Notes on fractal harmonic measure problem

Qile Yan

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1 Result on unit disk

In this section, we consider the domain Ω is a unit disk (see Fig. 1).

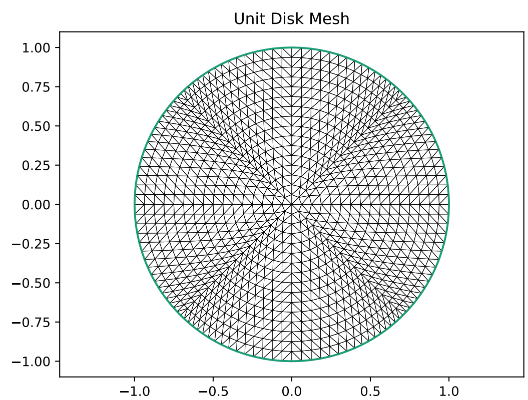


Figure 1: Unit disk.

We first test the firedrake solver on the PDE:

$$-\Delta u = f \quad \text{on } \Omega \tag{1.1}$$

with $u = g$ on $\partial\Omega$.

In Fig. 2 and Fig. 3, we plot the error in L^2 and H^1 norm with respect to mesh size h and degree of freedom. 2, the manufactured solution is $u(x, y) = 2+x+3y$. In Fig. 3, the manufactured solution is $u(x, y, z) = 2 + x^2 + y$. The Lagrange linear element is used. The uniform refinement is done by built-in function MeshHierarchy.

(a) Error vs mesh size (b) Error vs degree of freedom

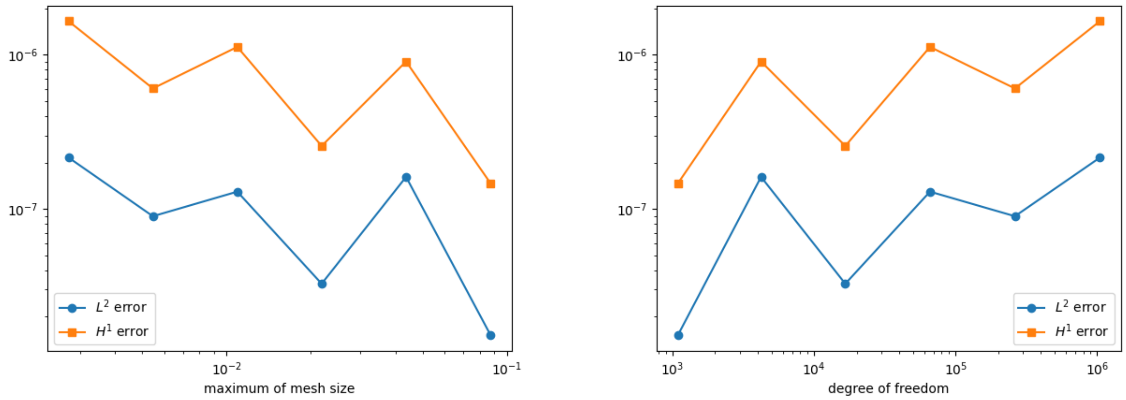


Figure 2: Unit disk: manufactured solution $u = x + 3y + 2$. Note that only round off error appears here since the solution is in the linear finite element space. Remark: Here, we use the Krylov subspace method with BoomerAMG in preconditioner in the solver. If we change the solver to be LU factorisation, the error here would be around $10^{-14} \sim\ 10^{-15}$.

(a) Error vs mesh size (b) Error vs degree of freedom

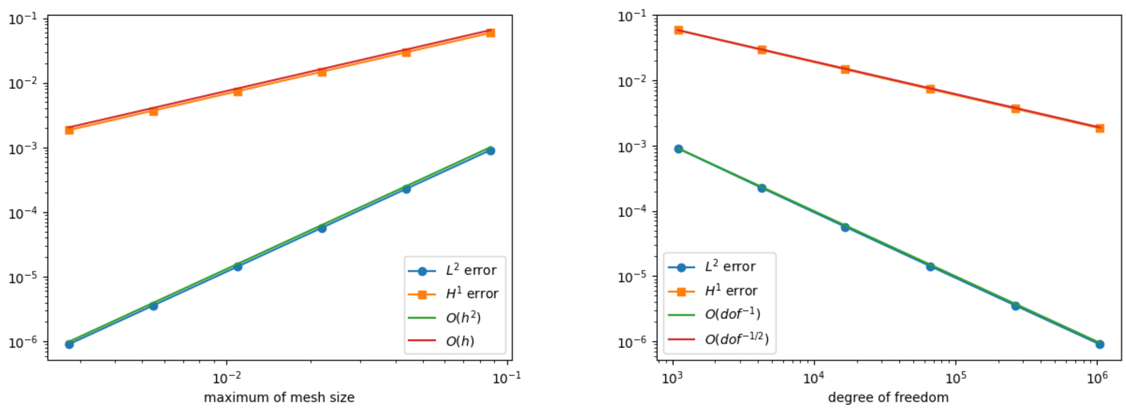


Figure 3: Unit disk: manufactured solution $u = x^2 + y + 2$.

Next, we solve the PDE

$$-\Delta u = f \quad \text{on } \Omega \tag{1.2}$$

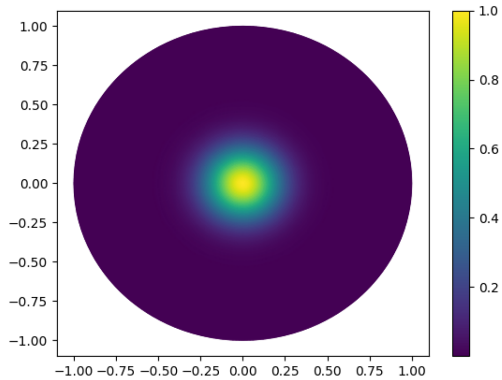
with $u = 0$ on the unit disk and evaluate the solution at points from center to boundary. In particular, we evaluate the solutions at

$$\mathbf{x}_i = (0, 1 - 1/2^i), \quad 0 \le i \le n_{max}. \tag{1.3}$$

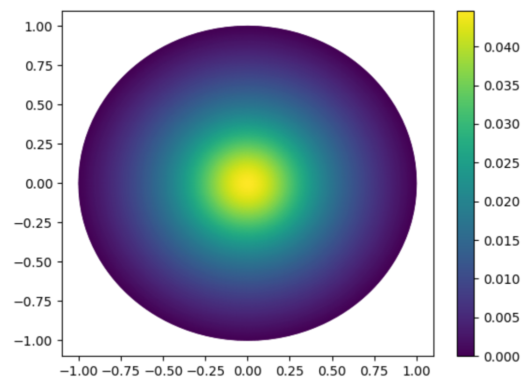
In Fig. 4, we show the results of the PDE (1.2) with a different f. That is, we define f as

$$f(x, y) = e^{-20(x^2+y^2)}.$$

(a) f



(b) Solution u



(c) Evaluation at $\{\mathbf{x}_i\}_{i=5}^{15}$.

