# PERCOLATION FOR COX POINT PROCESSES WITH CANYON SHADOWING

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## 1. Foreword

1.1. **Network model.** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and state space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, where $\mathcal{B}(\mathbb{R}^2)$ is the usual Borel $\sigma$-algebra of $\mathbb{R}^2$.

Let $\lambda_S > 0$ and $X_S$ be a homogeneous planar Poisson point process (PPP) in the state space $\mathbb{R}^2$ with intensity $\lambda_S$. Consider the Poisson-Voronoi tessellation (PVT) $S$ associated with $X_S$. In particular, $S$ is stationary. By analogy with a telecommunications network, $S$ will be called *street system* from now owards.

Denote by $E \coloneqq (e_i)_{i\geq 1}$ the edge-set of $S$ and by $V \coloneqq (V_i)_{i\geq 1}$ the vertex-set of $V$. Furthering the aforementioned analogy, the elements of $E$ (resp. $V$) will be called *street segments* (resp. *crossroads*).

Let $\Lambda$ be the stationary random measure defined such that:

- $\mathbb{E}\Lambda\,[0,1]^2 = 1$
- $\Lambda(dx) = \nu_1(S \cap dx)$, where $\nu_1$ denotes the 1-dimensional Hausdorff measure of $\mathbb{R}^2$. In other words, $\nu_1(S \cap B)$ is the total edge length of $S$ contained in the Borel observation window $B$ and so $\Lambda$ can be seen as a Lebesgue-like measure on the edges of $S$, rescaled in such a way that the total $\Lambda$-measure of a 1-area window is 1.

The key network parameters are:

- The user density $\lambda > 0$.
- The relay proportion $p \in (0, 1)$.
- The connectivity radius $r > 0$.

The users, equipped with mobile devices, are modelled by a Cox process $X^\lambda$ driven by the random intensity measure $\lambda\Lambda$. In other words, conditioned on a given realization of the street system $S$, $X^\lambda$ is a PPP with mean measure $\lambda\Lambda$. In particular, the number of users on a given street segment $e \in E$ is a Poisson random variable with mean $\nu_1(e)$ and the numbers of users in two disjoint subsets of $E$ are independent random variables.

The relays (either representing physical antennas or additional users not modelled through $X^\lambda$) are modelled by a doubly stochastic Bernoulli point process $Y$ on the set of crossroads $V$ with parameter $p$, so that one can write:

$$Y = \sum_i \mathbb{1}_{\{U_i \leq p\}} \delta_{V_i},$$

where $U_i \overset{\text{i.i.d.}}{\sim} \mathcal{U}(0,1)$ and where $\delta_x$ denotes the Dirac measure at $x$. In other words, conditioned on $\Lambda$ (or, equivalently, $S$) each crossroad is retained (resp. erased) independently from all others with probability $p$ (resp. $1-p$).
Moreover, we also assume that the processes of users and of relays are conditionnally independent given their random support, i.e. $X^\lambda \perp\!\!\!\perp Y \mid \Lambda$. We denote $Z := X^\lambda \cup Y$ the superposition of users and relays. The network is modelled by the *connectivity graph* $\mathcal{G}_{r,\lambda,p}$ defined in the following way:

- $\mathcal{G}_{r,\lambda,p}$ is undirected.
- The vertex set of $\mathcal{G}_{r,\lambda,p}$ are the points of $Z$.
- The edge $Z_i \leftrightsquigarrow Z_j, i \neq j$ is drawn if and only if $Z_i$ and $Z_j$ are located on the *same* street segment and of mutual Euclidean distance less than $r$. In other words:

$$\forall\, i \neq j,\ Z_i \leftrightsquigarrow Z_j \Leftrightarrow \begin{cases} \exists\, e \in E,\ Z_i \in e \text{ and } Z_j \in e \\ \|Z_i - Z_j\| \leq r \end{cases} \tag{1}$$

Main question : Percolation regime of the connectivity graph $\mathcal{G}_{r,\lambda,p}$? Are there critical values of the parameters $(r, \lambda, p)$ for which percolation of $\mathcal{G}_{r,\lambda,p}$ occurs with positive probability?

1.2. **Definitions and Notations.** We begin with introducing a few notations and definitions useful for the purposes of our developments.

**Notation 1.** For $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$, we denote as customary the Euclidean distance between $A$ and $B$ by:

$$\mathrm{dist}(A, B) := \inf\{\|x - y\|_2,\ x \in A,\ y \in B\}$$

**Notation 2.** For $x \in \mathbb{R}^2$, $n \in \mathbb{N} \setminus \{0\}$, we denote by

$$Q_n(x) := x + [-n/2, n/2]^2$$

the square of side $n$ centered at $x$. We note that this is exactly the definition of the closed ball with center $x$ and radius $n/2$ for the infinite norm of $\mathbb{R}$

**Notation 3.** For $n \in \mathbb{N} \setminus \{0\}$, we will write $Q_n$ to mean $Q_n(0)$.

**Notation 4.** We denote by $\mathbf{M}$ the space of Borel measures on $\mathbb{R}^2$, equipped with the evalutation $\sigma$-algebra [3, Section 13.1], i.e. the smallest $\sigma$-algebra making the mappings $\Xi \mapsto \Xi(B)$ measurable for all Borel sets $B \subset \mathbb{R}^d$.

