

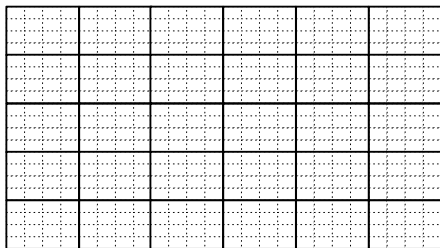
Matrix-vector multiplication in terms of dot-products

Let M be an $R \times C$ matrix.

Dot-Product Definition of matrix-vector multiplication: $M * \mathbf{u}$ is the R -vector \mathbf{v} such that $\mathbf{v}[r]$ is the dot-product of row r of M with \mathbf{u} .

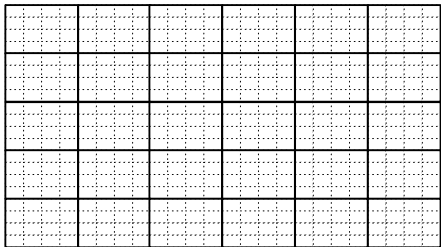
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 10 & 0 \end{bmatrix} * [3, -1] = \begin{bmatrix} [1, 2] \cdot [3, -1], & [3, 4] \cdot [3, -1], & [10, 0] \cdot [3, -1] \end{bmatrix} \\ = [1, 5, 30]$$

Applications of dot-product definition of matrix-vector multiplication: Downsampling



- ▶ Each pixel of the low-res image corresponds to a little grid of pixels of the high-res image.
- ▶ The intensity value of a low-res pixel is the *average* of the intensity values of the corresponding high-res pixels.

Applications of dot-product definition of matrix-vector multiplication: Downsampling



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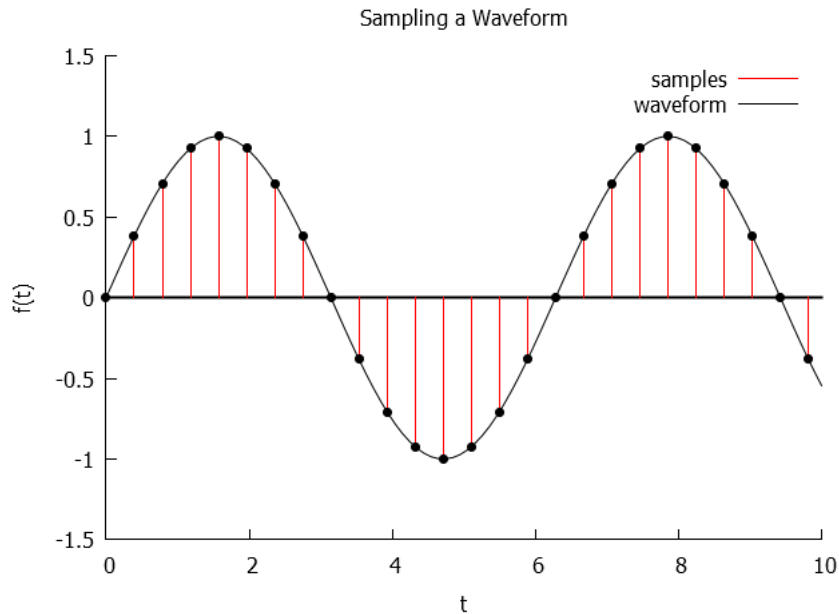
- ▶ Averaging can be expressed as dot-product.
- ▶ We want to compute a dot-product for each low-res pixel.
- ▶ Can be expressed as matrix-vector multiplication.

Applications of dot-product definition of matrix-vector multiplication: blurring



- ▶ To blur a face, replace each pixel in face with average of pixel intensities in its neighborhood.
- ▶ Average can be expressed as dot-product.
- ▶ By dot-product definition of matrix-vector multiplication, can express this image transformation as a matrix-vector product.
- ▶ Gaussian blur: a kind of weighted average

Applications of dot-product definition of matrix-vector multiplication: Audio search



Applications of dot-product definition of matrix-vector multiplication:

Audio search

Lots of dot-products!

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
2	7	4	-3	0	-1	-6	4	5	-8	-9												

5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
	2	7	4	-3	0	-1	-6	4	5	-8	-9											

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5	-6	9	-9	-5	-9	-5	5	-8	-5	-9	9	8	-5	-9	6	-2	-4	-9	-1	-1	-9	-3
						2	7	4	-3	0	-1	-6	4	5	-8	-9						

Applications of dot-product definition of matrix-vector multiplication: Audio search

Lots of dot-products!