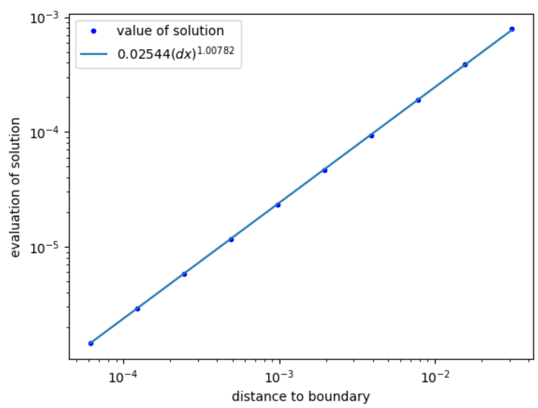


Figure 4: On unit disk. (a). forcing term f. (b) solution u. (c). Evaluation of solution at a sequence points $\{\mathbf{x}_i\}_{i=1}^{15}$ where $\mathbf{x}_i$ is defined as in (1.3). Notice that the slope is almost equal to $\frac{\partial u}{\partial n} \approx 0.025$.

$$-\int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_{\Omega} f = 2\pi \int_0^1 e^{-20r^2} r dr = \frac{-\pi}{20}(e^{-20} - 1)$$
$$\Rightarrow \frac{\partial u}{\partial n} = \frac{1}{40}(e^{-20} - 1) \approx -0.025$$

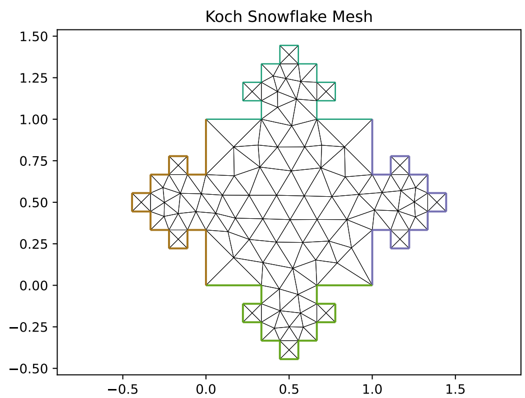
2 On snowflake boundary domain

In this note we solve the PDE

$$-\Delta u = f \quad \text{on } \Omega \tag{2.1}$$

with $u = 0$ on $\partial\Omega$. Here, Ω is a unit square where each edges are replaced by the Koch snowflake (See Fig. 5). In this note, we use the FEM python package firedrake solving the PDE.

(a) $n = 2$



(b) $n = 4$

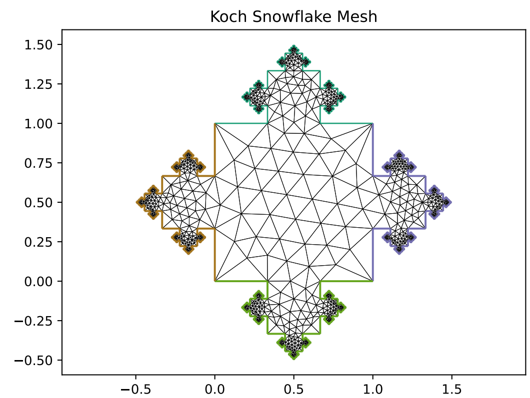
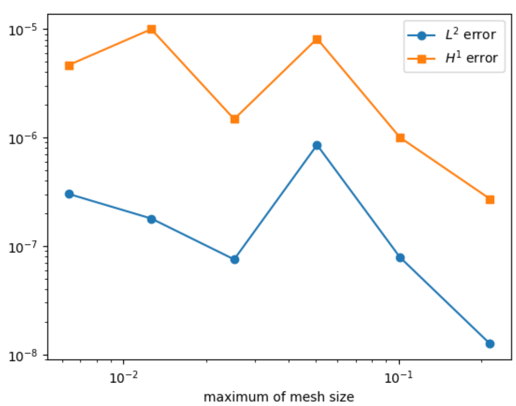


Figure 5: Unit square with each edges replaced by a Koch snowflake with n iterations. Meshsize is 0.5 for each vertices in .geo file.

In Fig. 6 and Fig. 7, we plot the error in L^2 and H^1 norm with respect to mesh size h and degree of freedom where domain is unit square with snowflake ($n = 4$ iterations). In Fig. 6, the manufactured solution is $u(x, y) = 2 + x + 3y$. In Fig. 7, the manufactured solution is $u(x, y, z) = 2 + x^2 + y$. The Lagrange linear element is used. The uniform refinement is done by built-in function MeshHierarchy.

(a) Error vs mesh size



(b) Error vs degree of freedom

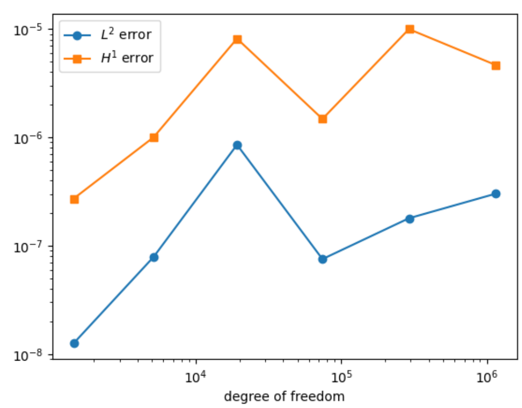


Figure 6: Unit square with $n = 8$ snowflake iterations: manufactured solution $u = x + 3y + 2$. Note that only round off error appears here since the solution is in the linear finite element space.

