Report on -div A grad

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1 Problem setting

Let $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a symmetric and uniformly elliptic field in the sense of

$$e \cdot A(x)e \geq \frac{1}{C_0}|e|^2, \quad |A(x)e| \leq |x|, \quad \text{for all } x, e \in \mathbb{R}^d.$$

In [1], they consider the acoustic operator

$$H_a := -\nabla \cdot A\nabla, \quad \text{on } L^2(\mathbb{R}^d).$$

where A is periodic, quasiperiodic or random.

What have been already done if the coefficient field A is random:

1. In Theorem 1.2 of [4], it was proved that when $d = 1$ and

$$A(x) = \frac{1}{1 + \sum_{n\in Z} f(x - n - d_n(\omega))}$$

where $\mathrm{supp}(f) \subset [-s, s], f \in L^1, 1 + f > 0, f \neq 0$ and the displacements $\{d_n(\omega)\}$ are i.i.d random variables taking values in $[-d_{max}, d_{max}]$ with $d_{max} + s < 1/2$ then the operator H_a almost surely has dense pure point spectrum with exponentially decaying eigenfunctions.

2. In [5], they consider the case that the coefficient A is a random perturbation of a periodic function A_0. They showed that the random operator H_a exhibits Anderson localization inside the gap in the spectrum of H_{a_0}.

3. In [1], the authors provides a strong control on the spatial spreading of the mass density of eigenstates in the lower spectrum. In particular, they proved a lower bound on the space width or localization length of an eigenstate ψ:

$$\ell_\theta(\psi) := \inf\left\{r \geq 0 : \|\psi\|_{\mathrm{L}^2(B_r)} \geq (1-\theta)\|\psi\|_{\mathrm{L}^2\left(\mathbb{R}^d\right)}\right\}, \quad 0 < \theta < 1/2.$$

With some assumptions on the random field, they showed that given $0 < \theta < 1/2$ and $\varepsilon > 0$, there exists a positive random variable $\lambda_{\theta,\varepsilon}$ such that: if H_a has an eigenvalue $\lambda \leq \lambda_{\theta,\varepsilon}$, any associated eigenstate ψ_λ satisfies

$$\ell_\theta(\psi_\lambda) \geq \begin{cases} \lambda^{\varepsilon - \frac{2}{3}} & : d = 1 \\ \lambda^{\varepsilon - \frac{1}{2}\left(\left\lfloor \frac{d}{2}\right\rfloor + 1\right)} & : d > 1 \end{cases}.$$

Some open problems raised from those forwarded emails :

1. Can one prove that there is always an interval of delocalization in 2D?

2. Is 3D indeed more delocalising?

2 Numerical experiment

2.1 1d random displacement model in [4]

The codes is under the folder:grad_A_div/1d

Consider solving

$$-(Au')' = \lambda u$$

on a closed interval $[x_0, x_1]$. Let $s = 1/4$ and define

$$f(x) = 1/8[\max\{(1-x^2/s^2)^3, 0\}(3x^2+1)]' = \begin{cases} \frac{6}{8}x(1-x^2/s^2)^3 - \frac{6}{8s^2}x(1-x^2/s^2)^2(3x^2+1) & x \in [-s, s] \\ 0, & x \notin [-s, s] \end{cases}$$

Then $\mathrm{supp}(f) \subset [-s, s], f \in C^1, 1 + f > 0$. See Fig.1 for the f.

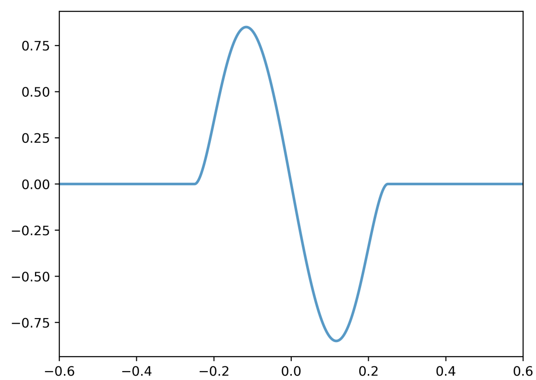

Figure 1: function f. code in grad_A_div/1d/test.py.

Let

$$A(x) = \frac{1}{1 + \sum_{n \in Z, x_0+1 \le n \le x_1-1} f(x - n - d_n(\omega))}$$

where the displacements $\{d_n(\omega)\}$ are i.i.d random variables taking values in $[-d_{max}, d_{max}]$ with $d_{max} = 1/5$. See Fig. 2 for one example.

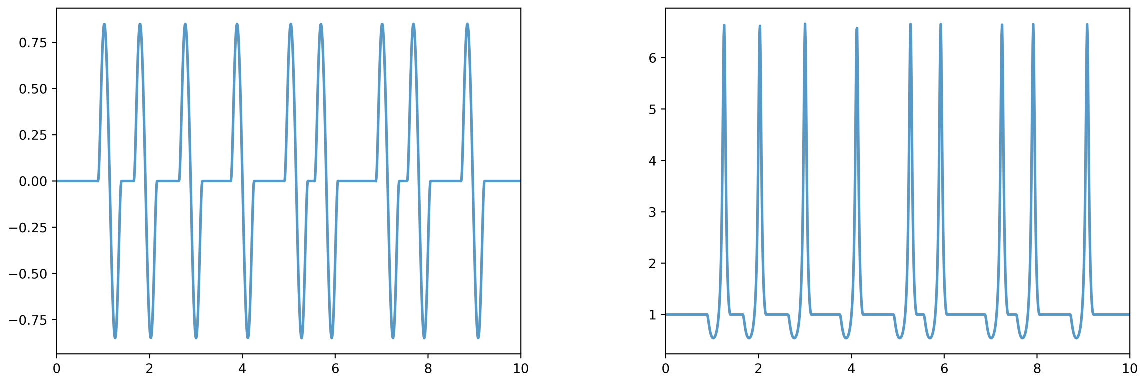

Figure 2: $x_0 = 0, x_1 = 10$. Left:$\sum_{n \in Z, x_0+1 \leq n \leq x_1-1} f(x - n - d_n(\omega))$. Right: $A(x)$. code in grad_A_div/1d/test.py.

