CS 232 Computer and Communication Networks
Assignment 1

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1. Consider two Exponentially distributed random variables X_1 and X_2, with rate $\lambda_1 = 6$ and $\lambda_2 = 3$, respectively.
   (a) Compute the probability $P(X_2 > 4)$.
   (b) Compute the probability $P(X_1 \leq 3)$.
   (c) Compute the probability $P(\min\{X_1, X_2\} \leq 5)$.
   (d) Compute the probability that X_2 is smaller than X_1.

Ans:

(a) The probability $P(X_2 > 4)$ is expressed as:

$$P(X_2 > 4) = \int_4^\infty \lambda_2 e^{-\lambda_2 x}\, dx = \int_4^\infty 3e^{-3x}\, dx.$$

By calculating the integral, we can obtain:

$$P(X_2 > 4) = \left[-e^{-3x}\right]_4^\infty = 0 - \left(-e^{-12}\right) = e^{-12}.$$

(b) The probability $P(X_1 \leq 3)$ can be expressed as:

$$P(X_1 \leq 3) = 1 - P(X_1 > 3) = 1 - \int_3^\infty \lambda_1 e^{-\lambda_1 x}\, dx = 1 - \int_3^\infty 6e^{-6x}\, dx.$$

By calculating the integral, we have:

$$P(X_1 > 3) = \left[-e^{-6x}\right]_3^\infty = 0 - \left(-e^{-18}\right) = e^{-18}.$$

Thus,

$$P(X_1 \leq 3) = 1 - e^{-18}.$$

(c) Since $\min\{X_1, X_2\} \leq 5$ can be regarded as an exponential distribution with rate $\lambda = \lambda_1 + \lambda_2 = 6 + 3 = 9$, we can write:

$$P(\min\{X_1, X_2\} \leq 5) = 1 - P(\min\{X_1, X_2\} > 5).$$

Thus we have:

$$P(\min\{X_1, X_2\} \leq 5) = 1 - \int_5^\infty 9e^{-9x}\, dx = 1 - e^{-45}.$$

(d) Recall that the exponential distribution has an important property that the distribution $\min\{X_1, ..., X_n\}$ is exponential $(\lambda_1 + ... + \lambda_n)$, and the probability that the minimum is X_i is $\frac{\lambda_i}{\lambda_1 + ... + \lambda_n}$. Thus the probability that $X_2 < X_1$ can be computed as:

$$P(X_2 < X_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{3}{6+3} = \frac{1}{3}.$$

2. A router sends out 45 packets every 3 seconds on average. Suppose that the time in between two packets sent out can be modeled as an exponential r.v. with the corresponding parameter.

   (a) What is the probability that a packet will be sent out in less than 3 seconds?

   (b) Suppose that at time $t = 0$ a packet was sent out, what is the probability that at time $t = 3$ no further packets were sent out?

   (c) Assume now 72 packets every 3 seconds are sent out on average. Is the probability of part (a) larger, smaller, or equal now?

Ans:

(a) The average number of packets sent out is 45 every 3 seconds, which means the average rate of packet transmission (λ) is:

$$\lambda = \frac{45}{3} = 15,$$

i.e., the time between packets is modeled as an exponential random variable X with rate $\lambda = 15$. The probability that a packet is sent out in less than

t seconds is given by the cumulative distribution function (CDF) of the exponential distribution:

$$P(X < t) = 1 - e^{-\lambda t}.$$

For $t = 3$, we have:

$$P(X < 3) = 1 - e^{-15\times 3} = 1 - e^{-45}.$$

(b) The time until the next packet is also an exponential random variable. According to the memoryless property of exponential distribution, the probability that no packet is sent out in $t = 3$ seconds can be computed as:

$$P(X > 3) = e^{-\lambda t} = e^{-15\times 3} = e^{-45}.$$

(c) In the case that 2 packets every 3 seconds are sent out on average, the rate becomes:

$$\lambda = \frac{72}{3} = 24.$$

The probability that a packet will be sent out in less than 3 seconds in (a) now becomes:

$$P(X < 3) = 1 - e^{-\lambda t} = 1 - e^{-24\times 3} = 1 - e^{-72}.$$

Since $e^{-72} < e^{-45}$, we can conclude that:

$$1 - e^{-72} > 1 - e^{-45}.$$

Thus, the probability becomes larger since the packet transmission rate increases.

3. A router is receiving packets from two different clients. Assume the time between the generation of two consecutive packets at each client is exponentially distributed with parameters $\lambda_1 = 2$ packets/second for node 1, and at $\lambda_2 = 3$ packets/second for node 2.

   (a) What is the probability that the next packet will come from node 2?

   (b) What is the probability that the router will not receive any packet in the next 3 seconds?

(c) Imagine that after the three seconds mentioned in part (b) elapsed, a third client enters the system and is now sending packets through the same router. Assume the time between the generation of two consecutive packets at each client is exponentially distributed as before for nodes 1 and 2, but with parameter $\lambda_3 = 4$ packets/second for node 3. What is the probability that the next packet will arrive from node 3?

