+ algebra.

ex. prove S is a vector space for $S \in \mathbb{R}^2$ so that $S = \{(x,y) \mid \forall x, y \in \mathbb{R},\ x = y\}$

- given $(x_1, x_2)\ (y_1, y_2) \in S \Rightarrow (x, x), (y, y) \in S$
  $(x, x) + (y, y) = (x+y,\ x+y)$, $x+y = x+y \Rightarrow (x+y,\ x+y) \in S$ additivity holds
- $\forall \lambda \in \mathbb{R}, \forall v \in S$, w/ $v = (x, x)$, $\lambda v = \lambda(x, x) = (\lambda x, \lambda x)$
  $\lambda x = \lambda x \Rightarrow \lambda v \in S$ scaling holds
- $\longrightarrow$ S is a vector space

ex. w/ $T = \{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$, is T a vector space?

- let's consider $u, v \in T$ such that $u = (x, x^2)$ and $v = (y, y^2)$
- if T is a vector space, $u + v \in T$ and $\forall \lambda \in \mathbb{R}, \lambda u \in T$
- $u + v = (x+y,\ x^2 + y^2)$ however $(x+y)^2 \neq x^2 + y^2$
  $\longrightarrow$ T is not a vector space

ex. prove that $\{(1,0), (0,1), (1,2)\}$ forms a GS of $\mathbb{R}^2$.

- Let's consider $(x, y) \in \mathbb{R}^2$, for the set to be a GS of $\mathbb{R}^2$
  $\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $(x, y) = \alpha_1(1,0) + \alpha_2(0,1) + \alpha_3(1,2)$
- we have a system $\begin{cases} x = \alpha_1 + \alpha_3 \\ y = \alpha_2 + 2\alpha_3 \end{cases} \Rightarrow \begin{cases} \alpha_1 = x - \alpha_3 \\ \alpha_2 = y - 2\alpha_3 \end{cases}$
  $\longrightarrow$ We have a GS of $\mathbb{R}^2$

ex. Prove the linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (2x + 2y, z, x - z)$

- given $v, w \in \mathbb{R}^3$ such that $v = (x, y, z)$ and $w = (\alpha, \beta, \gamma)$

$$f(v) + f(w) = (2x + 2y, z, x - z) + (2\alpha + 2\beta, \gamma, \alpha - \gamma)$$
$$= \big(2(x+\alpha) + 2(y+\beta),\ z + \gamma,\ (x+\alpha) - (z+\gamma)\big)$$
$$= f(v + w) \text{ additivity holds}$$

- $\forall \lambda \in \mathbb{R}$, $v \in \mathbb{R}^3$ $\lambda f(v) = \lambda(2x + 2y, z, x - z)$

$$= (2\lambda x + 2\lambda y\ \lambda z\ \lambda x - \lambda z)$$
$$= f(\lambda v) \text{ . scaling holds}$$

$\longrightarrow$ f is a linear map

ex. given the linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x+y, y+z, x+2y+z)$, determine a basis of $\ker(f)$ ① and $\operatorname{Im}(f)$ ②.

① - $\ker(f) = \{v \in \mathbb{R}^3 \mid f(v) = 0_{\mathbb{R}^3}\}$, $f(v) = 0_{\mathbb{R}}$ w/ $v = (x,y,z)$

$\Rightarrow \begin{cases} x+y=0 \\ y+z=0 \\ x+2y+z=0 \end{cases} \Rightarrow \begin{cases} x=-z \\ y=-z \end{cases} \Rightarrow \forall y \in \mathbb{R}, (-y, y, -y) \in \ker(f)$

$\Rightarrow (-1,1,-1)$ forms a GS of $\ker(f)$ and since $(-1,1,-1) \neq 0_V$ it forms a basis of $\ker(f)$

② we know $\dim(\mathbb{R}^3) = \dim(\ker(f)) + \operatorname{rk}(f) \Rightarrow \operatorname{rk} f = 2$

we associate to f the matrix $A_f = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, knowing $\operatorname{rk} f = 2$ we have to pick two columns of A_f obtained as images of $\mathbb{R}^3$'s canonical basis and prove they are independent. Let's pick A_{f_1} and A_{f_2} such that

$A_{f_1}^T = (1,1,0)$, $A_{f_2}^T = (0,1,1)$ form a GS of $\operatorname{Im}(f)$

$\forall \alpha_1, \alpha_2 \quad \alpha_1(1,1,0) + \alpha_2(0,1,1) = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$