**Notation 5.** For a (possibly random) Borel measure $\mu$ on $\mathbb{R}^2$ and $A \subset \mathbb{R}^2$, we denote the restriction of $\mu$ to $A$ by:

$$\mu_A(\cdot) := \mu(A \cap \cdot)$$

Finally, we will use the following convenient notation for the length of a street segment or a subset of street segment:

**Notation 6.** Let $e \in E$ and $s \subseteq e$. Then we denote the length of $s$ by $|s|$.

**Definition 1.** Let $\mu$ be a (possibly random) Borel measure on $\mathbb{R}^2$. The *support* of $\mu$ is the following set:

$$\operatorname{supp}(\mu) := \{x \in \mathbb{R}^d \ : \forall \varepsilon > 0,\ \mu(Q_\varepsilon(x)) > 0\}$$

We will then need the concepts of *stabilization* and *essential asymptotical connectedness* given in [2] for investigating spatial dependencies of random measures:

**Definition 2.** A random measure $\Xi$ on $\mathbb{R}^2$ is called *stabilizing* if there exists a random field of stabilisation radii $R = \{R_x\}_{x\in\mathbb{R}^2}$ defined on the same probability space as $\Xi$ such that:

(1) $(\Xi, R)$ are jointly stationary
(2) $\lim\limits_{n\uparrow\infty} \mathbb{P}\left(\sup\limits_{y\in Q_n\cap\mathbb{Q}^2} R_y < n\right) = 1$
(3) for all $n \geq 1$, the random variables

$$\left\{f\left(\Xi_{Q_n(x)}\right)\mathbb{1}\left\{\sup_{y\in Q_n(x)\cap\mathbb{Q}^2} R_y < n\right\}\right\}_{x\in\varphi}$$

are independent for all bounded measurable functions

$$f : \mathbf{M} \to [0, +\infty)$$

and finite $\varphi \subset \mathbb{R}^2$ such that $\forall\, x \in \varphi$, $\operatorname{dist}(x, \varphi \setminus \{x\}) > 3n$.

**Remark 1.** Throughout the rest of this paper, we assume the random variables $(R_x)_{x\in\mathbb{R}^2}$ to be $\Lambda$-measurable, as has been done by the authors of [2].

**Remark 2.** In the above definition, the random field of stabilization radii $R = \{R_x\}_{x\in\mathbb{R}^2}$ may not be unique, i.e. a random measure can be stabilizing for more than one field of stabilization radii.

**Remark 3.** Let $b > 0$ and assume that $\Xi$ is $b$-dependent, i.e.

$$\forall A, B \subset \mathbb{R}^d,\ \operatorname{dist}(A, B) > b \Rightarrow \Xi_A \perp\!\!\!\perp \Xi_B$$

Then $\Xi$ is stabilizing.

**Definition 3** (Essentially asymptotically connected random measure)**.** Let $\Xi$ be a random measure. Then $\Xi$ is *essentially asymptotically connected* if there exists a random field $R = \{R_x\}_{x\in\mathbb{R}^d}$ such that $\Lambda$ is stabilizing for $R$ and if for all $n \geq 1$, whenever $\sup\limits_{y\in Q_{2n}\cap\mathbb{Q}^2} R_y < n/2$, the following assertions are satisfied:

(1) $\operatorname{supp}(\Xi_{Q_n}) \neq \emptyset$
(2) $\operatorname{supp}(\Xi_{Q_n})$ is contained in a connected component of $\operatorname{supp}(\Xi_{Q_{2n}})$

The following result from [2] will turn out to be useful for our purposes:

**Proposition 1.** Let $\Lambda = \nu_1(S \cap dx)$, where $S$ is the PVT generated by an homogeneous stationary Poisson. Then $\Lambda$ is stabilizing and essentially asymptotically connected for the following stabilization field:

$$\forall x \in \mathbb{R}^2,\, R_x = \inf\{\|x - X_{S,i}\|,\, X_{S,i} \in X_S\},$$

where $X_S$ is the PPP having generated $S$.

We will need the following notation for convenience:

**Notation 7.** Assume that $\Xi$ is a stabilizing random measure for the stabilization field $R = \{R_x\}_{x\in\mathbb{R}^2}$. Then, for $x \in \mathbb{R}^2$ and $n \in \mathbb{N} \setminus \{0\}$ we denote:

$$R(Q_n(x)) \coloneqq \sup_{y\in Q_n(x)\cap\mathbb{Q}^2} R_y$$

Finally, we will adapt the usual definitions of openness and closedness of crossroads and parts of street segments (possibly the whole street segments themselves) in our model as follows:

**Definition 4** (Open/Closed crossroad)**.** Say a crossroad $V_i \in V$ is *open* if it is an atom of the point process $Y$, i.e. $Y(\{V_i\}) = 1$, or, equivalently, $U_i \le p$.
Say $V_i$ is *closed* if it is not open, i.e. $Y(\{V_i\}) = 0$, or, equivalently, $U_i > p$.

**Definition 5** (Open/Closed street segment)**.** Let $e \in E$ be a street segment and let $\emptyset \neq s \subseteq e$ be a non-empty subset of $e$.
Say $s$ is *open* if either of the two following set of conditions are satisfied:

(1) $|s| \le r$

**OR**

(2) $\begin{cases} |s| > r \\ \forall c \subset s,\ (|c| = r \text{ and } c \text{ topologically closed}) \Rightarrow X^\lambda(c) \ge 1 \end{cases}$

Say $s$ is *closed* if $s$ is not open, i.e.:
$\begin{cases} |s| > r \\ \exists\, c \subset s, \text{ such that } |c| = r \text{ and } c \text{ topologically closed and } X^\lambda(c) = 0 \end{cases}$

## 2. Results

Our results concern the phase transition of the connectivity graph corresponding to the model presented in Section 1.1 as well as a minimality condition on $p$ to make percolation to occur possible. Namely, we have the following:

**Theorem 1** (Minimality condition on $p$)**.** Let $p^*$ be the Voronoi percolation site threshold (a theoretical estimate is given in [5] and a numerical one is given in [1]). Let $p < p^*$. Then, for all $\lambda > 0$ and $r > 0$, the connectivity graph $\mathcal{G}_{r,\lambda,p}$ does not percolate with probability 1.

