

Notes on fractal harmonic measure problem

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1 Result on unit disk

In this section, we consider the domain Ω is a unit disk (see Fig. 1).

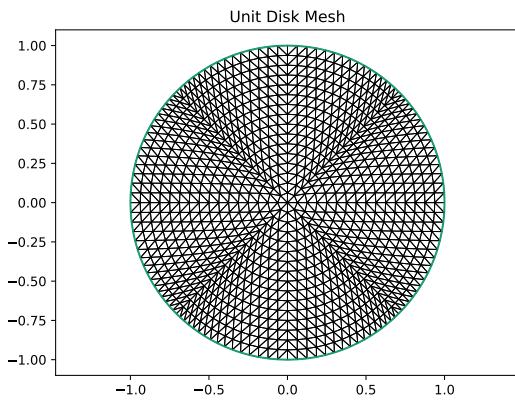


Figure 1: Unit disk.

We first test the firedrake solver on the PDE:

$$-\Delta u = f \quad \text{on } \Omega \tag{1.1}$$

with $u = g$ on $\partial\Omega$.

In Fig. 2 and Fig. 3, we plot the error in L^2 and H^1 norm with respect to mesh size h and degree of freedom. In Fig. 2, the manufactured solution is $u(x, y) = 2 + x + 3y$. In Fig. 3, the manufactured solution is $u(x, y, z) = 2 + x^2 + y$. The Lagrange linear element is used. The uniform refinement is done by built-in function `MeshHierarchy`.

(a) Error vs mesh size



(b) Error vs degree of freedom

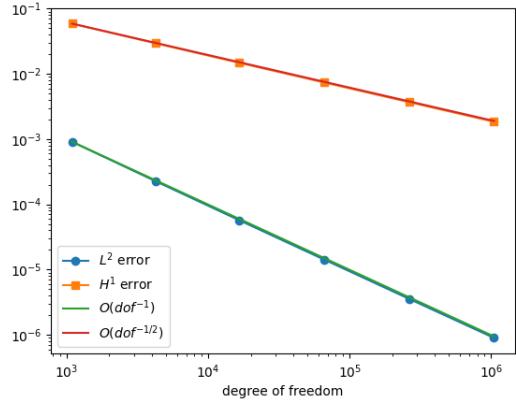


Figure 2: Unit disk: manufactured solution $u = x + 3y + 2$. Note that only round off error appears here since the solution is in the linear finite element space. Remark: Here, we use the Krylov subspace method with BoomerAMG in preconditioner in the solver. If we change the solver to be LU factorisation, the error here would be around $10^{-14} \sim 10^{-15}$.

(a) Error vs mesh size



(b) Error vs degree of freedom

Figure 3: Unit disk: manufactured solution $u = x^2 + y + 2$.

Next, we solve the PDE

$$-\Delta u = f \quad \text{on } \Omega \quad (1.2)$$

with $u = 0$ on the unit disk and evaluate the solution at points from center to boundary. In particular, we evaluate the solutions at

$$\mathbf{x}_i = (0, 1 - 1/2^i), \quad 0 \leq i \leq n_{max}. \quad (1.3)$$

In Fig. 4, we show the results of the PDE (1.2) with a different f . That is, we define f as

$$f(x, y) = e^{-20(x^2+y^2)}.$$



Figure 4: On unit disk. (a). forcing term f . (b) solution u . (c). Evaluation of solution at a sequence points $\{\mathbf{x}_i\}_{i=1}^{15}$ where \mathbf{x}_i is defined as in (1.3). Notice that the slope is almost equal to $\frac{\partial u}{\partial n} \approx 0.025$.

$$\begin{aligned} - \int_{\partial\Omega} \frac{\partial u}{\partial n} &= \int_{\Omega} f = 2\pi \int_0^1 e^{-20r^2} r dr = \frac{-\pi}{20}(e^{-20} - 1) \\ \Rightarrow \frac{\partial u}{\partial n} &= \frac{1}{40}(e^{-20} - 1) \approx -0.025 \end{aligned}$$

2 On snowflake boundary domain

In this note we solve the PDE

$$-\Delta u = f \quad \text{on } \Omega \tag{2.1}$$

with $u = 0$ on $\partial\Omega$. Here, Ω is a unit square where each edges are replaced by the Koch snowflake (See Fig. 5). In this note, we use the FEM python package firedrake solving the PDE.

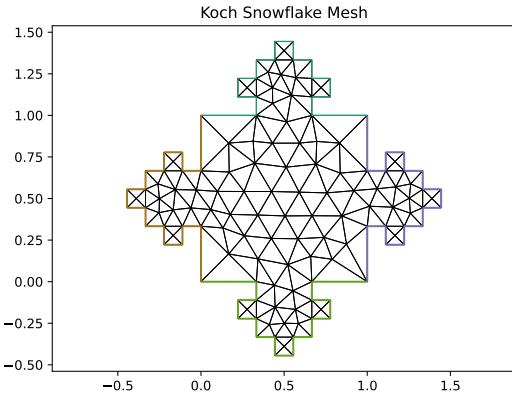
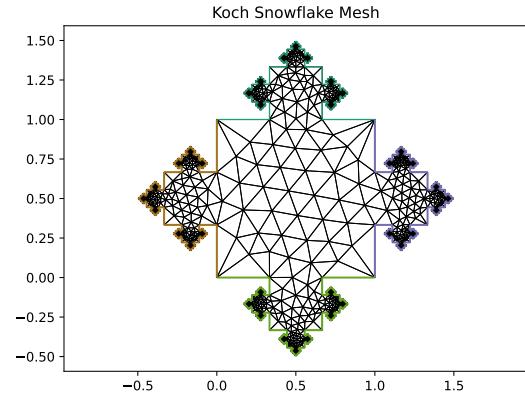
(a) $n = 2$ (b) $n = 4$ 