$A_{f_1}^T$ and $A_{f_2}^T$ are independent

$\longrightarrow (-1,1,-1)$ forms a basis of $\ker(f)$, $\{(1,1,0),(0,1,1)\}$ forms a basis of $\operatorname{Im}(f)$

ex. give the linear map $f: \mathbb{R}^3 \to \mathbb{R}^2$ $f(x,y,z) = (x, y+z)$, find the basis of $\ker(f)$ and $\operatorname{Im}(f)$

① $(x, y+z) = 0_{\mathbb{R}^2} \Rightarrow x = 0, y+z = 0 \Rightarrow x = 0, y = -z$

$\Rightarrow \ker(f) = \{(0,-z,z) \ \forall z \in \mathbb{R}\}$

$\Rightarrow z \cdot (0,-1,1) = 0$ iff $z = 0$ (linearly independent)

$\Rightarrow \{(0,-1,1)\}$ forms a basis of $\ker(f)$

② Since $\dim(\mathbb{R}^3) = \dim(\ker(f)) + \operatorname{rk} f \Rightarrow \operatorname{rk} f = 2$

we identify the matrix A associated with f according to the canonical basis of $\mathbb{R}^3$.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Im f contain the points $\forall x,y,z \in \mathbb{R}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z$

Finally $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly dependent and since $\operatorname{rk} f = 2$

$\{(0,1),(1,0)\}$ forms a basis of $\operatorname{Im}(f)$

proof $\forall \alpha, \beta \in \mathbb{R}$ $\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \Rightarrow \alpha = \beta = 0$

ex Is $C = \{(3,0),(0,3)\}$ an orthonormal basis of $\mathbb{R}^2$?

- prove C a GS of $\mathbb{R}^2$. $\forall v \in \mathbb{R}^2$, v is a GS of $\mathbb{R}^2$ if $\forall w \in \mathbb{R}^2$ it holds that
  $\exists \alpha, \beta \in \mathbb{R}$ such that $w = \alpha v_1 + \beta v_2$

  Let's consider $w = (x,y)$ $\forall x,y \in \mathbb{R}$ $\quad w = \alpha . v_1 + \beta . v_2$ for some α, β

  $\Rightarrow x = 3\alpha , y = 3\beta \Rightarrow C$ is a GS of $\mathbb{R}^2$

- prove C is a basis. $(3,0)$ and $(0,3)$ form a basis if they are linearly independent
  i.e. $\forall \alpha, \beta \in \mathbb{R}$ $\quad \alpha(3,0) + \beta(0,3) = 0_{\mathbb{R}^2}$ implies $\alpha = \beta = 0$

  we have the system $\begin{cases} 3\alpha = 0 \\ 3\beta = 0 \end{cases} \Rightarrow \alpha = \beta = 0 \Rightarrow C$ is a basis of $\mathbb{R}^2$

- prove C is orthonormal or not. $\{(3,0),(0,3)\}$ is orthonormal iff $(3,0) \perp (0,3)$ and $\|(3,0)\| = \|(0,3)\| = 1$

  if $(3,0) \perp (0,3)$, it implies $<\underbrace{(3,0)}_{v}, \underbrace{(0,3)}_{w}> = 0$

  $\Rightarrow \sum_{i=1}^{2} v_i w_i = 3 \times 0 + 0 \times 3 = 0 \quad \Rightarrow v \perp w$

  $\|(3,0)\| = \sqrt{<(3,0),(3,0)>} = \sqrt{3 \times 3 + 0 \times 0} = \sqrt{9} = 3 \neq 1 \Rightarrow C$ is not an orthonormal basis of $\mathbb{R}^2$

- obtain the orthonormal basis of $\mathbb{R}^2$.

  by multiplying each vector of v by the inverse of its norm, we can obtain a orthonormal basis of $\mathbb{R}^2$.

  Knowing $\|(3,0)\| = \|(0,3)\| = 3$ so $\frac{1}{3} \times C = \{\frac{1}{3}(3,0), \frac{1}{3}(0,3)\} = \{(1,0),(0,1)\}$

  The set $\{(1,0),(0,1)\}$ forms an orthonormal basis of $\mathbb{R}^2$

ex Prove that for $f \in \mathcal{L}(V,W)$, $Im(f)$ is a subvector space.