Ans:

(a) Assume X_1 and X_2 are the the time between the generation of two consecutive packets at agent 1 and agent 2, respectively. The probability that the next packet will come from node 2 can be expressed as $P(X_2 < X_1)$. Since the probability that the minimum is X_i is $\frac{\lambda_i}{\lambda_1+\ldots+\lambda_n}$. Thus $P(X_2 < X_1)$ can be given by:

$$P(X_2 < X_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{3}{2+3} = \frac{3}{5}.$$

(b) The probability that the router will not receive any packet in the next 3 seconds can be represented as $P(\min\{X_1, X_2\} > 3)$. Since $\min\{X_1, X_2\}$ can be regarded as an exponential distribution with rate $\lambda = \lambda_1 + \lambda_2 = 5$. Thus the probability that no packet is received in a certain time t is given by the survival function of the exponential distribution. The total rate of packet arrivals is $\lambda = \lambda_1 + \lambda_2 = 2 + 3 = 5$ packets per second. The probability that no packet arrives in the next 3 seconds is:

$$P(\min\{X_1, X_2\} > 3) = e^{-\lambda t} = e^{-5\times 3} = e^{-15}.$$

(c) Assume X_3 is the time between the generation of two consecutive packets at agent 3. Based on the memoryless property, the probability that the next packet will come from node 3 means X_3 is the minimum one among X_1, X_2 and X_3. Thus we have:

$$P(\min\{X_1, X_2, X_3\} = X_3) = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{4}{2+3+4} = \frac{4}{9}.$$

4. Consider a router with service rate μ pkt/s, and arrival rate λ pkt/s. At time $t = 0$ a packet (packet A) is being served. The next packet (packet B) arrives at the router buffer according to the exponential distribution.

(a) What is the probability that Packet B is arriving at the buffer while Packet A is still being served?

(b) What is the average time packet B spends in the buffer?

(c) What is the average time packet B spends in the system? Hint: divide the total time into two parts: T1 and R.

Ans:

(a) Assume X_A is the service time of packet A while X_B is the arrive time of packet B. The probability that Packet B arrives at the buffer before packet A finishes being served is equivalent to the probability that the inter-arrival time of packet B is less than the service time of packet A, i.e., $P(X_A > X_B)$. Thus we have:

$$P(X_B < X_A) = P(\min\{X_A, X_B\} = X_B) = \frac{\lambda}{\lambda + \mu}.$$

Thus, the probability that packet B arrives while packet A is still being served is $\frac{\lambda}{\lambda+\mu}$.

(b) The average time packet B spends in the buffer (denoted as W_B) is the remaining service time of packet A. The remaining service time of an exponentially distributed variable with rate μ is also exponentially distributed with the same rate. Thus, the average time Packet B spends in the buffer is:

$$\mathbb{E}[W_B] = \frac{1}{\mu}.$$

(c) First we can divded the system event into two branches. The first branch is: A out $\rightarrow$ B arrive $\rightarrow$ B out; the second branch is: B arrive $\rightarrow$ A out $\rightarrow$ B out. The total time packet B spends in the system T_B can be divided into two parts:

- T_1: the time until the first thing that happens (A out or B arrive)
- R: the rest of the time.

So we have $\mathbb{E}[T_B] = \mathbb{E}[T_1] + \mathbb{E}[R]$. The time until the first thing happens is an exponential distribution with parameter $\mu + \lambda$, so we have:

$$\mathbb{E}[T_1] = \frac{1}{\mu + \lambda}.$$

To compute $\mathbb{E}[R]$, we condition on what was the first thing to happen, either packet A finished service first or packet B arrived first. For the first case (A out $\rightarrow$ B arrive $\rightarrow$ B out), we have:

$$\mathbb{E}[R|\text{A out}] = P(X_A < X_B) \times (\frac{1}{\lambda} + \frac{1}{\mu}) = \frac{\mu}{\lambda + \mu} \times \frac{\lambda + \mu}{\lambda\mu} = \frac{1}{\lambda}.$$

For the second case (B arrive $\to$ A out $\to$ B out), we have:

$$\mathbb{E}[R|\text{B arrive}] = P(X_B < X_A) \times 2 \times \frac{1}{\mu} = \frac{\lambda}{\lambda+\mu} \times \frac{2}{\mu} = \frac{2\lambda}{\mu(\lambda+\mu)}.$$

Thus we can obtain:

$$\mathbb{E}[R] = \mathbb{E}[R|\text{A out}] + \mathbb{E}[R|\text{B arrive}] = \frac{1}{\lambda} + \frac{2\lambda}{\mu(\lambda+\mu)} = \frac{\mu\lambda + \mu^2 + 2\lambda^2}{\mu\lambda^2 + \lambda\mu^2}$$

So we have:

$$\mathbb{E}[T_B] = \mathbb{E}[T_1] + \mathbb{E}[R] = \frac{1}{\mu+\lambda} + \frac{\mu\lambda + \mu^2 + 2\lambda^2}{\mu\lambda^2 + \lambda\mu^2} = \frac{2\mu\lambda + \mu^2 + 2\lambda^2}{\mu\lambda^2 + \lambda\mu^2}.$$