Figure 5: Unit square with each edges replaced by a Koch snowflake with n iterations. Meshsize is 0.5 for each vertices in .geo file.

In Fig. 6 and Fig. 7, we plot the error in L^2 and H^1 norm with respect to mesh size h and degree of freedom where domain is unit square with snowflake ($n = 4$ iterations). In Fig. 6, the manufactured solution is $u(x, y) = 2 + x + 3y$. In Fig. 7, the manufactured solution is $u(x, y, z) = 2 + x^2 + y$. The Lagrange linear element is used. The uniform refinement is done by built-in function MeshHierarchy.

(a) Error vs mesh size

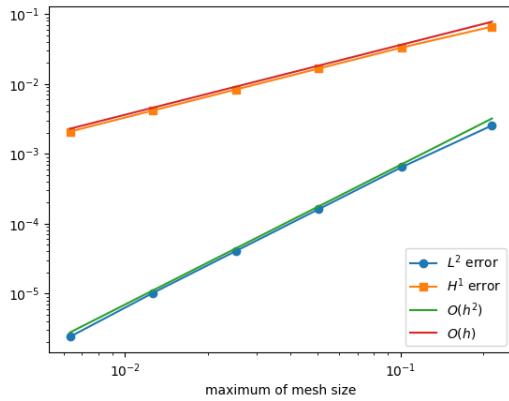


(b) Error vs degree of freedom

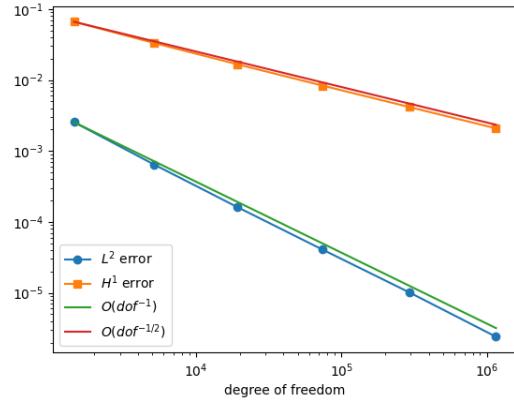


Figure 6: Unit square with $n = 8$ snowflake iterations: manufactured solution $u = x + 3y + 2$. Note that only round off error appears here since the solution is in the linear finite element space.

(a) Error vs mesh size

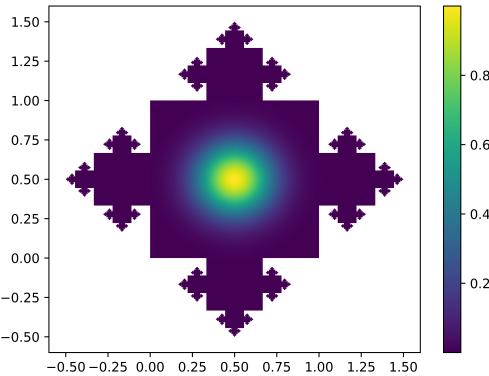
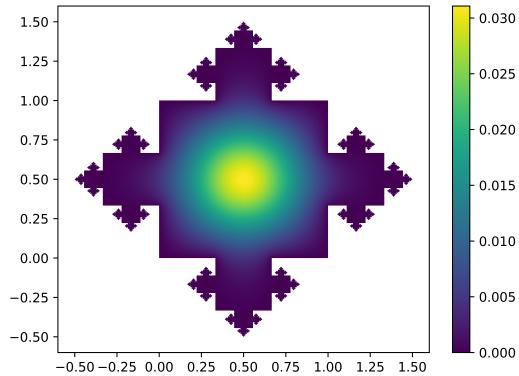


(b) Error vs degree of freedom

Figure 7: Unit square with $n = 8$ snowflake iterations: manufactured solution $u = x^2 + y + 2$.

In Fig. 8(b), we plot the solution of the PDE (2.1) where Ω is the square with 4 snowflake iterations as in Fig. 5(b) and f is defined as

$$f(x, y) = e^{-20((x-0.5)^2 + (y-0.5)^2)}.$$

(a) f (b) Solution u Figure 8: On unit square with $n = 8$ snowflake iterations. (a). forcing term f . (b) solution of the PDE (2.1).

In Fig. 10, we evaluate the solution along a random path with each points $\mathbf{x}_i, 0 \leq i \leq n$ is a center of the square from i -th snowflake iteration. In particular, we create the sequence of the points along the path as followings:

- 1). Set $\mathbf{x}_0 = (0.5, 0.5)$
- 2). Pick a random integer from 0 to 3. Each integer corresponds to one of four normal directions \vec{v}_1 . (upward, downward, left, right).
- 3). Set $\mathbf{x}_1 = \mathbf{x}_0 + h_0 \vec{v}_1$ with $h_0 = \frac{2}{3}$.