$Im(f) = \{w \in W \mid w = f(v) \ \exists v \in V\}$. assume $w_1, w_2 \in Im(f)$

$Im(f)$ is a subvector space iff 1) $\forall w_1, w_2 \in Im(f)$ $\quad w_1 + w_2 \in Im(f)$

2) $\forall w \in Im(f), \forall \lambda \in \mathbb{R}$ $\quad \lambda w \in Im(f)$

1) $w_1 + w_2 = f(v_1) + f(v_2)$ by definition $\Rightarrow w_1 + w_2 \overset{lin}{=} f(v_1 + v_2) \Rightarrow f(v_1 + v_2) \in Im(f)$

$\rightarrow \exists v_1, v_2 \in V$ such that $w_1 + w_2 = f(v_1 + v_2) \in Im(f)$

2) $\forall \lambda \in \mathbb{R}$, $\lambda w = \lambda f(v) \overset{lin}{=} f(\lambda v) \in Im(f)$

$\rightarrow \exists v \in V, \forall \lambda \in \mathbb{R}$ such that $\lambda w = f(\lambda v) \in Im(f)$

by definition $Im(f)$ is a subvector space

ex consider linear map $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x+y+2z, x-z, 3y)$
determine if f is invertible, and if yes determine f^{-1}.

1) f is invertible iff f is injective and surjective.

f is injective iff $\dim(\ker(f)) = 0$

" " surjective if $Im(f) = \mathbb{R}^3 \mid rk(f) = \dim(\mathbb{R}^3)$

Let's consider $f(x,y,z) = 0_{\mathbb{R}^3} \Rightarrow \begin{cases} x+y+2z=0 \\ x-z=0 \\ 3y=0 \end{cases} \Rightarrow \begin{cases} 3z=0 \\ x=z \\ y=0 \end{cases} \Rightarrow x=y=z=0$

$\ker(f) = 0_{\mathbb{R}^3}$

f is injective $\to rk(f) = 3 \to f$ is surjective $\to f$ is invertible

2) $f^{-1} = \begin{pmatrix} 1/3 & 2/3 & -1/9 \\ 0 & 0 & 1/3 \\ 1/3 & -1/3 & -1/9 \end{pmatrix}$

ex given $\langle v, w \rangle = v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2 + w_3v_3$, a map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$
show $\langle v, w \rangle$ is a scalar product.

- it is a scalar product if it is bilinear and symmetric.

1) given vectors $u, v, w \in \mathbb{R}^3$

$\langle u+v, w \rangle = (u_1+v_1)w_1 + (u_1+v_1)w_2 + (u_2+v_2)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3$

$= u_1w_1 + v_1w_1 + u_1w_2 + v_1w_2 + u_2w_1 + v_2w_1 + u_2w_2 + v_2w_2 + u_3w_3 + v_3w_3$

$= u_1w_1 + u_1w_2 + u_2w_1 + u_2w_2 + u_3w_3 + v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2 + v_3w_3$

$= \langle u, w \rangle + \langle v, w \rangle$

$\forall \lambda \in \mathbb{R}, \langle \lambda v, w \rangle = \lambda v_1w_1 + \lambda v_1w_2 + \lambda v_2w_1 + \lambda v_2w_2 + \lambda v_3w_3$

$= \lambda(v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2 + v_3w_3)$

$= \lambda \langle v, w \rangle$

$\Rightarrow \langle v, w \rangle$ is bilinear

2) $\langle v, w \rangle = v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2 + v_3w_3$

$= w_1v_1 + w_1v_2 + w_2v_1 + w_2v_2 + w_3v_3$

$= \langle w, v \rangle$

$\Rightarrow \langle v, w \rangle$ is symmetric

$\langle v, w \rangle$ is a scalar product ex prove. 1) $\|v\| \geq 0$ and $\|v\| = 0$ iff $\|v\| = 0_V$

2) $\|\lambda v\| = |\lambda| . \|v\|$ and $\|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\langle v, w\rangle$

3) Cauchy Schwartz inequality $= |\langle v, w\rangle| \leq \|v\| . \|w\|$

4) Triangle inequality $= \big|\|v\| - \|w\|\big| \leq \|v+w\| \leq \|v\| + \|w\|$

5) $\langle v, w\rangle = \frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right]$

1) by the definition of the positive definite (a dot product $\langle .,. \rangle : V \times V \to \mathbb{R}$ is a positive definite $\langle v, v\rangle > 0$ $\forall v \in V$ inducing a norm $\|.\|$ such as $\|v\| = \sqrt{\langle v, v\rangle}$)

