Calculus I: Lecture Notes

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Contents

Week I: Limits

We make the following notational conventions:

- For an arbitrary function $f(x)$, we will denote by $f^n(x)$ for positive n to be the n-fold product of f. We will denote by $f^{(n)}(x)$ as the nth derivative, and $f^{-1}(x)$ to be the inverse function to f. If we wish to write something of the form $\frac{1}{f(x)}$, more compactly, we will write $(f(x))^{-1}$.
- If $n > 0$, then $\sin^n x$ is to be the n-fold product of $\sin x$. In other words $\sin^2 x = (\sin x) \cdot (\sin x)$. If $n < -1$, we will never employ the notation $\sin^n x$, and if $n = -1$, then $\sin^{-1} x$ will be the inverse function to sin, often denoted arcsin. The same holds for other trigonometric functions.
- The functions $\csc x$, $\sec x$ and $\cot x$ denote $\frac{1}{\sin(x)}$, $\frac{1}{\cos x}$ and $\frac{1}{\cot x}$ respectively.
- The function $\log x$ will always refer to the base 10 logarithm, while $\ln x$ will be base $e \approx 2.718\ldots$; if we need to use logarithms in base a for some real number a, then will denote them by $\log_a x$.

1.1 Introduction and Motivation

Suppose you drop a ball off a building that is 500 meters tall, then results from physics tell us that the height of the ball as a function of time is given by:

$$y(t) = -10 \cdot t^2 + 500 \tag{1.1}$$

Note that the above function only makes physical sense on the interval $[0, t_0]$, where t_0 is the moment that the ball hits the ground. Since the height of the ball at t_0 is zero, we can find t_0 by setting (1.1) equal to zero:

$$-10 \cdot t^2 + 500 = 0 \Rightarrow 10t^2 = 500 \Rightarrow t^2 = 50 \Rightarrow t = \pm\sqrt{50} = \pm 5 \cdot \sqrt{2}$$

We also know that we should take the positive square root, as negative time does not make physical sense. It follows that our height function is physically defined on $[0, 5 \cdot \sqrt{5}]$. We now might ask ourselves a variety of different physical questions about our falling ball, such as: what is the average speed of the ball? This is a question we can answer with purely algebraic techniques; indeed we know the ball travels a total of $\Delta y = -500$[1] meters over a span of $\Delta t = 5 \cdot \sqrt{5}$ seconds, so the average speed, v_{avg},

[1] Here Δy is negative as $\Delta y = y_f - y_i = 0 - 500$

is given by:

$$v_{\text{avg}} = \frac{\Delta y}{\Delta t} = \frac{500}{5 \cdot \sqrt{s}} \approx -70.71\,\text{m/s}$$

However, what if we want to know the speed of the ball when it has traveled 250 meters? Or right before it crashes into the ground? Or at any point along it's trajectory? Answering these questions requires more sophisticated techniques, the techniques of calculus. In fact, the field of calculus was almost entirely motivated by questions of this form.

Our first step in answering such questions is to analyze the average velocity of our ball over a Δt geometrically. Suppose we want to calculate the speed of our ball at $t_1 = 2$, $y(t_1) = 460$, then a good place to start is to consider the average speed of the ball over an interval starting at t_0, say $[2, 5]$. In this case, v_{avg} is given by:

$$v_{\text{avg}} = \frac{\Delta y}{\Delta t} = \frac{y(t_1 + 3) - y(t_1)}{(t_1 + 3) - t_2} = \frac{250 - 460}{3} = -70\,\text{m/s}$$

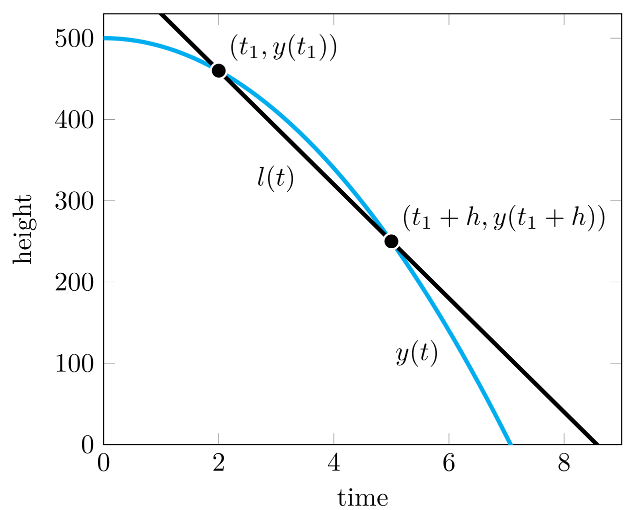
Now, v_{avg} is the slope of the line which goes through the points:

$$(t_1, y(t_1)) = (2, 460) \qquad \text{and} \qquad (t_1 + 3, y(t_1 + 3)) = (5, 250)$$

The function corresponding to this line[2] is given by:

$$l(t) = v_{\text{avg}} \cdot (t - 2) + 460$$

We can then draw the following graph:



The black line is $l(t)$, the blue line is $y(t)$, and we have marked two points on the graph, $(t_1, y(t_1))$ and $(t_1 + h, y(t_1 + h))$, where $h = 3$. The key insight is that if we could somehow take the average speed over the interval $[t_1, t_1]$, we would obtain the speed of the ball at $(t_1, y(t_1))$ because that interval consists of only a single point. We cannot do this naively though, as our formula for average velocity, i.e. the slope of the line passing through the end points of the interval would yield:

$$\frac{\Delta y}{\Delta t} = \frac{y(t_1) - y(t_1)}{t_1 - t_1} = \frac{0}{0}$$

which doesn't make mathematical or physical sense! So, how can we rectify the situation? The next key insight is that if we allow h to vary instead of being fixed, then as h gets closer and closer to zero,

[2] i.e. the function whose graph is this line.

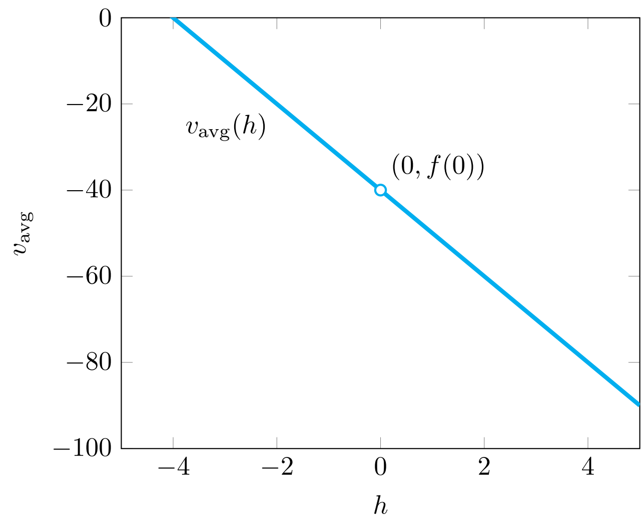
since the interval becomes smaller and smaller, we get average velocities which are closer and closer to the velocity at t_0. Allowing h to vary makes v_{avg} a function of h given by:

$$v_{\text{avg}}(h) = \frac{\Delta y}{\Delta t} = \frac{y(t_1+h)-y(t_1)}{t_1+h-t_1} = \frac{-10(t_1+h)^2+500-(-10t_1^2+500)}{h} = \frac{-10h^2-20h\cdot t_1}{h}$$

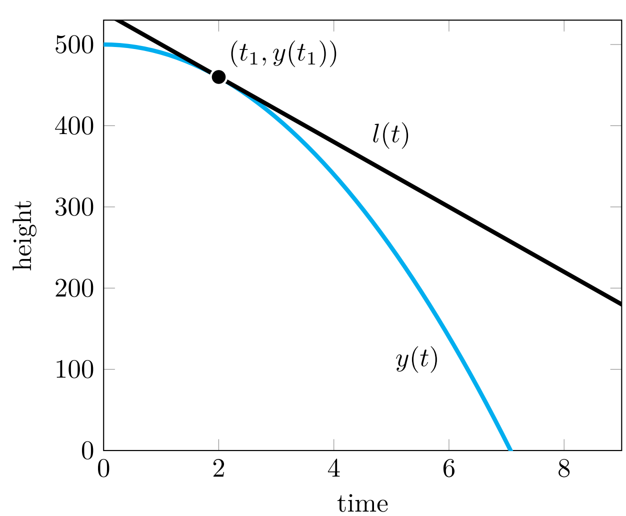
This is a *rational function*, and as such is not defined at $h=0$, however, for all $h \neq 0$, we have that:

$$v_{\text{avg}}(h) = -10h - 20\cdot t_1 \tag{1.2}$$

because each term has at least one power of h, and we can divide by h when h is nonzero. In particular, the graph of this function is the graph of the function $f(h) = -10h - 20\cdot t_1$ with a hole at $h=0$:



So while we can't actually plug $h=0$ into our v_{avg} to get the speed at t_1, it is clear from the above graph, that as h gets closer to 0, $v_{\text{avg}}(h)$ approaches the numerical $f(0) = -20\cdot t_1$, which in this case is -40 as $t_1 = 2$. It therefore makes intuitive sense to say that the speed of the ball at t_1 is $-60\,\text{m/s}$. Moreover, if we replace $l(t)$ in our original picture with the line going through the point $(-2, 460)$, at a slope $m = -60$, we have the following:



So the velocity of the ball at $t_1 = 2$ is also the slope of the line *tangent*[3] to the graph of $y(t)$ at $(t_1, y(t_1))$.

[3]By which we mean only glances the graph of the function at $(2, 440)$, instead of intersecting it.