Concerning the sub-critical phase, we have the following:

**Theorem 2** (Existence of a non-trivial sub-critical phase)**.** Assume $p = 1$. Then there exists $r_0 > 0$ such that, whenever $r \le r_0$, we have:

$$\lambda_c(r) \coloneqq \inf\{\lambda > 0 : \mathbb{P}(\mathcal{G}_{r,\lambda,1} \text{ percolates}) > 0\} > 0$$

In other words, there exists a subcritical phase for percolation of the connectivity graph when all crossroads are equipped with a relay if the connection radius is not too large.

A straightforward consequence of Theorem 2 is the following:

**Corollary 1.** Let $r_0$ be defined as in Theorem 2. Then, whenever $r \leq r_0$, for all $p \in (p^*, 1]$, we have:

$$\inf\{\lambda > 0 : \mathbb{P}(\mathcal{G}_{r,\lambda,p} \text{ percolates}) > 0\} > 0$$

Finally, we also were able to get a matching super-criticality result:

**Theorem 3** (Existence of a non-trivial super-critical phase)**.** For sufficiently large and finite $\lambda$ and sufficiently large $p > p^*$, $\mathcal{G}_{r,\lambda,p}$ percolates, i.e. there exists a non-trivial super-critical phase.

While Theorem 1 is quite straightforward, Theorems 2 and 3 require the use of renormalization techniques similar to the ones exposed in [2] and the domination by product measures theorem [4, Theorem 0.0].
We will carry the proofs of the former theorems in the rest of this section.

2.1. **Proof of Theorem 1.**
Let $p^*$ denote the usual Poisson-Voronoi site percolation threshold, as defined in [1, 5]. It is known that $p \in (0,1)$. Note that by stationarity, $p^*$ is independent of the intensity of the PPP generating the considered PVT. Hence, consider site percolation on the PVT $S$ with parameter $p \in (0,1)$ and denote by $\mathcal{G}_p$ the associated graph.
Now, note that for given $p \in [0,1]$, all $\lambda > 0$ and $r > 0$, the edge-set of the connectivity graph of our model $\mathcal{G}_{r,\lambda,p}$ is a subset of the edge-set of $\mathcal{G}_p$.
Hence: $\forall \lambda > 0, \forall r > 0$, $\mathcal{G}_p$ does not percolate $\Rightarrow$ $\mathcal{G}_{r,\lambda,p}$ does not percolate. Now, by definition of $p^*$, $\forall p < p^*$, $\mathcal{G}_p$ does not percolate. This concludes the proof of Theorem 1. $\square$

2.2. **Proof of Theorem 2.**
Proving Theorem 2 is equivalent to proving that there exists $r_0$ such that whenever $r \leq r_0$, $\mathcal{G}_{r,\lambda,1}$ does not percolate if $\lambda$ is sufficiently small but positive.
As customary in any continuum percolation problem and as has been done in [2], we will use a renormalization argument and introduce a discrete percolation model constructed in such a way that if the discrete model does not percolate, then neither does $\mathcal{G}_{r,\lambda,1}$. Proving the absence of percolation of the discrete model will then be done via appealing to [4, Theorem 0.0].

To this end, for $n \geq 1$, say a site $z \in \mathbb{Z}^2$ is *n-good* if the following conditions are satisfied:

(1) $R(Q_n(nz)) < n$
(2) $\forall e \in E$, $s_{z,e} := e \cap Q_n(nz)$ is closed

Say a site $z \in \mathbb{Z}^2$ is *n-bad* if it is not $n$-good.

Our first claim is the following:

**Lemma 1.** Percolation of $\mathcal{G}_{r,\lambda,1}$ implies percolation of the process of $n$-bad sites.

*Proof.* Assume $\mathcal{G}_{r,\lambda,1}$ percolates and denote by $\mathcal{C}$ a giant component of $\mathcal{G}_{r,\lambda,1}$. Since $\mathcal{C}$ is infinite, we have: $\#\{z : \mathcal{C} \cap Q_n(nz) \neq \emptyset\} = +\infty$.
Denote $\{z : \mathcal{C} \cap Q_n(nz) \neq \emptyset\} := \{z_i, i \geq 1\}$. Note that $\mathcal{C}$ is made of open street segments and open crossroads. Therefore, since $Q_n(nz_i)$ is crossed by $\mathcal{C}$ for all $i \geq 1$, $Q_n(nz_i)$ has to be crossed by some open non-empty $s_e$, for some $e \in E$. Hence $z_i$ is $n$-bad, and $\{z_i, i \geq 1\}$ is a infinite component of $n$-bad sites. Now, since $\mathcal{C}$ is connected, then so is $\{z_i, i \geq 1\}$. Hence, the process of $n$-bad sites percolates. □

Therefore, it suffices to prove that the process of $n$-bad sites does not percolate if $\lambda$ and $r$ are sufficiently small. This will be done via appealing to [4, Theorem 0.0].
The conditions of [4, Theorem 0.0] are valid for so-called $k$-dependent random fields:

**Definition 6.** Let $\mathbf{X} = (X_s)_{s \in \mathbb{Z}^d}$ be a discrete random field. Let $k \geq 1$. Then $\mathbf{X}$ is said to be $k$-dependent if for all $p \geq 1$ and all $\psi = \{s_1, \dots s_p\} \subset \mathbb{Z}^d$ finite with the property that $\forall i \neq j, \|s_i - s_j\|_\infty > k$, the random variables $(X_{s_i})_{1 \leq i \leq p}$ are independent.

As we shall see thereafter, the process of $n$-bad sites previously defined is 3-dependent:

**Lemma 2.** For $z \in \mathbb{Z}^2$, set $\zeta_z := \mathbb{1}\{z \,\text{is}\, n\text{-bad}\}$. Then $(\zeta_z)_{z \in \mathbb{Z}^2}$ is a 3-dependent random field.

