ex Let $X \hookrightarrow \mathcal{E}xp(\mu)$, $Y \hookrightarrow \mathcal{E}xp(\lambda)$ assuming $X \perp\!\!\!\perp Y$, compute $\mathcal{P}(X > Y)$

$$P(X>Y) = E[\mathbb{1}_{X>Y}] = \int_{x=0}^{+\infty}\int_{y=0}^{+\infty} \mathbb{1}_{X>Y}\, f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{x=0}^{+\infty}\int_{y=0}^{+\infty} \mathbb{1}_{x>y}\, f_x(x) \cdot f_y(y)\, dy\, dx \quad \text{because } X \perp\!\!\!\perp Y$$

$$= \int_{x=0}^{+\infty}\int_{y=0}^{+\infty} \mathbb{1}_{x>y}\, \mu e^{-\mu x} \cdot \lambda e^{-\lambda y}\, dy\, dx$$

$$= \int_{x=0}^{+\infty}\int_{y=0}^{x} \mu \cdot e^{-\mu x} \cdot \lambda e^{-\lambda y}\, dy\, dx = \int_{x=0}^{+\infty} \mu \cdot e^{-\mu x} \int_{y=0}^{x} \lambda e^{-\lambda y}\, dy\, dx$$

$$= \int_{x=0}^{+\infty} \mu e^{-\mu x} \left[\lambda \cdot \frac{-1}{\lambda} \cdot e^{-\lambda y}\right]_0^x dx = \int_{x=0}^{+\infty} \mu \cdot e^{-\mu x} \left(1 - e^{-\lambda x}\right) dx$$

$$= \int_{x=0}^{+\infty} \mu e^{-\mu x} - \mu e^{-(\mu+\lambda)x}\, dx = \int_{x=0}^{+\infty} \mu \cdot e^{-\mu x}\, dx - \int_{x=0}^{+\infty} \mu e^{-(\mu+\lambda)x}\, dx$$

$$= \left[\mu \cdot \frac{-1}{\mu} e^{-\mu x}\right]_0^{+\infty} - \left[\mu \cdot \frac{-1}{\mu+\lambda} e^{-(\mu+\lambda)x}\right]_0^{+\infty}$$

$$= \left[-e^{-\mu x}\right]_0^{+\infty} - \mu\left[\frac{-1}{\mu+\lambda} e^{-(\mu+\lambda)x}\right]_0^{+\infty}$$

$$= 1 - \mu \cdot \left(\frac{1}{\mu+\lambda}\right)$$

$$= 1 - \mu \cdot \frac{1}{\mu+\lambda} \quad \Rightarrow \quad \boxed{= \frac{\lambda}{\mu+\lambda}}$$

rk. $E[X] = \int X \cdot f(X)\, dx \quad \Rightarrow \quad E[g(X)] = \int g(X)\, f(X)\, dx$

transformation

$$\Rightarrow E[\mathbb{1}\, X] = \int \mathbb{1}\, X\, f\, X\, dx$$

ex we consider the density function $f(x,y) = \begin{cases} k \times \left(\frac{1}{x^2} + y^2\right) & \text{if } 1 \leqslant x \leqslant 5 \text{ and } -1 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

1) what should be the value of k

2) what are the marginal distributions?

3) are X and Y independent if $\binom{X}{Y}$ random vector whose distribution is given by f.

4) what is the distribution of X given $Y=0$. $\Rightarrow X_{|Y=0}$

1) given f a density function 1) $\forall x,y \in X(\Omega), Y(\Omega)$, $f(x,y) \geqslant 0$ which implies

