreover, using the universal property of the quotient, A/I also obviously satisfies the universal property of the co-equalizer hence:

$$A/I \cong A \otimes_{B \otimes_C B} B$$

hence the kernel of $\imath_A$ must be I. It follows that equalizer $E_{f,g}$ is the closed subscheme $\mathbb{V}(I) \subset \operatorname{Spec} A$.

[99] I.e. take the diagram from Definition 3.13.1 and flip all the arrows.

We are now ready to prove the main result of the section:

Theorem 3.13.1. *Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if f is quasiseparated, and f satisfies the uniqueness part of the valuative criteria.*

Proof. Suppose that f is separated, and that we have a valuative diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\imath"] \arrow[d, "\imath_\eta"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

Let $m : \operatorname{Spec} A \to X$ make the diagram commute then we have that m and l are morphisms in the category of Y schemes, so let $E_{m,l}$ be the equalizer of m and l in this category. Since X is separated over Y we have that by Lemma 3.13.2 that $E_{m,l}$ is a closed subscheme $\mathbb{V}(I) \subset \operatorname{Spec} A$. Since $m \circ \imath = l \circ i$, there is a unique morphism $\mu : \operatorname{Spec} k \to E_{m,l}$ such that $\operatorname{eq} \circ \mu = \imath$. In particular, we have that $\operatorname{eq}(\mu(\eta)) = \eta \in \operatorname{Spec} A$, hence $\mathbb{V}(I)$ contains the unique generic point and must be equal to $\operatorname{Spec} A$. By Lemma 3.13.1 we have that $m = l$. Since every separated morphism is quasiseparated we have the claim.

Suppose that f is quasiseparated and satisfies the uniqueness part of the valuative criteria, then by definition the diagonal morphism $\Delta : X \to X \times_Y X$ is a quasicompact locally closed immersion. It therefore suffices to show by Proposition 3.6.1 that Δ has closed image. In particular, if we can show that Δ satisfies the existence part of the valuative criteria, we will have that Δ is universally closed and thus has has closed image. Suppose we have a valuative diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\imath"] \arrow[d, "\imath_\eta"'] & X \arrow[d, "\Delta"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[ur, dashed, "l"] & X \times_Y X
\end{tikzcd}$$

The data of a morphism $j : \operatorname{Spec} A \to X \times_Y X$ is equivalent to the data of morphisms $a, b : \operatorname{Spec} A \to X$ making the following diagram commute:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[dr, "j"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "b"'] & & \\
& X \times_Y X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

By post composing Δ and j with π_X then f, we obtain a new diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\imath"] \arrow[d, "\imath_\eta"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "j'"] \arrow[ur, dashed, "l'"] & Y
\end{tikzcd}$$

where we claim setting $l' = a$ or b makes this diagram commute. Clearly the bottom triangle commutes by assumption of the properties of a and b. We want to show that $a \circ \imath_\eta = \imath$ and $b \circ \imath_\eta = \imath$, however this is clear as:

$$a \circ \imath_\eta = \pi_X \circ j \circ \imath_\eta = \pi_x \circ \Delta \circ \imath = \imath$$

and similarly for b. Since f satisfies the uniqueness part of the valuative criteria we have that $a = b$, so we claim that setting $l = a$ in the original valuative diagram makes the diagram commute. Indeed, since $\pi_X \circ \Delta = \mathrm{Id}$ and $a = b$, both both j and $\Delta \circ a$ make the same fibre product diagram commute and are thus equal giving us the lower triangle. The upper triangle commutes by our earlier work, and so Δ satisfies the existence part of the valuative criteria implying the claim. $\square$

Recall that in Example 3.6.5 we shoed that the scheme obtained by gluing two copies of $\mathbb{A}^1_{\mathbb{C}}$ along the open set U_x was not separated over $\operatorname{Spec}\mathbb{C}$ by explicitly showing that the diagonal map was not closed. Here we show that the analogue of this scheme over $\mathbb{Z}$ is not separated because it fails the uniqueness part of valuative criteria.

Example 3.13.5. Let X be the scheme obtained by gluing two copies of $\mathbb{A}^1_{\mathbb{Z}}$ along the open set U_x. Let $\phi : \mathbb{Z}[x] \to \mathbb{Q}(x)$ be the inclusion map, and $f : \mathbb{Q} \to \mathbb{A}^1_{\mathbb{Z}}$ the induced morphism of schemes. Compose this with the open embedding $\psi_1 : \mathbb{A}^1_{\mathbb{Z}} \to X$ Then the following diagram commutes trivially:

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{Q}(x) \arrow[r, "\imath"] \arrow[d, "\imath_\eta"] & X \arrow[d, "\pi"] \\
\operatorname{Spec}\mathbb{Q}[x]_{\langle x\rangle} \arrow[r, "j"] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

Note that $\mathbb{Q}[x]_{\langle x\rangle}$ is indeed a valuation ring as if $p/q \in \mathbb{Q}(x)$ and both p and q are divisible by a power of x then we can reduce the fraction so that only one of them or neither is divisible by x.

We can identify the points in X as as the points in $\mathbb{A}^1_{\mathbb{Z}}$ but with two copies of each prime ideal in $\mathbb{V}(x)$, in particular we have two copies of $\langle x\rangle$ which we denote by $\langle x\rangle_1$ and $\langle x\rangle_2$. Note that the morphism $\imath : \operatorname{Spec}\mathbb{Q} \to X$ is independent of our choice of the open embedding as $\eta \in U_x$. Since $\imath$ takes η to $\eta \in X$ it suffices to define a morphism $\mathbb{Z}[x] \to \mathbb{Q}[x]_{\langle x\rangle}$ which makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{Q}(x) & \mathbb{Z}[x] \arrow[l, "\phi"'] \arrow[dl, "l^\sharp"] \\
\mathbb{Q}[x]_{\langle x\rangle} \arrow[u, "\imath_\eta^\sharp"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

and the inclusion map sending x to x works. This yields two scheme morphisms $l_1, l_2 : \operatorname{Spec}\mathbb{Q}[x]_{\langle x\rangle} \to X$ given by taking the induced scheme morphism $l : \operatorname{Spec}\mathbb{Q}[x]_{\langle x\rangle} \to \mathbb{A}^1_{\mathbb{Z}}$ and post composing with the open embeddings $\psi_1, \psi_2 : \mathbb{A}^1_{\mathbb{Z}} \to X$. Clearly both l_1 and l_2 make the diagram commute, but l_1 takes $\langle x\rangle$ to $\langle x\rangle_1$ and l_2 take $\langle x\rangle$ to $\langle x\rangle_2$, hence X is not separated.

In Example 3.6.3 we showed the $\mathbb{P}^n_A$ was separated for any commutative ring A by brute computation. In the following example we use the valuative criteria to demonstrate that $\mathbb{P}^n_{\mathbb{Z}}$ is separated over $\operatorname{Spec}\mathbb{Z}$.

Example 3.13.6. Suppose we have a valuative diagram:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\imath"] \arrow[d, "\imath_\eta"] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "\pi"] \\
\operatorname{Spec} A \arrow[r, "j"] \arrow[ur, dashed, "l"] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

Let $\imath$ come from the morphism:

$$\begin{aligned}
\phi : \mathbb{Z}[x_0, \dots, x_n] &\to k \\
x_i &\mapsto a_i
\end{aligned}$$

which is unique up to rescaling $(a_0, \dots, a_n)$. Moreover, by our work in Example 3.12.4, we can take all the a_i to be in $A \subset k$, with at least one element, say a_j, equal to 1. We first claim that any morphism l making the above diagram commute has image in U_{x_j}. Suppose that $x_j \in l(\mathfrak{m})$, then for some $m \neq j$ we have that $l(\mathfrak{m}) \subset U_{x_m}$, in particular, the image of l is contained in U_{x_m} as A is local and so l comes from a morphism $l^\sharp : \mathbb{Z}[x_0/x_m, \dots, x_n/x_m] \to A$. If $x_j \notin l(\mathfrak{m})$ then $l^\sharp(x_j/x_m) \in \mathfrak{m}$, however by the commutativity of the diagram we must have that $l^\sharp(x_j/x_m) = 1/a_m$. It follows $l(\mathfrak{m}) \in U_{x_j}$, and two lifts $l_1, l_2 : \operatorname{Spec} A \to \mathbb{P}^n_{\mathbb{Z}}$ factor through the open embedding U_{x_j}. It thus suffices to show that if $B = \mathbb{Z}[x_0, \dots, x_n]$, and ϕ_1 and ϕ_2 make the following diagram commute:

$$\begin{tikzcd}
k & (B_{x_j})_0 \arrow[l, "\imath^\sharp_{U_{x_j}}"'] \arrow[dl, "\phi_1{,}\phi_2"'] \\
A \arrow[u, "\imath^\sharp_\eta"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

then $\phi_1 = \phi_2$. However, since $\imath^\sharp_\eta$ is just the inclusion, we must have that ϕ_1 and ϕ_2 agree with $\imath^\sharp_{U_{x_j}}$, hence they must agree. In more detail ϕ_1 and ϕ_2 must send x_i/x_j to a_i because $\imath^\sharp_{U_{x_j}}$ does. It follows that $\mathbb{P}^n_{\mathbb{Z}}$ is separated over $\operatorname{Spec} \mathbb{Z}$.

Note that finite type morphisms are quasicompact, hence we have the following formal consequence called the valuative criteria for being proper:

Theorem 3.13.2. *Let $f : X \to Y$ be a finite type quasiseparated morphism. Then f is proper if and only if it satisfies the existence and uniqueness parts of the valuative criteria.*

Proof. This follows easily from Theorem 3.12.2, Theorem 3.13.1, and the definition of proper maps. $\square$

Varieties: A Rosetta Stone

4.1 Some Commutative Algebra: Perfect Fields

4.2 Why Are There so Many Definitions of a Variety?

$\mathscr{O}_X$ Modules I: Towards Vector Bundles

5.1 Definitions and Examples over Ringed Spaces

In Section 1.2 and Section 1.3 we broadly discussed sheafs of rings, abelian groups, and sets over topological spaces. In this chapter, we will extend these ideas to the category of modules over a commutative ring. Let $\mathscr{F}$ be a sheaf of abelian groups over X , then prescribing an A-module structure on $\mathscr{F}(U)$ for each $U \subset X$ gives us a sheaf of a A-modules. However, what we would really like, is for the A-module structure to vary with respect to a sheaf of rings on X, i.e. we want $\mathscr{F}(U)$ to be an $\mathscr{O}_X(U)$ module for all $U \subset X$. We define this precisely now:

Definition 5.1.1. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathscr{F}$ a presheaf on X. Then $\mathscr{F}$ is a **presheaf of $\mathscr{O}_\mathbf{X}$ modules** if there exists a sheaf morphism:

$$m_{\mathscr{F}} : \mathscr{O}_X \times \mathscr{F} \longrightarrow \mathscr{F}$$

which makes $\mathscr{F}(U)$ an $\mathscr{O}_X(U)$ module for each $U \subset X$. A **sheaf of $\mathscr{O}_\mathbf{X}$ modules** or a **$\mathscr{O}_\mathbf{X}$ module** is a presheaf of $\mathscr{O}_X$ modules that is also a sheaf. A **morphism of presheaves of $\mathscr{O}_\mathbf{X}$ modules** is a presheaf morphism $F : \mathscr{F} \to \mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{O}_X \times \mathscr{F} \arrow[r, "m_{\mathscr{F}}"] \arrow[d, "\mathrm{Id}\times F"'] & \mathscr{F} \arrow[d, "F"] \\
\mathscr{O}_X \times \mathscr{G} \arrow[r, "m_{\mathscr{G}}"] & \mathscr{G}
\end{tikzcd}$$

A **morphism of sheaves of $\mathscr{O}_\mathbf{X}$ modules** is a morphism in the underlying category of presheaves of $\mathscr{O}_X$ modules. We denote the category of presheaves of $\mathscr{O}_X$ modules, and the category of sheaves of $\mathscr{O}_X$ modules by $\mathrm{Mod}^p_{\mathscr{O}_X}$ and $\mathrm{Mod}_{\mathscr{O}_X}$ respectively. At times, we will refer to sheaves of $\mathscr{O}_X$ modules simply as '$\mathscr{O}_X$ modules'.

Example 5.1.1. Letting $E \to X$ be a smooth vector bundle over a smooth manifold X, by Example 1.2.2 we have that $\Gamma(-, E)$ is a sheaf on M; we wish to show that this is a C^∞ module. For each open U, we define:

$$m_U : C^\infty(U) \times \Gamma(U, E) \to \Gamma(U, E)$$

to be the usual multiplication of a smooth function with a smooth section of E over U. If $(f, \phi) \in C^\infty(U) \times \Gamma(U, E)$, we need to show that:

$$f|_U \cdot \phi|_U = (f \cdot \phi)|_U$$

This is however true by construction, because f is an honest to god function on U with values in $\mathbb{R}$, and ϕ is an honest to map $U \to E|_U$. Moreover, a vector bundle morphism over X $F : E \to E'$ induces a morphism of the underlying C^∞ modules.

One readily checks that that $\mathrm{Mod}^p_{\mathscr{O}_X}$ and $\mathrm{Mod}_{\mathscr{O}_X}$ form abelian categories, and that the proof of Theorem 1.2.1 holds essentially verbatim when one replaces the words 'abelian group' with '$\mathscr{O}_X$ module'. We can also sheafify $\mathscr{O}_X$ modules, glue $\mathscr{O}_X$ modules and their morphisms, and take stalks at a point x to get $(\mathscr{O}_X)_x$ modules. These facts are all borderline immediate given the content covered in Section 1.2 and Lemma 5.1.1, so we elect to not reprove these results in this section as there are more pressing matters at hand. The main being that given a continuous map $f : X \to Y$, the inverse image functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, does not send $\mathscr{O}_Y$ modules to $\mathscr{O}_Y$ modules, but to $f^{-1}\mathscr{O}_Y$ modules. We will instead need to construct a different functor, called the pullback functor, which will combine tensor products, and the inverse image functor. We begin with showing that sheafification commutes with finite products:

Lemma 5.1.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X, then $(\mathscr{F}\times\mathscr{G})^\sharp$ is canonically isomorphic to $\mathscr{F}^\sharp\times\mathscr{G}^\sharp$. In particular, if $f : Y \to X$ is a continuos map, then $f^{-1}(\mathscr{F}\times\mathscr{G}) \cong f^{-1}\mathscr{F}\times f^{-1}\mathscr{G}$.*

Proof. One might imagine there is a slick proof of this fact exploiting the universal property of products, and sheafification, but as far as we can tell, there is no avoiding a direct computation with the definition of sheafification, and stalks, hence we show the isomorphism directly.

First note, that clearly we have a canonical isomorphism $(\mathscr{F}\times\mathscr{G})_x \cong \mathscr{F}_x\times\mathscr{G}_x$, hence we can consider elements of $(\mathscr{F}\times\mathscr{G})^\sharp$ to be sequence (s_x), where $s_x \in \mathscr{F}_x\times\mathscr{G}_x$. Let $\pi_{\mathscr{F}x}, \pi_{\mathscr{G}x}$ denote the projections on the level of stalks $\mathscr{F}_x\times\mathscr{G}_x \to \mathscr{F}_x$, $\mathscr{F}_x\times\mathscr{G}_x \to \mathscr{G}_x$ respectively induced by the projection morphisms on presheaves. Then for all U, we claim that the map:

$$\begin{aligned}(\mathscr{F}\times\mathscr{G})^\sharp(U) &\longrightarrow \prod_{x\in U}\mathscr{F}_x \times \prod_{x\in U}\mathscr{G}_x\\ (s_x) &\longmapsto ((\pi_{\mathscr{F}x}(s_x)), (\pi_{\mathscr{G}_x}(s_x)))\end{aligned}$$

as image in $\mathscr{F}^\sharp(U)\times\mathscr{G}^\sharp(U)$. Since doing the following for $\mathscr{F}$ will be the same as doing it for $\mathscr{G}$, we need only show that for each $x\in U$, there exists an open neighborhood V of x, and a section $t \in \mathscr{F}(U)$ such that $t_y = \pi_{\mathscr{F}y}(s_y)$ for all $y \in V$. This is however clear; since $(s_x) \in (\mathscr{F}\times\mathscr{G})^\sharp(U)$, we have that there exists an open neighborhood V of x and a section $t \in \mathscr{F}(U)\times\mathscr{G}(U)$ such that $t_y = s_y$. Now note that for all $y \in V$:

$$\pi_{\mathscr{F}}(t)_y = \pi_{\mathscr{F}_y}(t_y) = \pi_{\mathscr{F}y}(s_y)$$

so we have obtained a map:

$$F : (\mathscr{F}\times\mathscr{G})^\sharp(U) \longrightarrow \mathscr{F}^\sharp(U)\times\mathscr{G}^\sharp(U)$$

which clearly commutes restricts. This is also clearly an isomorphism on stalks, so F is an isomorphism as desired, which must be unique by abstract nonsense.

To prove the second claim, by the first it suffices to provide an isomorphism:

$$f_p^{-1}(\mathscr{F}\times\mathscr{G}) \cong f_p^{-1}\mathscr{F}\times f_p^{-1}\mathscr{G}$$

Our work in Proposition 1.3.5 demonstrates that $\mathscr{F} \mapsto f_p\mathscr{F}$ is a functor $\mathrm{PSh}(X) \to \mathrm{PSh}(Y)$, and so there are projection maps:

$$f_p^{-1}\pi_{\mathscr{F}} : f_p^{-1}(\mathscr{F}\times\mathscr{G}) \to f_p^{-1}\mathscr{F} \qquad f_p^{-1}\pi_{\mathscr{G}} : f_p^{-1}(\mathscr{F}\times\mathscr{G}) \to f_p^{-1}\mathscr{G}$$

and so by the universal property of the product, these determine a morphism:

$$F : f_p^{-1}(\mathscr{F}\times\mathscr{G}) \longrightarrow f_p^{-1}\mathscr{F}\times f_p^{-1}\mathscr{G}$$

given on an open set U by:

$$s \longmapsto (f_p^{-1}\pi_{\mathscr{F}}(s), f_p^{-1}\pi_{\mathscr{F}}\mathscr{G}(s))$$

To check that this is an isomorphism, it suffices to check that this is an isomorphism on stalks. Recall that there are natural isomorphisms:

$$f_p^{-1}(\mathscr{F} \times \mathscr{G})_y \cong (\mathscr{F} \times \mathscr{G})_{f(y)} \cong \mathscr{F}_{f(y)} \times \mathscr{G}_{f(y)}$$

and so if $s_y = [U, s]$, for $y \in U \subset Y$, then:

$$(f_p^{-1}\pi_{\mathscr{F}})_y(s_y) = [U, f_p^{-1}\pi_{\mathscr{F}}(s)]$$

However, if $s = [V, t]$, for $t \in \mathscr{F}(V) \times \mathscr{G}(V)$, and $f(U) \subset V$, then:

$$(f_p^{-1}\pi_{\mathscr{F}})([V, t]) = [V, \pi_{\mathscr{F}}(t)]$$

Under the isomorphism $(f_p^{-1}\mathscr{F})_y \cong \mathscr{F}_{f(y)}$ we have that:

$$[U, f_p^{-1}\pi_{\mathscr{F}}(s)] \longmapsto [V, \pi_{\mathscr{F}}(t)] = (\pi_{\mathscr{F}})_{f(y)}(t_y)$$

and similarly for $\pi_{\mathscr{G}}$. It follows that up to isomorphism, the stalk map:

$$(f_p^{-1}F)_y : f_p^{-1}(\mathscr{F} \times \mathscr{G})_y \longrightarrow f_p^{-1}\mathscr{F}_y \times f_p^{-1}\mathscr{G}_y$$

is the given by the map:

$$\begin{aligned} (\mathscr{F} \times \mathscr{G})_{f(y)} &\longrightarrow \mathscr{F}_{f(y)} \times \mathscr{G}_{f(y)} \\ t_{f(y)} &\longmapsto ((\pi_{\mathscr{F}})_{f(y)}(t_{f(y)}), (\pi_{\mathscr{G}})_{f(y)}(t_{f(y)})) \end{aligned}$$

which is an obvious isomorphism, implying the claim. □

Not only does Lemma 5.1.1 guarantee that the $\mathrm{Mod}_{\mathscr{O}_X}$ and f^{-1} behave mostly as expected, but it also allows us to quickly demonstrate the following failure:

Corollary 5.1.1. *Let $f : X \to Y$ be a morphism of ringed space, $\mathscr{F}$ an $\mathscr{O}_X$ module on X, and $\mathscr{G}$ an $\mathscr{O}_Y$ module on Y. Then $f_*\mathscr{F}$ is an $\mathscr{O}_Y$ module, and $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$ module.*

Proof. We give $f_*\mathscr{F}$ the structure of an $\mathscr{O}_Y$ module by setting:

$$\begin{aligned} m_U : \mathscr{O}_Y(U) \times (f_*\mathscr{F})(U) &\longrightarrow (f_*\mathscr{F})(U) \\ (s, \phi) &\longmapsto f_U^\sharp(s) \cdot \phi \end{aligned}$$

Since $f_U^\sharp \in (f_*\mathscr{O}_X)(U) = \mathscr{O}_X(f^{-1}(U))$, this makes $(f_*\mathscr{F})(U)$ an $\mathscr{O}_Y(U)$ module. Moreover, since the restriction maps on $f_*\mathscr{F}$ are inherited from those on $\mathscr{F}$, and thus respect multiplication, and since $f_U^\sharp$ commutes with restriction maps, the collection m_U determines a sheaf morphism, hence $f_*\mathscr{F}$ is an $\mathscr{O}_Y$ module.

We now need to construct a morphism:

$$f^{-1}\mathscr{O}_Y \times f^{-1}\mathscr{G} \longrightarrow f^{-1}\mathscr{G}$$

By Lemma 5.1.1, and Proposition 1.3.5, we have that the defining map:

$$\mathscr{O}_Y \times \mathscr{G} \to \mathscr{G}$$

induces a morphism:

$$f^{-1}\mathscr{O}_Y \times f^{-1}\mathscr{G} \to f^{-1}\mathscr{G}$$

which one each open set $U \subset X$ will make $f^{-1}\mathscr{G}(U)$ an $f^{-1}\mathscr{O}_Y(U)$ module as desired. □

Recall that if M_1 and M_2 are A modules, we can form their tensor product $M_1 \otimes_A M_2$. This tensor product satisfies the following universal property: for every A bilinear map $M_1 \oplus M_2 \to N$, there exists a unique A-linear map $M_1 \otimes_A M_2 \to N$ making the following diagram commute:

$$\begin{tikzcd} M_1 \oplus M_2 \arrow[r] \arrow[d] & N \\ M_1 \otimes_A M_2 \arrow[ur] & \end{tikzcd}$$

With this recollection in mind, we form the following definition:

Definition 5.1.2. A $\mathscr{O}_{\mathbf{X}}$ **bilinear** morphism of presheaves or sheaves of $\mathscr{O}_X$ modules, is a morphism of presheaves/sheaves $\mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$, such that for each $U \subset X$, $\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \to \mathscr{G}(U)$ is a bilinear map. We define the **tensor product presheaf** by:

$$(\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$$

Obviously, we need to check that this is a presheaf, and while we are at it, we might as well prove some desirable properties of the the presheaf.

Lemma 5.1.2. *Let $\mathscr{F}_1, \mathscr{F}_2$ be presheaves (or sheaves) of $\mathscr{O}_X$ modules, then the following hold:*

i) The tensor product presheaf $\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{G}$ is a presheaf.

ii) The tensor product presheaf satisfies the universal property of the tensor product in $\mathrm{Mod}_{\mathscr{O}_X}$.

iii) For all $x \in X$, there is a natural isomorphism $(\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x \cong (\mathscr{F}_1)_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{F}_2)_x$.

Proof. We obviously start with *i*). Let $V \subset U$ be open sets of X , we need to write down restriction maps:

$$\theta^U_V : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V)$$

Denote the restrictions maps for $\mathscr{F}_1$ and $\mathscr{F}_2$ by $(\theta_1)^U_V$ and $(\theta_2)^U_V$, then we have bilinear map:

$$\begin{aligned} \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \longrightarrow \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) &\longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \\ (s,t) \longmapsto (s|_V, t|_V) &\longmapsto (s|_V) \otimes (t|_V) \end{aligned}$$

and so by the universal property of the tensor product, we get well defined restriction maps:

$$\begin{aligned} \theta^U_V : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) &\longrightarrow \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \\ s \otimes t &\longmapsto (s|_V) \otimes (t|_V) \end{aligned}$$

which obviously satisfy $\theta^V_W \circ \theta^U_V = \theta^U_W$, making the tensor product presheaf, a presheaf.

For *ii*), we first need a bilinear sheaf morphism $\mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$. For each U, we have bilinear morphism:

$$\begin{aligned} \otimes^p_U : \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) &\longrightarrow \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \\ (s,t) &\longrightarrow s \otimes t \end{aligned}$$

We need only check that $\theta^U_V \circ \otimes^p_U = \otimes^p_V \circ \theta^U_V$, however this clear as:

$$\theta^U_V \circ \otimes^p_U(t,s) = \theta^U_V(t \otimes s) = t|_V \otimes s|_V = \otimes^p_V(t|_V, s_V) = \otimes^p_V \circ \theta^U_V(t,s)$$

so the assignment $U \mapsto \theta_U$ defines a sheaf morphism. Suppose that $F : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$ is a a bilinear sheaf morphism, then for each $U \subset X$, there is a unique Ψ_U which makes the following diagram commute:

$$\begin{tikzcd} \mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \\ \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur] & \end{tikzcd}$$

We need to show that $\theta^U_V \circ \Psi_U = \Psi_V \circ \theta^U_V$. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\theta^U_V \circ F_U"] \arrow[d] & \mathscr{G}(V) \\
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur] &
\end{tikzcd}$$

and note that:

$$\begin{aligned}
\Psi_V \circ \theta^U_V \circ \otimes^p_U =& \Psi_V \circ \otimes^p_V \circ \theta^U_V \\
=& F_V \circ \theta^U_V \\
=& \theta^U_V \circ F_U
\end{aligned}$$

while:

$$\theta^U_V \circ \Psi_U \circ \otimes^p_U = \theta^U_V \circ F_U$$

so both $\theta^U_V \circ \Psi_U$ and $\Psi_V \circ \theta^U_V$ make the diagram commute implying equality. It follows that $\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property of the tensor product in the category of presheaves.

For iii), we want to show $(\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x$ satisfies the universal property of the the tensor product. The morphism $\otimes^p : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2$ yields a stalk map $\otimes^p_x : (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \to (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x$. Now suppose that $F : (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \to M$ is an $\mathscr{O}_{X,x}$ bilinear map. Now this is equivalent to the data in the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta^U_V"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi_V"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "F"] & \\
& M &
\end{tikzcd}$$

where the ϕ_U is bilinear for each U[100]. In particular, each ϕ_U, gives a unique $\varphi_U : \mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\phi_U"] \arrow[d, "\otimes^p_U"] & M \\
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur, "\varphi_U"] &
\end{tikzcd}$$

We claim that these commute with restriction maps; indeed, consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[r, "\phi_V \circ \theta^U_V"] \arrow[d, "\otimes^p_U"] & M \\
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[ur, "\varphi_U"] &
\end{tikzcd}$$

then:

$$\varphi_U \circ \otimes^p_U = \phi_U = \phi_V \circ \theta^U_V$$

[100]This follows because it is true on the level of sets, and since F is bilinear, and ψ_U is linear, the ϕ_U must be bilinear. In particular they are defined by $\phi_U = F \circ \psi_U$.

so the diagram commutes. We also have that:

$$\begin{aligned}\varphi_V \circ \theta_V^U \circ \otimes_U^p =& \varphi_V \circ \otimes_V^p \circ \theta_V^U \\ =& \phi_V \circ \theta_V^U\end{aligned}$$

so by uniqueness of the morphism, we have that:

$$\varphi_U = \varphi_V \circ \theta_V^U$$

giving us a unique F' which makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}_1(U) \otimes_{\mathscr{O}_X(U)} \mathscr{F}_2(U) \arrow[rr, "\theta_V^U"] \arrow[dr, "\psi'_U"] \arrow[ddr, bend right, "\varphi_U"'] & & \mathscr{F}_1(V) \otimes_{\mathscr{O}_X(V)} \mathscr{F}_2(V) \arrow[dl, "\psi'_V"'] \arrow[ddl, bend left, "\varphi_V"] \\
& (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x \arrow[d, "\exists! F'"] & \\
& M &
\end{tikzcd}$$

We thus now need only check that $F' \circ \otimes_x^p = F$. It suffices to show that:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta_V^U"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\phi_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\phi_V"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "F' \circ \otimes_x^p"] & \\
& M &
\end{tikzcd}$$

Note that $\otimes_x^p$ is given by the following diagram:

$$\begin{tikzcd}
\mathscr{F}_1(U) \oplus \mathscr{F}_2(U) \arrow[rr, "\theta_V^U"] \arrow[dr, "\psi_U"] \arrow[ddr, bend right, "\psi'_U \circ \otimes_U"'] & & \mathscr{F}_1(V) \oplus \mathscr{F}_2(V) \arrow[dl, "\psi_V"'] \arrow[ddl, bend left, "\psi'_V \circ \otimes_U"] \\
& (\mathscr{F}_1)_x \oplus (\mathscr{F}_2)_x \arrow[d, "\otimes_x"] & \\
& (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)_x &
\end{tikzcd}$$

It follows that:

$$F' \circ \otimes_x^p \circ \psi_U = F' \circ \psi' \circ \otimes_U^p = \varphi_U \circ \otimes_U^p = \phi_U$$

implying the claim. □

The next obvious step is to construct a tensor product in the category $\mathrm{Mod}_{\mathscr{O}_X}$, and there is essentially one way to do this:

Definition 5.1.3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\mathscr{O}_X$ modules[101], then the **tensor product of $\mathscr{O}_\mathbf{X}$ modules** is:

$$\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2 := (\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2)^\sharp$$

[101] I.e. sheaves of $\mathscr{O}_X$ modules! This is the last reminder of this nomenclature.

We now wish to check that this is actually the tensor product in the category $\mathrm{Mod}_{\mathscr{O}_X}$, i.e. that $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ is satisfies the universal property.

Lemma 5.1.3. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\mathscr{O}_X$ modules, then $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property of the tensor product in $\mathrm{Mod}_{\mathscr{O}_X}$. Moreover, there is a natural isomorphism $(\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2)_x \cong (\mathscr{F}_1)_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{F}_2)_x$*

Proof. The second statement is an obvious consequence of Lemma 1.2.4. We obtain a morphism $\otimes : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$, by setting $\otimes = \mathrm{sh} \circ \otimes^p$. Now suppose that $F : \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{G}$ is a bilinear $\mathscr{O}_X$ morphism. Then by Lemma 5.1.2 there exists a unique $\mathscr{O}_X$-linear morphism $\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2 \to \mathscr{G}$. By the universal property of sheafification, there is then a unique $\mathscr{O}_X$ linear morphism $\tilde{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}_1 \oplus \mathscr{F}_2 \arrow[rr, "F"] \arrow[d, "\otimes^p"] & & \mathscr{G} \\
\mathscr{F}_1 \otimes^p_{\mathscr{O}_X} \mathscr{F}_2 \arrow[rr, "\mathrm{sh}"] \arrow[urr, "\exists! F'"] & & \mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2 \arrow[u, "\exists! \tilde{F}"']
\end{tikzcd}$$

In particular, we have that $\tilde{F} \circ \otimes = F$, and is the unique map making this diagram commute, hence $\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{F}_2$ satisfies the universal property as desired. □

Now note that if $f : X \to Y$ is a morphism of ringed spaces, we have a sheaf morphism $\hat{f} : f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$, which clearly makes $\mathscr{O}_X$ an $f^{-1}\mathscr{O}_X$ module. It follows that we can take the tensor product $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, which can now be viewed as an $\mathscr{O}_X$ module. Indeed, for each open set U, the map given on simple tensors:

$$\mathscr{O}_X(U) \times (f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X)(U) \longrightarrow (f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X)(U)$$
$$(s, \phi \otimes t) \longmapsto \phi \otimes (st)$$

commutes with restriction maps, and thus defines a morphism of sheaves. This morphism of sheaves clearly makes $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ a presheaf of $\mathscr{O}_X$ modules. We then compose with sheafification to obtain a morphism:

$$\mathscr{O}_X \times f^{-1}\mathscr{F} \otimes^p_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \longrightarrow f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

and using the universal property of sheafification obtain a sheaf morphism:

$$\mathscr{O}_X \times f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \longrightarrow f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

which makes $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ an $\mathscr{O}_X$ module.

Definition 5.1.4. Let $f : X \to Y$ be morphism of ringed spaces, and $\mathscr{F}$ an $\mathscr{O}_Y$ module on Y. Then the **pull back of** $\mathscr{F}$ is the $\mathscr{O}_X$ module $f^*\mathscr{F}$, defined by:

$$f^*\mathscr{F} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$$

Note that $f^* : \mathrm{Mod}_{\mathscr{O}_Y} \to \mathrm{Mod}_{\mathscr{O}_X}$ is obviously a functor by the fact that f^{-1} is a functor, and the universal property of the tensor product. Moreover, the stalk at x is canonically given by:

$$(f^*\mathscr{F})_x \cong \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$$

In the next section, using a sheaf theoretic version of the tensor-hom adjunction for modules, we will be able to show that f^* is left adjoint f_* in the category of $\mathscr{O}_X$ modules. For now we continue to prove general statements regarding pullbacks and tensor products.

Morally tangential to the pullback, is a sheaf theoretic extension of scalars. In particular, if $\mathscr{O}_X \to \mathscr{O}'_X$ is a morphism of a sheaf of rings, and $\mathscr{F}$ is an $\mathscr{O}_X$ module, we can make $\mathscr{F}$ and $\mathscr{O}_{X'}$ module via:

$$\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}'_X$$

We now prove the following basic statements regarding tensor products:

Proposition 5.1.1. *Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{O}_X$ modules, $\mathscr{H}'$ an $\mathscr{O}'_X$ module, and $\mathscr{G}$ is also an $\mathscr{O}'_X$ module. Then there are unique isomorphisms:*

a) $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \cong \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{F}$

b) $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \otimes_{\mathscr{O}'_X} \mathscr{H}' \cong \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{G} \otimes_{\mathscr{O}'_X} \mathscr{H}')$

c) $(\mathscr{F} \oplus \mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{H} \cong (\mathscr{F} \oplus \mathscr{H}) \otimes_{\mathscr{O}_X} (\mathscr{F} \oplus \mathscr{G})$

d) $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{F} \cong \mathscr{F}$.

Proof. Since sheafification is a functor $\mathrm{PMod}_{\mathscr{O}_X} \to \mathrm{Mod}_{\mathscr{O}_X}$, it suffices to show each statement this for the presheaf tensor product. This is however clear, as for each open $U \subset X$ we have the unique isomorphisms a), b), c) and d) just of the underlying $\mathscr{O}_X(U)$ modules $\mathscr{F}(U), \mathscr{G}(U), \mathscr{H}(U)$. By the universal property, all of these isomorphisms, will have to commute with restriction maps, and so they yield $\mathscr{O}_X$ linear isomorphisms of presheaves of $\mathscr{O}_X$ modules, implying the claim. $\square$

We wish to prove similar properties for the pull back, but we need the following lemma:

Lemma 5.1.4. *Let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_Y$ modules, and $f : X \to Y$ a morphism of locally ringed spaces. Then there is a canonical isomorphism:*

$$f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}$$

Proof. Note that there exists a bilinear map $\mathscr{F} \oplus \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}$ given by the tensor product. Applying the f^{-1} we get the following natural bilinear morphism:

$$f^{-1}\mathscr{F} \oplus f^{-1}\mathscr{G} \longrightarrow f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$$

which on stalks is given up to isomorphism by the tensor product map:

$$\otimes_{f(x)} : \mathscr{F}_{f(x)} \oplus \mathscr{G}_{f(x)} \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}$$

By the universal property of the tensor product, there is then $f^{-1}\mathscr{O}_Y$ linear module morphism:

$$F : f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G} \longrightarrow f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$$

By the universal property of the tensor product, the map on stalks must then be the unique one making the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}_{f(x)} \oplus \mathscr{G}_{f(x)} \arrow[r, "\otimes_{f(x)}"] \arrow[d, "\otimes_{f(x)}"] & \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \arrow[ur] &
\end{tikzcd}$$

which must be the identity. It follows that F is an isomorphism on stalks and thus an isomorphism. $\square$

We can now easily prove the following:

Proposition 5.1.2. *Let $\mathscr{F}$, and $\mathscr{G}$ be a $\mathscr{O}_Y$ modules, and $f : X \to Y$ a morphism of ringed spaces. Then we have the following natural isomorphisms:*

a) $f^*\mathscr{O}_Y \cong \mathscr{O}_X$

b) $f^*(\mathscr{F} \oplus \mathscr{G}) \cong f^*\mathscr{F} \oplus f^*\mathscr{G}$

c) $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}$

Moreover, if $\imath : U \to X$ is an open embedding, then $\imath^*\mathscr{F} \cong \mathscr{F}|_U$.

Proof. For a), we have that by d) of Proposition 5.1.1:

$$f^*\mathscr{O}_Y = f^{-1}\mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \cong \mathscr{O}_X$$

For b), we have that by Lemma 5.1.1, and c) of Proposition 5.1.1:

$$\begin{aligned} f^*(\mathscr{F} \oplus \mathscr{G}) =& f^{-1}(\mathscr{F} \oplus \mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\ \cong& (f^{-1}\mathscr{F} \oplus f^{-1}\mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\ \cong& (f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \oplus (f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \\ \cong& f^*\mathscr{F} \oplus f^*\mathscr{G} \end{aligned}$$

For c), by Lemma 5.1.3, and a), d) and b) of Proposition 5.1.1:

$$\begin{aligned} f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) =& f^{-1}(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \\ \cong& (f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}) \otimes_{f^{-1}\mathscr{O}_Y} (\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X) \\ \cong& f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \left[(f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X\right] \\ \cong& (f^{-1} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X) \otimes_{\mathscr{O}_X} f^*\mathscr{G} \\ \cong& f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G} \end{aligned}$$

For the final statement we have that by Corollary 1.3.2, d) of Proposition 5.1.1:

$$\begin{aligned} \imath^*\mathscr{F} =& \imath^{-1}\mathscr{F} \otimes_{\imath^{-1}\mathscr{O}_X} \mathscr{O}_U \\ \cong& \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{O}_U \\ \cong& \mathscr{F}|_U \end{aligned}$$

□

We now provide an elementary proof that the tensor product functor is right exact $- \otimes_A M$.

Lemma 5.1.5. *Let M be an A module, and :*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

be an exact sequence of A modules, then the following sequence is exact:

$$N_1 \otimes_A M \longrightarrow N_2 \otimes_A M \longrightarrow N_3 \otimes_A M \longrightarrow 0$$

Proof. Let f_i denote the morphism $N_i \to N_{i+1}$, and $f_i \otimes \mathrm{Id}_M$ the induced map:

$$N_i \otimes_A M \to N_{i+1} \otimes_A M$$

We first show that $f_2 \times \mathrm{Id}$ is still surjective. Let:

$$\beta = \sum_i n_i \otimes m_i \in N_3 \otimes M$$

then each $n_i = f(n_i')$ for some $n_i' \in N_2$, so the element:

$$\alpha : \sum_i n_i' \otimes m \in N_2 \otimes M$$

satisfies:

$$f_2 \otimes \mathrm{Id}(\alpha) = \beta$$

implying that $f_2 \otimes \mathrm{Id}$ is surjective as desired.

It is clear that $\mathrm{im}\, f_1 \otimes \mathrm{Id} \subset \ker(f_2 \otimes \mathrm{Id})$. Suppose that:

$$\beta = \sum_i n_i \otimes m_i \in \ker(f_2 \otimes \mathrm{Id})$$

and recall that since the original sequence is exact, $N_3 \cong N_2/f_1(N_1)$. Note there is a canonical isomorphism[102]

$$N_2/\,\mathrm{im}\, f_1 \otimes_A M \cong (N_2 \otimes_A M)/(\mathrm{im}(f_1 \otimes \mathrm{Id}))$$

as then $\beta \in \ker(f_2 \otimes \mathrm{Id})$ implies that up to some canonical isomorphism, $[\beta] = 0 \in (N_2 \otimes_A M)/(\mathrm{im}(f_1 \otimes \mathrm{Id}))$, so $\beta \in \mathrm{im}(f_1 \otimes \mathrm{Id})$ implying exactness. □

[102] After noting that clearly $\mathrm{im}\, f \otimes \mathrm{Id} = \mathrm{im}\, \imath \otimes \mathrm{Id}$, where $\imath : \mathrm{im}\, f_1 \to N_2$ is the inclusion map.

Using the above, we wish to extend this right exactness to a statement about $\mathscr{O}_X$ and $\mathscr{O}_Y$ modules:

Proposition 5.1.3. *Let*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

be an exact sequence of $\mathscr{O}_Y$ modules. Then for any $\mathscr{G}$ the following sequence is exact:

$$\mathscr{F}_1 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow \mathscr{F}_2 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow \mathscr{F}_3 \otimes_{\mathscr{O}_Y} \mathscr{G} \longrightarrow 0$$

In particular, $f^* : \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ is a right exact functor.

Proof. We will leverage Proposition 1.2.9 throughout this proof. The second statement will follow from the first, as if

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is an exact sequence, then:

$$0 \longrightarrow f^{-1}\mathscr{F}_1 \longrightarrow f^{-1}\mathscr{F}_2 \longrightarrow f^{-1}\mathscr{F}_3 \longrightarrow 0$$

must be exact, as on stalks it is given up to isomorphism by:

$$0 \longrightarrow (\mathscr{F}_1)_{f(x)} \longrightarrow (\mathscr{F}_2)_{f(x)} \longrightarrow (\mathscr{F}_3)_{f(x)} \longrightarrow 0$$

which are exact because the original sequence was exact. This holds for all $x \in X$, hence Proposition 1.2.9 implies that inverse image sequence is an exact sequence of $f^{-1}\mathscr{O}_Y$ modules. It follows that if the tensor product is right exact then:

$$f^*\mathscr{F}_1 \longrightarrow f^*\mathscr{F}_2 \longrightarrow f^*\mathscr{F}_3 \longrightarrow 0$$

is an exact sequence so f^* is a right exact functor.

To see that $- \otimes_{\mathscr{O}_Y} \mathscr{G}$ is right exact, note that on stalks we have:

$$(\mathscr{F}_1)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow (\mathscr{F}_2)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow (\mathscr{F}_3)_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y \longrightarrow 0$$

which is exact by Lemma 5.1.5. This holds for all $y \in Y$ so Proposition 1.2.9 implies the claim. ☐

We fix the notation that for any indexing set I, $\mathscr{F}^I$ is the direct sum over:

$$\mathscr{F}^I = \bigoplus_{i \in I} \mathscr{F}$$

In other words we want to take infinite coproducts, and not infinite direct products.[103] We now list some full subcategories of Mod_X with the following barrage of definitions:

Definition 5.1.5. Let $\mathscr{F}$ be an $\mathscr{O}_X$ module, then $\mathscr{F}$ is a **quasicoherent $\mathscr{O}_X$ module** if for every $x \in X$, there exists an open neighborhood U of x, and indexing sets I and J such that we have an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

We say that $\mathscr{F}$ is a **finite type** if for every point $x \in X$ there exists a neighborhood U of x such that there is a surjection:

$$\mathscr{O}_U^n \to \mathscr{F}|_U$$

for some $n \in \mathbb{N}$. We say that $\mathscr{F}$ is a **coherent $\mathscr{O}_\mathbf{X}$ module** if $\mathscr{F}$ is of finite type, and for any open set $U \subset X$, and every finite set $\{s_1, \dots, s_m\} \subset \mathscr{F}(U)$, the kernel of the induced map:

$$\mathscr{O}_U^m \to \mathscr{F}|_U$$

[103] In the category of abelian groups, these do not agree when I is infinite. In particular, the infinite coproduct consists of infinite sequences where all but finitely terms are nonzero, and the infinite product is all infinite sequences.

is of finite type. Finally, $\mathscr{F}$ is said to be **locally free** for every point in x there exists an open neighborhood U such that

$$\mathscr{F}|_U \cong \mathscr{O}_U^I$$

If I is finite, we say that $\mathscr{F}$ is **finite locally free**, and if we can always choose I to have cardinality n we say that $\mathscr{F}$ is **locally free of rank n**.

We are particularly interested in $\mathscr{O}_X$ modules which are quasicoherent, coherent, or locally free of rank n, which at times we will call vector bundles. We denote their respective categories by $\mathrm{QCoh}_{\mathscr{O}_X}$, $\mathrm{Coh}_{\mathscr{O}_X}$, and $\mathrm{Vec}_{\mathscr{O}_X}$.

Example 5.1.2. We briefly provide some justification for the term vector bundle. If $\pi : E \to X$ is an honest to god vector bundle over a smooth manifold, and $\{U_i\}$ is a trivializing cover so that:

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^n$$

then there exists a local frame $\{e_1, \ldots, e_r\}$ over $\pi^{-1}(U_i)$. In particular, this local frame induces an isomorphism of sheaves:

$$\Gamma(-, E)|_{U_i} \cong \mathscr{O}_{U_i}^r$$

hence locally free sheaves of $\mathscr{O}_X$ modules are our scheme analogue of vector bundles.

One might hope that we have the following chain of implications:

$$\mathscr{F} \text{ is locally free of rank } n \Rightarrow \mathscr{F} \text{ is coherent} \Rightarrow \mathscr{F} \text{ is quasicoherent}$$

however, as the next example shows, not every finite locally free module need be coherent. In fact the following example shows that there exist locally ringed spaces where $\mathscr{O}_X$ is not even coherent over itself!

Example 5.1.3. Let:

$$X = \operatorname{Spec} A = \operatorname{Spec}\left(k[x, y_1, y_2, \ldots]/\left\langle \{xy_i\}_{i=1}^\infty \right\rangle\right)$$

and take $[x] \in \mathscr{O}_X(X)$. Then the induced map:

$$\phi : \mathscr{O}_X \longrightarrow \mathscr{O}_X$$

given on opens by:

$$\begin{aligned} \mathscr{O}_X(U) &\longrightarrow \mathscr{O}_X(U) \\ s|_U &\longmapsto s|_U \cdot [x]|_U \end{aligned}$$

cannot have kernel of finite type. Indeed, if this were true, then for all $\mathfrak{p} \in X$, we would have that the stalk $(\ker \phi)_\mathfrak{p}$ is finitely generated $\mathscr{O}_{X,\mathfrak{p}}$ module. Let $\mathfrak{p} = \langle [x], [y_1], \ldots \rangle$, then $\mathscr{O}_{X,\mathfrak{p}} \cong A_\mathfrak{p}$, and $\phi_\mathfrak{p} : A_\mathfrak{p} \to A_\mathfrak{p}$ is given by:

$$\frac{[a]}{[s]} \longmapsto \frac{[a] \cdot [x]}{[s]}$$

We claim that $[y_i]/1 \in A_\mathfrak{p}$ is nonzero for all i. Indeed, if it were then there is some $[s] \notin \mathfrak{p}$ such that:

$$[s] \cdot [y] = 0 \Rightarrow s_i y_i \in \left\langle \{xy_i\}_{i=1}^\infty \right\rangle$$

implying that $s_i y_i$ has a factor of x, so $[s]$ has a factor of $[x]$ in it as well. In particular, the $[y_i]/1$ are nonzero in $A_\mathfrak{p}$, and obviously lie in $\ker \phi_\mathfrak{p}$ for all i, so $\langle [y_1]/1, \ldots \rangle \subset \ker \phi_\mathfrak{p}$. If $[a]/[s] \in \ker \phi_\mathfrak{p}$, then there exists some $[t] \notin \mathfrak{p}$ such that :

$$[t] \cdot [x] \cdot [a] = 0 \Rightarrow t \cdot (xa) \in \left\langle \{xy_i\}_{i=1}^\infty \right\rangle$$

However, since $[t] \notin \mathfrak{p}$, we have that t cannot not have a factor of x or y_i in it. It follows that $xa \in \left\langle \{xy_i\}_{i=1}^\infty \right\rangle$, and thus a must be a sum of elements which have factors of y_i in them. It follows that $\ker \phi_\mathfrak{p} = \langle [y_1]/1, \ldots \rangle$, and so $\ker \phi_\mathfrak{p}$ is not a finitely generated $\mathscr{O}_{X,\mathfrak{p}}$ module.

The desired implication is fixed if we require $\mathscr{O}_X$ to be coherent over itself. Indeed we have the following lemma:

Lemma 5.1.6. *Suppose that $\mathscr{O}_X$ is coherent over itself, then $\mathscr{F}$ is a coherent $\mathscr{O}_X$ module if and only if it is of finite presentation, i.e. for every x there exists an open neighborhood U, such that the following sequence is exact for some m and n:*

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Proof. Suppose $\mathscr{F}$ is coherent, then for every x there is an open neighborhood of x such that $\mathscr{F}|_U$ is finitely generated. In other words, if:

$$\phi : \mathscr{O}_U^n \to \mathscr{F}|_U$$

is the surjection, we have an exact sequence:

$$0 \longrightarrow \ker\phi \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Since $\mathscr{F}$ is coherent though, we have that $\ker\phi$ is of finite type, hence there is a neighborhood of x and open neighborhood V, which must be contained in U[104], such that we have a surjection:

$$\mathscr{O}_V^n \to \ker\phi|_V$$

It follows that we have the following exact sequence:

$$\mathscr{O}_V^n \longrightarrow \mathscr{O}_V^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

where the first map is the projection onto $\ker\phi|_V$ composed with the inclusion of $\ker\phi|_V$ into $\mathscr{O}_V^m$.

Now suppose that $\mathscr{F}$ is finitely presented Let $\{V_i\}$ be a cover of X such that we have an exact sequence:

$$\mathscr{O}_{V_i}^n \xrightarrow{\beta_i} \mathscr{O}_{V_i}^m \xrightarrow{\alpha_i} \mathscr{F}|_{V_i} \longrightarrow 0$$

We claim that $\mathscr{F}|_{V_i}$ is the cokernel of β_i. However this clear as there is a unique morphism:

$$\operatorname{coker}\alpha_i \to \mathscr{F}|_{V_i}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{O}_{V_i}^n \arrow[r, "\alpha_i"] & \mathscr{O}_{V_i}^m \arrow[rr, "\beta_i"] \arrow[dr, "\pi"'] & & \mathscr{F}|_{V_i} \arrow[r] & 0 \\
& & \operatorname{coker}\alpha_i \arrow[ur, "\theta"'] & &
\end{tikzcd}$$

On stalks we have the following diagram up to isomorphism:

$$\begin{tikzcd}
\mathscr{O}_{V_i,x}^n \arrow[r, "\alpha_{i,x}"] & \mathscr{O}_{V_i,x}^m \arrow[rr, "\beta_{i,x}"] \arrow[dr, "\pi_x"'] & & \mathscr{F}_x \arrow[r] & 0 \\
& & \operatorname{coker}\alpha_{i,x} \arrow[ur, "\theta_x"'] & &
\end{tikzcd}$$

Since $\beta_{i,x}$ is surjective we have that θ_x is surjective. Note that the $\ker\pi_x = \operatorname{im}\alpha_{i,x}$ by definition, and $\operatorname{im}\alpha_{i,x} = \ker\beta_{i,x}$ by assumption. We have $\ker\beta_{i,x} = \pi_x^{-1}(\ker\theta_x)$, so:

$$\ker\pi_x = \operatorname{im}\alpha_{i,x} = \pi_x^{-1}(\ker\theta_x)$$

In particular, $\pi_x^{-1}(0) = \pi_x^{-1}(\ker\theta_x)$, implying that:

$$0 = \ker\theta_x$$

[104]This is because $\ker\phi$ is only a sheaf on U.

because π_x is surjective. It follows that θ_x is an isomorphism, so θ is an isomorphism.

Accepting for the moment that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category,[105] it follows that $\mathscr{F}|_{V_i}$ is coherent for each i. Now let $\{s_1, \dots, s_l\} \subset \mathscr{F}(U)$, and $\phi : \mathscr{O}_U^l \to \mathscr{F}|_U$ the associated map. Then for each i, we have that $\phi|_{U\cap V_i} : \mathscr{O}_{U\cap V_i} \to \mathscr{F}|_{U\cap V_i}$ must have kernel of finite type as each $\mathscr{F}|_{V_i}$ is coherent. Since the V_i cover X, it follows that $\ker \phi$ must be of finite type hence $\mathscr{F}$ is coherent as desired. □

We have the immediate corollary:

Corollary 5.1.2. *Let X be a locally ringed space such that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$ module. Then we have the following chain of implications:*

$$\mathscr{F} \text{ is locally free of rank } n \Rightarrow \mathscr{F} \text{ is coherent} \Rightarrow \mathscr{F} \text{ is quasicoherent}$$

Proof. By Lemma 5.1.6 every coherent $\mathscr{O}_X$ module is finitely presented and thus quasicoherent. Now suppose that $\mathscr{F}$ is locally free of rank n, then $\mathscr{F}$ is locally isomorphic to $\mathscr{O}_U^n$, so again accepting for the moment that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, we have that there exists an open cover on which $\mathscr{F}|_U$ is coherent. The same argument at the end of Lemma 5.1.6 implies that $\mathscr{F}$ is coherent. □

Now $\mathrm{Vec}_{\mathscr{O}_X}$ has no hope of being an abelian category as the kernel of a vector bundle homomorphism between manifolds, is only a vector bundle when the map has constant rank on each fibre. Furthermore, $\mathrm{QCoh}_{\mathscr{O}_X}$ will be an abelian category when X is a scheme, but there are locally ringed spaces for which this is not true; we will not spend time delving into counter examples. What is always true is that the category of coherent modules over a ringed space is always abelian, a statement we will prove in this section. Before embarking on this endeavor, we first prove a few key results about the categories $\mathrm{QCoh}_{\mathscr{O}_X}$ and $\mathrm{Vec}_{\mathscr{O}_X}$. We will need the following lemma:

Lemma 5.1.7. *Suppose we have exact sequences of $\mathscr{O}_X$ modules:*

$$\mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

$$\mathscr{G}_1 \longrightarrow \mathscr{G}_2 \longrightarrow \mathscr{G}_3 \longrightarrow 0$$

Then there exists an exact sequence of the form:

$$(\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{G}_2) \oplus (\mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G}_1) \longrightarrow \mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G}_2 \longrightarrow \mathscr{F}_3 \otimes_{\mathscr{O}_X} \mathscr{G}_3 \longrightarrow 0$$

Proof. Denote by f_i the maps $\mathscr{F}_i \to \mathscr{F}_{i+1}$, and by g_i the maps $\mathscr{G}_i \to \mathscr{G}_{i+1}$. We construct the first map in the claimed exact sequence, which we denote by β, to be the direct sum of $f_1 \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes g_2$, and the second map to be the unique map $f_2 \otimes g_2$. All of these maps come from the obvious diagrams.

To show that this sequence is exact, it suffices to show this on stalks, and so we need only prove this in the category of A-modules. So replacing $\mathscr{F}_i$ with M_i, and $\mathscr{G}_i$ with N_i, and denoting the maps by the same notation, we want to show that the following sequence is exact:

$$(M_1 \otimes_A N_2) \oplus (M_2 \oplus N_1) \xrightarrow{\ \beta\ } M_2 \otimes_A N_2 \xrightarrow{f_2 \otimes g_2} M_3 \otimes_A N_3 \longrightarrow 0$$

The map $f_2 \otimes g_2$ is surjective: if $m_3 \otimes n_3 \in M_3 \otimes_A N_3$, then there exists some $m_2 \in M_2$ and $n_3 \in N_2$ such that $f_2(m_2) = m_3$ and $g_2(n_2) = n_3$, hence

$$f_2 \otimes g_2(m_2 \otimes n_2) = f_2(n_2) \otimes g_2(n_2) = n_3 \otimes m_3$$

Since f_2 and g_2 are surjective, we have that:

$$M_3 \cong M_2/\operatorname{im} f_1 \qquad \text{and} \qquad N_3 \cong N_2/\operatorname{im} g_1$$

hence:

$$M_3 \otimes_A N_3 \cong M_2/\operatorname{im} f_1 \otimes_A N_2/\operatorname{im} g_1 \cong (M_2 \otimes_A N_2)/(\operatorname{im}(f_1 \otimes \mathrm{Id}) + \operatorname{im}(\mathrm{Id} \otimes g_1))$$

The submodule we are quotienting out by is precisely $\operatorname{im} \beta$, hence the sequence is exact by the same argument in Lemma 5.1.5 □

[105] We prove this in Theorem 5.1.1.

Proposition 5.1.4. *Let $f : X \to Y$ be a morphism of locally ringed spaces, and $\mathscr{F}$ an $\mathscr{O}_Y$ module. The following hold:*

i) The categories $\mathrm{QCoh}_{\mathscr{O}_X}$, and $\mathrm{Vec}_{\mathscr{O}_X}$ are additive.

ii) $\mathrm{QCoh}_{\mathscr{O}_X}$ and $\mathrm{Vec}_{\mathscr{O}_X}$ are closed under tensor products.

iii) If $\mathscr{F}$ is quasicoherent, then so is $f^*\mathscr{F}$.

iv) If $\mathscr{F}$ is locally free of finite rank n then so is $f^*\mathscr{F}$.

Moreover, and if $f : Y \to X$ is a morphism of locally ringed spaces, pulling back induces functors $f^* : \mathrm{QCoh}_{\mathscr{O}_X} \to \mathrm{QCoh}_{\mathscr{O}_Y}$ and $f^* : \mathrm{Vec}_{\mathscr{O}_X} \to \mathrm{Vec}\mathscr{O}_Y$.

Proof. Since each category is a full subcategory, and the 0 object obviously lies in each, we need only show that the direct sums stay in their respective categories.

Suppose that $\mathscr{F}$ and $\mathscr{G}$ are locally free of rank n, then for each $x \in X$ there exists $U, V \subset X$ containing x such that:

$$\mathscr{F}|_U \cong \mathscr{O}_U^n \qquad \text{and} \qquad \mathscr{G}|_V \cong \mathscr{O}_V^m$$

It is then obvious that on $U \cap V$:

$$(\mathscr{F} \oplus \mathscr{G})|_{U\cap V} \cong \mathscr{F}|_{U\cap V} \oplus \mathscr{G}|_{U\cap V} \cong \mathscr{O}_{U\cap V}^n \oplus \mathscr{O}_{U\cap V}^m \cong \mathscr{O}_{U\cap V}^{n+m}$$

so $\mathscr{F} \oplus \mathscr{G}$ is locally free of rank $n + m$.

Supposing that $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent, we can via a similar argument above, find an open set U on which there exists indexing sets I, J, K, and L such that the following sequences are exact:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

$$\mathscr{O}_U^K \longrightarrow \mathscr{O}_U^L \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

hence the following sequence is exact:

$$\mathscr{O}_U^{I\cup K} \longrightarrow \mathscr{O}_U^{J\cup L} \longrightarrow (\mathscr{F} \oplus \mathscr{G})|_U \longrightarrow 0$$

implying that both $\mathrm{QCoh}_{\mathscr{O}_X}$ and $\mathrm{Vec}\mathscr{O}_X$ are additive proving i).