*Proof.* As a starting point, note that $\forall z \in \mathbb{Z}^2, \zeta_z = 1 - \mathbb{1}\{z \,\text{is}\, n\text{-good}\}$. It is therefore equivalent to prove that the process of $n$-good sites is 3-dependent.
For $z \in \mathbb{Z}^2$, set $\xi_z = \mathbb{1}\{z \,\text{is}\, n\text{-good}\}$. Let $\psi = \{z_1, \dots z_p\} \subset \mathbb{Z}^2$ be such that $\forall i \neq j, \|z_i - z_j\|_\infty > 3$. We want to show that the random variables $(\xi_{z_i})_{1 \leq i \leq p}$ are independent. Equivalently, since we are dealing with indicator functions, this amounts to showing that:

$$\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) = \prod_{i=1}^{p} \mathbb{E}(\xi_{z_i})$$

Now, we have:

$$\begin{aligned} \mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) &= \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i} \middle| \Lambda\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{p} \mathbb{1}\{R(Q_n(nz_i)) < n\} \prod_{i=1}^{p} \mathbb{1}\{\forall\, e \in E,\, s_{z_i,e} \,\text{is closed}\} \middle| \Lambda\right)\right] \qquad (2) \end{aligned}$$

By $\Lambda$-measurability of the random variables $\{R_x\}_{x \in \mathbb{R}^2}$, we can take the indicators of the form $\mathbb{1}\{R(Q_n(nz_i)) < n\}$, $1 \leq i \leq p$ out of the conditional expectation given $\Lambda$ in (2). This yields:

(3)
$$\mathbb{E}\left(\prod_{i=1}^{p}\xi_{z_i}\right) = \mathbb{E}\left[\prod_{i=1}^{p}\mathbb{1}\{R(Q_n(nz_i) < n\}\mathbb{E}\left(\prod_{i=1}^{p}\mathbb{1}\{\forall\, e\in E,\, s_{z_i,e} \text{ is closed}\}\middle|\Lambda\right)\right]$$

For $1 \le i \le p$, set $A_{z_i} := \{\forall\, e \in E,\, s_{z_i,e} \text{ is closed}\}, 1 \le i \le p$. According to Definition 5, for a given $1 \le i \le p$, the event $A_{z_i}$ only depends on the configuration of the random measure $\Lambda$ and of the Cox process $X^\lambda$ inside the square $Q_n(nz_i)$. Therefore, given $\Lambda$, the former events only depend on $X^\lambda \cap Q_n(nz_i)$ for $1 \le i \le p$. Since we have $\forall i \neq j, \|z_i - z_j\|_\infty > 3$, then the squares $Q_n(nz_i) =$ are disjoint. Now, we have that $X^\lambda$ is a Poisson Point Process given $\Lambda$. Thus, by Poisson independence property, the events $(A_{z_i})_{1\le i\le p}$ are conditionally independent given $\Lambda$. Hence (3) yields:

(4)
$$\mathbb{E}\left(\prod_{i=1}^{p}\xi_{z_i}\right) = \mathbb{E}\left[\prod_{i=1}^{p}\mathbb{1}\{R(Q_n(nz_i) < n\}\prod_{i=1}^{p}\mathbb{E}\left(\mathbb{1}\{\forall\, e\in E,\, s_{z_i,e} \text{ is closed}\}\middle|\Lambda\right)\right]$$

Now, setting, $f(\Lambda_{Q_n(x)}) := \mathbb{E}\left(\mathbb{1}\{\forall\, e\in E,\, s_{x,e} \text{ is closed}\}\middle|\Lambda\right)$, a deterministic, bounded and measurable function of $\Lambda_{Q_n(x)}$, we are left with:

$$\mathbb{E}\left(\prod_{i=1}^{p}\xi_{z_i}\right) = \mathbb{E}\left(\prod_{i=1}^{p} f(\Lambda_{Q_n(nz_i)})\mathbb{1}\{R(Q_n(nz_i)) < n\}\right) \tag{5}$$

Now, the set $\varphi := \{nz_1, \dots, nz_p\} \subset \mathbb{R}^d$ is finite and satisfies:

$$\forall i \neq j, \|nz_i - nz_j\|_\infty > 3n$$

Since the infinite norm is always upper bounded by the Euclidean norm, we have $\forall i \neq j, \|nz_i - nz_j\|_2 > 3n$, and so $\varphi$ satisfies:

$$\forall x \in \varphi,\ \mathrm{dist}(x, \varphi \setminus \{x\}) > 3n$$

We can therefore apply the definition of stabilization [2, Definition 2.3] to get that the random variables appearing in the right-hand side of (5) are independent. Hence:

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{p}\xi_{z_i}\right) &= \prod_{i=1}^{p}\mathbb{E}\left(f(\Lambda_{Q_n(nz_i)})\mathbb{1}\{R(Q_n(nz_i)) < n\}\right)\\
&= \prod_{i=1}^{p}\mathbb{E}\left[\mathbb{E}\left(\mathbb{1}\{\forall\, e\in E,\, s_{z_i,e} \text{ is closed}\}\middle|\Lambda\right)\mathbb{1}\{R(Q_n(nz_i)) < n\}\right]\\
&= \prod_{i=1}^{p}\mathbb{E}\left[\mathbb{E}\left(\mathbb{1}\{\forall\, e\in E,\, s_{z_i,e} \text{ is closed}\}\mathbb{1}\{R(Q_n(nz_i)) < n\}\middle|\Lambda\right)\right] \qquad (6)
\end{aligned}$$