The entire process outlined above is called 'taking the derivative at t_1'. In particular, the process of seeing what the value of $v_{\text{avg}}(h)$ is as h approaches 0 (even though $v_{\text{avg}}(h)$ is not defined at $h = 0$!) is called 'taking the limit of $v_{\text{avg}}(h)$ as h goes to zero'. We employ the following notation for this:

$$\lim_{h \to 0} v_{\text{avg}}(h)$$

In particular, if we let t_1 vary, we get a new function defined by:

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

By our earlier work, this is the same as:

$$y'(t) = \lim_{h \to 0} \frac{-10h^2 - 20h \cdot t}{h}$$

For all nonzero h this is equal to the equation (1.2), so the limit as h approaches zero is precisely $-20 \cdot t$. It follows that we have a function:

$$y'(t) = -20 \cdot t$$

This is called the derivative of $y(t)$, and for every t_1 in the interval $[0, 2 \cdot \sqrt{5}]$, when we plug in t_1 to $y'(t)$, we get the velocity of the ball at the time t_1, or equivalently the velocity of the ball at a height of $y(t_1)$ meters.

The rest of the course, and much of calculus in general, is about studying the properties of derivatives for various functions, but even when we delve deeper into abstraction, and away from the world of physics, we should keep the above picture of a ball falling off a building in mind.

1.2 Definitions and Examples

In the previous section, we motivated the idea of a derivative by examining a physical problem. However, the process for finding the speed of a ball as it falls from a building relied on the notion of 'taking a limit' of a function, and in fact the concept of a derivative relies heavily on this idea. Due to this we spend the next few sections, discussing limits, and their properties.

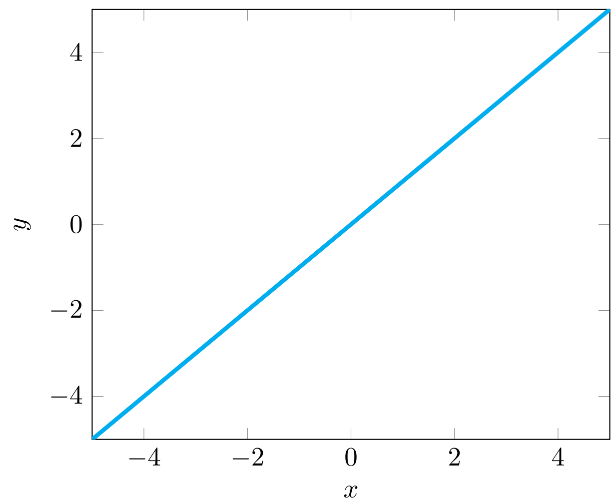
Limits essentially come in two flavors, we have limits as 'x goes to positive or negative infinity', and limits as 'x approaches a', where a is some finite real number. The former is a way of characterizing the long term behavior of a function $f(x)$, and the later analyzes the behavior of f near a point a, even if $f(a)$ is undefined. We begin with limits of the form x goes to positive or negative infinity, and employ the notation:

$$\lim_{x \to \infty} f(x) \qquad \text{and} \qquad \lim_{x \to -\infty} f(x)$$

Now limits of the above form can be a real number, can be 'positive or negative infinity', or they can be non existent. When limits above the form are a real number say a, this means that x gets bigger and bigger, (or more and more negative), that the value of $f(x)$ gets closer and closer to a. In other words, in this situation we have that as x approaches positive (or negative) infinity, $f(x)$ approaches a. The next option is that $f(x)$ 'blows up' as x approaches positive or negative infinity, by which we mean that x gets bigger and bigger (or more and more negative) the value of $f(x)$ continues to grow in either the positive or negative direction. In this case, we have the limit as x approaches positive or negative infinity is equal to positive or negative infinity, depending on which direction the function grows. The final option is that the limit may fail to exist, in which case we simply write DNE. This can happen if the long term behavior of the function is oscillatory like if $f(x) = \sin x$; in this case the function neither grows without bound, nor does it approach some constant, it just oscillates between 1 and -1 periodically, hence there is no value that $f(x)$ approaches as x approaches positive or negative infinity.

The quickest way of dealing with limits of these form is by using 'big number logic'. This is best taught via example:

Example 1.1. Let $f(x) = x$, then the graph of $f(x)$ is:



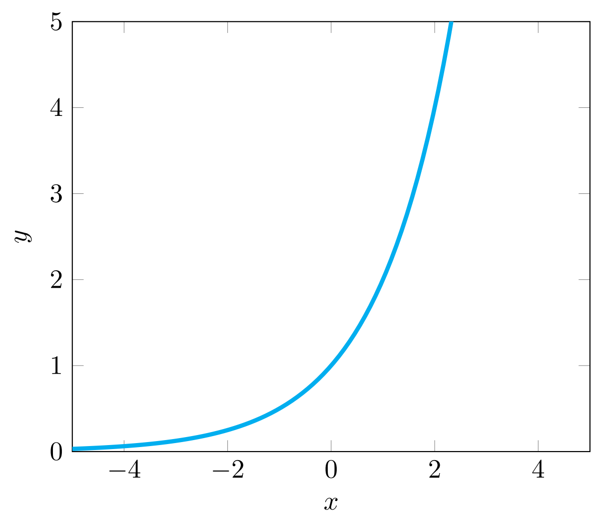
Now as we plug larger and larger and numbers into $f(x)$, we just return that same number since $f(x) = x$. In particular as x gets bigger $f(x)$ bigger, so the long term behavior of $f(x)$ is to get bigger and bigger. When this occurs, we write:

$$\lim_{x\to\infty} f(x) = \infty$$

because $f(x)$ will just keep growing as x grows. Similarly, when x gets more and more negative, $f(x)$ gets more and more negative, hence:

$$\lim_{x\to-\infty} f(x) = -\infty$$

Example 1.2. Let $f(x) = 2^x$, we can see from the graph of this function:



that the limit as x approaches positive infinity is infinity, and that the limit as x approaches negative infinity is 0. But how can we determine this with big number logic? The idea is that if x is getting bigger, i.e. as x approaches positive infinity, then 2^x also just gets bigger, as we are just taking larger and larger powers of two. From this we conclude that:

$$\lim_{x\to\infty} 2^x = \infty$$

However, if x is negative, then we are dividing 1 by larger and larger powers of 2. In particular, we have the following infinite sequence when x is a negative integer:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

So as we plug in more and more negative values for x, 2^x gets closer and closer to zero, because we get fractions with larger and larger denominators. It follows that:

$$\lim_{x \to \infty} 2^x = 0$$

Example 1.3. Let:

$$f(x) = \frac{5x^3 + 2x^2 - x + 1}{x^2 + 3x - 2}$$

We want to determine the limit as $x \to \infty$ and $x \to -\infty$. Unlike the previous two cases, this function is not easily graphed by hand, so we have to rely solely on big number logic. Let us first determine the limit as $x \to \infty$; the point is that as x gets very very large ,the terms which contribute most to the quotient:

$$\frac{5x^3 + 2x^2 - x + 1}{x^2 + 3x - 2}$$

are only the highest order terms in numerator and denominator. In other words, if x is very very large, then $5x^3 + 2x^2 - x + 1$ is very close to $5x^3$, because x^3 will be so much larger than $2x^2 - x + 1$. The same holds for the denominator, hence:

$$\begin{aligned}
\lim_{x \to \infty} \frac{5x^3 + 2x^2 - x + 1}{x^2 + 3x - 2} &= \lim_{x \to \infty} \frac{5x^3}{x^2} \\
&= \lim_{x \to \infty} 5x \\
&= \infty
\end{aligned}$$

Similarly, the same logic demonstrates that:

$$\begin{aligned}
\lim_{x \to -\infty} \frac{5x^3 + 2x^2 - x + 1}{x^2 + 3x - 2} &= \lim_{x \to \infty} \frac{5x^3}{x^2} \\
&= \lim_{x \to -\infty} 5x \\
&= -\infty
\end{aligned}$$

Note that if we change the denominator to be a cubic:

$$f(x) = \frac{5x^3 + 2x^2 - x + 1}{x^3 + 3x - 2}$$

then:

$$\begin{aligned}
\lim_{x \to \infty} \frac{5x^3 + 2x^2 - x + 1}{x^3 + 3x - 2} &= \lim_{x \to \infty} \frac{5x^3}{x^3} \\
&= \lim_{x \to \infty} 5 \\
&= 5
\end{aligned}$$

and:

$$\begin{aligned}
\lim_{x \to -\infty} \frac{5x^3 + 2x^2 - x + 1}{x^3 + 3x - 2} &= \lim_{x \to -\infty} \frac{5x^3}{x^3} \\
&= \lim_{x \to -\infty} 5 \\
&= 5
\end{aligned}$$

If we make the denominator a quartic:

$$f(x) = \frac{5x^3 + 2x^2 - x + 1}{x^4 + 3x - 2}$$

then:

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x^3 + 2x^2 - x + 1}{x^3 + 3x - 2} &= \lim_{x\to\infty} \frac{5x^3}{x^4} \\
&= \lim_{x\to\infty} \frac{5}{x} \\
&= 0
\end{aligned}$$

and:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{5x^3 + 2x^2 - x + 1}{x^3 + 3x - 2} &= \lim_{x\to-\infty} \frac{5x^3}{x^4} \\
&= \lim_{x\to-\infty} \frac{5}{x} \\
&= 0
\end{aligned}$$

In particular, big number logic gives us the following result:

Theorem 1.1. *If $p(x)$ and $q(x)$ are polynomials such that:*

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad \textit{and} \qquad q(x) = b_m x^m + \cdots + b_1 x + b_0$$

for some positive integers m and n, and real numbers $a_0, \ldots, a_n, b_0, \ldots b_n$, then:

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} \qquad \textit{and} \qquad \lim_{x\to-\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m}$$