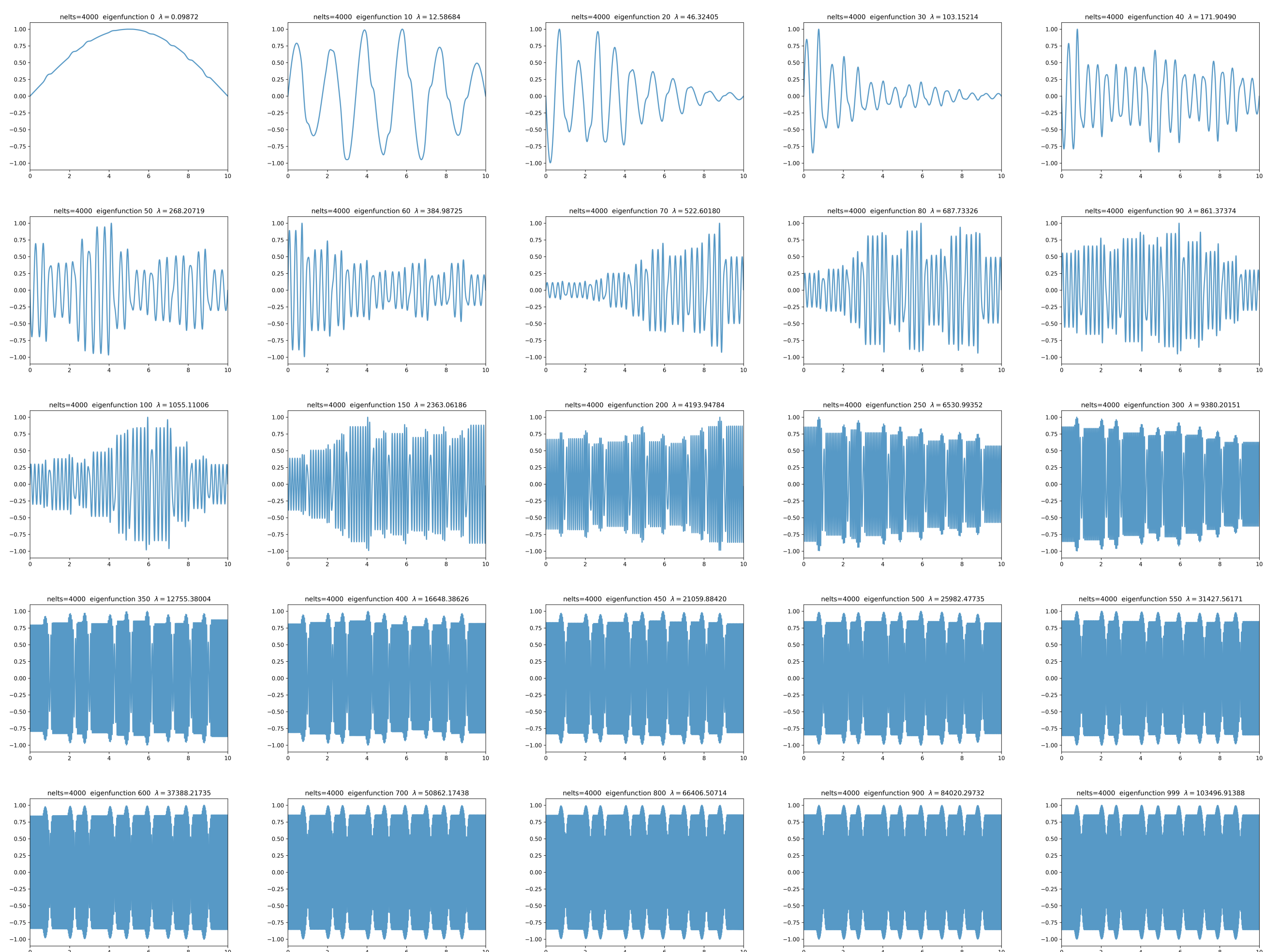

Figure 3: Over $[0, 10]$, Dirichlet, $d_{max} = 0.2$.