- 4). For $i = 2, \dots, n$, Pick a random integer from 0 to 2. Each integer corresponds to one of three normal directions \vec{v}_i (forward, left, right). Set

$$\mathbf{x}_i = \mathbf{x}_{i-1} + h_{i-1} \vec{v}_i.$$

where $h_{i-1} = \frac{2}{3^i}$.



Figure 9: Two random path on the unit square with $n = 8$ snowflake iterations.

Let $dx_i = (1/3)^i$. Then we would like to approximate $u(\mathbf{x}_i)$ as a function of dx_i in the form

$$u(\mathbf{x}_i) = c(dx_i)^\alpha$$

By taking the log on both ends,

$$\log u(\mathbf{x}_i) = \log c + \alpha \log(dx_i), \quad 0 \leq i \leq n.$$

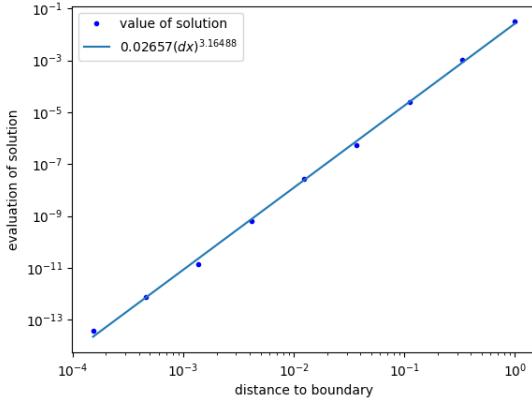
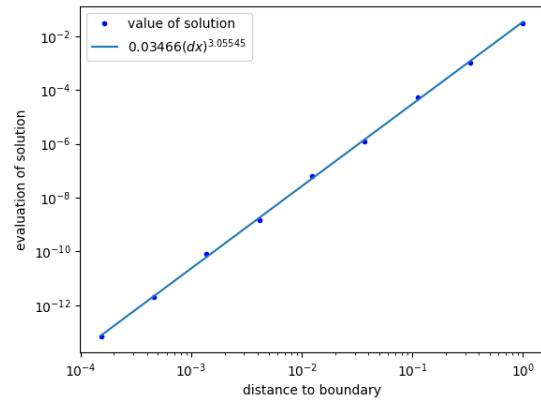
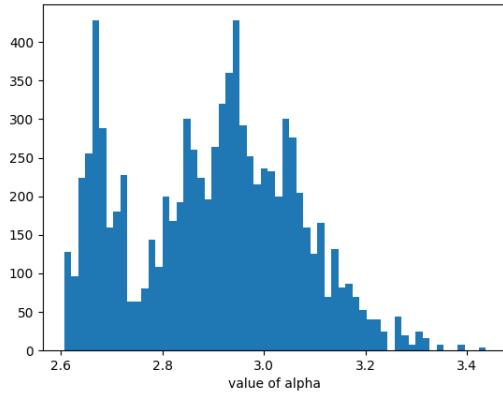
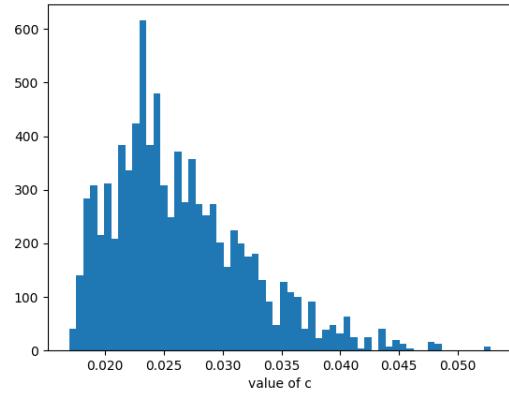
The coefficients c and α are found by least squares approximation.

Running over 500 random path for two times, we get:

| | Mean | Standard deviation |
|----------|---------|--------------------|
| c | 0.02598 | 0.005612 |
| α | 2.89311 | 0.16436 |

and

| | Mean | Standard deviation |
|----------|---------|--------------------|
| c | 0.02567 | 0.005760 |
| α | 2.90044 | 0.16815 |

(a) $c = 0.02657, \alpha = 3.16488$ (b) $c = 0.03466, \alpha = 3.05545$ Figure 10: On unit square with $n = 8$ snowflake iterations. Result of different random path(a) distribution of α (b) distribution of c .Figure 11: On unit square with $n = 8$ snowflake iterations. Result over all possible path. Total number of path is $4 * 3^{8-1} = 8748$ (a). distribution of α . (b). distribution of c .

2.1 On unit square

Code in fractal_boundary/Ex3_square_harmonic_v3.