$k \times (\frac{1}{x^2} + y^2) > 0 \Rightarrow k \geqslant 0$

2) $\int_{\mathbb{R}^2} f(x,y)\, dy\, dx = 1$

$$= \int_{x=1}^{5} \int_{y=-1}^{1} k \times \left(\frac{1}{x^2} + y^2\right) dy\, dx = k \int_{x=1}^{5} \int_{y=-1}^{1} \frac{1}{x^2} + y^2\, dy\, dx = k \int_{x=1}^{5} \left[\frac{1}{3}y^3 + \frac{1}{x^2}y\right]_{-1}^{1} dx$$

$$= k \int_1^5 \frac{1}{3} + \frac{1}{3} + \frac{1}{x^2} + \frac{1}{x^2}\, dx = 2k \int_1^5 \frac{1}{x^2} + \frac{1}{3}\, dx = 2k \cdot \left[-\frac{1}{x} + \frac{1}{3}x\right]_1^5 = 2k \cdot \left(-\frac{1}{5} + 1 + \frac{5}{3} - \frac{1}{3}\right)$$

$$= 2k \cdot \left(\frac{32}{15}\right) \qquad \frac{64}{15} k = 1 \quad \Rightarrow \quad \boxed{k = \frac{15}{64}}$$

2) $$MD_X = \int_{y=-1}^{1} \frac{15}{64}\left(\frac{1}{x^2} + y^2\right) dy = \frac{15}{64} \int_{y=-1}^{1} \frac{1}{x^2} + y^2\, dy = \frac{15}{64}\left(\left[\frac{1}{x^2}y\right]_{-1}^{1} + \left[\frac{1}{3} \times y^3\right]_{-1}^{1}\right)$$

$$= \frac{15}{64} \times \left(\frac{2}{x^2} + \frac{2}{3}\right) = \boxed{\frac{15}{32x^2} + \frac{5}{32}} \quad \forall x \in [1;5] \text{ otherwise } 0$$

$$MD_Y = \int_1^5 \frac{15}{64}\left(\frac{1}{x^2} + y^2\right) dx = \frac{15}{64} \int_1^5 \frac{1}{x^2} + y^2\, dx = \frac{15}{64}\left(\left[-\frac{1}{x}\right]_1^5 + \left[y^2 x\right]_1^5\right)$$

$$= \frac{15}{64} \times \left(\frac{4}{5} + 4y^2\right) = \boxed{\frac{3}{16} + \frac{15}{16}y^2} \quad \forall y \in [-1,1] \text{ otherwise } 0$$

3) $$f_{XY}(x,y) = \begin{cases} \frac{15}{64}\left(\frac{1}{x^2} + y^2\right) & \forall x \in [1,5], y \in [-1,1] \\ \text{otherwise } 0 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{15}{32x^2} + \frac{5}{32} & \forall x \in [1,5] \\ \text{otherwise } 0 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{3}{16} + \frac{15}{16}y^2 & \forall y \in [-1,1] \\ \text{otherwise } 0 \end{cases}$$

3) Two random variables X and Y forming a random vector $\binom{X}{Y}$ continuous are independent

iff $f_{XY}(x,y) = f_X(x) f_Y(y) \ \forall x,y \in \mathbb{R}$

$$f_X(x)\cdot f_Y(y) = \begin{cases} \left(\frac{15}{32x^2}+\frac{5}{32}\right)\left(\frac{3}{16}+\frac{15}{16}y^2\right) & \forall x\in[1,5],\ y\in[-1,1] \\ \text{otherwise } 0 \end{cases}$$

$$= \begin{cases} \frac{15(x^2+3)(5y^2+1)}{512x^2} & \forall x\in[1,5],\ y\in[-1,1] \\ \text{otherwise } 0 \end{cases}$$

$f_X(x)\cdot f_Y(y) \neq f_{XY}(x,y) \Rightarrow$ X and Y are not independent

4) The distribution of X given $(Y=0)$ is given by the density function such as

$$f_{X|Y=0} = \frac{f_{XY}(x,0)}{f_Y(0)} \quad \forall x\in[1,5]$$

$$= \frac{\frac{15}{64}\times\left(\frac{1}{x^2}+\cdot 0^2\right)}{64\times\left(\frac{3}{16}+0\right)} = \frac{15}{12x^2}$$