Example 1.4. Earlier we noted that $\lim_{x\to\infty} \sin x$ does not exist due to its oscillatory behavior. In this example, we examine a similar function:

$$f(x) = \frac{\sin x}{e^x}$$

Using big number logic, we see that as x gets very large, we are dividing numbersx between -1 and 1, i.e. $\sin x$, by an extremely large number e^x. It follows that even though the function is oscillating, it is getting closer and closer to zero as x approaches infinity, so:

$$\lim_{x\to\infty} \frac{\sin x}{e^x} = 0$$

However, as x gets more and more negative, we are dividing a number between -1 and 1, i.e. $\sin x$, by a number getting closer and closer to zero since $\lim_{x\to-\infty} e^x = 0$. It follows that $\sin x/e^x$ is oscillating between extremely large negative numbers and extremely large positive numbers, hence no limit exists, as it is not growing in a consistent direction. Therefore:

$$\lim_{x\to-\infty} \frac{\sin x}{e^x} \text{ does not exist}$$

We now begin our handling of limits as 'x approaches a' for some real number a. Instead of writing 'limit of $f(x)$ as x approaches a' we employ the notation:

$$\lim_{x\to a} f(x)$$

Just as the infinite limits, the 'result' of the above expression has three possibilities, all of which tell us something about the behavior of $f(x)$ near $x = a$. The first possibility is that:

$$\lim_{x \to a} f(x) = L$$

for some real number L; what this means is that as x approaches, or gets closer and closer to a, the values $f(x)$ get closer and closer to a. Now note that that x could approach a from the left or the right of a, so for the limit to be equal to L, $f(x)$ has to approach L in both directions; we will delve more into this later. An example of this case is our $v_{\text{avg}}(h)$ function from Section 1.1; as h approached 0 the value of $v_{\text{avg}}(h)$ approached the speed at which the ball was traveling at $t_0 = 2$. In particular:

$$\lim_{h \to 0} v_{\text{avg}}(h) = -40$$

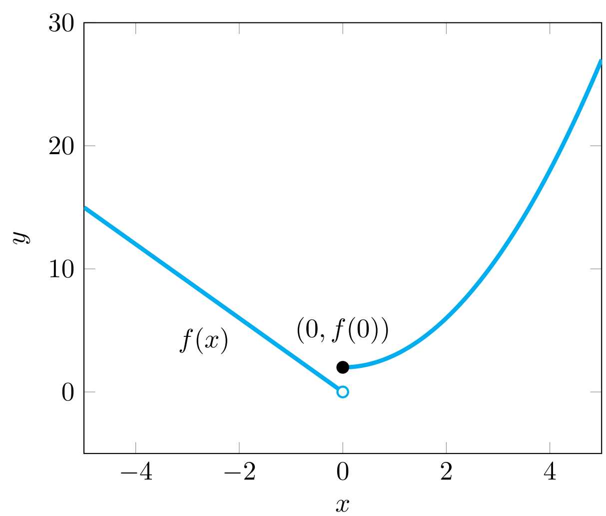
The next situation is that:

$$\lim_{x \to a} f(x) = \pm\infty$$

by which we mean that as x approaches a, $f(x)$ grows without bound in the positive or negative direction. In other words $f(x)$ gets larger and larger, or more and more negative as x approaches a. Finally, we can have that the limit of $f(x)$ as x approaches a fails to exist. This is most commonly found in the following situation, let:

$$f(x) = \begin{cases} x^2 + 2 & \text{for } x \geq 0 \\ -3x & \text{for } x < 0 \end{cases} \tag{1.3}$$

The above notation means that for $x < 0$, $f(x) = -3x$ and for $x \geq 0$ $f(x) = x^2 + 2$. The graph of this function is given by:



Now what should the limit of $f(x)$ be as x goes to zero? The problem is that if $x > 0$, then as x gets closer and closer to zero, $f(x)$ gets closer and closer to 2, but if $x < 0$ then as x get closer and closer to zero, $f(x)$ gets closer and closer to 0. We clearly don't have that $f(x)$ is growing in a consistent direction, and from our earlier discussion, $f(x)$ can't approach a consistent value L, so the limit as x approaches $f(x)$ does not fall into either of the previously discussed categories. In this case, we thus say the limit as x approaches a of $f(x)$ does not exist. Notation we say that:

$$\lim_{x \to a} f(x) \text{ does not exist}$$

Before delving into examples, we briefly formalize our analysis of the limit of $f(x)$ as defined in (1.3).

Definition 1.1. Let $f(x)$ be a function, and a a real number. We define the **limit as x approaches a from the left** as a limit of $f(x)$ where we only consider values of $x < a$. We denote this by:

$$\lim_{x \to a^-} f(x)$$

In other words, we only care if $f(x)$ approaches L, grows in a positive or negative direction, or does not exist while analyzing values of x which are less than a. Similarly we define the **limit as x approaches a from the right** as a limit of $f(x)$ where we only consider values of $x > a$. We denote this by:

$$\lim_{x \to a^+} f(x)$$

In particular, if $f(x)$ is as defined in (1.3) we have that:

$$\lim_{x \to a^-} f(x) = 0 \neq 2 = \lim_{x \to a^+} f(x)$$

We have the following result:

Theorem 1.2. *Let $f(x)$ be a function, and a a real number. Then the following are true:*

a) If $\lim_{x \to a} f(x) = L$, ∞, or $-\infty$, for some real number L, then $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) =$L, ∞ or $-\infty$ respectively.