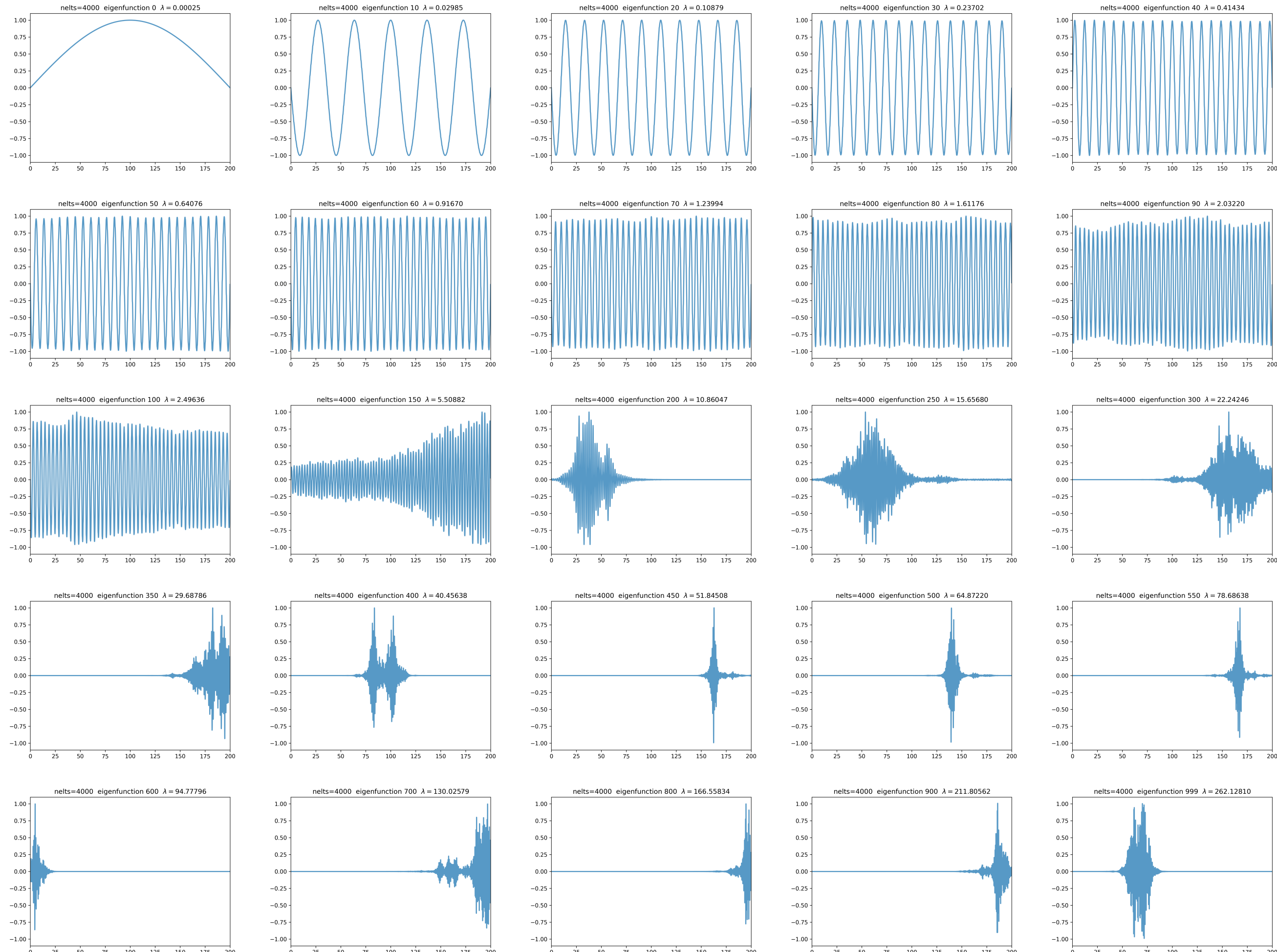

Figure 4: Over $[0, 200]$, Dirichlet, $d_{max} = 0.2$.