$$f_{X|Y=0} = \frac{5}{4x^2} \quad \forall x\in[1,5] \Rightarrow f_{X|Y=0} = \begin{cases} \frac{5}{4x^2} & \forall x\in[1,5] \\ \text{otherwise } 0 \end{cases}$$

ex 1) give the expression of ϕ_X when $X \hookrightarrow P(\lambda)$

2) we repeat n times an experimentation w/ 2 outputs (independent repetitions), we assume that $p = \frac{\lambda}{n}$, we denote by X_n the number of successes during n experiments

a) give distribution of X_n and its ϕ_X

b) prove that $\phi_n(t) \xrightarrow[n \to +\infty]{} \phi_X(t)$ when we precise X

1) $\phi_X(t) = E\left[e^{itx}\right] = \sum_{k=0}^{+\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{+\infty} \frac{(\lambda e^{it})^k}{k!}$

Taylor series $\sum_{n=0}^{+\infty} \frac{x^k}{k!} = e^x \Rightarrow E\left[e^{itx}\right] = e^{-\lambda} \cdot e^{\lambda e^{it}}$

$$\boxed{= e^{\lambda \cdot (e^{it} - 1)}}$$

2)a) $X_n \hookrightarrow \mathcal{B}\left(n, \frac{\lambda}{n}\right)$

$\phi_{X_n} = E\left[e^{itx}\right] = E\left[e^{it(Y_1 + Y_2 + \dots + Y_n)}\right]$ given $\{Y_1, Y_2, \dots, Y_n\}$ n independent Bernoulli random variables

$= E\left[\prod_{k=1}^{n} e^{itY_k}\right] = \prod_{k=1}^{n} E\left[e^{itY_k}\right]$

$E\left[e^{itY}\right] = \sum_{x \in X(\Omega)} e^{itx} \cdot p_X(x) = e^{it \cdot 1} \cdot \frac{\lambda}{n} + e^{it \cdot 0} \cdot \left(1 - \frac{\lambda}{n}\right)$

$= 1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^{it}$

$$\phi_{X_n}(t) = \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^{it}\right)^n = \left(1 + \frac{\lambda e^{it} - \lambda}{n}\right)^n = \boxed{\left(1 + \frac{\lambda \cdot (e^{it} - 1)}{n}\right)^n}$$

b) $\lim_{n \to +\infty} \phi_{X_n}(t) = \lim_{n \to +\infty} \left(1 + \frac{\lambda(e^{it} - 1)}{n}\right)^n = e^{\lambda(e^{it} - 1)}$

$\lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

given $X \hookrightarrow P(\lambda)$ and $\phi_X = e^{\lambda(e^{it} - 1)}$

$Y_n \hookrightarrow \mathcal{B}\left(n, \frac{\lambda}{n}\right)$ and $\phi_{Y_n} = \left(1 + \frac{\lambda(e^{it} - 1)}{n}\right)^n$

$$\boxed{n \to +\infty \Rightarrow \phi_{Y_n} = \phi_X}$$

ex Let $X \hookrightarrow \mathcal{N}(\mu, \sigma^2)$, What is $\phi_X(t)$?

Let $Y \hookrightarrow \mathcal{N}(0,1)$, $Y = \frac{X-\mu}{\sigma} \Rightarrow X = \mu + \sigma Y$

$$\phi_X(t) = E\left[e^{itX}\right] = E\left[e^{it(\mu+\sigma Y)}\right]$$

$$= \int_{\mathbb{R}} e^{itx}\, dF_X(x)$$

$$= \int_{\mathbb{R}} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= E\left(e^{it\mu} \cdot e^{it\sigma Y}\right) = e^{it\mu} \cdot E\left[e^{it\sigma Y}\right]$$

$$= e^{it\mu} \cdot \phi_Y(t\sigma)$$

$$= e^{it\mu} \cdot e^{-(t\sigma)^2/2} = e^{it\mu - \sigma^2 t^2/2}$$

$$\phi_Y(u) = e^{-\frac{u^2}{2}} = E\left[e^{iuY}\right]$$

$$\phi_X(t) = e^{it\mu - \sigma^2 t^2/2}$$

$\mu + \sigma Y \hookrightarrow \mathcal{N}(\mu, \sigma^2)$ $\quad E\left[e^{it(\mu+\sigma Y)}\right] = e^{it\mu} \cdot E\left[e^{it\sigma Y}\right]$