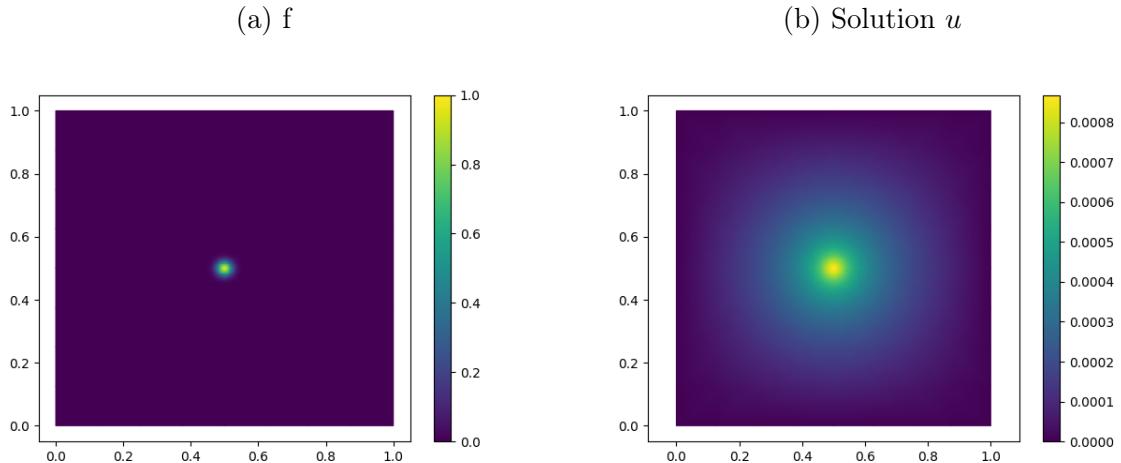


Figure 12: On unit square. (a). forcing term $f(x, y) = e^{-2000((x-0.5)^2 + (y-0.5)^2)}$. (b) solution of the PDE.

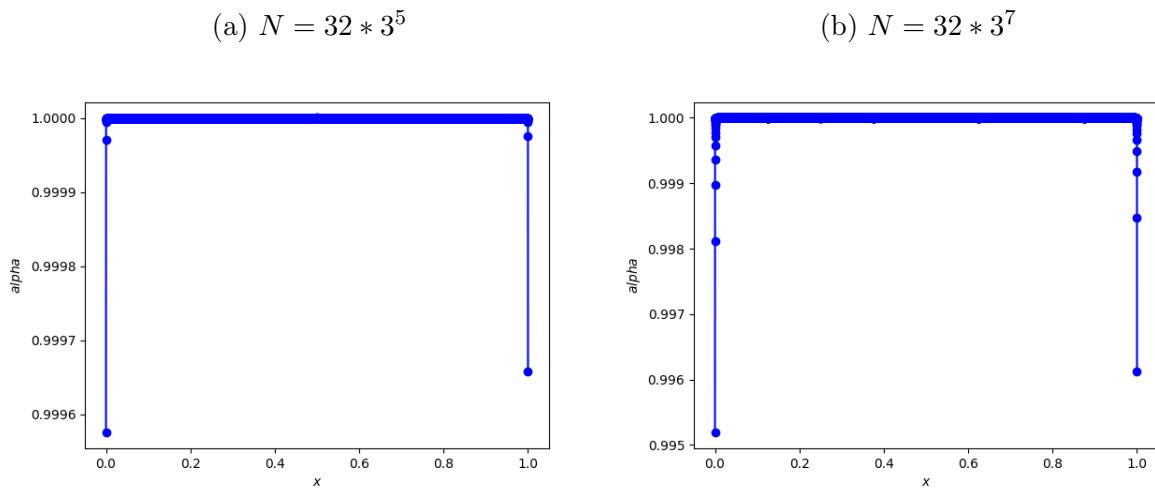


Figure 13: On unit square. Distribution of α w.r.t points on bottom boundary $[0, 1]$. N is the number of segments on bottom edge. Point for regression with distance to boundary $\frac{1}{2} \left(\frac{1}{3}\right)^7 \left(\frac{1}{2}\right)^j, j = 1, 2, \dots, 6$.

2.2 On square with snowflake

Code in fractal_boundary/Ex3_square_harmonic_adaptive.

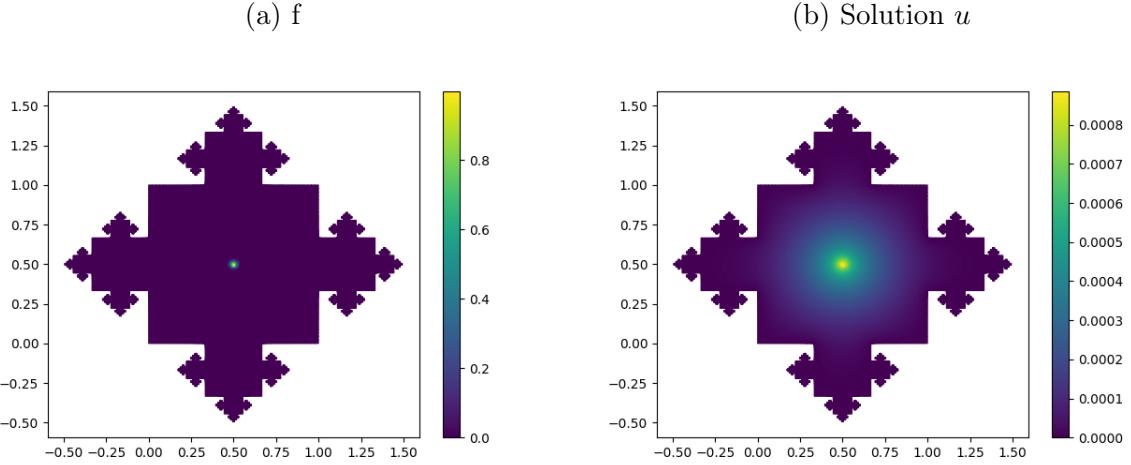


Figure 14: On unit square with $n = 4$ snowflake iterations. (a). forcing term $f(x, y) = e^{-2000((x-0.5)^2+(y-0.5)^2)}$. (b) solution of the PDE.