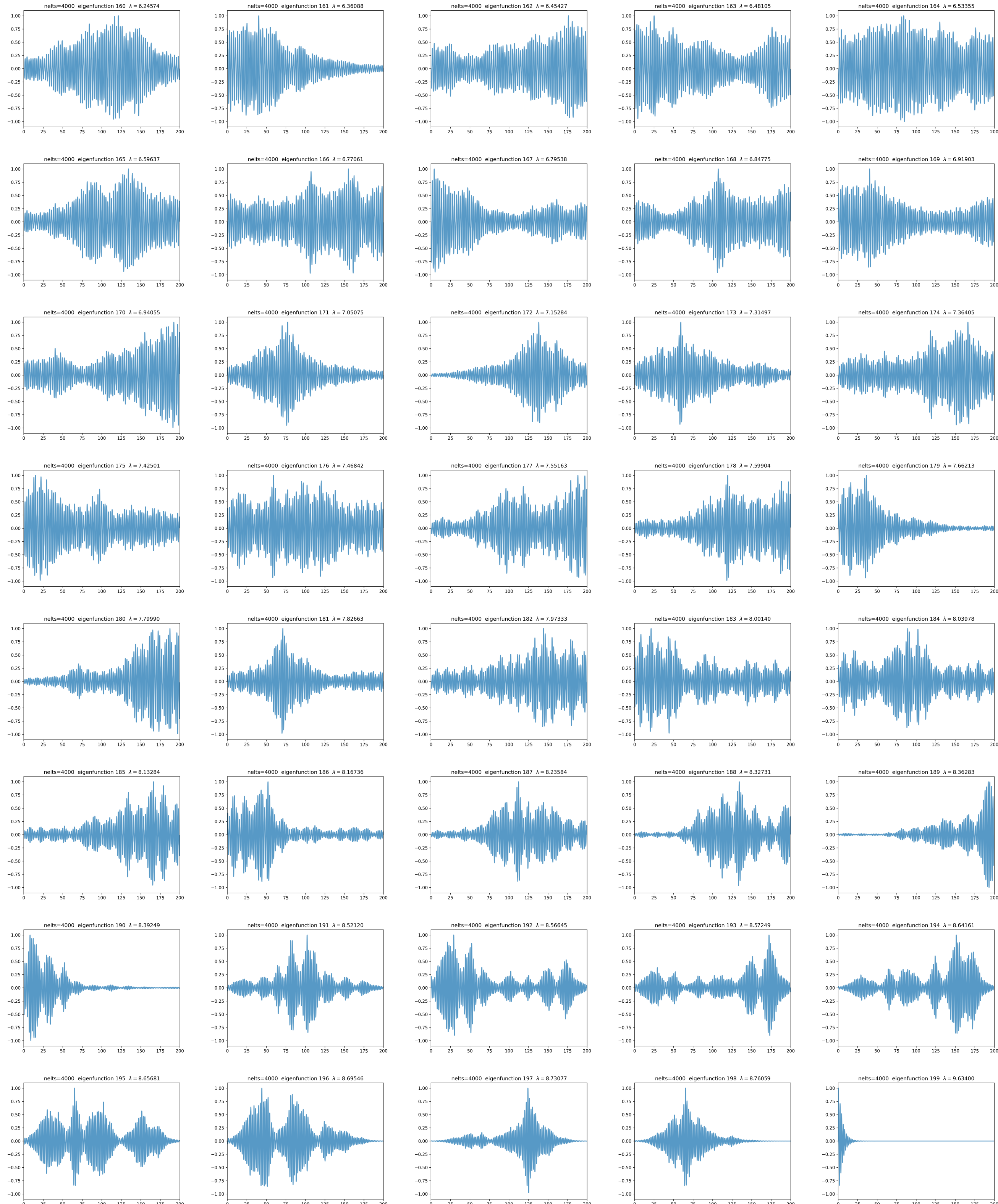

Figure 5: Over $[0, 200]$, Dirichlet, $d_{max} = 0.2$. modes 160 to 200, transition to localization

2.2 Periodic coefficients

1. In literature, it is a conjecture that the spectrum of the elliptic operator $-\text{div}(A\nabla)$ with periodic and sufficiently smooth coefficients must be absolutely continuous.

2. In [7], Filonov constructed an explicit example of A such that the operator admits a nontrivial and compactly-supported eigenfunction: For $d \geq 3$, there exists a real-valued function $u \in C_0^\infty(\mathbb{R}^d)$, $u \neq 0$, λ, and g possessing the property

$$\begin{aligned} & g(x) \geqslant c_0\mathbf{1} > 0 \quad \forall x \in \mathbb{R}^d \\ & |g(x) - g(y)| \leqslant c_1\varphi(|x-y|) \quad \forall x, y \in \mathbb{R}^d, |x-y| < \varepsilon \end{aligned}$$

   such that

$$-\operatorname{div}(g\nabla u) = \lambda u.$$

3. In [8] and [1], they proved that the operator has no L^2 eigenfunctions near the bottom: there exists $\lambda_0 > 0$ (depending on d, A) such that the operator admit no eigenvalue in $[0, \lambda_0]$.

4. In [8], Armstrong et al. proved the so called large-scale analyticity: A is 1-periodic. There exists $C < \infty$ such that, for every $m \in \mathbb{N}$ with $m \geq 0, R \in [2Cm, \infty)$ and solution $u \in H^1(Q_R)$ (Q_R is cube centered at origin with length R) of

$$-\text{div}(A\nabla u) = 0 \quad \text{in } Q_R,$$

   there exists $\psi \in \mathcal{A}_m^0$ such that, for every $r \in [Cm, R]$,

$$\|u - \psi\|_{\underline{L}^2(Q_r)} \leq \left(\frac{Cr}{R}\right)^{m+1} \|u\|_{\underline{L}^2(Q_R)}.$$

   Here $\mathcal{A}_m^0$ is a subset of the "Heterogeneous polynomials" space $\mathcal{A}_m$.

The Floquet-Bloch solution is written as $u(x) = e^{ikx}p(x)$ where $p(x)$ is periodic function.