(a) Error vs mesh size (b) Error vs degree of freedom

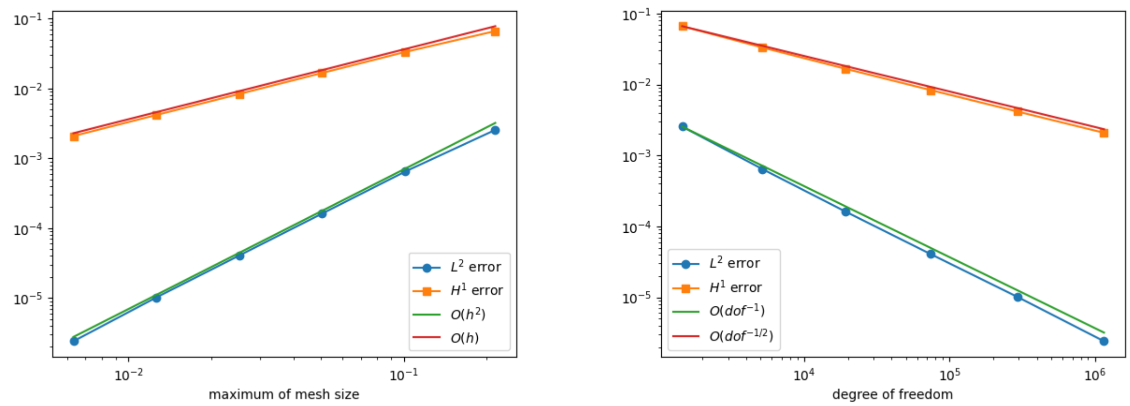


Figure 7: Unit square with $n = 8$ snowflake iterations: manufactured solution $u = x^2 + y + 2$.

In Fig. 8(b), we plot the solution of the PDE (2.1) where Ω is the square with 4 snowflake iterations as in Fig. 5(b) and f is defined as

$$f(x, y) = e^{-20((x-0.5)^2+(y-0.5)^2)}.$$

(a) f (b) Solution u

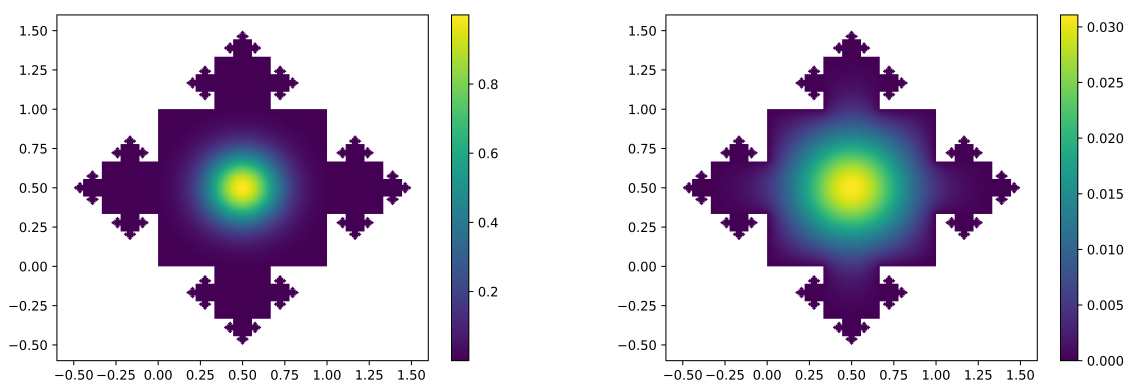


Figure 8: On unit square with $n = 8$ snowflake iterations. (a). forcing term f. (b) solution of the PDE (2.1).

In Fig. 10, we evaluate the solution along a random path with each points $\mathbf{x}_i, 0 \le i \le n$ is a center of the square from i-th snowflake iteration. In particular, we create the sequence of the points along the path as followings:

1). Set $\mathbf{x}_0 = (0.5, 0.5)$

2). Pick a random integer from 0 to 3. Each integer corresponds to one of four normal directions $\vec{v}_1$. (upward, downward, left, right).

3). Set $\mathbf{x}_1 = \mathbf{x}_0 + h_0\vec{v}_1$ with $h_0 = \frac{2}{3}$.

4). For $i = 2, ..., n$, Pick a random integer from 0 to 2. Each integer corresponds to one of three normal directions $\vec{v}_i$ (forward, left, right). Set

$$\mathbf{x}_i = \mathbf{x}_{i-1} + h_{i-1}\vec{v}_i.$$

where $h_{i-1} = \frac{2}{3^i}$.

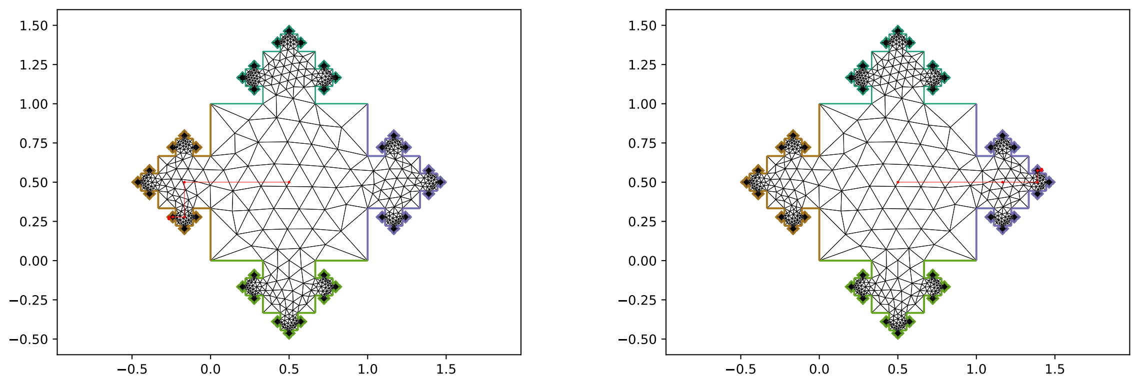


Figure 9: Two random path on the unit square with $n = 8$ snowflake iterations.

Let $dx_i = (1/3)^i$. Then we would like to approximate $u(\mathbf{x}_i)$ as a function of dx_i in the form

$$u(\mathbf{x}_i) = c(dx_i)^\alpha$$

By taking the log on both ends,

$$\log u(\mathbf{x}_i) = \log c + \alpha \log(dx_i), \quad 0 \leq i \leq n.$$

The coefficients c and α are found by least squares approximation.

