CS 232 Computer and Communication Networks Assignment 2

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1. Consider the queuing system in the figure, where packets whose service is completed by Servers 1 and 2 go to Server 3. The service time of Server 1, 2, and 3 is exponential with rate $\mu_1 = 1$ pks/s, $\mu_2 = 3$ pkt/s and μ_3= 4 pkt/s, respectively. At time $t = 0$, packet C arrives in the buffer of Server 1. When C arrived, Server 1 has packet A in service while Server 2 has packet B in service.
   (a) Compute the probability that A exits the system before any other packets move to server 3.
   (b) Compute the probability that B exits the system before any other packets move to server 3.
   (c) Compute the expected time T needed by Packet A to exit the system.

Ans:

(a) The probability that packet A completes service at Server 1 before packet B completes service at Server 2 is:

$$P_1 = \frac{\mu_1}{\mu_1 + \mu_2} = \frac{1}{1+3} = \frac{1}{4}.$$

The probability that A completes service at Server 3 before either B or C completes their services is:

$$P_2 = \frac{\mu_3}{\mu_2 + \mu_1 + \mu_3} = \frac{4}{3+1+4} = \frac{4}{8} = \frac{1}{2}.$$

Thus the overall probability that A exits the system before any other packets move to Server 3 (considering both Server 1 and Server 3) is given by:

$$P = P_1 \times P_2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

(b) The probability that packet B completes its service at Server 2 before packet A completes at Server 1 is:

$$P_1 = \frac{\mu_2}{\mu_1 + \mu_2} = \frac{3}{1+3} = \frac{3}{4}.$$

The probability that B completes service at Server 3 before A or C is:

$$P_2 = \frac{\mu_3}{\mu_1 + \mu_3} = \frac{4}{1+4} = \frac{4}{5}.$$

The overall probability that B exits the system before any other packets move to Server 3 is:

$$P = P_1 \times P_2 = \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}.$$

(c) First we can dived the system event into two branches. The first branch is: A to server 3 $\rightarrow$ A out; the second branch is: B to server 3, while there are also 2 sub-branches after B to server 3. Sub-branch 1 is: A to server 3 $\rightarrow$ B out $\rightarrow$ A out; sub-branch 2 is B out $\rightarrow$ A to server 3 $\rightarrow$ A out. The total time packet A spends in the system T_A can be divided into two parts:

- T: the time until the first thing that happens (A to server 3 or B to server 3 first)
- R: the rest of the time.

So we have $\mathbb{E}[T_A] = \mathbb{E}[T] + \mathbb{E}[R]$ and $\mathbb{E}[R] = \mathbb{E}[R|\text{A arrives server 3 first}] + \mathbb{E}[R|\text{B arrives server 3 first}]$. The time until the first thing happens is an exponential distribution with parameter $\mu_1 + \mu_2$, so we have:

$$\mathbb{E}[T] = \frac{1}{\mu_1 + \mu_2} = \frac{1}{4}.$$

To compute $\mathbb{E}[R]$, we condition on what was the first thing to happen, either packet A arrives server 3 first or packet B arrives server 3 first. For the first case (A arrives server 3), we have:

$$\mathbb{E}[R|\text{A arrives server 3 first}] = \frac{\mu_1}{\mu_1 + \mu_2} \times (\frac{1}{\mu_1} + \frac{1}{\mu_3}) = \frac{1}{4} \times \frac{5}{4} = \frac{5}{16}.$$

For the second case (B arrive server 3 first), since there are two sub-branches, we can agian separate the second branch into two parts:

- T_2: the time until the first thing that happens (A arrives server 3 or B out first)
- R_2: the rest of the time.

So we have $\mathbb{E}[R|\text{B arrives server 3 first}] = \frac{\mu_2}{\mu_1+\mu_2}(\mathbb{E}[T_2] + \mathbb{E}[R_2])$. The time until the first thing happens is an exponential distribution with parameter $\mu_1 + \mu_3$, so we have:

$$\mathbb{E}[T_2] = \frac{1}{\mu_1 + \mu_3} = \frac{1}{5}.$$

To compute $\mathbb{E}[R_2]$, we condition on what was the first thing to happen, either packet A arrives server 3 first or packet B out first. For the first case (A arrives server 3 first), we have:

$$\mathbb{E}[R_2|\text{A arrives server 3 first}] = \frac{\mu_1}{\mu_1 + \mu_3} \times (\frac{1}{\mu_3} + \frac{1}{\mu_3}) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}.$$

For the second case (B out first), we have:

$$\mathbb{E}[R_2|\text{B out first}] = \frac{\mu_3}{\mu_1 + \mu_3} \times (\frac{1}{\mu_1} + \frac{1}{\mu_3}) = \frac{4}{5} \times \frac{5}{4} = 1.$$

Thus we can obtain:

$$\mathbb{E}[R_2] = \frac{1}{10} + 1 = \frac{11}{10}.$$

and

$$\mathbb{E}[R|\text{B arrives server 3 first}] = \frac{3}{4}(\frac{1}{5} + \frac{11}{10}) = \frac{39}{40}.$$

Then $\mathbb{E}[R]$ can be calculated as:

$$\mathbb{E}[R] = \frac{5}{16} + \frac{39}{40} = \frac{25 + 78}{80} = \frac{103}{80}.$$

Thus we have:

$$\mathbb{E}[T_A] = \frac{1}{4} + \frac{103}{80} = \frac{123}{80} = 1.5375.$$

2. Consider a router receiving packets according to a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda = 3$ packets/second.