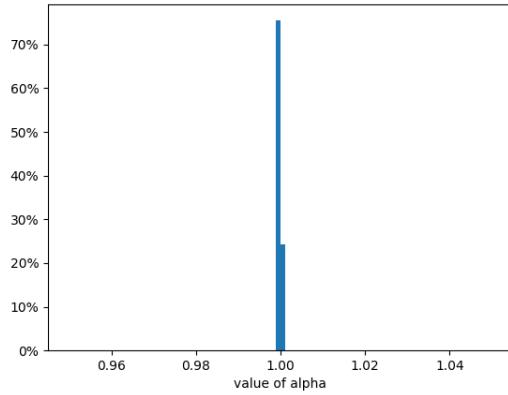
In the following the order α is estimated as:

1. Divide the each edge of bottom boundary into N pieces $[x_i, x_{i+1}]$.
2. Estimate the α_i at the middle point $\bar{x}_i = 0.5(x_i + x_{i+1})$.
3. Evaluate the solution at the points

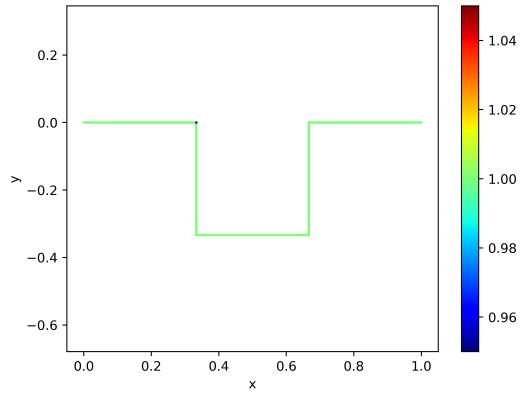
$$\tilde{x}_{i,j} = \bar{x}_i - \vec{n} \frac{1}{2} \left(\frac{1}{3}\right)^7 \left(\frac{1}{2}\right)^j, \quad j = 1, 2, \dots, 6.$$

where \vec{n} is the normal vector at boundary points \bar{x}_i , and $\frac{1}{2} \left(\frac{1}{3}\right)^7 \left(\frac{1}{2}\right)^j$ guarantees that those points are inside the smallest (n th, when $n \leq 7$) koch squares.

(a) distribution of α



(b) α at bottom boundary



(c) linear regression std for the slope

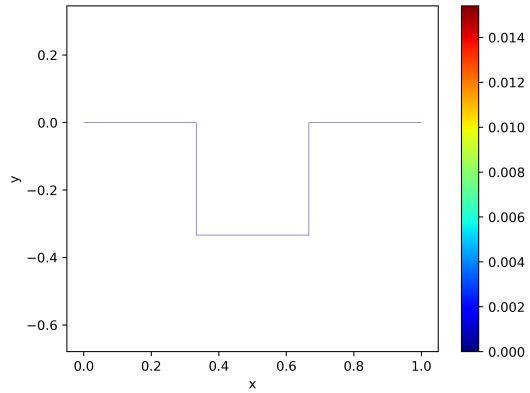
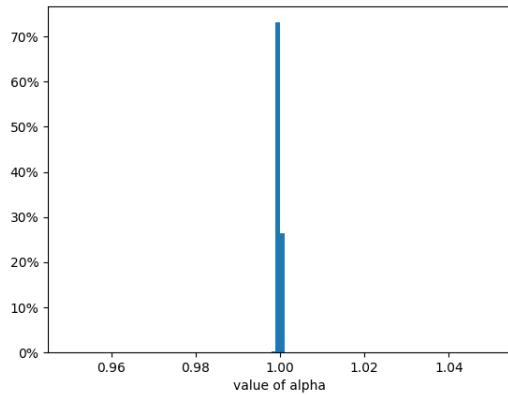
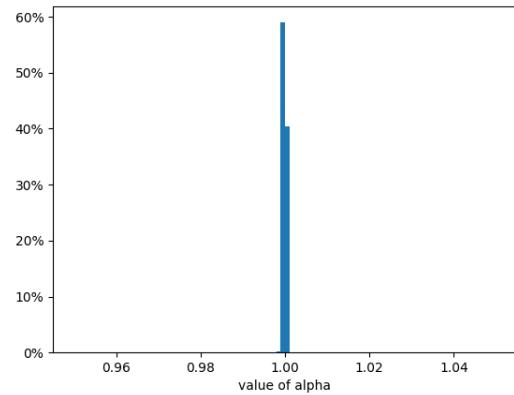


Figure 15: On unit square with $n = 1$ snowflake iterations. Distribution of α . The length of segments $l = \frac{1}{3^7} \frac{1}{32}$.