(where we have used $\Lambda$-measurability of the $R$'s in (6))

$$\begin{aligned}
&= \prod_{i=1}^{p}\mathbb{E}\left[\mathbb{1}\{\forall\, e\in E,\, s_{z_i,e} \text{ is closed}\}\mathbb{1}\{R(Q_n(nz_i)) < n\}\right]\\
&=: \prod_{i=1}^{p}\mathbb{E}(\xi_{z_i})
\end{aligned}$$

This concludes the proof of the lemma. □

Now that we have proven that the process of $n$-bad sites is 3-dependent, in order to apply [4, Theorem 0.0], it remains to prove that the probability for a site $z \in \mathbb{Z}^d$ to be $n$-bad is sufficiently small when $\lambda$ and $r$ are chosen sufficiently small. Equivalently, as has been done in [2], we need to prove the following:

$$\limsup_{n \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\lambda \downarrow 0} \mathbb{P}(z \text{ is } n\text{-bad}) = 0$$

By stationarity, it suffices to prove that:

$$\limsup_{n \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\lambda \downarrow 0} \mathbb{P}(0 \text{ is } n\text{-bad}) = 0$$

Now, note that we have:

$$\begin{aligned} \mathbb{P}(0 \text{ is } n\text{-bad}) &= \mathbb{P}\Big(\{R(Q_n) \geq n\} \cup \{\exists\, e \in E,\, e \cap Q_n \text{ is open}\}\Big) \\ &\leq \mathbb{P}\left(R(Q_n) \geq n\right) + \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) \end{aligned}$$

On the one hand, we note that the quantity $R(Q_n)$ does not depend on $\lambda$ and $r$ and by Definition 2, we have $\lim_{n \uparrow \infty} \mathbb{P}(R(Q_n) \geq n) = 0$.
Therefore, it remains to deal with the quantity $\mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right)$. We have:

$$\begin{aligned} \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) &= 1 - \mathbb{P}\left(\forall\, e \in E,\, e \cap Q_n \text{ is open}\right) \\ &= 1 - \mathbb{E}\left( \prod_{e \in E\,:\, e \cap Q_n \neq \emptyset} \mathbb{1}\{e \cap Q_n \text{ is open}\} \right) \\ &= 1 - \mathbb{E}\left[ \mathbb{E}\left( \prod_{e \in E\,: e \cap Q_n \neq \emptyset} \mathbb{1}\{e \cap Q_n \text{ is closed}\} \middle| \Lambda \right) \right] \end{aligned}$$

For $e \in E$, denote:

$$\begin{aligned} s_e &\coloneqq e \cap Q_n \\ A_e &\coloneqq \{\exists\, c \subset s_e,\, c \text{ topologically closed}, |c| = r, X^\lambda(c) = 0\} \end{aligned}$$

We have: $\mathbb{1}\{s_e \text{ is closed}\} = \mathbb{1}\{|s_e| > r\}\mathbb{1}\{A_e\}$. Thus:

$$\begin{aligned} \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) &= 1 - \mathbb{E}\left[ \mathbb{E}\left( \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\}\mathbb{1}\{A_e\} \middle| \Lambda \right) \right] \\ &= 1 - \mathbb{E}\left[ \prod_{e \in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \mathbb{E}\left( \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{1}\{A_e\} \middle| \Lambda \right) \right], \end{aligned}$$

by $\Lambda$-measurability of the events $\{|s_e > r|\},\ e \in E$.

Given $\Lambda$, $X^\lambda$ has the distribution of a Poisson point process with mean measure $\lambda\Lambda$ and $A_e$ only depends on $X^\lambda \cap e$ once $e$ is fixed. Therefore, given $\Lambda$, the events $\{A_e : e \in E\}$ depend on the number of Cox points on distinct edges and so, by the Poisson independence property, these events are conditionally independent given $\Lambda$. Thus:

$$\begin{aligned}
\mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) &= 1 - \mathbb{E}\left[\prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \mathbb{E}\left(\prod_{e \in E\,: s_e \neq \emptyset} \mathbb{1}\{A_e\} \middle| \Lambda\right)\right] \\
&= 1 - \mathbb{E}\left[\prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{E}\left(\mathbb{1}\{A_e\} | \Lambda\right)\right] \\
&= 1 - \mathbb{E}\left[\prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{P}\left(A_e | \Lambda\right)\right] \qquad (7)
\end{aligned}$$

For all $e \in E$ such that $s_e \neq \emptyset$, we have that $0 \leq \mathbb{P}(A_e|\Lambda) \leq 1$, and moreover the event $A_e$ is increasing in $\lambda$ with $\lim\limits_{\lambda \downarrow 0} \mathbb{1}\{A_e\} = 1$ a.s.. So, by monotone convergence, we have that for all $e \in E$ such that $s_e \neq \emptyset$, $\lim\limits_{\lambda \downarrow 0} \mathbb{P}(A_e|\Lambda) = 1$ a.s.. As a matter of fact:

$$\limsup_{\lambda \downarrow 0} \prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{P}\left(A_e | \Lambda\right) = \prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \quad \text{a.s.}$$

Noting that $\left|\prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\} \prod_{e \in E\,: s_e \neq \emptyset} \mathbb{P}\left(A_e | \Lambda\right)\right| \leq 1$, we can apply dominated convergence in (7), we get:

$$\begin{aligned}
\limsup_{\lambda \downarrow 0} \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) &= 1 - \mathbb{E}\left(\prod_{e\in E: s_e \neq \emptyset} \mathbb{1}\{|s_e| > r\}\right) \\
&= 1 - \mathbb{E}\left(\prod_{e \in E} \mathbb{1}\{|e \cap Q_n| > r\} \mathbb{1}\{e \cap Q_n \neq \emptyset\}\right) \qquad (8)
\end{aligned}$$