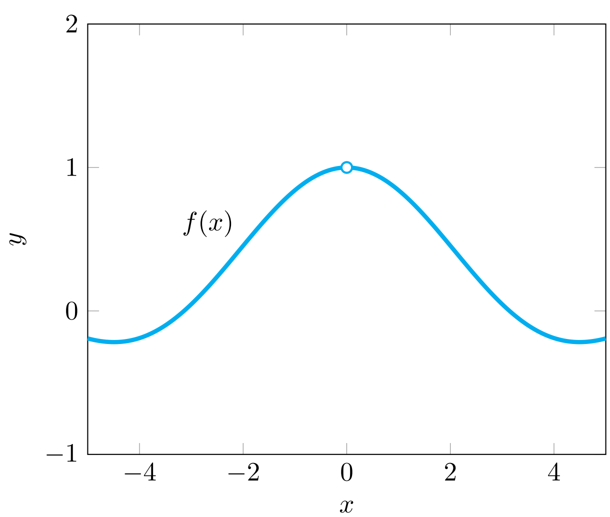
b) If $\lim_{x \to a^-} = \lim_{x \to a^+} = L$, ∞ or $-\infty$, then $\lim_{x \to a} f(x) = L$, ∞, or $-\infty$ respectively.

We now look at some examples:

Example 1.5. Let:

$$f(x) = \frac{\sin x}{x}$$

then the graph of $f(x)$ is given by:



From the graph of the function, it is easy to see that as x approaches 0 from the left, $f(x)$ approaches 1, and as x approaches 0 from the right, $f(x)$ approaches 1. We thus have that:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This was easy since we could see the graph, but graphing this function by hand is difficult (I know I wouldn't be able to do it!); on the first homework you will calculate this limit formally with appealing to the graph of the function.

1.3 Continuity

Note even if $f(a) = L$ it could be the case that $\lim_{x\to a} f(x) \neq L$. Indeed consider the next example:

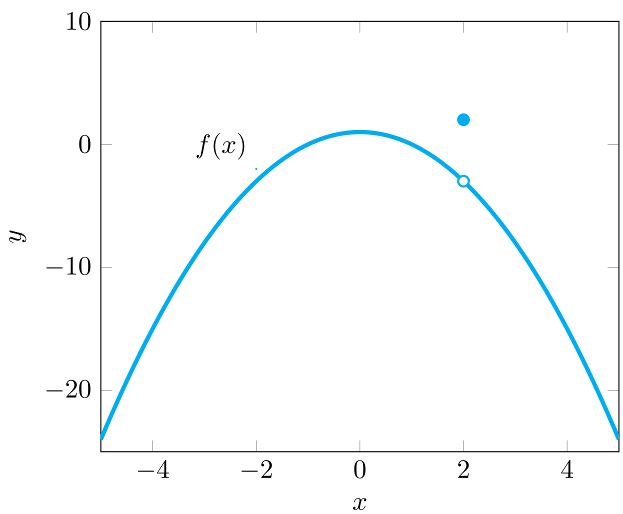
Example 1.6. Let:

$$f(x) = \begin{cases} -x^2 + 1 & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$$

Let us analyze the limit as x approaches 2 without appealing to a graphical argument. Since $x^2 + 1$ is a continuous function, i.e. we can draw it's graph without lifting up our pencil, we have that as x gets closer and closer to 2, $x^2 + 1$ gets closer and closer to 5. One can see this with the following chart:

x	$-x^2 + 1$
1.9	-2.61
1.99	-2.96
2.01	-3.04
2.1	-3.41

It follows that $\lim_{x\to 2} f(x) = -3$, however from the definition of $f(x)$, we have that $f(2) = 2$. Looking at the graph of this function, we see that the fact that $\lim_{x\to 2} f(x) \neq f(2)$ reflects the fact that $f(x)$ is not a continuous function:



With this example in mind, we give a different definition of a function being continuous:

Definition 1.2. Let $f(x)$ be a function, then $f(x)$ is **continuous at a** if:

$$\lim_{x\to a} f(x) = f(a)$$

A function $f(x)$ is **continuous on it's domain** if for every real number a in the domain of $f(x)$, $f(x)$ is continuous at a. A function is **continuous** if it is continuous for every real number. A **discontinuity of f(x)** is a real number a such that $f(x)$ is not continuous at a, or $f(x)$ is not defined at a. A discontinuity a of $f(x)$ is a **removable discontinuity**, if:

$$\lim_{x\to a} f(x) = L$$

for some real number L. A discontinuity a of $f(x)$ is a **jump discontinuity** if:

$$\lim_{x\to a^-} f(x) = L^- \neq L^+ = \lim_{x\to a^+} f(x)$$

for some real numbers L^- and L^+. A discontinuity a of $f(x)$ is an **infinite discontinuity** if:

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty$$

This definition, while more verbose and complicated than the 'drawing a graph without lifting up a pen' definition, mathematically captures the spirit of the concept of continuity, and is therefore the 'correct' definition for this concept. Trigonometric functions, exponential functions, logarithmic functions, radical functions[4] and rational functions are all continuous on their domains; that is they are continuous everywhere they are defined. Polynomials, exponential functions, $\sin x$ and $\cos x$ are examples of continuous functions, as they are defined everywhere.

