Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of a single nonzero vector $\mathbf{v}$:

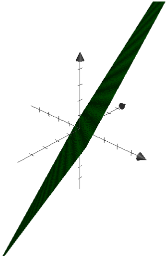
$$\text{Span } \{\mathbf{v}\} = \{\alpha\,\mathbf{v} \;:\; \alpha \in \mathbb{R}\}$$

This is the line through the origin and $\mathbf{v}$. *One-dimensional*

Span of the empty set:just the origin. *Zero-dimensional*

Span $\{[1, 2], [3, 4]\}$: all points in the plane. *Two-dimensional*

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:



Two-dimensional

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Is the span of k vectors always k-dimensional?
No.

- Span $\{[0,0]\}$ is 0-dimensional.
- Span $\{[1,3],[2,6]\}$ is 1-dimensional.
- Span $\{[1,0,0],[0,1,0],[1,1,0]\}$ is 2-dimensional.

Fundamental Question: How can we predict the dimensionality of the span of some vectors?

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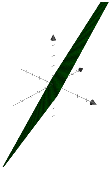
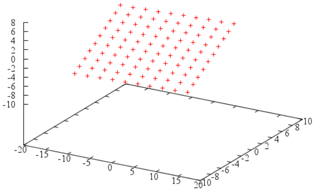


Two-dimensional

Useful for plotting the plane



$\{\alpha\,[1, 0.1.65] + \beta\,[0, 1, 1]\ :$
$\alpha \in \{-5, -4, \ldots, 3, 4\},$
$\beta \in \{-5, -4, \ldots, 3, 4\}\}$

Geometry of sets of vectors: span of vectors over $\mathbb{R}$

Span of two 3-vectors? Span $\{[1, 0, 1.65], [0, 1, 1]\}$ is a plane in three dimensions:

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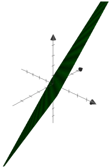

Perhaps a more familiar way to specify a plane:

$$\{(x, y, z) \; : \; ax + by + cz = 0\}$$

Using dot-product, we could rewrite as

$$\{[x, y, z] \; : \; [a, b, c] \cdot [x, y, z] = 0\}$$

Set of vectors satisfying a linear equation with right-hand side *zero*.

We can similarly specify a line in three dimensions:

$$\{[x, y, z] \; : \; \mathbf{a}_1 \cdot [x, y, z] = 0, \mathbf{a}_2 \cdot [x, y, z] = 0\}$$

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Geometry of sets of vectors: Two representations

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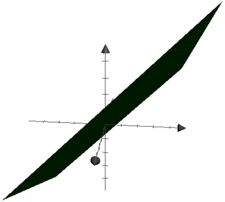
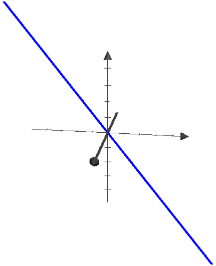


Span $\{[4, -1, 1], [0, 1, 1]\}$ $\quad\quad$ $\{[x, y, z] \ : \ [1, 2, -2] \cdot [x, y, z] = 0\}$



Span $\{[1, 2, -2]\}$ $\quad\quad$ $\{[x, y, z] \ :$
$[4, -1, 1] \cdot [x, y, z] = 0,$
$[0, 1, 1] \cdot [x, y, z] = 0\}$

Geometry of sets of vectors: Two representations

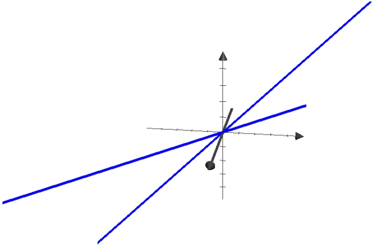
Two ways to represent a geometric object (line, plane, etc.) containing the origin:

- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

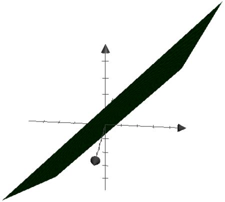
Each representation has its uses.

Suppose you want to find the plane containing two given lines

- First line is Span $\{[4, -1, 1]\}$.
- Second line is Span $\{[0, 1, 1]\}$.



- The plane containing these two lines is Span $\{[4, -1, 1], [0, 1, 1]\}$



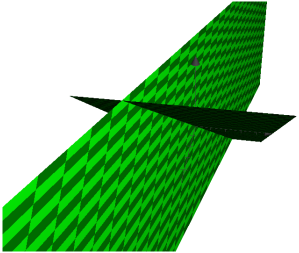
Geometry of sets of vectors: Two representations

Two ways to represent a geometric object (line, plane, etc.) containing the origin:

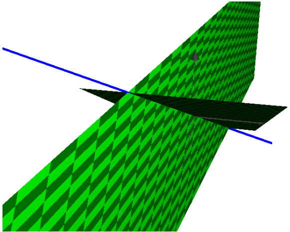
- Span of some vectors
- Solution set of some system of linear equations with zero right-hand sides

Each representation has its uses.

Suppose you want to find the intersection of two given planes:



- First plane is $\{[x, y, z] \ : \ [4, -1, 1] \cdot [x, y, z] = 0\}$.
- Second plane is $\{[x, y, z] \ : \ [0, 1, 1] \cdot [x, y, z] = 0\}$.



