

# CS 232 Computer and Communication Networks

## Assignment 2

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1. Consider the queuing system in the figure, where packets whose service is completed by Servers 1 and 2 go to Server 3. The service time of Server 1, 2, and 3 is exponential with rate  $\mu_1 = 1$  pks/s,  $\mu_2 = 3$  pkt/s and  $\mu_3 = 4$  pkt/s, respectively. At time  $t = 0$ , packet C arrives in the buffer of Server 1. When C arrived, Server 1 has packet A in service while Server 2 has packet B in service.
  - (a) Compute the probability that A exits the system before any other packets move to server 3.
  - (b) Compute the probability that B exits the system before any other packets move to server 3.
  - (c) Compute the expected time T needed by Packet A to exit the system.

**Ans:**

- (a) The probability that packet A completes service at Server 1 before packet B completes service at Server 2 is:

$$P_1 = \frac{\mu_1}{\mu_1 + \mu_2} = \frac{1}{1 + 3} = \frac{1}{4}.$$

The probability that A completes service at Server 3 before either B or C completes their services is:

$$P_2 = \frac{\mu_3}{\mu_2 + \mu_1 + \mu_3} = \frac{4}{3 + 1 + 4} = \frac{4}{8} = \frac{1}{2}.$$

Thus the overall probability that A exits the system before any other packets move to Server 3 (considering both Server 1 and Server 3) is given by:

$$P = P_1 \times P_2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

- (b) The probability that packet B completes its service at Server 2 before packet A completes at Server 1 is:

$$P_1 = \frac{\mu_2}{\mu_1 + \mu_2} = \frac{3}{1 + 3} = \frac{3}{4}.$$

The probability that B completes service at Server 3 before A or C is:

$$P_2 = \frac{\mu_3}{\mu_1 + \mu_3} = \frac{4}{1 + 4} = \frac{4}{5}.$$

The overall probability that B exits the system before any other packets move to Server 3 is:

$$P = P_1 \times P_2 = \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}.$$

- (c) First we can divide the system event into two branches. The first branch is: A to server 3  $\rightarrow$  A out; the second branch is: B to server 3, while there are also 2 sub-branches after B to server 3. Sub-branch 1 is: A to server 3  $\rightarrow$  B out  $\rightarrow$  A out; sub-branch 2 is B out  $\rightarrow$  A to server 3  $\rightarrow$  A out. The total time packet A spends in the system  $T_A$  can be divided into two parts:

- $T$ : the time until the first thing that happens (A to server 3 or B to server 3 first)
- $R$ : the rest of the time.

So we have  $\mathbb{E}[T_A] = \mathbb{E}[T] + \mathbb{E}[R]$  and  $\mathbb{E}[R] = \mathbb{E}[R|A \text{ arrives server 3 first}] + \mathbb{E}[R|B \text{ arrives server 3 first}]$ . The time until the first thing happens is an exponential distribution with parameter  $\mu_1 + \mu_2$ , so we have:

$$\mathbb{E}[T] = \frac{1}{\mu_1 + \mu_2} = \frac{1}{4}.$$

To compute  $\mathbb{E}[R]$ , we condition on what was the first thing to happen, either packet A arrives server 3 first or packet B arrives server 3 first. For the first case (A arrives server 3), we have:

$$\mathbb{E}[R|A \text{ arrives server 3 first}] = \frac{\mu_1}{\mu_1 + \mu_2} \times \left( \frac{1}{\mu_1} + \frac{1}{\mu_3} \right) = \frac{1}{4} \times \frac{5}{4} = \frac{5}{16}.$$

For the second case (B arrive server 3 first), since there are two sub-branches, we can again separate the second branch into two parts:

- $T_2$ : the time until the first thing that happens (A arrives server 3 or B out first)
- $R_2$ : the rest of the time.

So we have  $\mathbb{E}[R|B \text{ arrives server 3 first}] = \frac{\mu_2}{\mu_1 + \mu_2}(\mathbb{E}[T_2] + \mathbb{E}[R_2])$ . The time until the first thing happens is an exponential distribution with parameter  $\mu_1 + \mu_3$ , so we have:

$$\mathbb{E}[T_2] = \frac{1}{\mu_1 + \mu_3} = \frac{1}{5}.$$

To compute  $\mathbb{E}[R_2]$ , we condition on what was the first thing to happen, either packet A arrives server 3 first or packet B out first. For the first case (A arrives server 3 first), we have:

$$\mathbb{E}[R_2|A \text{ arrives server 3 first}] = \frac{\mu_1}{\mu_1 + \mu_3} \times \left( \frac{1}{\mu_3} + \frac{1}{\mu_3} \right) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}.$$

For the second case (B out first), we have:

$$\mathbb{E}[R_2|B \text{ out first}] = \frac{\mu_3}{\mu_1 + \mu_3} \times \left( \frac{1}{\mu_1} + \frac{1}{\mu_3} \right) = \frac{4}{5} \times \frac{5}{4} = 1.$$

Thus we can obtain:

$$\mathbb{E}[R_2] = \frac{1}{10} + 1 = \frac{11}{10}.$$

and

$$\mathbb{E}[R|B \text{ arrives server 3 first}] = \frac{3}{4} \left( \frac{1}{5} + \frac{11}{10} \right) = \frac{39}{40}.$$

Then  $\mathbb{E}[R]$  can be calculated as:

$$\mathbb{E}[R] = \frac{5}{16} + \frac{39}{40} = \frac{25 + 78}{80} = \frac{103}{80}.$$

Thus we have:

$$\mathbb{E}[T_A] = \frac{1}{4} + \frac{103}{80} = \frac{123}{80} = 1.5375.$$

2. Consider a router receiving packets according to a Poisson process  $\{N(t), t \geq 0\}$  with rate  $\lambda = 3$  packets/second.
  - (a) Compute the probability that the router will receive 2 packets in the next second.
  - (b) Compute the expected total number of packets the router would receive after 2 seconds.

