Calculus I

Problem Set II

April 6, 2025

Provide **handwritten** answers on a separate sheet of paper, and separate your challenge problems from the textbook problems. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources.

Textbook Problems:

Ch. 3:

Challenge Problems

i) In this problem we explore the derivative of the common trigonometric functions.

   a) Draw the graph of $\sin x$. By estimating the slope of the tangent line at each point, draw a graph of it's derivative. Do the same for $\cos x$. What do you notice?

   b) Using the limits you calculated in the previous challenge homework, and the limit definition of the derivative to show that:

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x$$

   c) Using the quotient rule find the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\csc x$.

ii) In mathematics, sometimes we don't want to write out all the terms in a sum, so we employ something called sigma summation notation. This works as follows, say I have a function of integers, i.e. for each positive integer n, I have a new number $a(n)$. If I want to sum all of these numbers from 0 to some big N, we have to write:

$$a(0) + a(1) + a(2) + a(3) + \cdots + a(N-2) + a(N-1) + a(N)$$

Since mathematicians are lazy, we have decide that instead of writing this out every time, we should just write:

$$\sum_{n=0}^{n} a(n) = a(0) + a(1) + a(2) + a(3) + \cdots + a(N-2) + a(N-1) + a(N)$$

Concretely, if $a(n) = n$, and $N = 5$ then:

$$\sum_{n=0}^{5} n = 0 + 1 + 2 + 3 + 4 + 5 = 15$$

Now, we can also use this to write down polynomials. For example:

$$\sum_{n}^{N} a(n) \cdot x^n = a(0) + a(1)x + a(2)x^2 + a(3)x^3 + \cdots + a(N-2)x^{N-2} + a(N-1)x^{N-1} + a(N)$$

With $a(n) = n$ and $N = 5$ again we have that:

$$\sum_{n=0}^{5} nx^n = x + 2x^2 + 3x^3 + 4x^4 + 5x^5$$

Some functions can be represented as *infinite sums* of polynomials. Indeed, we have that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{0.1}$$

by which we mean that if we plug a number, say 1 into e^x, then $e^1 = e$ is given by the infinite sum:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \cdots$$

Now in practice we can't sum up infinitely many things, so what this actually means is that as we increase the number of terms in our sum, we get closer and closer to e.

a) By using Desmos to graph $\sum_{n=0}^{N}(x^n/n!)$ for larger and larger N, convince yourself that e^x is really given by (1). Do the same for:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

b) Using the power rule, and differentiate each infinite polynomial term by term to obtain the derivatives of e^x, $\sin x$ and $\cos x$.

c) Recall that i is an imaginary number satisfying $i^2 = -1$. Using the summation formulas for $\sin x$, $\cos x$, and e^x, show that:

$$e^{ix} = \cos x + i \sin x$$

Deduce that:

$$e^{i\pi} + 1 = 0$$

iii) In this exercise we confirm some statements made in lecture.

a) Using the chain rule, and product rule, derive the quotient rule by writing f/g as $f \cdot (g)^{-1}$.

b) Suppose that f is differentiable at a point a. Show that f is continuos at this point by showing $\lim_{x \to a} f(x) = f(a)$.

iv) Find the the derivatives of the following functions:

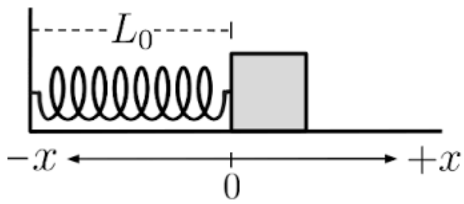
$$a)\ \log_a x \qquad b)\ \arcsin x$$

$$c)\ \arccos x \qquad d)\ \arctan x$$

$$e)\ \sinh x \qquad f)\ \cosh x$$

Hint: a) requires the chain rule, $b) - d)$ require inverse function theorem, and $e) - f)$ require only the derivative of e^x.

v) In this problem we consider a block of mass m attached to a spring which is then attached to a wall. When the spring is not compressed at all, it has a length of L_0, in particular we have the following picture:



If we stretch the string a distance of A in the positive x direction, and then let it go, results from physics tell us that the boxes distance from the $x = 0$ position is given as a function of time by:

$$x(t) = A \cdot \cos\left(t \cdot \sqrt{\frac{k}{m}}\right)$$

Here k is the spring constant which measures how hard it is to pull or push the string.

a) Find the velocity function $v(t)$.

b) Given any velocity function, $v(t)$, argue that the derivative of $v(t)$ should be an acceleration function, telling us the instantaneous acceleration at an time. Find $a(t)$ in this situation.

c) Newton's laws state that the force function is given by $F(t) = m \cdot a(t)$. Hooke's law states that the force function for a spring is given by $F(t) = -k \cdot x(t)$. Verify that $m \cdot a(t) = -k \cdot x(t)$,[1] i.e. that that the distance function of the box satisfies Hooke's Law.

[1]To do this just find $m \cdot a(t)$, and $-k \cdot x(t)$ and see the are exactly the same.