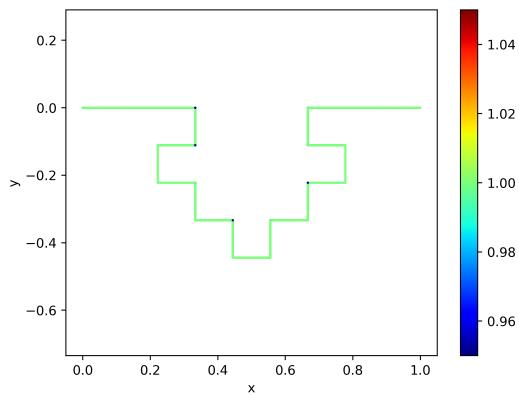
(a) distribution α at bottom



(b) distribution α for $y \leq -1/3$.



(c) α at bottom boundary



(d) linear regression std for the slope

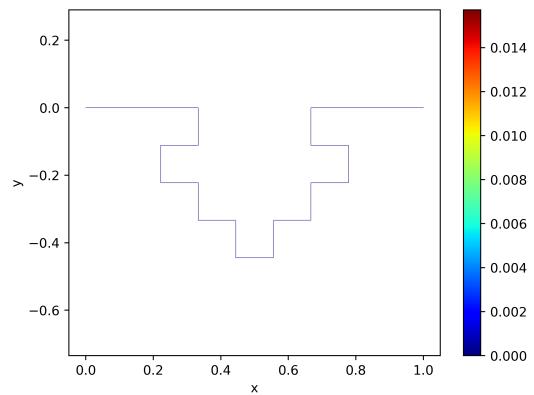
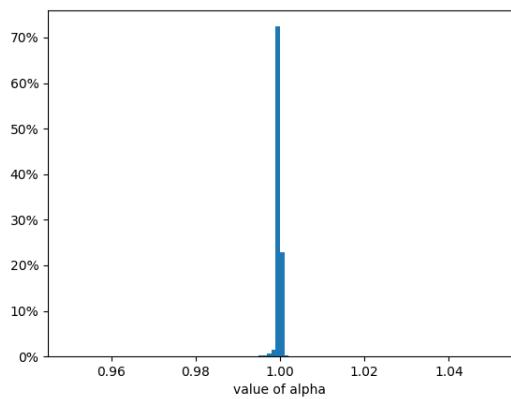
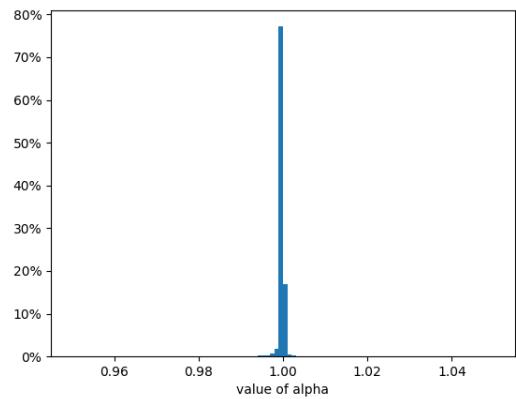
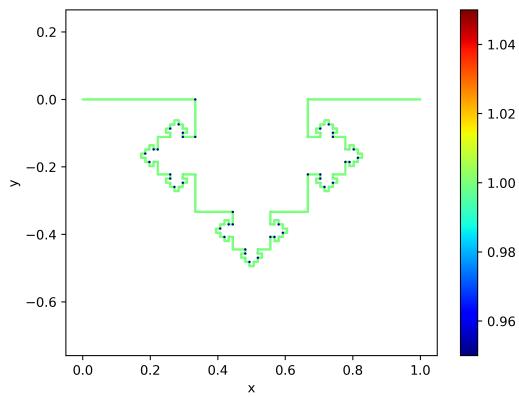


Figure 16: On unit square with $n = 2$ snowflake iterations. Distribution of α . The length of segments $l = \frac{1}{3^7} \frac{1}{32}$

(a) distribution α at bottom(b) distribution α for $\frac{2}{3} \leq x \leq \frac{7}{9}, y \leq -\frac{2}{9}$ (c) α at bottom boundary

(d) linear regression std for the slope

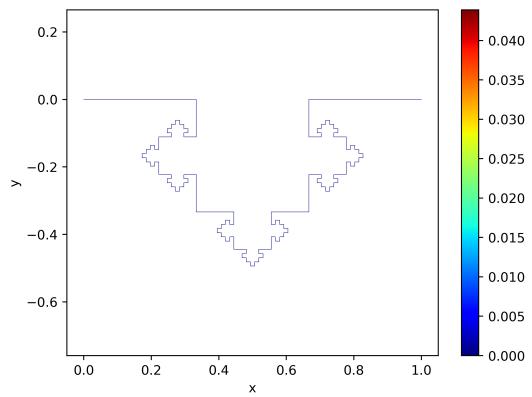


Figure 17: On unit square with $n = 4$ snowflake iterations. Distribution of α . The length of segments $l = \frac{1}{3^7} \frac{1}{32}$.