Let $\mathscr{F}$ and $\mathscr{G}$ be locally free of rank n and m respectively. Finding an open set U on which both are trivial, and letting $\imath : U \to X$ be the open embedding, we have that by inductively applying part c) of Proposition 5.1.1:

$$\begin{aligned}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U &\cong \imath^{-1}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \\ &\cong \imath^{-1}\mathscr{F} \otimes_{\imath^{-1}\mathscr{O}_X} \imath^{-1}\mathscr{G} \\ &\cong \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \\ &\cong \mathscr{O}_U^n \otimes_{\mathscr{O}_U} \mathscr{O}_U^m \\ &\cong \mathscr{O}_U^{n+m}\end{aligned}$$

as desired. We note that if $\mathscr{F}$ and $\mathscr{G}$ are not of finite rank, i.e. $\mathscr{F}|_U \cong \mathscr{O}_U^I$ and $\mathscr{G}|_U \cong \mathscr{O}_U^J$ then over U there is an isomorphism:

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U \cong \mathscr{O}_U^{I\times J}$$

Indeed, this is true on the level of stalks, so the induced map will be an isomorphism.

Supposing that $\mathscr{F}$ and $\mathscr{G}$ are quasicoherent, and finding an open set on which we have the exact sequences:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

$$\mathscr{O}_U^K \longrightarrow \mathscr{O}_U^L \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

By Lemma 5.1.7, we have the following short exact sequence

$$(\mathscr{O}_U^I \otimes_{\mathscr{O}_U} \mathscr{O}_U^L) \oplus (\mathscr{O}_U^J \otimes_{\mathscr{O}_U} \mathscr{O}_U^K) \longrightarrow \mathscr{O}_U^J \otimes_{\mathscr{O}_U} \mathscr{O}_U^L \longrightarrow (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})|_U \longrightarrow 0$$

hence $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasicoherent by the preceding result regarding tensor products of locally free/free sheaves proving ii).

For iii) let $\mathscr{F}$ be an $\mathscr{O}_Y$ module which is locally free. Then if $\mathscr{F}|_U \cong \mathscr{O}_U^n$ we have by part a) and b) of Proposition 5.1.2 :

$$f^*\mathscr{F}|_{f^{-1}(U)} \cong f^*\mathscr{O}_U^n \cong$$

where $\mathscr{O}_{f^{-1}(U)}$ is $\mathscr{O}_X$ restricted to $f^{-1}(U)$.

For iv), if $\mathscr{F}$ is a quasicoherent $\mathscr{O}_Y$ module, we have an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for some open $U \subset Y$. Since f^* is right exact by Proposition 5.1.3, we have by part a) of Proposition 5.1.2 that the following sequence is exact:

$$\mathscr{O}_{f^{-1}(U)}^I \longrightarrow \mathscr{O}_{f^{-1}(U)}^J \longrightarrow f^*\mathscr{F}|_{f^{-1}(U)} \longrightarrow 0$$

implying the claim. □

The goals for the the rest of this section are as follows: we wish to prove that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, that the tensor products of coherent $\mathscr{O}_X$ modules are coherent, and that the pullback of coherent $\mathscr{O}_Y$ modules is coherent under suitable conditions, namely that both $\mathscr{O}_Y$ and $\mathscr{O}_X$ are coherent over themselves. We begin with proving that $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category, an exercise we break into stages. We begin with showing kernels and cokernels are coherent.

Lemma 5.1.8. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of coherent $\mathscr{O}_X$ modules, then* $\ker f$ *and* $\mathrm{coker}\, f$ *are both coherent.*

Proof. Note that both $\mathscr{F}$ and $\mathscr{G}$ are of finite type; in particular, each x there exists a U such that:

$$\pi : \mathscr{O}_U^n \to \mathscr{F}|_U$$

is surjective. Since $\mathscr{G}$ is coherent, the kernel of the composition

$$f \circ \pi : \mathscr{O}_U^n \to \mathscr{G}|_U$$

is of finite type. We claim that the image of:

$$\pi \circ \imath : \ker(f \circ \pi) \hookrightarrow \mathscr{O}_U^n \to \mathscr{F}|_U$$

is $\ker f$. In particular, we claim that:

$$\ker(f|_U \circ \pi) \to \mathscr{F}|_U \to \mathscr{G}|_U$$

is exact at $\mathscr{F}|_U$. It suffices to prove this on stalks; clearly the composition is zero so that $\mathrm{im}\, \pi_x \circ \imath_x \subset \ker f_x$. Suppose that $s_x \in \ker f_x$, then by surjectivity there exists a $t_x \in \mathscr{O}_{U,x}^n$ such that $\pi_x(t_x) = s_x$, so $s_x \in \mathrm{im}\, \pi_x$. In particular, $t_x \in \ker f_x \circ \pi_x$ by definition, hence $s_x \in \mathrm{im}\, \pi_x \circ \imath_x$. We thus have a surjection:

$$\ker(f|_U \circ \pi) \longrightarrow \ker f|_U$$

and since $\ker(f|_U \circ \pi)$ is of finite type, we have that for all $x \in U$ there is some open neighborhood V of x and a surjection:

$$\mathscr{O}_V^m \to \ker(f \circ \pi)|_V \to \ker f|_V$$

so $\ker f$ is of finite type. Now $\ker f$ is a finite type sub $\mathscr{O}_X$ module of $\mathscr{F}$; let $\{s_1, \ldots, s_n\} \in \ker f(U)$, then the induced morphism:

$$\phi : \mathscr{O}_U^m \to \ker f|_U$$

must have kernel of finite type because $\ker f|_U$ injects into $\mathscr{F}|_U$. It follows that $\ker f$ is a coherent $\mathscr{O}_X$ module.

Now consider $\operatorname{coker} f$; since $\mathscr{G}$ surjects onto $\operatorname{coker} f$ we have that $\operatorname{coker} f$ must be of finite type. Let $\{s_1, \ldots, s_n\} \subset (\operatorname{coker} f)(U)$, and:

$$\phi : \mathscr{O}_U^n \to \operatorname{coker} f|_U$$

the induced morphism. Let $x \in U$, and consider $s_{1,x}, \ldots, s_{n,x} \in \operatorname{coker} f_x$; since $\operatorname{coker} f_x \cong \mathscr{G}_x / \operatorname{im} f_x$, we have lifts $t_{1,x}, \ldots, t_{n,x} \in \mathscr{G}_x$. By taking $2n$ intersections we obtain an open neighborhood of x, V, with sections $s_1', \ldots, s_n' \in \operatorname{coker} f(V)$ and lifts $t_1, \ldots, t_n \in \mathscr{G}(V)$ such that $\pi(s_i') = t_i$. By restricting to a smaller open set if necessary, we may assume that there is a surjection $\xi : \mathscr{O}_V^m \to \operatorname{im} f|_V$. We have that $t_1, \ldots, t_n$, and ξ determine a surjection:

$$\beta : \mathscr{O}_V^n \oplus \mathscr{O}_V^m \longrightarrow \mathscr{G}|_V$$

We thus can construct the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_V^m \arrow[r, "\imath_m"] \arrow[d, "\xi"] & \mathscr{O}_V^m \oplus \mathscr{O}_V^n \arrow[r, "\pi_n"] \arrow[d, "\beta"] & \mathscr{O}_V^n \arrow[r] \arrow[d, "\phi|_V"] & 0 \\
0 \arrow[r] & \operatorname{im} f|_V \arrow[r, "\imath"] & \mathscr{G}|_V \arrow[r, "\pi"] & \operatorname{coker} f|_V \arrow[r] & 0
\end{tikzcd}$$

where both rows are exact. The snake lemma, which applies in any abelian category, gives an exact sequence of the form:

$$0 \longrightarrow \ker \xi \longrightarrow \ker \beta \longrightarrow \ker \phi|_V \longrightarrow \operatorname{coker} \xi \longrightarrow \cdots$$

However, ξ is a surjection, hence we have that $\ker \beta$ surjects onto $\ker \phi|_V$ as $\operatorname{coker} \xi = 0$ by Proposition 1.2.8. It follows that since $\ker \beta$ is of finite type as $\mathscr{G}$ is coherent, that $\ker \phi|_V$ must be of finite type as well, implying the claim. □

We have the following corollary:

Corollary 5.1.3. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves of $\mathscr{O}_X$ modules, where $\mathscr{F}$ is of finite type, and $\mathscr{G}$ is coherent. Then $\ker f$ is of finite type.*

Proof. This follows by noticing that the part of the proof in Lemma 5.1.8 showing that $\ker f$ was of finite type, depended only on $\mathscr{F}$ be of finite type. □

The task of showing that $\mathrm{Coh}_{\mathscr{O}_X}$ has direct sums is surprisingly delicate as far as we can tell. In fact, it seems that the best path towards a proof of this is via the following lemma:

Lemma 5.1.9. *Let:*

$$0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$ modules. If any two of the three are coherent, then so is the third.

Proof. Note that if $\mathscr{G}$ and $\mathscr{H}$ are coherent, then $\mathscr{F}$ is the kernel of f and thus coherent by Lemma 5.1.8. If $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{H}$ is the cokernel of the morphism $\mathscr{F} \to \mathscr{G}$, and thus coherent by Lemma 5.1.8.

Now suppose that $\mathscr{F}$ and $\mathscr{H}$ are coherent. We first show that $\mathscr{G}$ is finite type; since $\mathscr{F}$ and $\mathscr{H}$ are finite type, we can find a common open set U such that $\mathscr{O}_U^n$ and $\mathscr{O}_U^m$ surject onto $\mathscr{F}|_U$ and $\mathscr{H}|_U$

respectively. Taking U to be small enough, the same argument in Lemma 5.1.8 demonstrates that we can take lifts of the sections which define the map $\mathscr{O}_U^m$. It follows that we obtain a morphism $\mathscr{O}_u^n \oplus \mathscr{O}_U^m \to \mathscr{G}|_U$ which manifestly makes the following diagrams commute:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_U^n \arrow[r] \arrow[d] & \mathscr{O}_U^n \oplus \mathscr{O}_U^m \arrow[r] \arrow[d] & \mathscr{O}_U^m \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}|_U \arrow[r] & \mathscr{G}|_U \arrow[r] & \mathscr{H}|_U \arrow[r] & 0 \end{tikzcd}$$

It suffices to check surjectivity of the middle morphisms on stalks, however this then follows from the surjectivity part of the five lemma, implying that $\mathscr{G}$ is of finite type.

Now let $\{s_1, \dots, s_n\} \subset \mathscr{G}(U)$ define the morphism:

$$\phi : \mathscr{O}_U^n \to \mathscr{G}|_U$$

Then $\ker g|_U \circ \phi$ is of finite type as $\mathscr{H}|_U$ is coherent. We thus have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathscr{O}_U^n \arrow[r, "\mathrm{Id}"] \arrow[d, "\phi"] & \mathscr{O}_U^n \arrow[r] \arrow[d, "g|_U \circ \phi"] & 0 \\ 0 \arrow[r] & \mathscr{F}|_U \arrow[r, "f|_U"] & \mathscr{G}|_U \arrow[r, "g|_U"] & \mathscr{H}_U \arrow[r] & 0 \end{tikzcd}$$

and so the snake lemma once again implies an exact sequence of the form:

$$0 \longrightarrow \ker \phi \longrightarrow \ker g|_U \circ \phi \longrightarrow \operatorname{coker} 0 \longrightarrow \cdots$$

However, $\operatorname{coker} 0 = \mathscr{F}$, hence $\ker \phi$ is the kernel of the morphism $\ker g|_U \circ \phi \to \mathscr{F}|_U$. The claim now follows from Corollary 5.1.3 ☐

We now prove the main result of the section:

Theorem 5.1.1. *Let $(X, \mathscr{O}_X)$ be a ringed space, then $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category.*

Proof. First note that $\mathrm{Coh}_{\mathscr{O}_X}$ is additive; indeed Lemma 5.1.9 implies that if $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{F} \oplus \mathscr{G}$ are coherent, because we have the following exact sequence:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \oplus \mathscr{G} \longrightarrow \mathscr{G} \longrightarrow 0$$

Moreover, Lemma 5.1.8 implies that kernels and cokernels of coherent modules are coherent.

We need to show that monomorphisms are kernels, and epimorphisms are cokernels. However, Theorem 1.2.1, shows that if $f : \mathscr{F} \to \mathscr{G}$ is a monomorphism between coherent $\mathscr{O}_X$ modules, then $(\mathscr{F}, f)$ is the kernel of:

$$\pi : \mathscr{G} \to \operatorname{coker} f$$

Since $\mathscr{G}$ is coherent, and $\operatorname{coker} f$ is coherent by Lemma 5.1.8, we have that f is the kernel of a morphism between coherent $\mathscr{O}_X$ modules, as desired. Similarly, if $f : \mathscr{F} \to \mathscr{G}$ is an epimorphism, then $(\mathscr{G}, f)$ is the cokernel of $\imath : \ker f \to \mathscr{F}$, which is a morphism of coherent $\mathscr{O}_X$ module by Lemma 5.1.8. It follows that epimorphisms are cokernels, and so $\mathrm{Coh}_{\mathscr{O}_X}$ is an abelian category. ☐

We end this section with the following result:

Proposition 5.1.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. The following hold:*

i) *$\mathrm{Coh}_{\mathscr{O}_X}$ is closed under tensor products.*

ii) *If $\mathscr{O}_X$ and $\mathscr{O}_Y$ are coherent modules, then f^* is a functor $\mathrm{Coh}_{\mathscr{O}_Y} \to \mathrm{Coh}_{\mathscr{O}_X}$.*

Proof. Note that clearly finitely presented $\mathscr{O}_X$ modules are closed under tensor products by part *ii*) of Proposition 5.1.4. Let $\mathscr{F}$ be of finite presentation, and $\mathscr{G}$ be coherent, then by right exactness of the tensor product, for some U we have:

$$\mathscr{O}_U^n \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{O}_U^m \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow 0$$

By parts a) and c) of Proposition 5.1.1, this can be rewritten as:

$$\mathscr{G}^n|_U \longrightarrow \mathscr{G}^m|_U \longrightarrow \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow 0$$

By Theorem 5.1.1, $\mathrm{Coh}_{\mathscr{O}_X}$ forms an abelian category, hence the first two terms are coherent $\mathscr{O}_X$ modules. It follows that $\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U$ is a cokernel of a morphism between coherent sheaves and is thus coherent. Since all such U cover X, we have that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is coherent.

By Lemma 5.1.6, since $\mathscr{O}_Y$ is coherent, we have that $\mathscr{F}$ being a coherent $\mathscr{O}_Y$ module is equivalent to $\mathscr{F}$ being of finite presentation. It follows by part *iv*) of Proposition 5.1.4 that $f^*\mathscr{F}$ is locally of finite presentation as well. Since $\mathscr{O}_X$ is coherent, the same lemma proves that $f^*\mathscr{F}$ is coherent, implying the claim. □

5.2 Tensor-Hom Adjunction for $\mathscr{O}_X$ Modules

In this section we continue to assume that $(X, \mathscr{O}_X)$ is an arbitrary ringed space, and develop an analogue of the tensor hom adjunction in the category of $\mathscr{O}_X$ modules. This will allow us to easily prove that f^* is the left adjoint of the direct image functor. We begin with a review of the statement and proof in the category of A-modules.

Theorem 5.2.1. *Let M and N be A modules, and P and N be B modules. There is a natural isomorphism of abelian groups:*

$$\mathrm{Hom}_B(M \otimes_A N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$$

Before proving this statement recall that the B module structure on $M \otimes_A N$ is given by:

$$b \cdot (m \otimes n) = m \otimes (bn)$$

and that the A module structure on $\mathrm{Hom}_B(N, P)$ is given by:

$$\begin{aligned}(a \cdot \phi) : N &\longrightarrow P \\ n &\longmapsto \phi(a \cdot n)\end{aligned}$$

We now begin the proof:

Proof. We first construct a map:

$$\Psi : \mathrm{Hom}_B(M \otimes_A N, P) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$$

Let $f \in \mathrm{Hom}_B(M \otimes_A N, P)$, and let $\otimes : M \oplus N \to M \otimes_A N$ be the tensor map. Let $\tilde{f} = f \circ \otimes$, then we claim that $\tilde{f}$ B linear in the second component. It is clear that the additivity condition holds; let $m \in M$, $n \in N$, and $b \in B$, then we have the following:

$$\tilde{f}(m, bn) = f(m \otimes bn) = f(b \cdot (m \otimes n)) = b \cdot f(m \otimes n)$$

as desired. For each m, we thus get a map $m \lrcorner \tilde{f}$ defined by:

$$(m \lrcorner \tilde{f})(n) = \tilde{f}(m, n)$$

which is B-linear. We want to see that the assignment $m \mapsto m \lrcorner \tilde{f}$ is A linear; let $m_1, m_2 \in M$ and $a_1, a_2 \in A$, then for all n in N:

$$\begin{aligned}(a_1m_1 + a_2m_2) \lrcorner \tilde{f}(n) =& \tilde{f}(a_1m_1 + a_2m_2, n)\\ =& f(a_1m_1 \otimes n + a_2m_2 \otimes n)\\ =& f(a_1m_1 \otimes n) + f(a_2m_2 \otimes n)\\ =& f(m_1 \otimes a_1n) + f(m_2 \otimes a_2n)\\ =& m_1 \lrcorner \tilde{f}(a_1n) + m_2 \lrcorner \tilde{f}(a_2n)\\ =& a_1 \cdot (m_1 \lrcorner \tilde{f})(n) + a_2 \cdot (m_2 \lrcorner \tilde{f})(n)\end{aligned}$$

It follows that we have obtained a map:

$$\begin{aligned}\Psi : \mathrm{Hom}_B(M \otimes_A N, P) &\longrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))\\ f &\longrightarrow (m \mapsto m \lrcorner \tilde{f})\end{aligned}$$

This is clearly a morphism of abelian groups, and is functorial/natural in N, and so the morphism is natural.

Suppose that $\Psi(f) = 0$, then for all m we have that $m \lrcorner \tilde{f} = 0$. In particular, for all simple tensors $m \otimes n$, we would have that:

$$f(m \otimes n) = (m \lrcorner \tilde{f})(n) = 0$$

Since f is a group homomorphism, it follows that f is identically zero on $M \otimes_A N$ and is thus the zero morphism. This shows that Ψ is injective.

Now let $\phi \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, P))$; then we obtain a map:

$$\begin{aligned}g : M \oplus N &\longrightarrow P\\ (m, n) &\longmapsto (\phi(m))(n)\end{aligned}$$

Note that g satisfies the following:

$$g(am, n) = (\phi(am))(n) = (a \cdot \phi(m))(n) = \phi(m)(an) = g(m, na)$$

and is additive in both entries. By the construction of the tensor product[106], these are the minimal requirements to get a well defined group homomorphism:

$$f : M \otimes_A N \to P$$

which satisfies $f \circ \otimes = g$, and is obviously B-linear. Clearly, the assignment $m \mapsto m \lrcorner \tilde{f}$ is then equal to the map ϕ. It follows that Ψ is an isomorphism implying the claim. □

The first stumbling block in extending the above result to the category of $\mathscr{O}_X$ modules, is that $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is not a sheaf, so an expression of the form:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathrm{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H}))$$

makes no sense. We fix this with the following definition:

Definition 5.2.1. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathscr{O}_X$ modules, then the **Hom sheaf**[107], denoted $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, is the sheaf defined on opens by:[108]

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

Note that since $\mathscr{O}_U$ modules form an abelian category, we have that this is a priori a presheaf of abelian groups.

[106]See for example, Atiyah Macdonald Chapter 2, Proposition 2.12.

[107]In any abelian category, a functor of this form is called an internal hom functor, as it is an analogue of the true Hom functor, but has value in the abelian category, rather than the category of abelian groups.

[108]This can obviously defined similarly for general sheaves, or sheaves of abelian groups, etc.

Lemma 5.2.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$ modules, then $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a sheaf of $\mathscr{O}_X$ modules.*

Proof. We first show that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a sheaf; the restriction maps are the obvious ones sending a natural transformation to the restricted natural transformation. These obviously satisfy the presheaf conditions. Let $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U)$, and $\{U_i\}$ be a cover for U such that $F|_{U_i}$ is the zero morphism. In particular, this implies that that the stalk map F_x is zero for all $x \in U$, hence F is the zero morphism.

Now suppose that $F_i \in \mathrm{Hom}_{\mathscr{O}_{U_i}}(\mathscr{F}|_{U_i},\mathscr{G}|_{U_i})$ so that $F_i|_{U_i\cap U_j} = F_j|_{U_i\cap U_j}$, then Proposition 1.2.11 implies that the F_i glue[109] together to yield a unique morphism $\mathscr{F}|_U \to \mathscr{G}|_U$ which restricts to F_i on U_i. It follows that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a sheaf.

We define a sheaf morphism

$$\mathscr{O}_X \times \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$$

as follows: let $U \subset X$ be arbitrary, then $(s,F) \in \mathscr{O}_X(U) \times \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U)$ is sent to the sheaf morphism $s \cdot F$, defined on opens $V \subset U$ by:

$$\begin{aligned}(s\cdot F)_V : \mathscr{F}(V) &\longrightarrow \mathscr{G}(V)\\ t &\longmapsto s|_V \cdot F_V(t)\end{aligned}$$

This clearly commutes with restrictions and so defines an element in $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U)$. The assignment $(s,F) \mapsto s\cdot F$ also clearly commutes with restrictions $\square$

If $\mathscr{F}$ is an $\mathscr{O}_X$ module, then we denote by $\mathscr{F}^*$ the dual sheaf $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X)$. One may hope that taking stalks commutes with the $\underline{Hom}$, i.e. for an isomorphism of the form:

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x \cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

however this is rarely the case:

Example 5.2.1. Let X be irreducible, and $\mathscr{O}_X$ be the constant sheaf with values in $\mathbb{Z}_{\mathfrak{p}}$ on X. Note since no finite intersection of open sets can be empty, this is the honest to god constant presheaf . Let $\mathscr{F}$ be the sky scraper sheaf of $\mathbb{Z}_{\mathfrak{p}}$ at x, see Lemma 1.2.7. Supposing x is a closed point, then we claim that:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{O}_U) = 0$$

for all $U \subset X$. If $x \notin U$ then $\mathscr{F}|_U = 0$ hence the claim; if $x \in U$, then $\mathscr{F}|_U$ is nonzero, however for $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{O}_U)$, and $s \in \mathscr{F}|_U(V)$, we claim that:

$$F_V(s) = 0$$

for all $V \subset U$, and $s \in \mathscr{F}(V)$. Indeed, the restriction maps $\theta^V_W : \mathscr{O}_U(V) \to \mathscr{O}_U(W)$ are the identity, so let $W = V \smallsetminus \{x\}$, then $s|_W = 0$, hence we have that:

$$0 = F|_W(s|_W) = \theta^V_W \circ F_V(s)$$

so by injectivity, we have that $F_V(s) = 0$. It follows that F is identically zero on all $V \subset U$, hence F is the zero morphism. We have thus shown that:

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_X)_x = 0 \ncong \mathbb{Z}_{\mathfrak{p}} = \mathrm{Hom}_{\mathbb{Z}_{\mathfrak{p}}}(\mathbb{Z}_{\mathfrak{p}},\mathbb{Z}_{\mathfrak{p}}) = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

We will eventually show that the remedy for this is when $\mathscr{F}$ is finitely presented, which will imply that:

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x \cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

[109] The morphisms gluing the $\mathscr{F}|_{U_i}$ together are just the identity morphisms, and similarly for the $\mathscr{G}|_{U_i}$.

Just as in the category of A-modules, we have that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},-)$ and $\underline{Hom}_{\mathscr{O}_X}(-,\mathscr{F})$ are functors. Indeed, we know where each should send objects, so let $F : \mathscr{G} \to \mathscr{H}$, then we have a morphism:

$$F^* : \underline{Hom}_{\mathscr{O}_X}(\mathscr{H},\mathscr{F}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})$$

given on opens by:

$$\begin{aligned} F_U^* : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{H}|_U,\mathscr{F}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}|_U,\mathscr{F}|_U) \\ G &\longmapsto G \circ F|_U \end{aligned}$$

which obviously commute with restriction maps. One easily checks that that $(F \circ G)^* = G^* \circ F^*$, and so $\underline{Hom}_{\mathscr{O}_X}(-,\mathscr{F})$ is a contravariant functor. Similarly, we have $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},-)$ is a covariant functor, sending F to:

$$F_* : \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{H})$$

given on opens by:

$$\begin{aligned} (F_*)_U : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{H}|_U) \\ G &\longmapsto F|_U \circ G \end{aligned}$$

Our goal is to show that these functors are exact, just as in the case of A modules.

Proposition 5.2.1. *Let $\mathscr{F}$ an $\mathscr{O}_X$ module, then the functors $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},-)$ and $\underline{Hom}_{\mathscr{O}_X}(-,\mathscr{F})$ are left exact.*[110]

Proof. Let:

$$0 \longrightarrow \mathscr{G}_1 \xrightarrow{f_1} \mathscr{G}_2 \xrightarrow{f_2} \mathscr{G}_3$$

be an exact sequence of $\mathscr{O}_X$ modules. We first show that:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}_1) \xrightarrow{f_{1*}} \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}_2) \xrightarrow{f_{2*}} \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}_3)$$

is exact. It suffices to show that this exact on every open set,[111] i.e. that the following sequence of abelian groups is exact for every U:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}_1|_U) \xrightarrow{(f_{1*})_U} \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}_2|_U) \xrightarrow{(f_{2*})_U} \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}_3|_U)$$

Suppose that $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}_1|_U)$ satisfies:

$$(f_{1*})_U(F) = f_1|_U \circ F = 0$$

In particular since $\ker f_1|_U = 0$, we must have that $\ker F = \mathscr{F}_U$, so F is the zero morphism, implying $(f_{1*})_U$ is injective. Now clearly if $F \in \mathrm{im}(f_{1*})_U$ then $(f_{2*})_U(F) = 0$; suppose that $F \in \ker(f_{2*})_U$, then we have that:

$$f_2|_U \circ F = 0$$

then we want to show that $F = f_1|_U \circ G$ for some $G \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}_2|_U)$. Note that since $\ker f_1|_U = 0$, we have that $\mathscr{G}_1|_U$ is canonically $\mathrm{im}\, f_1|_U = \ker f_2|_U$, i.e. $(\mathscr{G}_1|_U, f_1|_U)$ satisfies the universal property of the kernel. By the aforementioned, we have that there exists a unique map G such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}|_U \arrow[r, "F"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists! G"'] & \mathscr{G}_2|_U \arrow[r, "f_2|_U"] & G_3|_U \\ & \mathscr{G}_1|_U \arrow[u, dashed, "f_1|_U"'] & \end{tikzcd}$$

[110] A contravariant functor is left (right) exact if it takes right (left) exact sequences to left (right) exact sequences.

[111] Note that exact sequences of sheaves don't *need* to be exact on open sets, but if they are exact on open sets then they are exact.

implying the claim.

Now let:

$$\mathscr{G}_1 \xrightarrow{f_1} \mathscr{G}_2 \xrightarrow{f_2} \mathscr{G}_3 \longrightarrow 0$$

be an exact sequence of $\mathscr{O}_X$ modules. By the same argument to show that:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{G}_3, \mathscr{F}) \xrightarrow{f_2^*} \underline{Hom}_{\mathscr{O}_X}(\mathscr{G}_2, \mathscr{F}) \xrightarrow{f_1^*} \underline{Hom}_{\mathscr{O}_X}(\mathscr{G}_1, \mathscr{F})$$

is exact, it suffices to show that we have an exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_3|_U, \mathscr{F}|_U) \xrightarrow{(f_2^*)_U} \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_2|_U, \mathscr{F}|_U) \xrightarrow{(f_1^*)_U} \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_1, |_U, \mathscr{F}|_U)$$

Let $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_3|_U, \mathscr{F}|_U)$ be such that $F \circ f_2|_U = 0$. It follows that $\ker F = \mathscr{G}_3|_U$ as $\mathrm{im}\, f_2|_U = \mathscr{G}_3$, so F is the zero map, hence $F = 0$, and $(f_2^*)_U$ is injective.

Now let $F \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_2|_U, \mathscr{F}|_U)$, clearly if $F = G \circ f_2|_U$ then we have that $(f_1^*)_U(F) = 0$. Now suppose that F satisfies:

$$F \circ f_1|_U = 0$$

Note that since f_2 is surjective, we have that $(\mathscr{G}_3|_U, f_2|_U)$ is canonically $\mathrm{coker}\, f_1$, hence by the universal property of the cokernel, there exists a unique G such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{G}_1|_U \arrow[r, "f_1|_U"] & \mathscr{G}_2|_U \arrow[rr, "F"] \arrow[dr, "f_2|_U"'] & & \mathscr{F}|_U \\
& & \mathscr{G}_3|_U \arrow[ur, "G"'] &
\end{tikzcd}$$

Therefore $F = G \circ f_2|_U$ for a unique $G \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}_3, \mathscr{F}_U)$, hence we have proven exactness of the sequence. ☐

Using exactness, we will be able to show that the desired property holds on stalks in nice enough situations. First of all note that we have maps:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}_U) \longmapsto \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$
$$F \longmapsto F_x$$

which obviously commute with restrictions, hence there is a unique morphism Ψ_x making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U) \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{F}|_V, \mathscr{G}|_V) \arrow[dl] \arrow[ddl, bend left] \\
& \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \arrow[d, "\Psi_x"] & \\
& \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) &
\end{tikzcd}$$

We need the following lemma, which is an analogue of the result that $\mathrm{Hom}_A(A^I, M) \cong M^I$.

Lemma 5.2.2. *Let $\mathscr{F}$ be an $\mathscr{O}_X$ module, then:*

$$\underline{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{F}) \cong \mathscr{F}^n$$

as $\mathscr{O}_X$ modules.

Proof. This is essentially obvious, and it suffices to prove that there is a natural[112] isomorphism:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^n, \mathscr{F}|_U) \cong \mathscr{F}(U)^n$$

for all U. First note by the universal property of the coproduct, we have that naturally:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^n, \mathscr{F}|_U) \cong \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{F}|_U)^n$$

hence it suffices to show that:

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{F}|_U) \cong \mathscr{F}(U)$$

Now define a morphism:

$$\begin{aligned} \Phi_U : \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{F}|_U) &\longrightarrow \mathscr{F}(U) \\ F &\longmapsto F_U(1) \end{aligned}$$

where $1 \in \mathscr{O}_U(U)$ is the 'global' unit section. This is injective as if $F_U(1) = 0$, then for all $V \subset U$ and $s \in \mathscr{O}_U(V)$ we have, :

$$F_V(s) = s \cdot F_V(1) = s \cdot F_V(\theta_V^U(1)) = s \cdot F_U(1) = 0$$

implying that F is the zero morphism. This is surjective because if $a \in \mathscr{F}(U)$ then the map defined for all $V \subset U$:

$$\begin{aligned} F_V : \mathscr{O}_U(V) &\longrightarrow \mathscr{F}(V) \\ s &\longmapsto s \cdot a|_V \end{aligned}$$

defines a morphism of $\mathscr{O}_X$ modules. In particular $F_U(1) = a$, hence Φ_U is an isomorphism for all U, and clearly commutes with restrictions, implying the claim. □

We now have the following:

Proposition 5.2.2. *Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$ modules, and $x \in X$. If $\mathscr{F}$ is of finite type, then for all $\mathscr{O}_X$ modules $\mathscr{G}$, Ψ_x is injective. If $\mathscr{F}$ is in addition finitely presented, then for all $\mathscr{G}$ Ψ_x is an isomorphism.*

Proof. Let $[U, F] \in \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x$ and suppose that $\Psi_x([U, F]) = F_x = 0$. By shrinking U if we need to, there exist sections $s_1, \ldots, s_n$ of $\mathscr{F}(U)$ such that we have a surjection:

$$\mathscr{O}_U^n \longrightarrow \mathscr{F}|_U$$

In particular, the $s_{i,y}$ generate $\mathscr{F}_y$ as an $\mathscr{O}_{X,y}$ module for all $y \in U$. Since $F_x = 0$, we have that $F_x(s_{i,x}) = 0$ for all $i = 1, \ldots, n$. By taking n intersections, we can find an open neighborhood V of x such that $F_V(s_i|_V) = 0$ for all i. In particular, we have that for all $y \in V$ the $F_y = 0$ is the zero morphism, so $F|_V = 0$ as well. It follows that $[U, F] = [V, F|_V] = 0$, so Ψ_x is injective.

Now suppose that $\mathscr{F}$ is finitely presented. For every $x \in U$, we have an exact sequence:

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Since taking stalks is exact, we have that:

$$\mathscr{O}_{U,x}^n \longrightarrow \mathscr{O}_{U,x}^m \longrightarrow \mathscr{F}_x \longrightarrow 0$$

is exact. Now taking $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, \mathscr{G}_x)$ we obtain an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) \longrightarrow \mathscr{G}_x^m \longrightarrow \mathscr{G}_x^n$$

[112] I.e. will commute with restriction maps.

Now applying $\underline{Hom}_{\mathscr{O}_X}(-,\mathscr{G})$ to the initial exact sequence yields:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{G}^m \longrightarrow \mathscr{G}^n$$

by Lemma 5.2.2. Taking stalks we get the following exact sequence:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x \longrightarrow \mathscr{G}_x^m \longrightarrow \mathscr{G}_x^n$$

The map $\mathscr{G}_x^m \to \mathscr{G}_x^n$ is, up to isomorphism, the same in both instances; it follows that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})_x$ and $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$ are both the kernel of the same map and thus isomorphic. □

Now that we have successfully found conditions in which the stalks of $\underline{Hom}_{\mathscr{O}_X}$ behave as desired we are ready to move on to the main and final goal of this chapter: proving a tensor hom adjunction for sheaves.

Theorem 5.2.2. *Let $\mathscr{O}_X$ and $\mathscr{O}'_X$ be sheaves of commutative rings on X. Let $\mathscr{F},\mathscr{G} \in \mathrm{Mod}_{\mathscr{O}_X}$, and $\mathscr{G},\mathscr{H} \in \mathrm{Mod}_{\mathscr{O}'_X}$. Then there is a naturals isomorphism of sheaves of abelian groups:*

$$\underline{Hom}_{\mathscr{O}_X{}'}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G},\mathscr{H}) \cong \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H}))$$

Before we begin with the proof of the above statement, we briefly describe how $\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}$ is an $\mathscr{O}'_X$ module, and how $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H})$ is an $\mathscr{O}_X$ module. On the level of presheaves, we have a canonical morphism:

$$\mathscr{O}'_X \times (\mathscr{F}\otimes^p_{\mathscr{O}_X}\mathscr{G}) \longrightarrow \mathscr{F}\otimes^p_{\mathscr{O}_X}\mathscr{G}$$

given on opens by:

$$\begin{aligned} \mathscr{O}'_X(U) \times (\mathscr{F}(U)\otimes_{\mathscr{O}_X(U)}\mathscr{G}(U)) &\longrightarrow \mathscr{F}(U)\otimes_{\mathscr{O}_X(U)}\mathscr{G}(U) \\ (s, f\otimes g) &\longmapsto f\otimes(s\cdot g) \end{aligned}$$

This obviously makes $\mathscr{F}\otimes^p_{\mathscr{O}_X}\mathscr{G}$ a presheaf of $\mathscr{O}'_X$ modules, so by sheafifying and taking the induced morphism, $\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}$ an $\mathscr{O}'_X$ module. To show that $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H})$ is an $\mathscr{O}_X$ module, for each $s \in \mathscr{O}_X(U)$, we first define the morphism of $\mathscr{O}_U$ modules $\phi_s : \mathscr{G}|_U \to \mathscr{G}|_U$ given on opens by:

$$\begin{aligned} \mathscr{G}|_U(V) &\longrightarrow \mathscr{G}|_U(V) \\ f &\longmapsto s|_V \cdot f \end{aligned}$$

We thus define a morphism:

$$\mathscr{O}_X \times \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H}) \longrightarrow \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H})$$

on open sets by:

$$\begin{aligned} \mathscr{O}_X(U) \times \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{G}|_U,\mathscr{H}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{G}|_U,\mathscr{H}|_U) \\ (s,F) &\longmapsto F\circ\phi_s \end{aligned}$$

One easily checks that this assignment makes $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G}|_U,\mathscr{H}|_U)$ an $\mathscr{O}_X(U)$ module, and that these maps commute with restrictions, giving $\underline{Hom}_{\mathscr{O}'_X}(\mathscr{F},\mathscr{G})$ the structure of an $\mathscr{O}'_X$ module. We now proceed with the proof, it will be very similar to Theorem 5.1.1:

Proof. We first wish to define a morphism:

$$\Psi : \underline{Hom}_{\mathscr{O}'_X}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G},\mathscr{H}) \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F},\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H}))$$

On open sets, this should be a morphism of abelian groups:

$$\Psi_U : \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{F}|_U\otimes_{\mathscr{O}_U}\mathscr{G}|_U,\mathscr{H}|_U) \longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\underline{Hom}_{\mathscr{O}'_X}(\mathscr{G},\mathscr{H})|_U)$$

Given a morphism of $\mathscr{O}'_X$ modules $f : \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \to \mathscr{H}|_U$, we obtain the following morphism of abelian groups:

$$\tilde{f} : f \circ \otimes : \mathscr{F}|_U \oplus \mathscr{G}|_U \to \mathscr{H}|_U$$

Using the above, we need to define a morphism $\mathscr{F}|_U \to \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U$. Let $s \in \mathscr{F}(V)$, then we define a morphism $s \lrcorner \tilde{f}|_V : \mathscr{G}|_V \to \mathscr{H}|_V$ on opens by:

$$\begin{aligned} \mathscr{G}(W) &\longrightarrow \mathscr{H}(W) \\ t &\longmapsto \tilde{f}(s|_W, t) \end{aligned}$$

which is automatically a morphism of $\mathscr{O}'_V$ modules. The assignment $\mathscr{F}(V) \longrightarrow \mathrm{Hom}_{\mathscr{O}'_V}(\mathscr{G}|_V, \mathscr{H}|_V)$ is also clearly a morphism of $\mathscr{O}_U(V)$ modules, and so defines a morphism of $\mathscr{O}_X$ modules $\mathscr{F}_U \to \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U$ which we denote by $(-)\lrcorner \tilde{f}$. Hence Ψ_U is given on opens by:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{O}'_U}(\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U, \mathscr{H}|_U) &\longrightarrow \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U) \\ f &\longmapsto (-)\lrcorner \tilde{f} \end{aligned}$$

Suppose that $(-)\lrcorner \tilde{f} = 0$, then for all $V \subset U$ and $s \in \mathscr{F}(V)$ we have that $s \lrcorner \tilde{f}|_V : \mathscr{G}|_V \to \mathscr{H}|_V$ is the zero morphism. On global sections, this means that for all $s \in \mathscr{F}(V)$ and all $t \in \mathscr{G}(V)$, $\tilde{f}(s,t) = f \circ \otimes (s,t) = 0$. However this implies that the stalk map:

$$f_x : \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x \longrightarrow \mathscr{H}_x$$

is zero on simple tensors, hence f_x is zero. It follows that f is identically zero and Ψ_U is injective.

Now let $g \in \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \underline{Hom}_{\mathscr{O}'_X}(\mathscr{G}, \mathscr{H})|_U)$; we define a morphism:

$$\mathscr{F}|_U \oplus \mathscr{G}|_U \longrightarrow \mathscr{H}|_U$$

on open sets by:

$$\begin{aligned} \mathscr{F}(V) \oplus \mathscr{G}(V) &\longrightarrow \mathscr{H}(V) \\ (s,t) &\longmapsto (g_V(s))_V(t) \end{aligned}$$

Note that $g_V : \mathscr{F}(V) \longrightarrow \mathrm{Hom}_{\mathscr{O}'_V}(\mathscr{G}|_V, \mathscr{H}|_V)$, so $(g_V(s))_V : \mathscr{G}(V) \longrightarrow \mathscr{H}(V)$. As in Theorem 5.1.1, this morphism satisfies the minimal properties to factor through the tensor product over $\mathscr{O}_X$, namely being additive in both entries, and respecting the $\mathscr{O}_X$ module structure on $\mathscr{F}$ and $\mathscr{G}$. It follows that we get an induced morphism:

$$f : \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \longrightarrow \mathscr{H}|_U$$

After unraveling our definition of Ψ,, one easily checks that $\Psi_U(f)$ is equal to g, so Ψ_U is surjective, implying the claim. $\square$

Notice now that by the above, Lemma 5.2.2, and Theorem 1.3.1 we easily have that:

$$\begin{aligned} \underline{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) =& \underline{Hom}_{\mathscr{O}_X}(f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X, \mathscr{G}) \\ \cong& \underline{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \underline{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{G})) \\ \cong& \underline{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{F}, \mathscr{G}) \\ \cong& \underline{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G}) \end{aligned}$$

taking global sections we obtain:

Theorem 5.2.3. *Let $\mathscr{F}$ be an $\mathscr{O}_Y$ module, $\mathscr{G}$ an $\mathscr{O}_X$ module, and $f : X \to Y$ a morphism of ringed spaces. There is then a natural isomorphism:*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_*\mathscr{G})$$

In other words f^* is the left adjoint of f_*.

5.3 Some Commutative Algebra: Localization of Modules

In Section 1.1 we laid the ground work in commutative algebra, namely the localization of a ring, to construct the structure sheaf of an affine scheme in Section 1.4. In this section, we do something remarkably similar for modules over a fixed ring A, so that in the next section we can easily construction modules over affine schemes. In particular, our goal is to develop a theory of localization for modules, and explore their properties. Most of this section comes from Atiyah-Macdonald.

Lemma 5.3.1. *Let $S \subset A$ be a multiplicatively closed subset. There exists an exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$ which we call the localization of a module.*

Proof. We impose an equivalence relation on the set $M \times S$ as follows: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists a $t \in S$ such that:

$$t(s_2 m_1 - s_1 m_2) = 0$$

Essentially the same proof as in Proposition 1.1.2 shows that $M \times S/\sim$, which we denote by $S^{-1}M$ going forward, has the structure of an $S^{-1}A$ module. In particular, if $a/s \in S^{-1}A$, then we define:

$$[a, s] \cdot [m, t] = [am, st] \qquad \text{and} \qquad [m_1, t_1] + [m_2, t_2] = [t_2 m_1 + t_1 m_2, t_1 t_2]$$

which are easily checked to be well defined. We also denote the equivalence classes $[m, t]$ by m/t, and thus multiplication and addition are given by:

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st} \qquad \text{and} \qquad \frac{m_1}{t_1} + \frac{m_2}{t_2} = \frac{t_2 m_1 + t_1 m_2}{t_1 t_2}$$

Now let $\phi : M \to N$ be an A module morphism; we want to define an $S^{-1}A$ morphism $\phi' : S^{-1}M \to S^{-1}N$. Since any such morphism must satisfy:

$$\begin{aligned} \phi'\left(\frac{m}{t}\right) &= \phi'\left(\frac{1}{t} \cdot \frac{m}{1}\right) \\ &= \frac{1}{t} \cdot \phi'\left(\frac{m}{1}\right) \end{aligned}$$

there is essentially one way to define this morphism, and that is as:

$$\phi'\left(\frac{m}{t}\right) = \frac{\phi(m)}{t}$$

We check that this well defined: suppose that $m_1/t_1 = m_2/t_2$, then there is an s satisfying:

$$s(t_1 m_2 - t_2 m_1) = 0$$

Since ϕ is a morphism of A modules, it follows easily that:

$$s(t_1\phi(m_2) - t_2\phi(m_1)) = 0$$

implying that:

$$\frac{\phi(m_1)}{t_1} = \frac{\phi(m_2)}{t_2}$$

hence ϕ' is well defined. Let $\psi : N \to P$ be another morphism of modules, and $m/t \in S^{-1}M$, then:

$$(\psi \circ \phi)'\left(\frac{m}{t}\right) = \frac{\psi(\phi(m))}{t} = \psi'\left(\frac{\phi(m)}{t}\right) = \psi'\left(\phi'\left(\frac{m}{t}\right)\right)$$

hence $(\psi \circ \phi)' = \psi' \circ \phi'$. Since we clearly have that Id' is the identity morphism $S^{-1}M \to S^{-1}M$, it follows that the assignment $M \mapsto S^{-1}M$ and $\phi \mapsto \phi'$ defines a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$.

It remains to show that this functor is exact. Let:

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_2$$

be an exact sequence of A-modules, we claim that:

$$S^{-1}M_1 \xrightarrow{f_1'} S^{-1}M_2 \xrightarrow{f_2'} S^{-1}M_2$$

is exact. It is clear that $f_2' \circ f_1' = 0$, so we need only show that $\ker f_2' \subset \operatorname{im} f_1'$. Let $m_2/t_2 \in \ker f_2'$, then:

$$f_2'\left(\frac{m_2}{t_2}\right) = \frac{f_2(m_2)}{t_2} = 0$$

then there exists an $s \in S$ such that:

$$s \cdot f_2(m_2) = 0$$

In particular, we have that $f_2(s \cdot m_2) = 0$, so there is a unique element $m_1 \in M_1$ such that $f_1(m_1) = s \cdot m_2$. It follows that:

$$f_1'\left(\frac{m_1}{st_2}\right) = \frac{f_1(m_1)}{st_2} = \frac{s \cdot m_2}{s \cdot t_2} = \frac{m_2}{t_2}$$

hence the sequence is exact. $\square$

We call this functor localization, and as in the ring case if we S is the multiplicatively closed subset generated by $f \in A$, and if $S = \mathbb{A} \setminus \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$ we denote $S^{-1}M$ by M_g and $M_{\mathfrak{p}}$ respectively. Moreover, note that we have a well defined localization map $\pi : M \to S^{-1}M$, sending m to $m/1$.

Lemma 5.3.2. *The kernel of the map $\pi : M \to S^{-1}M$ is precisely:*

$$\{m \in M : \exists s \in S, s \cdot m = 0\}$$

In particular, if A is an integral domain, and M is torsion free, then $\ker \pi = 0$.

Proof. If $m/1 = 0$ then by definition there exists an $s \in S$ such that $s \cdot m = 0$. If A is an integral domain, and M has zero torsion, then for all $a \in A$ and all $m \in M$ we have that $a \cdot m = 0$ implies either m or a is equal to zero implying the claim. $\square$

We now show that localization behaves well with submodules, and quotients:

Lemma 5.3.3. *Let M be an A modules, N_1 and N_2 submodules of M, and $S \subset A$ a multiplicatively closed set. Then the following hold:*

i) *If $\pi : M \to S^{-1}M$ is the localization map then $S^{-1}N_1 = \langle \pi(N) \rangle \subset S^{-1}M$.*
ii) $S^{-1}(N_1 \cap N_2) = S^{-1}(N_1) \cap S^{-1}(N_2)$
iii) $S^{-1}(N_1 + N_2) = S^{-1}N_2 + S^{-1}N_2$
iv) *There is a natural isomorphism $S^{-1}(M/N_1) \cong S^{-1}M/S^{-1}N_1$.*

Proof. For i), note that $S^{-1}N_1$ is easily identified as a submodule of $S^{-1}M$ as the inclusion morphism $\imath : N_1 \to M$ gets sent to a morphism $\imath' : S^{-1}N_1 \to S^{-1}M$ satisfying:

$$\imath'\left(\frac{n}{s}\right) = \frac{\imath(n)}{s} = \frac{n}{s} \in S^{-1}M$$

so it is also an inclusion. now if $n/s \in S^{-1}N$ we have that

$$n/s = (1/s) \cdot (n/s) \in \langle \pi(N) \rangle$$

If $m/s \in \langle \pi(N) \rangle$, then for some $a/s \in S^{-1}A$ we have $m/s = (a/s) \cdot (n/1)$, however $a \cdot n \in N_1$ as N_1 is an A submodule/ It follows that $an/s \in S^{-1}N_1$ implying i).

For ii) if $n/s \in S^{-1}(N_1 \cap N_2)$, then by i) we can take $n/1$ to be such that $n \in N_1 \cap N_2$. In particular, $n/s \in S^{-1}N_1$ and $n/s \in S^{-1}N_2$ hence $n/s \in S^{-1}N_1 \cap S^{-1}N_2$. Conversely, if $n/s \in S^{-1}N_1 \cap S^{-1}N_2$, then $n/s \in S^{-1}N_i$ for each i. It follows that we can take n to be such that $n \in N_1 \cap N_2$ so $n/s \in S^{-1}(N_1 \cap N_2)$ by i), implying ii).

For *iii*), let $n/s \in S^{-1}(N_1 + N_2)$, then $n \in N_1 + N_2$ hence $n = n_1 + n_2$ for $n_i \in N_i$. It follows that $n/s = n_1/s + n_2/s \in S^{-1}N_1 + S^{-1}N_2$, giving us the first inclusion. If $n/s \in S^{-1}N_1 + S^{-1}N_2$ then we can write n/s as $n_1/s_1 + n_2/s_2$ where $n_i \in N_i$. Now:

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{s_2 n_1 + s_1 n_2}{s_1 s_2} \in S^{-1}(N_1 + N_2)$$

because $s_2 n_1 + s_1 n_1 \imath N_1 \cap N_2$.

Finally, for *iv*), we have an exact sequence:

$$0 \longrightarrow N_1 \longrightarrow M \longrightarrow M/N_1 \longrightarrow 0$$

so the functor S^{-1} gives us an exact sequence:

$$0 \longrightarrow S^{-1}N_1 \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N_1) \longrightarrow 0$$

implying that $S^{-1}(M/N_1) \cong S^{-1}M/S^{-1}N_1$ as desired. □

Alternatively to the construction in Lemma 5.3.1, we can view $S^{-1}M$ as a tensor product. Indeed, localization makes $S^{-1}A$ an A modules, so we could define $S^{-1}M$ as $M \otimes_A S^{-1}A$, one just has to check that this is an equivalent definition.

Proposition 5.3.1. *There is a natural isomorphism of $S^{-1}A$ modules:*

$$M \otimes_A S^{-1}A \cong S^{-1}M$$

Proof. Note that that we have an A bilinear morphism:

$$\begin{aligned} M \times S^{-1}A &\longrightarrow S^{-1}M \\ (m, a/s) &\longmapsto (m \cdot a)/s \end{aligned}$$

which then descends to an A linear morphism:

$$\phi : M \otimes_A S^{-1}A \longrightarrow S^{-1}M$$

This easily seen to be $S^{-1}A$ linear, where $M \otimes_A S^{-1}A$ has the obvious structure of an $S^{-1}A$ modules. Moreover, it is clearly surjective, as if $m/s \in S^{-1}M$, then we have that $\phi(m \otimes 1/s) = m/s$. Let:

$$\alpha = \sum_{i=1}^{n} m_i \otimes (a_i/s_i) \in \ker \phi$$

Then note that:

$$\alpha = \sum_{i=1}^{n} a \cdot m_i \otimes (\frac{1}{s_i})$$

If $t_i = s_1 \cdots \hat{s_i} \cdots s_n$, then $1/s_i = t_i/s$, so:

$$\alpha = \sum_{i=1}^{n} a \cdot m_i \otimes (t_i/s) = \left(\sum_{i=1}^{n} a \cdot t_i \cdot m_i \right) \otimes (1/s)$$

Let:

$$n = \sum_{i=1}^{n} a \cdot t_i \cdot m_i$$

then we have that:

$$n/s = 0$$

implying there is some $u \in S$ such that $u \cdot n = 0$. However, we can then write:

$$\alpha = n \otimes \frac{u}{su} = (un) \otimes \frac{1}{s} = 0$$

so ϕ is injective as well. □

Note that the above along with Lemma 5.1.1 implies that $S^{-1}A$ is a flat[113] A module. We also have the following obvious corollary:

Corollary 5.3.1. *Let M_1 and M_2 be A modules, then:*

$$S^{-1}(M_1 \otimes_A M_2) \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$$

Proof. By Proposition 5.3.1, we have that:

$$\begin{aligned} S^{-1}(M_1 \otimes_A M_2) \cong& (M_1 \otimes_A M_2) \otimes_A S^{-1}A \\ \cong& M_1 \otimes_A (M_2 \otimes_A S^{-1}A) \\ \cong& M_1 \otimes_A (S^{-1}A \otimes_{S^{-1}A} S^{-1}M_2) \\ \cong& (M_1 \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}M_2 \\ \cong& S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2 \end{aligned}$$

□

We end our short foray into commutative algebra by proving some 'local to global' properties of A modules:

Proposition 5.3.2. *Let M be an A module, then the following are equivalent:*

i) *M is the zero module.*

ii) *$M_{\mathfrak{p}}$ is the zero module for all $\mathfrak{p} \in \operatorname{Spec} A$.*

iii) *$M_{\mathfrak{m}}$ is the zero module for all $\mathfrak{m} \in |\operatorname{Spec} A|$*

Proof. Clearly $i) \Rightarrow ii)$, and $ii) \Rightarrow iii)$, so it suffices to show $iii) \Rightarrow i)$. Suppose that $M_{\mathfrak{m}}$ is the zero module for all $\mathfrak{m}$, and let $m \in M$. Let $I \subset A$ be the ideal defined by:

$$I = \{a \in A : a \cdot m = 0\}$$

If $I = A$ then $m = 0$, otherwise $I \subset \mathfrak{m}$ for some $\mathfrak{m} \in |\operatorname{Spec} A|$. In this case, we have that $m/1 = 0 \in A_{\mathfrak{m}}$, hence there is some $s \notin \mathfrak{m}$ satisfying $s \cdot m = 0$. This implies that $s \in I$, but $I \subset \mathfrak{m}$, so $I \not\subset \mathfrak{m}$ and thus $I = A$, $m = 0$. Since this holds for arbitrary m we have that M is the zero module. □

This then implies the following:

Corollary 5.3.2. *Let:*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

be a sequence of A modules, then the following are equivalent:

i) *The sequence of A modules is exact.*

ii) *For all $\mathfrak{p} \in \operatorname{Spec} A$ the localized sequence is exact.*

iii) *For all $\mathfrak{m} \in |\operatorname{Spec} A|$ the localized sequence is exact.*

In particular, a morphism of A modules is injective or surjective if and only if the localized morphism is injective or surjective for all $\mathfrak{m} \in |\operatorname{Spec} A|$.

Proof. We clearly have $i) \Rightarrow ii) \Rightarrow iii)$ since localization is an exact functor. Now suppose that that:

$$M_{1\mathfrak{m}} \xrightarrow{f_{1\mathfrak{m}}} M_{2\mathfrak{m}} \xrightarrow{f_{2\mathfrak{m}}} M_{3\mathfrak{m}}$$

is exact for all $\mathfrak{m}$. Since $f_{2\mathfrak{m}} \circ f_{1\mathfrak{m}} = 0$ for all $\mathfrak{m}$, and $f_{2\mathfrak{m}} \circ f_{1\mathfrak{m}} = (f_2 \circ f_1)_{\mathfrak{m}}$, we have that $\operatorname{im}(f_2 \circ f_1)_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, hence $\operatorname{im} f_2 \circ f_1 = 0$ by Proposition 5.3.2. It follows that $\operatorname{im} f_1 \subset \ker f_2$. In particular, we have that there is a well defined quotient $\ker f_2 / \operatorname{im} f_1$, and by Lemma 5.3.3 we have that

$$(\ker f_2/\operatorname{im} f_1)_{\mathfrak{m}} \cong (\ker f_2)_{\mathfrak{m}}/(\operatorname{im} f_1)_{\mathfrak{m}} \cong \ker f_{2m}/\operatorname{im} f_{1\mathfrak{m}} = 0$$

so by Proposition 5.3.2 we that $\ker f_2/\operatorname{im} f_1 = 0$ implying the claim. □

[113] I.e. the functor $\otimes_A S^{-1}A$ is exact.

A further consequence of the above is that flatness is a local property:

Proposition 5.3.3. *Let M be an A module then the following are equivalent:*

i) *M is a flat A module.*

ii) *For all $\mathfrak{p} \in \operatorname{Spec} A$ $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ module.*

iii) *For all $\mathfrak{m} \in |\operatorname{Spec} A|$ $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$ module.*

Proof. Suppose that M is a flat module, and $f : N \to P$ an injective morphism of $A_{\mathfrak{p}}$ modules. We want to show that the induced map

$$f' : N \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \longrightarrow P \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$$

is also injective. Now $\pi : A \to S^{-1}A$ gives every $A_{\mathfrak{p}}$ module an A module structure, such that f is also an A module morphism. Now observe that we have the following isomorphisms:

$$N \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \cong N \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A M) \cong N \otimes_A M$$

so up to isomorphism the morphism $N \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \to P \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ is the map $N \otimes_A M \to P \otimes_A M$ induced by tensoring with M. It follows that since M is flat that f' is flat hence $M_{\mathfrak{p}}$ is flat.

Clearly $ii) \Rightarrow iii)$. Assuming $iii)$ let $f : N \to P$ an injective morphism, and $f' : N \otimes_A M \to P \otimes_A M$ the induced map. By Corollary 5.3.1, for all $\mathfrak{m} \in |\operatorname{Spec} A|$, up to isomorphism $f'_{\mathfrak{m}} : (N \otimes_A M)_{\mathfrak{p}} \to (P \otimes_A M)_{\mathfrak{m}}$ is the morphism:

$$N_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \longrightarrow P_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$$

induced by $f_{\mathfrak{m}} : N_{\mathfrak{m}} \to P_{\mathfrak{m}}$ and tensoring with $M_{\mathfrak{m}}$. It follows that $\ker f'_{\mathfrak{m}} = (\ker f')_{\mathfrak{m}} = 0$. Since this holds for all $\mathfrak{m}$, we have that $\ker f' = 0$ so M is flat. □

We can also show the following result, which will be useful for future discussions of locally free sheaves:

Lemma 5.3.4. *Let $f : M \to N$ be a surjective morphism of free A modules. If M and N both have rank n, then f is an isomorphism*

Proof. It suffices to show that $\ker f_{\mathfrak{p}}$ is the zero module for all $\mathfrak{p} \in \operatorname{Spec} A$. Note that $f_{\mathfrak{p}}$ is surjective as tensoring is right exact. We have an exact sequence of the form:

$$0 \longrightarrow \ker f_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow 0$$

Since $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ module, the sequence is split, and so tensoring with any arbitrary $A_{\mathfrak{p}}$ module preserves the exactness of this specific exact sequence. In particular, if we tensor with any $A_{\mathfrak{p}}$ module, we obtain another split exact sequence. We tensor with $k = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, to obtain the exact sequence:

$$0 \longrightarrow \ker f_{\mathfrak{p}} \otimes k \longrightarrow M_{\mathfrak{p}} \otimes k \longrightarrow N_{\mathfrak{p}} \otimes k \longrightarrow 0$$

However, since $M_{\mathfrak{p}} \otimes k$ and $N_{\mathfrak{p}} \otimes k$ are now vector spaces of the same dimension, and $f_{\mathfrak{p}} \times \mathrm{Id}_k$ is surjective, we have that $\ker f_{\mathfrak{p}} \otimes k = 0$. Since:

$$\ker f_{\mathfrak{p}} \otimes k \cong \ker f_p / \mathfrak{m}_{\mathfrak{p}} \cdot \ker f_{\mathfrak{p}}$$

we have that $\ker f_{\mathfrak{p}} = \mathfrak{m}_{\mathfrak{p}} \cdot \ker f_{\mathfrak{p}}$. Therefore, as $\mathfrak{m}_{\mathfrak{p}}$ is the only maximal ideal of $A_{\mathfrak{p}}$, Lemma 3.10.1[114] implies that $\ker f_{\mathfrak{p}} = 0$. It follows that $\ker f_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$, so by Proposition 5.3.2 $\ker f = 0$, hence f is an isomorphism. □

[114] Specifically part *b)*

5.4 Quasicoherent Sheaves Over a Scheme

In this section we develop the theory of quasicoherent sheaves over a scheme. Since the scheme structure on X is generally fixed, we use $\mathrm{QCoh}(X)$ to refer to the category of quasicoherent $\mathscr{O}_X$ modules, breaking from our notation in the previous section. Our main goal in this section is to associate to each A module a quasicoherent sheaf over $\operatorname{Spec} A$, and show that every quasicoherent sheaf over X is locally of this form. Using this, we will show that $\mathrm{QCoh}(X)$ is an abelian category, prove desirable properties about $\mathrm{QCoh}(X)$, and explore a connection between quasicoherent sheaves of ideals of $\mathscr{O}_X$, and closed subschemes of X.