- The intersection is $\{[x, y, z] \ : \ [4, -1, 1] \cdot [x, y, z] = 0, [0, 1, 1] \cdot [x, y, z] = 0\}$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ satisfies

- Property V1 because

$$0\,\mathbf{v}_1+\cdots+0\,\mathbf{v}_n$$

- Property V2 because

$$\text{if } \mathbf{v}=\beta_1\,\mathbf{v}_1+\cdots+\beta_n\,\mathbf{v}_n \text{ then } \alpha\,\mathbf{v}=\alpha\,\beta_1\mathbf{v}_1+\cdots+\alpha\,\beta_n\,\mathbf{v}_n$$

- Property V3 because

$$\begin{gathered}\text{if } \mathbf{u}=\alpha_1\,\mathbf{v}_1+\cdots+\alpha_n\,\mathbf{v}_1\\ \text{and } \mathbf{v}=\beta_1\,\mathbf{v}_1+\cdots+\beta_n\,\mathbf{v}_n\\ \text{then } \mathbf{u}+\mathbf{v}=(\alpha_1+\beta_1)\mathbf{v}_1+\cdots+(\alpha_n+\beta_n)\,\mathbf{v}_n\end{gathered}$$

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Solution set $\{\mathbf{x}\ :\ \mathbf{a}_1\cdot\mathbf{x}=0,\quad\ldots,\quad\mathbf{a}_m\cdot\mathbf{x}=0\}$ satisfies

- Property V1 because
$$\mathbf{a}_1\cdot\mathbf{0}=0,\quad\ldots,\quad\mathbf{a}_m\cdot\mathbf{0}=0$$
- Property V2 because
$$\text{if } \mathbf{a}_1\cdot\mathbf{v}=0,\quad\ldots,\quad\mathbf{a}_m\cdot\mathbf{v}=0$$
$$\text{then } \mathbf{a}_1\cdot(\alpha\,\mathbf{v})=\alpha\,(\mathbf{a}_1\cdot\mathbf{v})=0,\cdots,\mathbf{a}_m\cdot(\alpha\,\mathbf{v})=\alpha\,(\mathbf{a}_m\cdot\mathbf{v})=0$$
- Property V3 because
$$\text{if } \mathbf{a}_1\cdot\mathbf{u}=0,\quad\ldots,\quad\mathbf{a}_m\cdot\mathbf{u}=0$$
$$\text{and } \mathbf{a}_1\cdot\mathbf{v}=0,\quad\ldots,\quad\mathbf{a}_m\cdot\mathbf{v}=0$$
$$\text{then } \mathbf{a}_1\cdot(\mathbf{u}+\mathbf{v})=\mathbf{a}_1\cdot\mathbf{u}+\mathbf{a}_1\cdot\mathbf{v}=0,\quad\ldots,\quad\mathbf{a}_m\cdot(\mathbf{u}+\mathbf{v})=\mathbf{a}_m\cdot\mathbf{u}+\mathbf{a}_m\cdot\mathbf{v}=0$$

Two representations: What's common?

Subset of $\mathbb{F}^D$ that satisfies three properties:

Property V1 Subset contains the zero vector $\mathbf{0}$

Property V2 If subset contains $\mathbf{v}$ then it contains $\alpha\,\mathbf{v}$ for every scalar α

Property V3 If subset contains $\mathbf{u}$ and $\mathbf{v}$ then it contains $\mathbf{u}+\mathbf{v}$

Any subset $\mathcal{V}$ of $\mathbb{F}^D$ satisfying the three properties is called a *vector space.*

Example: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{x} \,:\, \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ are vector spaces.

If $\mathcal{U}$ is also a vector space and $\mathcal{U}$ is a subset of $\mathcal{V}$ then $\mathcal{U}$ is called a *subspace* of $\mathcal{V}$.

Example: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{x} \,:\, \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$ are *subspaces* of $\mathbb{R}^D$

Possibly profound fact we will learn later: Every subspace of $\mathbb{R}^D$

- can be written in the form Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$
- can be written in the form $\{\mathbf{x} \,:\, \mathbf{a}_1 \cdot \mathbf{x} = 0, \quad \ldots, \quad \mathbf{a}_m \cdot \mathbf{x} = 0\}$

Abstract vector spaces

In traditional, abstract approach to linear algebra:

- We don't define vectors as sequences [1,2,3] or even functions $\{\texttt{a:1, b:2, c:3}\}$.
- We define a vector space over a field $\mathbb{F}$ to be any set $\mathcal{V}$ that is equipped with
  - an *addition* operation, and
  - a *scalar-multiplication* operation

  satisfying certain axioms (e.g. commutate and distributive laws) and Properties V1, V2, V3.

Abstract approach has the advantage that it avoids committing to specific structure for vectors.

I avoid abstract approach in this class because more concrete notion of vectors is helpful in developing intuition.

Geometry of sets of vectors: convex hull

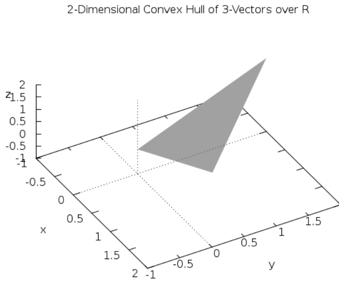
Earlier, we saw: The **u**-to-**v** line segment is

$$\{\alpha\,\mathbf{u} + \beta\,\mathbf{v}\ :\ \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\}$$

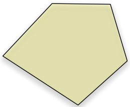
Definition: For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ over $\mathbb{R}$, a linear combination

$$\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

is a *convex combination* if the coefficients are all nonnegative and they sum to 1.



- Convex hull of a single vector is a point.
- Convex hull of two vectors is a line segment.
- Convex hull of three vectors is a triangle



Convex hull of more vectors? Could be higher-dimensional... but not necessarily.

For example, a convex polygon is the convex hull of its vertices