- (a) The length of segments $l = \frac{1}{3^7} \frac{1}{32}$. Point for regression with distance to boundary
 $\frac{1}{2} \left(\frac{1}{3}\right)^7 \left(\frac{1}{2}\right)^j, \quad j = 1, 2, \dots, 6.$
- (b) The length of segments $l = \frac{1}{3^8} \frac{1}{32}$. Point for regression with distance to boundary
 $\frac{1}{2} \left(\frac{1}{3}\right)^8 \left(\frac{1}{2}\right)^j, \quad j = 1, 2, \dots, 6.$

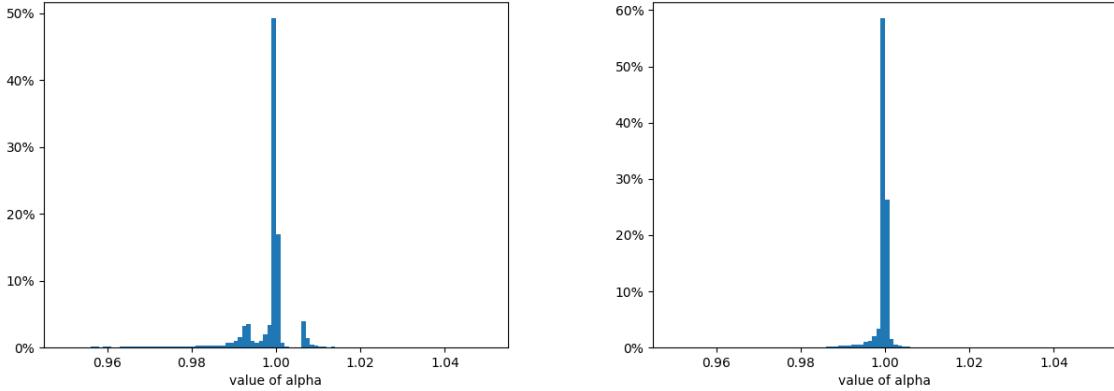


Figure 18: On unit square with $n = 6$ snowflake iterations. Distribution of α .

In the following, we check that whether

$$\omega\{l|\alpha_l > 1\} \approx \sum_{l:\alpha_l > 1} u(x_{l/2}) \approx \sum_{l:\alpha_l > 1} c_l |l/2|^{\alpha_l} \rightarrow 0$$

as $|l| \rightarrow 0$.

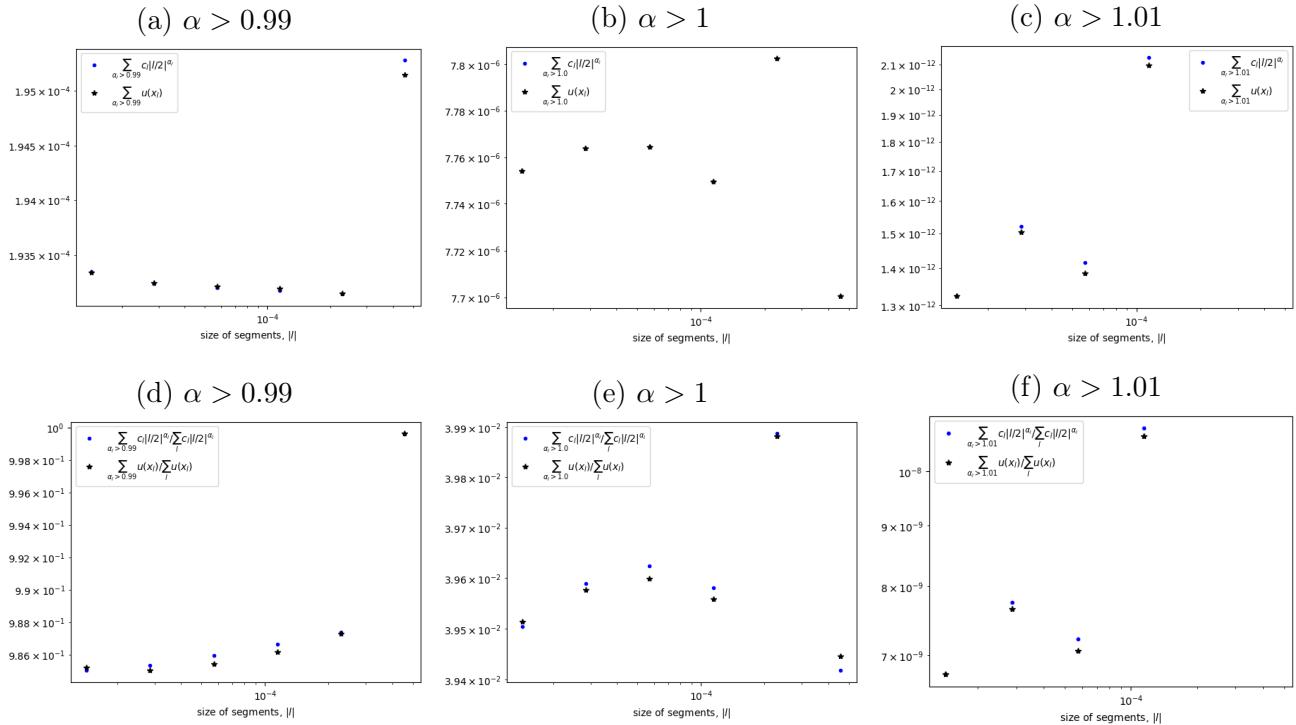


Figure 19: On unit square with $n = 4$ snowflake iterations. Estimation of harmonic measure of segments with $\alpha > 1$, i.e., $\omega\{l|\alpha_l > 1\}$. length of segments $l = \frac{1}{3^7} \frac{1}{2^i}$ where $i = [0, 1, 2, 3, 4, 5]$. script: hm.py