$\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0_V$

2) $\|\lambda v\| = \sqrt{\langle \lambda v, \lambda v\rangle} = \sqrt{\lambda^2 \langle v, v\rangle}$ by bilinearity $= |\lambda| . \|v\|$

$$\|v+w\|^2 = \left(\sqrt{\langle v+w, v+w\rangle}\right)^2 = \langle v+w, v+w\rangle$$
$$= \langle v, v+w\rangle + \langle w, v+w\rangle$$
$$= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle$$
$$= \|v\|^2 + \|w\|^2 + 2\langle v, w\rangle$$

3) $\forall v, w \in V$, $\forall \alpha, \beta \in \mathbb{R}$, $\|\alpha v + \beta w\|^2 = \alpha^2\|v\|^2 + \beta^2\|w\|^2 + 2\alpha\beta\langle v, w\rangle$

for $\alpha = \|w\|^2$, $\beta = -\langle v, w\rangle$ $\quad = \|w\|^4\|v\|^2 + \langle v, w\rangle^2\|w\|^2 - 2\|w\|^2\langle v, w\rangle^2$

$$0 \leq \|w\|^2\|v\|^2 - \langle v, w\rangle^2$$
$$\langle v, w\rangle^2 \leq \|w\|^2\|v\|^2$$
$$\Rightarrow \quad |\langle v, w\rangle| \leq \|v\|.\|w\|$$

4) $(\|v\| - \|w\|)^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \leq \|v\|^2 + \|w\|^2 + 2\langle v, w\rangle$
$$\leq \|v+w\|^2$$

$$\|v\|^2 + \|w\|^2 + 2\langle v, w\rangle \leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$$
$$\leq (\|v\| + \|w\|)^2$$

$$\Rightarrow (\|v\| - \|w\|)^2 \leq \|v+w\|^2 \leq (\|v\| + \|w\|)^2$$

$$\Rightarrow \big|\|v\| - \|w\|\big| \leq \|v+w\| \leq \|v\| + \|w\|$$

5) $\|v+w\|^2 - \|v-w\|^2 = \|v\|^2 + \|w\|^2 + 2<v,w> - \|v\|^2 - \|w\|^2 + 2<v,w>$

$= 4<v,w>$

$\Rightarrow \quad <v,w> = \frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right]$

ex. given $v_1, v_2 \in V$ and $w_1 \in V$ such that $w_1 = v_2 - P_{V_1}(v_2)$ w/ $P_{V_1}(v_2) = \frac{<v_1,v_2>}{\|v_1\|^2} v_1$

prove $w_1 \perp v_1$.

$w_1 \perp v_1 \rightarrow <v_1, w_1> = 0$

$<v_1, w_1> = <v_1, v_2 - P_{V_1}(v_2)> = <v_1, v_2> - <v_1, P_{V_1}(v_2)>$

$= <v_1, v_2> - <v_1, \frac{<v_1,v_2>}{\|v_1\|^2} v_1>$

$= <v_1, v_2> - \frac{<v_1,v_2>}{\|v_1\|^2} <v_1, v_1>$

$= <v_1, v_2> - <v_1, v_2>$

$= 0$

ex given $U \subset V$, $\dim(V) = N$, consider $\{u_1, \dots, u_R\}$ an orthonormal basis of U and $\dim(U) = R < N$ and $\forall i, j \in [1, R], i \neq j, u_i \perp u_j$. $\|u_i\| = 1$

prove $P_U(.) : V \to U$ is a vector space.

The orthogonal projection from V to U noted $P_U(.) : V \to U$ such that

$P_U(V) = \sum_{i=1}^{R} <v, u_i> u_i$ is a s.v.s if additivity and scaling hold

- given $v, w \in V$. $P_U(v+w) = \sum_{i=1}^{R} <v+w, u_i> u_i$

$= \sum_{i=1}^{R} \left[(<v,u_i> + <w,u_i>) u_i\right]$

$= \sum_{i=1}^{R} <v,u_i> u_i + \sum_{i=1}^{R} <w,u_i>$

$= P_U(v) + P_U(w)$ additivity holds

- given $v \in V$, $\forall \alpha \in \mathbb{R}$ $P_U(\lambda v) = \sum_{i=1}^{R} <\lambda v, u_i> u_i = \sum_{i=1}^{R} \lambda <v, u_i> u_i$