$$\phi_X(t) = E\left[e^{itX}\right] = E\left[e^{it(\mu+\sigma Y)}\right] = e^{it\mu}\, E\left[e^{it\sigma Y}\right]$$

$$= e^{it\mu} \cdot \phi_Y(t\sigma)$$

$$= e^{it\mu} \cdot e^{-\frac{1}{2}(\sigma t)^2}$$

$$= e^{it\mu - \frac{1}{2}(\sigma t)^2}$$

proof $\phi_Y(u) = e^{-\frac{1}{2}u^2}$ when $Y \hookrightarrow \mathcal{N}(0,1)$ see the page exc given (X_n) a sequence of iid r.v. $P(X_n = 0) = 1 - \frac{1}{n}$ and $P(X_n = \sqrt{n}) = \frac{1}{n}$

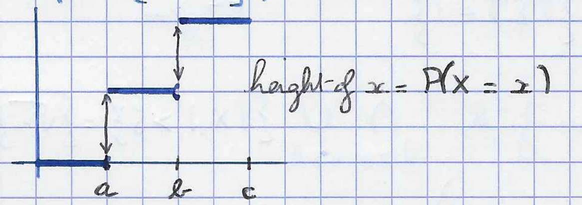
1 what are the convergences for X_n? 2 What is the limit of X_n?

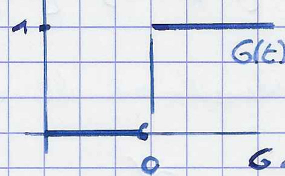
- X_n is a discrete variable. let $n \in \mathbb{N}$. $\Rightarrow F_X(t) = \begin{cases} 0 \text{ if } t < 0 \\ 1 - 1/n \text{ if } t \in [0; \sqrt{n}[\\ 1 \text{ if } t \geq \sqrt{n} \end{cases}$ $\rightarrow P(X \leq t)$

$$\Rightarrow F_n(t) \xrightarrow[n \to +\infty]{} G(t) = \begin{cases} 0 \text{ if } t < 0 \\ 1 \text{ if } t \geq 0 \end{cases}$$

$\rightarrow$ satisfies the property associated with a distribution function

if $(t) \geq \sqrt{n} \Rightarrow P\left(X_n \in \{0, \sqrt{n}\}\right) = P(X = 0) + P(X = \sqrt{n})$





$X(\Omega) = \{0\}$

G is a r.v. that takes only one r.v.

- $X_n \xrightarrow[n \to +\infty]{} X$ w/ X the r.v. always 0 i.e. constant.

probability ↓ implies → distribution

X converges in probability since X is always 0 a constant

↳ proof by definition. we have to see if $\forall \varepsilon > 0$, $P(|X_n| > \varepsilon) \xrightarrow[n \to +\infty]{} 0$

$P(|X_n| > \varepsilon) \underset{X_n \geq 0}{=} P(X_n > \varepsilon) \Rightarrow$

$P(X_n > \varepsilon) = \begin{cases} 0 \text{ if } \varepsilon > \sqrt{n} \\ \frac{1}{n} \text{ if } \varepsilon \leq \sqrt{n} \end{cases}$ ← X discrete

$\rightarrow P(X_n = \sqrt{n})$

$$\Rightarrow \forall \varepsilon > 0, P(|X_n| > \varepsilon) \leq \frac{1}{n}$$

$\Rightarrow$ we deduce that $\forall \varepsilon > 0$, $P(|X_n| > \varepsilon) \xrightarrow[n \to \infty]{} 0$ and $(X_n) \xrightarrow[p]{} (X = 0)$

