CS 232 Computer and Communication Networks
Assignment 4

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1. Consider an $M/M/1/3$ system with arrival rate $\lambda = 4$ pkt/s and service rate $\mu = 12$ pkt/s. Please note that the total number of packets include packets being served as well.
   (a) Compute the probability that the system has 1 packet being served and 1 packet waiting in the buffer.
   (b) Compute the expected time a packet needs to go through the system. Please note that the time includes queuing and service.

Ans:

(a) The system has a maximum of 3 packets (including the one in service). The steady-state probabilities are given by:

$$P_n = \rho^n P_0, \quad n = 0, 1, 2, 3.$$

To determine P_0, we use the normalization condition:

$$\sum_{n=0}^{3} P_n = 1.$$

Substitute $P_n = \rho^n P_0$, where $\rho = \frac{\lambda}{\mu} = \frac{1}{3}$:

$$P_0 \left(1 + \rho + \rho^2 + \rho^3\right) = 1.$$

The geometric series sum is:

$$P_0 \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3\right) = 1.$$

Simplify:

$$P_0 \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) = 1.$$

$$P_0\left(\frac{27}{27}+\frac{9}{27}+\frac{3}{27}+\frac{1}{27}\right)=1.$$

$$P_0\cdot\frac{40}{27}=1 \implies P_0=\frac{27}{40}.$$

Thus, the steady-state probabilities are:

$$P_n=\rho^n P_0=\left(\frac{1}{3}\right)^n\cdot\frac{27}{40},\quad n=0,1,2,3.$$

Substitute $n=2$:

$$P_2=\left(\frac{1}{3}\right)^2\cdot\frac{27}{40}=\frac{1}{9}\cdot\frac{27}{40}=\frac{27}{360}=\frac{3}{40}.$$

Thus, the probability is:

$$P_2=\frac{3}{40}.$$

(b) The expected time a packet spends in the system is given by Little's Law:

$$\mathbb{E}[T]=\frac{\mathbb{E}[N]}{\lambda(1-P_3)}.$$

The average number of packets in the system is:

$$\mathbb{E}[N]=\sum_{n=0}^{3} n\cdot P_n.$$

Substitute the probabilities:

$$\mathbb{E}[N]=0\cdot P_0+1\cdot P_1+2\cdot P(N=2)+3\cdot P_3.$$

$$\mathbb{E}[N]=1\cdot\left(\frac{1}{3}\cdot\frac{27}{40}\right)+2\cdot\left(\frac{1}{9}\cdot\frac{27}{40}\right)+3\cdot\left(\frac{1}{27}\cdot\frac{27}{40}\right).$$

Simplify:

$$\mathbb{E}[N]=\frac{81}{360}+\frac{54}{360}+\frac{27}{360}=\frac{162}{360}=\frac{9}{20}.$$

Since we have

$$P_3=\frac{1}{3}^3*\frac{27}{40}=\frac{1}{40}.$$

Using Little's Law, the expected time is:

$$\mathbb{E}[T]=\frac{\frac{9}{20}}{4(1-\frac{1}{40})}=\frac{9}{78}\,\mathrm{s}.$$

2. Consider an $M/M/1/\infty$ system with arrival rate $\lambda = 6$ pkt/s and service rate $\mu = 12$ pkt/s.
   (a) Compute the expected time a packet spends in the system.
   (b) Compute the time a packet is expected to wait in the buffer before getting served.

Ans:

(a) For an $M/M/1/\infty$ system, the average number of packets in the system is:

$$\mathbb{E}[N] = \frac{\lambda}{\mu - \lambda}.$$

Substituting the given values:

$$\mathbb{E}[N] = \frac{6}{12 - 6} = \frac{6}{6} = 1.$$

Using Little's Law:

$$\mathbb{E}[T] = \frac{\mathbb{E}[N]}{\lambda} = \frac{1}{6} \text{ s.}$$

(b) The average waiting time in the buffer is given by:

$$\mathbb{E}[N_q] = \frac{\mathbb{E}[T_q]}{\lambda}.$$

For an $M/M/1/\infty$ system, the average number of packets in the queue is:

$$\mathbb{E}[T_q] = \frac{\lambda^2}{\mu(\mu - \lambda)}.$$

Substituting the given values:

$$\mathbb{E}[T_q] = \frac{6^2}{12(12 - 6)} = \frac{36}{12 \times 6} = \frac{1}{2}.$$

Thus, the expected waiting time in the buffer is:

$$\mathbb{E}[T_q] = \frac{\mathbb{E}[N_q]}{\lambda} = \frac{\frac{1}{2}}{6} = \frac{1}{12} \text{ s.}$$

3. Consider a large system that is composed of five separate $M/M/1/\infty$ servers serving

five different applications, each application has $\lambda = 3$ pkt/s in packet generation rate and each server has $\mu = 6$ pkt/s in service rate.

(a) Which is better in terms of packet delay, having a dedicated server for each application with $\mu = 6$ per server or having a single powerful server with 5μ service rate serving all of the five applications simultaneously? Compute a proof for your answer (Hint: compute the packet delay $\mathbb{E}[T]$ for both cases and compare them).

(b) Assume that the separate five servers have been combined into a single powerful server with service rate 5μ. Compute the probability that there are 6 packets in the system.

Ans:

(a) For each $M/M/1/\infty$ server, the average packet delay is:

$$\mathbb{E}[T_{\text{independent}}] = \frac{1}{\mu - \lambda}.$$

Substituting $\mu = 6$ and $\lambda = 3$:

$$\mathbb{E}[T_{\text{independent}}] = \frac{1}{6-3} = \frac{1}{3} \text{ s.}$$

For the single powerful $M/M/1/\infty$ server, the average packet delay is:

$$\mathbb{E}[T_{\text{shared}}] = \frac{1}{\mu_{\text{total}} - \lambda_{\text{total}}}.$$

Substituting $\mu_{\text{total}} = 30$ and $\lambda_{\text{total}} = 15$:

$$\mathbb{E}[T_{\text{shared}}] = \frac{1}{30-15} = \frac{1}{15} \text{ s.}$$

Thus, the single powerful server has a lower packet delay and is the better choice.