$= \lambda \sum_{i=1}^{R} <v, u_i> u_i$

$= \lambda P_U(v)$ scaling holds

$P_U(.)$ is a s.v.s.

ex show that $\ker(P_U(.)) = U^\perp$

It implies 1) $\ker(P_U) \subseteq U^\perp$ 2) $U^\perp \subseteq \ker(P_U)$ } double inclusion ← to prove set equality

- assume $v \in \ker(P_U)$ i.e. $P_U(v) = 0_U \Rightarrow \sum_{i=1}^{R} \overbrace{\langle v, u_i \rangle}^{\alpha_i} u_i = 0$

$\Rightarrow \langle v, u_i \rangle = 0$ for all $i \Rightarrow \forall i \; v \perp u_i$

$\Rightarrow u \in U, \langle v, u \rangle = \langle v, \sum_{i=1}^{R} \alpha_i u_i \rangle \xRightarrow{\text{bilinearity}} \langle v, u \rangle = \underbrace{\sum_{i=1}^{R} \langle v, \alpha_i u_i \rangle}_{=0}$

$\Rightarrow v \in U^\perp$ w/ $U^\perp = \{v \in V, \langle v, u \rangle = 0 \; \forall u \in U\}$

- if $v \in U^\perp$, $P_U(v) = \sum_{i=1}^{R} \langle v, u_i \rangle u_i$ by definition $\langle v, u_i \rangle = 0$

$= 0 \Rightarrow U^\perp \subset \ker(P_U)$

$\longrightarrow \ker(P_U) = U^\perp$

$v = \underbrace{P_U(v)}_{\in U} + \underbrace{v - P_U(v)}_{\in U^\perp}$

ex prove $U \cap U^\perp = 0_V$

if $v \in U$ and $v \in U^\perp$ then $\forall u \in U, \langle v, u \rangle = 0$

since $v \in U$, $\langle v, v \rangle = \|v\|^2 = 0$

$\Rightarrow v = 0$ by positive definiteness

ex $\dim V = \dim(\ker(P_U)) + \dim(\mathrm{Im}(P_U)) = \dim(\ker(P_U)) + \mathrm{rk}(P_U)$

$= \dim U^\perp + \dim U$

ex. $\|v\|^2 = \|U + U^\perp\|^2 = \|u\|^2 + \|v^\perp\|^2 \underbrace{+ 2\langle v, v^\perp \rangle}_{0} = \|v\|^2 + \|v^\perp\|^2$

ex. prove $\langle A_i, A_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = 1 \end{cases}$

- $\mathrm{Im}(f_A) = \{w \in \mathbb{R}^N \mid w = A.v, \exists v \in \mathbb{R}^N\} = \{Av, \exists v \in \mathbb{R}^N\}$

$= A_1 v_1 + A_2 v_2 + \dots + A_n v_n, \exists v_1, v_2 \dots v_n \in \mathbb{R}^N\}$

- $\{A_1, A_2, \dots, A_n\}$ is a GS for $\mathrm{Im}(f_A)$ and since A invertible it forms a basis of $\mathbb{R}^N$

- consider the canonical basis of $\mathbb{R}^N$ $\{e_1, \dots, e_n\}$ e.g. $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

- $A^i = Ae_i \Rightarrow \langle A^i A^j \rangle = \langle Ae_i, Ae_j \rangle = (Ae_i)^T (Ae_j)$

$= A^T e_i^T A e_j = A^T A e_i^T e_j$

$= Id_N \, e_i^T e_j$

given $e_i \cdot e_j = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \Rightarrow \langle A_i, A_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

ex. prove $\langle v, w \rangle = \langle f_A(v), f_A(w) \rangle \; \forall v, w \in V$ $\qquad (AB)^T = B^T A^T$

$$v^T \cdot w = v^T \cdot Id_N \cdot w = v^T A^T A w = (Av)^T \cdot Aw = \langle f(v), f(w) \rangle$$

ex. given the eigenspace $V_\lambda = \{ v \in \mathbb{R}^N \mid Av = \lambda v \}$ the set of eigenvectors of V related to eigenvalue λ. $V\lambda$ is a v.s. Prove it.