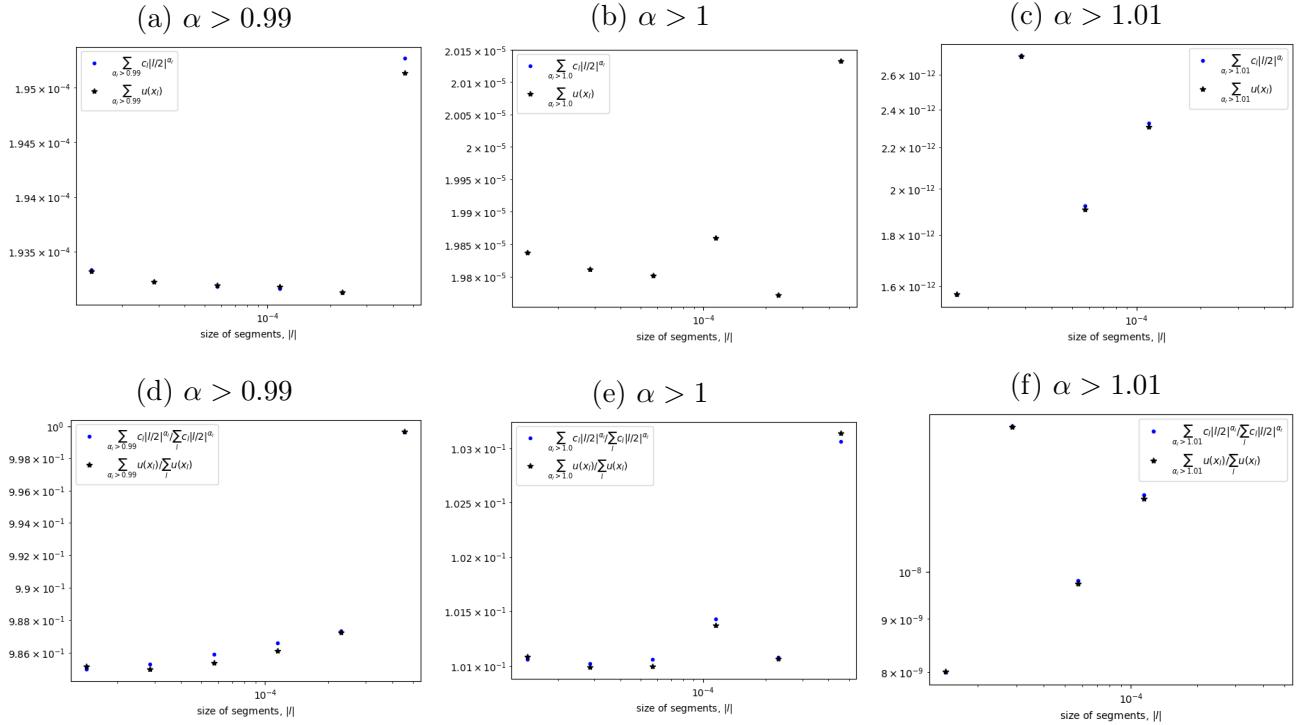


Figure 20: On unit square with $n = 6$ snowflake iterations. Estimation of harmonic measure of segments with $\alpha > 1$, i.e., $\omega\{l|\alpha_l > 1\}$. length of segments $l = \frac{1}{3^7} \frac{1}{2^i}$ where $i = [0, 1, 2, 3, 4, 5]$. script: hm.py

| n | $\sum u(x_l)$ | ratio | $\sum c_l l ^{\alpha_l}$ | ratio |
|---|----------------|---------------|---------------------------|---------------|
| 1 | 1.96268100e-04 | | 1.96294519e-04 | |
| 2 | 1.96244418e-04 | 0.99987933851 | 1.96278737e-04 | 0.9999196004 |
| 3 | 1.96240437e-04 | 0.99997971407 | 1.96276840e-04 | 0.99999033517 |
| 4 | 1.96238711e-04 | 0.99999120466 | 1.96275665e-04 | 0.99999401355 |
| 5 | 1.96237431e-04 | 0.99999347733 | 1.96274559e-04 | 0.99999436506 |
| 6 | 1.96237200e-04 | 0.99999882285 | 1.96274359e-04 | 0.99999898101 |

Table 1: On unit square with n snowflake iterations. Estimated harmonic measure of bottom boundary. length of segments $l = \frac{1}{3^7} \frac{1}{32^j}$, Point for regression with distance to boundary $\frac{1}{2} (\frac{1}{3})^7 (\frac{1}{2})^j$, $j = 1, 2, \dots, 6$. script:hm_all.py.