Again, by monotone convergence, we have for all $e \in E$:

$$\begin{aligned}
\lim_{r \downarrow 0} \mathbb{1}\{|e \cap Q_n| > r\} \mathbb{1}\{e \cap Q_n \neq \emptyset\} &= \mathbb{1}\{|e \cap Q_n| > 0\} \mathbb{1}\{e \cap Q_n \neq \emptyset\} \quad \text{a.s.} \\
&= \mathbb{1}\{e \cap Q_n \neq \emptyset\} \quad \text{a.s.}
\end{aligned}$$

So, applying dominated convergence again in (8) we get:

$$\limsup_{r \downarrow \infty} \limsup_{\lambda \downarrow 0} \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) = 1 - \mathbb{E}\left(\prod_{e \in E} \mathbb{1}\{e \cap Q_n \neq \emptyset\}\right) \qquad (9)$$

By monotone convergence, $\lim\limits_{n \uparrow \infty} \mathbb{1}\{e \cap Q_n \neq \emptyset\} = 1$ a.s. for all $e \in E$. Therefore, by dominated convergence in (9), we obtain:

$$\limsup_{n \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\lambda \downarrow 0} \mathbb{P}\left(\exists\, e \in E,\, e \cap Q_n \text{ is open}\right) = 1 - 1 = 0 \qquad (10)$$

To conclude, we have:

$$0 \leq \mathbb{P}(0 \,\text{is}\; n\text{-bad}) \leq \mathbb{P}(R(Q_n) \geq n) + \mathbb{P}(\exists\, e \in E,\, e \cap Q_n \,\text{is open})$$

Using Definition 2 on and (10), we finally get:

$$\limsup_{n \uparrow \infty} \limsup_{r \downarrow 0} \limsup_{\lambda \downarrow 0} \mathbb{P}(0 \,\text{is}\; n\text{-bad}) = 0$$

Hence, by [4, Theorem 0.0], the process of $n$-bad sites is stochastically dominated from above by a sub-critical Bernoulli process when $\lambda$ and $r$ are sufficiently small. In particular, the process of $n$-bad sites cannot percolate when $\lambda$ and $r$ are sufficiently small. In other words, there exists $r_0 > 0$ such that $\lambda_c(r) > 0$. This concludes the proof of Theorem 2. □

2.3. **Proof of Corollary 1.**
Let $r_0$ be defined as in Theorem 2. Let $r \leq r_0$ and $p \in (p^*, A]$. For fixed $\lambda > 0$, there are obviously fewer possible connections in the connectivity graph $\mathcal{G}_{r,\lambda,p}$ than in $\mathcal{G}_{r,\lambda,1}$. In other words, the vertex-set (resp. edge-set) of $\mathcal{G}_{r,\lambda,p}$ is a subset of the vertex-set (resp. edge-set) of $\mathcal{G}_{r,\lambda,1}$. Thus we have:

$$\mathcal{G}_{r,\lambda,p} \text{ percolates } \Rightarrow \mathcal{G}_{r,\lambda,1} \text{ percolates}$$

Therefore:

$$\inf\{\lambda > 0 : \mathbb{P}(\mathcal{G}_{r,\lambda,p} \text{ percolates}) > 0\} > \underbrace{\inf\{\lambda > 0 : \mathbb{P}(\mathcal{G}_{r,\lambda,1} \text{ percolates}) > 0\}}_{=:\lambda_c(r)}$$

By Theorem 2, we have that $\lambda_c(r) > 0$ whenever $r \leq r_0$. This concludes the proof of Corollary 1. □

2.4. **Proof of Theorem 3.**
As in the proof of Theorem 2, we will use a renormalization argument to prove that $\mathcal{G}_{r,\lambda,p}$ percolates if $\lambda$ and $p \in (p^*, 1]$ are chosen sufficiently large.
To this end, let us first define a discrete model in such a way that percolation of the discrete model will imply percolation of the continuum one. For $n \geq 1$, say a site $z \in \mathbb{Z}^2$ is $n$-good if the following conditions are satisfied:

(1) $R(Q_{6n}(nz)) < n/2$
(2) $E \cap Q_n(nz) \neq \emptyset$, i.e. the square $Q_n(nz)$ contains a *full* street segment (not just a subset of a street segment)
(3) There exists $e \in E \cap Q_n(nz)$ such that $e$ is open, in the sense of Definition 5. In other words, there exists an open edge which is fully included in the square $Q_n(nz)$.
(4) Every two open edges $e, e' \in E \cap Q_{3n}(nz)$ are connected by a path in $\mathcal{G}_{r,\lambda,1} \cap Q_{6n}(nz)$, i.e. $e$ and $e'$ are connected by a path of the connectivity graph staying inside of the larger square $Q_{6n}(nz)$ if all crossroads are open.
(5) $Y(Q_{6n}(nz)) = \#(V \cap Q_{6n}(nz))$, i.e. all crossroads in $Q_{6n}(nz)$ are open, in the sense of Definition 4.

The $n$-good sites have been defined so as to satisfy the following: Our first claim is the following:

**Lemma 3.** Percolation of the process of $n$-good sites implies percolation of the connectivity graph $\mathcal{G}_{r,\lambda,p}$.