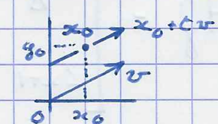
- $\forall v_1, v_2 \in V_\lambda$, $A(v_1 + v_2) = Av_1 + Av_2$
$= \lambda v_1 + \lambda v_2$
$= \lambda(v_1 + v_2) = A(v_1 + v_2)$ additivity holds
- $\forall \delta \in \mathbb{R}$, $v \in V$, $A(\delta v) = A \delta v = \delta A v = \delta A(v)$

ex. given the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{x^2 y}$, find the gradient of f.
(the gradient of f is the column vector which rows are formed by the partial derivatives of f w/ regards to each of its arguments.)

- $\dfrac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim\limits_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$
- $\dfrac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim\limits_{h \to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$

$$\Longrightarrow \left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 2xy \cdot e^{x^2 y} \\ \frac{\partial f}{\partial y}(x,y) &= x^2 e^{x^2 y} \end{aligned}\right\} \nabla f(x,y) = \begin{pmatrix} 2xy\, e^{x^2 y} \\ x^2 e^{x^2 y} \end{pmatrix}$$

ex. a direction is a vector in $\mathbb{R}^N$, $\|v\| = 1$, $x + t.v$ is a straight line passing through x in the direction of v, $\forall t \in \mathbb{R}$



$g : \mathbb{R} \to \mathbb{R}$ $g(t) = f(x_0 + tv)$ $\forall x_0 \in \mathbb{R}$, t fixed

$$\frac{\partial f}{\partial v}(x_0) = g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

In $\mathbb{R}^2$ we fix $V = (v_1, v_2)$

$$\frac{\partial f}{\partial v}(x_0, y_0) = \lim_{t \to 0} \frac{f((x_0 + y_0) + t(v_1, v_2)) - f(x_0, y_0)}{t}$$

$$= \lim_{t \to 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$$

If we choose $v = (1, 0)$

$$\frac{\partial f}{\partial v}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} = \frac{\partial f}{\partial x}(x_0, y_0)$$

ex. prove $Q^{-1} = Q^T$

$Q^{-1} = Q^T$ when colums of Q are $\perp$ and $\|\cdot\| = 1$

ex Show that $g'(t) = \frac{\partial f}{\partial v}(x+tv)$ knowing $g(t) = f(x+tv)$ and $\frac{\partial f}{\partial v}(x) = g'(0) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}$

$$g(t) = f(x_0+tv) \implies g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+(t+h)v) - f(x_0+tv)}{h}$$

$$= \lim_{h\to 0} \frac{f((x_0+tv)+th) - f(x_0+tv)}{h}$$

with $\overset{y_0}{u} = (x_0+tv)$

$$= \lim_{h\to 0} \frac{f(y_0+th) - f(y_0)}{h}$$

we know $\frac{\partial f}{\partial v}(y_0) = g'(0) = \lim_{t\to 0} \frac{f(y_0+tv) - f(y_0)}{t}$

thus $g'(t) = \frac{\partial f}{\partial v}(y_0)$

$$\implies \boxed{g'(t) = \frac{\partial f}{\partial v}(x_0+tv)} = \langle \nabla f(x_0+tv), v \rangle$$

__ex__ compute the derivative of $f(x,y) = x^2 y - e^{x+y}$ along the direction $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\langle \nabla f(x,y), v \rangle$

$$g(z) = f\left(x_0 + z\cdot\frac{1}{2},\ y_0 + z\cdot\frac{\sqrt{3}}{2}\right)$$

$$g'(z) = \frac{\partial g}{\partial z} = \frac{\partial f}{\partial x}\times\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\times\frac{\partial y}{\partial z} \quad w/\ x_0 + \frac{1}{2}z = x\ ;\ y_0 + \frac{\sqrt{3}}{2}z = y$$

$$= f_x(x,y)\times\frac{1}{2} + f_y(x,y)\times\frac{\sqrt{3}}{2}$$

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 y - e^{x+y}\right) = 2y.x - e^{x+y}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^2 y - e^{x+y}\right) = x^2 - e^{x+y}$$

$$\Rightarrow g'(z) = y.x - \frac{1}{2}e^{x+y} + \frac{\sqrt{3}}{2}x^2 - \sqrt{3}/2\ e^{x+y}$$

$$= x * y + \frac{-e^{x+y} + \sqrt{3}\left(x^2 - e^{x+y}\right)}{2} \ *$$

__correction__ w/ definition $f(x,y) = x^2 y - e^{x+y}$ | compute derivative of f along direction $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