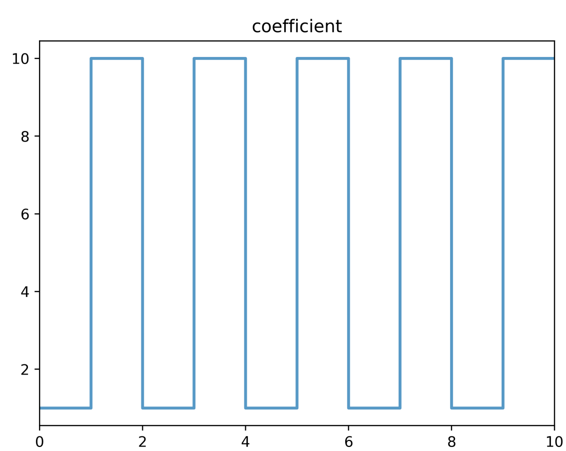

Figure 6: piecewise coefficients

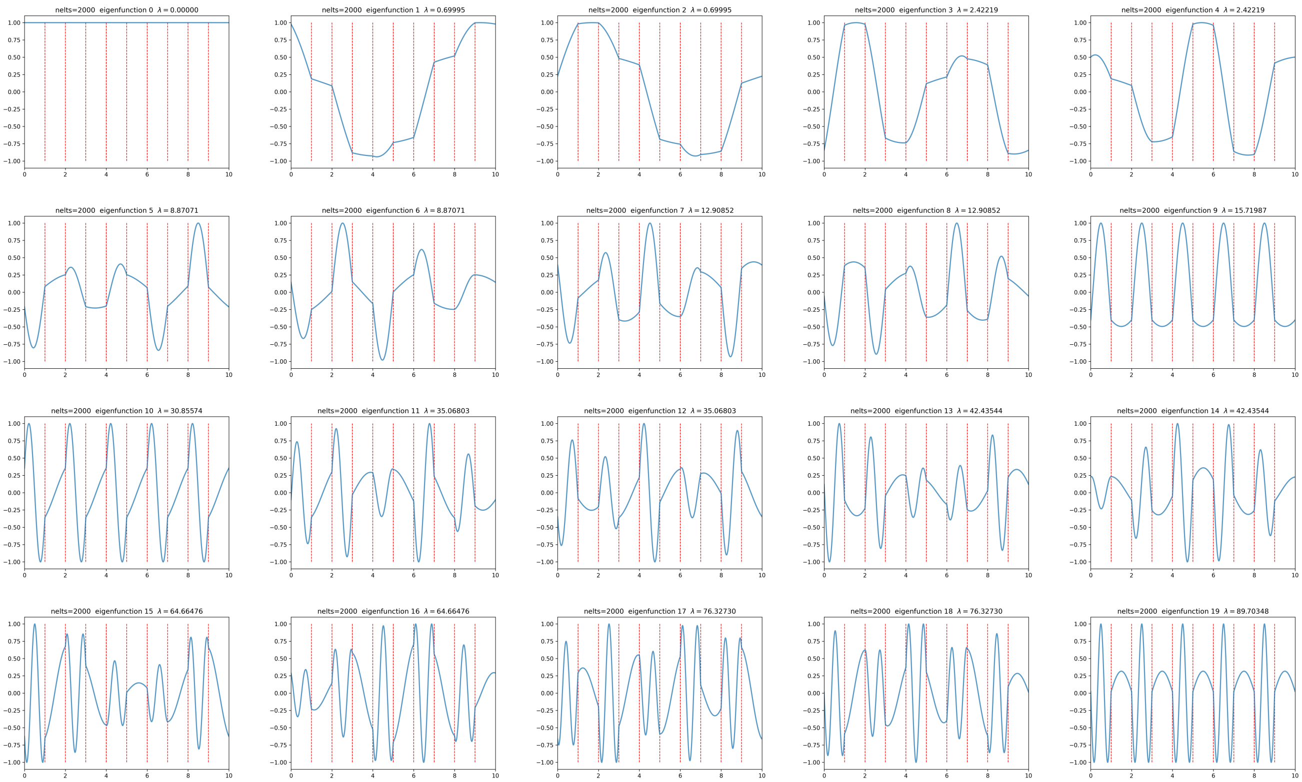

Figure 7: Over $[0, 10]$, periodic boundary. piecewise coefficients, first 20 modes.

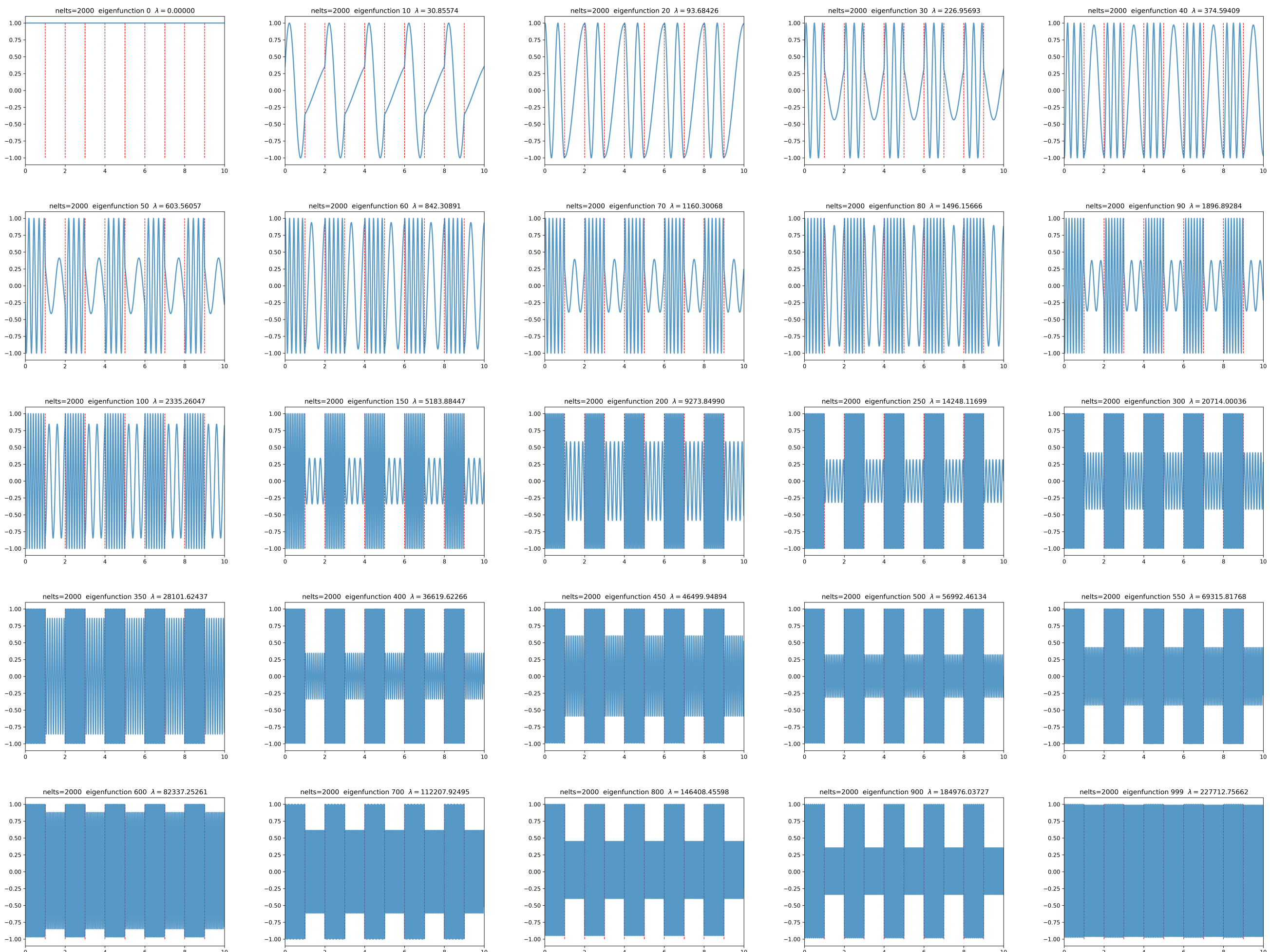

Figure 8: Over $[0, 10]$, periodic boundary. piecewise coefficients. eigen mode 0,10, 20,30,...