*Proof.* Let $\mathcal{C}$ be an infinite connected component of $n$-good sites. Consider a finite path $\{z_0, \dots, z_q\} \subset \mathcal{C}$ of $n$-good sites ($q \geq 1$). Then, by condition (2) in the definition of $n$-goodness, we can find an open edge $e_j \in E \cap Q_n(nz_j)$, for each $0 \leq j \leq q$. Let $0 \leq j \leq q-1$. Then $z_j = (a, b)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. Since $\mathcal{C}$ is connected in $\mathbb{Z}^2$, we have $z_{j+1} \in \{(a \pm 1, b), (a, b \pm 1)\}$. By symmetry, we can assume $z_{j+1} = (a+1, b)$. Thus:

$$\begin{aligned} Q_n(nz_j) &= [na - n/2, na + n/2] \times [nb - n/2, nb + n/2] \\ Q_n(nz_{j+1}) &= [na + n/2, na + 3n/2] \times [nb - n/2, nb + n/2] \\ Q_{3n}(nz_j) &= [na - 3n/2, na + 3n/2] \times [nb - 3n/2, nb + 3n/2] \\ Q_{6n}(nz_j) &= [na - 3n, na + 3n] \times [nb - 3n, nb + 3n] \end{aligned}$$

Therefore, we have $Q_n(nz_{j+1}) \subset Q_{3n}(nz_j)$ and so $e_{j+1} \in E \cap Q_n(nz_{j+1})$ implies $e_{j+1} \in E \cap Q_{3n}(nz_j)$. Since we also have $e_j \in E \cap Q_n(nz_j) \subset E \cap Q_{3n}(nz_j)$ and that $e_j$ and $e_{j+1}$ are both open, by condition (4) in the definition of,$n$-goodness, $e_j$ and $e_{j+1}$ are connected by a path $\mathcal{L}$ in $\mathcal{G}_{r,\lambda,1} \cap Q_{6n}(nz_j)$. But since $z_j$ is an $n$-good site, by condition (5) in the definition of $n$-goodness, all crossroads inside of $Q_{6n}(nz_j)$ are open. Therefore, the path $\mathcal{L}$ also connects $e_j$ and $e_{j+1}$ in $\mathcal{G}_{r,\lambda,p} \cap Q_{6n}(nz_j)$. Iterating this process gives rise to an infinite connected component in $\mathcal{G}_{r,\lambda,p}$. This concludes the proof of Lemma 3. □

As a matter of fact, proving Theorem 3 amounts to proving that the process of $n$-good sites percolates for sufficiently large $\lambda$ and $p$.
As has been done in the proof of Theorem 2, we will stochastically dominate the process of $n$-good sites by a Bernoulli process via the means of [4, Theorem 0.0]. To check that the conditions of this theorem are satisfied, we will need to use a slightly modified version of $n$-goodness more adapted to the use of the aforementioned theorem, as follows:

For $n \geq 1$, say a site $z \in \mathbb{Z}^2$ is weakly-$n$-good if the following conditions are satisfied:

(1') $R(Q_{6n}(nz)) < 6n$
(2) $E \cap Q_n(nz) \neq \emptyset$, i.e. the square $Q_n(nz)$ contains a *full* street segment (not just a subset of a street segment)
(3) There exists $e \in E \cap Q_n(nz)$ such that $e$ is open, in the sense of Definition 5. In other words, there exists an open edge which is fully included in the square $Q_n(nz)$.
(4) Every two open edges $e, e' \in E \cap Q_{3n}(nz)$ are connected by a path in $\mathcal{G}_{r,\lambda,1} \cap Q_{6n}(nz)$, i.e. $e$ and $e'$ are connected by a path of the connectivity graph staying inside of the larger square $Q_{6n}(nz)$ if all crossroads are open.
(5) $Y(Q_{6n}(nz)) = \#(V \cap Q_{6n}(nz))$, i.e. all crossroads in $Q_{6n}(nz)$ are open, in the sense of Definition 4.

Then the following is clear:

**Lemma 4.** $\forall\, z \in \mathbb{Z}^2$, $z$ is $n$-good $\Rightarrow$ $z$ is weakly-$n$-good. Therefore, we have the following: $\forall\, z \in \mathbb{Z}^2$, $\mathbb{P}(z \text{ is } n\text{-good}) \leq \mathbb{P}(z \text{ is almost-}n\text{-good})$

Moreover, since the condition (1) in the definition of $n$-goodness has not been used in the proof of Lemma 3, the following is also clear:

**Lemma 5.** Percolation of the process of weakly-$n$-good sites implies percolation of the connectivity graph $\mathcal{G}_{r,\lambda,p}$.

The reason why considering almost-$n$-good sites instead of $n$-good sites will turn out to be easier for our purposes is the following:

**Lemma 6.** For $z \in \mathbb{Z}^2$, set $\xi_z := \mathbb{1}\{z \text{ is weakly-}n\text{-good}\}$. Then $(\xi_z)_{z \in \mathbb{Z}^2}$ is an 18-dependent random field.

*Proof.* In the same way as in the proof of Lemma 1, it suffices to prove that for all finite $\psi = \{z_1, \dots z_p\} \subset \mathbb{Z}^2$ such that $\forall i \neq j, \|z_i - z_j\|_\infty > 18$, we have:

$$\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) = \prod_{i=1}^{p} \mathbb{E}(\xi_{z_i})$$

Denote respectively by $(A_z), (B_z), (C_z), (D_z), (F_z)$ the events $(1'), (2), (3), (4), (5)$ in the definition of weakly-$n$-goodness for $z \in \mathbb{Z}^2$. We thus have:

$$\forall z \in \mathbb{Z}^2,\ \xi_z = \mathbb{1}\{(A_z)\}\mathbb{1}\{(B_z)\}\mathbb{1}\{(C_z)\}\mathbb{1}\{(D_z)\}\mathbb{1}\{(F_z)\}$$

Note first that whenever $z \in \mathbb{Z}^2$, the indicators $\mathbb{1}\{(A_z)\}$ and $\mathbb{1}\{(B_z)\}$ are $\Lambda$-measurable. Thus, we have :

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) &= \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i} \middle| \Lambda\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\}\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i})\}\mathbb{1}\{(D_{z_i})\}\mathbb{1}\{(F_{z_i})\} \middle| \Lambda\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\}\mathbb{1}\{(B_{z_i})\}\mathbb{E}\left(\prod_{i=1}^{p} \mathbb{1}\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\} \middle| \Lambda\right)\right]
\end{aligned}$$