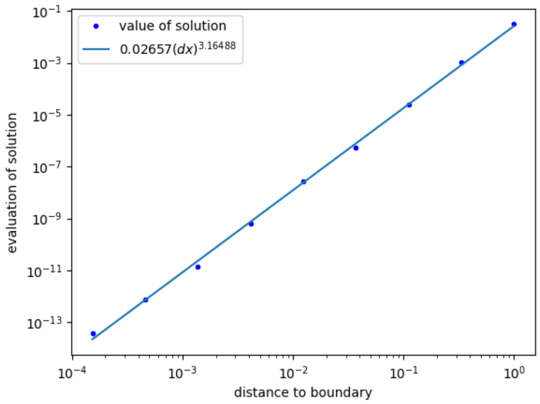
Running over 500 random path for two times, we get:

	Mean	Standard deviation
c	0.02598	0.005612
α	2.89311	0.16436

and

	Mean	Standard deviation
c	0.02567	0.005760
α	2.90044	0.16815

(a) $c = 0.02657$, $\alpha = 3.16488$



(b) $c = 0.03466$, $\alpha = 3.05545$

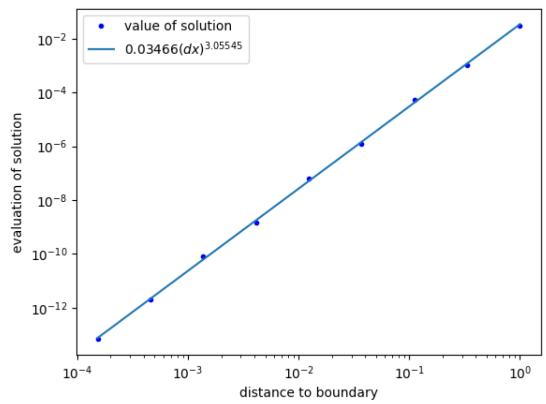
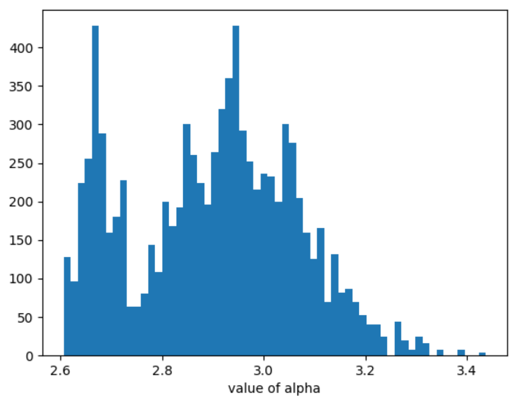


Figure 10: On unit square with $n = 8$ snowflake iterations. Result of different random path

(a) distribution of α



(b) distribution of c.

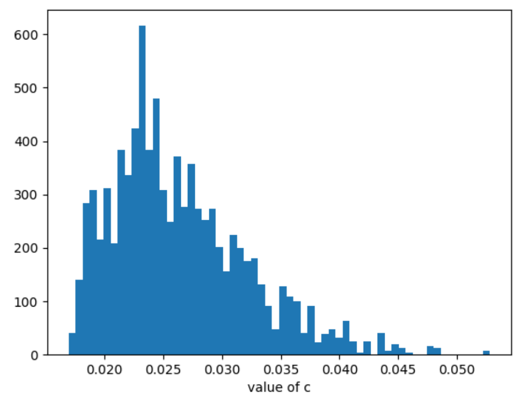


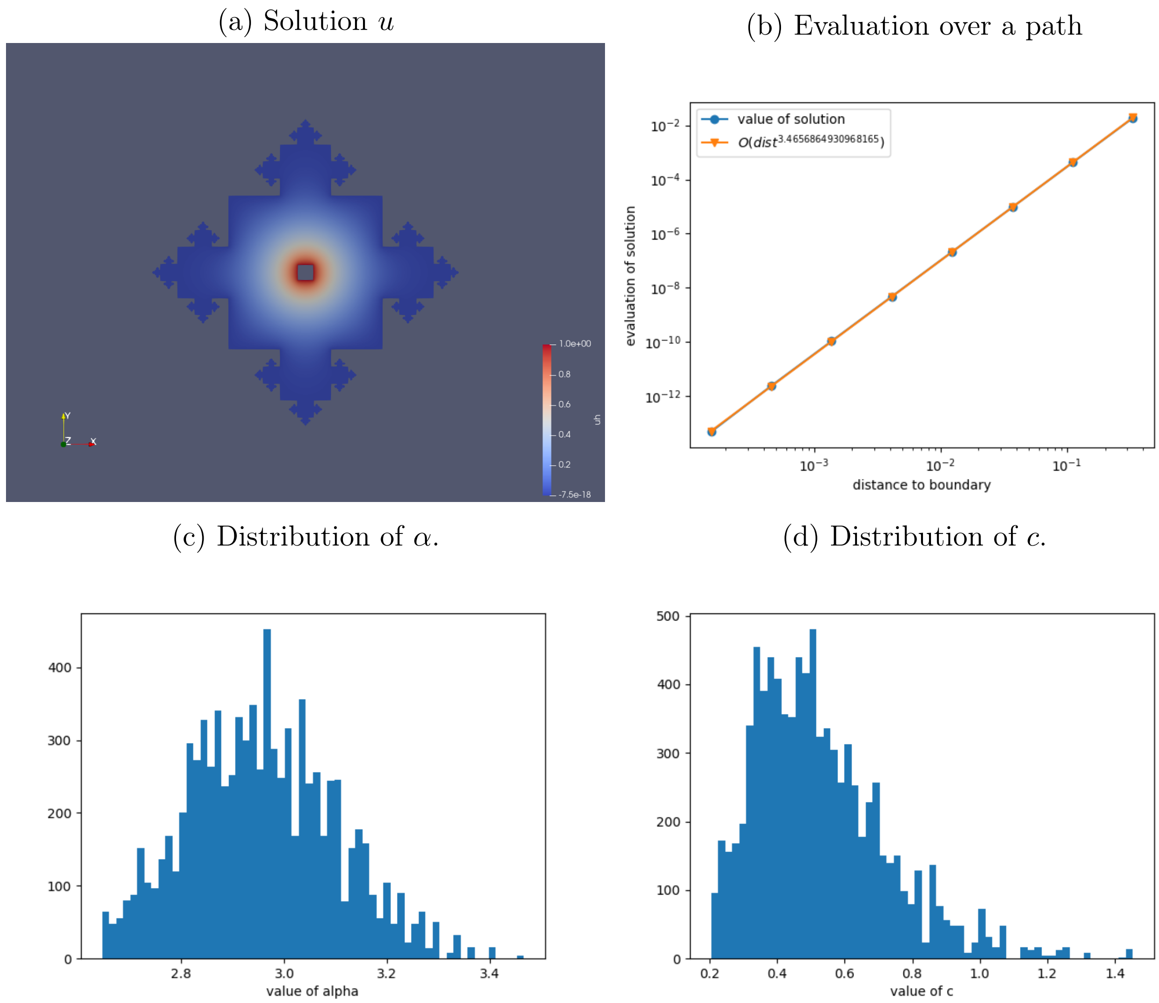
Figure 11: On unit square with $n = 8$ snowflake iterations. Result over all possible path. Total number of path is $4 * 3^{8-1} = 8748$ (a). distribution of α. (b). distribution of c.