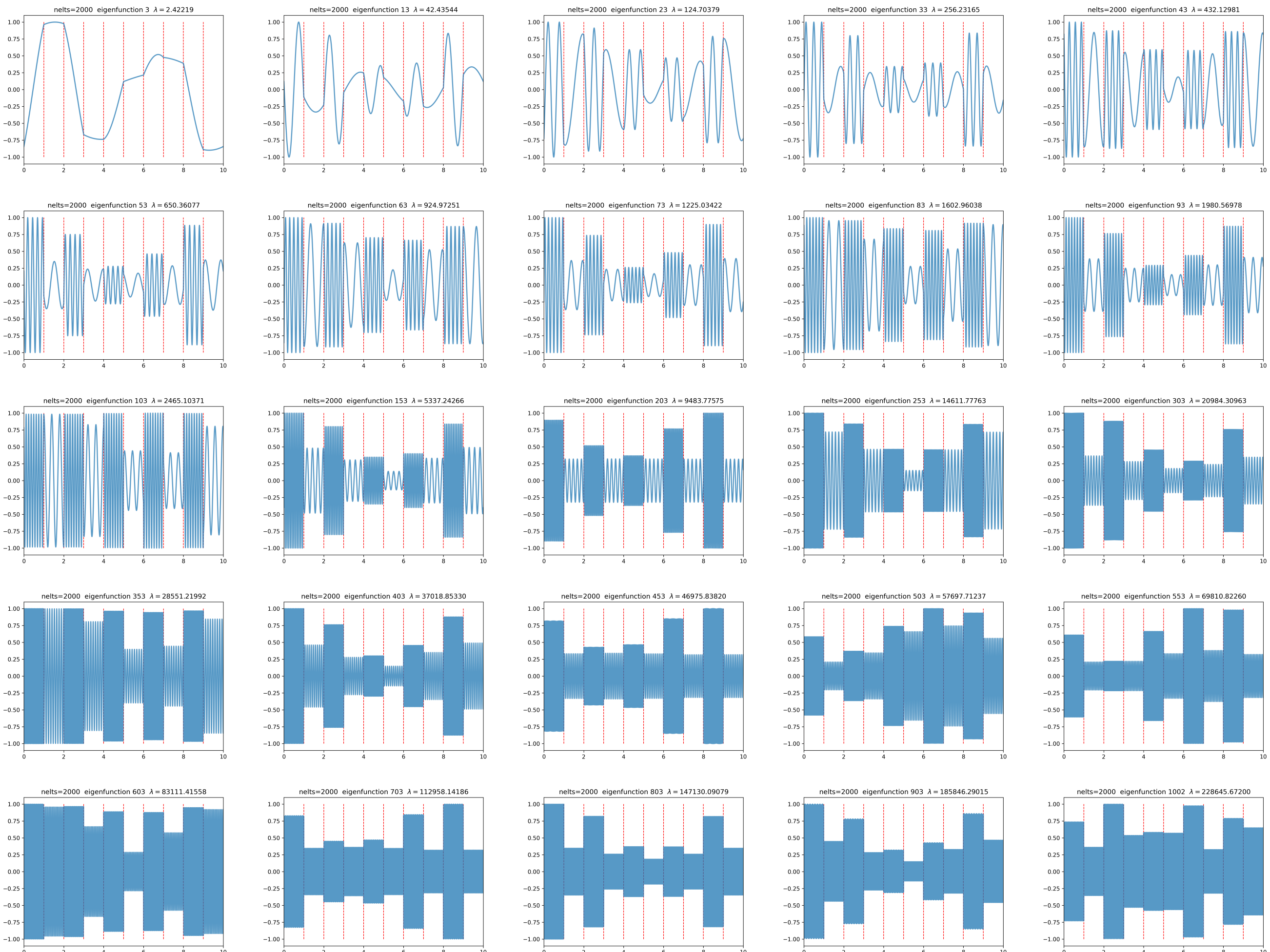

Figure 9: Over $[0, 10]$, periodic boundary. piecewise coefficients. eigen mode 3,13, 23,33,...