Lemma 5.4.1. *Let M be an A module, then there is a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ satisfying $\widetilde{M}(U_g) \cong M_g$. In particular $\widetilde{M}_\mathfrak{p} \cong M_\mathfrak{p}$, and the assignment $M \mapsto \widetilde{M}$ defines a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\operatorname{Spec} A}$.*

Proof. We define a sheaf on a basis by $\mathcal{F}(U_g) = M_g$. The restriction maps are those induced by identifying $M_g \cong M \otimes_A A_g$ and taking $\mathrm{Id} \otimes \theta_{U_h}^{U_g}$, where $\theta_h^g : A_g \to A_h$[115] are the usual restriction maps. It is clear that this defines a presheaf on a basis. Specifically, since $U_h \subset U_g$, we have that there exists an $k \in \mathbb{Z}^+$ and $a \in A$ such that $a \cdot h = g^k$, so these restriction maps are given by:

$$\begin{aligned} \theta_h^g : M_g &\longrightarrow M_h \\ \frac{m}{g^n} &\longmapsto \frac{m \cdot a^n}{h^{nk}} \end{aligned}$$

The same exact argument as in Proposition 1.4.3, but with replacing elements in A_g with elements in M_g demonstrates that this indeed defines a sheaf on a base. Due to the similarity of the argument, we elect to not reproduce it here.

To see that $\widetilde{M}_\mathfrak{p}$ is uniquely isomorphic to $M_\mathfrak{p}$, it suffices to show that $\mathcal{F}_\mathfrak{p}$ is isomorphic to $M_\mathfrak{p}$, however this argument is virtually identical to the one in Proposition 1.4.4, replacing $A_\mathfrak{p}$, and A_g with $M_\mathfrak{p}$ and M_g.

Finally, let $f : M \to N$ be a morphism of A modules, then on each distinguished open we get an induced morphism $M_g \to N_g$, given by $f \otimes \mathrm{Id}_{A_g}$. This map clearly commutes with restriction maps, hence by Theorem 1.4.1 we have a unique morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\tilde{f} : \widetilde{M} \to \widetilde{N}$. In particular, the stalk map $f_\mathfrak{p}$ is given by the induced map $M_\mathfrak{p} \to N_\mathfrak{p}$ up to a unique isomorphism. Moreover, since localization is a functor, we have that $\widetilde{(f \circ g)} = \tilde{f} \circ \tilde{g}$ hence the assignment $M \mapsto \widetilde{M}$ defines a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\operatorname{Spec} A}$ as desired. □

We have the following borderline immediate corollary:

Corollary 5.4.1. *For all A modules M, the sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules is quasicoherent. In particular, the assignment $M \mapsto \widetilde{M}$ is a functor $\mathrm{Mod}_A \to \mathrm{QCoh}(\operatorname{Spec} A)$.*

Proof. Let I be a possibly infinite indexing set such that:

$$f : \bigoplus_{i \in I} A \longrightarrow M$$

is a surjection. In particular, we could easily take I to have cardinality of M, take some bijection $h : I \to S$, and take f to be a direct sum of maps of the form:

$$\begin{aligned} f_i : A &\longrightarrow M \\ 1 &\longmapsto h(i) \end{aligned}$$

For the same reason, we easily obtain an indexing set J such that we have a surjection:

$$\bigoplus_{j \in J} A \longrightarrow \ker f$$

[115] We employ the same notation as in Proposition 1.4.3.

We thus have an exact sequence:

$$\bigoplus_{j\in J} A \longrightarrow \bigoplus_{i\in I} A \longrightarrow M \longrightarrow 0$$

Which induces a sequence of $\mathscr{O}_{\operatorname{Spec} A}$ modules by Lemma 5.3.1:

$$\mathscr{O}^J_{\operatorname{Spec} A} \longrightarrow \mathscr{O}^I_{\operatorname{Spec} A} \longrightarrow \widetilde{M} \longrightarrow 0$$

On stalks this given by:

$$\bigoplus_{j\in J} A_\mathfrak{p} \longrightarrow \bigoplus_{i\in I} A_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow 0$$

which is exact since localization is an exact functor by Lemma 5.2.1. It follows that the original sequence of $\mathscr{O}_{\operatorname{Spec} A}$ modules is exact, and the $\widetilde{M}$ is quasicoherent. □

Example 5.4.1. Let $I \subset A$ be any radical ideal, then I is by definition an A sub module of A. It follows that $\widetilde{I}$ is a quasicoherent sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules; in particular, one each distinguished open set U_g, we have that $\widetilde{I}(U_g) \cong I_g$. It follows that $\widetilde{I}$ is precisely the sheaf of ideals corresponding to the closed subset $\mathbb{V}(I)$.

If instead we start with any ideal I, then the closed subscheme $\mathbb{V}(I)$[116] has a scheme structure given by $\imath^{-1}\left(\mathscr{O}_{\operatorname{Spec} A}/\widetilde{I}\right)$.

Our first major goal of the section is to prove that this functor is an equivalence of categories. We begin with the following lemma:

Lemma 5.4.2. *Let M be an A module, and $\mathscr{F}$ a sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules. Any morphism $M \to \mathscr{F}(\operatorname{Spec} A)$ of A modules induces a morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\widetilde{M} \to \mathscr{F}$. Moreover, every such morphism of sheaves is induced by it's action on global sections, $M \to \mathscr{F}(\operatorname{Spec} A)$.*

Proof. Let $\phi : M \to \mathscr{F}(\operatorname{Spec} A)$ be an A module morphism. We define a morphism on distinguished opens by:

$$\begin{aligned} \psi_{U_g} : M_g &\longrightarrow \mathscr{F}(U_g) \\ \frac{m}{g^k} &\longrightarrow \frac{1}{g^k} \cdot (\phi(m)|_{U_g}) \end{aligned}$$

where we are using the fact that $\mathscr{F}(U_g)$ is an A_g module. If this morphism is well defined for each g, then it clearly commutes with restrictions, so suppose that $m/g^k = n/g^l$, then there exists an $L \in \mathbb{Z}^+$ such that:

$$g^L(g^l m - g^k n) = 0$$

Now note that:

$$\frac{1}{g^k} \cdot (\phi(m)|_{U_g}) - \frac{1}{g^k}(\phi(n)|_{U_g}) = \frac{1}{g^{k+l}} \cdot (\phi(g^l m - g^k_n))$$

Multiplying by $1 = g^L/g^L$ yields:

$$\frac{1}{g^{L+k+l}} \cdot (\phi(g^L(g^l m - g^k_n))) = 0$$

implying that ψ_{U_g} is well defined as desired. It follows from Theorem 1.4.1 that there is an induced morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules $\psi : \widetilde{M} \to \mathscr{F}$.

Now let $\psi : \widetilde{M} \to \mathscr{F}$ be a morphism of $\mathscr{O}_{\operatorname{Spec} A}$ modules, and set $\phi = \psi_{\operatorname{Spec} A}$. We need to show that:

$$\psi_{U_g}\left(\frac{m}{g^k}\right) = \frac{1}{g_k} \cdot \phi(m)|_{U_g}$$

[116] Which is not necessarily reduced!

Since ψ_{U_g} is a morphism of A_g modules, we have that:

$$\begin{aligned}\psi_{U_g}\left(\frac{m}{g^k}\right) =& \frac{1}{g^k}\cdot\psi_{U_g}(m/1)\\ =& \frac{1}{g^k}\cdot\psi_{U_g}(m|_{U_g})\\ =& \frac{1}{g^k}\cdot\phi(m)|_{U_g}\end{aligned}$$

implying the claim. $\square$

With this we can show the following:

Lemma 5.4.3. *Suppose that $\mathscr{F}$ is an $\mathscr{O}_{\operatorname{Spec} A}$ module such that there exists an exact sequence:*

$$\mathscr{O}^I_{\operatorname{Spec} A} \longrightarrow \mathscr{O}^J_{\operatorname{Spec} A} \longrightarrow \mathscr{F} \longrightarrow 0$$

Then there is an isomorphism $\widetilde{M} \cong \mathscr{F}$, where $M \cong \mathscr{F}(\operatorname{Spec} A)$.

Proof. We first note that $\mathscr{O}^I_{\operatorname{Spec} A}$ is the $\mathscr{O}_{\operatorname{Spec} A}$ module induced by A^I. Taking global sections, we get a morphism:

$$\phi_A : A^I \to A^J$$

and set $M = \operatorname{coker}\phi_A$. Since on global sections, we have that the composition:

$$A^J \longrightarrow A^I \longrightarrow F(\widetilde{\operatorname{Spec} A})$$

is equal to zero, there exists a unique morphism $\psi : M \to \mathscr{F}(\operatorname{Spec} A)$ which by Lemma 5.4.2 induces a unique morphism $\tilde{\psi} : \widetilde{M} \to F$. Since the morphism ϕ_A is the one which induces the morphism $\mathscr{O}^I_{\operatorname{Spec} A} \to \mathscr{O}^J_{\operatorname{Spec} A}$, we have that the following diagram commutes:

$$\begin{tikzcd} \mathscr{O}^I_{\operatorname{Spec} A} \arrow[r] & \mathscr{O}^J_{\operatorname{Spec} A} \arrow[rr] \arrow[dr] & & \mathscr{F} \\ & & \widetilde{M} \arrow[ur] & \end{tikzcd}$$

Talking stalks we obtain the following commutative diagram:

$$\begin{tikzcd} A^I_{\mathfrak{p}} \arrow[r] & A^J_{\mathfrak{p}} \arrow[rr] \arrow[dr] & & \mathscr{F}_{\mathfrak{p}} \\ & & M_{\mathfrak{p}} \arrow[ur] & \end{tikzcd}$$

Now, $M_{\mathfrak{p}}$ and $\mathscr{F}_{\mathfrak{p}}$ are both the cokernel of the morphism $A^I_{\mathfrak{p}} \to A^J_{\mathfrak{p}}$, so the morphism $M_{\mathfrak{p}} \to \mathscr{F}_{\mathfrak{p}}$ is the unique isomorphism which makes the above diagram commute. It follows that $\tilde{\psi}$ is an isomorphism implying the claim. $\square$

The preceding lemma demonstrates that for a very specific class of quasicoherent $\mathscr{O}_{\operatorname{Spec} A}$ modules, we have the desired claim. We now show that this holds in generality:

Proposition 5.4.1. *The functor* $\operatorname{Mod}_A \to \operatorname{QCoh}(\operatorname{Spec} A)$ *given by sending M to $\widetilde{M}$ is essentially surjective*[117]

[117]Recall this implies that any object $\mathscr{F} \in \operatorname{QCoh}(\operatorname{Spec} A)$ is isomorphic to $\widetilde{M}$ for some M.

Proof. Let $\mathscr{F} \in \mathrm{QCoh}(\operatorname{Spec} A)$, then every point has a neighborhood U such that there exists an exact sequence:

$$\mathscr{O}_U^I \longrightarrow \mathscr{O}_U^J \longrightarrow \tilde{F}|_U \longrightarrow 0$$

Without loss of generality, we can take $U = U_{f_i}$ for $f_i \in A$, and since $\operatorname{Spec} A$ is quasicompact, we can take finitely many to cover $\operatorname{Spec} A$. By Lemma 5.4.3, it follows that we have a cover $\{U_{f_i}\}_{i=1}^n$ of $\operatorname{Spec} A$ such that:

$$\mathscr{F}|_{U_{f_i}} \cong \widetilde{M_i}$$

These isomorphisms induce isomorphisms $\phi_{ij} : \widetilde{M_i}|_{U_{f_i f_j}} \to \widetilde{M_j}|_{U_{f_i f_j}}$ which trivially satisfy the cocycle condition. Denote by ψ_{ij} the isomorphisms $\widetilde{M_i}(U_{f_i f_j}) \to \tilde{M_j}(U_{f_i f_j})$ induced by taking global sections of ϕ_{ij}. Up to isomorphism we can view the global sections on $\mathscr{F}$ as:

$$\mathscr{F}(\operatorname{Spec} A) = \left\{ (m_i) \in \prod_{i=1}^n M_i : \psi_{ij}(m_i|_{U_{f_i f_j}}) = m_j|_{U_{f_i f_j}} \right\}$$

Note that for each i we naturally have:

$$\begin{aligned}
\widetilde{M_i}(U_{f_i f_j}) =& M_i \otimes_{A_{f_i}} A_{f_i f_j} \\
\cong& M_i \otimes_{A_{f_i}} A_{f_i} \otimes_A A_{f_j} \\
\cong& M_i \otimes_A A_{f_j} \\
\cong& (M_i)_{f_j}
\end{aligned}$$

so the restriction maps are localization maps, $m_i|_{U_{f_i f_j}} = m_i/1 \in (M_i)_{f_j}$, and the ψ_{ij} are isomorphisms $(M_i)_{f_j} \to (M_j)_{f_i}$. With this we have that up to isomorphism:

$$\mathscr{F}(\operatorname{Spec} A) = \left\{ (m_i) \in \prod_{i=1}^n M_i : \psi_{ij}(m_i/1) = m_j/1 \right\}$$

Moreover, we have that $\mathscr{F}(\operatorname{Spec} A)$ is the kernel of the morphism:

$$\begin{aligned}
\bigoplus_{i=1}^n M_i &\longrightarrow \bigoplus_{i,k=1}^n (M_i)_{f_k} \\
(m_i) &\longrightarrow (\psi_{ij}(m_i/1) - m_k/1)
\end{aligned}$$

Since localization is exact, and commutes with finite products, we have that if $M = \mathscr{F}(\operatorname{Spec} A)$, then:

$$M_{f_j} = \ker \left(\bigoplus_{i=1}^n (M_i)_{f_j} \longrightarrow \bigoplus_{i,k=1}^n (M_i)_{f_j f_k} \right)$$

where if $\psi'_{ik} : (M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ is the induced morphism, then the above map is given by:

$$(m_i/f_j^{l_i}) \longmapsto \left(\psi'_{ik} \left(\frac{m_i f_k^{l_i}}{(f_k f_j)^{l_i}} \right) - \frac{m_k f_i^{l_i}}{(f_j f_i)^{l_i}} \right)$$

It follows that:

$$M_{f_j} = \left\{ (m_i/f_j^{k_i}) \in \prod_i (M_i)_{f_j} : \psi'_{ik} \left(\frac{m_i f_k^{l_i}}{(f_k f_j)^{l_i}} \right) = \frac{m_k f_i^{l_i}}{(f_j f_i)^{l_i}} \right\}$$

Let ξ be the morphism $\widetilde{M} \to \mathscr{F}$ be the morphism induced by the identity map $M \to \mathscr{F}(\operatorname{Spec} A)$. The map $\xi_{U_{f_j}}$ is then given by:

$$\begin{aligned}
\xi_{U_{f_j}} : M_{f_j} &\longrightarrow M_j \\
(m_i/f_j^{k_i}) &\longrightarrow (1/f_j^{k_j}) \cdot m_j
\end{aligned}$$

Suppose that $(m_i/f_j^{k_i}) \mapsto 0$, and let $K = \max\{k_i\}$. Note that $(f_j^{K-k_i} m_i/1) \mapsto 0$ if and only if the original element does, so it suffices to consider an element in $\ker \xi_{U_{f_j}}$ of the form $(m_i/1)$.

Now let $m_j \in M_j$, we need to define elements in $(M_i)_{f_j}$, and do so by noting that there exist $m_i \in M_i$ and $k_i \in \mathbb{Z}^+$ such that:

$$\psi_{ji}(m_j/1) = \frac{m_i}{f_j^{k_i}} \in (M_i)_{f_j}$$

We claim that the sequence $(m_i/f_j^{k_i}) \in \prod (M_i)_{f_i}$ actually lies in M_{f_i}. Since the ϕ_{ij} satisfy the cocycle condition, we have that for all i, j, l:

$$\psi'_{jl} = \psi'_{il} \circ \psi'_{ji}$$

Now consider:

$$\begin{aligned}
\psi'_{il}\left(\frac{m_i f_l^{k_i}}{(f_l f_j)^{k_i}}\right) =& \psi'_{il}\left(\frac{m_i}{f_j^{k_i}}\Big|_{U_{f_i f_j f_l}}\right) \\
=& \psi'_{il}\left(\psi_{ji}(m_j/1)|_{U_{f_i f_j f_l}}\right) \\
=& \psi'_{il}\left(\psi'_{ji}\left(\frac{m_j}{1}\Big|_{U_{f_i f_j f_l}}\right)\right) \\
=& \psi'_{jl}\left(\frac{m_j}{1}\Big|_{U_{f_i f_j f_l}}\right) \\
=& \psi_{jl}\left(\frac{m_j}{1}\right)\Big|_{U_{f_i f_j f_l}} \\
=& \frac{m_l f_i^{kl}}{(f_i f_j)^{k_l}}
\end{aligned}$$

hence $(m_i/f_j^{k_i}) \in M_{f_i}$. It is clear that $(m_i/f_j^{k_i})$ maps to m_j, hence $\xi_{U_{f_j}}$ is surjective, and thus an isomorphism as desired. In particular, the sheaf morphism $\xi|_{U_{f_j}}$ is determined by $\xi_{U_{f_j}}$, and is thus an isomorphism. Since ξ is locally an isomorphism on an open cover, it follows that ξ is an isomorphism, and thus $\mathscr{F} \cong \widetilde{M}$ as desired. □

Example 5.4.2. Note that not every sheaf of $\mathscr{O}_{\operatorname{Spec} A}$ modules is of the form $\widetilde{M}$. As an example, the constant sheaf $\mathbb{Z}$ on $\operatorname{Spec} \mathbb{Z}$ is obviously not isomorphic to $\widetilde{Z} \cong \mathscr{O}_{\operatorname{Spec} \mathbb{Z}}$.

We can now prove our first main result of the section:

Theorem 5.4.1. *The functor* $\mathrm{Mod}_A \to \mathrm{QCoh}(\operatorname{Spec} A)$ *is an equivalence of categories. In particular* $\mathrm{QCoh}(\operatorname{Spec} A)$ *is an abelian category.*

Proof. It suffices to show that the functor is fully faithful, as it essentially surjective by the preceding proposition. Let M and N be A modules, then the morphism:

$$\begin{aligned}
\mathrm{Hom}_A(M, N) &\longrightarrow \mathrm{Hom}_{\operatorname{Spec} A}(\widetilde{M}, \widetilde{N}) \\
f &\longmapsto \tilde{f}
\end{aligned}$$

is surjective by Lemma 5.4.2. Suppose that $f \mapsto 0$, then by Lemma 5.4.1, we have that up to a unique isomorphism $\tilde{f}_{\mathfrak{p}} = f_{\mathfrak{p}}$, hence $f_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$. It follows that since $(\operatorname{im} f)_{\mathfrak{p}} = \operatorname{im} f_{\mathfrak{p}} = 0$, we have that f is the zero morphism by Proposition 5.3.2, implying the equivalence.

It is now obvious that $\mathrm{QCoh}(\operatorname{Spec} A)$ is an abelian category, as for all $\tilde{f} : \widetilde{M} \to \widetilde{N}$ we have that $\ker \tilde{f} \cong \widetilde{\ker f}$ and $\operatorname{coker} \tilde{f} \cong \widetilde{\operatorname{coker} f}$. □

An immediate corollary is that $\mathrm{QCoh}(X)$ is an abelian category when X is a scheme.

Corollary 5.4.2. *Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$ module, then $\mathscr{F}$ is quasicoherent if and only if for every affine open $U = \operatorname{Spec} A \subset X$ there exists an A module such that $\mathscr{F}|_U \cong \widetilde{M}$. Moreover* $\mathrm{QCoh}(X)$ *is an abelian category.*

Proof. If for every $U = \operatorname{Spec} A$, $\mathscr{F}|_U \cong \widetilde{M}$ for some A module M, $\mathscr{F}|_U$ is quasicoherent by Corollary 5.4.1, so $\mathscr{F}$ is quasicoherent as the affine open subschemes of X form a basis for the topology on X.

Now suppose that $\mathscr{F}$ is a quasicoherent, then the restriction to any affine open subscheme $U = \operatorname{Spec} A$, $\mathscr{F}|_U$, is a quasicoherent sheaf of $\mathscr{O}_U$ modules. By Proposition 5.4.1 it follows that there exists an A module M such that $\mathscr{F}|_U \cong \operatorname{QCoh}(X)$.

Now let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of quasicoherent sheaves of $\mathscr{O}_X$ modules. For any affine open $U = \operatorname{Spec} A$, we have that $(\ker f)|_U = \ker f|_U$ and $(\operatorname{coker} f)|_U \cong \operatorname{coker} f|_U$. By Theorem 5.4.1, $\ker f|_U$ and $\operatorname{coker} f|_U$ are quasicoherent sheaves of $\mathscr{O}_U$ modules, hence $\ker f$ and $\operatorname{coker} f$ are quasicoherent sheaves of $\mathscr{O}_X$ modules. It follows that $\operatorname{QCoh}(X)$ is an abelian category. ☐

Corollary 5.4.3. *If $\mathscr{G}$ is quasicoherent, and $\mathscr{F}$ is locally finitely presented then $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is quasicoherent.*

Proof. If $\mathscr{F}$ is finitely presented, then we have an open neighborhood where there exists an exact sequence:

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^m \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

Taking $\operatorname{Hom}_{\mathscr{O}_U}(-, \mathscr{G})$ gives an exact sequence:

$$0 \longrightarrow \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})|_U \longrightarrow \mathscr{G}|_U^n \longrightarrow \mathscr{G}|_U^m$$

hence $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})|_U$ is quasicoherent as it is the kernel of a morphism of quasicoherent modules. Since being quasicoherent is a local condition, it follows that $\underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is quasicoherent. ☐

In particular, the dual of a quasicoherent sheaf of $\mathscr{O}_X$ modules need not be quasicoherent unless $\mathscr{F}$ is finitely presented. Using the above, we also have an easier description of the category $\operatorname{Coh}(X)$ when X is a locally Noetherian scheme:

Proposition 5.4.2. *Let X be locally Noetherian, and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$ modules, then the following are equivalent:*

a) $\mathscr{F}$ is coherent.

b) $\mathscr{F}$ is finitely presented.

c) For any affine open $U = \operatorname{Spec} A$, $\mathscr{F}|_U \cong \widetilde{M}$ where M is a finitely generated A module.

d) There exists an affine cover $\{U_i = \operatorname{Spec} A_i\}$ so that $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$, with each M_i a finitely generated A_i module.

In particular, $\mathscr{O}_X$ is coherent over itself.

Proof. We have *a)* $\Rightarrow$ *b)* as every coherent module is finitely presented[118].

For *b)* $\Rightarrow$ *c)*, we have that $\mathscr{F}$ is in particular quasicoherent, so for any affine open $U = \operatorname{Spec} A$, we have that $\mathscr{F}|_U \cong \widetilde{M}$ where M is an A module. Since $\mathscr{F}$ is of finite presentation, $\mathscr{F}|_U$ is of finite presentation, hence there exists a finite cover of U by distinguished opens $\{U_{f_i}\}$ such that we have an exact sequence:

$$\mathscr{O}_{U_f}^{m_i} \longrightarrow \mathscr{O}_{U_f}^{n_i} \longrightarrow \widetilde{M}|_{U_{f_i}} \longrightarrow 0$$

which upon taking global sections yields an exact sequence:

$$A_{f_i}^{m_i} \longrightarrow A_{f_i}^{n_i} \longrightarrow M_{f_i} \longrightarrow 0$$

Note that the map $A_{f_i}^{n_i} \to M_{f_i}$ is fully determined by where it sends the elements $(0, \ldots, 1, \ldots, 0)$. By clearing denominators[119], we can take the morphism $A_{f_i}^{n_i} \to M_{f_i}$ to be of the form:

$$(0, \ldots, 1_{ij}, \ldots, 0) \longmapsto m_{ij}/1$$

[118] See for example Lemma 5.1.6.

[119] I.e. if $(0, \ldots, 1, \ldots, 0)$ gets sent to m/s we can take it to be instead $m/1$ and the morphism will still be surjective.

where the i index is parameterized by the f_i, and the j index determines the place of the 1. By taking elements in the preimage of the localization map, for each f_i we obtain a morphism:

$$A^{n_i} \longrightarrow M$$

induced by:

$$(0, \ldots, 1_{ij}, \ldots, 0) \longmapsto m_{ij}$$

We claim the direct sum of these morphisms:

$$\bigoplus_i A^{n_i} \to M$$

is surjective. For $\mathfrak{p} \in \operatorname{Spec} A$ we have that $\mathfrak{p} \in U_{f_i}$ for some i, hence the morphism:

$$\bigoplus_i A_{\mathfrak{p}}^{n_i} \to M_{\mathfrak{p}}$$

is surjective, as it contains the morphism $A_{\mathfrak{p}}^{n_i} \to M_{\mathfrak{p}}$ which is surjective as localization is exact. It follows by Corollary 5.3.2 that the morphism is surjective, hence M is finitely generated as there are finitely many f_i.

It is obvious that $c) \Rightarrow d)$.

For $d) \Rightarrow a)$, since X is locally Noetherian we have an open cover of X of affine opens $\{V_j = \operatorname{Spec} B_j\}$, where each B_i is Noetherian. Now if $\{U_i = \operatorname{Spec} A_i\}$ is the open cover on which $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$, such that M_i is a finitely generated A_i module. For each i and j, we have that $U_i \cap U_j$ can be covered by affine opens V_{ijk} which are distinguished in both U_i and U_j. Set $V_{ijk} = \operatorname{Spec}(B_j)_{f_{ik}} \cong \operatorname{Spec}(A_i)_{g_{jk}}$, then $\widetilde{M_i}|_{V_{ijk}} \cong \widetilde{(M_i)_{g_{jk}}}$. Via the isomorphism $(B_j)_{f_{ik}} \cong (A_i)_{g_{jk}}$, we can take $(M_i)_{g_{jk}}$ to be a finitely generated module over the Noetherian ring $(B_j)_{f_{ik}}$. It follows that we can assume the original cover $\{U_i = \operatorname{Spec} A_i\}$ is such that each A_i is Noetherian.

It now suffices to show that $\mathscr{F}|_{U_i}$ is a coherent module over $\operatorname{Spec} A_i$, hence it suffices to show that $\widetilde{M}$ is a coherent module over $\operatorname{Spec} A$ when M is finitely generated and A is Noetherian. Let $V \subset \operatorname{Spec} A$ be any open set, $s_1, \ldots, s_n \in \mathscr{O}_U(V)$, and $\phi : \mathscr{O}_V^n \to \widetilde{M}|_V$ the induced morphism. Let U_{f_i} be a cover of V by distinguished opens, then for each i we must have that:

$$\phi_{U_{f_i}} : A_{f_i}^n \longrightarrow M_{f_i}$$

has finitely generated kernel as A_{f_i} is Noetherian, and the direct sum of Noetherian modules is Noetherian. It follows that $\ker \phi$ is of finite type, as for each i there exists some n_i such that:

$$\mathscr{O}_{U_{f_i}}^{n_i} \longrightarrow \widetilde{M}|_{U_{f_i}}$$

is a surjection.

To see that $\mathscr{O}_X$ is Noetherian, note that $\mathscr{O}_X$ is globally of finite presentation, hence by $c)$ we have that $\mathscr{O}_X$ is coherent. $\square$

Note we have the obvious corollary:

Corollary 5.4.4. *Let $\mathscr{F}$ be a finite type, quasicoherent $\mathscr{O}_X$ module over a locally Noetherian scheme, then $\mathscr{F}$ is coherent.*

Example 5.4.3. Let $Y \subset X$ be a closed subset of X, and $I_{Y/X}$ be the sheaf of ideals corresponding to Y as in Definition 2.1.3. Then by our work on the induced reduced subscheme construction on Y, it follows that $I_{Y/X}$ is quasicoherent as over any affine open $U \subset I_{Y/X}$ we have that $I_{Y/X}|_U \cong \widetilde{I_{Y/X}(U)}$.

The above example hints at a connection between quasicoherent sheaves of ideals, and closed subschemes of X. We will explore this connection more precisely later, but first we prove that tensor products, and pullbacks of quasicoherent sheaves on a scheme X behave nicely.

Lemma 5.4.4. *Let M and N be A modules, P a B-module, and $f : \operatorname{Spec} A \to \operatorname{Spec} B$ a morphism of affine schemes. Then there are canonical isomorphisms:*

a) $\widetilde{M} \otimes_{\mathscr{O}_{\operatorname{Spec} A}} \widetilde{N} \cong \widetilde{M \otimes_A N}$

b) $f^*\widetilde{P} \cong \widetilde{P \otimes_B A}$

Proof. For a), note that since $M \otimes_A N$ satisfies the universal property of the tensor product in the category of A modules, that $\widetilde{M \otimes_A N}$ satisfies the universal property of the tensor product in $\operatorname{QCoh}(\operatorname{Spec} A)$. Since $\widetilde{M} \otimes_{\mathscr{O}_{\operatorname{Spec} A}} \widetilde{N}$ is the tensor product in $\operatorname{QCoh}(\operatorname{Spec} A)$, we must have that they are canonically isomorphic.

For b), there is an exact sequence:

$$\mathscr{O}^I_{\operatorname{Spec} B} \longrightarrow \mathscr{O}^J_{\operatorname{Spec} B} \longrightarrow \widetilde{P} \longrightarrow 0$$

which corresponds to an exact sequence:

$$B^I \longrightarrow B^J \longrightarrow P \longrightarrow 0$$

Since the tensor product is right exact, we obtain:

$$A^I \longrightarrow A^J \longrightarrow P \otimes_B A \longrightarrow 0$$

implying that $\widetilde{P \otimes_A B}$ is the cokernel of the induced morphism:

$$\mathscr{O}^I_{\operatorname{Spec} A} \to \mathscr{O}^J_{\operatorname{Spec} A}$$

However, by our work in Proposition 5.1.4 this morphism is the same as the one induced by pulling back the original sequence by f, hence $f^*\widetilde{P}$ is also the cokernel of the above morphism, implying the claim. $\square$

The following corollary now demonstrates that pullbacks and tensor products of quasicoherent $\mathscr{O}_X$ behave in an extraordinarily tractable way:

Corollary 5.4.5. *Let $\mathscr{F}$ and $\mathscr{G}$ be quasicoherent sheaves of $\mathscr{O}_X$ modules, $\mathscr{H}$ a quasicoherent sheaf of $\mathscr{O}_Y$ modules, and $f : X \to Y$ a morphism of schemes. If $U \subset X$ is an affine open, and V is an affine open containing $f(U)$, then we have the following isomorphisms*[120]*:*

a) $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}|_U \cong \mathscr{F}(U) \widetilde{\otimes_{\mathscr{O}_X(U)}} \mathscr{G}(U)$

b) $(f^*\mathscr{H})|_U \cong \mathscr{H}(V) \widetilde{\otimes_{\mathscr{O}_Y(V)}} \mathscr{O}_X(U)$

Proof. For a), let $\imath : U \to X$ be the open embedding, then by Lemma 5.1.4, Corollary 1.3.2, and Lemma 5.4.4:

$$\begin{aligned}
\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}|_U =& \imath^{-1}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \\
\cong& \imath^{-1}\mathscr{F} \otimes_{\imath^{-1}\mathscr{O}_X} \imath^{-1}\mathscr{G} \\
\cong& \mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U \\
\cong& \widetilde{\mathscr{F}(U)} \otimes_{\mathscr{O}_U} \widetilde{\mathscr{G}(U)} \\
\cong& \mathscr{F}(U) \widetilde{\otimes_{\mathscr{O}_X(U)}} \mathscr{G}(U)
\end{aligned}$$

implying the claim.

For b), we have that

$$(f^*\mathscr{H})|_U = \imath^*(f^*\mathscr{H}) = (f \circ \imath)^*\mathscr{H}$$

[120]Pardon the poor notation, the wide tilde command does not stretch far enough.

Since $f \circ \imath$ is a morphism $U \to V$, it follows by Lemma 5.4.4:

$$\begin{aligned}(f^*\mathscr{H})|_U \cong& (f\circ\imath)^*(\mathscr{H}|_V)\\ \cong& (f\circ\imath)^{-1}(\mathscr{H}|_V)\otimes_{(f\circ\imath)^{-1}\mathscr{O}_V}\mathscr{O}_U\\ \cong& \mathscr{H}(V)\otimes_{\widetilde{\mathscr{O}_Y(V)}}\mathscr{O}_X(U)\end{aligned}$$

as desired. □

If $f : X \to Y$ is a morphism of smooth manifolds, and $E \to Y$ is a vector bundle, we can construct the pullback bundle f^*E with fibres satisfying $(f^*E)|_x = E|_{f(x)}$. In particular, given a global section $s : Y \to E$, there is global section of f^*E given by precomposing s with f. This is called pulling back sections, and we wish to develop an analogue of this phenonomnen for quasicoherent sheaves of $\mathscr{O}_X$ modules.

Definition 5.4.1. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$ modules over a locally ringed space $(X, \mathscr{O}_X)$. The **fibre of** $\mathscr{F}$ at $x \in X$, denoted $\mathscr{F}|_x$ is the tensor product:

$$\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} k_x$$

Note that this is a k_x-linear vector space, and we define the **rank of $\mathscr{F}|_\mathbf{x}$** to be $\dim_{k_x}\mathscr{F}|_x$.

We have the following characterization of the fibre:

Lemma 5.4.5. *Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$ modules on a locally ringed space $(X, \mathscr{O}_X)$. Let $x \in X$ be a point, and equip $\{x\}$ with a 'sheaf' of rings k_x*[121]*. If $\imath_x : \{x\} \to X$ is the inclusion morphism, then:*

$$(\imath_x^*\mathscr{F})_x \cong \mathscr{F}|_x$$

Proof. This is essentially obvious, we have that:

$$(\imath_x^*\mathscr{F})_x \cong \mathscr{F}_{\imath_x(x)} \otimes_{\mathscr{O}_{X,\imath_x(x)}} k_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,\imath_x(x)}} k_x = \mathscr{F}|_x$$

□

This definition agrees with the smooth manifold one. In particular we have the following example:

Example 5.4.4. Let $\pi : E \to X$ be a smooth vector bundle over a smooth manifold, and consider the sheaf of smooth sections of E on X, denoted $\Gamma(-, E)$. We claim that there is a natural identification:

$$\pi^{-1}(x) \cong \Gamma(-, E)|_x$$

Indeed, we have that:

$$\Gamma(-,E)|_x = \Gamma(-,E)_x \otimes_{C_x^\infty} C_x^\infty/\mathfrak{m}_x \cong \Gamma(-,E)_x/(\mathfrak{m}_x\Gamma(-,E)_x)$$

We define a map:

$$\begin{aligned}\psi : \Gamma(-,E)_x &\longrightarrow \pi^{-1}(x)\\ [U,s] &\longmapsto s(x)\end{aligned}$$

which is easily seen to be surjective. Moreover, it's kernel is given by:

$$\{[U,s] \in \Gamma(-,E)_x : s(x) = 0\}$$

This clearly contains $\mathfrak{m}_x\Gamma(-,E)_x$, so suppose that $[U,s] \in \ker\psi$. By shrinking U, we can write $s = s^ie_i$ where e_i is a local frame for $E|_U$, and the s^i are smooth functions. In particular, we have have that $s^i(x) = 0$ for all i as $e_i(x)$ form a basis for $\pi^{-1}(x)$. It follows that:

$$[U,s] = \sum_i [U, s^ie_i]$$

[121] This is the sheaf which on $\{x\}$ is k_x and on the empty set is zero.

which is a sum of elements in $\mathfrak{m}_x\Gamma(-,E)_x$, hence $\ker\psi = \mathfrak{m}_x\Gamma(-,E)_x$. It follows that:

$$\pi^{-1}(x) \cong \Gamma(-,E)|_x$$

as desired.

One might hope that $f^*E|_x = E|_{f(x)}$ carries over verbatim to our scheme analogue, but this is unfortunately too much to ask for. Instead we have the following:

Lemma 5.4.6. *Let $f : X \to Y$ be morphism of schemes, and $\mathscr{F}$ a sheaf of $\mathscr{O}_Y$ modules, then:*

$$(f^*\mathscr{F})|_x \cong \mathscr{F}|_{f(x)} \otimes_{k_{f(x)}} k_x$$

If $\mathscr{F}$ is a locally free $\mathscr{O}_X$ module, X and Y are varieties over $k = \bar{k}$, f a morphism of k-schemes, and x a k-rational point[122] *then:*

$$(f^*\mathscr{F})|_x \cong \mathscr{F}|_{f(x)}$$

Proof. Note that the morphism $f_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ induces a field extension $\bar{f}_x : k_{f(x)} \hookrightarrow k_x$, as $f_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$. Note that:

$$\begin{aligned}(f^*\mathscr{F})|_x =&(f^*\mathscr{F})_x \otimes_{\mathscr{O}_X,x} k_x\\ =&(\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}) \otimes_{\mathscr{O}_X,x} k_x\\ =&\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} k_x\end{aligned}$$

Now we have that $k_x \cong k_{f(x)} \otimes_{k_{f(x)}} k_x$ hence:

$$\begin{aligned}(f^*\mathscr{F})|_x \cong&\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} k_{f(x)} \otimes_{k_{f(x)}} k_x\\ \cong&\mathscr{F}|_{f(x)} \otimes_{k_{f(x)}} k_x\end{aligned}$$

as desired.

Moreover, if X and Y are varieties, and x is a closed point, then by Proposition 3.5.2 $f(x)$ is a k-rational point. It follows that $k_x \cong k \cong k^I_{f(x)}$, hence:

$$f^*\mathscr{F}|_x = \mathscr{O}^I_{Y,f(x)} \otimes_{\mathscr{O}_Y,f(x)} k_x \cong k_x^I \cong k^I_{f(x)} \cong \mathscr{F}|_{f(x)}$$

□

By Lemma 5.2.2, we have that there is an isomorphism:

$$\underline{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y, \mathscr{F}) \cong \mathscr{F}$$

hence global sections of $\mathscr{F}$ are in one to one correspondence with sheaf morphisms $\mathscr{O}_Y \to \mathscr{F}$. Given a global section $s \in \mathscr{F}(Y)$, let $\phi_s : \mathscr{O}_Y \to \mathscr{F}$ be the morphism induced by s. By the functoriality of f^*, we obtain a morphism $f^*\phi_s : \mathscr{O}_X \to f^*\mathscr{F}$, which in turn determines a global section of $f^*\mathscr{F}$. We denote this section by f^*s.

Definition 5.4.2. Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}$ an $\mathscr{O}_Y$ module, and s a global section of $\mathscr{F}$. We define the **pullback of s** to be f^*s.

We now have the following result:

Proposition 5.4.3. *Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}$ an $\mathscr{O}_Y$ module, and s a global section of $\mathscr{F}$. Then:*

$$(f^*s)_x = s_{f(x)} \otimes 1 \in \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$$

Moreover, let $U = \operatorname{Spec} A \subset X$ map into $V = \operatorname{Spec} B \subset Y$, then if $\mathscr{F}$ is quasicoherent, and $\mathscr{F}|_V \cong \widetilde{M}$, we have that:

$$f^*s|_U = s|_V \otimes 1 \in \mathscr{F}(V) \otimes_{\mathscr{O}_Y(V)} \mathscr{O}_X(U)$$

[122] i.e. has residue field equal to k.

Proof. Let $\phi_s : \mathscr{O}_Y \to \mathscr{F}$, then the stalk map:

$$(\phi_s)_y : \mathscr{O}_{Y,y} \longrightarrow \mathscr{F}_y$$

sends 1 to s_y. On the level of stalks, up to isomorphism, f^* must take the above map to the morphism:

$$(f^*\phi_s)_x : \mathscr{O}_{X,x} \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}$$
$$1 \longmapsto s_{f(x)} \otimes 1$$

Since $f^*\phi_s$ is by definition the morphism which takes $1 \in \mathscr{O}_X(X)$ to $f^*s|_U \in (f^*\mathscr{F})(U)$, i.e. $f^*\phi_s = \phi_{f^*s} \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F})$, it follows that $(f^*s)_x = s_{f(x)} \otimes 1$ as desired.

Now let $\mathscr{F}$ be quasicoherent, $U = \operatorname{Spec} A \subset X$, $V = \operatorname{Spec} B \subset Y$ as stated, and $\mathscr{F}|_V \cong \widetilde{M}$. By Corollary 5.4.3, we have that $(f^*\mathscr{F})|_U \cong \widetilde{M \otimes_B A}$, and, by Lemma 5.4.4, we have that pulling back quasicoherent sheaves of $\mathscr{O}_{\operatorname{Spec} B}$ modules is equivalent to tensoring over B with A. In particular, the induced morphism:

$$\phi_{(f^*s)|_U} : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow \widetilde{M \otimes_B A}$$

is the one given on global sections by:

$$A \longrightarrow M \otimes_B A$$
$$1 \longmapsto s|_V \otimes 1$$

It follows that $(f^*s)|_U = s|_V \otimes 1$, implying the second claim. □

Foy any sheaf of $\mathscr{O}_X$ modules $\mathscr{F}$, we have an evaluation map $\mathscr{F}(U) \to \mathscr{F}|_x$, given by the composition:

$$\mathscr{F}(U) \longrightarrow \mathscr{F}_x \longrightarrow \mathscr{F}|_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} k_x \cong \mathscr{F}_x/(\mathfrak{m}_x\mathscr{F}_x) \tag{5.4.1}$$

which sends s to it's image in the quotient $\mathscr{F}_x/(\mathfrak{m}_x\mathscr{F}_x)$. We denote this evaluation by $s(x)$. When X is a smooth manifold, and $\mathscr{F} = \Gamma(-, E)$ this morphism, up to isomorphism, is the same as evaluating a section s at x. The following example provides a nice connection between the structure sheaf of the affine plane, and honest to god rational functions on k^n.

Example 5.4.5. Let $X = \mathbb{A}^n_k$, and $\mathscr{F} = \mathscr{O}_{\mathbb{A}^n_k}$ be the trivial rank 1 vector bundle with $k = \bar{k}$. We want to show that for any $s \in \mathscr{O}_{\mathbb{A}^n_k}(U_f)$, and $x \in U_f \cap |\mathbb{A}^n_k|$, that $s(x)$ is an actual evaluation map. In particular, $s = p/f^m$ where $p, f \in k[x_1, \dots, x_n]$, and x can be identified with a tuple $(u_1, \dots, u_n) \in k^n$ such that $f(u_1, \dots, u_n) \neq 0$, so we want to show that $s(x) = p(u_1, \dots, u_n)/f^m(u_1, \dots, u_n) \in k$. Since x is a closed point, we have that the residue field k_x is k, and since $\mathscr{O}_{A^n_k}$ is the trivial rank one vector bundle, we have that $\mathscr{O}_{A^n_k}|_x = k$. In particular, x corresponds to the maximal ideal $\mathfrak{m} = \langle x_1 - u_1, \dots, x_n - u_n\rangle$, so with $A = k[x_1, \dots, x_n]$, the the evaluation map (5.4.1) is given by:

$$A_f \longrightarrow A_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}/\mathfrak{m}' \longrightarrow k \tag{5.4.2}$$

where $\mathfrak{m}'$ is the unique maximal ideal $\langle \mathfrak{m} \rangle \subset A_{\mathfrak{m}}$. The last map is an isomorphism, induced by the map $A \to k$ given by evaluation at $(u_1, \dots, u_n)$. Since $\mathfrak{m}$ is the kernel of this map, everything outside of $\mathfrak{m}$ is invertible when sent to k, hence by the universal property of localization there is a unique morphism $A_{\mathfrak{m}} \to k$. The kernel of this morphism is precisely $\mathfrak{m}'$, hence we have the isomorphism $A_{\mathfrak{m}}/\mathfrak{m}' \to k$. Letting $s = p/f^m$ as before, we see that (5.4.2) is given by:

$$\frac{p}{f^m} \longmapsto \frac{p}{f^m} \in A_{\mathfrak{m}} \longmapsto \left[\frac{p}{f^m}\right] \longmapsto \frac{p(u_1, \dots, u_n)}{f^m(u_1, \dots, u_n)}$$

as desired.

The naive pullback of a section s in this context, given by $s \circ f \in \Gamma(X, f^*E)$, evaluates to $s(f(x)) \in E|_{f(x)}$, for all $x \in X$. In our more general context of $\mathscr{O}_X$ modules, we have the following analogue:

Corollary 5.4.6. *Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}$ a sheaf of $\mathscr{O}_Y$ modules, and $s \in \mathscr{F}(Y)$. Then for all $x \in X$, we have that:*

$$(f^*s)(x) = s(f(x)) \otimes 1 \in \mathscr{F}|_{f(x)} \otimes_{k_{f(x)}} k_x \cong (f^*\mathscr{F})|_x$$

Moreover, if X and Y are varieties over k, $\mathscr{F}$ is quasicoherent, and x is a k-rational point[123] *then:*

$$(f^*s)(x) = s(f(x)) \in \mathscr{F}|_{f(x)}$$

Proof. Note that $(f^*s)(x)$ is the image of $(f^*s)_x$ under the evaluation map $(f^*\mathscr{F})_x \to (f^*\mathscr{F})|_x$. By Lemma 5.4.6 this is given up to isomorphism by the obvious map:

$$\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x} \longrightarrow \mathscr{F}|_{f(x)} \otimes_{k_{f(x)}} k_x$$

By Proposition 5.4.3, we have that $(f^*s)_x = s_{f(x)} \otimes 1$, which is then obviously sent to $s(f(x)) \otimes 1$, hence we have the first claim.

The second claim now follows by identifying $(f^*\mathscr{F})|_x$ with $\mathscr{F}|_{f(x)}$ via the second part of Lemma 5.4.6 □

In Section 5.10 we will further explore the connection between this view of vector bundles, and sheaves of $\mathscr{O}_X$ modules, and viewing sections of locally free sheaves as honest to god morphisms from a base scheme to the total space.

In Example 5.4.3 we noted that if X is a scheme, and $Y \subset X$ is a closed subset, then the corresponding ideal sheaf $I_{Y/X}$, which induces the reduced subscheme structure on Y, is quasicoherent. We now show that this generalizes to any closed subscheme:

Lemma 5.4.7. *Let $\imath : Z \hookrightarrow X$ be a closed embedding. Then $I_{Z/X} := \ker \imath^\sharp$ is a quasicoherent sheaf of ideals on X. If X is locally Noetherian, then $\ker \imath^\sharp$ is coherent.*

Proof. It is obvious that $I_{Z/X}$ is a sheaf of ideals on X, and thus a sheaf of $\mathscr{O}_X$ modules. Now by Lemma 3.1.1 for every affine open $U = \operatorname{Spec} A$ of X, we have that $\imath^{-1}(U) \cong \operatorname{Spec} A/I$ for some ideal $I \subset A$. The sheaf morphism $\imath|_{\imath^{-1}(U)}$ is then the one induced by the projection $A \to A/I$. It is now easily seen that $I_{Z/X}|_U$ is the sheaf $\widetilde{I}$, as on each distinguished open, $\ker \imath^\sharp_{U_f}$ is the localized ideal I_f. By Corollary 5.4.2 it follows that $I_{Z/X}$ is quasicoherent.

If X is locally Noetherian, then we can find an affine open cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that each affine open is Noetherian. It follows that we obtain ideals $I_i \subset A_i$ such that $I_{Z/X}|_{U_i} \cong \widetilde{I}_i$. Since A_i is Noetherian, I_i is finitely generated, so by Proposition 5.4.2 we have that $I_{Z/X}$ is coherent. □

The next step is to show that this correspondence is actually one to one.

Theorem 5.4.2. *Let X be a scheme, then there is a one to one correspondence between closed subschemes of X, and quasicoherent sheaves of ideals on X.*

Proof. Let Z be a closed subscheme of X, then we have an equivalence class of closed embeddings $f : Z \to X$, where two closed embeddings f and g are equivalent if and only there is an isomorphism $F : Z \to Z$ such that $f \circ F = g$. In Lemma 5.4.7, we showed that a closed embedding induced a quasicoherent sheaf of ideals; we now need to show that any two equivalent closed embeddings induce the same quasicoherent sheaf of ideals.

Let f and g be equivalent closed embeddings $Z \to X$, and denote by $I_{Z/X,f}$ and $I_{Z/X,g}$ the kernels of $f^\sharp$ and $g^\sharp$ respectively. Fix an affine open $U = \operatorname{Spec} A \subset X$, then $f^{-1}(U) = \operatorname{Spec} A/I$ and $g^{-1}(U) = \operatorname{Spec} A/J$. It suffices to show that $I = J$. Since $f \circ F = g$, we have that $F^{-1}(f^{-1}(U)) = g^{-1}(U)$, thus $F|_{f^{-1}(U)}$ is a morphism $f^{-1}(U) \to g^{-1}(U)$. In particular, $F|_{f^{-1}(U)}$ comes from a ring isomorphism

[123] I.e. has residue field equal to k.

$\phi : A/J \to A/I$ which makes the following diagram commute:

$$\begin{tikzcd} & A \arrow[dl, "\pi_J"'] \arrow[dr, "\pi_I"] & \\ A/J \arrow[rr, "\phi"] & & A/I \end{tikzcd}$$

It follows that:

$$I = \ker \pi_I = \ker(\phi \circ \pi_J) = (\phi \circ \pi_J)^{-1}(0) = \pi_J^{-1}(\phi^{-1}(0)) = \pi_J^{-1}(0) = \ker \pi_J = J$$

implying any two equivalent closed embeddings induce the same quasicoherent sheaf of ideals.

Now suppose that Z and Y are two closed subschemes of X which induce the same quasicoherent sheaf of ideals on X. That is for closed embeddings $f : Z \to X$ and $g : Y \to X$, we have that $I_{Z/X} = I_{Y/X}$. We need to show that there is an isomorphism $F : Y \to Z$ such that $f \circ F = g$. Let $U = \operatorname{Spec} A$ be an open, then since $I_{Z/X} = I_{Z/Y}$ we have that $f^{-1}(U)$ is uniquely isomorphic to $g^{-1}(U)$ because both are of the form $A/I_{Z/X}(U)$. If we take an affine open cover $\{U_i\}$ of X, then these unique isomorphisms $F_i : g^{-1}(U_i) \to f^{-1}(U_i)$ must agree on overlaps, as they will both restrict to the obvious morphisms on a distinguished open cover of $U_i \cap U_j$. In other words, $F_i|_{U_g} = F_j|_{U_g}$ for any $U_g \subset U_i \cap U_j$ which is simultaneously distinguished in both $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$, because F_i and F_j are both induced by the quotient maps. It follows that the F_i glue together to give an isomorphism $F : Y \to Z$ satisfying $f \circ F = g$ by construction. Therefore, Z and Y represent the same equivalence class, so $Y = Z$ as closed subschemes.

We have so far shown that for every closed subscheme $Z \subset X$ there is a unique quasicoherent sheaf of ideals on X induced by Z. It remains to show that every quasicoherent sheaf of ideals comes from the kernel of a closed embedding. Let $\mathscr{I}$ be a sheaf of quasicoherent ideals, and define the following subset of X:

$$Z = \{x \in X : \mathscr{I}_x = \mathscr{O}_{X,x}\}^c$$

If $U = \operatorname{Spec} A$ is an affine open, then $\mathscr{I}|_U \cong \widetilde{I}$ for some $I \subset A$. We claim that $Z \cap U = \mathbb{V}(I) \subset \operatorname{Spec} A$; we have that:

$$Z \cap U = \{\mathfrak{p} \in \operatorname{Spec} A : I_{\mathfrak{p}} = A_{\mathfrak{p}}\}^c$$

Let $\pi_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ be the localization map, then $I_{\mathfrak{p}} = \langle \pi_{\mathfrak{p}}(I) \rangle$. If $I_{\mathfrak{p}} \neq A_{\mathfrak{p}}$, then no element in $\pi_{\mathfrak{p}}(I)$ is invertible. In particular, no $I \cap (A \smallsetminus \mathfrak{p}) = \emptyset$. It follows that $I \subset \mathfrak{p}$, so $\mathfrak{p} \in \mathbb{V}(I)$. Now suppose that $\mathfrak{p} \in \mathbb{V}(I)$, then $I \subset \mathfrak{p}$, hence $I_{\mathfrak{p}} \subset \mathfrak{m}_p \subsetneq A_{\mathfrak{p}}$. Since the affine opens form a basis for the topology, it follows that Z is closed.

We have an inclusion map $\imath : Z \to X$, but we need to make Z a scheme, and $\imath$ a closed embedding such that $\ker \imath^\sharp = \mathscr{I}$. We set $\mathscr{O}_Z := \imath^{-1}(\mathscr{O}_X/\mathscr{I})$, and claim that this makes Z a scheme. Let $U = \operatorname{Spec} A \subset X$, then the same argument at the end of Theorem 2.1.2 demonstrates that $(Z \cap U, \mathscr{O}_{Z \cap U})$ is isomorphic as a locally ringed space to $(\operatorname{Spec} A/\mathscr{I}(U), \mathscr{O}_{\operatorname{Spec} A/\mathscr{I}(U)})$, hence Z is a scheme. The natural morphism $\imath^\sharp : \mathscr{O}_X \to \imath_* \mathscr{O}_Z$ is then induced on open affines by the projection $A \to A/\mathscr{I}(U)$, hence $(\imath, \imath^\sharp)$ is a morphism of schemes making $\imath : Z \to X$ a closed embedding. By construction we have that $\ker \imath^\sharp = \mathscr{I}$, hence every quasicoherent sheaf of ideals comes from a closed embedding.

The map:

$$\{\text{closed subschemes of } X\} \to \{\text{quasicoherent sheaves of ideals on } X\}$$

is thus injective, and surjective, implying the claim. $\square$

5.5 Pushforwards of Quasicoherent Sheaves

In the first section, we saw that if $f : X \to Y$ was a morphism of locally ringed spaces, then f^* is a functor from quasicoherent sheaves on Y to quasicoherent sheaves on X. In fact, if $\mathscr{O}_X$ and $\mathscr{O}_Y$ were coherent, then f^* is a functor from $\mathrm{Coh}_{\mathscr{O}_Y} \to \mathrm{Coh}_{\mathscr{O}_X}$. In this section we discuss sufficient conditions on which f_* takes quasicoherent sheaves to quasicoherent sheaves, and the consequences of such a result. We will then treat the case of pushforwards of a closed embedding $\imath : Z \to X$ in detail, where we will be able to describe pushforwards and pull backs in a particularly satisfying way.

We need the following lemma:

Lemma 5.5.1. *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes, and M and A module. Then $f_*\widetilde{M} \cong \widetilde{M}_B$, where M_B is the abelian group M equipped with a B module structure induced by f. In particular f_* takes quasicoherent $\mathscr{O}_{\operatorname{Spec} A}$ modules to quasicoherent $\mathscr{O}_{\operatorname{Spec} B}$ modules.*

Proof. Let $\phi : B \to A$ be ring morphism inducing f. We have that on every distinguished open $U_b \subset \operatorname{Spec} B$:

$$f_*\widetilde{M}(U_b) = \widetilde{M}(U_{\phi(b)}) = M_{\phi(b)}$$

However, since B acts on M_B via ϕ, we have that $M_{\phi(b)}$ is $(M_B)_b$, which $\widetilde{M_B}(U_b)$. These identifications obviously commute with restrictions, hence we have an isomorphism $f_*\widetilde{M} \cong \widetilde{M}_B$. $\square$

Proposition 5.5.1. *Let $f : X \to Y$ be a qcqs morphism of schemes, and $\mathscr{F}$ be A quasicoherent sheaf of $\mathscr{O}_X$ modules. Then $f_*\mathscr{F}$ is a quasicoherent sheaf of $\mathscr{O}_Y$ modules.*

Proof. We already know that $f_*\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$ modules, so we need only show that $f_*\mathscr{F}$ is quasicoherent. A sheaf of $\mathscr{O}_Y$ modules is quasicoherent if and only if it's restriction $f_*\mathscr{F}|_U$ to each affine open is quasicoherent, and so it suffices to prove this when Y is an affine scheme.

Now X is quasicompact so choose an affine open cover $\{U_i\}_{i=1}^n$. Since X is quasiseparated over an affine scheme, we have that $U_i \cap U_j$ is quasicompact and thus admits an affine open cover $\{U_{ijk}\}_{i=1}^{m_{ij}}$. Denote by f_i and f_{ijk}, the morphism f restricted to U_i and U_{ijk} respectively. For any $V \subset Y$ open, have that by definition:

$$\begin{aligned}\mathscr{F}(f^{-1}(V)) &= \left\{ (s_i) \in \prod_{i=1}^n \mathscr{F}(f^{-1}(V) \cap U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \right\} \\ &= \left\{ (s_i) \in \prod_{i=1}^n \mathscr{F}(f^{-1}(V) \cap U_i) : s_i|_{U_{ijk}} = s_j|_{U_{ijk}} \forall i, j, k \right\}\end{aligned}$$

In particular, this is the kernel of the morphism:

$$\begin{aligned}F_V : \prod_{i=1}^n \mathscr{F}(f^{-1}(V) \cap U_i) &\longrightarrow \prod_{i,j,k} \mathscr{F}(f^{-1}(V) \cap U_{ijk}) \\ (s_i) &\longmapsto (s_i|_{U_{ijk}} - s_j|_{U_{ijk}})\end{aligned}$$

Note that F_V is actually a morphism:

$$F_V : \prod_{i=1}^n f_{i*}(\mathscr{F}|_{U_i})(V) \longrightarrow \prod_{i,j,k} f_{ijk*}(\mathscr{F}|_{U_{ijk}})(V)$$

These maps obviously commute with restriction maps, hence we get a morphism:

$$F : \prod_{i=1}^n f_{i*}(\mathscr{F}|_{U_i}) \longrightarrow \prod_{i,j,k} f_{ijk*}(\mathscr{F}|_{U_{ijk}})$$

Now F clearly satisfies $f_*\mathscr{F} = \ker F$; by Lemma 5.5.1 we have that $\prod_{i=1}^n f_{i*}(\mathscr{F}|_{U_i})$ and $\prod_{i,j,k} f_{ijk*}\mathscr{F}|_{U_{ijk}}$ are quasicoherent sheaves, hence $f_*\mathscr{F}$ is quasicoherent by Corollary 5.4.2. $\square$

The reason the qcqs hypothesis was needed above is that we don't necessarily have that infinite direct products of quasicoherent sheaves are quasicoherent because pulling back by an open embedding (i.e restricting the sheaf to an open set) does not commute with taking infinite direct products. We very much need to be able to make use finite covers for the above argument to hold.