2.1 Another example

Solver the PDE

$$\Delta u = 0 \quad \text{on } \Omega = Q \setminus Q_0$$

where Q the unit square with fractal boundary and $Q_0 = [0.45, 0.55] \times [0.45, 0.55]$. The boundary condition is $u = 0$ on ∂Q and $u = 1$ on ∂Q_0.



(a) Solution u

(b) Evaluation over a path

(c) Distribution of α.

(d) Distribution of c.

Figure 12: On unit square with $n = 8$ snowflake iterations. (a) solution of the PDE. (b) Evaluation over a path (c) Distribution of α. (d) Distribution of c.

2.2 On unit square

Code in fractal_boundary/Ex3_square_harmonic_v3.

(a) f (b) Solution u

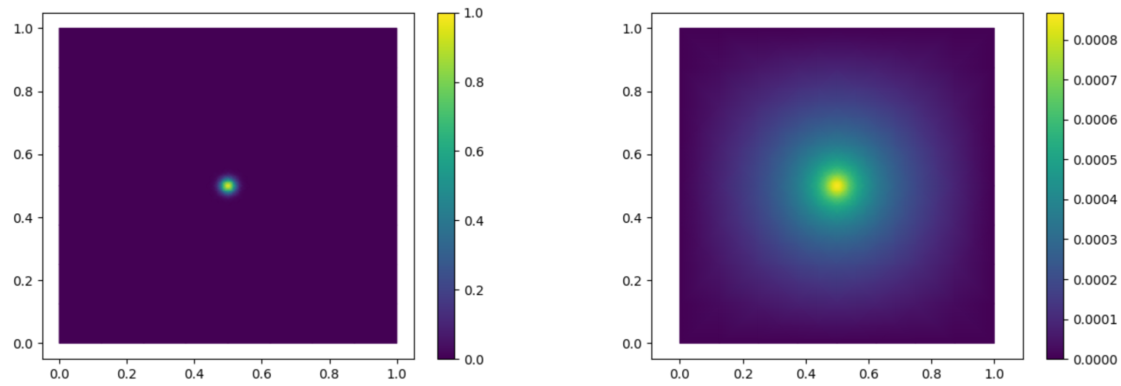


Figure 13: On unit square. (a). forcing term $f(x,y) = e^{-2000((x-0.5)^2+(y-0.5)^2)}$. (b) solution of the PDE.

(a) $N = 500$ (b) $N = 2000$

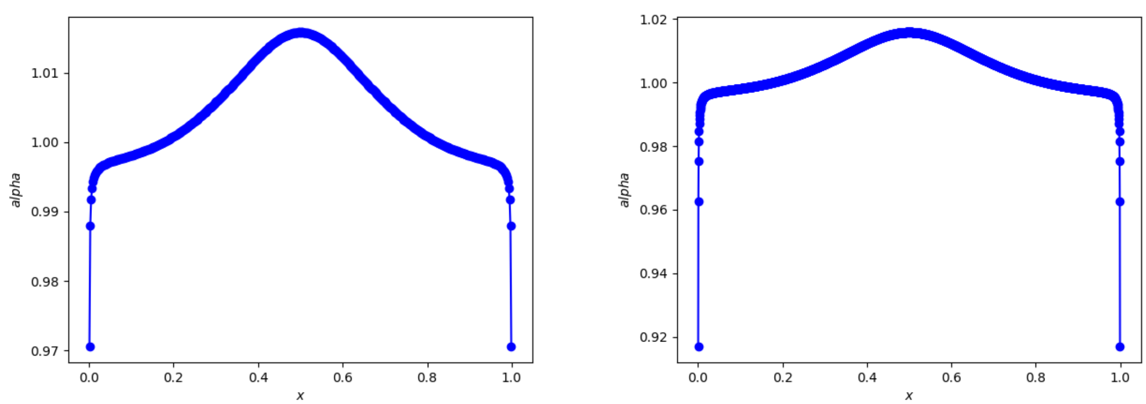


Figure 14: On unit square. Distribution of α w.r.t points on bottom boundary $[0,1]$. N is the number of segments on bottom edge. (a). $N = 500$. (b). $N = 2000$.

2.3 On square with snowflake

Code in fractal_boundary/Ex3_square_harmonic_adaptive.

(a) f

(b) Solution u

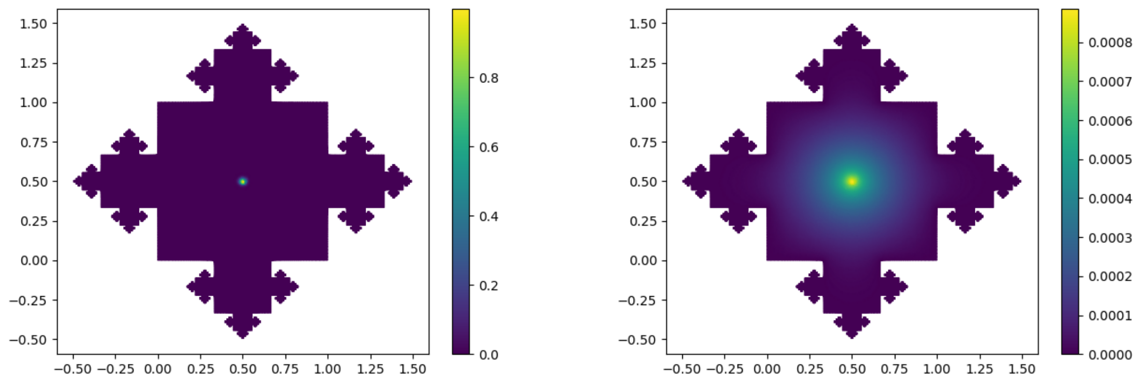


Figure 15: On unit square with $n = 4$ snowflake iterations. (a). forcing term $f(x, y) = e^{-2000((x-0.5)^2+(y-0.5)^2)}$. (b) solution of the PDE.