__solution 1__ $g(t) = f((x,y) + tv) = f\left((x,y) + t\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right)$

$$= f\left(x + \frac{t}{2},\ y + \frac{\sqrt{3}}{2}t\right)$$

we know that $\frac{\partial f}{\partial v}(x,y) = g'(0) \Rightarrow f\left(x + \frac{t}{2},\ y + \frac{\sqrt{3}}{2}t\right)$ *

$$* = \left(x + \frac{t}{2}\right)^2\left(y + \frac{\sqrt{3}}{2}t\right) - e^{\overbrace{x + \frac{t}{2} + y + \frac{\sqrt{3}}{2}t}^{\left(x + y + \frac{\sqrt{3}+1}{2}t\right)}}$$

$$g'(t) = \frac{2}{2}\left(x + \frac{t}{2}\right)\left(y + \frac{\sqrt{3}}{2}t\right) + \left(x + \frac{t}{2}\right)^2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}+1}{2}\cdot e^{x + \frac{t}{2} + y + \frac{\sqrt{3}}{2}t}$$

$$g(0) = xy + \frac{\sqrt{3}}{2}x^2 - \frac{\sqrt{3}+1}{2}e^{x+y} \ *$$

__sol 2__ $\frac{\partial f}{\partial v}(x,y) = \langle \nabla f(x,y), v \rangle$ | $\nabla f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} 2xy - e^{x+y} \\ x^2 - e^{x+y} \end{pmatrix}$

$$\langle \nabla f(x,y), v \rangle = xy - \frac{1}{2}e^{x+y} + \frac{\sqrt{3}}{2}x^2 - \frac{\sqrt{3}}{2}e^{x+y}$$

$$= xy + \frac{\sqrt{3}}{2}x^2 - \frac{\sqrt{3}+1}{2}e^{x+y} = g'(0)$$

<u>ex</u> given $A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$ compute the eigenvalues of A and the corresponding eigenvectors.

1) $\forall v \in \mathbb{R}^3 \quad x, y, z \in \mathbb{R}, \lambda \in \mathbb{C}$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 3-\lambda \end{pmatrix}$$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 3-\lambda \end{pmatrix}$$

2) find det $= 0$ $\rightarrow$ $\det(A - \lambda I_3) = (2-\lambda) \cdot \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$

$$= (2-\lambda) \cdot \left((3-\lambda)^2 - 1\right)$$

eigenvalues are solution to $\longrightarrow$ $= 0$

1) $2 - \lambda = 0 \rightarrow \lambda = 2$

2) $(3-\lambda)^2 - 1 = 0 \rightarrow (9 - 6\lambda + \lambda^2) - 1 = 0 \rightarrow \lambda^2 - 6\lambda + 8 = 0$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2}$$

$$= \{4, 2\}$$

eigenvalues $= \{2, 4\}$

3)

case 1

$$B = A - \lambda I = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

searching solution for $Bv = 0$

$\forall (x, y, z) \in \mathbb{R}^3, v = (x, y, z)$

$$Bv = 0 \Rightarrow \begin{cases} x - y + z = 0 \\ x - y + z = 0 \end{cases}$$

$\rightarrow y = x + z$

$$V_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y = x + z\}$$

$$= \{(x, x+z, z) \in \mathbb{R}^3 \mid \forall x, z \in \mathbb{R}\}$$

$$= \{x(1,1,0) + z(0,1,1) \mid x, z \in \mathbb{R}\}$$

$\{(1,1,0), (0,1,1)\}$ forms a GS of V_2

It also is a basis since $(1,1,0), (0,1,1)$ are linearly independent

case 2

$$C = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$Cv = 0$

$$Cv = \begin{cases} -x - y + z = 0 \\ -2y = 0 \\ -x + y + z = 0 \end{cases} \Rightarrow \begin{cases} x = z \\ y = 0 \\ x = z \end{cases}$$

$$V_4 = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0, x = z\}$$

$$= \{x(1,0,1) \in \mathbb{R}^3 \mid \forall x \in \mathbb{R}\}$$

$\{(1,0,1)\}$ forms a GS of V_4 and is a basis as $\forall \alpha \in \mathbb{R}$ $\alpha(1,0,1) = 0_3 \Rightarrow \alpha = 0$ linearly independent