The fact that pushforwards by qcqs morphisms will actually allow to deduce some incredibly desirable results about quasicoherent sheaves on a qcqs scheme. In particular, let X be qcqs; $\operatorname{Spec} \mathscr{O}_X(X)$ is also qcqs because every affine scheme is qcqs. If we let $f : X \to \operatorname{Spec} \mathscr{O}_X(X)$ be the natural morphism induced by the identity map $\mathscr{O}_X(X) \to \mathscr{O}_X(X)$[124], then Corollary 3.11.1 implies that f is qcqs. This will allow us to prove the following:

Corollary 5.5.1. *Let X be qcqs, and $\mathscr{F}$ a quasicoherent sheaf on X. Then for all $a \in \mathscr{O}_X(X)$, we have that the natural map:*

$$\mathscr{O}_X(X)_a \longrightarrow \mathscr{O}_X(X_a)$$

is an isomorphism. In generality, if $\mathscr{F}$ is quasicoherent, then the induced map:

$$\mathscr{F}(X)_a \longrightarrow \mathscr{F}(X_a)$$

is an isomorphism.

Proof. Note that:

$$X_a = \{x \in X : a_x \notin \mathfrak{m}_x \subset \mathscr{O}_{X,x}\}$$

and by our work in Proposition 2.1.2 we have that if $f : X \to \operatorname{Spec} \mathscr{O}_X(X)$ is the natural morphism discussed earlier, then $f^{-1}(U_a) = X_a$. In particular, the sheaf morphism is given on distinguished opens by the morphism:

$$\mathscr{O}_X(X)_a \longrightarrow \mathscr{O}_X(X_a)$$

which is induced by the restriction map $\theta^X_{X_a}$, and the universal property of localization. Note that $\mathscr{O}_X(X_a) = f_*\mathscr{O}_X(U_a)$; since $f_*\mathscr{O}_X$ is quasicoherent by Proposition 5.5.1, we have that $(f_*\mathscr{O}_X)(U_a) = \mathscr{O}_X(X)_a$, hence the first claim.

The second claim follows from the same observation:

$$(f_*\mathscr{F})(U_a) = \mathscr{F}(X_a) = \mathscr{F}(X)_a$$

and that the morphism $\mathscr{F}(X)_a \to \mathscr{F}(X_a)$ is induced by the ring morphism. □

5.6 Examples Over Projective Schemes

Let A be a $\mathbb{Z}^{\geq 0}$ graded ring; in this section we discuss how to take a graded module over A and construct a quasicoherent sheaf over $\operatorname{Proj} A$. A case of particular interest will be projective schemes, and projective space. First we recall the definition of a graded module:

Definition 5.6.1. Let M be a module over a positively graded ring A, then M is a **graded A module** if:

$$M = \bigoplus_i M_i$$

and $A_i \cdot M_j \subset M_{i+j}$ for all i and j. A **morphism of graded A of degree d** is a morphism of A modules which maps M_i into M_{i+d}.

Many analogues from Section 2.2 hold in this setting, and we state these results, and cite the results they come from:

Lemma 5.6.1. *Let M be a graded module A, N a submodule generated by homogeneous elements, and $S \subset A$ a multiplicatively closed set consisting of homogeneous elements. Then the following hold:*

[124] See Proposition 2.1.2.

a) *N is a graded A module such that $N_i \subset M_i$.*

b) *M/N is a graded A module, with grading given by $(M/N)_i = M_i/N_i$.*

c) *$S^{-1}M$ is a graded A module, with grading induced on homogeneous elements by $\deg(m/s) = \deg m - \deg s$.*

d) *If $f, g \in A$ are homogeneous elements of positive degree, then there is a unique isomorphism:*

$$(M_{fg})_0 \cong ((M_f)_0)_h$$

where $h = g^{\deg f}/f^{\deg g}$. In particular, if $h^{-1} = f^{\deg g}/g^{\deg g} \in (A_g)_0$ there is a unique isomorphism:

$$((M_f)_0)_h \cong ((M_g)_0)_{h^{-1}}$$

Proof. A similar from Lemma 2.2.2 proves *a*). A similar argument as in Lemma 2.2.3 proves *b*). A similar argument as in Lemma 2.2.4 proves *c*), and a similar argument as in Lemma 2.2.7 proves *d*). □

We now have the main result of the section, after which we will spend some time calculating examples:

Theorem 5.6.1. *Let M be graded A module, then there exists a quasicoherent sheaf of $\mathscr{O}_{\operatorname{Proj} A}$ modules, denoted $\mathscr{M}$, such that for each distinguished $U_f \subset \operatorname{Proj} A$, we have that $\mathscr{M}|_{U_f} \cong \widetilde{(M_f)_0}$. In particular, the assignment $M \mapsto \mathscr{M}$ is an exact functor from the category of graded modules*[125] *to* $\operatorname{QCoh}(\operatorname{Proj} A)$.

Proof. We emulate the proof of Theorem 2.2.1. Let A_+^{hom} be the set of homogeneous elements of A of positive, degree and consider the open cover $\{U_f\}_{f \in A_+^{\text{hom}}}$. For each U_f, we define $\mathscr{F}_f := \widetilde{(M_f)_0}$. We thus need only define sheaf morphism $\phi_{fg} : \mathscr{F}_f|_{U_{fg}} \to \mathscr{F}_g|_{U_{fg}}$ which satisfy the cocycle condition on triple overlaps. Let $h = g^{\deg f}/f^{\deg g}$; by Theorem 5.4.1, it suffices to define module isomorphisms:

$$\psi_{fg} : ((M_f)_0)_h \longrightarrow ((M_f)_0)_{h^{-1}}$$

such that the induced morphisms:

$$\psi_{fg,l} : ((M_f)_0)_{h'} \longrightarrow ((M_f)_0)_{(h')^{-1}}$$

where $h' = (gl)^{\deg f}/f^{\deg gl}$, satisfy:

$$\psi_{fl,g} = \psi_{gl,f} \circ \psi_{fg,l} \tag{5.6.1}$$

We define these to be the natural isomorphisms from part *d*) of Lemma 5.6.1. However, by the uniqueness of these isomorphisms, we must have that (5.6.1) holds, as $\psi_{gl,f} \circ \psi_{fg,l}$ will make the same tensor product diagram commute as $\psi_{fl,g}$. It follows that the $\mathscr{F}_f$ glue together to yield a sheaf we denote by $\mathscr{M}$. Now note that $\mathscr{M}$ is obviously quasicoherent, and satisfies $\mathscr{M}|_{U_f} \cong \widetilde{(M_f)_0}$ by construction, proving the claim.

Now let $\phi : M \to N$ be a morphism of graded modules, since localization exact for all $f \in A_+^{\text{hom}}$ we get a graded morphism $M_f \to N_f$, which in turn yields morphisms $\phi_f : (M_f)_0 \to (N_f)_0$, giving us sheaf morphism $\Phi_f : \mathscr{M}|_{U_f} \to \mathscr{N}|_{U_f}$. By localizing at $h = g^{\deg f}/f^{\deg g}$ we obtain the module morphism $\phi_{fg} : ((M_f)_0)_h \to ((N_f)_0)_h$ which induces the sheaf morphisms $\Phi_f|_{U_f \cap U_g} : \mathscr{M}|_{U_f \cap U_g} \to \mathscr{N}|_{U_f \cap U_g}$. We need to check that $\Phi_f|_{U_f \cap U_g} = \Phi_g|_{U_f \cap U_f}$. Since these morphisms were all uniquely determined by ϕ, the module morphisms $\phi_{fg} : ((M_f)_0)_h \to ((N_f)_0)_h$ and $\phi_{gf} : ((M_g)_0)_{h^{-1}} \to ((N_g)_0)_{h^{-1}}$ agree up to the isomorphisms $((M_f)_0)_h \cong ((M_g)_0)_{h^{-1}}$ and $((M_g)_0)_h \cong ((M_f)_0)_{h^{-1}}$. Since these are the isomorphisms which glue $\mathscr{M}$ and $\mathscr{N}$ together, it follows that $\Phi_f|_{U_f \cap U_g} = \Phi_g|_{U_f \cap U_f}$, hence we obtain a morphism $\Phi : \mathscr{M} \to \mathscr{N}$; this process clearly commutes with composition as localization is functorial, hence the assignment $M \mapsto \mathscr{M}$ is indeed a functor, which is exact because localization is exact, hence on an affine open covering it is exact. □

Note that $\mathscr{O}_{\operatorname{Proj} A} = \mathscr{A}$.

[125] Here the morphisms are module morphisms $\phi : M \to N$ such that $\phi(M_d) \subset N_d$.

Example 5.6.1. Let A be a graded ring and $n \in \mathbb{Z}$, then we define $A(n)$ to be the graded A-module whose underlying abelian group is equal to A, but with grading given by $A_i := A_{i+n}$. We set $\mathscr{O}_{\operatorname{Proj} A}(n)$ to be the quasicoherent sheaf of $\mathscr{O}_{\operatorname{Proj} A}$ modules induced by $A(n)$. In the case that $A = B[x_0, \ldots, x_m]$, then $\operatorname{Proj} A = \mathbb{P}^m_B$, and we denote this sheaf of $\mathscr{O}_{\mathbb{P}^m_B}$ modules by $\mathscr{O}_{\mathbb{P}^m_B}(n)$.

Lemma 5.6.2. *Let $n \in \mathbb{Z}$, then $\mathscr{O}_{\mathbb{P}^m_A}(n)$ is a rank 1 vector bundle, such that:*

$$\left(\mathscr{O}_{\mathbb{P}^m_A}(n)\right)(\mathbb{P}^m_A) \cong A[x_0, \ldots, x_m]_n$$

Proof. Let $B = A[x_0, \ldots, x_m]$. We have an affine open cover given by $U_{x_i} \cong \operatorname{Spec}(B_{x_i})_0$. To see that $\mathscr{O}_{\mathbb{P}^m_A}(n)$ is a rank 1 vector bundle it suffices to show that $\mathscr{O}_{\mathbb{P}^m_A}(n)|_{U_{x_i}} \cong \mathscr{O}_{\mathbb{P}^m_A}|_{U_{x_i}}$ for each i. In particular, we have that:

$$\mathscr{O}_{\mathbb{P}^m_A}(n)|_{U_{x_i}} \cong (\widetilde{B(n)_{x_i}})_0$$

The degree 0 elements of $B(n)_{x_i}$ are the degree n elements of B_{x_i} by definition. It thus suffices to provide a module isomorphism:

$$(B_{x_i})_0 = (A[x_0, \ldots, x_m]_{x_i})_0 \longrightarrow (A[x_0, \ldots, x_m]_{x_i})_n = (B(n)_{x_i})_0$$

Every element in $(B_{x_i})_0$ is a degree 0 polynomial, hence we define our morphism by:

$$\frac{p}{x_i^k} \longmapsto \frac{p(x_0, \ldots, x_m)}{x_i^k} \cdot x_i^n$$

where $p \in A[x_0, \ldots, x_m]$. This is injective as x_i^n is always an integral[126] element of $A[x_0, \ldots, x_m]_{x_i}$. Let $p/x_i^k \in (B(n)_{x_i})_0$, then $\deg p - k = n$, so $\deg p = k + n$. It follows that $p/x_i^{k+n} \in (B_{x_i})_0$, so the morphism is surjective as well and thus an isomorphism. It follows that we get an induced isomorphism $\mathscr{O}_{\mathbb{P}^m_A}(n)|_{U_{x_i}} \cong \mathscr{O}_{\mathbb{P}^m_A}|_{U_{x_i}}$, implying that $\mathscr{O}_{\mathbb{P}^m_A}(n)$ is a rank one vector bundle as desired.

Now by Theorem 1.2.2 we have that the global sections of $\mathscr{O}_{\mathbb{P}^m_A}(n)$ are given by:

$$\left\{ (s_i) \in \prod_i (B(n)_{x_i})_0 : s_j|_{U_{x_l \cdot x_j}} = s_l|_{U_{x_l \cdot x_j}} \right\}$$

Now, suppose that:

$$s_l = \frac{p}{x_l^{k_l}} \qquad \text{and} \qquad s_j = \frac{q}{x_j^{k_j}}$$

We identify $\mathscr{O}_{\mathbb{P}^m_A}(n)(U_{x_l \cdot x_j})$ as $((B(n)_{x_l})_0)_{x_j/x_l}$, then the restriction of s_l to $U_{x_l \cdot x_j}$ is given by:

$$\frac{p}{x_l^{k_l}} \in ((B(n)_{x_l})_0)_{x_j/x_l}$$

whilst the restriction of s_j to $U_{x_l \cdot x_j}$ is given by:[127]

$$\frac{q}{x_l^{k_j}} \cdot \left(\frac{x_j}{x_l}\right)^{-k_j} \in ((B(n)_{x_l})_0)_{x_j/x_l}$$

The compatibility condition then ensures that:

$$\frac{p}{x_l^{k_l}} = \frac{q}{x_l^{k_j}} \cdot \left(\frac{x_j}{x_l}\right)^{-k_j}$$

[126] In this proof we use the term 'integral element' of B to mean an element $b \in B$ such that if $b \cdot a = 0$ then $a = 0$. This is in contrast to when we have ring morphism $A \to B$,and b can be *integral over* A, in the sense that is the root of a monic polynomial in $A[x]$.

[127] Here we must localize to $((B(n)_{x_j})_0)_{x_l/x_j}$, and then apply the isomorphism from Lemma 5.6.1

which is the same as:

$$\frac{p \cdot x_j^{k_j}}{x_l^{k_l+k_j}} = \frac{q}{x_l^{k_j}} \in ((B(n)_{x_l})_0)_{x_j/x_l}$$

For these to be equal, there must be some K such that:

$$\left(\frac{x_j}{x_l}\right)^K \cdot \left(\frac{p \cdot x_j^{k_j}}{x_l^{k_l+k_j}} - \frac{q}{x_l^{k_j}}\right) = 0 \in (B(n)_{x_l})_0$$

However, x_j/x_l is an integral element of B_{x_l} , thus we must have that:

$$\frac{p \cdot x_j^{k_j}}{x_l^{k_l+k_j}} - \frac{q}{x_l^{k_j}} = 0 \in (B(n)_{x_l})_0$$

This is the same as stating that:

$$\frac{p \cdot x_j^{k_j} - q \cdot x_l^{k_l}}{x_l^{k_j+k_l}} = 0 \in (B(n)_{x_l})_0$$

hence there must be some K' such that:

$$x_l^{K'}\left(p \cdot x_j^{k_j} - q \cdot x_l^{k_l}\right) = 0 \in B(n)$$

Since x_l is an integral element of B we have that:

$$p \cdot x_j^{k_j} - q \cdot x_l^{k_l} = 0$$

hence:

$$p \cdot x_j^{k_j} = q \cdot x_l^{k_l}$$

It follows that the polynomial p can be written as $p' \cdot x_l^{k_l}$, and that q can be written as $q' \cdot x_j^{k_j}$ for some $q', p' \in B$. We thus have that:

$$p' \cdot x_j^{k_j} \cdot x_l^{k_l} = q' \cdot x_j^{k_j} \cdot x_l^{k_l} \Rightarrow p' = q'$$

Note that $\deg p' = \deg q' = n$, so we have that the compatibility condition implies that each $s_i \in (B(n)_{x_i})_0$ can be written as $p/1$ for some unique $p \in (A[x_0, \ldots, x_m])_n$. There is then an obvious isomorphism:

$$\left\{(s_i) \in \prod_i (B(n)_{x_i})_0 : s_j|_{U_{x_l \cdot x_j}} = s_l|_{U_{x_l \cdot x_j}}\right\} \cong (A[x_0, \ldots, x_m])_n$$

given by sending (s_i) to p, implying the claim. □

Note, that if n is negative then there are no global sections.

Example 5.6.2. In this example we consider $\mathscr{O}_{\mathbb{P}_k^m}(n)$, where $k = \bar{k}$. For any homogeneous $f \in k[x_0, \ldots, x_m]$, and any $x = [u_0, \ldots, u_m] \in |\mathbb{P}_k^m| \cap U_f$, we want to show that the evaluation map:

$$\mathscr{O}_{\mathbb{P}_k^m}(n)(U_f) \longrightarrow \mathscr{O}_{\mathbb{P}_k^m}(n)|_x$$

is given by:

$$\begin{aligned} (k[x_0, \ldots, x_m]_f)_0 &\longrightarrow k \\ \frac{p}{f^l} &\longmapsto \frac{p(u_0, \ldots, u_m)}{f^l(u_0, \ldots, u_m)} \end{aligned}$$

We first want to double check that $f(u_0, \dots, u_m) \neq 0$. Since the U_{x_i} cover $\mathbb{P}^m_k$ we have that $x \in U_{x_i} \cap U_f$ for some x_i; without loss of generality set $i = 0$, and let $\deg f = \alpha$. Note that $U_{x_i} \cap U_f = U_{f/x_0^\alpha} \subset \operatorname{Spec} k[x_1/x_0, \dots, x_m/x_i]$. If:

$$f = \sum_{i_0+\cdots+i_m} a_{i_0\cdots i_m} x_0^{i_0} \cdots x_m^{i_m}$$

then:

$$\begin{aligned} f/x_0^\alpha &= \sum_{i_0+\cdots+i_m} a_{i_0\cdots i_m} \frac{x_0^{i_0} \cdots x_m^{i_m}}{x_0^\alpha} \\ &= \sum_{i_0+\cdots+i_m} a_{i_0\cdots i_m} (x_1/x_0)^{i_0} \cdots (x_m/x_0)^{i_m} \end{aligned}$$

Moreover, x corresponds to the maximal ideal:

$$\mathfrak{m} = \langle x_1/x_0 - u_1/u_0, \dots, x_m/x_0 - u_m/u_0 \rangle$$

Since $x \in U_{f/x_0^\alpha}$, we have that $f/x_0^\alpha \notin \mathfrak{m}$, hence:

$$(f/x_0^\alpha)(u_1/u_0, \dots, u_m/u_0) = \sum_{i_0+\cdots+i_m} a_{i_0\cdots i_m} (u_1/u_0)^{i_0} \cdots (u_m/u_0)^{i_m} \neq 0$$

Multiplying throughout by u_0^α, we obtain:

$$\sum_{i_0+\cdots+i_m=\alpha} a_{i_0\cdots i_m} u_0^{i_0} \cdots u_m^{i_m} \neq 0$$

but this is just $f(u_0, \dots, u_m)$, hence $f(u_0, \dots, u_m) \neq 0$.

Now $\mathscr{O}_{\mathbb{P}^m_k}(n)(U_f)$ is $(k[x_0, \dots x_m]_f)_n$, hence p is a degree $l \cdot \alpha + n$ homogeneous polynomial. In particular, restricting p/f^l to $U_f \cap U_{x_0}$ is the given by the localization map $(k[x_0, \dots, x_m]_f)_n \to ((k[x_0, \dots, x_m]_f)_n)_{x_0^\alpha/f}$. The isomorphism:

$$((k[x_0, \dots, x_m]_f)_n)_{x_0^\alpha/f} \to ((k[x_0, \dots, x_m]_{x_0})_n)_{f/x_0^\alpha}$$

sends:

$$\frac{p}{f^l} \longmapsto \frac{p}{x_0^{l\cdot\alpha}} \cdot \left(\frac{f}{x_0^\alpha}\right)^{-l}$$

and identifies $U_{x_0} \cap U_f$ with $U_{f/x_0^\alpha} \subset \operatorname{Spec} k[x_1/x_0, \dots, x_m/x_0] = \mathbb{A}^{m-1}_k$. Let :

$$p = \sum_{j_0+\cdots+j_m=l\cdot\alpha} b_{j_0\cdots j_m} x_0^{j_0} \cdots x_m^{j_m}$$

then:

$$p/x_0^{l\cdot\alpha} = \sum_{j_0+\cdots+j_m=l\cdot\alpha} b_{j_0\cdots j_m} (x_1/x_0)^{j_0} \cdots (x_m/x_0)^{j_m}$$

Now note that we have:

$$\frac{p}{f^l}(x) = \frac{p}{f^l}\Big|_{U_{x_0}\cap U_f}(x) = \frac{p}{x_0^{l\cdot\alpha}} \cdot \left(\frac{f}{x_0^\alpha}\right)^{-l} (u_1/u_0, \dots, u_m/u_0)$$

Therefore, by Example 5.4.5, we have that:

$$\begin{aligned}\frac{p}{f^l}(x) &= \left(\sum_{j_0+\cdots+j_m} b_{j_0\cdots j_m}(u_1/u_0)^{j_0}\cdots(u_m/u_0)^{j_m}\right)\cdot\left(\sum_{i_0+\cdots+i_m} a_{i_0\cdots i_m}(u_1/u_0)^{i_0}\cdots(u_m/u_0)^{i_m}\right)^{-l}\\ &= \left(\frac{1}{u_0^{l\cdot\alpha}}\sum_{j_0+\cdots+j_m=l\cdot\alpha} b_{j_0\cdots j_m}u_0^{j_0}\cdots u_m^{j_m}\right)\cdot\left(\frac{1}{u_0^{\alpha}}\cdot\sum_{i_0+\cdots+i_m=\alpha} a_{i_0\cdots i_m}u_0^{i_0}\cdots u_m^{i_m}\right)^{-l}\\ &= \left(\sum_{j_0+\cdots+j_m=l\cdot\alpha} b_{j_0\cdots j_m}u_0^{j_0}\cdots u_m^{j_m}\right)\cdot\left(\sum_{i_0+\cdots+i_m=\alpha} a_{i_0\cdots i_m}u_0^{i_0}\cdots u_m^{i_m}\right)^{-l}\\ &= \frac{p(u_0,\dots,u_m)}{f^l(u_0,\dots,u_m)}\end{aligned}$$

as desired.

To continue, we want a choice free way of identifying the stalks $\mathscr{M}_\mathfrak{p}$; it would seem reasonable that $\mathscr{M}_\mathfrak{p} = (M_\mathfrak{p})_0$, however there is not a consistent grading on $M_\mathfrak{p}$ as $A \setminus \mathfrak{p}$ may contain non homogeneous elements. We thus instead examine $M_{\mathfrak{p}^{\text{hom}}}$, which is M localized by:

$$S = A^{\text{hom}} \cap \mathfrak{p}^c$$

We take the zero degree sections of this module and denote it by $M_{(\mathfrak{p})}$. Note that $M_{(\mathfrak{p})}$ is canonically $(M \otimes_A A_{\mathfrak{p}^{\text{hom}}})_0$.

Lemma 5.6.3. *Let $\mathfrak{p} \in \operatorname{Proj} A$, and M a graded A module. Then there is a canonical isomorphism $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}} \cong A_{(\mathfrak{p})}$, and $M_{(\mathfrak{p})}$ is canonically isomorphic to $\mathscr{M}_\mathfrak{p}$ as $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$ modules. In particular, the unique maximal ideal of $A_{(\mathfrak{p})}$ is given by:*

$$\mathfrak{m}_\mathfrak{p} = \left\{\frac{p}{a} \in A_{(\mathfrak{p})} : p \in \mathfrak{p}^{\text{hom}}\right\}$$

Proof. Note that since the U_f for $f \in A_+^{\text{hom}}$ form a basis for the topology on $\operatorname{Proj} A$, we can take the stalk $\mathscr{M}_\mathfrak{p}$ to be the colimit over U_f containing $\mathfrak{p}$.[128] In other words, we have that:

$$\mathscr{M}_\mathfrak{p} = \varinjlim_{\mathfrak{p}\in U_f} (M_f)_0$$

ordered by $U_f < U_g$ if $U_g \subset U_f$. There is a slight subtlety here, the restriction maps $(M_f)_0 \to (M_g)_0$ do not come from the natural localization map $M_f \to M_g$, as these do not restrict to the degree zero part well. Indeed, if $\deg g$ is not divisible by $\deg f$, and we set $g^k = a \cdot f$,[129] then the inverse of f will not lie in $(M_g)_0$. The solution is to see that if we replace g with $g^{\deg f}$ then $U_{g^{\deg f}} = U_g$ and $(M_{g^{\deg f}})_0 \cong (M_g)_0$; the restriction maps then work out fine as $\deg g$ is divisible by $\deg f$. This in fact exactly what the isomorphisms in part d) of Lemma 5.6.1 are taking care of when we glue the sheaf $\mathscr{M}$ together; they are identifying isomorphic modules to ensure compatibility. With that being said, it suffices to take the colimit over $U_g \subset U_f$ and $\deg g$ is divisible by $\deg f$, where the restriction morphisms $(M_f)_0 \to (M_g)_0$ do come from the induced morphism $M_f \to M_g$.[130]

We first prove this in the case that $\mathscr{M} = \mathscr{O}_{\operatorname{Proj} A}$ so as to actually establish that both $\mathscr{M}_\mathfrak{p}$ and $M_{(\mathfrak{p})}$ are $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$ modules. In particular, we want to show that as rings $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}} \cong A_{(\mathfrak{p})}$. There are obvious maps $A_f \to A_{\mathfrak{p}^{\text{hom}}}$ which restrict correctly to the degree zero part of both rings. In particular, these maps are given by $a/f^k \mapsto a/f^k$, where the second quotient is just taken in the larger localization. It is then obvious that we obtain morphism $(A_f)_0 \to A_{(\mathfrak{p})}$ which commute with the restriction maps $(A_f)_0 \to (A_g)_0$. In particular, by the universal property of the colimit, there is a unique morphism:

$$\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}} \longrightarrow A_{(\mathfrak{p})}$$

[128] In particular, Proposition 1.4.2 implies that we can treat $\mathscr{M}$ as coming from the sheaf on a basis $\mathcal{B}$ consisting of distinguished opens. The stalk is then given by this aforementioned colimit due to Theorem 1.4.1 and Proposition 1.4.1

[129] Note that we can always take a to be homogeneous because g and f are homogeneous, so any homogeneous part of a that isn't of degree $k \deg g - \deg f$ must multiply to zero with f.

[130] It is easy to check that in this case the degree zero elements map to degree zero elements.

given by sending the equivalence class $[U_f, a/f^k]$ to $a/f^k \in A_{(\mathfrak{p})}$. This is obviously surjective, as we can take any $a/s^k \in A_{(\mathfrak{p})}$ to have denominator with degree greater than zero[131], hence a/s^k is in the image of $(A_s)_0 \to A_{(\mathfrak{p})}$. If $[U_f, a/f^k]$ maps to zero it follows that there is some $u \notin \mathfrak{p}$ such that:

$$u \cdot a = 0$$

By taking powers of u we can take u to have degree divisible by $\deg f$. Moreover, $u \cdot f \cdot a = 0$, so a/f^k maps to zero under the restriction map $U_f \to U_{uf}$ hence $[U_f, a/f^k] = 0$. It follows that $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$ is canonically isomorphic to $A_{(\mathfrak{p})}$ as desired.

Now the same argument shows that we get a unique morphism of $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$ modules $(\mathscr{M}_f)_0 \to M_{(\mathfrak{p})}$, where $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$ acts on $M_{(\mathfrak{p})}$ via the above isomorphism. This map is given by $[U_f, m/f^k] \mapsto m/f^k$, and it is an isomorphism for the same reason that map $\mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}} \to A_{(\mathfrak{p})}$ was.

For the last claim, since $A_{(\mathfrak{p})} \cong \mathscr{O}_{\operatorname{Proj} A,\mathfrak{p}}$, we know that $A_{(\mathfrak{p})}$ is a local ring. Hence we need only show that:

$$I = \left\{ \frac{p}{a} \in A_{(\mathfrak{p})} : p \in \mathfrak{p}^{\text{hom}} \right\}$$

is a maximal ideal. First not that it cannot be the whole ring since $1 \notin I$. Secondly, note that it is clearly and ideal, as it is an abelian group, and swallows multiplication. Suppose that J is another ideal such that $I \subset J$; if $J \neq I$ then there is some $b/a \in J$ such that $b \notin \mathfrak{p}^{\text{hom}}$ but this implies that b/a is invertible, hence $J = A_{(\mathfrak{p})}$. It follows that I is maximal, and thus the unique maximal ideal of $A_{(\mathfrak{p})}$. □

We can now construct the following trivial vector bundle on $\mathbb{P}(V)$.

Example 5.6.3. Let V be an $n+1$ dimensional vector space over $k = \bar{k}$. We want to construct a trivial vector bundle E over $\mathbb{P}(V) = \operatorname{Proj} \operatorname{Sym} V^*$ such that at closed points $E|_\ell$ is canonically V. We do so as follows, we set:

$$M = V \otimes_k \operatorname{Sym} V^*$$

and give it the following grading:

$$M_d = V \otimes_k (\operatorname{Sym} V^*)_d$$

We claim that $E = \mathscr{M}$ is the desired $\mathscr{O}_{\mathbb{P}(V)}$ module/vector bundle. First note that if we choose a basis, then:

$$M = (\operatorname{Sym} V^*)^{n+1}$$

and so $\mathscr{M} \cong \mathscr{O}_{\mathbb{P}(V)}^{n+1}$ implying that $\mathscr{M}$ is indeed a trivial vector bundle. Moreover, we have that:

$$\mathscr{M}|_\ell = \mathscr{M}_\ell \otimes_{\mathscr{O}_{\mathbb{P}(V),\ell}} k_\ell$$

where $k_\ell = k$ as $\mathbb{P}(V)$ is a variety over an algebraically closed field. Now let $\mathfrak{p}_\ell$ be the homogeneous prime ideal corresponding to ℓ, then we have the following chain of canonical isomorphisms:

$$\begin{aligned}
\mathscr{M}_\ell \cong& M_{(\mathfrak{p}_l)} \\
\cong& (M \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^*_{\mathfrak{p}_l^{\text{hom}}})_0 \\
\cong& (V \otimes_k \operatorname{Sym} V^* \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^*_{\mathfrak{p}_l^{\text{hom}}})_0 \\
\cong& (V \otimes_k \operatorname{Sym} V^*_{\mathfrak{p}_l^{\text{hom}}})_0 \\
\cong& V \otimes_k \operatorname{Sym} V^*_{(\mathfrak{p}_l)}
\end{aligned}$$

as the grading on V is trivial. Since $\mathscr{O}_{\mathbb{P}(V,\ell)} \cong \operatorname{Sym} V^*_{(\mathfrak{p}_l)}$ we have that

$$\mathscr{M}|_\ell \cong V \otimes_k O_{\mathbb{P}(V,\ell)} \otimes_{\mathscr{O}_{\mathbb{P}(V,\ell)}} k \cong V \otimes_k k \cong V$$

[131] I.e. by multiplying by f/f where $\deg f > 0$

hence the fibre is canonically V as desired. In particular, the global sections of this vector bundle are also canonically V, and evaluating sections at a closed point ℓ on a distinguished open U_f is given on simple tensors by:

$$v \otimes p/f^m \longmapsto v \cdot p(u)/f^m(u)$$

where u is any vector in ℓ.[132]

Example 5.6.4. In this example we consider $\mathscr{O}_{\mathbb{P}(V)}(-1)$, still with V an $n+1$ dimensional vector space over $k = \bar{k}$; by choosing a basis it is clear that $\mathscr{O}_{\mathbb{P}(V)}(-1)$ is a rank 1 vector bundle, as any basis allows us to identify $\mathbb{P}(V)$ with $\mathbb{P}^n_k$, and $\mathscr{O}_{\mathbb{P}(V)}(-1)$ with $\mathscr{O}_{\mathbb{P}^n_k}(-1)$. In other words, any basis for V induces a dual basis $\omega_0, \dots, \omega_n$ of V^*, and so the irrelevant ideal of $\operatorname{Sym} V^*$ is generated by $\omega_0, \dots, \omega_n$. It follows that U_{ω_i} cover $\mathbb{P}(V)$, and that $\mathscr{O}_{\mathbb{P}(V)}|_{U_{\omega_i}} \cong \widetilde{M}$, where:

$$M = (k[\omega_0, \dots, \omega_n]_{\omega_i})_{-1}$$

which as a module is isomorphic to $k[\omega_0/\omega_i, \dots, \omega_n/\omega_i]$. We want to show that for any closed point $\ell \in \mathbb{P}(V)$, where $\ell \subset V$ is a a one dimensional linear subspace of V, we have:

$$\mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell \cong \ell \subset V$$

In other words, we want to show that $\mathscr{O}_{\mathbb{P}(V)}(-1)$ is the algebraic geometry equivalent of the tautological bundle $\gamma^n \to \mathbb{P}(V)$ defined by:

$$\gamma^n = \{(\ell, v) \in \mathbb{P}(V) \times V : v \in \ell\}$$

To do this, we first construct a morphism $\mathscr{O}_{\mathbb{P}(V)}(-1) \hookrightarrow E$, where E is the vector bundle from Example 5.6.3. By Theorem 5.6.1 it suffices to construct a graded morphism $\operatorname{Sym} V^*(-1) \to V \otimes_k \operatorname{Sym} V^*$, and we do this by noting that:

$$V \otimes_k \operatorname{Sym} V^* = (V \otimes_k V^*) \oplus V \otimes_k \left(\bigoplus_{i \neq 1} (\operatorname{Sym} V^*)_i \right)$$

There is an element in $V \otimes_k \operatorname{Sym} V^*$ which canonically corresponds to the identity map $V \to V$ when one identifies $V \otimes_k V^*$ with $\operatorname{End}(V)$. We call this element ξ, and define our morphism as follows:

$$\begin{aligned} \operatorname{Sym} V^*(-1) &\longrightarrow V \otimes_k \operatorname{Sym} V^* \\ \omega &\longmapsto \xi \cdot \omega \end{aligned}$$

This is a module homomorphism, and it sends $\operatorname{Sym} V^*(-1)_d = (\operatorname{Sym} V^*)_{d-1}$ to $(\operatorname{Sym} V^*)_d$ as $\deg \xi = 1$. It is also clearly injective, and since localization is exact, we obtain an injective morphism of vector bundles $\imath : \mathscr{O}_{\mathbb{P}(V)}(-1) \hookrightarrow E$. The natural stalk map $\imath_\ell : \mathscr{O}_{\mathbb{P}(V)}(-1)_\ell \hookrightarrow E_\ell$ obviously induces an map on the fibres $\imath|_\ell : \mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell \to E|_\ell$ which is injective since the stalks are free $\mathscr{O}_{\mathbb{P}(V),\ell}$ modules. This map makes the following diagram commute for any U:

$$\begin{tikzcd} (\mathscr{O}_{\mathbb{P}(V)}(-1))(U) \arrow[r] \arrow[d] & E(U) \arrow[d] \\ \mathscr{O}_{\mathbb{P}(V)}|_\ell \arrow[r, "\imath|_\ell"] & E|_\ell \end{tikzcd}$$

where the vertical morphisms are evaluation maps. We can view $\mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell$ as a quotient of $\mathscr{O}_{\mathbb{P}(V)}(-1)_\ell$, so let $[s_\ell] \in \mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell \cong \mathscr{O}_{\mathbb{P}(V)}(-1)_\ell/\mathfrak{m}_\ell \cdot \mathscr{O}_{\mathbb{P}(V)}(-1)_l$. By the commutativity of the diagram, since $s_\ell = [U_\omega, s]$ for some $s \in \mathscr{O}_{\mathbb{P}(V)}(U_\omega)$, we must have that when we identify $E|_\ell$ with V we obtain:

$$[s_l] \longmapsto (\xi \cdot s)(\ell) \in E|_\ell \cong V$$

[132]Note this is clearly independent of the chosen u as both p and f^m are homogeneous of the same degree.

Now by our work in Example 5.6.3 this evaluation is equal to $\xi(u)\cdot s(u)$ for some $u \in \ell$. In particular, $s(u) \in k$ and $\xi(u) = u$ as it is the identity element. It follows that $\imath|_\ell(\mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell) \subset \ell$, and since $\imath|_\ell$ is injective, and ℓ and $\mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell$ are one dimensional over k, we have that:

$$\mathscr{O}_{\mathbb{P}(V)}(-1)|_\ell \cong \ell$$

Thus $\mathscr{O}_{\mathbb{P}(V)}(-1)$ is our algebraic geometric analogue of the tautological bundle over $\mathbb{P}^n$.

In the above two examples we could also take V be to be a free module over $\mathbb{Z}$, and obtain the same maps; we would however lose the nice identifications we had with the fibres. Moreover, if we instead use the Grothendieck convention $\mathbb{P}(V) = \operatorname{Proj}\operatorname{Sym}V$, still with V a vector space over $k = \bar{k}$, then $\mathscr{O}_{\mathbb{P}(V)}(1)$ is the vector bundle induced by $\operatorname{Sym}V(1)$, and we can get a trivial vector bundle in the same way as Example 5.6.3 by taking E to be the sheaf induced by $V \otimes_k \operatorname{Sym}V$. We get a map $E \to \mathscr{O}_{\mathbb{P}(V)}(1)$ induced by multiplication $V \otimes_k \operatorname{Sym}V \to \operatorname{Sym}V(1)$, which is obviously surjective. A closed point of $\mathbb{P}(V)$ in this convention is a one dimensional quotient $\pi : V \to U$, then at ϕ, we have that the induced map:

$$E|_\pi = V \longrightarrow \mathscr{O}_{\mathbb{P}(V)}(1)|_\pi$$

has kernel equal to $\ker\pi$. In other words, in this formalism we realize $\mathscr{O}_{\mathbb{P}(V)}(1)|_\pi$ as the one dimensional quotient $\pi : V \to U$, so $\mathscr{O}_{\mathbb{P}(V)}(1)$ is naturally the universal quotient bundle in this formalism.

Note that when we take tensor products of graded modules we don't, a priori, have a grading on our new module. To rectify this, we fix the following convention for grading a tensor products:

$$(M \otimes_A N)_d = \left\{ \sum_{e,f} m_f \otimes n_e : m_e \in M_e, n_f \in N_e, e + f = d \right\}$$

This clearly makes $M \otimes_A N$ into a graded A module which satisfies the following universal property: for every bilinear map $\psi : M \oplus N \to Q$ of graded A modules such that $\psi(M_d \oplus N_f) \subset Q_{d+f}$, there is a unique graded A module morphism $M \otimes_A N \to Q$. An equivalent definition of the grading is to define $(M \otimes_A N)_d$ as the cokernel of the following morphism:

$$\begin{aligned} \bigoplus_{t+r+s=d} M_d \otimes_{A_0} \otimes A \otimes_{A_0} N &\longrightarrow \bigoplus_{u+v=d} M_u \otimes_{A_0} N_v \\ m \otimes a \otimes n &\longmapsto am \otimes n - m \otimes an \end{aligned}$$

We now wish to show the following reasonable fact:

Lemma 5.6.4. *Let M and N be graded A-modules, $P = M \otimes_A N$ with and $X = \operatorname{Proj} A$, then there is an $\mathscr{O}_X$ module morphism:*

$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \longrightarrow \mathscr{P}$$

which is an isomorphism if A is generated in degree 1.

Proof. By the universal property of the tensor product, we need only define a morphism:

$$\mathscr{M} \oplus \mathscr{N} \longrightarrow \mathscr{P}$$

To do so, it suffices to define a morphism:

$$(M_f)_0 \oplus (N_f)_0 \longrightarrow ((M \otimes_A N)_f)_0$$

for any $f \in A_+^{\text{hom}}$. We do so as follows:

$$(m/f^k, n/f^l) \longmapsto \frac{m \otimes n}{f^{k+l}}$$

This induces a sheaf morphism:

$$\lambda_f : (\mathscr{M} \oplus \mathscr{N})|_{U_f} \longrightarrow \mathscr{P}|_{U_f}$$

We need to check that these agree on overlaps. Now let $g \in A_+^{\mathrm{hom}}$ $\lambda_f|_{U_{f\cdot g}}$ is induced by localization, as is $\lambda_g|_{U_{f\cdot g}}$. In particular, by replacing f and g with $f^{\deg g}$ and $g^{\deg f}$ we may assume that $\deg g = \deg f$. Then with $h = g/f$ and $h' = g/f$ we have that:

$$\lambda_f|_{U_{f\cdot g}} : ((M_f)_0 \oplus (N_f)_0)_h \longrightarrow (((M \otimes_A N)_f)_0)_h$$

is given by:

$$(m/f^k, n/f^l) \cdot h^{-p} \longmapsto \frac{m \otimes n}{f^{k+l}} \cdot h^{-2p}$$

While:

$$\lambda_g|_{U_{f\cdot g}} : ((M_g)_0 \oplus (N_g)_0)_{h'} \longrightarrow (((M \otimes_A N)_g)_0)_{h'}$$

is given by:

$$(m/g^k, n/g^l) \cdot (h')^{-p} \longmapsto \frac{m \otimes n}{g^{k+l}} \cdot (h')^{-2p}$$

It now suffices to check that the following diagram commutes:

$$\begin{tikzcd}
((M_f)_0)_h \oplus ((N_f)_0)_h \arrow[r, "\lambda_f|_{U_{fg}}"] \arrow[d, "\psi_{fg}"] & (((M \otimes_A N)_f)_0)_h \arrow[d, "\phi_{fg}"] \\
((M_g)_0)_{h'} \oplus ((N_g)_0)_{h'} \arrow[r, "\lambda_g|_{U_{fg}}"] & ((M \otimes_A N)_g)_0)_{h'}
\end{tikzcd}$$

where:

$$\psi_{fg} : (m/f^k, n/f^l) \cdot h^{-p} \longmapsto (mf^l/g^{k+l}, nf^k/g^{k+l}) \cdot (h')^{-l-k+p}$$

and:

$$\phi_{fg} : \frac{m \otimes n}{f^k} \cdot h^{-p} \longmapsto \frac{m \otimes n}{g^k} \cdot (h')^{p-k}$$

Now consider the composition $\lambda_g|_{U_g} \circ \psi_{fg}$, this sends $(m/f^k, n/f^l) \cdot h^{-p}$ to an element of the form:

$$\frac{f^{k+l} m \otimes n}{g^{2k+2l}} \cdot \left(\frac{f}{g}\right)^{-2l-2k+2p} = \frac{m \otimes n}{g^{k+l}} \cdot \left(\frac{f}{g}\right)^{-l-k+p}$$

whilst the composition $\phi_{fg} \circ \lambda_f|_{U_g}$ sends the same element to

$$\frac{m \otimes n}{g^{k+l}} \cdot \left(\frac{f}{g}\right)^{-l-k+p}$$

hence the diagram commutes. It follows that the λ_f agree on overlaps, thus yielding a morphism:

$$\Lambda : \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \longrightarrow \mathscr{P}$$

as desired.

Now suppose that A is generated in degree 1, then $\operatorname{Proj} A$ is covered by distinguished opens U_f where $\deg f = 1$. It follows that if we can show that λ_f is an isomorphism when $\deg f = 1$, that Λ will be an isomorphism. Note that for all integers d and e, we obtain morphisms:

$$M_d \oplus N_e \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

given by sending (m, n) to $m/f^d \otimes n/f^e$. The direct sum of these maps, gives us a morphism of abelian groups

$$M \oplus N \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

Since f is degree one, we have that the map:

$$A \longrightarrow (A_f)_0$$

defined on a homogeneous element $a \in A_d$ by:

$$a \longmapsto a/f^d$$

is a ring homomorphism. We thus have that $(M_f)_0 \otimes_{(A_f)_0} (N_f)_0$ has the structure of an A module making the above map a bilinear morphism of A modules. By the universal property of the tensor product we get a unique morphism:

$$M \otimes_A N \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

Since the image of f under the morphism $A \to (A_f)_0$ is 1, which is trivially invertible, we get a unique morphism $A_f \to (A_f)_0$ which just takes any homogeneous a/f^k to $a/f^{\deg a}$. This gives $(M_f)_0 \otimes_{(A_f)_0} (N_f)_0$ the structure of an A_f module, hence there is obvious bilinear A module morphism:

$$(M \otimes_A N) \oplus A_f \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

This then yields an A module homomorphism:

$$(M \otimes_A N) \otimes_A A_f = (M \otimes_A N)_f \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

which is also an obvious A_f module homomorphism. This is given on an homogeneous element of the form $m \otimes n/f^k$ by:

$$\frac{m \otimes n}{f^k} \longmapsto \frac{m}{f^{\deg m}} \otimes \frac{n}{f^{\deg n}}$$

We restrict to the degree zero part to get a morphism:

$$\alpha_f : ((M \otimes_A N)_f)_0 \longrightarrow (M_f)_0 \otimes_{(A_f)_0} (N_f)_0$$

Now $\Lambda|_{U_f}$ is induced by the $(A_f)_0$ module morphism:

$$\begin{aligned} \beta_f : (M_f)_0 \otimes_{(A_f)_0} (N_f)_0 &\longrightarrow ((M \otimes_A N)_f)_0 \\ m/f^k \otimes n/f^l &\longmapsto m \otimes n/f^{k+l} \end{aligned}$$

We want to show that $\beta_f \circ \lambda_f = \text{Id}$ and $\lambda_f \circ \beta_f = \text{Id}$, and it suffices to do so on simple tensors. We have the following:

$$\begin{aligned} \alpha_f \circ \beta_f(m/f^k \otimes n/f^l) =& \alpha_f \left(\frac{m \otimes n}{f^{k+l}} \right) \\ =& \frac{m}{f^{\deg m}} \otimes \frac{n}{f^{\deg n}} \end{aligned}$$

but since $m/f^k \in (M_f)_0$ and $n/f^l \in (N_f)_0$ we must have that $\deg m = k$ and $\deg n = l$. It follows that $\alpha_f \circ \beta_f = \text{Id}$; for the other direction:

$$\begin{aligned} \beta_f \circ \alpha_f \left(\frac{m \otimes n}{f^{k+l}} \right) =& \beta_f \left(\frac{m}{f^{\deg m}} \otimes \frac{n}{f^{\deg n}} \right) \\ =& \frac{m \otimes n}{f^{\deg m + \deg n}} \end{aligned}$$

The same argument demonstrates that $\deg m + \deg n = k + l$ hence $\beta_f \circ \alpha_f = \text{Id}$. It follows that $\Lambda|_{U_f}$ is an isomorphism, hence Λ is an isomorphism as desired. $\square$

We have the following example:

Example 5.6.5. Let $m, n \in \mathbb{Z}$, then we want to show $E = \mathscr{O}_{\mathbb{P}(V)}(n) \otimes_{\mathscr{O}_{\mathbb{P}(V)}} \mathscr{O}_{\mathbb{P}(V)}(m)$ is isomorphic to the vector bundle $\mathscr{O}_{\mathbb{P}(V)}(m+n)$, where V is a free A module. By Lemma 5.6.4, we have that E is isomorphic to the sheaf induced by $\operatorname{Sym} V^*(n) \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^*(m)$. Note that if we forget the grading, we have a natural isomorphism $\operatorname{Sym} V^* \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^* \cong \operatorname{Sym} V^*$, sending $\omega \otimes \eta \to \omega \cdot \eta$. We claim that this map induces an isomorphism of graded modules $\operatorname{Sym} V^*(n) \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^*(m) \cong \operatorname{Sym} V^*(n+m)$. Let:

$$\sum_i \eta_i \otimes \omega_i \in (\operatorname{Sym} V^*(n) \otimes_{\operatorname{Sym} V^*} \operatorname{Sym} V^*(m))_d$$

Then $\deg \eta_i + \deg \omega_i = m + n + d$, hence under the aforementioned isomorphism, get a degree d element of $\operatorname{Sym} V^*(n+m)$. It follows that the isomorphism respects the grading implying the claim. Note this implies there $\mathscr{O}_{\mathbb{P}(V)}(n)$ and $\mathscr{O}_{\mathbb{P}(V)}(-n)$ tensor together to get the structure sheaf $\mathscr{O}_{\mathbb{P}(V)}$.

To see why we need A to be generated in degree 1 consider the following example:

Example 5.6.6. Let $A = k[x, y, z]$, where $\deg x = 1$, $\deg y = 2$, and $\deg z = 3$. We have that $A(1) \otimes_A A(2) \cong A(3)$; if $X = \operatorname{Proj} A$ then we will show that:

$$\mathscr{O}_X(1) \otimes_{\mathscr{O}_X} \mathscr{O}_X(2) \not\cong \mathscr{O}_X(3)$$

Indeed, we have that:

$$(A_z)_0 = k[x^3/z, xy/z, y^3/z^2] = k[u, v, w]/\left\langle uw - v^3 \right\rangle$$

Now one easily[133] checks that the following map

$$\begin{aligned} (A_z)_0 \oplus (A_z)_0 &\longrightarrow (A(1)_z)_0 \\ (p, q) &\longmapsto p \cdot x + q \cdot (y^2/z) \end{aligned}$$

is a surjection hence $(A(1)_z)_0$ is free of rank 2. Similarly, we have a surjection

$$\begin{aligned} (A_z)_0 \oplus (A_z)_0 &\longrightarrow ((A)(2)_z)_0 \\ (p, q) &\longmapsto p \cdot x^2 + q \cdot y \end{aligned}$$

However, for $(A(3)_z)_0$ we have a surjection

$$\begin{aligned} (A_z)_0 &\longrightarrow ((A)(3)_z)_0 \\ p &\longmapsto p \cdot z \end{aligned}$$

so $(A(3)_z)_0$ is free of rank 1. If we look at the fibre at $x = \left\langle x^3/z, xy/z, y^3/z^2 \right\rangle$, then clearly $k_x = k$, and the fibre of $\mathscr{O}_X(1) \otimes_{\mathscr{O}_X} \mathscr{O}_X(2)$ is $k^2 \otimes_k k^2 \cong k^4$, while the fibre of $\mathscr{O}_X(3)$ is k, hence the sheaves cannot be isomorphic.

Our work throughout this section should lead one to ask the following question: is $\operatorname{QCoh}(\operatorname{Proj} A)$ equivalent to the category of graded modules over A? The answer is a resounding no. Indeed, consider a module M and a module N such that for a $d \geq n$ we have that $M_d = N_d$. These modules need not be isomorphic; suppose that $M_d \neq 0$ for $d < n$, then we could let N be $\bigoplus_{d \geq n} M_d$, then N and M can't be isomorphic as graded modules, as $M_d \mapsto 0$ for all $d \leq n$. However, since M and N agree for high enough degree we have that $\mathscr{M} \cong \mathscr{N}$. For simplicity sake take A to be finitely generated in degree 1, then some degree one elements $a_1, \dots, a_m$ generate the irrelevant ideal. In particular, $U_{a_1}, \dots, U_{a_m}$ cover $\operatorname{Proj} A$, however so do $U_{a_1^n}, \dots, U_{a_m^m}$. On the latter open cover, we have that $\mathscr{M}|_{U_{a_i^n}} = \widetilde{(M_{a_i^n})_0}$ and $\mathscr{N}|_{U_{a_i^n}} = \widetilde{(N_{a_1^n})_0}$. Elements of $(M_{a_i^n})_0$ are of the form $m/(a_i^n)^l$, where $\deg m = n \cdot l \geq n$. Since M and N agree in degree $d \geq n$ it follows that $(M_{a_i^n})_0 = (N_{a_1^n})_0$ thus the identify map induces an isomorphism $\mathscr{N}|_{U_{a_i^n}} \cong \mathscr{M}|_{U_{a_i^n}}$ for all i. These isomorphisms glue to yield an isomorphism $\mathscr{N} \cong \mathscr{M}$, even though N and M are not isomorphic as graded modules. In the next section, we explore this problem further, as there is a relatively satisfying fix when $\operatorname{Proj} A$ is quasicompact.

[133] If not tediously.

5.7 Line Bundles and Invertible Sheaves

In this section we explore and develop some results regarding locally free sheaves of rank one and their connections to invertible sheaves. We begin with the following definition, which fixes some terminology:

Definition 5.7.1. A **line bundle** over a locally ringed space X is a locally free $\mathscr{O}_X$ module of rank one. An $\mathscr{O}_X$ module $\mathscr{F}$ is an **invertible sheaf**, or just **invertible**, if there exists an $\mathscr{O}_X$ module $\mathscr{F}^{-1}$ satisfying $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1} \cong \mathscr{O}_X$.

Note that if $\mathscr{G}$ and $\mathscr{F}^{-1}$ satisfy $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1} \cong \mathscr{O}_X$, and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \cong \mathscr{O}_X$ then $\mathscr{G} \cong \mathscr{F}^{-1}$. Indeed, we have that:

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1} \cong \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$$

so applying $\mathscr{F}^{-1}$ on both sides yields:

$$\mathscr{F}^{-1} \cong \mathscr{G}$$

Our initial exploration of locally free sheaves of rank one will culminate by showing the above definitions are actually equivalent. We begin with the following lemma:

Lemma 5.7.1. *Let X be a locally ringed space, and $\mathscr{F}$ an $\mathscr{O}_X$ module, and $s \in \mathscr{F}(X)$ a global section. Then the support of s:*

$$\operatorname{Supp}(s) = \{x \in X : s_x \neq 0 \in \mathscr{O}_{X,x}\}$$

is closed. Moreover, if $\mathscr{F}$ is locally free of finite rank, then the vanishing locus of s:

$$\mathbb{V}(s) = \{x \in X : s(x) = 0\}$$

is closed. Equivalently, the compliment:

$$X_s = \{x \in X : s(x) \neq 0\}$$

is open.

Proof. We can tell that $\operatorname{Supp}(s)$ is closed because it's compliment is open. Indeed, if $s_x = 0$ then on a open neighborhood $s = 0$, hence s_y for points in an open neighborhood of x. It follows that every point $x \in X \smallsetminus \operatorname{Supp}(s)$ has an open neighborhood and is thus open, hence $\operatorname{Supp}(s)$ is closed.

Now assume that $\mathscr{F}$ is locally free of finite rank. Recall that $s(x) = 0$ is equivalent to the condition that $s_x \in \mathfrak{m}_x \cdot \mathscr{F}_x$. Choosing an open neighborhood U of x such that $\mathscr{F}|_U$ is trivial, we have that we can write $\mathscr{F}|_U$ as $\mathscr{O}_U^n$ for some n. Choose a basis e_i for $\mathscr{O}_U^n$ so that $s|_U = a^i e_i$ with $a^i \in \mathscr{O}_U(U)$. On the level of stalks, we have that:

$$\mathfrak{m}_x \cdot \mathscr{F}_x \cong \mathfrak{m}_x \cdot \mathscr{O}_{X,x}^n = \underbrace{\mathfrak{m}_x \oplus \cdots \mathfrak{m}_x}_{n\text{-times}}$$

It follows that $s_x = a_x^i \cdot e_{i_x}$ lies in $\mathfrak{m}_x \cdot \mathscr{F}_x$ if and only if each $a_x^i \in \mathfrak{m}_x$. So suppose that $s_x \notin \mathfrak{m}_x \cdot \mathscr{F}_x$, then we have that at least one $a_x^j \notin \mathfrak{m}_x$, however this implies that a_x^j is invertible. Therefore, there is a potentially smaller open neighborhood $V \subset U$ of x on which $a^j|_V$ is invertible, and thus $a_y^j \notin \mathfrak{m}_y$ for all $y \in V$. It follows that for all $y \in V$ we have that $s_y \notin \mathfrak{m}_y \cdot \mathscr{F}_y$, hence X_s is open, as every point contains an open neighborhood. It follows that $\mathbb{V}(s)$ is closed. ☐

We note that we can also define:

$$\operatorname{Supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}$$

but this is in general not a closed set. The following example demonstrates that there are quasicoherent examples to $\mathbb{V}(s)$ being closed if $\mathscr{F}$ is not locally free of finite rank.

Example 5.7.1. Let X be an integral scheme which is not a single point, and consider a closed point x. The inclusion of the point is by $\operatorname{Spec} k_x \hookrightarrow X$, which is an affine morphism as the preimage of any open affine in X is $\operatorname{Spec} k_x$ or empty. It follows that the morphism is qcqs, hence the pushforward of the structure sheaf $\mathscr{O}_{\operatorname{Spec} k_x}$ is quasicoherent.

Take the global section 1, we claim that:

$$X_s = \{x\}$$

Indeed, $\imath_* \mathscr{O}_{\operatorname{Spec} k_x}$ is the sky scrape sheaf $x_* k_x$, hence it's stalk is only nonzero at x. It follows that the only point where 1 does not vanish is $\{x\}$, which is closed by assumption.

We also prove the following general result:

Lemma 5.7.2. *Let X be a locally ringed space, and $\mathscr{F}$ a locally free sheaf of rank n on X. An open subset $U \subset X$ trivializes $\mathscr{F}$, i.e. $\mathscr{F}|_U \cong \mathscr{O}_U^n$, if and only if there exist $s_1, \ldots, s_n \in \mathscr{F}(U)$ such that for all $x \in U$ the set $B = \{s_1(x), \ldots, s_n(x)\}$ forms a basis of $\mathscr{F}|_x$.*

Proof. Supposing that such $s_1, \ldots, s_n$ exist, we obtain a morphism:

$$F : \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U$$

given by sending $(a_1, \ldots, a_n) \in \mathscr{O}_U(V)$ to $\sum_i a_i s_i|_V$. On the level of stalks, we have a morphism:

$$F_x : \mathscr{O}_{X,x}^n \longrightarrow \mathscr{F}_x$$

which we can compose with the evaluation map $\operatorname{ev}_x : \mathscr{F}_x \to \mathfrak{m}_x \cdot \mathscr{F}_x$. If $a \in \mathfrak{m}_x \cdot \mathscr{O}_{X,x}^n$, then we obviously have that $a \in \ker \operatorname{ev}_x \circ F_x$. It follows that there exists a unique morphism:

$$\bar{F}_x : k_x^n \cong \mathscr{O}_{X,x}^n / \mathfrak{m}_x \cdot \mathscr{O}_{X,x}^n \longrightarrow \mathscr{F}|_x$$

given by:

$$([a_1], \ldots, [a_1]) \longmapsto \sum_i [a_i] \cdot s_i(x) = \sum_i [a_i \cdot s_{i_x}]$$

Note that we can take the a_i to comes from sections of $\mathscr{O}_X(V)$ for some small enough open set V, then $\bar{F}_x$ is the map taking the tuple $(a_1(x), \ldots, a_n(x)) \in k_x^n$ to:

$$\sum_i a_i(x) \cdot s_i(x) = \sum_i (a_i \cdot s_i|_V)(x)$$

Since the $s_i(x)$ form a basis for $F|_x$ it follows that the map $\bar{F}_x$ is surjective, and so since $\mathfrak{m}_x$ is the only maximal ideal of $\mathscr{O}_{X,x}$ we have that F_x is surjective by Lemma 3.10.1.[134] Since $\mathscr{F}_x$ is free of finite rank over $\mathscr{O}_{X,x}$ it follows by Lemma 5.3.4 that F_x is an isomorphism, hence F is an isomorphism of sheaves.

