# LINMA2361 - Nonlinear dynamical systems

## Language dynamics

Quentin Lété

January 8, 2018

Ecole polytechnique de Louvain

# Language death - Key facts

- $\sim$ 7,000 languages in the world
- Half of them are endagered
- One language extinction every **two weeks**
- At this rate, 90% will disappear with the current generation

# Outline

# Basic models

# Abrams and Strogatz's model

## Notations

- Two competing languages : $X$ and $Y$
- Proportions of speakers : $x$ and $y$
- Perceived status $= s$

## System

$$\dot{x} = yP_{YX}(x,s) - xP_{XY}(x,s)$$

with

$$P_{YX}(x,s) = cx^a s \qquad P_{XY}(x,s) = c(1-x)^a(1-s)$$

$a$ : case dependent. In practice,

$$a = 1.31 \pm 0.25$$

# Bilinguism

## Notations

- Proportion of bilinguals : $b$
- Similarity : $k$

## System

$$\begin{cases} \dot{x} = yP_{YX} + bP_{BX} - x(P_{XY} + P_{XB}) \\ \dot{y} = xP_{XY} + bP_{BY} - y(P_{YX} + P_{YB}) \\ \dot{b} = xP_{XB} + yP_{YB} - b(P_{BX} + P_{BY}) \end{cases} \tag{1}$$

with

$$\begin{cases} P_{XB} = c \cdot k(1-s)(1-x)^a \\ P_{YB} = c \cdot ks(1-y)^a \\ P_{BX} = P_{YX} = c \cdot (1-k)s(1-y)^a \\ P_{BY} = P_{XY} = c \cdot (1-k)(1-s)(1-x)^a \end{cases}$$

# Temporal extension

# Langauge revival

**System**

$$\begin{cases} \dot{x} = c\big((1-x)sx^a - x(1-s)(1-x)^a\big) \\ \dot{s} = S(x,s) \end{cases} \tag{2}$$

**Requirements**

- $S(x,s) > 0$ when s is small, $S(x,s) < 0$ when s is large
- Keep the system well-defined : $s(t) \in [0;1] \quad \forall t$
- $S(x,s) = -S(1-x, 1-s)$

# First status dynamic

$$S(x,s) = (\frac{1}{x} - \frac{1}{y})s(1-s) = (\frac{1}{x} + \frac{1}{x-1})s(1-s) \tag{3}$$

On domain $A = [\gamma, 1-\gamma] \times [0;1]$

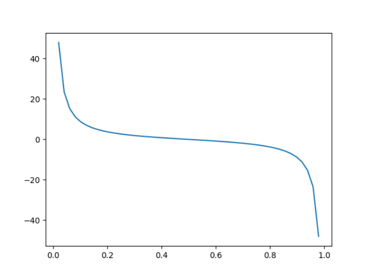



**Figure 1:** Function $\frac{1}{x} + \frac{1}{x-1}$ on $[0;1]$

# Analysis

## Positive invariance

**Theorem**

*The vector field f of system (2) together with (3) is Lipschitz continuous on* $A = [\gamma, 1-\gamma] \times [0;1]$

**Theorem**

*If* $S$ *is given by equation (3) and if* $a > 0$*, then the set* $A = [\gamma, 1-\gamma] \times [0;1]$ *for* $\gamma > 0$ *sufficiently small is positively invariant for system (2).*

## Equilibrium

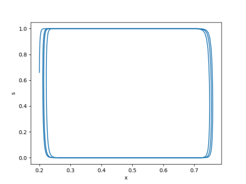
- Only one inside $A$ : $(\frac{1}{2}, \frac{1}{2})$
- Eigenvalues :

$$\begin{cases} \lambda_1 = 0.194602 + 1.08263i \\ \lambda_2 = 0.194602 - 1.08263i \end{cases}$$

- Stability : Unstable

### Theorem

*Poincarré-Bendixson* Let $\dot{x} = f(x)$ be a dynamical system with $x \in \mathbb{R}^2$. Let $R$ be a closed bounded subset of $\mathbb{R}^2$ that contains no equilibrium points. Let $t \to x(t)$ be a trajectory that stays in $R$ for all $t > 0$ and suppose that $f \in C^1$ in an open subset that contains $R$.
Then, $x(t)$ is a periodic solution or $x(t)$ tends towards a closed orbit ad $t \to \infty$.