- (c) Imagine that 2 packets arrived in the time interval from 0 to the end of 3s, compute the probability that between the start time of  $t = 4$  and at the end of time  $t = 6$  exactly 3 packets arrive at the router (Hint: Consider the expression  $P(N(6) - N(4)) = 3$ )).

**Ans:**

- (a) The number of packets received in time  $t$  follows a Poisson distribution given by:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

In this case, we have  $\lambda = 3$  packets/second,  $t = 1$  second, and we want to find the probability of receiving  $k = 2$  packets. Thus, we have:

$$P(N(1) = 2) = \frac{(3 \cdot 1)^2 e^{-3 \cdot 1}}{2!} = \frac{3^2 e^{-3}}{2!} = \frac{9e^{-3}}{2}.$$

- (b) For a Poisson process, the expected number of events in a time interval  $t$  is given by:

$$E[N(t)] = \lambda t.$$

In this case, with  $\lambda = 3$  packets/second and  $t = 2$  seconds:

$$E[N(2)] = 3 \times 2 = 6.$$

- (c) The stationary increments  $N(t) - N(s)$  is a Poisson distribution with mean  $\lambda(t - s)$ . For  $t = 6$  and  $s = 4$  we have:

$$P(N(6) - N(4)) = 3 = P(N(2) = 3) = \frac{(3 \cdot 2)^3 e^{-3 \cdot 2}}{3!} = \frac{6^3 e^{-6}}{6} = 36e^{-6}.$$

3. A router sends out 20 packets every 10 seconds on average. Suppose that the time in between two packets sent out can be modeled as an exponential random variable.
- What is the probability that the next packet will be sent out after 5 seconds?
  - What is the probability that exactly 5 packets will leave in the next second?
  - What is the probability that exactly 3 packets will leave in the next 2 seconds?
  - What is the probability that more than 2 packets, but less than 5 packets will leave in the next 2 seconds?

**Ans:**

- (a) Since the router sends out 20 packets every 10 seconds on average. We can obtain the rate  $\lambda$  of the exponential distribution:

$$\lambda = \frac{20}{10} = 2 \text{ packet/second.}$$

The time between packets is modeled as an exponential random variable with rate  $\lambda = 2$ . The cumulative distribution function (CDF) for the exponential distribution is given by:

$$P(X > t) = e^{-\lambda t}.$$

The probability that the next packet will be sent out after 5 seconds is the complementary probability:

$$P(X > 5) = e^{-5\lambda}.$$

- (b) The number of packets sent in a given time is a Poisson process. In this case, the probability of sending exactly  $k = 5$  packets in  $t = 1$  second is:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Substituting  $\lambda = 2$ ,  $t = 1$ , and  $k = 5$ :

$$P(N(1) = 5) = \frac{(2 \cdot 1)^5 e^{-2 \cdot 1}}{5!} = \frac{2^5 e^{-2}}{120} = \frac{4}{15} e^{-2}.$$

- (c) the probability that exactly 3 packets will leave in the next 2 seconds can be calculated as:

$$P(N(2) = 3) = \frac{(4)^3 e^{-4}}{3!} = \frac{64 e^{-4}}{6} = \frac{32}{3} e^{-4}.$$

- (d) This probability can be expressed as:

$$P(3 \leq N(2) < 5) = P(N(2) = 3) + P(N(2) = 4)$$

We already calculated  $P(N(2) = 3)$  in (c). Now we need to compute  $P(N(2) = 4)$ :

$$P(N(2) = 4) = \frac{(4)^4 e^{-4}}{4!} = \frac{256 e^{-4}}{24} = \frac{32}{3} e^{-4}.$$

By summing these probabilities, we have:

$$P(3 \leq N(2) < 5) = P(N(2) = 3) + P(N(2) = 4) = \frac{64}{3} e^{-4}.$$

4. router is receiving packets from two different clients. Assume the time between the generation of two consecutive packets at each client is exponentially distributed with parameters  $\lambda_1 = 2$  packets/second for client 1, and at  $\lambda_2 = 3$  packets/second for client 2
- What is the probability that the next packet will come from node 1?
  - What is the probability that the router will receive exactly 2 packets in the next 3 seconds?
  - Imagine that at time  $t = 3$ , two packets have arrived at the router. At  $t = 5$ , what is the probability that at least 1 more packet will arrive?

**Ans:**

- The probability that the next packet will come from node 1 actually is the probability  $P(\min\{X_1, X_2\} = X_1)$ . Since the minimum of two exponential distributions also follows an exponential distribution with rate  $\lambda_1 + \lambda_2$ , thus we have:

$$P(\text{next packet from client 1}) = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{2}{5}.$$

- The total number of packets received from both clients in a time interval follows a Poisson distribution with rate  $\lambda = 2 + 3 = 5$  packets/second. Thus the probability of receiving exactly  $k = 2$  packets in the next 3 seconds is:

$$P(N(3) = 2) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{(5 \times 3)^2 e^{-15}}{2!} = \frac{225}{4} e^{-15}.$$

- The probability that at least 1 more packet will arrive equals to:

$$P(N(2) \geq 1) = 1 - P(N(2) = 0).$$

Since  $P(N(2) = 0)$  is:

$$P(N(2) = 0) = \frac{(10)^0 e^{-10}}{0!} = e^{-10}.$$

Thus we have

$$P(N(2) \geq 1) = 1 - e^{-10}.$$