Dual representations of vector spaces

Two important ways to represent a vector space $\mathcal{V}$:

$\mathcal{V}$ =solution set of homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$

Equivalently, $\mathcal{V} = \text{Null} \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

$\mathcal{V} = \text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

Equivalently, $\mathcal{V} = \text{Row} \begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_n \end{bmatrix}$

How to transform between these two representations?

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_n$ for solution set

Equivalently, given matrix A, find B such that Row B = Null A

From right to left: Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_n$, find system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

Equivalently, given matrix B, find matrix A such that Null A = Row B

Dual representations of vector spaces

Two important ways to represent a vector space:

As the solution set of homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$$

Equivalently, Null $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

As Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Equivalently,

Row $\begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_k \end{bmatrix}$

How to transform between these two representations?

From left to right: Given homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

From right to left:
Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$,
find homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

The dual of a vector space

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Solution set is the set of vectors $\mathbf{u}$ such that $\mathbf{a}_1 \cdot \mathbf{u} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{u} = 0$

$$\underbrace{\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Equivalent: Given rows of a matrix A, find generators for Null A

$$\begin{array}{c} \text{rows of a matrix } A \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for Null } A \end{array}$$

If $\mathbf{u}$ is such a vector then

$$\mathbf{u} \cdot (\alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_m \, \mathbf{a}_m) = 0$$

for any coefficients $\alpha_1, \ldots, \alpha_m$.

Definition: The set of vectors $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$ is called the *dual* of $\mathcal{V}$. Dual is written as $\mathcal{V}^*$.

Example: The dual of Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is the solution set for $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$

$$\begin{array}{c} \text{generators for a vector space } \mathcal{V} \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for dual space } \mathcal{V}^* \end{array}$$

Dual of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *dual space* of $\mathcal{V}$, written $\mathcal{V}^*$, is

$$\mathcal{V}^* = \{\mathbf{u} \in \mathbb{F}^n \ : \ \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^* = \text{Span } \{[1,0,-1]\}$:

- Note that $[1,0,-1] \cdot [1,0,1] = 0$ and $[1,0,-1] \cdot [0,1,0] = 0$.
  Therefore $[1,0,-1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- For any scalar β,
  $$\beta\,[1,0,-1] \cdot \mathbf{v} = \beta\,([1,0,-1] \cdot \mathbf{v}) = 0$$
  for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Which vectors $\mathbf{u}$ satisfy $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$?
  Only scalar multiples of $[1,0,-1]$.

Example over $GF(2)$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^* = \text{Span } \{[1,0,1]\}$:

- Note that $[1,0,1] \cdot [1,0,1] = 0$ (remember $GF(2)$ addition) and $[1,0,1] \cdot [0,1,0] = 0$.
- Therefore $[1,0,1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Of course $[0,0,0] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- $[1,0,1]$ and $[0,0,0]$ are the only such vectors.

Dual space

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^* = \text{Span } \{[1,0,-1]\}$
$\dim \mathcal{V} + \dim \mathcal{V}^* = 3$

Example over $GF(2)$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^* = \text{Span } \{[1,0,1]\}$.
$\dim \mathcal{V} + \dim \mathcal{V}^* = 3$

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1,0],[0,1,0,1]\}$.
Then $\mathcal{V}^* = \text{Span } \{[1,0,-1,0],[0,1,0,-1]\}$. $\dim \mathcal{V} + \dim \mathcal{V}^* = 4$

Dual Dimension Theorem: $\dim \mathcal{V} + \dim \mathcal{V}^* = n$

Proof: Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be generators for $\mathcal{V}$.

Let $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

Then $\mathcal{V}^* = \text{Null } A$.

Rank-Nullity Theorem states that

$$\begin{array}{ccccc} \text{rank } A & + & \text{nullity } A & = & n \\ \dim \mathcal{V} & + & \dim \mathcal{V}^* & = & n \end{array}$$

QED

The dual of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *dual space* of $\mathcal{V}$, written $\mathcal{V}^*$, is

$$\mathcal{V}^* = \{\mathbf{u} \in \mathbb{F}^n \;:\; \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

$$\begin{array}{ccc}
\text{rows of a matrix } A & & \text{generators for a vector space } \mathcal{V} \\
\downarrow & & \downarrow \\
\boxed{\text{Algorithm X}} & = & \boxed{\text{Algorithm X}} \\
\downarrow & & \downarrow \\
\text{generators for Null } A & & \text{generators for dual space } \mathcal{V}^*
\end{array}$$

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Algorithm X solves left-to-right problem....

what about right-to-left problem?

The dual of a vector space

From left to right: Given system
$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

$$\begin{array}{c}\text{generators for a vector space } \mathcal{V} \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for dual space } \mathcal{V}^* \end{array}$$

What happens if we apply Algorithm X to generators for dual space $\mathcal{V}^*$?

$$\begin{array}{c}\text{generators for dual space } \mathcal{V}^* \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for dual of dual space } (\mathcal{V}^*)^* \end{array}$$

From right to left: Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$, find system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

$$\begin{array}{c}\text{generators for dual space } \mathcal{V}^* \\ \downarrow \\ \boxed{\text{Algorithm Y}} \\ \downarrow \\ \text{generators for original space } \mathcal{V} \end{array}$$

Theorem: $(\mathcal{V}^*)^* = \mathcal{V}$ (The dual of the dual is the original space.)

Theorem shows:

$$\text{Algorithm X} = \text{Algorithm Y}$$

We still must prove the Theorem...

Dual space

Theorem: $(\mathcal{V}^*)^* = \mathcal{V}$ (The dual of the dual is the original space.)

Proof:
Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be a basis for $\mathcal{V}$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be a basis for $\mathcal{V}^*$.
Since $\mathbf{b}_1 \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$,

$$\mathbf{b}_1 \cdot \mathbf{a}_1 = 0, \mathbf{b}_1 \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_1 \cdot \mathbf{a}_m = 0$$

Similarly $\mathbf{b}_i \cdot \mathbf{a}_1 = 0, \mathbf{b}_i \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_i \cdot \mathbf{a}_m = 0$ for $i = 1, 2, \ldots, k$.

Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^*}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^*)^*$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^*)^*$, $\mathbf{a}_4$ is in $(\mathcal{V}^*)^*$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^*)^*$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^*)^*$.

Thus $\underbrace{\text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^*)^*$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^*)^*$.

Dual space

Theorem: $(\mathcal{V}^*)^* = \mathcal{V}$ (The dual of the dual is the original space.)

Proof:
Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^*}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^*)^*$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^*)^*$, $\mathbf{a}_4$ is in $(\mathcal{V}^*)^*$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^*)^*$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^*)^*$.

Thus $\underbrace{\text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^*)^*$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^*)^*$.

By Dual Dimension Theorem, $\dim \mathcal{V} + \dim \mathcal{V}^* = n$.

By Dual Dimension Theorem applied to $\mathcal{V}^*$, $\dim \mathcal{V}^* + \dim(\mathcal{V}^*)^* = n$.

Together these equations show $\dim \mathcal{V} = \dim(\mathcal{V}^*)^*$ QED