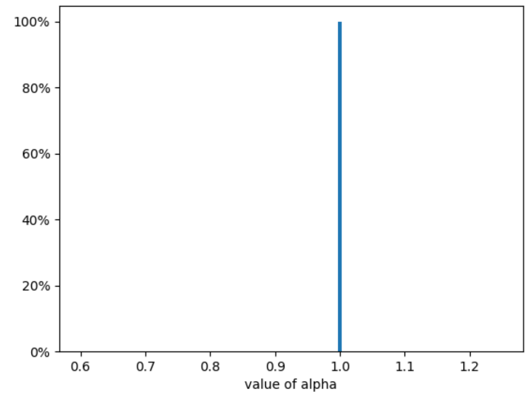
In the following the order α is estimated as:

1. Divide the each edge of bottom boundary into N pieces $[x_i, x_{i+1}]$.
2. Estimate the α_i at the middle point $\bar{x}_i = 0.5(x_i + x_{i+1})$.
3. Evaluate the solution at the points

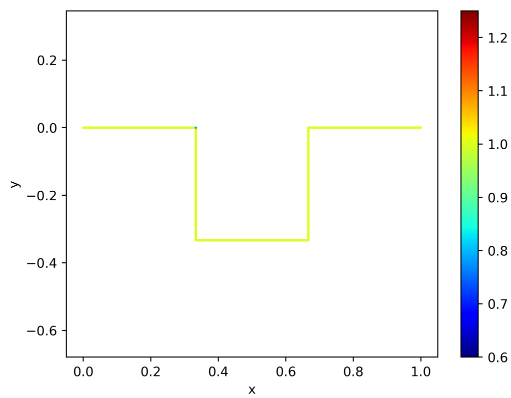
$$\tilde{x}_{i,j} = \bar{x}_i - \vec{n}\frac{1}{2}\left(\frac{1}{3}\right)^7\left(\frac{1}{2}\right)^j, \quad j = 1, 2, ..., 6.$$

   where $\vec{n}$ is the normal vector at boundary points $\bar{x}_i$, and $\frac{1}{2}\left(\frac{1}{3}\right)^7\left(\frac{1}{2}\right)^j$ guarantees that those points are inside the smallest (nth, when $n \leq 7$) koch squares.

(a) distribution of α



(b) α at bottom boundary



(c) linear regression std for the slope

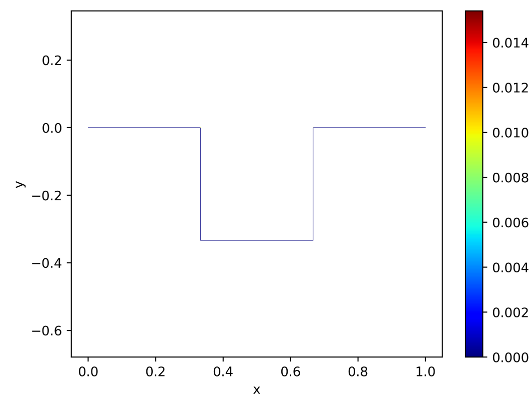
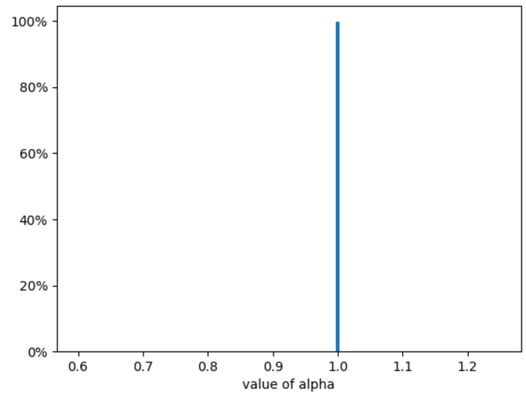
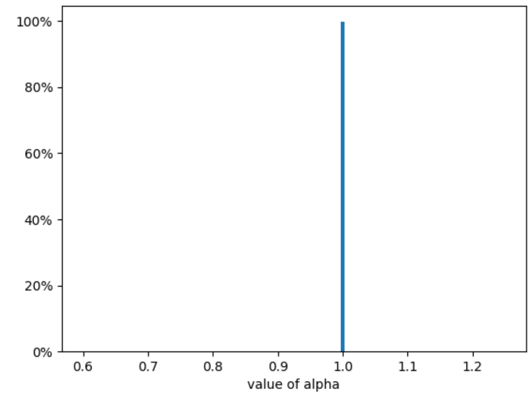


Figure 16: On unit square with $n = 1$ snowflake iterations. Distribution of α. The length of segments $l = \frac{1}{3^7}\frac{1}{32}$.

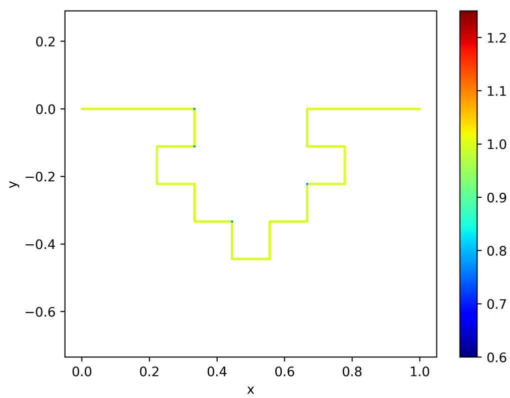
(a) distribution α at bottom



(b) distribution α for $y \leq -1/3$.