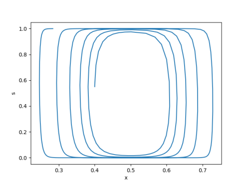




**Figure 2:** Trajectories for the system 2 with status dynamic 3. Starting points are respectively $[x_0, s_0] = [0.2, 0.66]$ and $[0.4, 0.55]$.

# Second status dynamic

$$S(x, s) = \alpha(\frac{1}{x+1} + \frac{1}{x-2})(1-s)s \tag{4}$$

where $\alpha \in \mathbb{R}_+$.

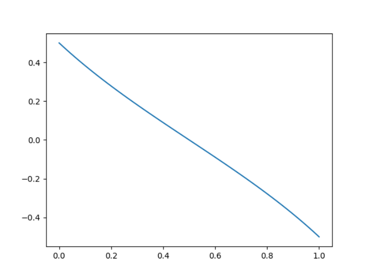



**Figure 3:** Function $\frac{1}{x+1} + \frac{1}{x-2}$ on $[0; 1]$

# Extinction ?

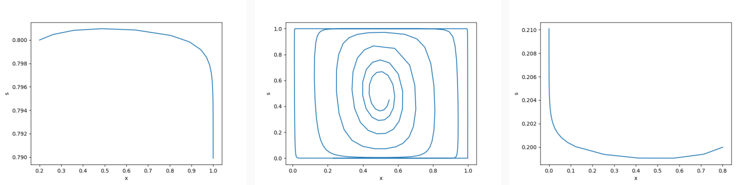



**Figure 4:** Trajectories for the system 2 with status dynamic 4. Starting points are respectively $[x_0, s_0] = [0.2, 0.8], [0.55, 0.45]$ and $[0.8, 0.2]$. The value of $\alpha$ is respectively $10^{-3}$, 1 and $10^{-3}$.

# Analysis

## Positive invariance

**Theorem**

*If $S$ is given by equation (4) and if $a > 0$, then the set $B = [0;1] \times [0;1]$ is positive invariant for system (1).*

## Stability

Nullclines :

$$n_x = \{(x,s)|x=0\} \cup \{(x,s)|x=1\} \cup \{(x,s)|sx^{a-1} = (1-s)(1-x)^{a-1}\}$$

$$n_s = \{(x,s)|s=0\} \cup \{(x,s)|s=1\} \cup \{(x,s)|x = \frac{1}{2}\}$$

Eigenvalues :

| | $(0,0)$ | $(1,0)$ | $(1,0)$ | $(1,1)$ | $(\frac{1}{2},\frac{1}{2})$ |
|---|---|---|---|---|---|
| $\lambda_1$ | 0.31 | -1 | 0 | -1 | $0.194602 + 0.310758i$ |
| $\lambda_2$ | 0.5 | -0.5 | -0.5 | 0.5 | $0.194602 - 0.310758i$ |

# Basin of attraction

## What does it depend on ?

- Strength parameter $\alpha$
- Intial status $s_0$

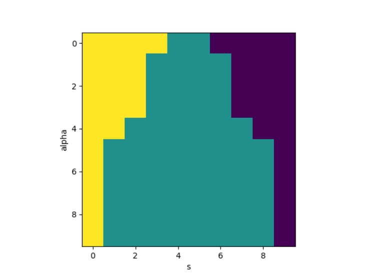



**Figure 5:** Heat map of the long term behavior of the system with respect to the initial status and the parameter $\alpha$. In yellow, the trajectories that converged to $(0, 1)$. In purple, the trajectories that converged to $(1, 0)$. In blue, the trajectories that did not converged to any of the fixed points.