$\dim(V_2) = 2 \quad \dim(V_4) = 1$

<u>ex.</u> with $g(x,y) = x^2 y + xy$, compute ∇g and Hg

$$g(x,y) = x^2 y + xy = x(xy + y) = (x^2 + x)y$$

Using the L'Hopital rule : $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$

$$g_x(x,y) = \lim \frac{[(x+h)^2 + (x+h)]y - (x^2+x)y}{h} = 0 \text{ F.I}$$

$$\overset{DH}{=} \lim_{h \to 0} \frac{(2(x+h)+1)y}{1} = \lim_{h \to 0} (2(x+h)+1)y = (2x+1)y$$

$$= \partial g / \partial x (x,y)$$

$$\Rightarrow \nabla g(x,y) = \begin{pmatrix} (2x+1)y \\ x^2+x \end{pmatrix}$$

$$Hg = \begin{bmatrix} 2y & 2x+1 \\ 2x+1 & 0 \end{bmatrix}$$

$\forall t \in \mathbb{R}$, $g(t) = f(x_0 + t.v)$, what is the relationship between $g'(t)$, $g''(t)$ and ∇f?

$g'(0) = \frac{\partial f}{\partial v}(x_0) = < \nabla f(x_0), v >$, In general: $g'(t) = \frac{\partial f}{\partial v}(x + tv)$

$= < \nabla f x_0 + tv, v >$

$\nabla f(x_0 + tv) \in \mathbb{R}^N$, $< \nabla f(x_0 + tv), v > = \sum_{i=1}^{N} f_{x_i}(x_0 + tv).v_i$

$\Longrightarrow g'(t) = \sum_{i=1}^{N} f_{x_i}(x_0 + tv).v_i$

$\Longrightarrow g''(t) = \frac{\partial}{\partial t} g'(t) = \sum_{i=1}^{N} \frac{\partial}{\partial t}\left(f_{x_i}(x_0 + tv)\right).v_i$ $\longrightarrow \sum_{i=1}^{N} < \nabla f_{x_i}(x_0 + tv), v > v_i$

if we set $h(t) = f_{x_i}(x_0 + tv) \Longrightarrow \frac{\partial h}{\partial t}(t) = \frac{\partial h}{\partial v}(x_0 + tv)$

$\frac{\partial h}{\partial t}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + tv)\, v_i\, v_j$

With quadratic form $v^T A v = \sum_{i=1}^{N}\sum_{j=1}^{N} v_i v_j A_{ij} \Longrightarrow v^T Hf(x_0 + tv) v$

$\Longrightarrow g''(0) = v^T Hf(x_0) v$ in particular

w/ $g(t) = f(x_0 + tv)$, $g''(0) = < \nabla f(x_0), v > = v^T Hf(x_0) v$

If g admits first and second order derivatives in $t = 0$ (and we assume it does) and $t = 0$ is a local minimum, then $g'(0) = 0 \Rightarrow < f(x_0), v > = 0$

$g''(0) > 0$ positive $\Rightarrow v^T Hf(x_0) v > 0$

$\forall v \in \mathbb{R}^N$

$\Longrightarrow \nabla f(x_0) = 0_{\mathbb{R}^N}$

$\Longrightarrow Hf(x_0)$ is positive definite

conclusion: If x_0 is a local minimum for $f : \mathbb{R}^N \to \mathbb{R}$, $x_0 \in \mathring{A} \subset \mathbb{R}^N$

$\Longrightarrow \nabla f(x_0) = 0$

$Hf(x_0)$ is positive definite

taylor: $g(t) = g(0) + g'(0)t + \frac{1}{2} g''(0) t^2 . R(t)$ It is known that $R(t) = o(t^2)$

$\Rightarrow \lim_{t \to 0} R(t)/t^2 = 0$

set $x = x_0 + tv \Rightarrow tv = x - x_0$

$$g(t) = f(x_0) + \langle \nabla f(x_0), x - x_0 \rangle + \frac{1}{2}(x - x_0)^T Hf(x_0)(x - x_0) + o(t^2)$$

w/ $o(t^2) = o(\|x - x_0\|^2)$

$$o(t^2) = o((t\|v\|)^2) \text{ w/ } \|v\| = 1$$
$$= o(\|x - x_0\|^2)$$

ex find local minimum/Maximum for $f(x,y) = x^2 + 2y^2$.

1) $f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = 2x$

$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = 4y$

$$\nabla f(x,y) = \begin{pmatrix} 2x \\ 4y \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = y = 0$$

$(0,0)$ is a stationary point for f

2) $Hf(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow$ eigenvalues $= \{2, 4\}$

$\Rightarrow Hf(x,y) = Hf(0,0)$ positive definite

$\Rightarrow (0,0)$ is a local minimum for f

if negative definite = maximum