   (a) Compute the probability that the router will receive 2 packets in the next second.

   (b) Compute the expected total number of packets the router would receive after 2 seconds.

(c) Imagine that 2 packets arrived in the time interval from 0 to the end of 3s, compute the probability that between the start time of $t = 4$ and at the end of time $t = 6$ exactly 3 packets arrive at the router (Hint: Consider the expression $P(N(6) - N(4)) = 3)$).

Ans:

(a) The number of packets received in time t follows a Poisson distribution given by:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

In this case, we have $\lambda = 3$ packets/second, $t = 1$ second, and we want to find the probability of receiving $k = 2$ packets. Thus, we have:

$$P(N(1) = 2) = \frac{(3 \cdot 1)^2 e^{-3 \cdot 1}}{2!} = \frac{3^2 e^{-3}}{2!} = \frac{9e^{-3}}{2}.$$

(b) For a Poisson process, the expected number of events in a time interval t is given by:

$$E[N(t)] = \lambda t.$$

In this case, with $\lambda = 3$ packets/second and $t = 2$ seconds:

$$E[N(2)] = 3 \times 2 = 6.$$

(c) The stationary increments $N(t) - N(s)$ is a Poisson distribution with mean For $t = 2$ and $k = 3$ we have:

$$P(N(6) - N(4)) = 3) = P(N(2) = 3) = \frac{(3 \cdot 2)^3 e^{-3 \cdot 2}}{3!} = \frac{6^3 e^{-6}}{6} = 36e^{-6}.$$

3. A router sends out 20 packets every 10 seconds on average. Suppose that the time in between two packets sent out can be modeled as an exponential random variable.
   (a) What is the probability that the next packet will be sent out after 5 seconds?
   (b) What is the probability that exactly 5 packets will leave in the next second?
   (c) What is the probability that exactly 3 packets will leave in the next 2 seconds?
   (d) What is the probability that more than 2 packets, but less than 5 packets will leave in the next 2 seconds?

Ans:

(a) Since the router sends out 20 packets every 10 seconds on average. We can obtain the rate λ of the exponential distribution:

$$\lambda = \frac{20}{10} = 2 \ \textit{packet/second}.$$

The time between packets is modeled as an exponential random variable with rate $\lambda = 2$. The cumulative distribution function (CDF) for the exponential distribution is given by:

$$P(X > t) = e^{-\lambda t}.$$

The probability that the next packet will be sent out after 5 seconds is the complementary probability:

$$P(X > 5) = e^{-5\lambda} = e^{-10}.$$

(b) The number of packets sent in a given time is a Poisson process. In this case, the probability of sending exactly $k = 5$ packets in $t = 1$ second is:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Substituting $\lambda = 2$, $t = 1$, and $k = 5$:

$$P(N(1) = 5) = \frac{(2 \cdot 1)^5 e^{-2\cdot 1}}{5!} = \frac{2^5 e^{-2}}{120} = \frac{4}{15} e^{-2}.$$

(c) the probability that exactly 3 packets will leave in the next 2 seconds can be calculated as:

$$P(N(2) = 3) = \frac{(4)^3 e^{-4}}{3!} = \frac{64e^{-4}}{6} = \frac{32}{3} e^{-4}.$$

(d) This probability can be expressed as:

$$P(3 \leq N(2) < 5) = P(N(2) = 3) + P(N(2) = 4)$$

We already calculated $P(N(2) = 3)$ in (c). Now we need to compute $P(N(2) = 4)$:

$$P(N(2) = 4) = \frac{(4)^4 e^{-4}}{4!} = \frac{256e^{-4}}{24} = \frac{32}{3} e^{-4}.$$

By summing these probabilities, we have:

$$P(3 \leq N(2) < 5) = P(N(2) = 3) + P(N(2) = 4) = \frac{64}{3} e^{-4}.$$

4. router is receiving packets from two different clients. Assume the time between the generation of two consecutive packets at each client is exponentially distributed with parameters $\lambda_1 = 2$ packets/second for client 1, and at $\lambda_2 = 3$ packets/second for client 2

   (a) What is the probability that the next packet will come from node 1?

   (b) What is the probability that the router will receive exactly 2 packets in the next 3 seconds?

   (c) Imagine that at time t = 3, two packets have arrived at the router. At t = 5, what is the probability that at least 1 more packet will arrive?

Ans:

(a) The probability that the next packet will come from node 1 actually is the probability $P(\min\{X_1, X_2\} = X_1)$. Since the minimum of two exponential distributions also follows an exponential distribution with rate $\lambda_1 + \lambda_2$, thus we have:

$$P(\text{next packet from client 1}) = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{2}{5}.$$

(b) The total number of packets received from both clients in a time interval follows a Poisson distribution with rate $\lambda = 2 + 3 = 5$ packets/second. Thus the probability of receiving exactly $k = 2$ packets in the next 3 seconds is:

$$P(N(3) = 2) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{(5 \times 3)^2 e^{-15}}{2!} = \frac{225}{2} e^{-15}.$$

(c) The probability that at least 1 more packet will arrive equals to:

$$P(N(2) \geq 1) = 1 - P(N(2) = 0).$$

Since $P(N(2) = 0)$ is:

$$P(N(2) = 0) = \frac{(10)^0 e^{-10}}{0!} = e^{-10}.$$

Thus we have

$$P(N(2) \geq 1) = 1 - e^{-10}.$$