If U is a trivializing neighborhood for $\mathscr{F}$, then we have an isomorphism:

$$\mathscr{O}_U^n \longrightarrow \mathscr{F}|_U$$

The $s_i \in \mathscr{F}(U)$ can be taken to be the image of any basis for $\mathscr{O}_U^n(U)$. It is clear that any such such choice satisfies the conditions that $s_1(x), \ldots, s_n(x)$ for a basis for $\mathscr{F}|_x$ for all $x \in U$. □

Corollary 5.7.1. *Let $\mathscr{L}$ be a line bundle, and $s \in \mathscr{L}(X)$. Then X_s is a trivializing open set for $\mathscr{L}$. In particular, any isomorphism $\mathscr{L}|_{X_s} \to \mathscr{O}_{X_s}$ sends $s|_{X_s}$ to an invertible element in $\mathscr{O}_X(X_s)$.*

Proof. The first statement is clear. If $\phi : \mathscr{L}|_{X_s} \to \mathscr{O}_{X_s}$ is an isomorphism, then, for all $x \in X_s$, the condition that $s(x) \neq 0$ is equivalent to $s_x \notin \mathfrak{m}_x \cdot \mathscr{L}_x$. In particular if $\phi_x(s_x) \in \phi_x(\mathfrak{m}_x \cdot \mathscr{L}_x)$, then $s_x \in \phi_x^{-1}(\phi_x(\mathfrak{m}_x \cdot \mathscr{L}_x)) = \mathfrak{m}_x \cdot \mathscr{L}_x$ because ϕ_x is an isomorphism for all $x \in X_s$. It follows by the contrapositive that

$$\phi_x(s_x) \notin \phi_x(\mathfrak{m}_x \cdot \mathscr{L}_x)$$

[134] Specifically part d).

and since ϕ_x is an isomorphism we have that $\phi_x(\mathfrak{m}_x \cdot \mathscr{L}_x) = \mathfrak{m}_x \cdot \phi_x(\mathscr{L}_x) = \mathfrak{m}_x$. Hence $\phi_{X_s}(s|_{X_s})_x \notin \mathfrak{m}_x$ for all $x \in X_s$, so our work in Proposition 2.1.2 implies that $\phi_{X_s}(s|_{X_s})$ is indeed invertible.

□

Note that for locally free sheaf of finite rank n, we have that if U is a trivializing open set for $\mathscr{F}$ then:

$$\mathscr{F}^*|_U = \underline{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)|_U = \underline{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{O}_U) \cong \underline{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^n, \mathscr{O}_U)$$

By Lemma 5.2.2 we have that:

$$\mathscr{F}^*|_U \cong \underline{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{O}_U)^n \cong \mathscr{O}_U^n$$

In particular, the dual sheaf of a locally free sheaf of finite rank is locally free of the same rank. Moreover, they admit the same trivializing open sets.

Proposition 5.7.1. *Let X be a locally ringed space, then any line bundle is invertible.*

Proof. Suppose that $\mathscr{L}$ is a line bundle. We claim that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^*$ is isomorphic to $\mathscr{O}_X$. Indeed, there is a natural bilinear morphism of $\mathscr{O}_X$ modules:

$$\mathscr{L} \oplus \mathscr{L}^* \longrightarrow \mathscr{O}_X$$

given on open sets by:

$$\begin{aligned} \mathscr{L}(U) \oplus \mathscr{L}^*(U) &\longrightarrow \mathscr{O}_X(U) \\ (s, \omega) &\longmapsto \omega_U(s) \end{aligned}$$

hence the universal property of the tensor product yields a homomorphism:

$$m : \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^* \longrightarrow \mathscr{O}_X$$

Note that since $\mathscr{L}$ is in particular of finite presentation, we have that by Proposition 5.2.2 $(\mathscr{L}^*)_x$ is naturally $(\mathscr{L}_x)^*$, hence there is no confusion when writing $\mathscr{L}_x^*$.. In particular, we can choose a single basis element s_x for $\mathscr{L}_x$, which induces a dual basis element ω_x for $\mathscr{L}_x^*$, satisfying $\omega_x(s_x) = 1$. It follows that on stalks, we can write any element in $\mathscr{L}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x^*$ as $a_x \cdot s_x \otimes \omega_x$, for $a_x \in \mathscr{O}_{X,x}$. We thus have that m_x is obviously an isomorphism for all $x \in X$, hence $\mathscr{L}$ is invertible. □

We wish to show the converse to the above statement. We will need the following lemma about finite type and finitely presented modules.

Lemma 5.7.3. *Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$ modules. Then the following hold:*

a) *If $\mathscr{G}$ is locally finite type, and f_x is surjective then there is and open neighborhood V of x such that $f|_V$ is a surjection.*
b) *If $\mathscr{G}$ is locally of finite type and $\mathscr{G}_x = 0$ then there is an open neighborhood V of x such that $\mathscr{G}|_V = 0$.*
c) *If $\mathscr{G}$ is finitely presented, $\mathscr{F}$ is finite type, and f is surjective then* $\ker f$ *is finite type.*
d) *If $\mathscr{G}$ is finitely presented, and $\mathscr{G}_x \cong \mathscr{O}_{X,x}^n$ then there exists an open neighborhood V of x such that $\mathscr{G}|_V \cong \mathscr{O}_V^n$.*

Proof. We will prove $a)$, then show that $a) \Rightarrow b)$, after which we will prove $c)$ and then show that $a) \wedge b) \wedge c) \Rightarrow d)$. For $a)$, suppose that f_x is a surjection, and note that we have a surjection $\alpha : \mathscr{O}_U^n \to \mathscr{G}|_U$ for some U containing x. In particular, if s_i are the images of the obvious basis elements e_i in $\mathscr{O}_U^n(U)$, then since the e_{i_x} generate $\mathscr{O}_{U,x}^n$ for all $x \in U$, we have that the s_{i_x} generate $\mathscr{G}_x$ for all $x \in U$ as well. Fix $x \in U$, then since f_x is surjective, there are elements $t_{i_x} \in \mathscr{F}_x$ such that $\alpha_x(t_{i_x}) = s_{i_x}$. In particular, by taking a finite number of intersections of open neighborhoods of x, we can assume that the t_{i_x} all come from sections $t_i \in \mathscr{F}(V)$ for some open neighborhood V of x. Then we have that

$$f_x([V, t_i]) = [V, f_V(t_i)] = s_{i_x} = [V, s_i|_V]$$

By definition, there is then an open neighborhood $W_i \subset V$ of x such that $f_{W_i}(t_i|_{W_i}) = s_i|_{W_i}$. Taking W to be the intersection of the t_i, we have that $f_W(t_i|_W) = s_i|_W$ for all i. Since s_{i_y} generate $\mathscr{G}_y$ for all $y \in W$, we have that f_y is surjective for all $y \in W$, hence $f|_W$ is surjective.

For b), if $\mathscr{G}_x = 0$, then the zero morphism $0 \to \mathscr{G}$ is surjective at x, and the claim follows from a).

For c), by taking an intersection of where $\mathscr{F}$ is of finite type, and where there exists a finite presentation of $\mathscr{G}$, we obtain the following diagram:

$$\begin{tikzcd} & & \mathscr{O}_U^l \arrow[d, dashed, "\psi"] & \\ & & \mathscr{F}|_U \arrow[d, dashed, "f|_U"] & \\ \mathscr{O}_U^m \arrow[r, dashed, "\beta"] & \mathscr{O}_U^n \arrow[r, dashed, "\alpha"] & \mathscr{G}|_U \arrow[r] & 0 \end{tikzcd}$$

where α, ψ, and f are surjective, and β surjects onto the kernel of α. Note that $f \circ \psi$ is surjective, and let e_i be a natural basis for $\mathscr{O}_U^l(U)$. By the same arguments as in a), since α is surjective, we can shrink U if necessary to find a sections $t_i \in \mathscr{O}_U^n(U)$ such that $\alpha_U(t_i) = f_U \circ \psi_U(e_i)$. This in turn determines a morphism $\phi : \mathscr{O}_U^l \to \mathscr{O}_U^n$ such that $\alpha \circ \phi = f|_U \circ \psi$. By potentially shrinking U again, we can do the same thing for α, , and obtain a morphism $\omega : \mathscr{O}_U^n \to \mathscr{O}_U^l$ such that $\alpha = f|_U \circ \psi \circ \omega$. We claim that the morphism:

$$g : \mathscr{O}_U^m \oplus \mathscr{O}_U^l \longrightarrow \mathscr{F}|_U$$

given by the direct sum of $\psi \circ \omega \circ \beta$ and $\psi \circ (\mathrm{Id} - \omega \circ \phi)$ surjects onto the kernel of f. Since surjectivity is stalk local by definition, it suffices to show g_x surjects onto $\ker f_x$. Suppose that $s_x \in \ker f_x$, then there exists some $m_x \in \mathscr{O}_{U,x}^l$ such that $\psi_x(m_x) = s_x$; in particular, $m_x \in \ker(f_x \circ \psi_x)$. Now since $\alpha_x \circ \phi_x = f_x \circ \psi_x$, we have that $\phi(m_x)$ is in the kernel of α_x, hence there exists some $n_x \in \mathscr{O}_U^m$ such that $\beta_x(n_x) = \phi_x(m_x)$. It follows that:

$$\begin{aligned} g_x(n_x, m_x) =& \psi_x \circ \omega_x \circ \beta_x(n_x) + \psi(m_x) - \psi_x \circ \omega_x \circ \phi_x(m_x) \\ =& \psi_x \circ \omega_x \circ \phi_x(m_x) + s_x - \psi_x \circ \omega_x \circ \phi_x(m_x) \\ =& s_x \end{aligned}$$

It is clear that $f_x \circ g_x = 0$, hence $\operatorname{im} g_x = \ker f_x \circ \psi_x$ implying that g surjects onto $\ker f$ as desired.

Now suppose that $\mathscr{G}$ is finitely presented with $\mathscr{G}_x \cong \mathscr{O}_{X,x}^n$ for some n, and $x \in X$. Let e_{i_x} be the natural basis for $\mathscr{O}_{X,x}^n$, and denote the isomorphism $\mathscr{O}_{X,x}^n \to \mathscr{G}_x$ by g. Let $s_{i_x} = g(e_{i_x})$, then we can assume that each s_{i_x} comes from a section s_i of $\mathscr{G}(V)$ for some open neighborhood V of x. Since the e_{i_x} come from the obvious basis of $\mathscr{O}_X^n(X)$, we have that the morphism $f : \mathscr{O}_V^n \to \mathscr{G}|_V$ induced on global sections by $e_i|_V \to s_i|_V$ satisfies $f_x = g$. Since g is surjective, we can shrink V so that $f|_V$ is a surjection by a). By c), since $\mathscr{O}_W^n$ is of finite type, we have that $\ker f|_V$ of finite type. Since $\ker f_x = \ker g_x = 0$, by b) we can shrink V again to obtain that $\ker f|_V = 0$. It follows that $f|_V : \mathscr{O}_V^n \to \mathscr{G}|_V$ is both surjective and injective and thus an isomorphism as desired. $\square$

With the above lemma, it suffices to show that any invertible sheaf $\mathscr{F}$ is finitely presented and satisfies $\mathscr{F}_x \cong \mathscr{O}_{X,x}$ for all $x \in X$.

Lemma 5.7.4. *Let $\mathscr{F}$ be an invertible sheaf on a locally ringed space X, then $\mathscr{F}$ is finitely presented.*

Proof. Let $\mathscr{F}^{-1}$ be the inverse to $\mathscr{F}$, and let $\psi : \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1} \to \mathscr{O}_X$ be an isomorphism. Consider the bilinear morphism:

$$\mathscr{F} \oplus (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1}) \longrightarrow \mathscr{F}$$

given on opens by:

$$\begin{aligned} \mathscr{F}(U) \oplus (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1}) &\longrightarrow \mathscr{F}(U) \\ (s, t) &\longmapsto s \cdot \psi(t) \end{aligned}$$

The universal property of the tensor product then yields a morphism:

$$f : \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1} \longrightarrow \mathscr{F}$$

which on stalks is given by the map:

$$\begin{aligned} f_x : \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x^{-1} &\longmapsto \mathscr{F}_x \\ t_x \otimes t'_x \otimes s_x &\longmapsto t_x \cdot \psi_x(t'_x \otimes s_x) \end{aligned}$$

extended linearly. Note that f_x is an isomorphism with inverse given by sending $t_x \in \mathscr{F}_x$ to $t_x \otimes \psi_x^{-1}(1)$. Since this holds for all x, we have that f is an isomorphism of sheaves. Moreover, consider the isomorphism:

$$\mathrm{Id} \oplus \psi^{-1} : \mathscr{F} \oplus \mathscr{O}_X \longrightarrow \mathscr{F} \oplus (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1})$$

By post composing with the tensor product morphism we get a morphism of sheaves:

$$\mathscr{F} \oplus \mathscr{O}_X \longrightarrow \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1})$$

which is bilinear. The universal property of the tensor product then gives us the morphism of sheaves:

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X \longrightarrow \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1})$$

which on stalks is given by the map:

$$\begin{aligned} (\mathrm{Id} \otimes \psi^{-1})_x : \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} &\longrightarrow \mathscr{F} \otimes_{\mathscr{O}_{X,x}} (\mathscr{F} \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x^{-1}) \\ t_x \otimes a_x &\longmapsto t_x \otimes \psi_x^{-1}(a_x) \end{aligned}$$

extended linearly. This is again obviously an isomorphism as both Id and ψ^{-1} are isomorphisms. By precomposing with the natural isomorphism $\mathscr{F} \cong \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X$, and then post composing with the natural isomorphism which comes from swapping the first two factors, we obtain an isomorphism:

$$g : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1}$$

We give a concrete description of this map on the level of stalks: for $x \in X$, if $\psi_x^{-1}(1) = \sum_i s_{i,x} \otimes t_{i,x}$ for $s_{i,x} \in \mathscr{F}_x$ and $t_{i,x} \in \mathscr{F}_x^{-1}$, then the stalk map is given by:

$$\begin{aligned} g_x : \mathscr{F}_x &\to \mathscr{F}_x \otimes_{\mathscr{O}_X} \mathscr{F}_x \otimes_{\mathscr{O}_X} \mathscr{F}_x^{-1} \\ s_x &\longmapsto \sum_i s_{i,x} \otimes s_x \otimes t_{i,x} \end{aligned}$$

Now, by post composing with f we get an automorphism $\mathscr{F} \to \mathscr{F}$ which on the level of stalks is given by:

$$s_x \longmapsto \sum_{i=1}^{n} s_{i,x} \cdot \psi(s_x \otimes t_{i,x})$$

Fixing $x \in X$, by taking a finite number of intersections, we can assume that $s_{i,x}$ and $t_{i,x}$ all come from elements $s_i \in \mathscr{F}(U)$ and $t_i \in \mathscr{F}^{-1}(U)$ where U is an open neighborhood of x. In particular, $s_i \otimes t_i$ is an element of the presheaf $(\mathscr{F} \otimes^p_{\mathscr{O}_X} \mathscr{F}^{-1})(U)$, but defines an element in the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{F}^{-1}(U)$ via the sheafification morphism, which we abusively denote by $s_i \otimes t_i$ as well. By further shrinking U, since $\sum_i \psi_x(s_{i,x} \otimes t_{i,x})$ is the unit element, we can assume that $\sum_i s_i \otimes t_i = \psi_U^{-1}(1)$[135]. With this, we have that on U, $(f \circ g)_U$ is given by:

$$s \longmapsto \sum_i s_i \psi(s \otimes t_i)$$

[135]The $\sum_i \psi_x(s_{i,x} \otimes t_{i,x}) - 1$ is zero, and the stalk $s_i \otimes t_i$ is $s_{i,x} \otimes t_{i,x}$ by construction, so there must be an open neighborhood of x for which this is true. We take the intersection of this open neighborhood with U.

Moreover for any $V \subset U$ we have that $(f \circ g)_V$ is given by:

$$s \longmapsto \sum_i s_i|_V \psi(s \otimes t_i|_V)$$

hence the restricted morphism $(f \circ g)|_U : \mathscr{F}|_U \to \mathscr{F}|_U$ factors as follows:

$$\mathscr{F}|_U \longrightarrow \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U$$

which on an open subset $V \subset U$ is given by:

$$s \longmapsto (\psi_V(s \otimes t_1|_V), \ldots, \psi_V(s \otimes t_n|_V)) \longmapsto \sum_i s_i|_V \psi(s \otimes t_i|_V)$$

Denote by α the morphism $\mathscr{O}_U^n \to \mathscr{F}|_U$, and β the morphism $\mathscr{F}|_U \to \mathscr{O}_U^n$. Since $\alpha \circ \beta$ is an isomorphism, it follows that α must be surjective, hence $\mathscr{F}$ is of finite type. Let $\xi = \beta \circ (\alpha \circ \beta)^{-1}$, then $\alpha \circ \xi = \mathrm{Id}$ hence α admits a right inverse. Since $\mathrm{Mod}_{\mathscr{O}_X}$ is an abelian category, it follows that the exact sequence:

$$0 \longrightarrow \ker \alpha \longrightarrow \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

splits, and so $\mathscr{O}_U^n \cong \ker \alpha \oplus \mathscr{F}|_U$. In particular, $\mathscr{O}_U^n$ surjects onto $\ker \alpha$ via the isomorphism post composed with the projection map, hence $\ker \alpha$ is finite type, and we have an exact sequence of the form:

$$\mathscr{O}_U^n \longrightarrow \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

implying that $\mathscr{F}$ is finitely presented as desired.

□

To show that second condition, that $\mathscr{F}_x \cong \mathscr{O}_{X,x}$ we will need the following lemmas flat modules, and finitely presented modules:

Lemma 5.7.5. *Every direct summand of a flat A module is flat, and every direct sum of flat A modules is flat.*

Proof. Let $\{M_i\}_{i \in I}$ be a potentially infinite family of flat A modules, $Q \to P$ an injective morphism of A modules. Set $N = \bigoplus_i M_i$, then we need to show that:

$$Q \otimes_A N \to P \otimes_A N$$

is injective. We first claim that $Q \otimes N \cong \bigoplus_i (Q \otimes_A M_i)$. Indeed, let $\imath_i : M_i \to N$ be the map taking $m \in M_i$ to the sequence with a zero in all entries $j \neq i$ and an m in the ith entry. Now consider the bilinear map:

$$\begin{aligned} Q \oplus M_i &\longrightarrow Q \otimes_A N \\ (q, m) &\longmapsto q \otimes \imath_i(m) \end{aligned}$$

From the universal property of the tensor product we get a morphism $\psi_i : Q \otimes_A M_i \to Q \otimes_A N$ which on simple tensors sends $q \otimes m$ to $q \otimes \imath_i(m)$. Taking the direct sum of all such morphisms, we obtain am morphism:

$$\begin{aligned} \psi : \bigoplus_i (Q \otimes_A M_i) &\longrightarrow Q \otimes_A N \\ (\omega_i)_{i \in I} &\longmapsto \sum_i \psi_i(\omega_i) \end{aligned}$$

To show this map is surjective it suffices to do so on simple tensors. In particular, let $q \otimes n \in Q \otimes_A N$, then n is a sequence of elements $(m_i)_{i \in I}$ such that finitely many m_i are non zero. The sequence $(q \otimes m_i)_{i \in I} \in \bigoplus_i Q \otimes_A M_i$ then maps to the element:

$$\sum_i q \otimes \imath_i(m_i) = q \otimes \left(\sum_i \imath_i(m_i) \right) = q \otimes n$$

hence ψ is surjective. Now let $(\omega_i)_{i\in I} \in \bigoplus_i (Q \otimes_A M_i)$ and suppose that $\psi((\omega_i)_{i\in I}) = 0$. In particular, this means that $\psi_i(\omega_i) = 0$ for all i, hence it suffices to show that ψ_i is injective. Consider the map $\pi_i : N \to M_i$ given by projecting onto the, then bilinear map:

$$\begin{aligned} Q \oplus N &\longrightarrow Q \otimes_A M_i \\ (q, n) &\longmapsto q \otimes \pi_i(n) \end{aligned}$$

induces an A linear map $\alpha_i : Q \otimes_A N \to Q \otimes_A M_i$. We then obviously have that $\alpha_i \circ \psi_i$ is the identity on $Q \otimes_A M_i$ hence ψ_i must be injective, and ψ is an isomorphism as desired.

It follows that up to isomorphism, the map $Q \otimes_A N \to P \otimes_A N$ is the direct sum of the morphisms $Q \otimes_A M_i \to P \otimes_A M_i$. Each of these morphisms is injective as M_i is flat, hence $Q \otimes_A N \to P \otimes_A N$ is injective as well.

Now suppose that N is a flat A-module, and $M_1, M_2 \subset N$ sub modules satisfying $N \cong M_1 \oplus M_2$. Let $Q \to P$ be an injective morphism of A modules, then we have that the induced morphism $Q \otimes_A N \to P \otimes_A N$ is injective. This then implies that the induced morphism:

$$(Q \otimes_A M_1) \oplus (Q \otimes_A M_2) \longrightarrow (P \otimes_A M_1) \oplus (Q \otimes_A M_2)$$

is also injective. However, this map is the direct sum of the induced morphisms $Q \otimes_A M_i \to P \otimes_A M_i$, which must be injective as the total map is injective. It follows that each M_i is flat, implying the claim. ☐

Lemma 5.7.6. *Let:*

$$0 \longrightarrow M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_3 \longrightarrow 0$$

be an exact sequence of A modules. Then the following hold:

a) If M_3 is finitely presented, and M_2 is finitely generated, then M_1 is finitely generated.

b) If M_3 is flat, then for any A module N the induced map $M_1 \otimes_A N \to M_2 \otimes_A N$ is injective.

Proof. For a), choose a presentation fo M_3:

$$A^m \xrightarrow{\beta} A^n \xrightarrow{\alpha} M_3 \longrightarrow 0$$

We define a map $\pi : A^n \to M_2$ as follows: let e_i be the ith basis element of A^n then since both α and f are surjective there must exist and $x_i \in M_2$ such that $\beta(e_i) = f(x_i)$. We thus set:

$$\pi(a_1, \dots, a_n) = \sum_i a_i x_i$$

We then clearly have the following commutative diagram:

$$\begin{tikzcd} & A^m \arrow[r, "\beta"] & A^n \arrow[r, "\alpha"] \arrow[d, "\pi"] & M_3 \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r, "g"] & M_2 \arrow[r, "f"] & M_3 \arrow[r] & 0 \end{tikzcd}$$

We define a map $\eta : A^m \to M_1$ as follows: since the above diagram commutes, and the rows are exact, we must have that for all $x \in A^m$, $\pi \circ \beta(x) \in \ker f = \operatorname{im} g$. Since g is injective there is a unique $y \in M_1$ such that $\pi \circ \beta(x) = g(y)$; we set $\eta(x) = y$. This is obviously well defined and yields an A-module morphism, and so we obtain the following commutative diagram:

$$\begin{tikzcd} & A^m \arrow[r, "\beta"] \arrow[d, "\eta"] & A^n \arrow[r, "\alpha"] \arrow[d, "\pi"] & M_3 \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r, "g"] & M_2 \arrow[r, "f"] & M_3 \arrow[r] & 0 \end{tikzcd}$$

Applying the Snake lemma then gives an exact sequence:

$$\ker \mathrm{Id} \longrightarrow \operatorname{coker} \eta \longrightarrow \operatorname{coker} \pi \longrightarrow \operatorname{coker} \mathrm{Id}$$

hence $\operatorname{coker}\eta \cong \operatorname{coker}\pi$. Since M_2 is finitely generated, $\operatorname{coker}\pi \cong M_2/\operatorname{im}\pi$ is finitely generated, hence $M_1/\operatorname{im}\eta$ is finitely generated. Note that $\operatorname{im}\eta$ is generated by $\eta(e_1), \dots, \eta(e_m)$. Let $[m_1], \dots, [m_k]$ generate $\operatorname{coker}\eta$, then we claim that $\{\eta(e_1), \dots, \eta(e_m), m_1, \dots, m_k\}$ is a generating set for M_1. Indeed, let $m \in M$, then:

$$[m] = \sum_i a_i[m_i]$$

This implies there is some $x \in \operatorname{im}\eta$ such that:

$$m = \sum_i a_i[m_i] + x$$

Since x can be written as:

$$\sum_j a_j \eta(e_j)$$

we have proven a).

For b), we employ the notation $A^I = \bigoplus_{i \in I} A$, and note that there is always some potentially infinite set I for which A^I surjects on N. Denote this map by π, then we have an exact sequence of the form:

$$0 \longrightarrow \ker\pi \longrightarrow A^I \longrightarrow N \longrightarrow 0$$

By tensoring our original exact sequence with N, A^I, and $\ker\pi$, since A^I is flat we obtain the following diagram:

$$\begin{tikzcd}
 & M_1 \otimes_A N \arrow[r, "g_N"] & M_2 \otimes_A N \arrow[r, "f_N"] & M_3 \otimes_A N \arrow[r] & 0 \\
0 \arrow[r] & (M_1)^I \arrow[r, "g_I"] & (M_2)^I \arrow[r, "f_I"] & (M_3)^I \arrow[r] & 0 \\
 & M_1 \otimes_A \ker\pi \arrow[r, dashed, "g_\pi"] & M_2 \otimes_A \ker\pi \arrow[r, dashed, "f_\pi"] & M_3 \otimes_A \ker\pi \arrow[r] & 0
\end{tikzcd}$$

Now that the ith column of this diagram looks like the exact sequence obtained by tensoring the exact sequence corresponding to N with M_i, hence we can fill in the diagram as follows:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & M_1 \otimes_A N \arrow[r, "g_N"] \arrow[u] & M_2 \otimes_A N \arrow[r, "f_N"] \arrow[u] & M_3 \otimes_A N \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & (M_1)^I \arrow[r, "g_I"] \arrow[u, dashed, "\pi_1"] & (M_2)^I \arrow[r, "f_I"] \arrow[u, dashed, "\pi_2"] & (M_3)^I \arrow[r] \arrow[u, dashed, "\pi_3"] & 0 \\
 & M_1 \otimes_A \ker\pi \arrow[r, dashed, "g_\pi"] \arrow[u, dashed, "\imath_1"] & M_2 \otimes_A \ker\pi \arrow[r, dashed, "f_\pi"] \arrow[u, dashed, "\imath_2"] & M_3 \otimes_A \ker\pi \arrow[r] \arrow[u, dashed, "\imath_3"] & 0 \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

where all rows and columns are exact (with the last column being exact by flatness of M_3). We claim that this diagram is commutative. First note that since $\ker\pi \subset A^I$, the elements of $\ker\pi$ are sequences $(a_i)_{i \in I}$ with all but finitely many $a_i = 0$. It follows that $\imath_j$ acts on simple tensors by:

$$m \otimes (a_i)_{i \in I} \longmapsto (m \cdot a_i)_{i \in I} \in (M_j)^I$$

Similarly, π_j sends a sequence $(m_i)_{i \in I}$ to the tensor $\sum_i m_i \otimes \pi_i(e_i)$, where $e_i \in A^I$ is the ith basis element of A^I. It now suffices to check that each quadrant is commutative. Let $(m_i)_{i \in I} \in (M_1)^I$, then we have that

$$\pi_2 \circ g_I((m_i)_{i \in I}) = \pi_2(g(m_i)_{i \in I}) = \sum_i g(m_i) \otimes \pi_i(e_i)$$

whilst:

$$g_N \circ \pi_1((m_i)_{i\in I}) = \sum_i g_N(m_i \otimes \pi_i(e_i)) = \sum_i g(m_i) \otimes \pi_i(e_i)$$

so the upper left quadrant commutes. The same computation with g_N and g_I replaced with f_N and f_I respectively demonstrates that the upper right quadrant commutes. It suffices to check that the lower quadrants commute on simple tensors, so let $m \otimes (a_i)_{i\in I} \in M_1 \otimes_A \ker \pi$. We have that:

$$g_I \circ \imath_1(m \otimes (a_i)_{i\in I}) = (g(a_i m))_{i\in I} = (g(m) \cdot a_i)_{i\in I}$$

while:

$$\imath_2 \circ g_\pi(m \otimes (a_i)_{i\in I}) = \imath_2(g(m) \otimes (a_i)_{i\in I}) = (g(m) \cdot a_i)_{i\in I}$$

hence the lower left quadrant commutes. The same computations with g_π and g_I replaces by f_π and f_I demonstrates that the lower right quadrant commutes as well.

Our goal is to now show that the top row is actually exact. Suppose that $x \in M_1 \otimes_A N$ satisfies $g_N(x) = 0$. Note that by the surjectivity of π_1 there is a $y \in (M_1)^I$ such that $\pi_1(y) = x$. By the commutativity of the diagram it follows that $\pi_2(g_I(y)) = 0$, hence $g_I(y) \in \operatorname{im} \imath_2$ by exactness of the column. Therefore, there exists a $z \in M_2 \otimes_A \ker \pi$ such that such that $\imath_2(z) = g_I(y)$. Since $f_I \circ g_I = 0$, we have that $f_I \circ \imath_2(z) = 0$, hence $\imath_3 \circ \circ f_\pi(z) = 0$. However, $\imath_3$ is injective, so we have that $f_\pi(z) = 0$, thus there by exactness of the bottom row there is a $w \in M_1 \otimes_A \ker \pi$ such that $g_\pi(w) = z$. In particular, we have that $\imath_2 \circ g_\pi(w) = g_I(y)$ so the commutativity of the diagram implies that $g_I(\imath_1(w)) = g_I(y)$. Since g_I is injective it follows that $\imath_1(w) = y$ so $y \in \operatorname{im} \imath_1$ and thus $\pi_1(y) = 0$ hence $x = 0$ as desired. It follows that g_N is injective implying the claim.

□

Lemma 5.7.7. *Let A be a local ring, and M be a finitely presented, flat A module. Then M is free.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of A, and $k_{\mathfrak{m}} = A/\mathfrak{m}$ the residue field. Note that $M \otimes_A k_{\mathfrak{m}} \cong M/\mathfrak{m}M$ is a vector space over of $k_{\mathfrak{m}}$, and is finite dimensional as M is finitely presented. Let $\{m_1, \ldots, m_l\}$ be elements of M which map to a basis of $M/\mathfrak{m}M$. Nakayama's lemma, i.e. Lemma 3.10.1[136], then implies that $\{m_1, \ldots, m_l\}$ generate M. Let $\phi : A^l \to M$ be the map induced by the set $\{m_1, \ldots, m_l\}$, then we claim that $\ker \phi$ is finite. Indeed, we have a short exact sequence of the form:

$$0 \longrightarrow \ker \phi \longrightarrow A^l \longrightarrow M \longrightarrow 0$$

where we know that M is finitely presented. Since A^l is obviously finitely generated we have that Lemma 5.7.6 *a*) implies that $\ker \phi$ is finitely generated. Part *b*) of Lemma 5.7.6 implies that the following sequence is exact:

$$0 \longrightarrow \ker \phi \otimes_A k_{\mathfrak{m}} \longrightarrow A^l \otimes_A k_{\mathfrak{m}} \longrightarrow M \otimes_A k_{\mathfrak{m}} \longrightarrow 0$$

Note that $A^l \otimes_A k_{\mathfrak{m}} \cong k_{\mathfrak{m}}^l$, $M \otimes_A k_{\mathfrak{m}} \cong M/\mathfrak{m}M$. More the induced map $\phi' : k_{\mathfrak{m}}^l \to M/\mathfrak{m}M$ them is the one sending the standard basis of $k_{\mathfrak{m}}^l$ to the basis $\{[m_1], \ldots, [m_l]\}$ of $M/\mathfrak{m}M$, and is thus an isomorphism. It follows that $\ker \phi \otimes_A k_{\mathfrak{m}} = 0$, and since $\ker \phi \otimes_A k_{\mathfrak{m}} \cong \ker \phi / \mathfrak{m} \cdot \ker \phi$, we find that $\ker \phi = \mathfrak{m} \cdot \ker \phi$. Nakayama's lemma[137], then implies that $\ker \phi = 0$ hence $A^l \cong M$ and M is free as desired. □

With the above lemmas we can now prove the desired result:

Lemma 5.7.8. *Let $\mathscr{F}$ be an invertible sheaf on a locally ringed space X, then for all $x \in X$ we have that $\mathscr{F}_x \cong \mathscr{O}_{X,x}$.*

[136] In particular version *e*).
[137] In particular part *b*)

Proof. Let $\mathscr{F}^{-1}$ be the inverse module to $\mathscr{O}_X$, and $\psi : \mathscr{F} \otimes_{\mathscr{O}_{X,x}} \mathscr{F}^{-1} \to \mathscr{O}_X$ an isomorphism. Fixing $x \in X$, then from our argument in Lemma 5.7.4 we can find a neighborhood U of x for which an automorphism $\mathscr{F} \to \mathscr{F}$ factors as:

$$\mathscr{F}|_U \to \mathscr{O}_U^n \to \mathscr{F}|_U$$

Denote by α the morphism $\mathscr{O}_U^n \to \mathscr{F}|_U$ then the work in Lemma 5.7.4 also demonstrated that $\mathscr{O}_U^n$ splits as $\mathscr{F}|_U \oplus \ker \alpha$. It follows that $\mathscr{O}_{X,x}^n$ then also splits as $\mathscr{F}_x \oplus \ker \alpha_x$. In particular, $\mathscr{F}_x$ is a direct summand of $\mathscr{O}_{X,x}^n$ and is thus flat by Lemma 5.7.5. Moreover, $\mathscr{F}_x$ is finitely presented because $\mathscr{F}$ is finitely presented. Since $\mathscr{O}_{X,x}$ is a local ring, Lemma 5.7.7 implies that $\mathscr{F}_x$ is free. The same exact argument can be made with $\mathscr{F}$ replaced by $\mathscr{F}^{-1}$, thus we have that for some positive $n, m \in \mathbb{Z}$:

$$\mathscr{O}_{X,x} \cong \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x^{-1} \cong \mathscr{O}_{X,x}^n \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x}^m \cong \mathscr{O}_{X,x}^{n \cdot m}$$

It follows that $n \cdot m = 1$ hence $n = m = 1$ and so $\mathscr{F}_x \cong \mathscr{O}_{X,x}$. □

We can prove our original goal:

Corollary 5.7.2. *Every invertible sheaf on a locally ringed space is a line bundle.*

Proof. By Lemma 5.7.8, and Lemma 5.7.5 we have that $\mathscr{F}$ is a finitely presented $\mathscr{O}_X$ module with $\mathscr{F}_x \cong \mathscr{O}_{X,x}$ for all $x \in X$. Part *d)* of Lemma 5.7.3 then implies that every point has an open neighborhood where $\mathscr{F}|_U \cong \mathscr{O}_U$, hence $\mathscr{F}$ is a line bundle. □

5.8 Line Bundles and Projective Space

In this section our goal is twofold; first we want to demonstrate that when $\operatorname{Proj} A$ is quasicompact every quasicoherent sheaf on $\operatorname{Proj} A$ comes from a graded A module, albeit not uniquely. Secondly, we want to explore the relationship between line bundles over a scheme, and morphisms from that scheme to projective space. We first fix the following notation: let X be scheme and $\mathscr{L}$ a line bundle on X, then for $n \in \mathbb{Z}$ we set:

$$\mathscr{L}^{\otimes n} = \begin{cases} \mathscr{L} \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \mathscr{L} & \text{if } n > 0 \\ \mathscr{L}^{-1} \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \mathscr{L}^{-1} & \text{if } n < 0 \\ \mathscr{O}_X & \text{if } n = 0 \end{cases}$$

where it should be understood that there are n terms in the two non zero cases. The use of $\mathscr{L}^{-1}$ makes sense here since every line bundle is invertible. In particular, it should always be understood that $\mathscr{L}^{-1} = \mathscr{L}^*$ on the nose. Clearly, for all $n, m \in \mathbb{Z}$ we have:

$$\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \cong \mathscr{L}^{\otimes (n+m)}$$

This endows the set[138] of isomorphism classes of line bundles on X with an abelian group structure, called the *Picard group* of X. We also set:

$$\mathscr{L}_\bullet(X) := \bigoplus_{n=0}^{\infty} \mathscr{L}^{\otimes n}(X)$$

and if $\mathscr{F}$ is an $\mathscr{O}_X$ module:

$$\mathscr{F}_\bullet(\mathscr{L}, X) := \bigoplus_{n \in \mathbb{Z}} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^n)(X)$$

This brings us to the following lemma:

Lemma 5.8.1. *Let X be a scheme, and $\mathscr{L}$ a line bundle. Then $\mathscr{L}_\bullet(X)$ has the natural structure of a graded ring*[139] *such that for every $\mathscr{O}_X$ module, $\mathscr{F}_\bullet(\mathscr{L}, X)$ is a graded $\mathscr{L}_\bullet(X)$ module.*

[138] We take the fact that the category of line bundles on a scheme has a set of isomorphism classes for granted.
[139] Recall all rings are commutative unless otherwise stated.

Proof. We fix the sheaves:

$$\mathscr{L}_\bullet = \bigoplus_{n=0}^{\infty} \mathscr{L}^{\otimes n} \qquad \text{and} \qquad \mathscr{F}_\bullet(\mathscr{L}) = \bigoplus_{n\in\mathbb{Z}} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$$

It suffices to then prove the slightly stronger statement that $\mathscr{L}_\bullet$ is a sheaf of rings on X, and $\mathscr{F}_\bullet(\mathscr{L})$ is an $\mathscr{L}_\bullet$ module. We first need to show the following: for all n and m greater than zero, we claim that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{L}^{\otimes n} \oplus \mathscr{L}^{\otimes m} \arrow[r, "\phi"] \arrow[d, "\psi_{nm}"] & \mathscr{L}^{\otimes m} \otimes \mathscr{L}^{\otimes n} \arrow[d, "\alpha_{nm}"] \\
\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \arrow[r, "\alpha_{mn}"] & \mathscr{L}^{\otimes(n+m)}
\end{tikzcd}$$

Note that ψ_{nm} is just the natural bilinear tensor product map, and ϕ the natural bilinear tensor product map, post composed with the swap isomorphism. When $n = 0$ or $m = 0$, the diagram commutes trivially because α_{0m}, α_{n0} and α_{00} are the isomorphisms $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \cong \mathscr{L}^{\otimes m}$, $\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{O}_X$, and $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{O}_X$. Assuming that $n, m > 0$, α_{nm} and α_{mn} essentially serve as changes of notation; they act as us forgetting the splitting of the $n + m$ fold tensor product along n and m.

By Proposition 1.2.2 it suffices to check that this diagram commutes on the level of stalks. Letting $x \in X$, it then suffices to prove this diagrams on simple tensors of the form $s = s_1 \otimes \cdots \otimes s_n \in \mathscr{L}_x^{\otimes n}$ and $t = t_1 \otimes \cdots \otimes t_m \in \mathscr{L}_x^{\otimes m}$, where $s_i, t_j \in \mathscr{L}_x$. The commutativity of the diagram is then the statement that in $\mathscr{L}_x^{\otimes(n+m)}$:

$$t_1 \otimes \cdots \otimes t_m \otimes s_1 \otimes \cdots \otimes s_n = s_1 \otimes \cdots \otimes s_n \otimes t_1 \otimes \cdots \otimes t_m$$

Now note that $\mathscr{L}_x \cong \mathscr{O}_{X,x}$, so there exists some $e \in \mathscr{L}_{X,x}$ such that every $s \in \mathscr{L}_{X,x}$ can be written as $s = a \cdot e$ for some unique $a \in \mathscr{O}_{X,x}$. If we set $s_i = a_i e$ and $t_i = b_i e$, then:

$$t_1 \otimes \cdots \otimes t_m \otimes s_1 \otimes \cdots \otimes s_n = b_1 \cdots b_m a_1 \cdots a_n (e \otimes \cdots \otimes e)$$

while:

$$s_1 \otimes \cdots \otimes s_n \otimes t_1 \otimes \cdots \otimes t_m = a_1 \cdots a_n b_1 \cdots b_m (e \otimes \cdots \otimes e)$$

Since $\mathscr{O}_{X,x}$ is a commutative ring, it follows that the two are equal, and thus the diagram commutes.

If $s \in \mathscr{L}_\bullet$, then we denote by s_i component of s of tensor rank i. We define a multiplication map:[140]

$$\mu : \mathscr{L}_\bullet \times \mathscr{L}_\bullet \longrightarrow \mathscr{L}_\bullet$$

given on open sets by:

$$\mu_U : (s, t) \longmapsto \sum_{i,j} (\alpha_{ij} \circ \psi_{ij})_U (s_i, t_j)$$

Since $\alpha_{ij} \circ \psi_{ij}$ is a sheaf morphism μ is also a sheaf morphism. Since $\alpha_{ij} \circ \psi_{ij}$ is bilinear, μ_U is bilinear, hence we have the distribution axioms of a ring are satisfied automatically. The associativity property is inherited from associativity of the tensor product, and the commutativity property follows from our above computation. Since $\mathscr{L}_\bullet(U)$ is already an abelian group, the only axiom we need to check is that there is a unit. We claim that $1 \in \mathscr{O}_X(U) \subset \mathscr{L}_\bullet(U)$ is the unit. Indeed, by definition we have $\alpha_{0j} \circ \psi_{0j}(1, t_j) = t$ so commutativity of μ implies that 1 is the unit. It follows that μ make $\mathscr{L}_\bullet$ into a sheaf of commutative rings as desired. It is then clear that $\mathscr{L}_\bullet$ is actually a sheaf of graded rings, with grading $\mathscr{L}_\bullet(U)_n = \mathscr{L}^{\otimes n}(U)$; in particular this grading is obviously compatible with restriction maps.

[140]We switch to $\times$ notation instead of $\oplus$ as we are defining a multiplication map to treat $\mathscr{L}_\bullet$ as a sheaf of rings instead of an $\mathscr{O}_X$ module.

Note now that there is an obvious morphism:

$$\beta_{nm} : \mathscr{L}^{\otimes n} \times (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m}) \longrightarrow \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(m+n)}$$

given by the composition:

$$\mathscr{L}^{\otimes n} \times (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m}) \to \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(m+n)}$$

If $t \in \mathscr{F}_\bullet(\mathscr{L}, U) := \mathscr{F}_\bullet(\mathscr{L})(U)$, denote by t_i the component of t living in $\mathscr{F} \otimes \mathscr{L}^{\otimes i}(U)$. We then define a sheaf morphism:

$$m : \mathscr{L}_\bullet \times \mathscr{F}_\bullet(\mathscr{L}) \longrightarrow \mathscr{F}_\bullet(\mathscr{L})$$

given on open sets by:

$$m_U : (s,t) \longmapsto \sum_{i,j} \beta_{ij}(s_i, t_j)$$

Since all of the multiplication is happening in the $\mathscr{L}^{\otimes(m+n)}$ component of $\mathscr{F}_\bullet(\mathscr{L})$, this map trivially makes $\mathscr{F}_\bullet(\mathscr{L})$ an $\mathscr{L}_\bullet$ module. It is also clearly a sheaf of graded modules, with grading given by $\mathscr{F}_\bullet(\mathscr{L}, U)_n = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}(U)$. In particular this grading is compatible with the restriction maps.

It follows that $\mathscr{L}_\bullet(X)$ is a graded ring, and $\mathscr{F}_\bullet(\mathscr{L}, X)$ is a graded $\mathscr{L}_\bullet(X)$ module. $\square$

Recall Corollary 5.5.1, which told us that if X is quasicompact and quasiseperated, and $\mathscr{F}$ is a quasicoherent sheaf on X then for all $a \in \mathscr{O}_X(X)$ there are isomorphisms:

$$\mathscr{O}_X(X)_a \cong \mathscr{O}_X(X_a) \qquad \text{and} \qquad \mathscr{F}(X)_a \cong \mathscr{F}(X_a)$$

We wish to generalize this result to our current setting. Let $\mathscr{L}$ be a line bundle on X, and $s \in \mathscr{L}(X)$, then Corollary 5.7.1 implies that the set:

$$X_s = \{x \in X : s(x) \neq 0\}$$

is open and trivializes $\mathscr{L}$. Fix an isomorphism $\phi : \mathscr{L}|_{X_s} \to \mathscr{O}_{X_s}$. For all n, there is then an isomorphism:

$$\mathscr{F} \otimes_{\mathscr{O}_X X} \mathscr{L}^{\otimes n}|_{X_s} \to \mathscr{F}|_{X_s}$$

induced by ϕ, and then multiplying. By taking global sections of the above isomorphism, we get an isomorphism:

$$\psi_n : (\mathscr{F} \otimes_{\mathscr{O}_X X} \mathscr{L}^{\otimes n})(X_s) \longrightarrow \mathscr{F}(X_s)$$

We thus obtain a map:

$$\begin{aligned} \psi : \mathscr{F}_\bullet(\mathscr{L}, X) &\longrightarrow \mathscr{F}(X_s) \\ t &\longmapsto \sum_i \psi_i(t_i|_{X_s}) \end{aligned}$$

where t_i is the degree i part of t. We then obtain a bilinear $\mathscr{L}_\bullet(X)$ map:

$$\begin{aligned} \mathscr{F}_\bullet(\mathscr{L}, X) \oplus \mathscr{L}_\bullet(X)_s &\longrightarrow \mathscr{F}(X_s) \\ (t, \alpha/s^n) &\longmapsto \psi(t) \cdot \phi_s(\alpha|_{X_s} \cdot s|_{X_s}^{-n}) \end{aligned}$$

where ϕ_s denotes the isomorphism on global sections induced by ϕ. We finally obtain a morphism of $(\mathscr{L}_\bullet(X)_s)_0$ modules:

$$\Psi : (\mathscr{F}_\bullet(\mathscr{L}, X)_s)_0 \longrightarrow \mathscr{F}(X_s)$$

induced by the universal property of the tensor product, and then restricting to the degree zero component. We now prove the following:

Lemma 5.8.2. *Let X be a scheme, $\mathscr{L}$ a line bundle, and $\mathscr{F}$ a quasicoherent sheaf. If X is qcqs then the morphism Ψ constructed above is an isomorphism. In particular, with $\mathscr{F} = \mathscr{O}_X$ we obtain that:*

$$(\mathscr{L}_\bullet(X)_s)_0 \cong \mathscr{O}_X(X_s)$$

Proof. Choose a finite open affine covering of X, $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$ such that $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$ for all i.[141] Denote the isomorphisms $\mathscr{L}|_{U_i} \to \mathscr{O}_{U_i}$ by ϕ_i, the isomorphism $\mathscr{L}|_{X_s} \to \mathscr{O}_{X_s}$ by ϕ, and the induced isomorphism on global sections by ϕ_s. Let $f_i = \phi_{i,U_i}(s|_{U_i}) \in A_i$, then we claim that $X_s \cap U_i = U_{f_i}$. Indeed, if $x \in U_{f_i}$, then $(f_i)_x \notin \mathfrak{m}_x$ as otherwise f_i lies in the prime ideal corresponding to $\mathfrak{m}_x$. Since $(f_i)_x = \phi_{i,x}(s_x)$, it follows that $\phi_{i,x}(s_x) \notin \mathfrak{m}_x$. The fact that $\phi_{i,x}$ is an isomorphism then implies that $s_x \notin \mathfrak{m}_x\mathscr{L}_x$ hence $x \in X_s \cap U_i$. Now suppose that $x \in X_s \cap U_i$, then it suffices to show that $(f_i)_x \notin \mathfrak{m}_x$, but this follows because $s_x \notin \mathfrak{m}_x \cdot \mathscr{L}_x$, hence $\phi_{i,x}(s_x) \notin \mathfrak{m}_x$. Therefore $X_s \cap U_i = U_{f_i}$.

We now show that Ψ is injective, which we note only depends on the quasicompactness of X. Let $t/s^n \in \ker \Psi$, and note that this implies that $t \in (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n})(X)$, as the domain of Ψ is the degree zero part of $\mathscr{L}_\bullet(X)_s$. Let ϕ_s be our isomorphism $\mathscr{L}(X_s) \to \mathscr{O}_X(X_s)$, then since t is of degree n we have that:

$$\Psi(t/s^n) = \psi_n(t|_{X_s}) \cdot \phi_s(s|_{X_s})^{-n} = 0$$

Since $\phi_s(s|_{X_s})^{-n}$ is invertible in $\mathscr{O}_X(X_s)$, and ψ_n is an isomorphism we must have that $t|_{X_s} = 0$. This implies that $(t|_{U_i})|_{U_{f_i}} = 0$. We can view $t|_{U_i}$ as an element of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}(U_i)$; the isomorphism ϕ_i then induces isomorphisms $\alpha_{i,n} : \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}|_{U_i} \to \mathscr{F}|_{U_i}$, which on global sections satisfies $(\alpha_{i,n})_{U_i}(t \cdot s|_{U_i}^k) = (\alpha_{i,n-k})_{U_i}(t) \cdot f_i^k$. Since U_i is an affine scheme and thus qcqs, we have that by Corollary 5.5.1:

$$\mathscr{F}|_{U_i}(U_{f_i}) \cong \mathscr{F}|_{U_i}(U_i)_{f_i} \tag{5.8.1}$$

The conditions that $(t|_{U_i})|_{U_{f_i}} = 0$ implies that that $(\alpha_{i,n})_{U_i}(t|_{U_i})|_{U_{f_i}} = 0$, and thus that there is an integer m_i such that $f_i^{m_i} \cdot (\alpha_{i,n})_{U_i}(t|_{U_i}) = 0$. Let m be the largest of these integers, then we claim that $t \cdot s^m \in \mathscr{F}_\bullet(\mathscr{L}, X)$ restricts to zero on each U_i. Indeed, we have that for each i:

$$(\alpha_{i,n+m})_{U_i}((t \cdot s^m)|_{U_i}) = (\alpha_{i,n})_{U_i}(t|_{U_i}) \cdot f_i^m = 0$$

Since the $\alpha_{i,n+m}$ is an isomorphism, it follows that $(t \cdot s^m)|_{U_i} = 0$ for all i, hence $t \cdot s^m = 0$. Since $t \cdot s^m/s^{n+m} = t/s^n$ we have that $t/s^n = 0$ hence the map is injective.

We now show surjectivity of Ψ. Fix $t \in \mathscr{F}(X_s)$, and let $U_{ij} = U_i \cap U_j$; note that each U_{ij} is quasicompact as X is quasiseperated. Via the isomorphism (5.8.1), for each i we can identify $t|_{U_{f_i}}$ with $t_i/f_i^{m_i} \in \mathscr{F}|_{U_i}(U_i)_{f_i}$, for some $t_i \in \mathscr{F}|_{U_i}(U_i)$ and some $m_i \in \mathbb{N}$. Let m be the larges of all such m_i, and let $q_i \in \mathscr{L}(U_i)$ be the inverse image of 1 under the isomorphism $\phi_{i,U_i} : \mathscr{L}(U_i) \to \mathscr{O}_X(U_i)$. We then set:

$$u_i = (f_i^{m-m_i} \cdot t_i) \cdot q_i^m \in (\mathscr{F}_\bullet(\mathscr{L}, U_i))_m = (\mathscr{F} \otimes \mathscr{L}^{\otimes m})(U_i)$$

Note that:

$$f_i \cdot q_i f_i \cdot \phi_{i,U_i}^{-1}(1) = \phi_{i,U_i}^{-1}(f_i) = s|_{U_i} \tag{5.8.2}$$

We claim that $u_i|_{U_{ij} \cap X_s} = u_j|_{U_{ij} \cap X_s}$. Note that since f_i is invertible over U_{f_i}, we have that:

$$(f_i^{m-m_i} \cdot t_i)|_{U_{f_i}} = t_i|_{U_{f_i}} \cdot f_i|_{U_{f_i}}^{-m_i} \cdot f_i|_{U_{f_i}}^m \in \mathscr{F}|_{U_i}(U_{f_i})$$

The isomorphism:

$$\eta : \mathscr{F}(U_i)_{f_i} \longrightarrow \mathscr{F}(U_{f_i})$$

is given by sending a/f_i^k to $a|_{U_i} \cdot f_i|_{U_{f_i}}^{-k}$, hence we have that:

$$\eta(t_i/f_i^{m_i}) = t_i|_{U_{f_i}} \cdot f_i|_{U_{f_i}}^{-m_i}$$

[141] This can be done by first taking a trivializing cover of $\mathscr{L}$, and then covering each open set by open affines.

but we defined $t_i/f_i^{m_i}$ to be the unique section corresponding to $t|_{U_{f_i}}$ under η, hence we have:

$$t_i|_{U_{f_i}} \cdot f_i|_{U_{f_i}}^{-m_i} = t|_{U_{f_i}}$$

The above result, and (5.8.2) then demonstrate:

$$u_i|_{U_i \cap X_s} = u_i|_{U_{f_i}} = t|_{U_{f_i}} \cdot s^m|_{U_{f_i}} \in (\mathscr{F}_\bullet(\mathscr{L}, U_{f_i}))_m = (\mathscr{F} \otimes \mathscr{L}^{\otimes m})(U_{f_i})$$

Similarly we have that:

$$u_j|_{U_j \cap X_s} = t|_{U_{f_j}} \cdot s^m|_{U_{f_j}}$$

By further restricting to $U_{ij} \cap X_s$ we obviously obtain the desired equality.

Let $Y = U_{ij}$, and note that $Y_{s|_Y} = X_s \cap U_{ij}$. Then, since U_{ij} is quasicompact, we know that the map:

$$\Psi' : ((\mathscr{F}|_Y)_\bullet(\mathscr{L}|_Y, Y)_{s|_Y})_0 \longrightarrow \mathscr{F}|_Y(Y_{s|_Y})$$

is injective. In particular, considering $u_i|_{U_{ij}}/s|_{U_{ij}}^m$, and $u_j|_{U_{ij}}/s|_{U_{ij}}^m$ in $((\mathscr{F}|_Y)_\bullet(\mathscr{L}|_Y, Y)_{s|_Y})_0$, we have that:

$$\begin{aligned}\Psi'(u_i|_{U_{ij}}/s|_{U_{ij}}^m) =& \psi_m(u_i|_{U_{ij}\cap X_s}) \cdot \phi_{U_{ij}\cap X_s}(s|_{U_{ij}\cap X_s})^{-m} \\ =& \psi_m(u_j|_{U_{ij}\cap X_s}) \cdot \phi_{U_{ij}\cap X_s}(s|_{U_{ij}\cap X_s})^{-m} \\ =& \Psi'(u_j|_{U_{ij}}/s|_{U_{ij}}^m)\end{aligned}$$

Injectivity then implies that there is some m_{ij} such that:

$$u_j|_{U_{ij}} \cdot s|_{U_{ij}}^{m_{ij}} = u_i|_{U_{ij}} \cdot s|_{U_{ij}}^{m_{ij}}$$

Doing this for all i and j we can let m' be the largest of such m_{ij} and obtain the fact that:

$$u_i|_{U_{ij}} \cdot s|_{U_{ij}}^{m'} = u_j|_{U_{ij}} \cdot s|_{U_{ij}}^{m'}$$

for all i and j. The second sheaf axiom then gives of an element $u \in \mathscr{F}_\bullet(\mathscr{L}, X)$ of degree $m' + m$ which restricts to u_i on U_i. We claim that $u/s^{m+m'} \in (\mathscr{F}_\bullet(\mathscr{L}, X)_s)_0$ maps to t under Ψ. It suffices to show that $\Psi(u)|_{U_i \cap X_s} = t|_{U_i \cap X_s}$ for all i; recalling that $U_i \cap X_s = U_{f_i}$, this follows easily from the fact that:

$$\begin{aligned}u|_{U_i \cap X_s} =& u|_{U_i}|_{U_i \cap X_s} \\ =& (u_i \cdot s|_{U_i}^{m'})|_{U_i \cap X_s} \\ =& u_i|_{U_{f_i}} \cdot s|_{U_{f_i}}^{m'} \\ =& t|_{U_{f_i}} \cdot s|_{U_{f_i}}^{m+m'}\end{aligned}$$

Recall that $\psi_{m+m'}$ comes from take global sections of a sheaf isomorphism, so denote by $\psi_{m+m', U_{f_i}}$ the isomorphism induced by the same sheaf isomorphism but on sections over U_{f_i}. We then obtain that:

$$\begin{aligned}\Psi(u/s^{m+m'})|_{U_{f_i}} =& \left(\psi_{m+m'}(u_i|_{X_s})\phi_s(s|_{X_s})^{-m-m'}\right)|_{U_{f_i}} \\ =& \psi_{m+m', U_{f_i}}(t|_{U_{f_i}} \cdot s|_{U_{f_i}}^{m+m'}) \cdot \phi_{U_{f_i}}(s|_{X_s})^{-m-m'} \\ =& t|_{U_{f_i}} \cdot \phi_{U_{f_i}}(s|_{U_{f_i}})^{m+m'} \cdot \phi_{U_{f_i}}(s|_{X_s})^{-m-m'} \\ =& t|_{U_{f_i}}\end{aligned}$$

The first sheaf axiom then implies that $\Psi(u/s^{m+m'}) = t$ and so Ψ is surjective, as desired. □

If $X = \operatorname{Proj} A$, recall that $\mathscr{O}_X(n)$ is the quasicoherent sheaf of $\mathscr{O}_X$ modules induced by the graded A module $A(n)$. When $A = B[x_0, \ldots, x_n]$ we know that these are always line bundles; our first goal is to determine a criterion for when $\mathscr{O}_X(n)$ is a vector bundle in this more general setting.

Lemma 5.8.3. *Let $X = \operatorname{Proj} A$, then $\mathscr{O}_X(n)|_{U_f}$ is trivial whenever $\deg f|n$. In particular, if $X = \{U_f\}_{f\in A_d}$ then $\mathscr{O}_X(dm)$ is a line bundle for all $m \in \mathbb{Z}$.*

Proof. Suppose such a cover exists, then it suffices to show that $\mathscr{O}_X(n)|_{U_f}$ is trivial when f is of degree d which divides n. Since $U_f = \operatorname{Spec}(A_f)_0$, it suffices to show that $(A(n)_f)_0 \cong (A_f)_0$ as $(A_f)_0$ modules. Note that $(A(n)_f)_0$ is just the degree n component of A_f, hence we need only prescribe an isomorphism:

$$(A_f)_n \longrightarrow (A_f)_0$$

Every element $a/f^m \in (A_f)_n$ satisfies $\deg a - m \cdot d = n$, and we know by divisibility there exists some l such that f^l has degree n, so we define a map $\psi : (A_f)_n \longrightarrow (A_f)_0$ by sending a/f^m to a/f^{m+l}, which has degree zero. This is obviously an $(A_f)_0$ module homomorphism, and has an obvious inverse given by multiplying $f^l/1$, hence $\mathscr{O}_X(n)|_{U_f}$ is trivial.

□

The next example shows that the converse of the above claim is not true:

Example 5.8.1. Let:

$$A = k[x, y, z]/\left\langle x^6 + y^3 + z^2\right\rangle$$

with grading given by $\deg x = 1$, $\deg y = 2$, and $\deg z = 3$. Clearly $X = \operatorname{Proj} A$ cannot be covered by distinguished opens corresponding to elements of degree one as the only such sets are all equal to $U_{[x]}$. We claim that $\mathscr{O}_X(1)$ is a line bundle. If $\mathfrak{p} \in U_{[x]}$ then $\mathscr{O}_X(1)_\mathfrak{p}$ is obviously trivial as $\mathscr{O}_X(1)|_{U_{[x]}}$ is the trivial bundle by the above lemma. Suppose that $[x] \in \mathfrak{p}$, then we claim that neither $[y]$ nor $[z]$ lie in $\mathfrak{p}$. If $[y] \in \mathfrak{p}$, then we have that $[x^6 + y^3] \in \mathfrak{p}$ hence $[z^2] \in \mathfrak{p}$, and thus $[x]$, $[y]$, and $[z]$ are all in $\mathfrak{p}$. It follows that $\mathfrak{p}$ contains the irrelevant ideal, contradicting the fact that $\mathfrak{p} \in X$, so $[y] \notin \mathfrak{p}$. The same argument demonstrates that $[z] \notin \mathfrak{p}$ as well. Since both $[y]$ and $[z]$ do not lie in $\mathfrak{p}$ we have that $[y]^2/[z] \in A(1)_{(\mathfrak{p})} \cong \mathscr{O}_X(1)_\mathfrak{p}$ has an inverse in the larger ring $A_{\mathfrak{p}^{\text{hom}}}$. The $A(1)_{(\mathfrak{p})}$ module map:

$$\begin{aligned} A(1)_{(\mathfrak{p})} &\longrightarrow A_{(\mathfrak{p})} \\ \alpha &\longmapsto \alpha \cdot [z]/[y]^2 \end{aligned}$$

is then an isomorphism with inverse given by multiplication by $[y]^2/[z]$. It follows that $\mathscr{O}_{X,\mathfrak{p}} \cong \mathscr{O}_X(1)_\mathfrak{p}$ for all $\mathfrak{p} \in X$.