# Bilinguism

# System

$$\begin{cases} \dot{x} = c(1-x)\Big((1-k)s(1-y)^a - (1-s)x(1-x)^{a-1}\Big) \\ \dot{y} = c(1-y)\Big((1-k)(1-s)(1-x)^a - sy(1-y)^{a-1}\Big) \\ \dot{s} = \alpha\left(\frac{1}{x+1} - \frac{1}{y+1}\right)s(1-s) \end{cases} \quad (5)$$

## Nullclines

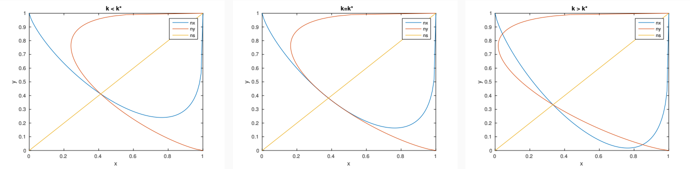
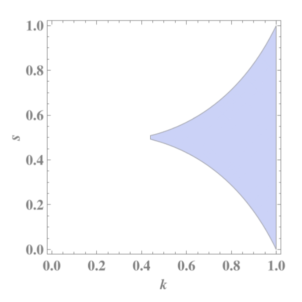



**Figure 6:** Nullclines of the system in the plane $s = \frac{1}{2}$

## Coexistence region and numerical simulation



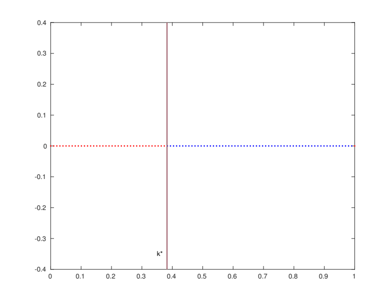




**Figure 7:** (Left) Coexistence region in function of $s$ and $k$. (Right) Numerical simulation of the stability of the equilibrium for system 5. Red corresponds to unstability while blue corresponds to stability. The vertical line represent $k = k^*$.

# Spatial analysis

# Basics

### Notations

- $K$ regions of similar surface
- Proportion of $X$ speakers in region $k$ : $x_k$
- $\mathcal{N}_k(x)$ : mean of $x_i$ in all neighbors $i$ of $k$

### System

$$\dot{x}_k = c\big((1 - x_k)s\mathcal{N}(x)^a - x_k(1 - s)(1 - \mathcal{N}(x))^a\big) \qquad \forall k \in \{1, ..., K\}$$

# Test on a real dataset : The Irish case

## Dataset

- Region = electoral district
- Reshape the country into a $135 \times 135$ square for simplicity
- Suppose no bilinguals (strong)

## Results

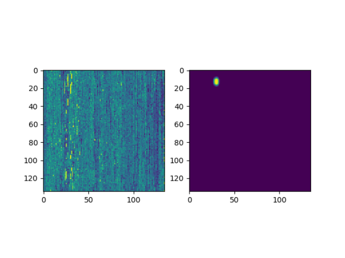



**Figure 8:** First and last frame of the simulation of the spatial model.

# Slower dynamics

## What does it tell us ?

- Quickly binary shape
- The dynamic is slower !

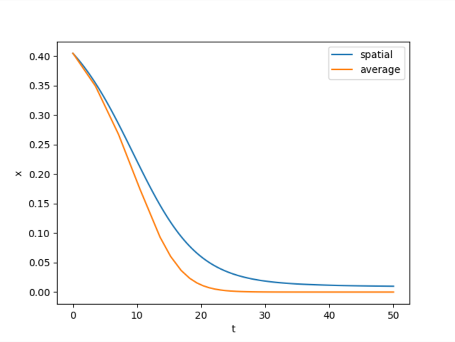



**Figure 9:** Average proportion of Irish speakers throughout the country using both dynamics.

# Conclusion

**Questions?**