(b) For an $M/M/1/\infty$ system, the steady-state probabilities are given by:

$$P_n = \rho^n P_0,$$

where $\rho = \frac{\lambda_{\text{total}}}{\mu_{\text{total}}}$ and $P_0 = 1 - \rho$.

Substitute $\lambda_{\text{total}} = 15$, $\mu_{\text{total}} = 30$:

$$\rho = \frac{\lambda_{\text{total}}}{\mu_{\text{total}}} = \frac{15}{30} = \frac{1}{2}, \quad P_0 = 1 - \rho = \frac{1}{2}.$$

The probability of 6 packets in the system is:

$$P_6 = \rho^6 \cdot P_0 = (\frac{1}{2})^6 \cdot \frac{1}{2} = \frac{1}{128}.$$

4. Suppose you and your friends begin a new start-up together and the service it provides requires that you set up a web server to service client requests which follows an $M/M/1/K$ queuing model (where $K = 2$). What would be the average number of customers that you expect your server to have to handle (i.e. expect in your system) in the following phases of your business?

   (a) In the beginning, the business is not very popular and your web server's service rate, $\mu_{startup} = 10$, is greater than the inter-arrival times of service requests, $\lambda_{startup} = 2$.

   (b) As you become more popular, your adolescent business begins to see more requests and now $\lambda_{adolescent} = 8$ and $\mu_{adolescent} = 10$.

   (c) A viral interview with your CEO leads to a spike in requests causing $(\lambda_{popular} = 15) > (\mu_{popular} = 10)$.

   (d) You are the CTO, and upon seeing the interview reach 1 million views, are worried about the spike. You ask the CFO about your budget for higher performance servers and the CFO tells him/her that you don't yet have enough money for them yet. You do have enough money for extra queue/buffer storage though and your CTO asks you if the company should double the queue size to 2K. What do you tell the CTO regarding the success or failure of this plan? And what general rule of networking does your response imply?

Ans:

(a) For an $M/M/1/K$ system with $K = 2$, we have:

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-\frac{1}{5}}{1-\left(\frac{1}{5}\right)^3} = \frac{\frac{4}{5}}{1-\frac{1}{125}} = \frac{\frac{4}{5}}{\frac{124}{125}} = \frac{100}{124} = \frac{25}{31}.$$

$$P_1 = \rho \cdot P_0 = \frac{1}{5} \cdot \frac{25}{31} = \frac{5}{32}, \quad P_2 = \rho^2 \cdot P_0 = \left(\frac{1}{5}\right)^2 \cdot \frac{25}{31} = \frac{1}{31}.$$

Thus, the average number of customers is:

$$\mathbb{E}[N] = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = 0 + \frac{5}{32} + 2 \cdot \frac{1}{31} = \frac{7}{31}.$$

(b) In this case, we have:

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-\frac{4}{5}}{1-\left(\frac{4}{5}\right)^3} = \frac{\frac{1}{5}}{1-\frac{64}{125}} = \frac{\frac{1}{5}}{\frac{61}{125}} = \frac{125}{305} = \frac{25}{61}.$$

$$P_1 = \rho \cdot P_0 = \frac{4}{5} \cdot \frac{25}{61} = \frac{100}{305}, \quad P_2 = \rho^2 \cdot P_0 = \left(\frac{4}{5}\right)^2 \cdot \frac{25}{61} = \frac{16}{25} \cdot \frac{25}{61} = \frac{16}{61}.$$

The average number of customers is:

$$\mathbb{E}[N] = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 = 0 + \frac{100}{305} + 2 \cdot \frac{16}{61} = \frac{100}{305} + \frac{160}{305} = \frac{260}{305} = \frac{52}{61}.$$

(c) When $\lambda > \mu$, the system becomes overloaded, meaning it cannot process incoming requests at the same rate they arrive. As a result, once the buffer reaches its capacity K, additional packets are dropped. The average number of customers in the system is constrained by the buffer size, and the system frequently operates at full capacity, leading to significant packet loss and increased delays. Therefore, the average number of customers can be regarded as $K = 2$.

(d) I will tell the CTO that this plan will fail. Increasing the queue size from $K = 2$ to $2K = 4$ does not address the root issue of $\rho > 1$. The system remains overloaded, and requests will continue to accumulate beyond the buffer size. The general networking principle is: increasing buffer size does not resolve congestion caused by an arrival rate exceeding the service rate. The only viable solution is to enhance the server's service capacity.

5. Consider an $M/M/c/c$ system where $c = 3$, the arrival rate $\lambda = 2$, and the service rate $\mu = 4$.
   (a) What is the probability that at time $t = 1$ the number of customers in the system is 3.
   (b) What does the probability in part (a) represent with respect to the system's state?

Ans:

(a) For an $M/M/c/c$ system, we have $\rho = \frac{\lambda}{\mu} = 0.5$. Thus we can calculate P_0 by:

$$P_0 = (1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!})^{-1} = \frac{48}{79}.$$

Thus we calculate P_3 as follows:

$$P_3 = \frac{\rho^3}{3!} P_0 = \frac{1}{48} P_0 = \frac{1}{48} \times \frac{48}{79} = \frac{1}{79}.$$

(b) The probability in (a) represents the proportion of time that the system is full. Since the system cannot serve new customers in this case, it also represent the discard probability of the system.