Example 1.7. Let $f(x)$ be the function from Example 1.6, then $f(x)$ is continuous everywhere but $x = 2$. Indeed, at $x = 2$ we have that $\lim_{x \to 2} f(x) = -3$ but $f(2) = 2$. It follows that 2 is a discontinuity because $f(x)$ is not continuous at 2. In particular, 2 is a removable discontinuity, because the limit as x approaches 2 exists and is finite. This example demonstrates why it is called a removable discontinuity, because we can alter the value of the function at one point to make $f(x)$ continuous there.

Example 1.8. Let:

$$f(x) = \frac{x^2 - 9}{x + 3}$$

then $f(x)$ is not defined at $x = -3$ as we will divide by zero. However, for all values $x \neq -3$, we have that:

$$\frac{x^2 - 9}{x + 3} = \frac{(x+3)(x-3)}{x+3} = x - 3$$

It follows that $\lim_{x \to -3} = -6$, so $f(x)$ is not continuos at $x = -3$, because $f(x)$ is not defined at $x = -3$ but it's limit exists, so -3 is a removable discontinuity. Indeed, if we define:

$$g(x) = \begin{cases} \frac{x^2-9}{x+3} & \text{if } x \neq -3 \\ -6 & \text{if } x = -3 \end{cases}$$

then $g(-3) = -6$, and $\lim_{x \to -3} g(x) = -6$ so $g(x)$ is continuous. We have in a sense removed the discontinuity with $g(x)$.

Example 1.9. Consider the function $f(x)$ as defined in Equation 1.3. Then the limit as x approaches 0 of $f(x)$ does not exist, so $f(x)$ is not continuous at $x = 0$. It follows that 0 is a discontinuity point, and it is a jump discontinuity because $\lim_{x \to 0^+} f(x) = 2$ and $\lim_{x \to 0^-} f(x) = 0$. The graph of $f(x)$ demonstrates why we call such a discontinuity a jump discontinuity, because $f(x)$ 'jumps' from one value to the next at $x = 0$.

Example 1.10. Let:

$$f(x) = \frac{1}{x}$$

then $f(0)$ is undefined as we would be dividing by zero. As we approach 0 from the left, we are dividing by negative numbers closer and closer to zero, hence $f(x)$ is approaching negative infinity. As we approach 0 from the right, we are dividing by smaller and smaller positive numbers, so $f(x)$ is 'blowing up' and approaching positive infinity. It follows that:

$$\lim_{x \to 0^+} f(x) = \infty \neq -\infty = \lim_{x \to 0^-} f(x)$$

So we have that the limit as x approaches zero does not exist, that $x = 0$ is a discontinuity point, and in particular it is an infinite discontinuity. If instead:

$$f(x) = \frac{1}{x}^2$$

then we have that the limit as x approaches 0 is ∞ because the left and right handed limits agree, however 0 is still an infinite discontinuity of $f(x)$.

[4] i.e. any function of the form x^a where a is not a whole number

We end the section with the following result on limits, known as the limit rules, and then use them to compute some examples.

Theorem 1.3. *Let a be a real number, and $f(x)$ and $g(x)$ defined for all $x \neq a$ on some open interval containing a. Moreover, suppose that*

$$\lim_{x\to a} f(x) = L \qquad \textit{and} \qquad \lim_{x\to a} g(x) = M$$

for some real numbers L and M, then the following results hold:

a) $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = L + M.$

b) $\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = L - M.$

c) $\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = L \cdot M$

d) If $M \neq 0$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M}$$

e) If $f(x)$ is continuous at M, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(M)$$

The above rules, and your challenge homework imply that polynomials are continuous for all real numbers. We now show that rational functions are continuous on their domain; showing that radical functions are continuous on their domain is a fact we take for granted as it is harder to show for *all* real numbers.

Example 1.11. Let $f(x)$ be a rational function, then $f(x)$ is of the form:

$$\frac{p(x)}{q(x)}$$

for some polynomials $p(x)$ and $q(x)$. The domain of $f(x)$ is all real numbers such that $q(x) \neq 0$, so for any a such that $q(a) \neq 0$, we have that:

$$f(a) = \frac{p(a)}{q(a)}$$

Since polynomials are continuous, we have that by d) of the limit rules:

$$\lim_{x\to a} f(x) = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} = \frac{p(a)}{q(a)}$$

so $f(x)$ is continuous on it's domain because it is continuous at every real number for which it is defined.