L^p

Let $p \in [1; +\infty[$, if $(X_n) \xrightarrow[L^p]{} Y \Rightarrow X_n \xrightarrow[p]{} Y$

we have to compute $E(|X_n|^p) = E[X_n^p] = 0^p \times P(X_n = 0) + (\sqrt{n})^p \times P(X = \sqrt{n})$

↖ X a positive r.v.

$$= \sqrt{n}^{\,p} \times \frac{1}{n} = \frac{n^{p/2}}{n} = \frac{1}{n^{1 - \frac{p}{2}}}$$

if $1 - \frac{p}{2} \geq 0 \Rightarrow E[|X_n|^p] \xrightarrow[n \to +\infty]{} 0$ and it converges in L^p

X converges in L^p for $p \in [0, 2]$

almost surely/ The only possibility is a convergence almost surely toward 0.

$\limsup (|X_n| > \delta) = 0$. we have convergence as if $\delta > \sqrt{n}$ then $\{|X_n| > \delta\} = \phi$. if $\sqrt{n} > \delta$ $\{|X_n| > \delta\} = \{\sqrt{n}\}$

$$\limsup (|X_n| > \delta) = \bigcap_{k \geq 0} \bigcup_{n \geq k} \{|X_n| > \delta\} \quad w/ \quad X(\Omega) = \{0, \sqrt{n}\}$$

$\forall n \geq k$, $\{|X_n| > \delta\} = \{\sqrt{n}\} \supset \{\lceil \delta \rceil + 1\}$ $\lceil \cdot \rceil$: floor

intersection over all values of k. $\bigcap_{k \geq 0} \bigcup_{n \geq l} \{|X_n| > \delta\} = \{\sqrt{n}\} \supset \{\lceil \delta \rceil + 1\}$

$$\Rightarrow P\left(\limsup \{|X_n| > \delta\}\right) = \frac{1}{n} \neq 0$$

$$\frac{1}{n} \geq \frac{1}{\lceil \delta \rceil + 1} \neq 0$$

X does not converge almost surely/

ex.

$X \hookrightarrow \mathcal{N}(\mu, \sigma^2)$ prove $\phi_X(t) = e^{-\frac{1}{2}t^2}$ when $\mu = 0$, $\sigma^2 = 1$

$$\phi_X(t) = E[e^{itx}]$$

$$= E[\cos(tx)] + i\,E[\sin(tx)] \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \int_{-\infty}^{\infty} \cos(tx) f(x)\,dx + i \int_{-\infty}^{+\infty} \sin(tx) f(x)\,dx$$

(1) $$= \int_{-\infty}^{+\infty} \cos(tx) f(x)\,dx + 0$$ since $\sin(tx) f(x)$ is an odd function. $\int_{-a}^{a} \text{odd} f^n = 0$

$$\frac{\partial}{\partial t}\phi_X(t) = \frac{\partial}{\partial t} E[e^{itX}] = E\left[\frac{\partial}{\partial t} e^{itX}\right] = E[iX e^{itX}]$$

$$= i\,E[X\cos(tX)] - E[X\sin(tX)]$$

$$= i\int_{-\infty}^{\infty} x\cos(tx) f(x)\,dx - \int_{-\infty}^{\infty} x\sin(tx) f(x)\,dx$$

$$= -\int_{-\infty}^{\infty} x\sin(tx) f(x)\,dx \qquad \text{since } x\cos(tx) \text{ is odd}$$

$$= -\int_{-\infty}^{\infty} x\sin(tx) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= -\int_{-\infty}^{\infty} x\sin(tx) \frac{\partial}{\partial x}\left[\frac{-1}{\sqrt{2\pi}\,x} e^{-\frac{x^2}{2}}\right] dx$$

$$= \int_{-\infty}^{\infty} \sin(tx) \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right] dx = \int_{-\infty}^{+\infty} \sin(tx) \frac{\partial}{\partial x} f_X(x)\,dx$$

$$= \Big[\sin(tx) f_X(x)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} t\cos(tx) f_X(x)\,dx \quad *$$