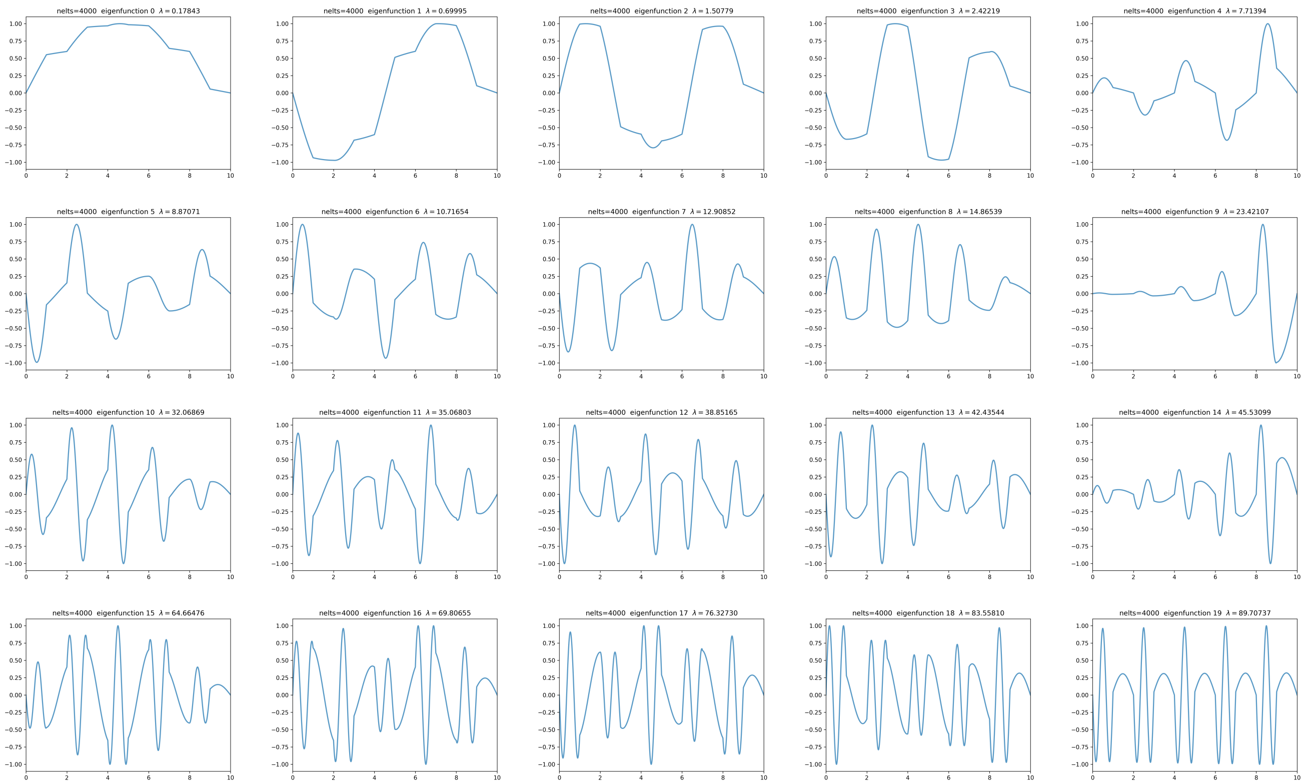

Figure 10: Over $[0, 10]$, dirichlet boundary. piecewise coefficients, first 20 modes.

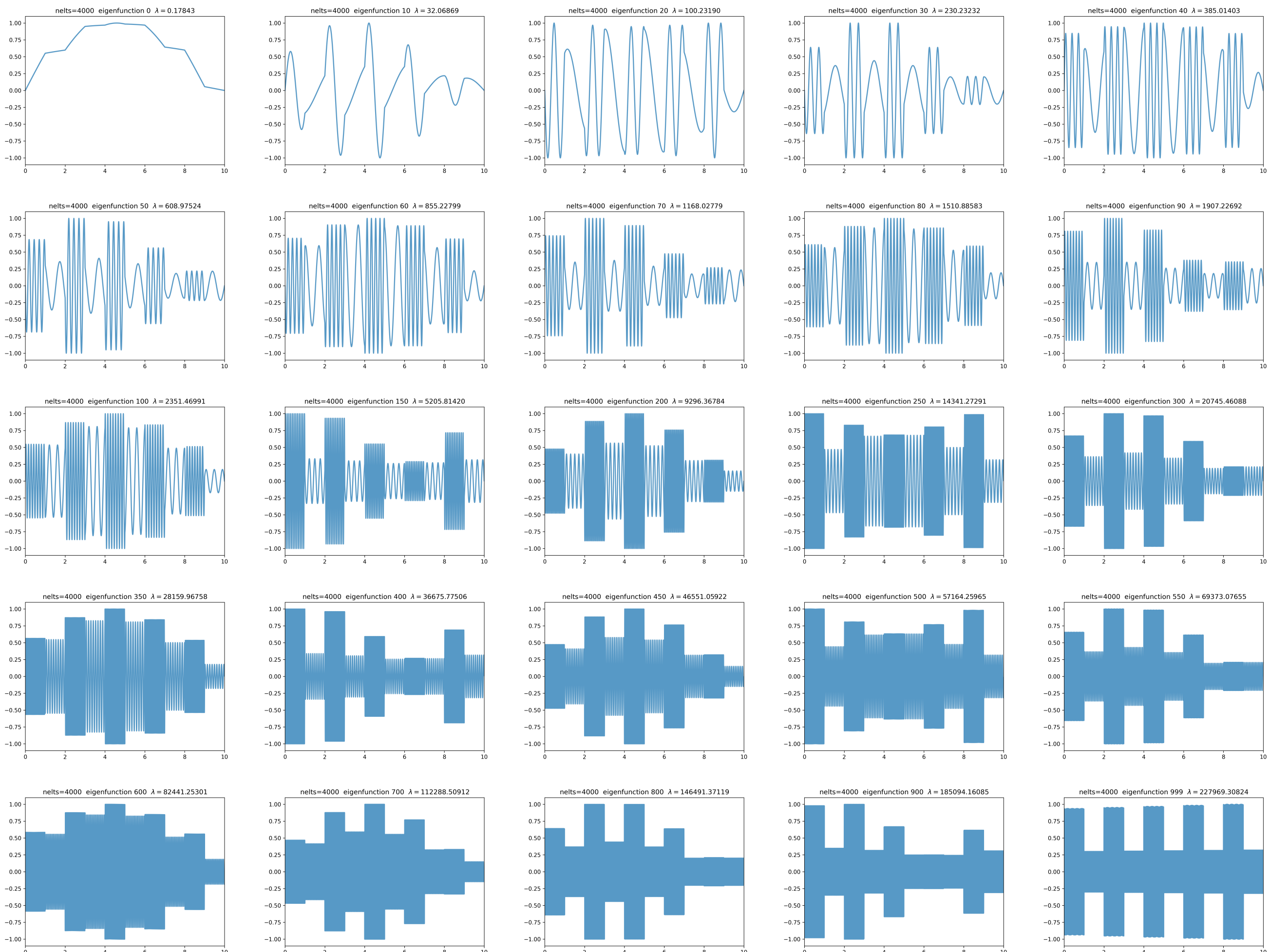

Figure 11: Over $[0, 10]$, dirichlet boundary. piecewise coefficients. eigen mode 0,10, 20,30,...