Now note that:

$$\mathscr{O}_X(1)|_{U_{[x]}} \cong \widetilde{(A_{[x]})_1} \qquad \mathscr{O}_X(1)|_{U_{[y]}} \cong \widetilde{(A_{[y]})_1} \qquad \mathscr{O}_X(1)|_{U_{[z]}} \cong \widetilde{(A_{[z]})_1}$$

Each of the modules is obviously finitely generated over their respective ring, and the sets $U_{[x]}$, $U_{[y]}$ and $U_{[z]}$ cover X. Since X is obviously Noetherian, Proposition 5.4.2 then implies that $\mathscr{O}_X(1)$ is coherent; in particular, $\mathscr{O}_X(1)$ is finitely presented and is trivial on each stalk, hence must be a line bundle by Lemma 5.7.3 part *d*).

Continuing with the hypothesis of the previous lemma, we have the following useful fact:

Lemma 5.8.4. *Let $X = \operatorname{Proj} A$ admit a covering of basic opens $\{U_f\}$, where every f has degree d. Then for all $a, b \in \mathbb{Z}$, the natural maps*

$$\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd) \longrightarrow \mathscr{O}_X((a+b)d)$$

are isomorphisms. In particular $\mathscr{O}_X(nd) \cong \mathscr{O}_X(d)^{\otimes n}$.

Proof. Note that clearly we have an isomorphism:

$$\begin{aligned} A(ad) \otimes_A A(bd) &\longrightarrow A((a+b)d) \\ f \otimes g &\longmapsto f \cdot g \end{aligned}$$

Let $\mathscr{M}$ be the $\mathscr{O}_X$ module associated to $A(ad) \otimes_A A(bd)$, then Theorem 5.6.1 implies there exists an isomorphism:

$$\mathscr{M} \longrightarrow \mathscr{O}_X((a+b)d)$$

given on distinguished opens by:

$$\begin{aligned} \mathscr{M}(U_h) &\longrightarrow \mathscr{O}_X((a+b)d)(U_h) \\ (f \otimes g)/h^k &\longrightarrow (f \cdot g)/h^k \end{aligned}$$

where we are implicitly using the identifications $\mathscr{M}(U_h) \cong ((A(ad) \otimes_A A(bd))_h)_0$, and $\mathscr{O}_X((a+b)d)(U_h) \cong (A((a+b)d)_h)_0$. From Lemma 5.6.4 we also obtain a morphism:

$$\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd) \longrightarrow \mathscr{O}_X((a+b)d)$$

given on distinguished opens by:

$$\begin{aligned} (\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd))(U_h) &\longrightarrow \mathscr{M}(U_h) \\ f/h^l \otimes g/h^m &\longrightarrow (f \otimes g)/f^{k+l} \end{aligned}$$

where we are again implicitly using the identification

$$(\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd))(U_h) \cong (A(ad)_h)_0 \otimes_{(A_h)_0} (A(bd)_h)_0$$

and the previous $\mathscr{M}(U_h)$ identification. By composing these two morphisms we obtain a morphism:

$$F : \mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd) \longrightarrow \mathscr{O}_X((a+b)d)$$

given on affine opens by:

$$\begin{aligned} F : \mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd)(U_h) &\longrightarrow \mathscr{O}_X((a+b)d)(U_h) \\ f/h^l \otimes g/h^m &\longrightarrow (f \cdot g)/f^{k+l} \end{aligned}$$

Since X is covered by distinguished opens of the form U_h with h of degree d it suffices to check that $F|_{U_h}$ is an isomorphism for all h. Since U_h is affine for each h, it suffices to check that on global sections $F|_{U_h}$ induces an isomorphism:

$$(\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd))(U_h) \cong \mathscr{O}_X((a+b)d)(U_h)$$

By Lemma 5.8.3 both $\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd)$ and $\mathscr{O}_X((a+b)d)$ are line bundles that are trivial over U_h. In particular both $(\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd))(U_h)$ and $\mathscr{O}_X((a+b)d)(U_h)$ are free $(A_h)_0$ modules of rank 1, so by Lemma 5.3.4 it suffices to show that F_{U_h} is surjective.

By our work in Lemma 5.8.3, we have that h^{a+b} induces an isomorphism:

$$\mathscr{O}_X(U_h) \longrightarrow \mathscr{O}_X((a+b)d)(U_h)$$

thus any element in $\mathscr{O}_X((a+b)d)(U_h)$ can be written as $a/h^k \cdot h^{a+b}$ where a/h^k has degree zero. In particular, the element $a/h^k \cdot (h^a \otimes h^b) \in (\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd))(U_h)$ clearly maps to $a/h^k \cdot h^{a+b}$ hence F_{U_h} is surjective, as desired.

For the second part of the claim, we have that if $n \geq 1$ then:

$$\mathscr{O}_X(nd) \otimes_{\mathscr{O}_X} \mathscr{O}_X(d) \cong \mathscr{O}_X((n+1)d)$$

so induction on n gives that $\mathscr{O}_X(d)^{n+1} \cong \mathscr{O}_X((n+1)d)$. If $n < 0$ then the same argument holds as soon as we show that $\mathscr{O}_X(-d) \cong \mathscr{O}_X(d)^*$. We know that:

$$\mathscr{O}_X(-d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(d) \cong \mathscr{O}_X$$

by the above, so $\mathscr{O}_X(-d)$ and $\mathscr{O}_X(d)^*$ are both the inverse of $\mathscr{O}_X(d)$ and are thus isomorphic implying the claim. $\square$

If $\mathscr{F}$ is a quasicoherent sheaf on $X = \operatorname{Proj} A$, then we fix the notation

$$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$$

and set:

$$M = \bigoplus_{n \in \mathbb{Z}} \mathscr{F}(n)(X)$$

We will often refer to M as the *graded A module associated to $\mathscr{F}$*, and as such need to show that M is indeed a graded A module. We do this now:

Lemma 5.8.5. *Let $X = \operatorname{Proj} A$, $\mathscr{F}$ a quasicoherent sheaf, and M be as above. Then M has the structure of a graded A module, and there is morphism of $\mathscr{O}_X$ modules:*

$$\mathscr{M} \longrightarrow \mathscr{F}$$

Proof. We first need to give M the structure of a graded A module, and to do so we want to show that:

$$\mathscr{O}_{X\bullet} := \bigoplus_{n \in \mathbb{Z}}^{\infty} \mathscr{O}_X(n)$$

is a sheaf of graded rings on X. Note that the morphism:

$$\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd) \longrightarrow \mathscr{O}_X((a+b)d)$$

can be constructed regardless of whether or not the the above sheaves are line bundles. In particular, for any m and n we have a morphism:

$$\mu_{mn} : \mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \longrightarrow \mathscr{O}_X(m+n)$$

given by actual multiplication on distinguished opens. These morphisms then induce a multiplication map:

$$\mu : \mathscr{O}_{X\bullet} \times \mathscr{O}_{X\bullet} \longrightarrow \mathscr{O}_{X\bullet}$$

which clearly makes $\mathscr{O}_{X\bullet}$ into a sheaf of graded rings. The $\mathscr{O}_X$ module:

$$\mathscr{F}_\bullet := \bigoplus_{n=0}^{\infty} \mathscr{F}(n) = \bigoplus_{n=0}^{\infty} \mathscr{F} \otimes \mathscr{O}_X(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{X\bullet}$$

then admits an obvious $\mathscr{O}_{X\bullet}$ structure induced by μ. Taking global sections gives M the structure of a graded $\mathscr{O}_{X\bullet}(X)$ module, and so it suffices to define a graded ring homomorphism $A \to \mathscr{O}_{X\bullet}(X)$. Note that any element of A_0 induces an element $\mathscr{O}_X(X)$ by sending $a \in A_0$ to $a/1 \in \mathscr{O}_X(U_f)$ for all $f \in A^+_{\text{hom}}$. These sections obviously agree on overlaps because they all have denominator one, and thus glue to a global section of $\mathscr{O}_X(X)$. Similarly, any $a \in A_n$ induces a section in $\mathscr{O}_X(n)(X)$ by the same process. Sending an element of A to the associated global section in $\mathscr{O}_{X\bullet}(X)$ gives us the map:

$$\Psi : A \to \mathscr{O}_{X\bullet}(X)$$

This map preserves grading by construction; let $a \in A_n$ and $b \in A_m$. Then $\Psi(a+b) = \Psi(a) + \Psi(b)$ as they obviously agree when restricted to distinguished opens. Similarly, $\Psi(ab)$ and $\Psi(a) \cdot \Psi(b)$ agree when restricted to distinguished opens, so $\Psi(ab) = \Psi(a) \cdot \Psi(b)$. It follows that Ψ is a graded ring homomorphism, and thus M has the structure of a graded A modules.

We now define the map:

$$\Phi : \mathscr{M} \longrightarrow \mathscr{F}$$

and we do so on distinguished opens. We need to provide maps $\phi_f : \mathscr{M}|_{U_f} \to \mathscr{F}|_{U_f}$, and then show that they agree on overlaps. Elements $(M_f)_0$ are of the form s/f^l where $s \in \mathscr{F}(n)$ for some n and l

satisfies $n = l \cdot \deg f$. Note that $s|_{U_f} \in \mathscr{F}(n)(U_f) \subset \mathscr{F}_\bullet(U_f)$, which has our previously defined graded $\mathscr{O}_{X\bullet}$ module structure. In particular, we have that $1/f^l \in \mathscr{O}_X(-n)(U_f)$, and so we can obtain the element $s|_{U_f} \cdot f^{-l} \in \mathscr{F}(U_f)$, which is the degree zero part of $\mathscr{F}_\bullet(U_f)$. We want to define ϕ_f to be the sheaf morphism induced by the map:

$$\begin{aligned} (M_f)_0 &\longrightarrow \mathscr{F}(U_f) \\ s/f^l &\longmapsto s|_{U_f} \cdot f^{-l} \end{aligned}$$

We need to check that this is well defined; suppose that $s/f^l = t/f^m$ then we need to show that $s|_{U_f} \cdot f^{-l} = t|_{U_f} \cdot f^{-t}$. Well $f/1$ is an invertible element in $\mathscr{O}_{X\bullet}(U_f)$, and we know that there exists some L such that:

$$f^L(f^m s - f^l t) = 0 \in M$$

For $N = (L+m)\deg f + \deg s$ we have that the above expression lies in the direct summand $\mathscr{F}(N)(X)$; by restricting to U_f we obtain that:

$$(f^{L+m}s)|_{U_f} = (f^{L+l}t)|_{U_f}$$

By unravelling the A-module structure on $\mathscr{F}_\bullet(X) = M$, and the $\mathscr{O}_{X\bullet}$ module structure on $\mathscr{F}_\bullet$, we have that $f|_{U_f}$ is precisely our invertible section $f/1 \in \mathscr{O}_{X\bullet}(U_f)$ from before. We can thus eliminate f^L, and cross multiply to obtain that:

$$s|_{U_f} \cdot f^{-l} = t|_{U_f} \cdot f^{-m}$$

so ϕ_f is well defined.

We now want to show that ϕ_f and ϕ_g agree on over laps $U_f \cap U_g$. By replacing f and g with $f^{\deg g}$ and $g^{\deg f}$ it suffices to show this when $\deg f = \deg g$. The module map:

$$\phi_{f,U_{fg}} : \mathscr{M}(U_{fg}) \longrightarrow \mathscr{F}(U_{fg})$$

is induced by the module map:

$$\begin{aligned} ((M_f)_0)_h &\longrightarrow (\mathscr{F}(U_f))_h \\ s/f^l \cdot h^{-m} &\longmapsto s|_{U_f} \cdot f^{-l} \cdot h^{-m} \end{aligned}$$

where $h = g/f$. Since U_f is affine, and thus qcqs, we know that $(\mathscr{F}(U_f))_h \cong \mathscr{F}(U_{fg})$ by sending $t \cdot h^{-m}$ to $t|_{U_{fg}} \cdot f^2/(gf)$.[142] We thus have that up to the identification $\mathscr{M}(U_{fg}) \cong ((M_f)_0)_h$, the map $\phi_{f,U_{fg}}$ is given by:

$$\begin{aligned} \alpha_f : ((M_f)_0)_h &\longrightarrow \mathscr{F}(U_{fg}) \\ (s/f^l) \cdot h^{-m} &\longmapsto (s|_{U_{fg}}) \cdot f^{2m}/(fg)^m \cdot g^l/(gf)^l \end{aligned}$$

where $g^l/(gf)^l$ is $f^{-l}|_{U_{fg}}$. Note that $f^{2m}/(fg)^m \cdot g^l/(gf)^l \in \mathscr{O}_X(-n)(U_{fg})$.

Let $h' = f/g$ then we have the isomorphism:

$$\begin{aligned} \beta_{fg} : ((M_f)_0)_h &\longrightarrow ((M_g)_0)_{h'} \\ s/f^l \cdot h^{-m} &\longmapsto s/g^l \cdot (h')^{m-l} \end{aligned}$$

Showing that ϕ_f and ϕ_g agree on overlaps now boils down to showing that $\alpha_g \circ \beta_{fg} = \alpha_f$. We have that:

$$\begin{aligned} \alpha_g \circ \beta_{fg}(s/f^l \cdot h^{-m}) =& \alpha_g(s/g^l \cdot (h')^m \cdot (h')^{-l}) \\ =& s|_{U_{fg}} \cdot f^{2m}/(gh)^m \cdot g^l/(gh)^l \end{aligned}$$

implying the claim. □

[142] This is because $g/f \in \mathscr{O}_X(U_f)$ restricts to element $g^2/(gf)$ in $\mathscr{O}_X(U_{fg}) \cong (A_{fg})_0$.

We can now show the first main result of this section:

Theorem 5.8.1. *If $X = \operatorname{Proj} A$ is quasicompact then the functor from graded modules to* $\operatorname{QCoh}(\operatorname{Proj} A)$ *described in Theorem 5.6.1 is essentially surjective.*

Proof. To show that the functor is essentially surjective, it suffices to construct a graded A module M for every $\mathscr{F}$ such that $\mathscr{M} \cong \mathscr{F}$.

As X is quasicompact we can cover X with finitely many distinguished opens $\{U_{f_i}\}_{i=1}^m$. By taking d to be the least common multiple multiples of the set $\{\deg f_i\}_{i=1}^n$, and replacing each f_i with $f_i^{n_i}$, where n_i is an integer satisfying $n_i \cdot \deg f_i = d$, we may assume that all f_i are of degree d. Lemma 5.8.3 and Lemma 5.8.4 imply that $\mathscr{O}_X(ad)$ is a line bundle for all $a \in \mathbb{Z}$, and that the multiplication maps $\mathscr{O}_X(ad) \otimes_{\mathscr{O}_X} \mathscr{O}_X(bd) \to \mathscr{O}_X((a+b)d)$ are isomorphisms for all $a, b \in \mathbb{Z}$.

Fix $\mathscr{L} = \mathscr{O}_X(d)$. Note that each f_i corresponds to a global section s_i of $\mathscr{L}$ which restricts to $f_1/1 \in \mathscr{L}(U_{f_j}) \cong (A_{f_j})_d$ for each each j. We claim that $X_{s_i} = U_{f_i}$; clearly $U_{f_i} \subset X_{s_i}$, so suppose that $\mathfrak{p} \in X_{s_i}$, then $s_{i,\mathfrak{p}} \notin \mathfrak{m}_\mathfrak{p} \cdot \mathscr{L}_\mathfrak{p}$. Now, under the identification $\mathscr{L}_\mathfrak{p} \cong (A_{\mathfrak{p}^{\text{hom}}})_d$, we have $s_{i,\mathfrak{p}}$ is precisely the element $f_i/1 \in (A_{\mathfrak{p}^{\text{hom}}})_d$, and by Lemma 5.6.3:

$$\mathfrak{m}_\mathfrak{p} = \left\{ \frac{p}{a} \in A_{(\mathfrak{p})} : p \in \mathfrak{p}^{\text{hom}} \right\}$$

The statement is therefore that $f_i/1$ is not of the form p/a for some $p \in \mathfrak{p}^{\text{hom}}$ of degree $d + \deg a$. Obviously this implies $f_i \notin \mathfrak{p}$ as other wise, $f_i/1$ is clearly of the prescribed form, hence $\mathfrak{p} \in U_{f_i}$.

Let M be the graded A module associated to $\mathscr{F}$, and let:

$$\Phi : \mathscr{M} \longrightarrow \mathscr{F}$$

be the morphism constructed in Lemma 5.8.5. We are done if we can show that Φ is an isomorphism, and it suffices to show that the maps:

$$\begin{aligned} \phi_{f_i, U_{f_i}} : (M_{f_i})_0 &\longrightarrow \mathscr{F}(U_f) \\ t/f_i^n k &\longmapsto t|_{U_f} \cdot f^{-n} \end{aligned}$$

are isomorphisms. We first claim that $(\mathscr{F}_\bullet(\mathscr{L}, X)_{s_i})_0 = (M_{f_i})_0$. Indeed, since $\mathscr{O}_X(d)^n \cong \mathscr{O}_X(nd)$ we have that up to this identification:

$$\mathscr{F}_\bullet(L, X) = \bigoplus_{n \in \mathbb{Z}} \mathscr{F}(dn) \subset M = \bigoplus_{n \in \mathbb{Z}} \mathscr{F}(n)$$

The A-module structure on M is given by identifying f with $s_i \in \mathscr{L}(X)$, and so inverting f_i is the same as inverting s_i. Hence we obtain that:

$$F_\bullet(\mathscr{L}, X)_{s_i} \subset M_{f_i}$$

Now both s_i and f_i are degree d, hence the only degree zero elements of M_{f_i} have to be elements of the form a/f_i^k where $a \in \mathscr{F}(kd)$, hence:

$$(F_\bullet(\mathscr{L}, X)_{s_i})_0 = (M_{f_i})_0$$

Now Lemma 5.8.2 tells us that we have an isomorphism:

$$\Psi : (\mathscr{F}_\bullet(\mathscr{L}, X)_{s_i})_0 \longrightarrow \mathscr{F}(X_{s_i})$$

given by:

$$t/s_i^n \longmapsto \psi_n(t|_{X_{s_i}}) \cdot \phi_{s_i}(s_i|_{X_{s_i}})^{-n}$$

where ψ_n is the isomorphism $\mathscr{F}(nd)(X_s) \to \mathscr{F}(X_s)$, and ϕ_{s_i} is the isomorphism $\mathscr{L}(X_s) \to \mathscr{O}_X(X_s)$. We want to show that Ψ is exactly $\phi_{f, U_{f_i}}$, and in this setting we can be extremely explicit about what ϕ_{s_i} and ψ_n actually are. Since $X_{s_i} = U_{f_i}$, and we know that t is a global section of $\mathscr{F}(dn)$, we have

$t|_{U_{f_i}} \in \mathscr{F}(U_{f_i}) \otimes_{(A_{f_i})_0} (A_{f_i})_{dn}$. By the linearity of the module map, it suffices to show this when $t|_{U_{f_i}}$ is a simple tensor of the form $t' \otimes f^n$, where $t' \in F(U_{f_i})$. The map ψ_n then takes $t' \otimes f^n$ to t', because ϕ_{s_i} is the map which sends f_i to 1. Since ϕ_{s_i} sends f_i to 1 we then have that:

$$\Psi(t/s_i^n) = t'$$

Meanwhile, with $t/s_i^n = t/f_i^n$, we have that $\phi_{f,U_{f_i}}(t/f_i^n) = t|_{U_{f_i}} \cdot f_i^{-n}$. Now, by the same assumption, $t|_{U_{f_i}} = t' \otimes f_i^n$. Since multiplication is defined via the tensor product, we have that $t|_{U_{f_i}} \cdot f_i^{-n} = (t' \otimes f_i^n) \cdot f_i^{-n} = t'$. It follows that $\phi_{f_i,U_{f_i}}$ is precisely the isomorphism Ψ, and thus Φ is an isomorphism as desired. $\square$

We now demonstrate a fundamental connection between $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$ and line bundles over a scheme. We fix the notation $\mathscr{O}_{\mathbb{P}^n_S}(d) := \pi^*_{\mathbb{P}^n_{\mathbb{Z}}} \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(d)$.

Definition 5.8.1. A **degree n linear system** over X, denoted $(\mathscr{L}, (s_1, \ldots, s_n))$, is a choice of a line bundle $\mathscr{L}$ over X, and n global sections s_i of $\mathscr{L}$. A linear system is called **base point free** if the sections do not simultaneously vanish, i.e.

$$\bigcap_{i=1}^{n} \mathbb{V}(s_i) = \emptyset$$

A morphism of linear systems $(\mathscr{L}, (s_0, \ldots, s_n)) \to (\mathscr{L}', (t_0, \ldots, t_n))$ is a morphism $\phi : \mathscr{L} \to \mathscr{L}'$ such that $\phi_X(s_i) = t_i$.

We define a functor $\mathbf{P}^n_S : \mathrm{Sch}/S \to \mathrm{Set}$ by sending an S-scheme X to the following set:

$$\mathbf{P}^n_S(X) = \{\text{isomorphism classes of degree } n+1 \text{ base point free linear systems over } X\}$$

In other words $[(\mathscr{L}, (s_0, \ldots, s_n))] \in \mathbf{P}^n_S(X)$ is the equivalence class of base point free linear systems under the equivalence relation:

$$(\mathscr{L}, (s_0, \ldots, s_n)) \sim (\mathscr{L}', (t_0, \ldots, t_n)) \Leftrightarrow (\mathscr{L}, (s_0, \ldots, s_n)) \cong (\mathscr{L}', (t_0, \ldots, t_n))$$

where the isomorphism is as linear systems.

We need to specify where morphisms go. If $f : X \to Y$ is a morphism of S-schemes, then $\mathbf{P}^n_S(f)$ is the morphism defined:

$$[(\mathscr{L}, (s_0, \ldots, s_n))] \longmapsto [(f^*\mathscr{L}, (f^*s_0, \ldots f^*s_n))]$$

We need to check that this is well defined.

Lemma 5.8.6. *Let $f : X \to Y$ be a morphism of S-schemes, then the map $\mathbf{P}^n_S(f)$ is well defined.*

Proof. Let $[(\mathscr{L}, (s_0, \ldots, s_n))] \in \mathbf{P}^n_S(Y)$, then we need to check that:

$$(\mathscr{L}, (s_0, \ldots, s_n)) \cong (\mathscr{L}', (t_0, \ldots, t_n)) \Rightarrow (f^*\mathscr{L}, (f^*s_0, \ldots, f^*s_n)) \cong (f^*\mathscr{L}', (f^*t_0, \ldots, f^*t_n))$$

Let $\phi : (\mathscr{L}, (s_0, \ldots, s_n)) \to (\mathscr{L}', (t_0, \ldots, t_n))$ be an isomorphism, then as f^* is a functor we know that $f^*\phi : f^*\mathscr{L} \to f^*\mathscr{L}'$ is an isomorphism, hence it suffices to check that $(f^*\phi)_X(f^*s_i) = f^*t_i$. Note that the section f^*s_i is defined to be the section determined by the morphism $\mathscr{O}_Y \to \mathscr{L}$ induced by s_i. In other words, if $\psi_s : \mathscr{O}_Y \to \mathscr{L}$ is given by $(\psi_{s_i})_Y(1) = s_i$, then we have that $f^*s_i = (f^*\psi_{s_i})_X$. It follows that:

$$\begin{aligned}(f^*\phi)_X(f^*s_i) =& (f^*\phi)_X \circ (f^*\psi_{s_i})_X(1) \\ =& f^*(\phi \circ \psi_{s_i})_X(1)\end{aligned}$$

By assumption we have that $\phi \circ \psi_{s_i} = \psi_{t_i}$, where $\psi_{t_i} : \mathscr{O}_Y \to \mathscr{L}'$ sends 1 to t_i. It follows that:

$$(f^*\phi)_X(f^*s_i) = f^*(\psi_{t_1})(1) = f^*t_i$$

hence $\mathbf{P}^n_S(f)$ is well defined. $\square$

We now prove the following result:

Theorem 5.8.2. *The functor $\mathbf{P}^n_S$ is represented by $\mathbb{P}^n_S$. In particular the universal object of $\mathbf{P}^n_S$ is the pair consisting of the scheme $\mathbb{P}^n_S$ and the linear system $[(\mathscr{O}_{\mathbb{P}^n_S}(1),(s_0,\dots,s_n))]$, where s_i is the pull back of the global section $x_i \in \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) = (\mathbb{Z}[x_0,\dots,x_n])_1$.*

Proof. We set $\mathbf{P}^n := \mathbf{P}^n_{\operatorname{Spec}\mathbb{Z}}$, and note that clearly $\mathbf{P}^n_S = \mathbf{P}^n \circ U_S$, where U_S is the forgetful functor. If we can prove that $\mathbf{P}^n$ is representable by $\mathbb{P}^n_{\mathbb{Z}}$, with universal object the pair $\mathbb{P}^n_{\mathbb{Z}}$ and $[\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1),(x_0,\dots,x_n)]$ then the rest of the claim follows from Lemma 2.5.3.

We define a natural transformation $F : \mathbf{P}^n \to h_{\mathbb{P}^n_{\mathbb{Z}}}$. Fix a scheme X, and let $(\mathscr{L},(s_0,\dots,s_n))$ be a base point free linear system. Since the system is base point free, we have that the sets X_{s_i} where each s_i are invertible cover X. We first define maps $f_i : X_{s_i} \to U_{x_i} \subset \mathbb{P}^n_{\mathbb{Z}}$. By Proposition 2.1.2 (or rather Example 2.5.1) it suffices to provide a ring homomorphism:

$$\mathbb{Z}[x_1/x_i,\dots,x_n/x_i] \longrightarrow \mathscr{O}_{X_{s_i}}(X_{s_i})$$

Note that $\mathscr{L}|_{X_{s_i}}$ is trivial by Corollary 5.7.1, so let $\psi_i : \mathscr{L}|_{X_{s_i}} \to \mathscr{O}_{X_{s_i}}$ be the isomorphism which identifies 1 with s_i. We denote by a_{ij} the element $(\psi_i)_{X_{s_i}}(s_j|_{X_{s_i}})$. Define our morphism f_i to be the one induced by:

$$\begin{aligned}\mathbb{Z}[x_1/x_i,\dots,x_n/x_i] &\longrightarrow \mathscr{O}_{X_{s_i}}(X_{s_i})\\ x_j/x_i &\longmapsto a_{ij}\end{aligned}$$

Let $\imath_i : U_{x_i} \to \mathbb{P}^n_{\mathbb{Z}}$, then we want to glue the $\imath_i \circ f_i : X_{s_i} \to \mathbb{P}^n_{\mathbb{Z}}$ together. Denote by X_{ij} the intersection $X_{s_i} \cap X_{s_j}$, then we need to check that $\imath_i \circ f_i|_{X_{ij}} = \imath_j \circ f_j|_{X_{ij}}$. Denote by g_{ij} the restricted morphism $\imath_i \circ f_i|_{X_{ij}}$, then we claim that $g_{ij}(X_{ij}) \subset U_{x_i \cdot x_j} \subset \mathbb{P}^n_{\mathbb{Z}}$. Indeed, note that a priori g_{ij} has image in U_{x_i}, and comes from the ring map:

$$\begin{aligned}\phi_{ij} : \mathbb{Z}[x_1/x_i,\dots,x_n/x_i] &\longrightarrow \mathscr{O}_{X_{ij}}(X_{ij})\\ x_k/x_i &\longmapsto a_{ik}|_{X_{ij}}\end{aligned}$$

However, the section $a_{ij}|_{X_{ij}}$ is invertible as it does not vanish on X_{s_j} by definition. Now, recall that:

$$g_{ij}(x) = \phi_{ij}^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$$

where π_x is the natural map $\mathscr{O}_{X_{ij}}(X_{ij}) \to \mathscr{O}_{X,x}$. Now, for all $g_{ij}(x)$, we know that $x_j/x_i \notin g_{ij}(x)$ as that would imply that $a_{ij,x} \in \mathfrak{m}_x$ which is impossible as $a_{ij}|_{X_{ij}}$ is invertible on X_{s_j}. It follows that $g_{ij}(X_{ij}) \subset U_{x_i \cap x_j}$ as desired. In particular, viewing the morphism g_{ij} as a map $X_{ij} \to U_{x_i x_j}$, we have that g_{ij} must come from the ring homomorphism:

$$\begin{aligned}\phi'_{ij} : (\mathbb{Z}[x_1,\dots,x_n]_{x_i x_j})_0 &\longrightarrow \mathscr{O}_{X_{ij}}(X_{ij})\\ (x_k \cdot x_l)/(x_i x_j) &\longmapsto a_{ik} \cdot a_{il}|_{X_{ij}} \cdot a_{ij}|_{X_{ij}}^{-1}\end{aligned}$$

Similarly, we denote by a_{jk} the element $(\psi_j)_{X_{s_j}}(s_k|_{X_{s_j}})$, then $a_{ji}|_{X_{ij}}$ is invertible by the same argument, and g_{ji} must from the ring homomorphism:

$$\begin{aligned}\phi'_{ji} : (\mathbb{Z}[x_1,\dots,x_n]_{x_i x_j})_0 &\longrightarrow \mathscr{O}_{X_{ij}}(X_{ij})\\ (x_k \cdot x_l)/(x_i x_j) &\longmapsto a_{jk} \cdot a_{jl}|_{X_{ij}} \cdot a_{ji}|_{X_{ij}}^{-1}\end{aligned}$$

Both $(\psi_i)_{X_{ij}}$ and $(\psi_j)_{X_{ij}}$ are isomorphisms $\mathscr{L}(X_{ij}) \to \mathscr{O}_{X_{ij}}(X_{ij})$. We claim that there exists a $b \in \mathscr{O}_{X_{ij}}(X_{ij})^\times$ such that $(\psi_j)_{X_{ij}}(s) = b \cdot (\psi_i)_{X_{ij}}(s)$ for all $s \in \mathscr{L}(X_{ij})$. Indeed, we have that $(\psi_i)_{X_{ij}}(s_j) = a_{ij}|_{X_{ij}}$, whilst $(\psi_j)_{X_{ij}}(s_j|_{X_{ij}}) = 1$, hence we claim that:

$$(\psi_j)_{X_{ij}}(s) = a_{ij}|_{X_{ij}}^{-1} \cdot (\psi_i)_{X_{ij}}(s)$$

Indeed, since every element $s \in \mathscr{L}(X_{ij})$ can be written uniquely as $s_j|_{X_{ij}} \cdot c$ for some $c \in \mathscr{O}_{X_{ij}}(X_{ij})$, we can write the write hand side of the above equality as:

$$\begin{aligned} a_{ij}|_{X_{ij}}^{-1} \cdot (\psi_i)_{X_{ij}}(s) =& a_{ij}|_{X_{ij}}^{-1} \cdot (\psi_i)_{X_{ij}}(s_j|_{X_{ij}\cdot c}) \\ =& a_{ij}|_{X_{ij}}^{-1} \cdot a_{ij}|_{X_{ij}} \cdot c \\ =& c \end{aligned}$$

while the left hand satisfies $(\psi_j)_{X_{ij}}(s) = c$ by definition. It follows:

$$\begin{aligned} a_{jk} \cdot a_{jl}|_{X_{ij}} \cdot a_{ji}|_{X_{ij}}^{-1} =& (\psi_j)_{X_{ij}}(s_k|_{X_{ij}}) \cdot (\psi_j)_{X_{ij}}(s_l|_{X_{ij}}) \cdot (\psi_j)_{X_{ij}}(s_i|_{X_{ij}}^{-1}) \\ =& \left[a_{ij}|_{X_{ij}}^{-1} \cdot (\psi_i)_{X_{ij}}(s_k|_{X_{ij}})\right] \cdot \left[a_{ij}|_{X_{ij}}^{-1} \cdot (\psi_i)_{X_{ij}}(s_l|_{X_{ij}})\right] \cdot a_{ij}|_{X_{ij}} \\ =& a_{ik} \cdot a_{il}|_{X_{ij}} \cdot a_{ij}|_{X_{ij}}^{-1} \end{aligned}$$

We have shown that the ring maps ϕ'_{ij} and ϕ'_{ji} agree, hence so $g_{ij} = g_{ji}$, and we obtain a global scheme morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$.

Suppose that $(\mathscr{L}', (t_0, \ldots, t_n))$ is another linear system isomorphic to $(\mathscr{L}, (s_0, \ldots, s_n))$. We need to check that the map $g : X \to \mathbb{P}^n_{\mathbb{Z}}$, obtained from applying the above procedure to $(\mathscr{L}', (t_0, \ldots, t_n))$, is equal to f. Let ξ be the isomorphism of linear systems, then note that $X_{s_i} = X_{t_i}$ as ξ is an isomorphism and sends s_i to t_i. It thus suffices to show that $f|_{X_i} = g|_{X_i}$, and since these morphisms have image in the affine scheme U_{x_i} we can check that the induced ring homomorphisms:

$$\phi_{s_i}, \phi_{t_i} : \mathbb{Z}[x_1/x_i, \ldots, x_n/x_i] \longrightarrow \mathscr{O}_{X_{s_i}}(X_{s_i})$$

are equal. Let ψ_{s_i} and ψ_{t_i} be the isomorphisms sending $s_i|_{X_{s_i}}$ to 1, and $t_i|_{X_{s_i}}$ to 1. Then we have that:

$$\phi_{s_i}(x_j/x_i) = \psi_{s_i}(s_j|_{X_{s_i}}) \qquad \text{and} \qquad \phi_{t_i}(x_j/x_i) = \psi_{t_i}(t_j|_{X_{s_i}})$$

Note that we can write $t_j = \xi_X(s_j)$, hence:

$$\phi_{t_i}(x_j/x_i) = \psi_{t_i} \circ \xi_{X_{s_i}}(s_j|_{X_{s_i}})$$

Since $\psi_{t_i} \circ \xi_{X_{s_i}}(s_i|_{X_{s_i}}) = \psi_{t_i}(t_i|_{X_{s_i}}) = 1$ we conclude that $\psi_{t_i} \circ \xi_{X_{s_i}} = \psi_{s_i}$, hence $\phi_{s_i} = \phi_{t_i}$ as desired.

We thus define the set map $F_X : \mathbf{P}^n(X) \to h_{\mathbb{P}^n_{\mathbb{Z}}}(X)$ by sending a class representative of $[(\mathscr{L}, (s_0, \ldots, s_n))]$ to the morphism f obtained by applying the above procedure to said class representative. As we have just shown, such a map is well defined. We check that this map is injective; suppose that $(\mathscr{L}, (s_0, \ldots, s_n))$ and $(\mathscr{L}', (t_0, \ldots, t_n))$ induce the same map f; then we need to check that there is an isomorphism $\xi : \mathscr{L} \to \mathscr{L}'$ which sends s_i to t_i. Note that by construction we have that $X_{s_i} \subset f^{-1}(U_{x_i})$; let $x \in f^{-1}(U_{x_i})$, then since the linear system is base point free there is a k such that $x \in X_{s_k}$. Let $\psi_{s_k} : \mathscr{L}|_{X_{s_k}} \to \mathscr{O}_{X_{s_k}}$ be the isomorphism sending $s_k|_{X_1}$ to 1, then the ring map:

$$\begin{aligned} \phi_{s_k} : \mathbb{Z}[x_1/x_k, \ldots, x_n/x_k] &\longrightarrow \mathscr{O}_{X_{s_k}}(X_{s_k}) \\ x_l/x_k &\longmapsto (\psi_{s_k})_{X_{s_k}}(s_l|_{X_{s_k}}) \end{aligned}$$

induces the morphism $f|_{X_{s_k}}$. We can thus identify $f(x)$ with the prime ideal $\phi_k^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$ in U_{x_j}. Since this must land in U_{x_i} as well, we must have that $x_i/x_k \notin \phi_k^{-1}(\pi_x^{-1}(\mathfrak{m}_x))$, implying that $(\psi_k)_{X_{s_k}}(s_i|_{X_{s_k}})_x \notin \mathfrak{m}_x$. Since ψ_k is an $\mathscr{O}_{X_{s_k}}$ module morphism it follows that $s_{i,x} \notin \mathfrak{m}_x \cdot \mathscr{L}_x$ hence $f^{-1}(U_{x_i}) = X_{s_i}$. Since this applies equally well for X_{t_i} we know that $X_{t_i} = X_{s_i}$. Now the fact the linear systems induce the same map implies That

$$(\psi_{s_i})_{X_{s_i}}(s_l|_{X_{s_i}}) = (\psi_{t_i})_{X_{s_i}}(t_l|_{X_{s_i}})$$

In particular, it follows that $\psi_{t_i} \circ \psi_{s_i}^{-1}$ is an isomorphism $\xi_i : \mathscr{L}|_{X_{s_i}} \to \mathscr{L}'|_{X_{s_i}}$ which sends $s_j|_{X_{s_i}}$ to $t_j|_{X_{s_i}}$ for all j. If we can show that $\xi_i|_{X_{s_i} \cap X_{s_j}} = \xi_j|_{X_{s_i} \cap X_{s_j}}$ then we are done, but this obvious because $s_i|_{X_{s_i} \cap X_{s_j}}$ is a trivializing section for $\mathscr{L}|_{X_{s_i} \cap X_{s_j}}$ and both $\xi_i|_{X_{s_i} \cap X_{s_j}}$ and $\xi_j|_{X_{s_i} \cap X_{s_j}}$ send $s_i|_{X_{s_i} \cap X_{s_j}}$ to

$t_i|_{X_{s_i}\cap X_{s_j}}$. Since they are $\mathscr{O}_X$ module isomorphisms, they must then agree on all over section over every open set of $X_{s_i} \cap X_{s_j}$.

Now let f be a scheme morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$. We take out line bundle to be $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$, and our linear system to be $(f^*x_0, \ldots, f^*x_n)$. We first claim that $f^{-1}(U_i) = X_{f^*x_i}$. Corollary 5.4.6 implies that for any $x \in X$:

$$(f^*x_i)(x) = x_i(f(x)) \otimes 1 \in \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)|_{f(x)} \otimes_{k_{f(x)}} k_x$$

Let α_i be the isomorphism $\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)|_{U_{x_i}} \to \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}|_{U_{x_i}}$. We then have that $\alpha_{i,f(x)}$, and the field extension $\bar{f}_x : k_{f(x)} \to k_x$ induces an isomorphism:

$$\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)|_{f(x)} \otimes_{k_{f(x)}} k_x \longrightarrow k_{f(x)} \otimes_{f(x)} k_x \longrightarrow k_x$$

given by:

$$x_i(f(x)) \otimes 1 \longmapsto \xi_{i,f(x)}(x_i(f(x))) \otimes 1 \longmapsto \bar{f} \circ \xi_{i,f(x)}(x_i(f(x)))$$

It follows tht $x_i(f(x)) \otimes 1$ is zero if and only if $\bar{f} \circ \xi_{i,f(x)}(x_i(f(x)))$ is zero in k_x. However this is zero if and only if $x_i(f(x))$ is zero, because both maps are injective. If $f(x) = \mathfrak{p}$ for some homogeneous prime, then $x_i(f(x)) = 0$ if and only if $x_i \in \mathfrak{p}$. It follows that $f^*x_i(x) \neq 0$ if and only if $f(x) \in U_{x_i}$, hence $X_{f^*x_i} = f^{-1}(U_{x_i})$. We can now view $f|_{X_{f^*x_i}}$ a morphism $X_{f_i} \to U_{x_i}$, meaning it is induced by the map on global sections:

$$f^\sharp_{U_{x_i}} : \mathbb{Z}[x_0/x_i, \ldots, x_n/x_i] \longrightarrow \mathscr{O}_{X_{f^*x_i}}(X_{f^*x_i})$$

It therefore suffices to show that the isomorphism $\psi_i : f^*\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(X_{f^*x_i}) \to \mathscr{O}_{X_{f^*x_i}}(X_{f^*x_i})$, induced by sending $f^*x_i|_{X_{f^*x_i}}$ to one, satisfies:

$$\psi_i(f^*x_j|_{X_{f^*x_i}}) = f^\sharp_{U_{x_i}}(x_j/x_i)$$

Note that on stalks, we have that:

$$\begin{aligned} \psi_{i,x} : \mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)_f(x) \otimes_{\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}},f(x)}} \mathscr{O}_{X,x} &\longrightarrow \mathscr{O}_{X,x} \\ s \otimes t &\longmapsto f_x \circ \alpha_{i,x}(s) \cdot t \end{aligned}$$

We note that $(f^*x_j)_x = x_j/1 \otimes 1$, where $x_j/1 \in \mathbb{Z}[x_0, \ldots, x_n](1)_{(f(x))}$.[143] The isomorphism to $\mathbb{Z}[x_0, \ldots, x_n]_{(f(x))}$ then sends $x_j/1$ to x_j/x_i. It follows by Lemma 1.3.1 that:[144]

$$\psi_i(f^*x_j|_{X_{f^*x_i}})_x = \psi_{i,x}((f^*x_j)_x) = f_x(x_j/x_i) = f^\sharp_{U_{x_i}}(x_j/x_i)_x$$

Since these agree on stalks, we have the desired equality on sections. It follows that the map induced by $(f^*\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1), (f^*x_0, \ldots, f^*x_n))$ is equal to f, hence F_X is surjective, and thus an isomorphism.

To show that F is a natural transformation, we need to show that for any $g : X \to Y$ the following square commutes:

$$\begin{tikzcd} \mathbf{P}^n(Y) \arrow[r, "F_X"] \arrow[d, "\mathbf{P}^n(g)"'] & h_{\mathbb{P}^n_{\mathbb{Z}}}(Y) \arrow[d, "g^*"] \\ \mathbf{P}^n(X) \arrow[r, "F_Y"] & h_{\mathbb{P}^n_{\mathbb{Z}}}(X) \end{tikzcd}$$

Let $[(\mathscr{L}, (s_0, \ldots, s_n))] \in \mathbf{P}^n(Y)$, and f be the induced map. Then we need to show that the map h induced by $(g^*\mathscr{L}, (g^*s_0, \ldots, g^*s_n))$ is equal to $f \circ g$. The same argument as before demonstrates that $g^{-1}(Y_{s_i}) = X_{g^*s_i}$, so we have that $f \circ g|_{X_{g^*s_i}}$ can be written as the composition:

$$X_{g^*s_i} \longrightarrow Y_{s_i} \longrightarrow U_{x_i}$$

[143] Here we treating $f(x)$ as a homogeneous prime ideal of $\mathbb{Z}[x_0, \ldots, x_n]$.

[144] Abuse of notation alert! We are using the notation x_j/x_i to refer to elements in $\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(U_{x_i})$, and those in the stalk, with the hope that the astute reader understands what is meant.

This morphism is then induced by the ring homomorphism

$$\begin{aligned}\mathbb{Z}[x_1/x_i, \dots, x_n/x_i] &\longrightarrow \mathscr{O}_{X_{g^*s_i}}(X_{g^*s_i})\\ x_j/x_i &\longmapsto g^\sharp_{X_{g^*s_i}} \circ \psi_{s_i}(s_j|_{Y_{s_i}})\end{aligned}$$

Meanwhile, the morphism h is induced by the ring homomorphism:

$$\begin{aligned}\mathbb{Z}[x_1/x_i, \dots, x_n/x_i] &\longrightarrow \mathscr{O}_{X_{g^*s_i}}(X_{g^*s_i})\\ x_j/x_i &\longmapsto \psi_{g^*s_i}(g^*s_j|_{X_{g^*s_i}})\end{aligned}$$

Now on stalks, we have that $(g^*s_j)_x = s_{j,g(x)} \otimes 1 \in \mathscr{L}_{g(x)} \otimes_{\mathscr{O}_{Y_{g(x)}}}$, and that $(\psi_{g^*z_i})_x$ takes $s \otimes t$ to $g_x \circ \psi_{s_i,x}(s) \cdot t$. It follows that:

$$\psi_{g^*s_i}(g^*s_j|_{X_{g^*s_i}})_x = (\psi_{g^*z_i})_x((g^*s_j)_x) = g_x \circ \psi_{s_i,x}(s_{j,x}) = (g^\sharp_{X_{g^*s_i}} \circ \psi_{s_i}(s_j|_{Y_{s_i}}))_x$$

Since the two sections agree on stalks, we have that they are equal, and so the ring homomorphisms are equal as they are determined fully by where x_j/x_i goes. We thus have that $f \circ g$ and h agree on an over cover and so must agree implying the claim.

The work in Lemma 2.4.4 implies that the pair consisting of the scheme $\mathbb{P}^n_{\mathbb{Z}}$, and isomorphism class $F^{-1}_{\mathbb{P}^n_{\mathbb{Z}}}(\mathrm{Id})$ is the universal object. Clearly, the bundle $(\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}, (x_0, \dots, x_n))$ induces the identity morphism on $\mathbb{P}^n_{\mathbb{Z}}$, hence $F^{-1}_{\mathbb{P}^n_{\mathbb{Z}}}(\mathrm{Id}) = [(\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}, (x_0, \dots, x_n))]$. The second part of the claim now follows again from Lemma 2.5.3.

□

Example 5.8.2. Recall Lemma 3.12.6 where we identified the k points of $\mathbb{P}^n_{\mathbb{Z}}$ with $(k^{n+1} \smallsetminus 0)/k^\times$. We can now provide a much slicker proof this fact. Note that up to isomorphism there is only one line bundle over $\operatorname{Spec} k$, hence our morphisms from $\operatorname{Spec} k$ to $\mathbb{P}^n_{\mathbb{Z}}$ are in bijection with $n+1$ tuples $(a_0, \dots, a_n)$ not all zero, up to isomorphisms of k^{n+1} fixing coordinate positions. Every such isomorphism is given by scalar multiplication by an element of $k^\times$, hence we have that every morphism from $\operatorname{Spec} k \to \mathbb{P}^n_{\mathbb{Z}}$ must come from an equivalence class $[a_0, \dots, a_n] \in (k^{n+1} \smallsetminus 0)/k^\times$.

Note that this identification holds with any base scheme S so long as $\operatorname{Spec} k$ admits the structure of an S-scheme. Moreover note that $\mathrm{Hom}_{\mathrm{Sch}/A}(\operatorname{Spec} A, \mathbb{P}^n_A)$ contains morphisms coming from elements of $(A^{n+1} \smallsetminus 0)/A^\times$[145] but not every morphism need be of this form as $\operatorname{Spec} A$ may admit non trivial line bundles. This is even more so true for arbitrary base schemes S.

Example 5.8.3. To specify a scheme morphism $\mathbb{P}^n_S \to \mathbb{P}^m_S$ it suffices to specify $m+1$ non simultaneously vanishing sections of $\mathscr{O}_{\mathbb{P}^n_S}(d)$ for some d. If $\{U_i = \operatorname{Spec} A_i\}$ is an affine open cover of S, then $\mathbb{P}^n_{A_i}$ is an open cover[146] for $\mathbb{P}^n_S$. We have that:

$$\mathscr{O}_{\mathbb{P}^n_S}(d)(\mathbb{P}^n_{A_i}) = \mathscr{O}_{\mathbb{P}^n_{A_i}}(d)(\mathbb{P}^n_{A_i}) = (A_i[x_0, \dots, x_n])_d$$

It follows that global sections of $\mathscr{O}_{\mathbb{P}^n_S}(d)$ are sequences (p_i) of degree d polynomials in $n+1$ variables with coefficients in A_i which agree on overlaps. In other words, global sections are degree d polynomials with coefficients in $\mathscr{O}_S(S)$:

$$\mathscr{O}_{\mathbb{P}^n_S}(d)(\mathbb{P}^n_S) = (\mathscr{O}_S(S)[x_0, \dots, x_n])_d$$

To provide a scheme morphism, it thus suffices to specify $n+1$ non simultaneously vanishing degree d polynomials in $n+1$ variables, with coefficients in $\mathscr{O}_S(S)$. When $S = \operatorname{Spec} k$, these are exactly what we would expect our morphisms to be. Note that these are not all of the morphisms, but just the morphisms coming from line bundles of the prescribed form.

[145] These correspond to base point free linear systems of the form $(\mathscr{O}_{\operatorname{Spec} A}, (a_0, \dots, a_n))$.
[146] Though not affine!

5.9 Vector Bundles and Grassmannians

5.10 The Total Space of a Vector Bundle

5.11 The Projective Bundle and Projective Morphisms

5.12 Some Commutative Algebra: Kähler Differentials

5.13 The Sheaf of Differentials

Dimension and Smoothness

6.1 Some Commutative Algebra: Krull Dimension

Let M be a smooth manifold, and recall that the dimension of M (if M is of pure dimension that is) is defined to be the dimension of the Euclidean space it is locally homeomorphic to. That is, if U is an open neighborhood of $p \in M$ and $\phi : U \to V \subset \mathbb{R}^n$, is a coordinate chart, then the dimension of M is n. In particular, we also have that the dimension as a vector space over $\mathbb{R}$ of the tangent space at a point is equal to the dimension of M for al $p \in M$.

We wish to develop a theory of dimension for schemes which mimics the above behavior in the category of smooth manifolds; that is for 'nice enough' schemes [147] we want our notion of dimension to be determined by the dimension of an open affine, as well as by the stalk at a closed point $x \in X$. In particular, we will also want single point schemes to have dimension zero, and our classical examples, $\mathbb{P}^n_k$ and $\mathbb{A}^n_k$, to have dimension n.

Since the category of affine schemes is anti-equivalent to the category of commutative rings, we will first develop the dimension theory for commutative rings.

Definition 6.1.1. Let A be a commutative ring; a strictly increasing finite chain of prime ideals:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

has **length n**[148]. Let $L(A) \subset \mathbb{N}$ be the ordered set consisting of the lengths of all finite increasing chains of prime ideals; we define the **Krull dimension** of a commutative ring A, denoted $\dim A$, to be $\sup L(A)$ if it exists, and to be infinite if there is no least upper bound[149].

One might quickly jump to the conclusion that all rings of finite dimension are Noetherian, or, equivalently, that any non-Noetherian ring will have infinite dimension. While the study of Krull dimension of Noetherian rings will prove a fruitful endeavor, as the next example shows, the former is not the case.

Example 6.1.1. Let $A = k[x_0, x_1, \dots,]/\left\langle x_0^2, x_1^2, \dots \right\rangle$, then we claim that A contains only one prime ideal. Note that A is clearly not Noetherian. The prime ideals of A are in bijection with prime ideals containing $I = \left\langle x_0^2, x_1^2, \dots \right\rangle$. That is every prime ideal can be identified with a prime ideal of A lying in the closed set $\mathbb{V}(I) \subset \operatorname{Spec} A$. We have that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, and that each $x_i \in \sqrt{I}$ as $x_i^2 \in I$. It follows that $\langle x_0, x_1, \dots \rangle \subset \sqrt{I}$, so $\sqrt{I} = \langle x_0, x_1, \dots, \rangle$ and is thus maximal. Therefore, $V(I)$ consists of a single point, and thus $A = k[x_0, x_1, \dots,]/\left\langle x_0^2, x_1^2, \dots \right\rangle$ has one prime ideal, so $\dim A = 0$.

We would also like to show the existence of a Noetherian ring of infinite dimension. However, the construction of such a ring was historically an elusive endeavor, and requires more machinery than we have on hand. Therefore, such an example appears later on in the section[150], but we stress that there Noetherian does not imply finite dimensional.

Example 6.1.2. We want to determine the dimension of $\mathbb{Z}$. Every non zero prime ideal in $\mathbb{Z}$ is is maximal, hence the only prime that can possible be contained in a non zero prime is the zero ideal. It

[147] To be defined later.
[148] We are essentially counting number of inclusions, not the number of prime ideals.
[149] Note that since $L(A) \subset \mathbb{N}$, if $\sup L(A)$ exists, then $\sup A \in L(A)$.
[150] See Example 6.1.3.

follows that every chain of increasing prime ideals is of one of two forms:

$$\langle 0\rangle \qquad \text{or} \qquad \langle 0\rangle \subset p\mathbb{Z}$$

where p is prime. It follows that $L(\mathbb{Z}) = \{0, 1\}$ which has least upper bound 1 hence $\dim \mathbb{Z} = 1$.

In particular, if A is a PID, then every non-zero prime ideal is maximal, so $L(A) = \{0, 1\}$ hence $\dim A = 1$. Note that for any field $k[x]$ is a PID, hence $\dim k[x] = 1$.

In order to determine the dimension of more complicated rings it will be convenient to determine equivalent definitions of dimension.

Definition 6.1.2. Let $\mathfrak{p} \in \operatorname{Spec} A$, and let $L(\mathfrak{p})$ be the set consisting of the lengths of all strictly increasing chains of prime ideals ending with $\mathfrak{p}$. We define the **height of** $\mathfrak{p}$, denoted $\operatorname{ht}(\mathfrak{p})$, to be $\sup L(\mathfrak{p})$ if it exists, and infinite otherwise.

We have the following characterization of height zero prime ideals:

Lemma 6.1.1. *Let $\mathfrak{p} \in \operatorname{Spec} A$, then $\operatorname{ht}(\mathfrak{p}) = 0$ if and only if $\mathfrak{p}$ is minimal*[151] *over $\langle 0\rangle$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal over 0, then by definition, if $\mathfrak{q} \subset \mathfrak{p}$, we have that $\mathfrak{q} = \mathfrak{p}$, hence the only chain of prime ideals ending with $\mathfrak{p}$ is the trivial chain $\mathfrak{p}$. It follows that $\operatorname{ht}(\mathfrak{p}) = 0$.

Let $\operatorname{ht}(\mathfrak{p}) = 0$, and suppose $\mathfrak{q} \subset \mathfrak{p}$. If this inclusion is strict, then we have that $\operatorname{ht}(\mathfrak{p}) \geq 1$, hence we must have that $\mathfrak{q} = \mathfrak{p}$ implying that $\mathfrak{p}$ is minimal over $\langle 0\rangle$. □

While we have used localization throughout this text, we have not yet had need to determine what $\operatorname{Spec} S^{-1}A$ is in terms of of prime ideals of A. We do so now:

Lemma 6.1.2. *Let S be a multiplicatively closed set, then their exists a bijection between $\operatorname{Spec} S^{-1}A$ and prime ideals of A such that $S \cap \mathfrak{p} = \emptyset$.*

Proof. This is entirely analogous to Proposition 1.1.3. We define the maps, and leave the rest of the proof as an exercise to the reader.

Let $\mathfrak{P}$ denote the set of prime ideals of A such that $S \cap \mathfrak{p} = \emptyset$; we define a set map:

$$\begin{aligned} f : \mathfrak{P} &\longrightarrow \operatorname{Spec} S^{-1}A \\ \mathfrak{p} &\longmapsto \langle \pi(\mathfrak{p})\rangle \end{aligned}$$

where $\pi : A \to S^{-1}A$ is the localization map. For this map to be well defined, we need to check that this is a prime ideal. Let $a/s, b/t \in S^{-1}A$ such that $ab/(ts) \in f(\mathfrak{p})$, it follows that:

$$\frac{ab}{ts} = \frac{p}{u}$$

for some $u \in S$, and some $p \in \mathfrak{p}$. There then exists an element $v \in S$ such that:

$$v(uab - pts) = 0$$

It follows that $abuv = ptsv \in \mathfrak{p}$, so $ab \in \mathfrak{p}$, hence either a or $b \in \mathfrak{p}$ implying either a/s or $b/t \in f(\mathfrak{p})$, so $f(\mathfrak{p})$ is prime.

We define an inverse map by:

$$\begin{aligned} g : \operatorname{Spec} S^{-1}A &\longrightarrow \mathfrak{P} \\ \mathfrak{q} &\longmapsto \pi^{-1}(\mathfrak{q}) \end{aligned}$$

This is clearly prime, so we need to check that $g(\mathfrak{q}) \cap S = \emptyset$. Suppose other wise, then there is some $s \in S$ such that $s \in \pi^{-1}(\mathfrak{q})$. It follows that $s/1 \in \pi(\pi^{-1}(\mathfrak{q})) \subset \mathfrak{q}$, implying that $\mathfrak{q} = S^{-1}A$ a contradiction.

□

[151] As in Theorem 3.4.3

Note that these maps are inclusion preserving, so if $\mathfrak{p} \subset \mathfrak{q} \in \mathfrak{P}$, then $f(\mathfrak{p}) \subset f(\mathfrak{q})$, and similarly for g. With the above characterization we can show the following:

Proposition 6.1.1. *Let* $\mathfrak{p} \in \operatorname{Spec} A$*, then* $\operatorname{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$.

Proof. We first show that $\operatorname{ht}(\mathfrak{p})$ is infinite if and only if $\dim A_{\mathfrak{p}}$ is infinite. Suppose that $\operatorname{ht}(\mathfrak{p})$ is infinite, then for any strictly increasing finite chain of prime ideals ending with $\mathfrak{p}$ of length n:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

we can find a sequence:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{p}$$

such that $m > n$. Each of these ideals is contained in $\mathfrak{p}$, hence $(A \setminus \mathfrak{p}) \cap \mathfrak{q}_i = \emptyset$, and similarly for each $\mathfrak{p}_i$. It follows that

$$f(\mathfrak{p}_0) \subset f(\mathfrak{p}_1) \subsetneq \cdots \subsetneq f(\mathfrak{p}_n) = \mathfrak{m}_{\mathfrak{p}}$$

is a chain of prime ideals of length n in $A_{\mathfrak{p}}$. Here f is the map from the preceding lemma, and $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal in $A_{\mathfrak{p}}$.

Suppose that $\dim A_{\mathfrak{p}}$ is finite, then $\sup L(A_{\mathfrak{p}})$ exists, then there exists an $n \in L(A)$, such that that for all $m \in L$, we have that $n \geq m$. In particular, n corresponds to a chain of prime ideals:

$$\mathfrak{q}'_0 \subsetneq \cdots \subsetneq \mathfrak{q}'_n = \mathfrak{m}_{\mathfrak{p}}$$

where we must end with $\mathfrak{m}_{\mathfrak{p}}$ as otherwise there exists a chain of length $n+1$ due to the fact that $\mathfrak{m}_{\mathfrak{p}}$ contains every ideal of $A_{\mathfrak{p}}$. It follows that:

$$g(\mathfrak{q}'_0) \subset \cdots \subsetneq g(\mathfrak{q}'_n) = \mathfrak{p}$$

is a chain of length n. However, since $\operatorname{ht}(\mathfrak{p})$ is infinite, we can take $m > n$ and find a chain:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{p}$$

Then:

$$f(\mathfrak{q}_0) \subset f(\mathfrak{q}_1) \subsetneq \cdots \subsetneq f(\mathfrak{q}_m) = \mathfrak{m}_{\mathfrak{p}}$$

is a chain of prime ideals in $A_{\mathfrak{p}}$ of length $m > n$, hence there exists $m \in L(A)$ such that $m > n$ a contradiction. It follows that if $\operatorname{ht}(\mathfrak{p})$ is infinite, then $\dim A_{\mathfrak{p}}$ is infinite as well.