- ▶ Represent as a matrix-vector product.
- ▶ One row per dot-product.

To search for $[0, 1, -1]$ in $[0, 0, -1, 2, 3, -1, 0, 1, -1, -1]$:

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} * [0, 1, -1]$$

Formulating a system of linear equations as a matrix-vector equation

Recall the *sensor node* problem:

- ▶ In each of several test periods, measure total power consumed:

$$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$$

- ▶ For each test period, have a vector specifying how long each hardware component was operating during that period:

$$\text{duration}_1, \text{duration}_2, \text{duration}_3, \text{duration}_4, \text{duration}_5$$

- ▶ Use measurements to calculate energy consumed per second by each hardware component.

Formulate as system of linear equations

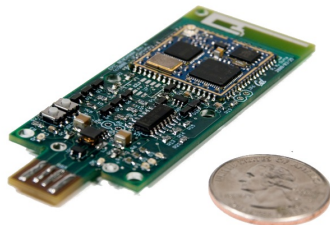
$$\text{duration}_1 \cdot \mathbf{x} = \beta_1$$

$$\text{duration}_2 \cdot \mathbf{x} = \beta_2$$

$$\text{duration}_3 \cdot \mathbf{x} = \beta_3$$

$$\text{duration}_4 \cdot \mathbf{x} = \beta_4$$

$$\text{duration}_5 \cdot \mathbf{x} = \beta_5$$



Formulating a system of linear equations as a matrix-vector equation

Linear equations

$$\mathbf{a}_1 \cdot \mathbf{x} = \beta_1$$

$$\mathbf{a}_2 \cdot \mathbf{x} = \beta_2$$

$$\vdots$$

$$\mathbf{a}_m \cdot \mathbf{x} = \beta_m$$

Each equation specifies the value of a dot-product.

Rewrite as

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} * \mathbf{x} = [\beta_1, \beta_2, \dots, \beta_m]$$

Matrix-vector equation for sensor node

Define $D = \{ \text{'radio'}, \text{'sensor'}, \text{'memory'}, \text{'CPU'} \}$.

Goal: Compute a D-vector \mathbf{u} that, for each hardware component, gives the current drawn by that component.

Four test periods:

- ▶ total milliampere-seconds in these test periods $\mathbf{b} = [140, 170, 60, 170]$
- ▶ for each test period, vector specifying how long each hardware device was operating:
 - ▶ $\text{duration}_1 = \text{Vec}(D, \text{'radio'}:.1, \text{'CPU'}:.3)$
 - ▶ $\text{duration}_2 = \text{Vec}(D, \text{'sensor'}:.2, \text{'CPU'}:.4)$
 - ▶ $\text{duration}_3 = \text{Vec}(D, \text{'memory'}:.3, \text{'CPU'}:.1)$
 - ▶ $\text{duration}_4 = \text{Vec}(D, \text{'memory'}:.5, \text{'CPU'}:.4)$

To get \mathbf{u} , solve $A * \mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} \text{duration}_1 \\ \text{duration}_2 \\ \text{duration}_3 \\ \text{duration}_4 \end{bmatrix}$

Triangular matrix

Recall: We considered *triangular* linear systems, e.g.

$$\begin{aligned} \begin{bmatrix} 1 & 0.5 & -2 & 4 \end{bmatrix} \cdot \mathbf{x} &= -8 \\ \begin{bmatrix} 0 & 3 & 3 & 2 \end{bmatrix} \cdot \mathbf{x} &= 3 \\ \begin{bmatrix} 0 & 0 & 1 & 5 \end{bmatrix} \cdot \mathbf{x} &= -4 \\ \begin{bmatrix} 0 & 0 & 0 & 2 \end{bmatrix} \cdot \mathbf{x} &= 6 \\ \begin{bmatrix} 0 & 0 & 0 & 2 \end{bmatrix} \cdot \mathbf{x} &= 6 \end{aligned}$$

We can rewrite this linear system as a matrix-vector equation:

$$\begin{bmatrix} 1 & 0.5 & -2 & 4 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix} * \mathbf{x} = [-8, 3, -4, 6]$$

The matrix is a *triangular* matrix.

