Change of basis

Suppose we have a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for some vector space $\mathcal{V}$.
How do we go

- from a vector $\mathbf{b}$ in $\mathcal{V}$
- to the coordinate representation $\mathbf{u}$ of $\mathbf{b}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$?

By linear-combinations definition of matrix-vector multiplication,

$$\left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{u} \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right]$$

By Unique-Representation Lemma,
$\mathbf{u}$ is the *only* solution to the equation

$$\left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right]$$

so we can obtain $\mathbf{u}$ by using a matrix-vector equation solver.

Function $f : \mathbb{F}^n \longrightarrow \mathcal{V}$ defined by $f(\mathbf{x}) = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right]$ is

- *onto* (because $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are generators for $\mathcal{V}$)
- *one-to-one* (by Unique-Representation Lemma)

so f is an invertible function.

Change of basis

Now suppose $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is one basis for $\mathcal{V}$ and $\mathbf{c}_1, \cdots, \mathbf{c}_k$ is another.

Define $f(\mathbf{x}) = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right]$ and define $g(\mathbf{y}) = \left[\begin{array}{c|c|c} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{array}\right] \left[\begin{array}{c} \mathbf{y} \end{array}\right]$.

Then both f and g are invertible functions.

The function $f^{-1} \circ g$ maps

- from coordinate representation of a vector in terms of $\mathbf{c}_1, \ldots, \mathbf{c}_k$
- to coordinate representation of a vector in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$

In particular, if $\mathcal{V} = \mathbb{F}^m$ for some m then

f invertible implies that $\left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right]$ is an invertible matrix.

g invertible implies that $\left[\begin{array}{c|c|c} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{array}\right]$ is an invertible matrix.

Thus the function $f^{-1} \circ g$ has the property

$$(f^{-1} \circ g)(\mathbf{x}) = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array}\right]^{-1} \left[\begin{array}{c|c|c} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \end{array}\right]$$

Change of basis

Proposition: If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{c}_1, \ldots, \mathbf{c}_k$ are bases for $\mathbb{F}^m$ then multiplication by the matrix

$$B = \left[\begin{array}{c|c|c} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{array} \right]^{-1} \left[\begin{array}{c|c|c} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{array} \right]$$

maps

- from the representation of a vector with respect to $\mathbf{c}_1, \ldots, \mathbf{c}_k$
- to the representation of that vector with respect to $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Conclusion: Given two bases of $\mathbb{F}^m$, there is a matrix B such that multiplication by B converts from one coordinate representation to the other.

Remark: Converting between vector itself and its coordinate representation is a special case:

- Think of the vector itself as coordinate representation with respect to standard basis.

Change of basis: simple example

Example: To map

from coordinate representation with respect to $[1, 2, 3], [2, 1, 0], [0, 1, 4]$

to coordinate representation with respect to $[2, 0, 1], [0, 1, -1], [1, 2, 0]$

multiply by the matrix

$$\left[\begin{array}{c|c|c} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right]^{-1} \left[\begin{array}{c|c|c} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 4 \end{array}\right]$$

which is

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{4}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \left[\begin{array}{c|c|c} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 4 \end{array}\right]$$

which is

$$\begin{bmatrix} -1 & 1 & -\frac{5}{3} \\ -4 & 1 & -\frac{17}{3} \\ 3 & 9 & \frac{10}{3} \end{bmatrix}$$