Now suppose that $\dim A_{\mathfrak{p}}$ is infinite, then as before, given $m \in L(A_{\mathfrak{p}})$, we can always find an $n \in L(A_{\mathfrak{p}})$ such that $n > m$. Suppose that $\operatorname{ht}(\mathfrak{p})$ is finite, and let $n = \sup L(\mathfrak{p})$. This corresponds to a chain of prime ideals ending with $\mathfrak{p}$ of length n:

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

It follows that:

$$f(\mathfrak{p}_0) \subsetneq \cdots \subsetneq f(\mathfrak{p}_n) = \mathfrak{m}_{\mathfrak{p}}$$

is a chain of prime ideals of length n in $A_{\mathfrak{p}}$. Since $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$, and $\dim A_{\mathfrak{p}}$ is infinite, we have that there exists a chain of prime ideals of length $m > n$ ending with $m_{\mathfrak{p}}$:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{m}_{\mathfrak{q}}$$

so:

$$g(\mathfrak{q}_0) \subsetneq \cdots \subsetneq g(\mathfrak{q}_m) = \mathfrak{p}$$

is a chain of prime ideals in A terminating with $\mathfrak{p}$ of length m. It follows that $m \in L(\mathfrak{p})$, and is greater than n hence we must have that no such n is complete.

Now suppose that $\dim A_{\mathfrak{p}}$ is finite, then by the above we equivalently have that $\operatorname{ht}\mathfrak{p}$ is finite as well. Let $\dim A_{\mathfrak{p}} = n$ and $\operatorname{ht}\mathfrak{p} = m$. Suppose that:

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_m = \mathfrak{p}$$

is a strictly in increasing chain of prime ideals of length m terminating with $\mathfrak{p}$, then we have that:

$$f(\mathfrak{p}_0) \subsetneq \cdots \subsetneq f(\mathfrak{p}_m) = \mathfrak{m}_{\mathfrak{p}}$$

is a strictly increasing chain of prime ideals of length m in $A_{\mathfrak{p}}$. It follows that $m \leq n$. Now let:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$$

be a strictly increasing chain of prime ideals of length n in $A_{\mathfrak{p}}$. We know that $\mathfrak{q}_n = \mathfrak{m}_{\mathfrak{p}}$, as otherwise there exists a chain of length $n+1$. It follows that:

$$g(\mathfrak{q}_0) \subsetneq \cdots \subsetneq g(\mathfrak{q}_n) = \mathfrak{p}$$

is a strictly increasing chain of prime ideals in A terminating with $\mathfrak{p}$ of length n. It follows that $n \leq m$, hence we must have that $m = n$ implying the claim. □

Now let $H(A)$ be the set defined by:

$$H(A) = \{\operatorname{ht}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec} A\}$$

where if $\operatorname{ht}(\mathfrak{p})$ is infinite, we replace it with the symbol ∞. Note that $\mathbb{N} \cup \{\infty\}$ carries a total order by declaring that $\infty > m$ for all $m \in \mathbb{N}$. It follows that $H(A) \subset \mathbb{N} \cup \{\infty\}$ carries a natural ordering, and that $\sup H(A) = \infty$ if and only if there exists a prime ideal of infinite height, or $\sup H(A)$ contains only prime ideals of finite height, but $H(A)$ is unbounded as a subset of $\mathbb{N}$. Our next result will characterize the Krull dimension of a ring in terms of the heights of prime ideals.

Proposition 6.1.2. *The Krull dimension of A is finite if and only if $\sup H(A) \neq \infty$. In particular if $\sup H(A) \neq \infty$, or $\dim A$ is finite, then $\dim A = \sup H(A)$.*

Proof. For the first claim, we will instead show the contrapositive; i.e. that $\dim A$ is infinite if and only if $\sup H(A) = \infty$.

Suppose that $\sup H(A) = \infty$, then there either exists a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ such that $\operatorname{ht}(\mathfrak{p})$ is infinite, or every prime ideal of A has finite height, but $H(A)$ is infinite. In the first case, it follows that that for all $n \in L(A)$ we can find an increasing chain of prime ideals of length $m > n$ which terminates with $\mathfrak{p}$, so $\dim A$ cannot be finite. In the latter case, it follows that for any $n \in L(A)$ we can find a $\mathfrak{q} \in \operatorname{Spec} A$ such that $\operatorname{ht}(\mathfrak{q}) > n$, but then $\operatorname{ht}(\mathfrak{q}) \in L(A)$ so $\sup L(A)$ does not exist, and $\dim A$ cannot be finite.

Now suppose $\dim A$ is infinite. For all $n \in H(A)$, we can find an increasing chain of prime ideals:

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m = \mathfrak{q}$$

where $m > n$. It follows that $\operatorname{ht}(\mathfrak{q})$ is either infinite, in which case $\sup H(A) = \infty$ and we are done, or $\operatorname{ht}(\mathfrak{q})$ is finite but greater than n. In the latter case, since n was arbitrary, we have that $H(A)$. so by definition $\sup H(A) = \infty$.

To prove the second claim, suppose that either $\sup H(A) \neq \infty$, or $\dim A$ is finite. In both cases, by the first statement we have that $\sup H(A) = m$ and $\dim A = n$ for $m, n \in \mathbb{N}$. Now if $\sup H(A) = m$, we have that that $\operatorname{ht}(\mathfrak{p}) \leq m$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Since $\dim A = n$, we have that there exists a chian of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$, which clearly has height n. It follow that $\operatorname{ht}(\mathfrak{p}) = n \leq m$. Now, similarly, we know that there exists a prime $\mathfrak{p}$ of height m, but this must also be less than or equal to n, hence $n \leq m$ and $m \leq n$ implying the claim. □

Now note that if we take $H_{\mathfrak{m}}(A) \subset H(A)$ to be the subset of heights of maximal ideals then same result holds. We now prove the following lemma:

Lemma 6.1.3. *Let* $\dim A = n$*, and* $\mathfrak{p} \in \operatorname{Spec} A$*, then:*

i) The quotient ring $A/\mathfrak{p}$ *is finite dimensional and satisfies:*

$$\dim A/\mathfrak{p} \leq \dim A - \operatorname{ht}(\mathfrak{p})$$

with equality if every maximal chain of prime ideals has the same length.

ii) If every maximal chain of prime ideals in A *has the same length, and every partial chain be extended to maximal one, then* $A/\mathfrak{p}$ *is a ring where every maximal ideal of prime ideals has the same length, and every partial*

iii) If every maximal chain of prime ideals has the same length, and $\mathfrak{p}$ *is a maximal ideal then* $\dim A_{\mathfrak{p}} = \dim A$*.*

Proof. Note that there is an inclusion preserving bijection $\operatorname{Spec} A/\mathfrak{p} = \mathbb{V}(\mathfrak{p})$. Moreover, $\mathbb{V}(\mathfrak{p})$ consists of all prime ideals which contain $\mathfrak{p}$, hence every chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$ in $A/\mathfrak{p}$ can be viewed as a chain of prime ideals $\mathfrak{p} \subset \mathfrak{p}_0 \subset \cdots \mathfrak{p}_k$ in A, which must have length less than or equal to n. It follows that at minimum that $A/\mathfrak{p}$ has finite dimension less than or equal to n.

Now let $\operatorname{ht}(\mathfrak{p}) = k$, then we have that there is a chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}$ of length k. Furthermore, let $\dim A/\mathfrak{p} = l$, then by the above discussion there is a chain of prime ideal $\mathfrak{p} = \mathfrak{q}_0 \subset \cdots \mathfrak{q}_l$ in A of length l. We can glue these chains together to get a chain in A of length $l + k$ which must be less than or equal to n. It follows that:

$$\dim A/\mathfrak{p} + \operatorname{ht}(\mathfrak{p}) \leq \dim A$$

implying the inequality.

Suppose that every maximal chain of prime ideals is the same length m; in particular we then have that $H_{\mathfrak{m}}(A) = \{m\}$, $\dim A = m$, and $\operatorname{ht}(\mathfrak{m}) = m$ for all maximal ideals of A. Let $\dim A/\mathfrak{p} = l$, then there exists a chain of prime ideals containing $\mathfrak{p}$, $\mathfrak{q}_0 \subset \cdots \mathfrak{q}_l$, which must have $\mathfrak{q}_0 = \mathfrak{p}$, and $\mathfrak{q}_l = \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, as otherwise we would have $\dim A/\mathfrak{p} > l$. We can extend this to a maximal chain of prime ideals in A:

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_k = \mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_l = \mathfrak{m}$$

where $\operatorname{ht}(\mathfrak{p}) \geq k$, and by assumption $k + l = m$. It follows that:

$$\dim A \geq \operatorname{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} \geq k + l = \dim A$$

hence $\operatorname{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A$, implying *i*).

For *ii*), suppose that there was a maximal chain of prime ideals of length $k < \dim A/\mathfrak{p} = l$. Then this corresponds to a chain of prime ideals containing $\mathfrak{p}$, $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$, such that $\mathfrak{q}_0 = \mathfrak{p}$ and $\mathfrak{q}_k$ is maximal. We can extend this to a maximal chain of prime ideals for $\mathfrak{p}_0 \subset \cdots \mathfrak{p}_n = \mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_k$ which must satisfy $k + n = \dim A$ since every maximal chain in A has the same length. It follows that since $\dim A = \dim A/\mathfrak{p} + \operatorname{ht}(\mathfrak{p})$, that $n > \operatorname{ht}(\mathfrak{p})$ a contradiction. Clearly, there can't be a maximal chain of prime ideals of length greater than the dimension, implying *ii*).

For *iii*), we have that if every maximal chain of prime ideals has length m, then $H_{\mathfrak{m}}(A) = m$, hence $\operatorname{ht}(\mathfrak{m}) = \dim A_{\mathfrak{m}} = m$ for all $\mathfrak{m}$. It follows that $\dim A_{\mathfrak{m}} = \dim A$ for all $\mathfrak{m}$ as well, as desired. ☐

Before moving onwards, where we will consider dimension theory in more restrictive cases, we begin our construction of an infinite dimensional Noetherian ring. We first need the following lemma from Atiyah and Bott:

Lemma 6.1.4. *Let* A *be a ring such that* $A_{\mathfrak{m}}$ *is Noetherian for all all maximal ideals* $\mathfrak{m}$*, and for all* $a \neq 0 \in A$*, we have that* a *lies in finitely many* $\mathfrak{m}$*. Then* A *is Noetherian.*

Proof. Let $I \subset A$ be an ideal, then their exist finitely many maximal ideals $\{\mathfrak{m}_i\}_{i=1}^n$ which contain I, as otherwise there would be some a which lies in infinitely many $\mathfrak{m}$. Let $a \in A$, then for all $1 \leq i \leq n$ we have that $a \in \mathfrak{m}_i$, and that there exist finitely many more $\mathfrak{m}_i$ such that $a \in \mathfrak{m}_i$ for all $1 \leq i \leq n+k$ for some k. We thus obtain the set $\{\mathfrak{m}_i\}_{i=1}^{n+k}$. Choose elements $b_j \in I$ such that $b_j \notin \mathfrak{m}_{n+j}$ for $1 \leq j \leq k$; moreover we have that if $\pi_{\mathfrak{m}_i} : A \to A_{\mathfrak{m}_i}$ is the localization map, then $\langle \pi_{\mathfrak{m}_i}(I) \rangle$ is finitely generated from all i. For each i, there thus exist elements $\{c_{j_i}\}_{j_i=1}^{n_i}$ in A whose image generate $\langle \pi_{\mathfrak{m}_i}(I) \rangle$. Let:

$$J = \langle a, b_l, c_{j_i} : 1 \leq l \leq k, 1 \leq i \leq n, 1 \leq j_i \leq n_i \rangle$$

We wish to show that $\langle \pi_{\mathfrak{m}}(I) \rangle = \langle \pi_{\mathfrak{m}}(J) \rangle$ for all maximal ideals $\mathfrak{m}$. Set $I_{\mathfrak{m}} = \langle \pi_{\mathfrak{m}}(I) \rangle$, and $J_{\mathfrak{m}} = \langle \pi_{\mathfrak{m}}(J) \rangle$. For $1 \leq i \leq n$, this is true as J contains elements which map to the generators of $I_{\mathfrak{m}}$ by construction. For each $\mathfrak{m}_{n+j}$, $1 \leq j \leq k$ this is true as both I and J contain elements (namely b_j) which map to invertible elements in $A_{\mathfrak{m}_{n+j}}$, so the ideals $I_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ are the whole ring. For any other maximal ideal, $\mathfrak{m}$, we have that $a \notin \mathfrak{m}$ so again the ideals $I_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ are the whole ring, and it follows that for all $\mathfrak{m} \in |\operatorname{Spec} A|$, we have that $I_{\mathfrak{m}} = J_{\mathfrak{m}}$.

Consider the identity map $\mathrm{Id} : A \to A$. This clearly descends to an A-module homomorphism $\imath : J \to I$. Moreover, for each maximal ideal, we have that Id induces the identity map $A_{\mathfrak{m}} \to A_{\mathfrak{m}}$, which again induces a unique, well defined $A_{\mathfrak{m}}$ module homomorphism $\imath : J_{\mathfrak{m}} \to I_{\mathfrak{m}}$. This map is clearly injective as the it comes from the restriction of an injective map, and moreover, it is surjective as $J_{\mathfrak{m}} = I_{\mathfrak{m}}$, and $\imath$ is an A-module homomorphism, so this map is an isomorphism for all $\mathfrak{m} \in |\operatorname{Spec} A|$. Note that $\imath$ is also injective, as $J \subset I$ by construction, so we need only show that $\imath$ is surjective. Consider the following exact sequence:

$$J \to I \to \operatorname{coker} \imath \to 0$$

This then gives rise to an exact sequence on stalks:

$$J_{\mathfrak{m}} \to I_{\mathfrak{m}} \to (\operatorname{coker} \imath)_{\mathfrak{m}} \to 0$$

but here $\imath_{\mathfrak{m}}$ is surjective, so $(\operatorname{coker} \imath)_{\mathfrak{m}} = \operatorname{coker} \imath_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$.

Now suppose for the sake of contradiction that $\operatorname{coker} \imath \neq 0$. Let $x \in \operatorname{coker} \imath$, and define the ideal:

$$I' = \{a \in A : a \cdot x = 0\}$$

We have that I' is contained in some maximal ideal $\mathfrak{m}$, so consider $I'_{\mathfrak{m}}$. Then, $x/1 \in (\operatorname{coker} \imath)_{\mathfrak{m}}$, but this must be equal to zero as $(\operatorname{coker} \imath)_{\mathfrak{m}} = 0$. This implies that there exists a $y \in A \setminus \mathfrak{m}$ such that $x \cdot y = 0$. However, this means that $y \in I'$ by definition, a contradiction as $I' \subset \mathfrak{m}$. It follows that $\operatorname{coker} \imath = 0$, so $\imath$ is surjective, and thus the restriction of the identity map to J takes J to I, implying $I = J$. Therefore, I is finitely generated, and since I was arbitrary A is Noetherian. ☐

We will also need the following result, known as the prime avoidance lemma.

Lemma 6.1.5. *Let $I \subset A$ be an ideal, and $I \subset \bigcup_{i \in L} \mathfrak{p}_i$ for some finite indexing set L. Then for some i we have that $I \subset \mathfrak{p}_i$.*

Proof. We first assume that $L = \{1, \ldots, n\}$, is arbitrary and proceed by induction. The case where $N = 1$ is obvious. Now suppose that $n = 2$, and that $I \not\subset \mathfrak{p}_1$ and $I \not\subset \mathfrak{p}_2$. Then there exists $a, b \in I$ such that $a \notin \mathfrak{p}_1$ and $b \notin \mathfrak{p}_2$, consequently, we have that $a \in \mathfrak{p}_2$ and $b \in \mathfrak{p}_1$ as otherwise $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2$ and we are done. We claim that $a + b \notin \mathfrak{p}_1$ and $a + b \notin \mathfrak{p}_2$. Indeed, if $a + b \in \mathfrak{p}_1$, then $a + b - b \in \mathfrak{p}_1$ so $a \in \mathfrak{p}_1$, and similarly for $\mathfrak{p}_2$. It follows that $I \not\subset \mathfrak{p}_1 \cup \mathfrak{p}_2$, so by the contrapositive we have that $I \subset \mathfrak{p}_1$ or $I \subset \mathfrak{p}_2$.

Now let $L = \{1, \ldots, n\}$, and assume the result holds for $L' = \{1, \ldots, n-1\}$. If the product:

$$I \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_{n-1} = \langle a \cdot p_1 \cdots p_{n-1} : a \in I, p_1 \in \mathfrak{p}_1, \ldots, p_{n-1}\mathfrak{p}_n \rangle$$

is contained in $\mathfrak{p}_n$, then we have that $I \subset \mathfrak{p}_n$ or $P = (\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1}) \subset \mathfrak{p}_n$. Indeed, if $a \in I$ and $p \in P$ such that $a, p \notin \mathfrak{p}_n$, then their product is in $I \cdot P \subset \mathfrak{p}_n$, contradicting the fact that $\mathfrak{p}_n$ is prime. If $I \subset \mathfrak{p}_n$

then we are done. If $P \subset \mathfrak{p}_n$, then we have that by induction $\mathfrak{p}_i \subset \mathfrak{p}_n$ for some i. If this is the case, then $I \subset \bigcup_{j \neq i \in L} \mathfrak{p}_j$, so by the inductive hypothesis we are done. We may thus assume that $I \cdot P \not\subset \mathfrak{p}_n$.

Furthermore, if for all $a \in I$ we have that $a \in \mathfrak{p}_i$ for some $i \in L'$, then $I \subset \bigcup_{i \in L'} \mathfrak{p}_i$ hence by the inductive hypothesis we are done. So we may further suppose that there exists an element $a \in I$ such that $a \notin \mathfrak{p}_i$ for all $i \in L'$. Now, if $a \notin \mathfrak{p}_n$, then we have that $I \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ so by the contrapositive we are done. Hence we may also assume that $a \in \mathfrak{p}_n$.

Suppose that $a \in \mathfrak{p}_n$, and $I \cdot P \not\subset \mathfrak{p}_n$, but $I \not\subset \mathfrak{p}_i$ for all i. Take an element $b \in I \cdot P$ such that $b \notin \mathfrak{p}_n$; then we claim that $a + b \notin \mathfrak{p}_i$ for all $i \in L$. Indeed, if $a + b \in \mathfrak{p}_i$ for some $i \in L'$, then since $b \in \mathfrak{p}_i$ for all $i \in L'$, we have that $a + b - b \in \mathfrak{p}_i$ a contradiction. Similarly, if $a + b \in \mathfrak{p}_n$ then $a + b - a \in \mathfrak{p}_n$, another contradiction. It, follows that $a + b \notin \mathfrak{p}_i$ for all $i \in L$, hence $I \not\subset \bigcup_{i \in L} \mathfrak{p}_i$, so by the contrapositive we have that $I \subset \mathfrak{p}_i$ for some $i \in L$, implying the claim.

□

We now finally construct our example:

Example 6.1.3. Let $A = k[x_0, x_1, \dots]$, and define the prime ideals:

$$\mathfrak{p}_i = \langle x_{2^i+1}, x_{2^i+2}, \dots, x_{2^{i+1}} \rangle$$

for all $i > 0$. We set:

$$S = \bigcap_{i \geq 1} (A \smallsetminus \mathfrak{p}_i)$$

Note that S is multiplicatively closed, indeed if $a, b \in S$, then $a, b, \in A \smallsetminus \mathfrak{p}_i$ for all i. Since $A \smallsetminus \mathfrak{p}_i$ is multiplicatively closed, we have that $a \cdot b \in A \smallsetminus \mathfrak{p}_i$.

We first claim that $S^{-1}A$ is infinite dimensional. Note that for any i we have the following chain:

$$\langle 0 \rangle \subset \langle x_{2^i+1} \rangle \subset \langle x_{2^i+1}, x_{2^i+2} \rangle \subset \cdots \subset \langle x_{2^i+1}, x_{2^i+2}, \dots, x_{2^{i+1}} \rangle = \mathfrak{p}_i$$

We claim that this is of length $2^{i+1} - 2^i$. Indeed, there are $2^{i+1} - 2^i$ elements which generate $\mathfrak{p}_i$, thus there are $2^{i+1} - 2^i - 1$ inclusions in the above chain ignoring the zero ideal, and when we include the zero ideal inclusion, we get $2^{i+1} - 2^i$ as the length. It follows that $\mathrm{ht}(\mathfrak{p}_i) > 2^{i-1} - 2^i$, which is a strictly increasing sequence of integers. We obtain that for $n \in \mathbb{N}$, we can find and i such that $\mathrm{ht}(\mathfrak{p}_i) > n$, so, via the inclusion preserving bijection from Lemma 6.1.2, we obtain that $\dim S^{-1}A = \infty$.

We will now make use of Lemma 6.1.4 and Lemma 6.1.5 to show that $S^{-1}A$ is Noetherian. Set $S^{-1}\mathfrak{p}_i = \langle \pi(\mathfrak{p}_i) \rangle$, where π is the localization map. We first show that any $f \in k[x_0, x_1, \dots]$ is contained in finitely many $\mathfrak{p}_i$. Indeed, since f is a finite sum of polynomials, there is a maximum j such that x_j appears in the polynomial f. We claim that $f \notin \mathfrak{p}_k$ for any $k \geq j$. Indeed, if $f \in \mathfrak{p}_k$, then there must be a $2^k + m$ for some m such that x_{2^k+m} appears in the polynomial f. However $2^k + m > j$, hence f cannot lie in $\mathfrak{p}_k$. Since there are finitely many $\mathfrak{p}_i$ such that $i < j$, it follows that f must lie in finitely many, perhaps 0, prime ideals of the form $\mathfrak{p}_i$.

Now let $f/g \in S^{-1}A$, and suppose that f/g lies in infinitely many prime ideals of the form $S^{-1}\mathfrak{p}_i$. It follows that $f/1$ lies in infinitely many $S^{-1}\mathfrak{p}_i$, so f lies in infinitely many $\mathfrak{p}_i$, a clear contradiction. It follows that all f/g must lie in finitely many $S^{-1}\mathfrak{p}_i$.

We first claim that:

$$(S^{-1}A)_{S^{-1}\mathfrak{p}_j} \cong A_{\mathfrak{p}_j}$$

for all j. Indeed, note that:

$$S = A \smallsetminus \bigcup_{i \geq 1} \mathfrak{p}_i$$

So consider the localization $\pi_j : A \to A_{\mathfrak{p}_i}$. If $s \in S$, then $s \notin \bigcup_{i \geq 1} \mathfrak{p}_i$, so $s \notin\in \mathfrak{p}_i$ for all i; in particular, $s \notin \mathfrak{p}_j$ so the image of is invertible. It follows that there is a unique map such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\pi"] & A_{\mathfrak{p}_j} \\ S^{-1}A \arrow[ur, "\phi"] & \end{tikzcd}$$

We claim that $A_{\mathfrak{p}_j}$ satisfies the universal property of localization with the localization map given by ϕ. Indeed, let $\psi : S^{-1}A \to B$ be a homomorphism such that for all $b \in \psi(S^{-1}A \setminus S^{-1}\mathfrak{p}_j)$, we have that b is invertible. By the universal property, there is then a unique map $A \to B$ that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\pi"] & B \\ S^{-1}A \arrow[ur, "\psi"] & \end{tikzcd}$$

Let $a \in A \setminus \mathfrak{p}_j$, then $a/1 \in S^{-1}A \setminus S^{-1}\mathfrak{p}_j$, hence $\beta(a) = \psi(a/1)$ is invertible. It follows there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\pi_j"] & B \\ A_{\mathfrak{p}_j} \arrow[ur, "\alpha"] & \end{tikzcd}$$

Now, we need only check that $\alpha \circ \phi = \psi$. Recall that the localizations π and π_j are epimorphisms. In particular, we have that:

$$\alpha \circ \phi \circ \pi = \alpha \circ \pi_j = \beta$$

whilst:

$$\psi \circ \pi = \beta$$

hence $\alpha \circ \phi = \psi$ as desired. It follows that $A_{\mathfrak{p}_j}$ is canonically isomorphic to $(S^{-1}A)_{S^{-1}\mathfrak{p}_j}$.

Now let $k(\mathfrak{p}_j^c)$ be the field of fractions of $k[x_i : x_i \notin \mathfrak{p}_j]$. Moreover, set

$$k(\mathfrak{p}_j^c)[\mathfrak{p}_j] = k(\mathfrak{p}_j^c)[x_{2^j+1}, \ldots, x_{2^{j+1}}]$$

We claim that:

$$A_{\mathfrak{p}_j} \cong (k(\mathfrak{p}_j^c)[\mathfrak{p}_j])_{\mathfrak{p}'_j}$$

where

$$\mathfrak{p}'_j = \langle x_{2^j+1}, \ldots, x_{2^{j+1}} \rangle \subset k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$$

There is an obvious inclusion $\imath_A : A \to k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, so compose this with the localization map $\pi_c : k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \to (k(\mathfrak{p}_j^c)[\mathfrak{p}_j])_{\mathfrak{p}'_j}$. Note that $\imath^{-1}(\mathfrak{p}'_j) \subset \mathfrak{p}_j$, hence if $a \notin \mathfrak{p}_j$, we have that $\imath_A(a) \notin \mathfrak{p}'_j$. It follows that $\pi_c \circ \imath_A(a)$ is invertible, hence there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_c \circ \imath_A"] \arrow[d, "\pi_j"] & k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}'_j} \\ A_{\mathfrak{p}_j} \arrow[ur, "\alpha"] & \end{tikzcd}$$

Note that $k(\mathfrak{p}_j^c) = k[\mathfrak{p}_j^c]_{\langle 0 \rangle}$. There is a canonical map $k[\mathfrak{p}_j^c] \to A_{\mathfrak{p}_j}$ given by inclusion, composed with localization. Every nonzero element in $k[\mathfrak{p}_j^c]$ then maps to invertible element of $A_{\mathfrak{p}_j}$ hence we obtain the following unique diagram:

$$\begin{tikzcd} k[p_j^c] \arrow[r, "\pi_j \circ \imath_k"] \arrow[d, "\pi_0"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c) \arrow[ur, "\beta"'] & \end{tikzcd}$$

Note that β is injective as $k(\mathfrak{p}_j^c)$ is a field. Now, we adjoin the variables $\{x_{2^j+1}, \ldots, x_{2^{j+1}}\}$, and obtain a unique map β', such the that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\imath_A"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \arrow[ur, "\beta'"'] & \end{tikzcd}$$

Note that β' restricted to the subfield $k(\mathfrak{p}_j^c)$ is just β, and that $\beta'(x_{2^j+m}) = \pi_j(x_{2^j+m})$. Moreover, we have that the unique maximal ideal $\mathfrak{m}_{\mathfrak{p}_j} \subset A_{\mathfrak{p}_j}$ is generated by $\{x_{2^j+1}/1, \ldots, x_{2^{j+1}}/1\}$. It follows that $(\beta')^{-1}(\mathfrak{m}_{\mathfrak{p}_j}) \subset \mathfrak{p}'_j$, hence if $c \notin \mathfrak{p}'_j$, then $\beta'(c) \notin \mathfrak{m}_{\mathfrak{p}_j}$. Therefore, there exists a unique map ξ such the the following square commutes:

$$\begin{tikzcd} A \arrow[r, "\pi_j"] \arrow[d, "\imath_A"'] & A_{\mathfrak{p}_j} \\ k(\mathfrak{p}_j^c)[\mathfrak{p}_j] \arrow[ur, "\beta'"'] \arrow[r, "\pi_c"] & k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}'_j} \arrow[u, "\xi"'] \end{tikzcd}$$

We check that $\alpha \circ \xi = \mathrm{Id}$. Using the fact that localization maps are epimorphisms, we have that:

$$\alpha \circ \xi \circ \pi_c = \alpha \circ \beta'$$

By identifying $\imath_A$ as the tensor product of two epimorphisms $\pi_0 \otimes \mathrm{Id} : k[\mathfrak{p}_j^c] \otimes_k k[\mathfrak{p}_j] \to k(\mathfrak{p}_j^c) \otimes_k k[\mathfrak{p}_j] \cong k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, conclude that $\imath_A$ is an epimorphism as well.

$$\alpha \circ \beta' \circ \imath_A = \alpha \circ \pi_j = \pi_c \circ \imath_A$$

whilst:

$$\mathrm{Id} \circ \pi_c \circ \imath_A = \pi_c \circ \imath_A$$

hence $\alpha \circ \xi = \mathrm{Id}$ as desired. To see that $\xi \circ \alpha = \mathrm{Id}$, we examine:

$$\xi \circ \alpha \circ \pi_j = \xi \circ \pi_c \circ \imath_A = \beta' \circ \imath_A = \pi_j$$

whilst:

$$\mathrm{Id} \circ \pi_j = \pi_j$$

hence $\xi \circ \alpha = \mathrm{Id}$ as desired. It follows that:

$$(S^{-1}A)_{S^{-1}\mathfrak{p}_j} = k(\mathfrak{p}_j^c)[\mathfrak{p}_j]_{\mathfrak{p}'_j}$$

which is the localization of the Noetherian ring $k(\mathfrak{p}_j^c)[\mathfrak{p}_j]$, so $(S^{-1}A)_{S^{-1}\mathfrak{p}_j}$ is indeed Noetherian.

We now check that each $S^{-1}\mathfrak{p}_i$ is maximal. We claim that $S^{-1}A/S^{-1}\mathfrak{p}_i$ is a field for all i. We need only check that every nonzero element $[a/s] \in S^{-1}A/S^{-1}\mathfrak{p}_i$ has an inverse. Since $[a/s]$ is non zero, we may assume that $a/s \notin S^{-1}\mathfrak{p}_i$, in particular $a/1 \notin S^{-1}\mathfrak{p}_i$. It follows that $a \notin \mathfrak{p}_i$. If a contains a monomial

cx_{2^i+1}, with $c \in k$ non zero, then $a \notin \mathfrak{p}_j$ for any $j \neq i$ as well, hence $a \in S$. If a contains no such polynomial, then we consider $a + x_{2^i+1}$ which cannot lie in $\mathfrak{p}_i$ as this would imply a does, and clearly cannot lie in $\mathfrak{p}_j$ for any j, so $a + x_{2^i+1} \in S$. If $a \in S$, then $a/1$ is invertible, so $[a/s]$ is invertible as well. If $a \notin S$, then $a + x_{2^i+1} \in S$, hence we see that:

$$[a/s] \cdot [s/(a + x_{2^i+1})] = [(a + x_{2^i+1})/s] \cdot [s/(a + x_{2^i+1})] = [1]$$

so every nonzero $[a/s]$ is invertible. It follows that $S^{-1}\mathfrak{p}_i$ is maximal for all i.

Finally we show that $S^{-1}\mathfrak{p}_i$ are the only maximal ideals of $S^{-1}A$. Suppose that $\mathfrak{m}$ is a maximal ideal of $S^{-1}A$, then, in particular, we have that $\mathfrak{m}$ corresponds to a prime ideal $\mathfrak{q}$ such that $S \cap \mathfrak{q} = \emptyset$. In other words, we have that $\mathfrak{q} \subset \bigcup_i \mathfrak{p}_i$. We need to now prove a generalized form of Lemma 6.1.5, i.e. that this implies that $\mathfrak{q} \subset \mathfrak{p}_i$ for some i. Our approach will mimic this lemma; that is, we will assume that $\mathfrak{q} \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ for any finite L, and show that this implies $\mathfrak{q} \not\subset \bigcup_i \mathfrak{p}_i$. For all $f \in A$, consider the following set:

$$L_f = \{n \in \mathbb{N} : f \in \mathfrak{p}_n\}$$

By our earlier work, we know this is a finite set; let $f \in \mathfrak{q}$, if $L_f \cap L_g \neq \emptyset$ for all $g \in \mathfrak{q}$, then we claim that $\mathfrak{q} \subset \bigcup_{L_f} \mathfrak{p}_i$. Indeed, if $g \in \mathfrak{q}$, and $L_g \cap L_f \neq \emptyset$, then there exist such $i \in L_f$ such that $g \in \mathfrak{p}_i$, hence $\mathfrak{q} \subset \bigcup_{L_f} \mathfrak{p}_i$. It follows, by the contrapositive, that if $\mathfrak{q} \not\subset \bigcup_{i \in L_f} \mathfrak{p}_i$ for all f there must exist some h such that $L_f \cap L_h = \emptyset$. Let $n \in L_h$, and m be the degree of the highest power monomial in f. Then we claim that $f + x_{2^n+1}^{m+1} h \notin \mathfrak{p}_i$ for all i. Note that $L_{x_{2^n+1}^{m+1} h} = L_h$ as $x_{2^n+1}^{m+1} \in \mathfrak{p}_n$, and $h \in \mathfrak{p}_n$, so $x_{2^n+1}^{m+1} h \in \mathfrak{p}_i$ for all i such that $h \in \mathfrak{p}_i$. Note that for all $i \in L_f \cup L_h$, we have that $f + x_{2^n+1}^{m+1} h \notin \mathfrak{p}_i$, as if $f + x_{2^n+1}^{m+1} h \in \mathfrak{p}_i$ for some $i \in L_f \cup L_h$, then i in either L_f or L_h, hence either $f \in \mathfrak{p}_i$ or $x_{2^n+1}^{m+1} h \in \mathfrak{p}_i$, and in either case we obtain that both f and $x_{2^n+1}^{m+1} h$ are in $\mathfrak{p}_i$ contradicting the fact that $L_f \cap L_h = \emptyset$. Now suppose that $i \notin L_f \cup L_h$; since $x_{2^n+1}^{m+1}$ is one degree higher than the highest degree monomial in f, there can be no combination of monomials in the sum $f + x_{2^n+1}^{m+1} h$. Since $i \notin L_f \cup L_h$, it follows that there must be a monomial in f which does not lie in $\mathfrak{p}_i$, and since there is no combination of monomials, we must have that the same monomial appears in $f + x_{2^n+1}^{m+1} h$. Therefore, $f + x_{2^n+1}^{m+1} h \notin \mathfrak{p}_i$, as $g \in \mathfrak{p}_i$ implies that each monomial of $g \in \mathfrak{p}_i$ because $\mathfrak{p}_i$ is generated by monomials. It follows that if $\mathfrak{q} \not\subset \bigcup_{i \in L} \mathfrak{p}_i$ for any finite set, then $\mathfrak{q} \not\subset \bigcup_i \mathfrak{p}_i$. By the contrapositive, if $q \subset \bigcup_i \mathfrak{p}_i$, then there exists some finite set L such that $\mathfrak{q} \subset \bigcup_{i \in L} \mathfrak{p}_i$. Lemma 6.1.5 then implies that $\mathfrak{q} \subset \mathfrak{p}_i$ for some i, hence $\mathfrak{m} = S^{-1}\mathfrak{q} \subset S^{-1}\mathfrak{p}_i$, but $S^{-1}\mathfrak{p}_i$ is not the whole ring, so $\mathfrak{m} = S^{-1}\mathfrak{p}_i$.

In conclusion, we have shown that the maximal ideals of $S^{-1}A$ are precisely $S^{-1}\mathfrak{p}_i$, that every element $f/g \in S^{-1}A$ is contained in finitely many $S^{-1}\mathfrak{p}_i$, and that $S^{-1}A_{S^{-1}\mathfrak{p}_i}$ is Noetherian for all i. By Lemma 6.1.4, we have that $S^{-1}A$ is Noetherian, hence $S^{-1}A$ is a Noetherian infinite dimensional ring.

To actually begin calculating the dimensions of our favorite rings, i.e. polynomial rings over a field and their quotients, we will need to study Noether normalization. In particular, this will eventually allow us to calculate the dimension of finitely generated k-algebras. We first review some field theory.

Let K/k be a field extension, i.e. $k \subset K$. Recall that a field extension is algebraic if every element in K is the root of some polynomial in $k[x]$. Moreover, an extension is finite if K is a finite dimensional k vector space. Many field extensions in nature are algebraic, however there exist plenty of non algebraic extensions of import, which are known as transcendental extensions. Indeed, consider the field extension $\mathbb{Q}(\pi)$, that is the smallest field containing $\mathbb{Q}$ and π. This is not an algebraic field extension as π is not the root of any polynomial in $\mathbb{Q}[x]$[152]. We also see that $\mathbb{Q}(\pi)/\mathbb{Q}$ is an infinite field extension. Indeed, we claim that $\{\pi, \ldots, \pi^n\}$ is a $\mathbb{Q}$-linear independent set for all n. Suppose there exist not identically zero $a_i/b_i \in \mathbb{Q}$ such that:

$$\frac{a_1}{b_1}\pi + \cdots + \frac{a_n}{b_n}\pi^n = 0$$

but this now implies that π is the root of the polynomial:

$$p(x) = \frac{a_1}{b_1}x + \cdots + \frac{a_n}{b_n}x^n$$

[152]This a hard fact to prove

which is false, so it follows that $a_i/b_i = 0$ for all i. Therefore, $\{\pi, \dots, \pi^n\}$ is a linearly independent set for all values of n, and $\mathbb{Q}(\pi)$ can clearly not be finite dimensional.

Corollary 6.1.1. *Let K/k be a field extension which is not algebraic. Then K is an infinite dimensional vector space.*

Proof. Since K/k is not an algebraic extension, there exists some $\alpha \in K$ such that α is not the root of any polynomial in $k[x]$. The argument for $\mathbb{Q}(\pi)$ applies to K proves the claim. □

Despite transcendental extensions being infinite dimensional vector spaces over the base field, we can still obtain finite numbers from them. Let $K/L/k$ and $K/F/k$ be intermediate field extensions, and denote by $L \cdot F$ the smallest field extension of k which contains both L and F. We write that $L \sim F$ if $L \cdot F$ is an algebraic extension of L and F.

Definition 6.1.3. Let K/k be a field extension; a **transcendence basis** for K is a set of algebraically independent[153] elements S, such that the smallest field extension of k containing S, denoted $k(S)$, satisfies $k(S) \sim K$. The **transcendence degree** K, denoted $\operatorname{tdeg}_k K$, is the cardinality of S.

Assuming that all of this well defined for the moment, we move to the following example:

Example 6.1.4. We immediately see that if K/k is algebraic, then $K \sim k$, so clearly $\operatorname{tdeg}_k K = 0$. Similarly, if $K = \mathbb{Q}(\pi)$ and $k = \mathbb{Q}$, then $\operatorname{tdeg}_{\mathbb{Q}} \mathbb{Q}(\pi) = 1$. Further, if $K'/K/k$, and K' is algebraic over K, then we have $\operatorname{tdeg}_k K = \operatorname{tdeg}_k K'$.

Example 6.1.5. Let $A = k[x_1, \dots, x_n]$, and $K = \operatorname{Frac}(A)$, it's field of fractions. We claim that $\operatorname{tdeg}_k K = n$; let $S = \{x_1, \dots, x_n\}$, then these are algebraically independent over k essentially by the definition of the polynomial ring. We need only check that that $K/k(S)$ is an algebraic extension, but this is easily seen to be true as $k(S) = K$. Indeed, the smallest field which contains S must also contain every polynomial in the x_i, hence every element of A must be invertible in $k(S)$, so $k(S) = A_\eta = K$.

In a sense, the transcendence degree of a field extension is measuring how much the field extension fails to be algebraic. We also note that the transcendence degree of A_η, is what we would expect the Krull dimension of A to be; this connection between transcendence degree and Krull dimension will be made clear with the results to come, but we first we check that all of this makes sense.

Lemma 6.1.6. *Let K/k be a field extension, then a transcendence basis S exists.*

Proof. First note that if there is no nonempty algebraically independent subset of K then k is algebraic. Indeed, this would imply that $\{x\}$ is not an algebraically independent subset, hence the homomorphism:

$$\begin{aligned} k[y] &\longrightarrow K \\ y &\longmapsto x \end{aligned}$$

is not injective, so x is algebraic. It follows that every element of K is algebraic so K/k is an algebraic field extension. In this case, $S = \emptyset$ is a transcendence basis.

Supposing that K/k is not algebraic, let S be an algebraically independent set, and T a subset of K containing S which generates K/k (as a k algebra). Let:

$$\mathscr{B} = \{B \subset K : B \text{ is algebraically independent and } S \subset B \subset T\}$$

We have that $\mathscr{B}$ is partially order by inclusion, and $\mathscr{B}$ is non empty as it contains S. Take any totally ordered subset $\mathscr{B}' \subset \mathscr{B}$, and set:

$$T' = \bigcup_{B \in \mathscr{B}'} B$$

We see that T' contains S, and is contained in T. We claim T' is algebraically independent, as if it isn't then we the map:

$$k[x_t : t \in T'] \longrightarrow K$$

[153] $S \subset K$ is algebraically independent if the map $k[y_s : s \in S] \to K$ given by $y_s \mapsto s$ is injective.

is not injective, so some $p \in k[x_t : t \in T']$ maps to zero. Since polynomial rings consist of finite sums of finite monomials, it follows that there exists $\{t_1, \dots, t_n\} \subset T'$ such that $p(t_1, \dots, t_n) = 0$. However, since this set is finite, and $\mathscr{B}'$ is totally ordered, we must have that there is some $B \in \mathscr{B}'$ which contains each t_i. This would imply B is not algebraically independent though, hence T' must be algebraically independent as well. It follows that every chain in $\mathscr{B}$ has an upper bound, so by Zorn's lemma there exists a maximal element $S' \in \mathscr{B}$.

We claim that $K/k(S')$ is algebraic; if it was not, then there is some $\alpha \in K$ which is not the root of a polynomial in $k(S')[x]$. It follows that $S' \cup \{\alpha\}$ is then algebraically independent, contradicting the maximality of S', so no such element can exist implying the claim. $\square$

Lemma 6.1.7. *Let K/k be a field extension, then the relation $\sim$ is an equivalence relation, and* $\operatorname{tdeg}_k K$ *is well defined.*

Proof. It is clear that for $K/L/k$, and $K/F/k$, we have that $L \sim F \Leftrightarrow F \sim L$, and $L \sim L$, hence the relation is both symmetric and reflexive. We check that this relation is transitive, and thus an equivalence relation.

First note that $L \sim F$ if and only if every element of L is algebraic over F, and every element of F is algebraic over L. Indeed, suppose $L \sim F$, then $x \in L \subset L \cdot F$ is algebraic over F and vice versa. Now, conversely, suppose that every element of L is algebraic over F, and every element of F is algebraic over L, and let $x \in L \cdot F$. In particular, since $L \cdot F$ is the smallest field extension of k containing L and F, hence x can be written as a sum $\sum_i l_i f_i$, where $l_i \in L$ and $f_i \in F$. Each l_i is algebraic over F by assumption, hence each $l_i f_i$ is algebraic over F, so x is algebraic over F. Similarly x is algebraic over L, hence $L \cdot F$ is an algebraic extension of L and F as desired.

Let $L \sim F \sim E$. Let $x \in L$, then x is algebraic over F so must be algebraic over E, hence every element in L is algebraic over E, and vice versa. It follows that $\sim$ is an equivalence relation as desired.

Suppose that $k(S) \sim K$, and S is not finite. If S' is any other transcendence basis, then we must have that $k(S') \sim k(S)$ by the transitivity property of $\sim$. For each $s' \in S'$, there must be a finite set $S_{s'} \subset S$ such that s' is algebraic over $k(S_{s'})$. Set:

$$T = \bigcup_{s' \in S} S_i$$

We have that $T \subset S$; suppose that $S \not\subset T$, then there is some $s \in S \setminus T$ which is algebraic over $k(S')$. However, by construction, $k(S')$ is algebraic over $k(T)$, hence s is algebraic over T. There is then some polynomial $p \in k(T)[x]$ such that $p(s) = 0$, but since $T \subset S$, this implies that S is not algebraically independent. If S' is finite then T is a finite collection of finite sets, and thus finite, so S is finite as well, hence S' is not finite. We then have that:

$$|S| = \left| \bigcup_{s' \in S'} S_{s'} \right| = |S'|$$

as S' is infinite and each $S_{s'}$ is finite.

Now note that if S is finite, then by the above argument we must have that S' is also finite. Let $S = \{s_1, \dots, s_n\}$, and $S' = \{t_1, \dots, t_m\}$, and without loss of generality suppose that $m \leq n$. We proceed via induction on m. If $m = 0$, then S' is empty, and K/k is algebraic, hence $n = 0$ as well. If $m > 0$, then $k(S) \sim k(S')$ so every element of S is algebraic over S'. It follows that we must have that there is an irreducible[154] polynomial $p \in k[y_1, \dots, y_{n+1}]$ such that $p(s_1, \dots, s_n, t_m) = 0$. Since t_m is not algebraic over k, p cannot be a polynomial entirely in y_{n+1}, so assume that p uses y_n without loss of generality. Let $T = (s_1, \dots, s_{n-1}, t_m)$, then we claim that $K/k(T)$ is algebraic. To do so, note that $k(T, s_n)/k(T)$ is algebraic as s_n is the root of $p(s_1, \dots, s_{n-1}, \cdot, t_m) \in k(T)[y_n]$, and that $K/k(T, s_n)$ is algebraic as $S \subset T \cup \{s_n\}$. We thus have the following chain of algebraic extensions:

$$K/k(T, s_n)/k(T)$$

[154] If it was not irreducible, S or S' would not be algebraically independent.

implying that $K/k(T)$ is algebraic. We want to show that T is a transcendence basis; if T is not algebraically independent, then there would be an irreducible polynomial $q \in k[y_1, \dots, y_n]$ such that $q(s_1, \dots, s_{n-1}, t_m) = 0$ which must involve y_n as $\{s_1, \dots, s_{n-1}\}$ is algebraically independent. This implies that t_m is algebraic over $k(s_1, \dots, s_{n-1})$, so:

$$k(T, s_n)/k(T)/k(s_1, \dots, s_{n-1})$$

is a chain of algebraic extensions. This implies that s_n is algebraic over $k(s_1, \dots, s_{n-1})$ which is obviously impossible as S is algebraically independent.

Since T is algebraically independent we have that T is a transcendence basis for K. Now consider $K/k(t_m)$, then we have that $k(S') = (k(t_m))(t_1, \dots, t_{m-1})$ and $K/K(S')$ is algebraic so $\{t_1, \dots, t_{m-1}\}$ is a transcendence basis for $K/k(t_m)$. Furthermore, we have that $k(T) = (k(t_m))(s_1, \dots, s_{n-1})$, and $K/k(T)$ is algebraic, so $\{s_1, \dots, s_{n-1}\}$ is a a transcendence basis for $K/k(t_m)$. By the inductive hypothesis $n - 1 = m - 1$, hence $n = m$ and we must have $|S| = |S'|$ in the finite case as well.

It follows that $\operatorname{tdeg}_k K$ is independent of our choice of transcendence basis as desired. □

For towers of finite field extensions $K/L/k$ we have that degree multiplies:

$$\dim_k K = \dim_L K \cdot \dim_k L$$

however in the case of transcendence degree we have that transcendence degree adds:

Lemma 6.1.8. *Let $K/L/k$ be field extensions with finite transcendence degrees, then:*

$$\operatorname{tdeg}_k K = \operatorname{tdeg}_L K + \operatorname{tdeg}_k L$$

Proof. Let $S \subset L$ be a transcendence basis for L/k, and $T \subset K$ be a transcendence basis for K/L. We claim that $S \cup T$ is a transcendence basis for K/k. We first show that $K/k(S \cup T)$ is algebraic; examine the following tower of field extensions:

$$K/L(T)/k(S \cup T)$$

Note that $K/L(T)$ is algebraic, so we need only show that $L(T)/k(S \cup T)$ is algebraic. Any element in $L(T)$ can be written as:

$$\sum_i l_i t_i$$

where $l_i \in L$, and $t_i \in T$. Since $L/k(S)$ is algebraic, we have that each l_i is the root of some polynomial in $k(S)[x]$. However, this polynomial also exists in $k(S \cup T)[x]$, hence each l_i, viewed as an element of $L(T)$, is algebraic over $k(S \cup T)$. Each t_i is also algebraic, as the polynomial $x - t_i \in k(S \cup T)[x]$ is a polynomial which has t_i as a root. It follows that any element of $L(T)$ is the sum of products of algebraic elements, and is thus algebraic, so $L(T)/k(S \cup T)$ is an algebraic extension. Since towers of algebraic extensions are algebraic, $K/k(S \cup T)$ is algebraic.

Now let $S = \{s_1, \dots, s_n\}$, and $T = \{t_1, \dots, t_m\}$, and consider the homomorphism:

$$\begin{aligned} k[x_1, \dots, x_n, y_1, \dots, y_m] &\longrightarrow K \\ x_i &\longmapsto s_i \\ y_i &\longmapsto t_i \end{aligned}$$

If $S \cup T$ is not algebraically independent, then there is some polynomial p in $k[x_1, \dots, x_n, y_1, \dots, y_n]$ which this map sends to zero. Consider the polynomial:

$$q' = p(s_1, \dots, s_n, \cdot, \dots, \cdot)$$

i.e. the polynomial $q' \in K[y_1, \dots, y_m]$ given by evaluating p on $\{s_1, \dots, s_m\}$. The coefficients of q' are multiples of elements of k with elements of L, hence the coefficients of q' lie in L, meaning we have that

$q' \in L[y_1, \ldots, y_m] \subset K[y_1, \ldots, y_m]$. If q' is identically zero, then the polynomial $q'' = p(\cdot, \ldots, \cdot, 1, \ldots, 1) \in k[x_1, \ldots, x_n]$ has a root at $(s_1, \ldots, s_n)$, implying that S is not algebraically independent, a contradiction. It follows that q' is not identically zero, however, q' then has $(t_1, \ldots, t_m)$ as a root so T is not algebraically independent, another contradiction. We thus see that no such q can exist, hence $S \cup T$ is algebraically independent as desired.

By the above, we have that $S \cup T$ is a transcendence basis, hence:

$$\operatorname{tdeg}_k K = |S \cup T| = |S| + |T| = \operatorname{tdeg}_L K + \operatorname{tdeg}_k L$$

as desired. □

The following lemma will prove useful, and generalizes Example 6.1.5:

Lemma 6.1.9. *Let A be an integral*[155] *finitely generated k algebra. Then if $K = \operatorname{Frac}(A)$, $\operatorname{tdeg}_k K$ is finite.*

Proof. Since A is finitely generated, and an integral domain, there exists a $\mathfrak{p} \in \mathbb{A}^m_k$ such that:

$$A = k[t_1, \ldots, t_m]/\mathfrak{p}$$

Denote by a_i the image of t_i in A, then clearly $K = k(a_1, \ldots, a_m)$. Let[156]:

$$n = \max\{|B| : B \subset \{a_1, \ldots, a_m\},\ B \text{ is an algebraically independent set over } k\}$$

We claim that $\operatorname{tdeg}_k K = n$; indeed without loss of generality we can take $\{a_1, \ldots, a_n\}$ to be an algebraically independent set, and since for any $i \neq 1, \ldots, n$, the set $\{a_1, \ldots, a_n, a_i\}$ is algebraically dependent, we have that $k(a_1, \ldots, a_n, a_{n+1}, \ldots, a_m)/k(a_1, \ldots, a_n)$ is algebraic. It follows that $\{a_1, \ldots, a_n\}$ is transcendence basis for K, and thus $\operatorname{tdeg}_k K = n$. □

With this notion of transcendence degree we prove the following theorem, known as the Noether Normalization:

Theorem 6.1.1. *Let A be an integral finitely generated k algebra, and K it's field of fractions, as in Lemma 6.1.9. If $\operatorname{tdeg}_k K = n$, then there exists an algebraically independent subset $\{\alpha_1, \ldots, \alpha_n\} \subset A$ over k, such that A is a finite extension of $k[y_1, \ldots, y_n]$.*

Proof. Since A is a finitely generated k algebra, and an integral domain, we can write:

$$A = k[t_1, \ldots, t_m]/\mathfrak{p}$$

for some $\mathfrak{p} \in \mathbb{A}^m_k$, and $m \geq 0$. Denote by a_i the image of t_i under the above projection. By Lemma 6.1.9 we have that $n \leq m$; we proceed by induction on m. The base case, $m = n$, immediately implies that $\{a_1, \ldots, a_m\}$ is a transcendence basis for K, thus $\mathfrak{p} = \langle 0 \rangle$, and so A is trivially a finite extension of $k[y_1, \ldots, y_{n=m}]$.

Now suppose that $m > n$, and we have proven that if $B = k[u_1, \ldots, u_{m-1}]/\mathfrak{q}$, and $\operatorname{tdeg}_k \operatorname{Frac}(B) = n$ then B is a finite extension of $k[y_1, \ldots, y_n]$. Since $n < m$, we have that the map:

$$\begin{aligned} k[t_1, \ldots, t_m] &\longrightarrow A \\ p &\longmapsto p(a_1, \ldots, a_m) \end{aligned}$$

is not injective. Let p lie in the kernel of the above homomorphism, and be of the form:

$$p = \sum_{i_1 \cdots i_k} k_{i_1 \cdots i_m} y_1^{i_1} \cdots y_m^{i_m}$$

[155] As in A is an integral domain, not that A is integral over k.

[156] Note that such an $n \leq m$ as there are only finitely many subsets of $a_1, \ldots, a_m$, the cardinality of each being bounded above by m.

Let α be the positive number defined by:

$$\xi = \max\{i_1 + \cdots + i_m : k_{i_1 \cdots i_m} \neq 0\}$$

Since p is not of degree zero we must have that $\xi > 0$, as ξ is the degree of p. Let T be the set defined as:

$$T = \{i_1 \cdots i_k : k_{i_1 \cdots i_m} \neq 0, i_1 + \cdots + i_k = \xi\}$$

We order this set with the *lexicographic order*, i.e. we say that:

$$i_1 \cdots i_m \leq j_1 \cdots j_m \Leftrightarrow \text{at the first index where } i_l \neq j_l,\ i_l < j_l$$

It is well known, and easily checked that this is a total order on the set of partitions of ξ and thus a total order on T. In particular, since T is finite there is a maximal element $j_1 \cdots j_m$.

For $i \neq m$ we define $r_i = (\xi + 1)^{m-1-i}$. We set $b_i = a_i - a_m^{r_i}$, and let B the k algebra generated by $\{b_1, \ldots, b_{m-1}\}$. Note that A is finitely generated as a B algebra since the map $B[x] \to A$ given by sending x to a_m is a surjection. We wish to show that A is integral over B and thus finite by Proposition 3.9.1. Since A is generated by a_m as a B algebra, it suffices to show that a_m is integral over B by Corollary 3.9.1.

We first claim that the polynomial:

$$\tilde{p}(x) = p(b_1 + x^{r_1}, \ldots, b_{m-1} + x^{r_{m-1}}, x) = \sum_{i_1 \cdots i_k} k_{i_1 \cdots i_m} (b_1 + x^{r_1})^{i_1} \cdots (b_{m-1} + x^{r_{m-1}})^{i_{m-1}} x_m^{i_m}$$

which clearly has coefficients in B, has it's highest degree element contained entirely in the term:

$$k_{j_1 \cdots j_m} \cdot (b_1 + x^{r_1})^{j_1} \cdots (b_{m-1} + x^{r_{m-1}})^{j_{m-1}} x_m^{j_m}$$

Indeed, we see that for all $i_1 \ldots i_k$, the highest degree element contained in the corresponding term has degree:

$$(\xi + 1)^{m-2} \cdot i_1 + \cdots + i_{m-1} + i_m$$

Let l be the first place where $j_l \neq i_l$, then $j_l > i_l$, and so we have that $j_l \cdot (\xi+1)^{m-1-l} > i_l \cdot (\xi+1)^{m-1-l}$. Each subsequent term consists of lower powers of $(\alpha + 1)$, and so we must have that:

$$(\xi + 1)^{m-2} \cdot j_1 + \cdots + j_{m-1} + j_m > (\xi + 1)^{m-2} \cdot i_1 + \cdots + i_{m-1} + i_m$$

It follows that the highest degree polynomial of $\tilde{p}(x)$ is really contained in the $j_1 \cdots j_k$ term. Therefore, we have that:

$$q(x) = \frac{1}{k_{j_1 \cdots j_m}} \tilde{p}(x) = \frac{1}{k_{j_1 \cdots j_m}} p(b_1 + x^{r_1}, \ldots, b_{m-1} + x^{r_{m-1}}, x)$$

is a monic polynomial satisfying:

$$q(a_m) = \frac{1}{k_{j_1 \cdots j_m}} p(b_1 + a_m^{r_1}, \ldots, b_{m-1} + a_m^{r_{m-1}}, a_m) = p(a_1, \ldots, a_m) = 0$$

hence A is integral over B, and thus finite as discussed.

Since B is a subalgebra of an integral domain, B is an integral domain. Let:

$$\begin{aligned} k[u_1, \ldots, u_{m-1}] &\longrightarrow B \subset A \\ u_i &\longmapsto b_i \end{aligned}$$

then $B \cong k[u_1, \ldots, u_{m-1}]/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$. In particular, $\mathrm{Frac}(B) = k(b_1, \ldots, b_{m-1})$. In fact, we have that:

$$K = k(a_1, \ldots, a_m)/k(b_1, \ldots, b_{m-1})$$

is algebraic, as $q \in k(b_1, \ldots, b_{m-1})[x]$ as well. It follows by Lemma 6.1.8 that:

$$\operatorname{tdeg}_k K = \operatorname{tdeg}_{\operatorname{Frac}(B)} K + \operatorname{tdeg}_k \operatorname{Frac}(B) = \operatorname{tdeg}_k \operatorname{Frac}(B)$$

hence by the inductive hypothesis there exists a finite extension:

$$k[y_1, \ldots, y_n] \to B$$

By Lemma 3.9.1 we have that the composition:

$$k[y_1, \ldots, y_n] \to B \to A$$

makes A a finite $k[y_1, \ldots, y_n]$ algebra. Since each map is injective, A is a finite extension of $k[y_1, \ldots, y_n]$, and letting α_i denote the image of y_i provides us with an algebraically independent subset of A, implying the claim. ☐

We will need the following lemma to make the connection between transcendence degree, and the Krull dimension of finitely generated integral domains.

Lemma 6.1.10. *Let $\phi : B \to A$ be an integral extension, then* $\dim A = \dim B$.

Proof. The fact that ϕ is an integral extension, means that Going Up (i.e. Lemma 3.10.4) holds for the induced map $\operatorname{Spec} A \to \operatorname{Spec} B$. In particular, if we have a chain of prime ideals of B:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

then by inductively applying Going Up, we obtain a chain of prime ideals in A:

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$$

where $\phi^{-1}(\mathfrak{q}_i) = \mathfrak{p}_i$. Note that $\mathfrak{q}_i \neq \mathfrak{q}_{i+1}$, hence $\dim B \leq \dim A$.