Now, note that conditioned on $\Lambda$, for each $1 \leq i \leq p$, the event $(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})$ only depends on the following:

- The configuration of $X^\lambda$ inside of the square $Q_{6n}(nz_i)$
- The configuration of $Y$ inside of the square $Q_{6n}(nz_i)$

Since $\psi$ satisfies $\forall i \neq j, \|z_i - z_j\|_\infty > 18$, then we have $\forall i \neq j, \|nz_i - nz_j\|_\infty > 18n$. As a matter of fact, the squares $\{Q_{6n}(nz_i))\ :\ 1 \leq i \leq p\}$ are disjoint, i.e.

$$\forall i \neq j, Q_{6n}(nz_i) \cap Q_{6n}(nz_j) = \emptyset$$

As a matter of fact, the events $\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\ :\ 1 \leq i \leq p\}$ depend on disjoint Borel sets. Now, conditioned on $\Lambda$, $X^\lambda$ has the distribution of a Poisson point process, $Y$ has the distribution of a Bernoulli point process and $X^\lambda \perp\!\!\!\perp Y\,|\Lambda$. By the independence property for Poisson and Bernoulli point processes, we have that the events $\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\ :\ 1 \leq i \leq p\}$ are conditionally independent given $\Lambda$. Hence:

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) &= \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\}\mathbb{1}\{(B_{z_i})\}\mathbb{E}\left(\prod_{i=1}^{p} \mathbb{1}\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\} \middle| \Lambda\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\}\mathbb{1}\{(B_{z_i})\} \prod_{i=1}^{p} \mathbb{E}\left(\mathbb{1}\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\} \middle| \Lambda\right)\right]
\end{aligned}$$

Again, by $\Lambda$-measurability of the events $\{(B_{z_i})\ :\ 1 \leq i \leq p\}$, we can put the indicators $\mathbb{1}\{(B_{z_i})\}$ back into the conditional expectation given $\Lambda$, so as to get:

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{p} \xi_{z_i}\right) &= \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\} \prod_{i=1}^{p} \mathbb{E}\left(\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\} \middle| \Lambda\right)\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{1}\{(A_{z_i})\} \prod_{i=1}^{p} \mathbb{E}\left(\mathbb{1}\{(B_{z_i}) \cap (C_{z_i}) \cap (D_{z_i}) \cap (F_{z_i})\} \middle| \Lambda\right)\right]
\end{aligned}$$

Now, setting $f(\Lambda_{Q_{6n}(x)}) := \mathbb{E}\left(\mathbb{1}\{(B_x)\cap(C_x)\cap(D_x)\cap(F_x)\}\,|\,\Lambda\right)$, a bounded measurable deterministic function of $\Lambda_{Q_{6n}(x)}$, we are left with:

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^p \xi_{z_i}\right) &= \mathbb{E}\left[\prod_{i=1}^p \mathbb{1}\{(A_{z_i})\}\prod_{i=1}^p \mathbb{E}\left(\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i})\cap(D_{z_i})\cap(F_{z_i})\}\middle|\Lambda\right)\right]\\ &= \mathbb{E}\left[\prod_{i=1}^p \mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}f(\Lambda_{Q_{6n}(nz_i)})\right] \qquad (11)\end{aligned}$$

Now, the set $\varphi := \{nz_1, \ldots, nz_p\} \subset \mathbb{R}^d$ is finite and satisfies:

$$\forall i \neq j, \|nz_i - nz_j\|_\infty > 18n$$

Since the infinite norm is always upper bounded by the Euclidean norm, we have $\forall i \neq j, \|nz_i - nz_j\|_2 > 18n$, and so $\varphi$ satisfies:

$$\forall x \in \varphi,\ \mathrm{dist}(x, \varphi \setminus \{x\}) > 18n = 3 \times 6n$$

We can therefore apply the definition of stabilization [2, Definition 2.3] (with $n$ replaced by $6n$ in the third condition) to get that the random variables appearing in the right-hand side of (11) are independent. Hence:

$$\mathbb{E}\left(\prod_{i=1}^p \xi_{z_i}\right) = \prod_{i=1}^p \mathbb{E}\left[\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}f(\Lambda_{Q_{6n}(nz_i)})\right] \qquad (12)$$

Now, for each $1 \leq i \leq p$, note that:

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}f(\Lambda_{Q_{6n}(nz_i)})\right]\\ &= \mathbb{E}\left[\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}\mathbb{E}\left(\mathbb{1}\{(B_{z_i})\cap(C_{z_i})\cap(D_{z_i})\cap(F_{z_i})\}\,|\,\Lambda\right)\right]\\ &= \mathbb{E}\left[\mathbb{E}\left(\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i})\}\mathbb{1}\{(D_{z_i})\}\mathbb{1}\{(F_{z_i})\}\middle|\Lambda\right)\right],\end{aligned}$$

using $\Lambda$-measurability of $\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}$

$$\begin{aligned}&= \mathbb{E}\left(\mathbb{1}\{R(Q_{6n}(nz_i) < 6n\}\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i})\}\mathbb{1}\{(D_{z_i})\}\mathbb{1}\{(F_{z_i})\}\right)\\ &= \mathbb{E}\left(\mathbb{1}\{(A_{z_i})\}\mathbb{1}\{(B_{z_i})\}\mathbb{1}\{(C_{z_i})\}\mathbb{1}\{(D_{z_i})\}\mathbb{1}\{(F_{z_i})\}\right)\\ &=: \mathbb{E}(\xi_{z_i})\end{aligned}$$

Thus, (12) yields:

$$\mathbb{E}\left(\prod_{i=1}^p \xi_{z_i}\right) = \prod_{i=1}^p \mathbb{E}(\xi_{z_i}),$$

as needed. This concludes the proof of Lemma 6. □

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