Calculus I

Challenge Homework Set I

March 28, 2025

Provide **handwritten** answers on a separate sheet of paper. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources.

Problem I: In this problem we examine common precalculus pitfalls. Let x, y, z be real numbers. For each of the following expressions, determine whether the equality is true or false. In both cases you must explain your reasoning, and if the equality is false then change the right side of the expression so that the equality is true.

$a)(x+y)^2 = x^2 + y^2$

$b)\dfrac{x^2+y^2+z^2}{x^2+y^2} = 1 + \dfrac{z^2}{x^2+y^2}$

$c)xyz - (x^2 - z^3 + 3y) = xyz - x^2 - z^3 - 3y$

$d)\cos(x+y) = \cos x + \cos y$

$e)z^{x+y} = z^x + z^y$

$f)\log_z(xy) = \log_z x \cdot \log_z y$

Problem II: Using the squeeze theorem, and limit laws, calculate the following limits[1]:

$a)\ \lim_{x\to 0} \dfrac{\sin x}{x}$

$b)\ \lim_{x\to 0} \dfrac{1-\cos x}{x}$

$c)\ \lim_{x\to 0} x^2 \sin \dfrac{1}{x}$

$d)\ \lim_{x\to 0} \dfrac{\sin 5x \cos x}{x}$

Now suppose that $f(x)$ is an arbitrary function, satisfying $\lim_{x\to a} f(x) = L$. For n a positive whole number, and c a real number, show the following[2]:

$e)\ \lim_{x\to a} (f(x))^n = L^n$

$f)\ \lim_{x\to a} c \cdot f(x) = c \cdot L$

[1]Hint: Your textbook covers two of these examples in great detail, and the other two are similar.
[2]Hint: Use the limit laws, the fact that the constant function $g(x) = c$ is continuous, and the fact that $h(x) = x^n$ is continuous.

Problem III: Let $f(x)$ and $g(x)$ be functions. We say that $f(x) < g(x)$ if:

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$$

Using this, order the following functions: $x!$, x, $\ln x$, x^e, e^x, $\sqrt{x}$, i.e. determine whether $\ln x < x$ or $x < \ln x$ for each function. Note that $x!$ is only defined for positive integers, and not all real numbers but we can still take limits like this if we restrict to the positive integers. What does this order tell you about the comparative growth rates of the functions?

Problem IV: In class we went over how one can use limits to calculate the speed of a ball falling off a building at a point t_0 in time. In this question, we consider a space ship flying away from earth; it's distance from earth for any $t > 0\,\mathrm{s}$ is given in kilometers by the function:

$$x(t) = t^3 + 2t^2 + t$$

Calculate the speed of the spaceship when it is $10\,\mathrm{km}$ away from earth. Do not use the power rule, but instead mimic the procedure done on the first day of class.

Problem V: An object with an initial temperature of T_0 is placed in an environment of ambient temperature T_s. The temperature of the object as a function of time $t \geq 0$ is given by:

$$T(t) = T_s + (T_s - T_0)e^{-kt}$$

where k is some positive real number.

a) When does the temperature of the object reach one half of the temperature of the ambient environment?

b) Evaluate the limit $\lim_{x\to\infty} T(t)$.

c) Physically interpret your answer to *b*).