Now note that given a chain of prime ideals in A:

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$$

we obtain a chain of prime ideals in B given by:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

where $\mathfrak{p}_i = \phi^{-1}(\mathfrak{q}_i)$. If we can show that $\phi^{-1}(\mathfrak{q}_i) \neq \phi^{-1}(\mathfrak{q}_{i+1})$ for all i, then we will have $\dim A \leq \dim B$ and be done.

For any $\mathfrak{p} \in \operatorname{Spec} B$ we have that the fibre is:

$$f^{-1}(\mathfrak{p}) = \operatorname{Spec} k_{\mathfrak{p}} \otimes_B A$$

Note that if $\mathfrak{q}_i, \mathfrak{q}_{i+1} \in \operatorname{Spec} A$ satisfy $\phi^{-1}(\mathfrak{q}_i) = \phi^{-1}(\mathfrak{q}_{i+1})$ then both $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ lie in the above fibre for some $\mathfrak{p}$. If $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ lie in the same fibre then $\dim k_{\mathfrak{p}} \otimes_B A > 0$ by definition, hence it suffices to show that $\dim k_{\mathfrak{p}} \otimes_B A = 0$. By Proposition 3.9.2, integral morphisms are preserved by base change, hence $k_{\mathfrak{p}} \to k_{\mathfrak{p}} \otimes_B A$ is integral. Moreover, it is injective as $k_{\mathfrak{p}}$ is a field. It thus suffices to show that any integral extension $k \to A$ implies $\dim A = 0$. Let $\mathfrak{p} \subset A$ be a prime, then we claim that $A/\mathfrak{p}$ is a field, and thus every prime is maximal. Note that the composition $k \to A/\mathfrak{p}$ is now an integral extension of k into an integral domain. Let $[a] \in A/\mathfrak{p}$ be nonzero, then we have that there exists a monic polynomial of smallest possible degree with coefficients in k satisfying:

$$[a]^n + c_{n-1}[a]^{n-1} + \cdots + c_0 = 0$$

Since $A/\mathfrak{p}$ is an integral domain, we then have that $c_0 = 0$, as otherwise $[a] = 0$, or the polynomial is not of smallest degree. In particular we have that:

$$1 = -c_0^{-1}([a]^n + c_{n-1}[a]^{n-1} + \cdots + c_1[a])$$

so:

$$[a]^{-1} = -c_0^{-1}([a]^{n-1} + c_{n-2}[a]^{n-1} + \cdots + c_1)$$

implying that $A/\mathfrak{p}$ is a field. It follows that every prime ideal is maximal and thus $\dim A = 0$. In particular, we have that $k_\mathfrak{p} \otimes_B A$ is zero dimensional ring, so if $\phi^{-1}(\mathfrak{q}_i) = \phi^{-1}(\mathfrak{q}_{i+1})$ then we cannot have $\mathfrak{q}_i \subset \mathfrak{q}_{i+1}$ as this would imply that $\dim k_\mathfrak{p} \otimes_B A$ has dimension greater than zero. $\square$

The following result is our *entire motivation* of going over the notion of transcendence degree:

Theorem 6.1.2. *Let A be an integral finitely generated k algebra, and K it's field of fractions as in Lemma 6.1.9; then* $\dim A = \operatorname{tdeg}_k K$.

Proof. We prove this on induction of $\operatorname{tdeg}_k K$. If $\operatorname{tdeg}_k K = 0$, then we have by Noether Normalization a finite extension $k \to A$. Proposition 3.9.1 and Lemma 6.1.10 then imply the base case.

Supposing this holds for transcendence degrees less than n, let $\operatorname{tdeg}_k K = n$. Again by Noether Normalization, we have a finite extension $k[x_1, \ldots, x_n] \to A$. It thus suffices to show that $\dim k[x_1, \ldots, x_n] = n$ by Lemma 6.1.10. Note that $\dim k[x_1, \ldots, x_n] \geq n$ as we always have the following chain of ideals:

$$\langle 0 \rangle \subset \langle x_1 \rangle \subset \cdots \subset \langle x_1, \ldots, x_n \rangle$$

Now suppose there exists a chain of prime ideals:

$$\langle 0 \rangle \subset \mathfrak{p}_1 \cdots \subset \mathfrak{p}_m$$

where $m > n$. Then take an irreducible element $f \in \mathfrak{p}_1$, and construct the chain of prime ideals:

$$\langle 0 \rangle \subset \langle f \rangle \cdots \subset \mathfrak{p}_m$$

We see that $\dim k[x_1, \ldots, x_n]/\langle f \rangle$ has dimension at least $m - 1 \geq n$. We claim this is a contradiction, as $\operatorname{tdeg}_k \operatorname{Frac}(k[x_1, \ldots, x_n]/\langle f \rangle) = n - 1$. Indeed, without loss of generality assume that x_n occurs in f, then with $B = k[x_1, \ldots, x_n]/\langle f \rangle$, and $b_i = [x_i]$ we claim that the set $\{[x_i]\}_{i=1}^{n-1}$ is a transcendence basis for $\operatorname{Frac}(B)/k$. Consider the map:

$$\begin{aligned} k[y_1, \ldots, y_{n-1}] &\longrightarrow B \subset \operatorname{Frac}(B) \\ y_i &\longmapsto b_i \end{aligned}$$

and suppose that $p \mapsto 0 \in B$. If

$$p = \sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} y_1^{i_1} \cdots y_n^{i_{n-1}}$$

we have that:

$$\sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} [x_1]^{i_1} \cdots [x_{n-1}]^{i_{n-1}} = 0 \Rightarrow \sum_{i_1 \cdots i_{n-1}} k_{i_1 \cdots i_{n-1}} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \in \langle f \rangle$$

which is impossible by construction, hence $\{b_i\}_{i=1}^{n-1}$ is an algebraically independent set. We claim that $\operatorname{Frac}(B)/k(b_1, \ldots, b_{n-1})$ is algebraic. Indeed, if:

$$f = \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

let $g \in k(b_1, \ldots, b_{n-1})[x]$ be given by:

$$g = \sum_{i_1 \cdots i_n} c_{i_1 \cdots i_n} b_1^{i_1} \cdots b_{n-1}^{i_{n-1}} x^{i_n}$$

then clearly $g(b_n) = 0$, so $\operatorname{Frac}(B)/k(b_1, \ldots, b_{n-1})$ is indeed algebraic. It follows that $\{b_1, \ldots, b_{n-1}\}$ is a transcendence basis, thus $\dim B = n - 1$, contradicting the existence of a chain of prime ideals in $k[x_1, \ldots, x_n]$ of length $m > n$. Therefore, $\dim k[x_1, \ldots, x_n] \leq n$, implying equality, and so $\dim A = n$ as well. $\square$

We now have a slick proof Zariski's lemma:

Theorem 6.1.3. *Let A be a finitely generated k algebra, and $\mathfrak{m} \in |\operatorname{Spec} A|$. Then $k_{\mathfrak{m}}/k$ is a finite extension of k.*

Proof. Note that the residue field $k_{\mathfrak{m}}$ is given precisely by $A/\mathfrak{m}$. In particular, we know that $k_{\mathfrak{m}}$ has dimension 0 as it is a field, and that $k_{\mathfrak{m}}$ is a finitely generated k algebra, via the composition:

$$k \hookrightarrow A \to k_{\mathfrak{m}}$$

The field of fractions of $k_{\mathfrak{m}}$ is then obviously $k_{\mathfrak{m}}$, hence we have that $\operatorname{tdeg}_k k_{\mathfrak{m}} = 0$. In particular, $k_{\mathfrak{m}}/k$ is an algebraic, i.e. integral extension, and is finitely generated, hence by Proposition 3.9.2 we have that $k_{\mathfrak{m}}/k$ is finite. $\square$

We end with the following result, which we could have proved at any point, and is nuanced version of the fact that if k is algebraically closed then every maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $\langle x_1 - a_1, \ldots, x_1 - a_n \rangle$.

Proposition 6.1.3. *Let k be an algebraically closed field, and I be a potentially infinite set such that $|I| < |k|$.*[157] *Then every maximal ideal of the polynomial ring $k[\{x_i\}_{i \in I}]$ is of the form $\langle \{x_i - a_i\}_{i \in I} \rangle$ for $a_i \in k$.*

Proof. Let $A = k[\{x_i\}_{i \in I}]$, and $\mathfrak{m} \in |\operatorname{Spec} A|$. Set $K = A/\mathfrak{m}$, then as vector spaces:

$$\dim_k K \leq \dim_k A = |I|$$

Suppose now that $t \in K$ is transcendental over k, then we claim that the elements $B = \{1/(t-a)\}_{a \in k}$ are linearly independent. Indeed, suppose that there exists $b_1, \ldots, b_n$ such that:

$$\frac{b_1}{t - a_1} + \cdots + \frac{b_n}{t - a_n} = 0$$

This then implies that:

$$b_1 \cdot (t - a_2) \cdots (t - a_n) + \cdots + b_n (t - a_1) \cdots (t - a_{n-1}) = 0$$

If the b_i are all non zero then t is not transcendental, hence we must have that $b_i = 0$ and B forms a linearly independent set. However, the cardinality of B is equal to the cardinality of k, implying that the dimension of A as a k vector space is greater than $|I|$, a contradiction, so K/k must be an algebraic extension. However, K is algebraically closed, hence $K = k$, and so we know that $\mathfrak{m}$ must be the kernel of a ring homomorphism $A \to k$.

Note that a quotient map $A \to A/\mathfrak{m} \cong k$ is obviously a k-algebra homomorphism Since A is the free object on I in the category k-algebras we therefore have that the map $A \to k$ is entirely determined by an element of the form $(a_i)_{i \in I}$ where each $a_i \in I$. It follows that for some choice of a_i, $\mathfrak{m}$ contains the ideal $\langle \{x_i - a_i\}_{i \in I} \rangle$ which is obviously maximal, hence $\mathfrak{m} = \langle \{x_i - a_i\}_{i \in I} \rangle$. $\square$

6.2 Dimension of Schemes

[157] Note that every algebraically closed field is of infinite cardinality.

$\mathscr{O}_X$ Modules II: Sheaf Cohomology

7.1 Some Homological Algebra: Derived Functors

In 1957, Grothendieck revolutionized the field of homological algebra with the publishing of his Tôhoku paper. In this paper, Grothendieck not only allowed formalized the connections between various categories, now all known as abelian categories, but he also brought many seemingly different constructions in homological algebra under the same umbrella with the concept of the derived functor. We already know about abelian categories, but what is a derived functor? We motivate the idea by looking at the relative homology of singular chains, the Tor construction, and the Ext construction. If the reader is unfamiliar with any of these, do not worry; we will explicitly construct Tor and Ext in this chapter, and will have no need to think about relative homology of singular chains again.

In each set up we start with a short exact sequence. Let X be at topological space, and $Z \subset X$ a subspace. Then we obtain a short exact sequence of singular chains:

$$0 \longrightarrow C_\bullet(Y) \longrightarrow C_\bullet(X) \longrightarrow C_\bullet(X,Y) \longrightarrow 0$$

Taking the zeroth homology of each complex is a right exact functor[158] $\mathrm{Ch}_+(\mathrm{Ab}) \to \mathrm{Ab}$, so we obtain the exact sequence:

$$H_0(Y) \longrightarrow H_0(X) \longrightarrow H_0(X,Y) \longrightarrow 0$$

which can be extended indefinitely to the left via the higher homology groups, and connecting morphisms $\delta_n : H_n(X,Y) \to H_{n-1}H(Y)$:

$$\cdots \longrightarrow H_1(X,Y) \longrightarrow H_1(X,Y) \xrightarrow{\delta_1} H_0(Y) \longrightarrow H_0(X) \longrightarrow H_0(X,Y) \longrightarrow 0$$

The higher homology functors $H_n : \mathrm{Ch}_+(\mathrm{Ab}) \to \mathrm{Ab}$ will be the *left derived functors* of the zeroth homology functor.

Suppose we have an exact sequence of A-modules:

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

then applying $- \otimes_A Q$ gives an exact sequence:

$$M \otimes_A Q \longrightarrow N \otimes_A Q \longrightarrow P \otimes_A Q \longrightarrow 0$$

which can be extended indefinitely to the left with the functors $\mathrm{Tor}_i^A(-,Q) : A - \mathrm{Mod} \to \mathrm{Ab}$:

$$\cdots \longrightarrow \mathrm{Tor}_1^A(N,Q) \longrightarrow \mathrm{Tor}_1^A(P,Q) \longrightarrow \mathrm{Tor}_0^A(M,Q) \longrightarrow \mathrm{Tor}_0^A(N,Q) \longrightarrow \mathrm{Tor}_0^A(P,Q) \longrightarrow 0$$

where $\mathrm{Tor}_A^0(-,Q) = - \otimes_A Q$. The higher Tor functors will be *left derived functors* of the tensor product.

Given the same exact sequence of A modules, suppose we apply $\mathrm{Hom}_A(P,-)$ to obtain a left exact sequence:

$$0 \longrightarrow \mathrm{Hom}_A(P,M) \longrightarrow \mathrm{Hom}_A(P,N) \longrightarrow \mathrm{Hom}_A(P,Q)$$

[158] $\mathrm{Ch}_+(\mathscr{C})$ for any abelian category $\mathscr{C}$ denotes the category of chain complexes bounded below by zero. We will give a precise definition of this later.

which can be extended indefinitely to the right with the functors $\operatorname{Ext}^i_A(P, -) : A - \operatorname{Mod} \to \operatorname{Ab}$:

$$0 \longrightarrow \operatorname{Ext}^0_A(P, M) \longrightarrow \operatorname{Ext}^0_A(P, N) \longrightarrow \operatorname{Ext}^0_A(P, Q) \longrightarrow \operatorname{Ext}^1_A(P, M) \longrightarrow \operatorname{Ext}^1_A(P, N) \longrightarrow \cdots$$

where $\operatorname{Ext}^0_A(P, -) = \operatorname{Hom}_A(P, -)$. We will eventually view the higher Ext functor as the *right derived functors* of $\operatorname{Hom}_A(P, -)$.

In all of the above situations, we had an exact sequence in some abelian category, applied a right or left exact functor, and then extended the exact sequence indefinitely to the left or right respectively to obtain a long exact sequence. With the concept of derived functors, we will make the procedures above systematic, and then eventually define sheaf cohomology to be the right derived functor of the global section functor, which is left exact.

Before we begin our treatment of derived functors, we have two categorical concepts that we are mostly used to but have to make precise. Namely we need to know what it means to be an exact sequence in a general abelian category, and we need to know how to deal with limits and colimits in full generality. The first issue rectifies a graver sin: we have used repeatedly that the snake lemma, the splitting lemmas, and the five lemma hold in arbitrary abelian categories, but these results require the concept of an exact sequence in an abelian category. Though we will not prove the aforementioned results in generality, we begin remedying this situation with a categorical definition of the image:

Definition 7.1.1. Let $\mathscr{C}$ be a category, and $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$, then the **image** of f, an object $\operatorname{im} f$ equipped with a monomorphism $\imath : \operatorname{im} f \to B$, and a morphism $\pi : A \to \operatorname{im} f$ such that for any morphism $g : A \to C$, and monomorphism $h : C \to B$ such that $h \circ g = f$ there exists a unique morphism $u : \operatorname{im} f \to C$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi"] \arrow[ddr, bend right, "g"'] & & B \\
& \operatorname{im} f \arrow[ur, "\imath"] \arrow[d, "\exists ! u"] & \\
& C \arrow[uur, bend right, "h"'] &
\end{tikzcd}$$

We now show that images always exist in an abelian category:

Lemma 7.1.1. *Let $\mathscr{C}$ be an abelian category, and $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$, then $\operatorname{im} f$ exists, and is unique up to unique isomorphism. In particular $\operatorname{im} f \cong \ker(\operatorname{coker}(f))$.*[159]

Proof. If $\operatorname{im} f$ exists, as it defined via a universal property, it will obviously be unique up to unique isomorphism. We will show that if π_f is the morphism $B \to \operatorname{coker} f$ then $\ker(\pi_f)$ is the image of f. In particular we note that in the category of A-modules we have that $\operatorname{coker} f = B / \operatorname{im} f$, where $\operatorname{im} f$ is the obvious A submodule. It follows that $\ker(\pi_f) = \operatorname{im} f$ essentially by definition, hence this is the natural thing to expect.

We have the following two diagrams for any C:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] & B \arrow[r, "g"] \arrow[d, "\pi_f"] & C \\
& \operatorname{coker} f \arrow[ur, dashed, "\exists ! \theta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "h"] \arrow[rr, bend left, "0"] \arrow[d, dashed, "\exists ! \mu"'] & B \arrow[r, "\pi_f"] & \operatorname{coker} \pi_f \\
\ker \pi_f \arrow[ur, "\imath"'] & &
\end{tikzcd}$$

We get the morphism $\imath : \ker \pi_f \to B$, and the fact that it is a monomorphism for free. Letting $C = A$, and $h = f$ in the right diagram we get the unique morphism $\pi : A \to \ker \pi_f$.

Now suppose that $g : A \to C$ and $h : C \to B$ satisfy $h \circ g = f$, with h monic. Since h is monic, we have that there is some $j : B \to J$ such that (C, h) satisfies the universal property of $\ker j$. In particular,

[159] Where by $\ker(\operatorname{coker}(f))$ we mean $\ker((B \to \operatorname{coker} f))$.

it suffices to show that $\jmath \circ \imath = 0$ as then we have the following diagram:

$$\begin{tikzcd}
\ker \pi_f \arrow[r, "\imath"] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists! u"'] & B \arrow[r, "m"] & J \\
& C \arrow[u, "h"'] &
\end{tikzcd}$$

We have that $j \circ f = j \circ h \circ g = 0$, so the universal property of $\operatorname{coker} f$ implies that there is a unique morphism $\beta : \operatorname{coker} f \to C$ such that $\beta \circ \pi_f = j$. We thus have that $j \circ \imath = \beta \circ \pi_f \circ \imath = 0$, thus the above diagram implies the existence of a unique $u : \ker \pi_f \to C$ satisfying $h \circ u = i$. Any other such u' satisfying $h \circ u' = i$ would have to make the above diagram commute, so the universal property of the kernel implies that u is unique as desired. ☐

Now suppose we have a sequence of the form:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $g \circ f = 0$. The universal property of $\imath_g : \ker g \to B$ then yields a morphism $\alpha : A \to \ker g$ such that $\imath_g \circ \alpha = f$. The universal property of $\operatorname{im} f$ then implies the existence of a unique morphism $\operatorname{im} f \to \ker g$.

Definition 7.1.2. Let $\mathscr{C}$ be an abelian category, then a sequence in $\mathscr{C}$:

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is said to be **exact at B** if $g \circ f = 0$ and the unique morphism $\operatorname{im} f \to \ker g$ is an isomorphism. A sequence in $\mathscr{C}$ is called an **exact sequence** if at every object it is exact. Short exact sequences, and long exact sequences are defined exactly the same as they are in $A - \mathrm{Mod}$.

We note , but do not prove, that with this definition it can be shown that both the contravariant and covariant Hom functors are left exact in any category. We do however prove the following useful fact:

Lemma 7.1.2. *Suppose we have a commutative diagram in an abelian category of the form:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
D \arrow[r, "h"] & E \arrow[r, "i"] & F
\end{tikzcd}$$

If the top row is exact, and the vertical morphisms are isomorphisms, then the bottom row is exact.

Proof. Note that the commutativity of the diagram implies that:

$$i \circ h = c \circ g \circ b^{-1} \circ b \circ f \circ a^{-1} = 0$$

because $g \circ f = 0$.

Let $\alpha : A \to \ker g$, and $\alpha' : D \to \ker i$ be the unique morphisms so that if $\imath_g : \ker g \to B$ and $\imath_i : \ker i \to E$ are the inclusions we have that $\imath_g \circ \alpha = f$, $\imath_i \circ \alpha' = h$. The morphisms $u : \operatorname{im} f \to \ker g$ and $u' : \operatorname{im} h \to \ker i$ then fit into the following commutative diagrams:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi_f"] \arrow[ddr, "\alpha"'] & & B \\
& \operatorname{im} f \arrow[ur, "\imath_f"] \arrow[d, "u"] & \\
& \ker g \arrow[uur, "\imath_g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, "\pi_h"] \arrow[ddr, "\alpha'"'] & & E \\
& \operatorname{im} h \arrow[ur, "\imath_h"] \arrow[d, "u'"] & \\
& \ker i \arrow[uur, "\imath_i"'] &
\end{tikzcd}$$

In particular, since the top row is exact we know that u is an isomorphism.

We define a morphism $\psi : \operatorname{im} f \to \operatorname{im} h$ to be the one fitting into the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi_f"] \arrow[ddr, bend right, "\pi_h \circ a"'] & & B \\
& \operatorname{im} f \arrow[ur, "\imath_f"] \arrow[d, "\psi"] & \\
& \operatorname{im} h \arrow[uur, bend right, "b^{-1} \circ i_h"'] &
\end{tikzcd}$$

We claim this is an isomorphism. Indeed, there define ψ^{-1} to be the morphism fitting into the following diagram:

$$\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, "\pi_h"] \arrow[ddr, bend right, "\pi_f \circ a^{-1}"'] & & E \\
& \operatorname{im} h \arrow[ur, "\imath_h"] \arrow[d, "\psi^{-1}"] & \\
& \operatorname{im} f \arrow[uur, bend right, "b \circ i_f"'] &
\end{tikzcd}$$

Now note that:

$$\psi \circ \psi^{-1} \circ \pi_h = \psi \circ \pi_f \circ a^{-1} = \pi_h \circ a \circ a^{-1} = \pi_h$$

whilst:

$$b^{-1} \circ i_h \circ \psi \circ \psi^{-1} = i_f \Rightarrow i_h \circ \psi \circ \psi^{-1} = b \circ i_f$$

It follows that $\psi \circ \psi^{-1}$ makes the following diagram commute:

$$\begin{tikzcd}
D \arrow[rr, "h"] \arrow[dr, "\pi_h"] \arrow[ddr, bend right, "\pi_h"'] & & E \\
& \operatorname{im} h \arrow[ur, "\imath_h"] \arrow[d, "\psi \circ \psi^{-1}"] & \\
& \operatorname{im} h \arrow[uur, bend right, "i_h"'] &
\end{tikzcd}$$

but so does the identity, hence $\psi \circ \psi^{-1} = \mathrm{Id}$. The same argument shows that $\psi^{-1} \circ \psi = \mathrm{Id}$ as well.

We define a morphism $\phi : \ker g \to \ker i$ to be the morphism fitting into the following diagram:

$$\begin{tikzcd}
\ker g \arrow[rr, "b \circ \imath_g"] \arrow[dr, "\phi"'] & & E \arrow[r, "i"] & D \\
& \ker i \arrow[ur, "\imath_i"'] & &
\end{tikzcd}$$

Note that this makes sense as $i \circ b = c \circ g$, and $g \circ \imath_g = 0$. We define ϕ^{-1} to be the morphism fitting into the following diagram:

$$\begin{tikzcd}
\ker i \arrow[rr, "b^{-1} \circ \imath_i"] \arrow[dr, "\phi^{-1}"'] & & E \arrow[r, "i"] & D \\
& \ker g \arrow[ur, "\imath_g"'] & &
\end{tikzcd}$$

which again commutes for the same reason. We have that $\phi \circ \phi^{-1}$ satisfies $\imath_i \circ \phi \circ \phi^{-1} = b \circ i_g \circ \phi^{-1} = i_i$. Since $\imath_i$ is a monomorphism, and the identity also satisfies $\imath_i \circ \mathrm{Id} = \imath_i$, it follows that $\phi \circ \phi^{-1} = \mathrm{Id}$. Similarly for $\phi^{-1} \circ \phi$ hence ϕ is an isomorphism.

We claim that $u' = \phi \circ u \circ \psi^{-1}$. Consider the morphism $\phi \circ u \circ \psi^{-1} \circ \pi_h$. By post composing with $\imath_i$ we find that:

$$\begin{aligned}
\imath_i \circ \phi \circ u \circ \psi^{-1} \circ \pi_h =& b \circ i_g \circ u \circ \psi^{-1} \circ \pi_h \\
=& b \circ \imath_f \circ \psi^{-1} \circ \pi_h \\
=& i_h \circ \pi_h \\
=& h
\end{aligned}$$

However α' is the unique morphism so that $\imath_i \circ \alpha' = h$. It follows that $\alpha' = \phi \circ u \circ \psi^{-1} \circ \pi_h$, and so the left triangle of the diagram defining u' commutes. For the right triangle we have that:

$$\imath_i \circ \phi \circ u \circ \psi^{-1} = i_h$$

by the same computation. It follows that $\phi \circ u \circ \psi^{-1}$ makes the diagram defining u' commute so by uniqueness $u' = \phi \circ u \circ \psi^{-1}$. Since u' is the composition of isomorphisms, it also must be an isomorphism and so the sequence is exact as desired. ☐

We now turn our attention towards our second general categorical concept. Potentially we could get away without making precise the concept of limits and colimits, but when you can only work on the level of morphisms, objects, and diagrams, this is a bit akin to blindfolding yourself, and tying two hands be your back.

Definition 7.1.3. Let $F : J \to \mathscr{C}$ be a covariant functor. A **cone** of (J, F) is the data of an object N in $\mathscr{C}$ along with morphisms $\psi_i : N \to F(i)$ for all objects i in J such that for any morphism $f : i \to j$ we have that $F(f) \circ \psi_i = \psi_j$. We denote the data of a cone by (N, ψ). The **limit** of (J, F) is a cone (L, ϕ) such that for any cone N there is a unique morphism $\alpha : N \to L$ such that the following diagram commutes:

$$\begin{tikzcd}
 & N \arrow[d, "\exists ! \alpha"] \arrow[ddl, bend right, "\psi_i"'] \arrow[ddr, bend left, "\psi_j"] & \\
 & L \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\
F(i) \arrow[rr, "F(f)"] & & F(j)
\end{tikzcd}$$

A **cocone** of (J, F) is the data of an object N in $\mathscr{C}$ along with morphisms $\psi : F(i) \to N$ for all objects i in J such that for any morphism $f : i \to j$ we have that $\psi_j \circ F(f) = \psi_i$. We denote the data of a cocone by (N, ψ) as well. The **colimit** of (J, F) is a cocone (L, ϕ) such that any cocone N there is a unique morphism $\alpha : L \to N$ making the following diagram commute:

$$\begin{tikzcd}
F(i) \arrow[rr, "F(f)"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & F(j) \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
 & L \arrow[d, "\exists ! \alpha"] & \\
 & N &
\end{tikzcd}$$

We call (co)limits **small** if J is a small category[160], and **finite** if J is a finite category.[161] When the J is at small or finite, we at times call the (J, F) a *diagram of shape* J.

We note that we can also formulate all of these concepts with contravariant functors by switching the arrows in J. Equivalently, given a contravariant functor $F : J \to \mathscr{C}$, one can just take the (co)limits of (J, F) to be the (co)limits of (J^{op}, F), as any contravariant functor $J \to \mathscr{C}$ is a covariant functor $J^{\mathrm{op}} \to \mathscr{C}$.

[160]The objects of J are a set, and the Hom sets in J are sets.
[161]The objects of J are finite set, and the Hom sets are finite sets.

Moreover, note that we need not take into account identity morphisms $\mathrm{Id} : i \to i$ when discussing limits and colimits, as any object $\mathscr{C}$ which is a (co)cone when only taking into account non identity morphisms, will immediately be a (co)cone in the usual sense. We briefly look at some examples:

Example 7.1.1. Let X be a topological space, and J_x the sub category of $\mathscr{C}(X)$ consisting only of those open sets which contain x. A presheaf $\mathscr{F}$ is a contravariant functor from $\mathscr{C}(X)$ to $\mathrm{Ab}, \mathrm{Set}$, or Ring, and so the colimit of $(J, \mathscr{F})$ is, as mentioned above, the colimit of $(J^{\mathrm{op}}, \mathscr{F})$. If one writes out the diagram, and unravels the definition laid out in Definition 1.2.2, this obviously gives us the stalk $\mathscr{F}_x$.

More concretely, if I is a partially ordered set, then I forms a small category where $\mathrm{Hom}_I(i,j)$ is empty if $i \not< j$ and consists of a single arrow if $i < j$. If $F : I \to \mathscr{C}$ is a functor then the colimit of (I, F) is the direct limit over I of F:

$$\mathrm{colim}(I, F) = \varinjlim_{i \in I} F(i)$$

whilst the inverse limit over I of F is given by:

$$\lim(I, F) = \varprojlim_{i \in I} F(i)$$

Example 7.1.2. Consider the category J consisting of two objects 1 and 2, where $\mathrm{Hom}_J(1,2)$ and $\mathrm{Hom}_J(2,1)$ are empty, and $\mathrm{Hom}_J(i,i)$ consists only of the identity morphism. Let $F : J \to \mathscr{C}$ the functor sending 1 to A and 2 to B for some objects A and B. A cone of (J, F) is then the data of an object N equipped with morphisms $N \to A$ and $N \to B$. The limit of (J, F), if it exists in $\mathscr{C}$, is then the direct product $A \times B$. Similarly the colimit of of (J, F), if it exists in $\mathscr{C}$, is the coproduct $A \coprod B$.

Let J be the category consisting of three objects 1, 2, and 3. Let $\mathrm{Hom}_J(1,2)$ and $\mathrm{Hom}_J(2,1)$ be empty, $\mathrm{Hom}_J(i,i)$ consist only of the identity, and $\mathrm{Hom}_j(1,3)$ and $\mathrm{Hom}_J(2,3)$ consist only of a singular arrow. Let $F(1) = A$, $F(2) = B$, and $F(3) = C$, and let $f : A \to C$ and $g : \to C$ be the morphisms coming from $\mathrm{Hom}_J(1,3)$ and $\mathrm{Hom}_J(2,3)$. A cone of (J, F) then consists of an object N, along with morphisms $p_A : N \to A$, $p_B : N \to B$, and $h : N \to C$ such that $f \circ p_a = g \circ p_B = h$. Note that the final condition implies specifying h is superfluous as it is already determined p_A and p_B. It is then easy to see that:

$$\lim(J, F) = A \times_C B$$

i.e. the limit of this functor is the fibre product. The colimit is the fibre coproduct, which can be recognized in the category of rings as the tensor product.

As we are about to show, in an abelian category, finite limits and colimits always exist, which is our main motivation for discussing this concept. Before proving this however, we recall some general facts/conventions in a general abelian category:

- If $f : A \to B$ is a morphism, then we say f is injective if $\ker f = 0$ and surjective if $\operatorname{coker} f = 0$.[162]
- f is a monomorphisms if and only if it is injective, and f is an epimorphism if and only if it is surjective.
- As in the category of sheaves of abelian groups, every monomorphism is the kernel morphism of it's own cokernel morphism, and every epimorphism is the cokernel morphism of it so own kernel morphism.
- If $(f, g) : A \oplus B \to D$ is a morphism coming from the universal property of the coproduct, then $(f, g) = f \circ \pi_A + g \circ \pi_B$.
- if $f \times g : D \to A \oplus B$ is the morphism coming from the universal property of the direct product, then $f \times g = \imath_A \circ f + \imath_B \circ g$.
- If f and g are morphisms, then $f \circ (-g) = (-f) \circ g = -(f \circ g)$

All of the above are easy to prove in any of the abelian categories we care about, and are not much harder to prove in an arbitrary abelian category. With these fact/conventions out of the way, we prove the following result:

[162] Equivalently, Lemma 7.1.1 implies that f is surjective if and only if the the induced morphism $\operatorname{im} f \to B$ is an isomorphism.

Lemma 7.1.3. *Let $\mathscr{C}$ be an abelian category, then finite limits and colimits exist.*

Proof. Let J be a finite category, and $F : J \to \mathscr{C}$ be a functor. If J is empty then both the colimit and limit of (J, F) is the zero object. Supposing that J is finite, identify the objects of J with the set $\{1, \ldots, n\}$, and set $A_i = F(i)$. The Hom sets are also finite, hence the set:

$$S = \{\text{all non identity morphisms between objects in } J\}$$

is a finite set. Let i_f and j_f denote the source and target of any morphism $f \in S$, then set:

$$B = \bigoplus_{i=1}^{n} A_i \qquad \text{and} \qquad C = \bigoplus_{f \in S} A_{i_f}$$

To construct a morphism $C \to B$, it suffices to specify morphisms $A_{i_f} \to B$. Each A_{i_f} corresponds to a morphism $A_{i_f} \to A_{j_f}$, and $j_f \in \{1, \ldots, n\}$ hence if $\imath_{j_f} : A_{j_f} \to B$ is the inclusion map, we have that $\imath_{j_f} \circ f$ is a morphism $A_{i_f} \to B$, giving us a morphism $g : C \to B$. By the same logic there we also obtain a morphism $h : C \to B$ induced by the morphisms $\imath_{i_f} : A_{i_f} \to B$. Note that the universal property of the coproduct implies that $g \circ \imath_{i_f} = \imath_{j_f} \circ F(f)$ and $h \circ \imath_{i_f} = \imath_{i_f}$.

We claim that the colimit of (J, F) is $L = \operatorname{coker}(g - h)$. Let $\pi : B \to L$ be the cokernel morphism, and $\imath_i : A_i \to B$ the natural inclusion. We then obtain a morphisms $\phi_i : A_i \to L$ given by $\pi \circ \imath_i$. If $f : i \to j$ is a morphism in J then we need to check that $\phi_j \circ F(f) = \phi_i$. We have that:

$$\phi_j \circ F(f) = \pi \circ \imath_j \circ F(f) = \pi \circ g \circ \imath_{i_f}$$

Since:

$$\pi \circ (g - h) = 0$$

we have that $\pi \circ g = \pi \circ h$ and thus:

$$\phi_j \circ F(f) = \pi \circ h \circ \imath_{i_f} = \pi \circ \imath_{i_f} = \phi_i$$

as desired. It follows that (L, ϕ) is a cocone of (J, F), so let (N, ψ) be another cocone. We need to construct a unique morphism $L \to N$ such that the relevant diagram commutes. The data of morphisms $\psi_i : A_i \to N$ induces a map $\theta : B \to N$ satisfying $\theta \circ \imath_i = \psi_i$. Note that we can write g and h as the sums:

$$g = \sum_{f \in S} \imath_{j_f} \circ F(f) \circ \pi_{i_f} \qquad \text{and} \qquad h = \sum_{f \in S} \imath_{i_f} \circ \pi_{i_f}$$

π_{i_f} is the natural projection. hence:

$$\begin{aligned}
\theta \circ (g - h) &= \sum_{f \in S} \theta \circ i_{j_f} \circ F(f) \circ \pi_{i_f} - \theta \circ \imath_{i_f} \circ \pi_{i_f} \\
&= \sum_{f \in S} \psi_{j_f} \circ F(f) \circ \pi_{i_f} - \psi_{i_f} \circ \pi_{i_f} \\
&= \sum_{f \in S} \psi_{i_f} \circ \pi_{i_f} - \psi_{i_f} \circ \pi_{i_f} \\
&=0
\end{aligned}$$

The universal property of the cokernel then implies that there exists a unique morphism $\alpha : L \to N$ such that $\alpha \circ \pi = \theta$.

We need to check that $\alpha \circ \phi_i = \psi_i$, however $\phi_i = \pi \circ \imath_i$, hence:

$$\alpha \circ \phi_i = \theta \circ \imath_i = \psi_i$$

Note that if any other morphism $\beta : L \to N$ satisfies $\beta \circ \phi_i = \psi_i$ then we obtain that:

$$\begin{aligned}\beta \circ \pi =& \beta \circ \sum_i \pi \circ \imath_i \circ \pi_i \\ =& \sum_i \psi_i \circ \pi_i \\ =& \theta\end{aligned}$$

hence β would also make the relevant cokernel diagram commute and thus be equal to α. It follows that α is unique, and thus L is the colimit of (J, F) as desired.

We now set:

$$D = \bigoplus_{f \in S} A_{j_f}$$

and define a morphism $g : B \to D$ by the universal property of the product applied to the morphisms $F(f) \circ \pi_{i_f} : B \to A_{j_f}$. Similarly, we obtain a morphism $h : B \to D$ induced by the morphisms $\pi_{j_f} : B \to A_{j_f}$ as $j_f \in \{1, \ldots n\}$. Note that $\pi_{j_f} \circ g = F(f) \circ \pi_{i_f}$ and $\pi_{j_f} \circ h = \pi_{j_f}$.

We claim that the limit of (J, F) is $L = \ker(g - h)$. We need to construct morphisms $\phi_i : L \to A_i$ such that for any morphism $f : i \to j$ we have that $F(f) \circ \phi_i = \phi_j$. Let $\imath : L \to B$ the kernel morphism, and $\pi_i : B \to A_i$, then we define ϕ_i by $\pi_i \circ \imath$. Since $g - h \circ \imath = 0$ by definition, we have that for any morphism $f : i \to j$:

$$F(f) \circ \phi_i = F(f) \circ \pi_i \circ \imath = \pi_j \circ g \circ \imath = \pi_j \circ h \circ \imath = \pi_j \circ \imath = \phi_j$$

hence (L, ϕ) is a cone of (J, F).

Now let (N, ψ) be another cone of (J, F). The data of morphisms $\psi_i : N \to A_i$ yields a morphism $\theta : N \to B$ satisfying $\pi_i \circ \theta = \psi_i$. As before we can write g and h as sums:

$$g = \sum_{f \in S} \imath_{j_f} \circ F(f) \circ \pi_{i_f} \qquad \text{and} \qquad h = \sum_{f \in S} \imath_{j_f} \circ \pi_{j_f}$$

where the projection and inclusion morphisms now have opposite sources and targets as in the colimit case. It follows that:

$$\begin{aligned}g - h \circ \theta =& \sum_{f \in S} \imath_{j_f} \circ F(f) \circ \pi_{i_f} \circ \theta - \imath_{i_f} \circ \pi_{i_f} \circ \theta \\ =& \sum_{f \in S} \imath_{j_f} \circ F(f) \circ \psi_{i_f} - \imath_{j_f} \circ \psi_{j_f} \\ =& \sum_{f \in S} \imath_{j_f} \circ \psi_{j_f} - \imath_{j_f} \circ \psi_{j_f} \\ =& 0\end{aligned}$$

The universal property of the kernel implies that there is a unique morphism $\alpha : N \to L$ such that $\imath \circ \alpha = \theta$.

We need to check that $\phi_i \circ \alpha = \psi_i$. However $\phi_i = \pi_i \circ \imath$ hence:

$$\phi_i \circ \alpha = \pi_i \circ \theta = \psi_i$$

Finally note that if any other morphism $\beta : L \to N$ satisfies $\phi_i \circ \beta = \psi_i$, then we have that:

$$\begin{aligned}\imath \circ \beta =& \left(\sum_i \imath_i \circ \pi_i \circ \imath \right) \circ \beta \\ =& \sum_i \imath_i \circ \psi_i \\ =& \theta\end{aligned}$$

hence β would also make the relevant kernel diagram commute and thus be equal to α. It follows that α is unique, and so L is the limit of (J, F) as desired. $\square$

In the above proof, one should notice that the colimit and limit arguments are essentially identical in strategy, one is just dual to the other.[163] Going forward, when this is obviously the case, we will only prove one of the statements and claim that the other follows from a ‘similar but dual argument’. We do this to in order to halve the size of our proofs, but we will switch whether we prove the dual or ‘regular’ statement as we please so to not get overly comfortable with one direction of argument.

The above lemma also implies that fibre products, and fibre coproducts exist in any abelian category.[164] We call commutative squares coming from a fibre coproduct *cocartesian squares*. In this specific case we can actually simplify the description of these (co)limits:

Example 7.1.3. Suppose we have a cartesian square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

By the preceding lemma we know that A is isomorphic to the kernel of some collection of maps between two finite direct sums. We claim that there is simpler description of A as the kernel of the morphism:

$$(i, -h) : B \oplus C \longrightarrow D$$

whee $(i, -h)$ is the morphism induced by the universal property of the coproduct. We will show that $\ker(i, -h)$ satisfies the universal property of $C \times_D B$. If π_B and π_C are the projection $B \oplus C \to B$ and $B \oplus C \to C$, and $\imath : \ker(i, -h) \hookrightarrow B \oplus C$, we define morphisms $f' : \ker(i, -h) \to B$ and $g' : \ker(i, -h) \to C$ via the compositions $\pi_B \circ \imath$ and $\pi_C \circ \imath$ respectively. Now note that:

$$i \circ f' - h \circ g' = i \circ \pi_B \circ \imath - h \circ \pi_C \circ \imath = (i - h) \circ \imath = 0$$

hence $i \circ f' = h \circ g'$ and we have the following commutative diagram:

$$\begin{tikzcd} \ker(i, -h) \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "i"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

Now let $p_B : Q \to B$ and $p_C : Q \to B$ be morphisms which make the relevant fibre product diagram commute. In particular, if $p_B \times p_C : Q \to B \oplus C$ is the morphism coming from the universal property of the direct product, then:

$$(i, -h) \circ p_B \times p_C = i \circ p_B - h \circ p_C = 0$$

The universal property of the kernel implies there is a unique morphism $\alpha : Q \to \ker(i, -h)$ which satisfies $\imath \circ \alpha = p_B \times p_C$. By construction we have that $f' \circ \alpha = \pi_B \circ p_B \times p_C = p_B$, and similarly that $g' \circ \alpha = p_C$. It follows that the following diagram commutes:

$$\begin{tikzcd} Q \arrow[dr, "\alpha"] \arrow[drr, bend left, "p_C"] \arrow[ddr, bend right, "p_B"'] & & \\ & \ker(i, -h) \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "i"] \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

[163]i.e. all of the arrows are flipped, use of kernel instead of cokernel etc.
[164]These are often called pullbacks, and push outs in the literature.

Suppose that some other $\beta : Q \to \ker(i, -h)$ makes the above diagram commute. Then since $\pi_B \circ \imath \circ \beta = p_B$ and $\pi_C \circ \imath \circ \beta = p_C$ we have that $\imath \circ \beta = p_B \times p_C$, hence $\beta = \alpha$ by the universal property of the kernel. It follows that α is unique, and $B \times_D C$ is uniquely isomorphic to $\ker(i, -h)$.

A similar but dual argument shows that if:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

is cocartesian, and we denote by $i \times (-h) : A \to B \oplus C$ the morphism coming from the universal property of the direct product, then $D \cong \operatorname{coker}(i \times (-h))$.

While the direct product, and coproduct are naturally isomorphic in any abelian category, the statement is obviously not true of fibre products and fibre coproducts. What we could ask instead is whether or not every cartesian square is also cocartesian and vice versa. This is again too much to ask for, but in special situations it is true:

Lemma 7.1.4. *Let $\mathscr{C}$ be an abelian category, and suppose we have the following commutative diagram in $\mathscr{C}$:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

Then the following hold:

a) *The diagram is cartesian if and only if the following sequence is exact:*

$$0 \longrightarrow A \xrightarrow{f \times g} B \oplus C \xrightarrow{(i,-h)} D$$

b) *The diagram is cocartesian if and only if the following sequence is exact:*

$$A \xrightarrow{f \times (-g)} B \oplus C \xrightarrow{(i,h)} D \longrightarrow 0$$

c) *If the above square is cartesian, and i is an epimorphism, then so is g, and the diagram is cocartesian.*

d) *If the above square is cocartesian, and f is a monomorphism then h is a monomorphism and the square is cartesian.*

Proof. We first prove a). Suppose the sequence is exact, then in particular we know that $\operatorname{im}(f \times g) \cong \ker(i, -h)$. From our work in Example 7.1.3 we then need only show that $\operatorname{im}(f \times g) \cong A$. Since $f \times g$ is a monomorphism as $\ker(f \times g) = 0$, we have that $(A, f \times g)$ satisfies the universal property of the kernel of the morphism $\pi : B \oplus C \to \operatorname{coker}(f \times g)$. However $\operatorname{im}(f \times g) \cong \ker(\pi)$ by Lemma 7.1.1 and so the diagram is cartesian.

Now suppose that the diagram is cartesian. We indeed have a sequence of morphisms:

$$0 \longrightarrow A \xrightarrow{f \times g} B \oplus C \xrightarrow{(i,-h)} D$$

To show it's exact it suffices to show that $\ker(f \times g) = 0$, $(i, -h) \circ f \times g = 0$, and the induces morphism $\operatorname{im}(f \times g) \to \ker(i, -h)$ is an isomorphism. To show that $\ker(f \times g) = 0$ it suffices to show that $f \times g$ is a monomorphism. Suppose that $\alpha : E \to A$ satisfies $f \times g \circ \alpha = 0$, then in particular $\pi_C \circ f \times g \circ \alpha = 0$, and $\pi_B \circ f \times g \circ \alpha = 0$. It follows that $f \times g \circ \alpha$ is the unique morphism associated to the zero morphisms $E \to C$ and $E \to B$. However, the zero morphism $E \to A$ makes the relevant fibre product diagram commute, hence by uniqueness $\alpha = 0$. It follows that $f \times g$ is a monomorphism, and thus $\ker(f \times g) = 0$.

Now now that:

$$(i, -h) \circ f \times g = (i \circ \pi_B - h \circ \pi_C) \circ f \times g = i \circ f - h \circ g = 0$$

hence $(i, -h) \circ f \times g = 0$. We now need to check that the natural map $\phi : \operatorname{im}(f \times g) \to \ker(i, -h)$. Note that if $\alpha : A \to \ker(i, -h)$ satisfying $\imath \circ \alpha = f \times g$, then ϕ makes the following diagram commute:

$$\begin{tikzcd}
A \arrow[rr, "f\times g"] \arrow[dr, "\pi'"] \arrow[ddr, "\alpha"'] & & B \oplus C \\
& \operatorname{im} f \times g \arrow[ur, "\beta"] \arrow[d, "\phi"] & \\
& \ker(i, -h) \arrow[uur, "\imath"'] &
\end{tikzcd}$$

By our work above, we know that π' must be an isomorphism, hence it suffices to show $\alpha : A \to \ker(i, -h)$ is an isomorphism. However, α comes from the universal property of the $\ker(i, -h)$ applied to the morphism $f \times g$. Our work in Example 7.1.3 then implies that α is the unique morphism $A \to \ker(i, -h)$ induced by the universal property of the fibre product $B \times_D C$ applied to the morphisms f and g. However, that morphism must be an isomorphism as the diagram is cartesian. It follows that the sequence is exact, implying a).

A similar but dual argument gives us b).

Now suppose that the square is cartesian, and i is an epimorphism. We have the following exact sequence by a):

$$0 \longrightarrow A \xrightarrow{f \times g} B \oplus C \xrightarrow{(i,-h)} D$$

And we want to show that we have the following exact sequence:

$$A \xrightarrow{f \times (-g)} B \oplus C \xrightarrow{(i,h)} D \longrightarrow 0$$

Well note that $(i, -h)$ is an epimorphism, as if $\alpha : D \to E$ satisfies $\alpha \circ (i, -h) = 0$, then $\alpha \circ (i, -h) \circ \imath_B = 0$, hence $\alpha \circ i = 0$. Since i is an epimorphism it follows that $\alpha = 0$, implying the claim. It follows that our original exact sequence is short exact as $(i, -h)$ is surjective. We claim we have a commutative diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f\times g"] \arrow[d, "\mathrm{Id}_A"] & B \oplus C \arrow[r, "(i,-h)"] \arrow[d, "(\imath_B, -\imath_C)"] & D \arrow[r] \arrow[d, "\mathrm{Id}_D"] & 0 \\
0 \arrow[r] & A \arrow[r, "f\times(-g)"] & B \oplus C \arrow[r, "(i,h)"] & D \arrow[r] & 0
\end{tikzcd}$$

It suffices to check that each square commutes. We have that:

$$\begin{aligned}
(\imath_B, -\imath_C) \circ f \times g =& (\imath_B, -\imath_C) \circ (\imath_B \circ f + \imath_C \circ g) \\
=& \imath_B \circ f - \imath_C \circ g \\
=& f \times (-g)
\end{aligned}$$

so the left square commutes. Similarly:

$$\begin{aligned}
(i, h) \circ (\imath_B, -\imath_C) =& (i, h) \circ (\imath_B \circ \pi_B - \imath_C \circ \pi_C) \\
=& i \circ \pi_B - h \circ \pi_C \\
=& (i, -h)
\end{aligned}$$

It follows that the diagram is commutative. We also have that:

$$\begin{aligned}
(\imath_B, -\imath_C) \circ (\imath_B, -\imath_C) =& (\imath_B, -\imath_C) \circ \imath_B \circ \pi_C - \imath_C \circ \pi_C \\
=& \imath_B \circ \pi_B + \imath_B \circ \imath_C \\
=& (\imath_B, \imath_C)
\end{aligned}$$

so since $(\imath_B, \imath_C) = \mathrm{Id}_{B\oplus C}$ as they make the same diagrams commute, we have that each vertical arrow in the diagram is an isomorphism. The sequence is therefore exact by Lemma 7.1.2, and so b) implies that the square is cocartesian.

To see that g is an epimorphism, consider the morphism $\phi : \operatorname{coker} g \to \operatorname{coker} i$ given by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] & C \arrow[rr, "\pi_i \circ h"] \arrow[dr, "\pi_g"'] & & \operatorname{coker} i \\
& & \operatorname{coker} g \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Note this diagram makes sense as $\pi_i \circ h \circ g = \pi_i \circ i \circ = 0$. The diagram is cocartesian so consider $\alpha : D \to \operatorname{coker} g$ coming from the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \arrow[ddr, bend left, "0"] & \\
C \arrow[r, "h"] \arrow[drr, bend right, "\pi_g"'] & D \arrow[dr, "\alpha"] & \\
& & \operatorname{coker} g
\end{tikzcd}$$

We thus obtain a unique morphism $\phi^{-1} : \operatorname{coker} i \to \operatorname{coker} g$ making the following diagram commute:

$$\begin{tikzcd}
B \arrow[r, "g"] & D \arrow[rr, "\alpha"] \arrow[dr, "\pi_i"'] & & \operatorname{coker} g \\
& & \operatorname{coker} i \arrow[ur, "\phi^{-1}"'] &
\end{tikzcd}$$

We see that:

$$\phi^{-1} \circ \phi \circ \pi_g = \phi^{-1} \circ \pi_i \circ h = \alpha \circ h = \pi_g$$

Since π_g is an epimorphism it follows that $\phi^{-1} \circ \phi = \mathrm{Id}$.

To check that $\phi \circ \phi^{-1}$ is the identity, consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \arrow[ddr, bend left, "0"] & \\
C \arrow[r, "h"] \arrow[drr, bend right, "\phi\circ\pi_g"'] & D \arrow[dr, "\pi_i"] & \\
& & \operatorname{coker} i
\end{tikzcd}$$

We claim that $\phi \circ \phi^{-1} \circ \pi_i$ also makes this diagram commute. Indeed we obviously have that $\phi \circ \phi^{-1} \circ \pi_i \circ i = 0$, because $\pi_i \circ i = 0$. Secondly, we have that:

$$\phi \circ \phi^{-1} \circ \pi_i \circ h = \phi \circ \alpha \circ h = \phi \circ \pi_g = \pi_i \circ h$$

It follows that $\phi \circ \phi^{-1} \circ \pi_i = \pi_i$ and so $\phi \circ \phi^{-1} = \mathrm{Id}$. In particular we hae that $\operatorname{coker} g \cong \operatorname{coker} i$. Since i is an epimorphism it follows that $\operatorname{coker} g = 0$ and so g is an epimorphism as well implying c)

Part d) follows from a similar but dual argument. $\square$

Definition 7.1.4. Let $\mathscr{C}$ be an abelian category, then an object P in $\mathscr{C}$ is **projective** if the functor $\mathrm{Hom}_{\mathscr{C}}(P, -)$ is exact. An object I in $\mathscr{C}$ is **injective** if the functor $\mathrm{Hom}_{\mathscr{C}}(-, P)$ is exact.

We now prove the following:

Lemma 7.1.5. *Let I be an object in $\mathscr{C}$, then the following are equivalent:*

a) *I is injective.*

b) *Every short exact sequence starting with I splits.*

c) *For every monomorphism $f : A \to B$, and every $g : A \to I$ there exists an $h : B \to I$ making the following diagram commute:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[dl, dashed, "\exists h"] \\ I & \end{tikzcd}$$

Similarly, if P is an object in $\mathscr{C}$ then the following are equivalent:

d) *P is projective.*

e) *Every short exact sequence ending with P splits.*

f) *For every epimorphism $f : A \to B$, and every $g : P \to B$ there exists an $h : P \to A$ making the following diagram commute:*

$$\begin{tikzcd} & A \arrow[d, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"] & B \end{tikzcd}$$

Proof. We show that *a)* implies *b)*; suppose we have an exact sequence of the form:

$$0 \longrightarrow I \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

then since I is injective, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}}(N, I) \xrightarrow{g} \mathrm{Hom}_{\mathscr{C}}(M, I) \xrightarrow{f^*} \mathrm{Hom}_{\mathscr{C}}(I, I) \longrightarrow 0$$

In particular, since f^* is surjective, there is a morphism $\alpha : M \to I$ such that $\alpha \circ f = \mathrm{Id}_I$. It follows that the short exact sequence splits, as every short exact sequence which admits a section splits.

We show that *b)* $\Rightarrow$ *c)*. Suppose we have a diagram of the following form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ I & \end{tikzcd}$$

where f is a monomorphism. We have that the following cocartesian square exists:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "i"] \\ I \arrow[r, "b"] & P \end{tikzcd}$$

Then Lemma 7.1.4 implies that since b is a monomorphism as f is one. As such we obtain an exact sequence of the form:

$$0 \longrightarrow I \xrightarrow{b} P \xrightarrow{\pi} \operatorname{coker} h \longrightarrow 0$$

Since this exact sequence splits by assumption, there exists a morphism $\alpha : P \to I$ such that $\alpha \circ b = \mathrm{Id}$. Define h to $\alpha \circ i$, and note that $\alpha \circ i \circ f = \alpha \circ h \circ g = g$ implying *c)*.

Now let:

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

be an exact sequence. Assuming $c)$, we want to show that the exact sequence:

$$0 \longrightarrow \mathrm{Hom}(C, I) \xrightarrow{\quad b^* \quad} \mathrm{Hom}(B, I) \xrightarrow{\quad a^* \quad} \mathrm{Hom}(A, I)$$

is actually short exact. It suffices to show that a^* is surjective. Suppose that $g : A \to I$ is a morphism, and note that a is a monomorphism. It follows that there is a morphism $h : B \to I$ such that $h \circ a = g$. It follows that $a^* h = g$ hence a^* is surjective and the sequence is exact, so $c) \Rightarrow a)$.

Properties $d) - f)$ follow from a similar but dual argument. □

In the category of A modules there is a nice characterization of projective modules:

Example 7.1.4. Let $\mathscr{C} = A - \mathrm{Mod}$, then we claim that every projective object in $\mathscr{C}$ is a direct summand of a free module. Indeed, we can write any A-module M as a quotient $\pi : A^I \to M$, where as usual A^I is an arbitrary coproduct. We thus have a short exact sequence:

$$0 \longrightarrow \ker \pi \xrightarrow{\quad \imath \quad} A^I \xrightarrow{\quad \pi \quad} M \longrightarrow 0$$

If M is projective then the exact sequence splits hence $A^I \cong M \oplus \ker \pi$. Moreover if M is a direct summand of A^I, i.e. $A^I \cong M \oplus Q$ for some Q, then M is projective. Suppose there was an exact sequence:

$$0 \longrightarrow D \xrightarrow{\quad a \quad} B \xrightarrow{\quad b \quad} C \longrightarrow 0$$

then we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(A^I, D) \xrightarrow{\quad a_* \quad} \mathrm{Hom}(A^I, B) \xrightarrow{\quad b_* \quad} \mathrm{Hom}(A^I, C)$$

Note that $\mathrm{Hom}(A^I, D) \cong D^I$, and so this sequence is short exact as is isomorphic to the short exact sequence:

$$0 \longrightarrow D^I \xrightarrow{\quad a^I \quad} B^I \xrightarrow{\quad b^I \quad} C^I \longrightarrow 0$$

Note this implies that A^I is projective. However, we have that $A^I \cong M \oplus Q$, hence $\mathrm{Hom}(A^I, D) \cong \mathrm{Hom}(M, D) \oplus \mathrm{Hom}(Q, D)$. We thus obtain a short exact sequence of the form:

$$0 \to \mathrm{Hom}(M, D) \oplus \mathrm{Hom}(Q, D) \to \mathrm{Hom}(M, B) \oplus \mathrm{Hom}(Q, B) \to \mathrm{Hom}(M, B) \oplus \mathrm{Hom}(Q, B) \to 0$$

Since each map is a direct sum of maps, it is then obvious that the sequence:

$$0 \longrightarrow \mathrm{Hom}(M, D) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow 0$$

is exact as well hence M is projective.

Injective objects in the category of A-modules are a bit less well behaved, however we will become well acquainted with them in good time. Note that the above, along with Lemma 5.7.5 implies that each projective module module is flat as it is a direct summand of a flat module (namely a free module).

Lemma 7.1.6. *Let $\{A_i\}_{i \in I}$ be a potentially infinite family of object in $\mathscr{C}$ an abelian category. The following hold:*

a) If each A_i is injective then $\prod_{i \in I} A_i$ is injective if it exists.

b) If each A_i projective then $\bigoplus_{i \in I} A_i$ is projective if it exists.

Proof. Suppose that both the arbitrary direct product and coproduct over I exist. This follows because for any object B, we have that:

$$\mathrm{Hom}(B, \prod_{i \in I} A_i) \cong \prod_{i \in I} \mathrm{Hom}(B, A_i) \qquad \text{and} \qquad \mathrm{Hom}(\bigoplus_{i \in I} A_i, B) \cong \bigoplus_{i \in I} \mathrm{Hom}(A_i, B)$$

It follows that if each A_i injective because, given any exact sequence we will obtain a diagram of the

$$0 \longrightarrow D \longrightarrow B \longrightarrow C \longrightarrow 0$$

we will obtain the diagram:

$$0 \longrightarrow \prod_{i\in I} \mathrm{Hom}(C, A_i) \longrightarrow \prod_{i\in I} \mathrm{Hom}(B, A_i) \longrightarrow \prod_{i\in I} \mathrm{Hom}(D, A_i) \longrightarrow 0$$

which is exact because on each component it is exact.[165]. The similar but dual argument proves b) $\square$

Definition 7.1.5. Let $\mathscr{C}$ be an abelian category. We say that $\mathscr{C}$ **has enough invectives** if every object A has a monomorphism $A \to I$ with I injective. Similarly, $\mathscr{C}$ **has enough projectives** if every object A has an epimorphism $P \to A$ with P projective.

Example 7.1.5. Clearly $A-\mathrm{Mod}$ has enough projectives.

We wish to spend the next few moments proving that $A-\mathrm{Mod}$ has enough invectives as well.

Example 7.1.6. First consider the category of $\mathbb{Z}$-modules, i.e. abelian groups. We claim that $\mathbb{Q}/\mathbb{Z}$ is injective. Indeed, suppose that we have a short exact sequence starting with $\mathbb{Q}/\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\quad f \quad} G \longrightarrow H \longrightarrow 0$$

It suffices to show that there exists a map $s : G \to \mathbb{Q}/\mathbb{Z}$ such that $s \circ f = \mathrm{Id}$.

7.2 Sheaf Cohomology as a Derived Functor

[165] This is not hard to see now that we are in the category of abelian groups