Definition: An $n \times n$ *upper triangular* matrix A is a matrix with the property that $A_{ij} = 0$ for $j > i$. Note that the entries forming the triangle can be zero or nonzero.

We can use backward substitution to solve such a matrix-vector equation.

Triangular matrices will play an important role later.

Computing sparse matrix-vector product

To compute matrix-vector or vector-matrix product,

- ▶ could use dot-product or linear-combinations definition.
(You'll do that in homework.)
- ▶ However, using those definitions, it's not easy to exploit sparsity in the matrix.

“Ordinary” Definition of Matrix-Vector Multiplication: If M is an $R \times C$ matrix and \mathbf{u} is a C -vector then $M * \mathbf{u}$ is the R -vector \mathbf{v} such that, for each $r \in R$,

$$v[r] = \sum_{c \in C} M[r, c] u[c]$$

Computing sparse matrix-vector product

“Ordinary” Definition of Matrix-Vector Multiplication: If M is an $R \times C$ matrix and \mathbf{u} is a C -vector then $M * \mathbf{u}$ is the R -vector \mathbf{v} such that, for each $r \in R$,

$$v[r] = \sum_{c \in C} M[r, c] u[c]$$

Obvious method:

```
1 for i in R:
2   v[i] :=  $\sum_{j \in C} M[i, j] u[j]$ 
```

But this doesn't exploit sparsity!

Idea:

- ▶ Initialize output vector \mathbf{v} to zero vector.
- ▶ Iterate over nonzero entries of M , adding terms according to ordinary definition.

```
1 initialize  $\mathbf{v}$  to zero vector
2 for each pair  $(i, j)$  in sparse representation,
3   v[i] = v[i] +  $M[i, j] u[j]$ 
```

Algebraic properties of matrix-vector multiplication

Proposition: Let A be an $R \times C$ matrix.

- ▶ For any C -vector \mathbf{v} and any scalar α ,

$$A * (\alpha \mathbf{v}) = \alpha (A * \mathbf{v})$$

- ▶ For any C -vectors \mathbf{u} and \mathbf{v} ,

$$A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v}$$

Algebraic properties of matrix-vector multiplication

To prove

$$A * (\alpha \mathbf{v}) = \alpha (A * \mathbf{v})$$

we need to show corresponding entries are equal:

Need to show

$$\text{entry } i \text{ of } A * (\alpha \mathbf{v}) = \text{entry } i \text{ of } \alpha (A * \mathbf{v})$$

Proof:

$$\text{Write } A = \left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array} \right].$$

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } A * (\alpha \mathbf{v}) &= \mathbf{a}_i \cdot \alpha \mathbf{v} \\ &= \alpha (\mathbf{a}_i \cdot \mathbf{v}) \end{aligned}$$

by homogeneity of dot-product

By definition of scalar-vector multiply,

$$\begin{aligned} \text{entry } i \text{ of } \alpha (A * \mathbf{v}) &= \alpha (\text{entry } i \text{ of } A * \mathbf{v}) \\ &= \alpha (\mathbf{a}_i \cdot \mathbf{v}) \end{aligned}$$

by dot-product definition of
matrix-vector multiply

QED

Algebraic properties of matrix-vector multiplication

To prove

$$A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v}$$

we need to show corresponding entries are equal:

Need to show

$$\text{entry } i \text{ of } A * (\mathbf{u} + \mathbf{v}) = \text{entry } i \text{ of } A * \mathbf{u} + A * \mathbf{v}$$

Proof:

$$\text{Write } A = \left[\begin{array}{c} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{array} \right].$$

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } A * (\mathbf{u} + \mathbf{v}) &= \mathbf{a}_i \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{a}_i \cdot \mathbf{u} + \mathbf{a}_i \cdot \mathbf{v} \end{aligned}$$

by distributive property of dot-product

By dot-product def. of matrix-vector mult,

$$\begin{aligned} \text{entry } i \text{ of } A * \mathbf{u} &= \mathbf{a}_i \cdot \mathbf{u} \\ \text{entry } i \text{ of } A * \mathbf{v} &= \mathbf{a}_i \cdot \mathbf{v} \end{aligned}$$

so

$$\text{entry } i \text{ of } A * \mathbf{u} + A * \mathbf{v} = \mathbf{a}_i \cdot \mathbf{u} + \mathbf{a}_i \cdot \mathbf{v}$$

QED