(2) $$= -t\int_{-\infty}^{+\infty} \cos(tx) f_X(x)\,dx$$

*integration by parts

$$\int_a^b u(x)\,v'(x)\,dx = \Big[u(x)\,v(x)\Big]_a^b - \int_a^b u'(x)\,v(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

we find that $$\int_{-\infty}^{+\infty} \cos(tx) f_X(tx)\,dx = -\frac{1}{t} \times \left(-t\int_{-\infty}^{+\infty} \cos(tx) f_X(tx)\,dx\right)$$

$$\Rightarrow \phi_X(t) = -\frac{1}{t}\frac{\partial}{\partial t}\phi_X(t) \quad \Rightarrow -t\,\phi_X(t) = \frac{\partial}{\partial t}\phi_X(t)$$

$$\Rightarrow \boxed{\phi_X(t) = e^{-\frac{1}{2}t^2}}$$ conditional on $\phi_X(0) = E[e^{i0X}] = 1$

ex Let $X \hookrightarrow \mathcal{E}xp(\mu)$, $Y \hookrightarrow \mathcal{E}xp(\lambda)$, $X \perp\!\!\!\perp Y$, assume $M = \max(X,Y)$

$T = \min(X,Y)$ - 1) distribution of M, 2) distribution of T.

1) we use the distribution function, Let $t \in \mathbb{R}$ $P(\max(X,Y) \leqslant t)$

$\max(X,Y) \leqslant t \implies \{X \leqslant t\} \cap \{Y \leqslant t\}$

$\forall t \in \mathbb{R}$, $P(\max(X,Y) \leqslant t) = P(X \leqslant t, Y \leqslant t)$

$= P(X \leqslant t) \times P(Y \leqslant t)$ because $X \perp\!\!\!\perp Y$

$$f_X(x) = \begin{cases} \mu e^{-\mu x} & \forall x \in \mathbb{R}^{*+} \\ \text{otherwise } 0 \end{cases} = \mathbb{1}_{x \geqslant 0}\, \mu \cdot e^{-\mu x}$$

$$P(\max(X,Y) \leqslant t) = \int_{-\infty}^{t} f_X(x)\,dx \cdot \int_{-\infty}^{t} f_Y(x)\,dx$$

$$= \int \mathbb{1}_{x \geqslant 0}\, \mu \cdot e^{-\mu x}\,dx \cdot \int \mathbb{1}_{x \geqslant 0}\, \lambda \cdot e^{-\lambda x}$$

$F_M(t)$

$$= \mathbb{1}_{t \geqslant 0} \left(\left[\mu e^{-\mu x}\right]_{-\infty}^{t} \cdot \left[\lambda e^{-\lambda x}\right]_{-\infty}^{t} \right)$$

$$F_M(t) = P(\max(X,Y) \leqslant t) = \mathbb{1}_{t \geqslant 0} (1 - \mu e^{-\mu x})(1 - \lambda e^{-\lambda x})$$

we deduce $f_M(t) = \begin{cases} \mu e^{-\mu t}(1 - e^{-\lambda t}) + \lambda e^{-t\lambda}(1 - e^{-\mu t}) & \text{if } t > 0 \\ \text{otherwise } 0 \end{cases}$

2) $P(T) = 1 - P(\min(X,Y) > t) = 1 - (P(X > t) \cdot P(Y > t))$ as $X \perp\!\!\!\perp Y$

$= 1 + ((1 - P(X \leqslant t)) \cdot (1 - P(Y \leqslant t)))$

$= 1 - (e^{-\mu t} \cdot e^{-\lambda t})$

$= 1 - e^{-(\mu + \lambda)t}$ $\qquad F_T(t) = \begin{cases} 1 - e^{-(\mu+\lambda)t} & \text{if } t \geqslant 0 \\ \text{otherwise } 0 \end{cases}$

we deduce $f_T(t) = \begin{cases} (\mu + \lambda) e^{-(\mu+\lambda)t} & \text{if } t \geqslant 0 \\ \text{otherwise } 0 \end{cases}$