(c) α at bottom bondary



(d) linear regression std for the slope

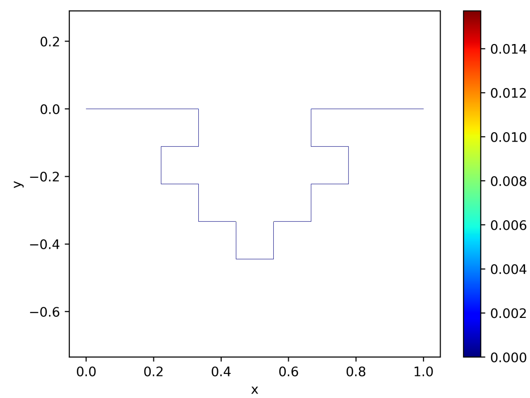
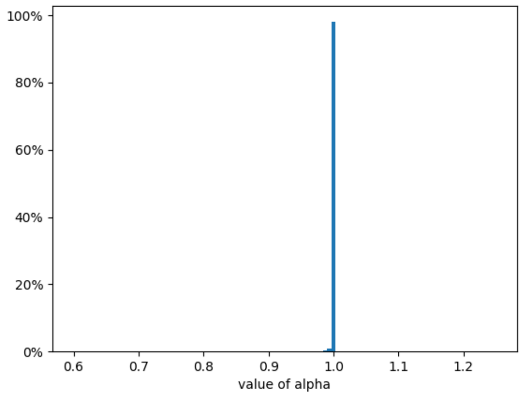
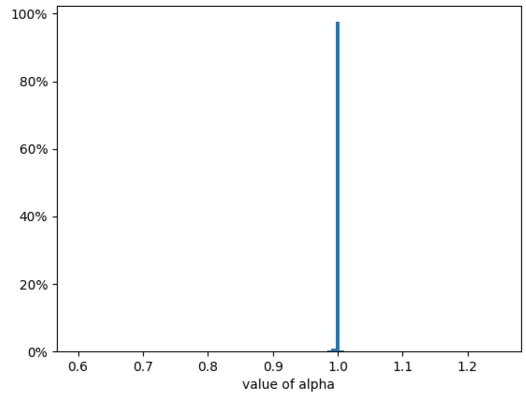


Figure 17: On unit square with $n = 2$ snowflake iterations. Distribution of α. The length of segments $l = \frac{1}{3^7}\frac{1}{32}$

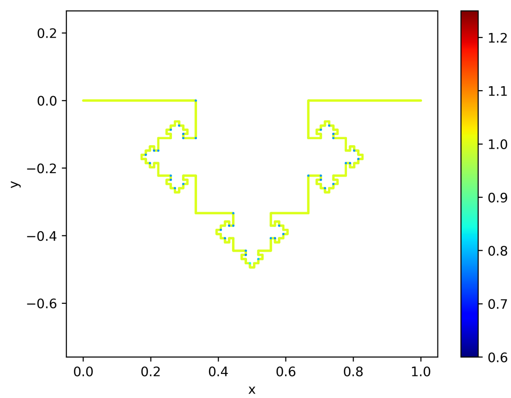
(a) distribution α at bottom



(b) distribution α for $\frac{2}{3} \leq x \leq \frac{7}{9}, y \leq -\frac{2}{9}$



(c) α at bottom boundary



(d) linear regression std for the slope

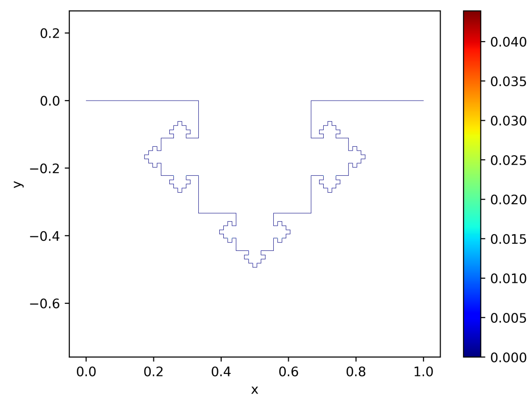


Figure 18: On unit square with $n = 4$ snowflake iterations. Distribution of α. The length of segments $l = \frac{1}{3^7}\frac{1}{32}$.

(a) distribution α, $n = 6$.

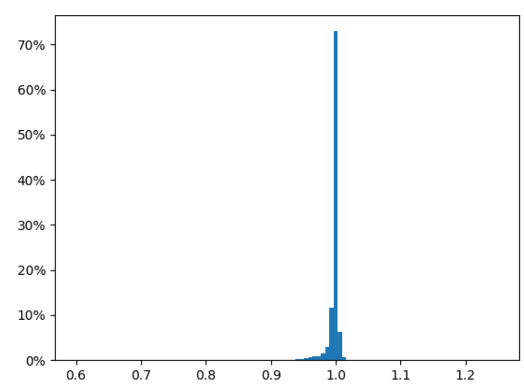


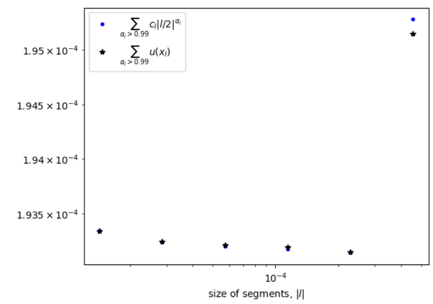
Figure 19: On unit square with $n = 6, 8$ snowflake iterations. Distribution of α. The length of segments $l = \frac{1}{3^7}\frac{1}{32}$.

In the following, we check that whether

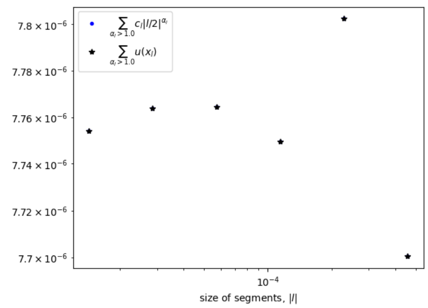
$$\omega\{l|\alpha_l > 1\} \approx \sum_{l:\alpha_l>1} u(x_{l/2}) \approx \sum_{l:\alpha_l>1} c_l|l/2|^{\alpha_l} \to 0$$

as $|l| \to 0$.