We claimed earlier that $\sin x$ and $\cos x$ were continuous functions by appealing to their graphs. However, every graph we draw is really only over some interval (a, b), so we have only shown that $\sin x$ and $\cos x$ are continuous over some some small interval, usually including 0. In the following example, we show that $\sin x$ is continuous at all real numbers:

Example 1.12. Let a be a real number, then we need to show that $\lim\limits_{x\to a} \sin x = \sin a$. Note that $\sin x = \sin((x-a)+a)$ because $x = x - a + a$. Using the trigonometric identity:

$$\sin(\theta+\gamma) = \sin\theta\cos\gamma + \cos\theta\cos\gamma$$

we find that:

$$\sin x = \sin(x-a)\cos a + \cos(x-a)\sin a$$

Using the first limit rule, we find that:

$$\lim_{x\to a} \sin x = \lim_{x\to a} (\sin(x-a)\cos a) + \lim_{x\to a} (\cos(x-a)\sin a)$$

We can view $\sin a$ and $\cos a$ as constant functions, hence using c) of Theorem 1.3, and the fact that constant functions are continuous, we have:

$$\lim_{x\to a} \sin x = \cos a \cdot \lim_{x\to a} \sin(x-a) + \sin a \cdot \lim_{x\to a} \cos(x-a)$$

Since $\lim\limits_{x\to a} x - a$ is equal to zero as $x - a$ is a continuous function, and $\sin x$ and $\cos x$ are continuous at zero[5], we have that by e):

$$\begin{aligned}\lim_{x\to a} \sin x &= \cos a \cdot \sin\left(\lim_{x\to a} x - a\right) + \sin a \cdot \cos\left(\lim_{x\to a} x - a\right) \\ &= \cos a \cdot \sin 0 + \sin a \cdot \cos 0 \\ &= \sin a\end{aligned}$$

meaning that $\sin x$ is continuous!

Using the limit laws, and properties of continuous functions, we can calculate many limits, but what about when the limit laws don't apply? For example, so suppose that $f(x)$ and $g(x)$ are functions, satisfying $\lim\limits_{x\to a} f(x) = 0$, and $\lim\limits_{x\to a} g(x) = 0$? Then if we naively try to apply the limit laws to their quotient we get:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

which doesn't make any sense. We have already seen how to deal with problem in certain cases such as Example 1.8, but we now provide an example of a more complicated situation.

Example 1.13. Let:

$$f(x) = \frac{x-2}{\sqrt{8-x^2}-2}$$

Note that their are two constraints on the domain of this function, namely that $8 - x^2$ is greater than or equal to zero, and that $\sqrt{8-x^2}$ does not equal to 2. It follows that the domain of this function is given by[6]:

$$\left[-2\sqrt{2}, -2\right) \bigcup \left(-2, 2\right) \bigcup \left(2, 2\sqrt{2}\right]$$

We want to find the limit of $f(x)$ as x approaches 2. Both the top function and the bottom function are continuous at $x = 2$, but if we naively apply the limit laws, then we end up with $0/0$, which as we mentioned earlier is no good. Instead, we should algebraically manipulate the equation by noticing we can 'rationalize the denominator'. Recall the difference of squares formula:

$$(a-b)(a+b) = a^2 - b^2$$

If we set $a = \sqrt{8-x^2}$, and $b = -2$, then we have that:

$$(\sqrt{8-x^2}-2)(\sqrt{8-x^2}+2) = 8 - x^2 - 4 = 4 - x^2$$

[5] You can draw their graph on the interval $(-\pi, \pi)$ to see this!

[6] We do this by first finding the interval on which $8 - x^2 \geq 0$, and then removing the solution to $\sqrt{8-x^2} = 2$ from said interval.

It follows that for all x:

$$
\begin{aligned}
f(x) =& \frac{x-2}{\sqrt{8-x^2}-2} \cdot \frac{\sqrt{8-x^2}+2}{\sqrt{8-x^2}+2} \\
=& \frac{(x-2)(\sqrt{8-x^2}+2)}{4-x^2} \\
=& \frac{(x-2)(\sqrt{8-x^2}+2)}{(2-x)(2+x)} \\
=& \frac{(x-2)(\sqrt{8-x^2}+2)}{-(x-2)(2+x)}
\end{aligned}
$$

where in the third step we have applied the difference of squares formula again, and in the final step we pulled out a negative 2. For all $x \neq 2$ we can set $(x-2)/(x-2)$ equal to one, so this simplifies to:

$$
f(x) = \frac{(\sqrt{8-x^2}+2)}{-(2+x)}
$$

Now we can apply the limit laws as both the top and the bottom functions have non zero limits x approaches 2. Therefore:

$$
\begin{aligned}
\lim_{x \to 2} f(x) =& \frac{\lim_{x \to a} \left(\sqrt{8-x^2}+2\right)}{\lim_{x \to a} -(2+x)} \\
=& \frac{\sqrt{8-4}+2}{-(2+2)} \\
=& -1
\end{aligned}
$$

This process for solving limits is called rationalizing the denominator.

1.4 The Squeeze Theorem