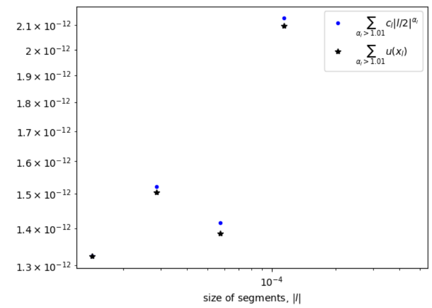
(a) $\alpha > 0.99$



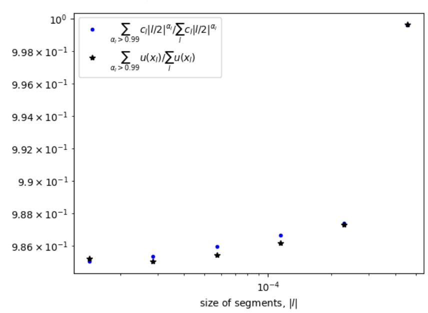
(b) $\alpha > 1$



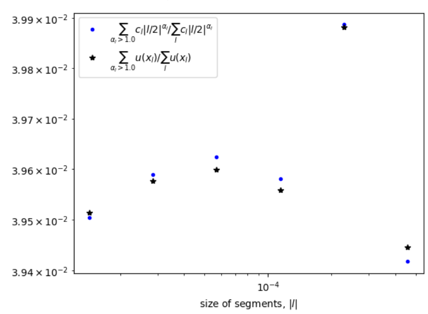
(c) $\alpha > 1.01$



(d) $\alpha > 0.99$



(e) $\alpha > 1$



(f) $\alpha > 1.01$

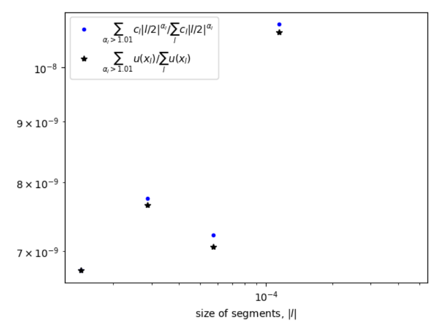


Figure 20: On unit square with $n = 4$ snowflake iterations. Estimation of harmonic measure of segments with $\alpha > 1$, i.e., $\omega\{l|\alpha_l > 1\}$. length of segments $l = \frac{1}{3^7}\frac{1}{2^i}$ where $i = [0, 1, 2, 3, 4, 5]$. script: hm.py

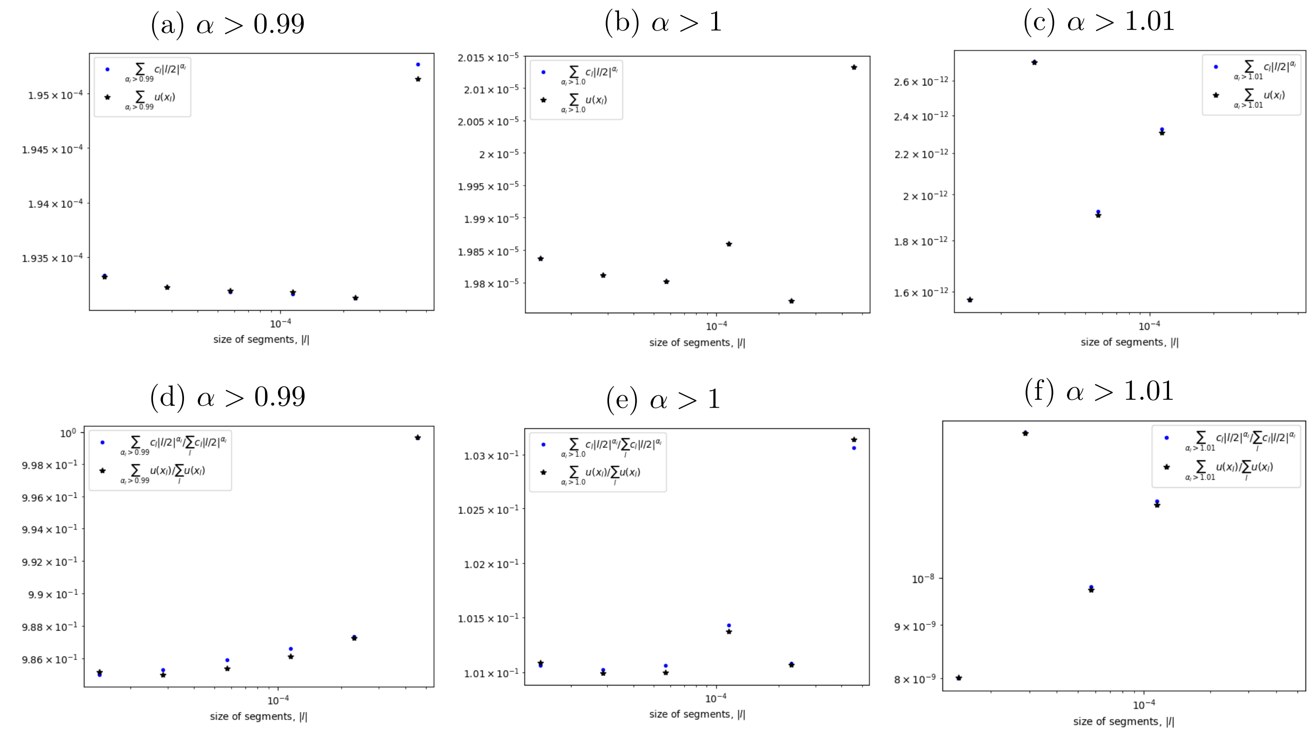


Figure 21: On unit square with $n = 6$ snowflake iterations. Estimation of harmonic measure of segments with $\alpha > 1$, i.e., $\omega\{l|\alpha_l > 1\}$. length of segments $l = \frac{1}{3^7}\frac{1}{2^i}$ where $i = [0, 1, 2, 3, 4, 5]$. script: hm.py

n	$\sum u(x_l)$	ratio	$\sum c_l\|l\|^{\alpha_l}$	ratio
1	1.96268100e-04		1.96294519e-04	
2	1.96244418e-04	0.99987933851	1.96278737e-04	0.9999196004
3	1.96240437e-04	0.99997971407	1.96276840e-04	0.99999033517
4	1.96238711e-04	0.99999120466	1.96275665e-04	0.99999401355
5	1.96237431e-04	0.99999347733	1.96274559e-04	0.99999436506
6	1.96237200e-04	0.99999882285	1.96274359e-04	0.99999898101

Table 1: On unit square with n snowflake iterations. Estimated harmonic measure of bottom boundary. length of segments $l = \frac{1}{3^7}\frac{1}{32}$, script:hm_all.py.