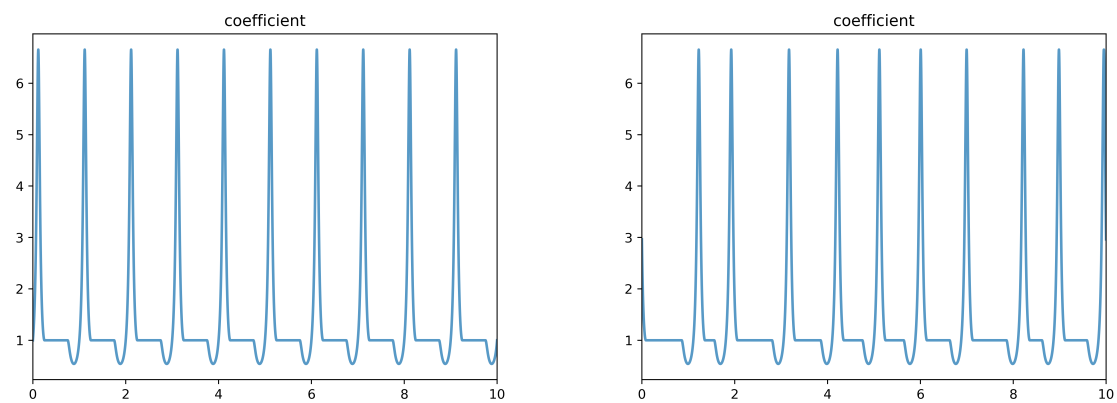

Figure 12: Coefficients on $[0, 10]$. (a). $d_{max} = 0$, i.e., no randomness and 1-periodic. (b) $d_{max} = 0.2$.

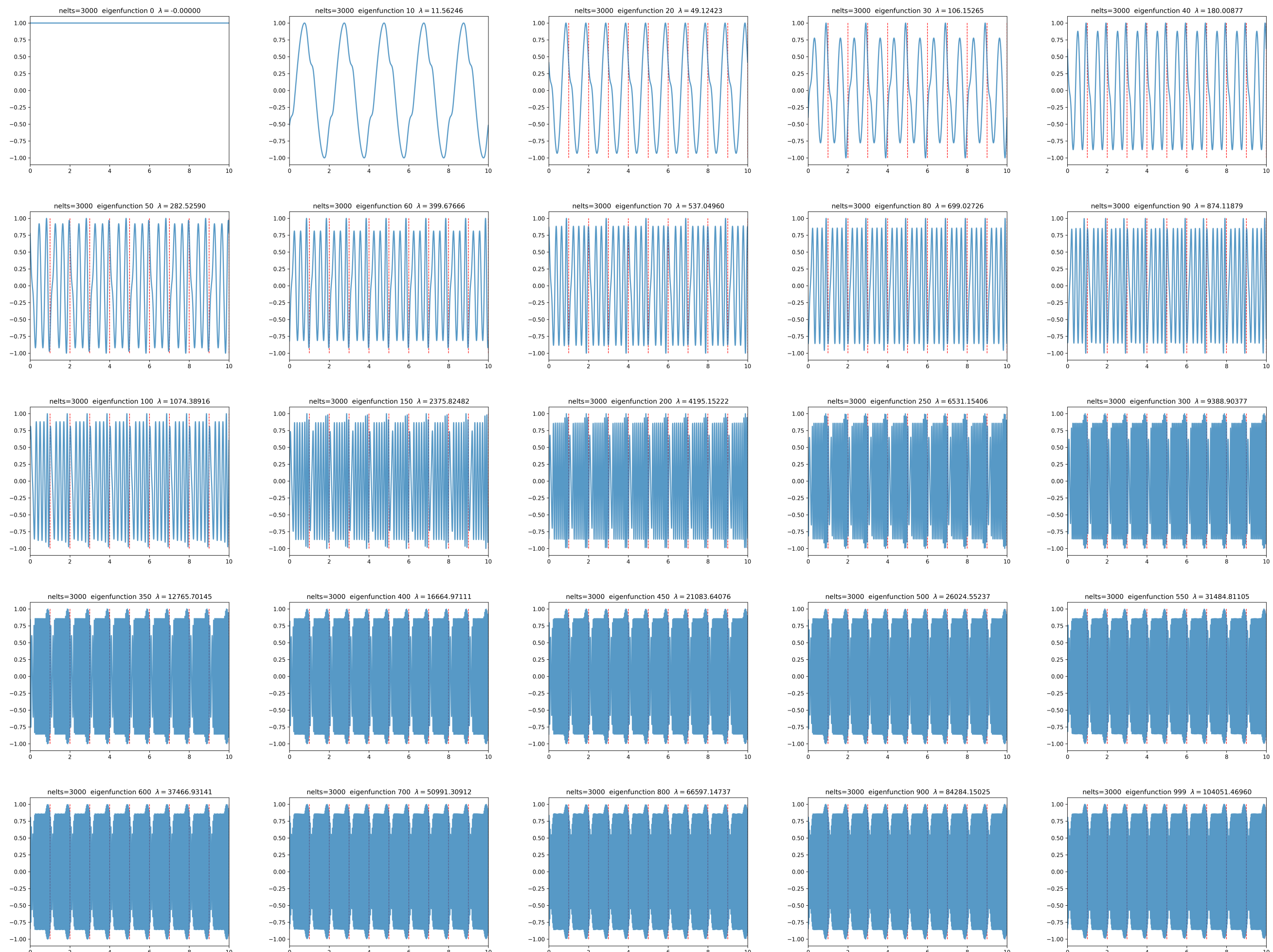

Figure 13: Over $[0, 10]$, periodic boundary. $d_{max} = 0$, i.e., no randomness and 1-periodic. eigen mode 0,10, 20,30,...

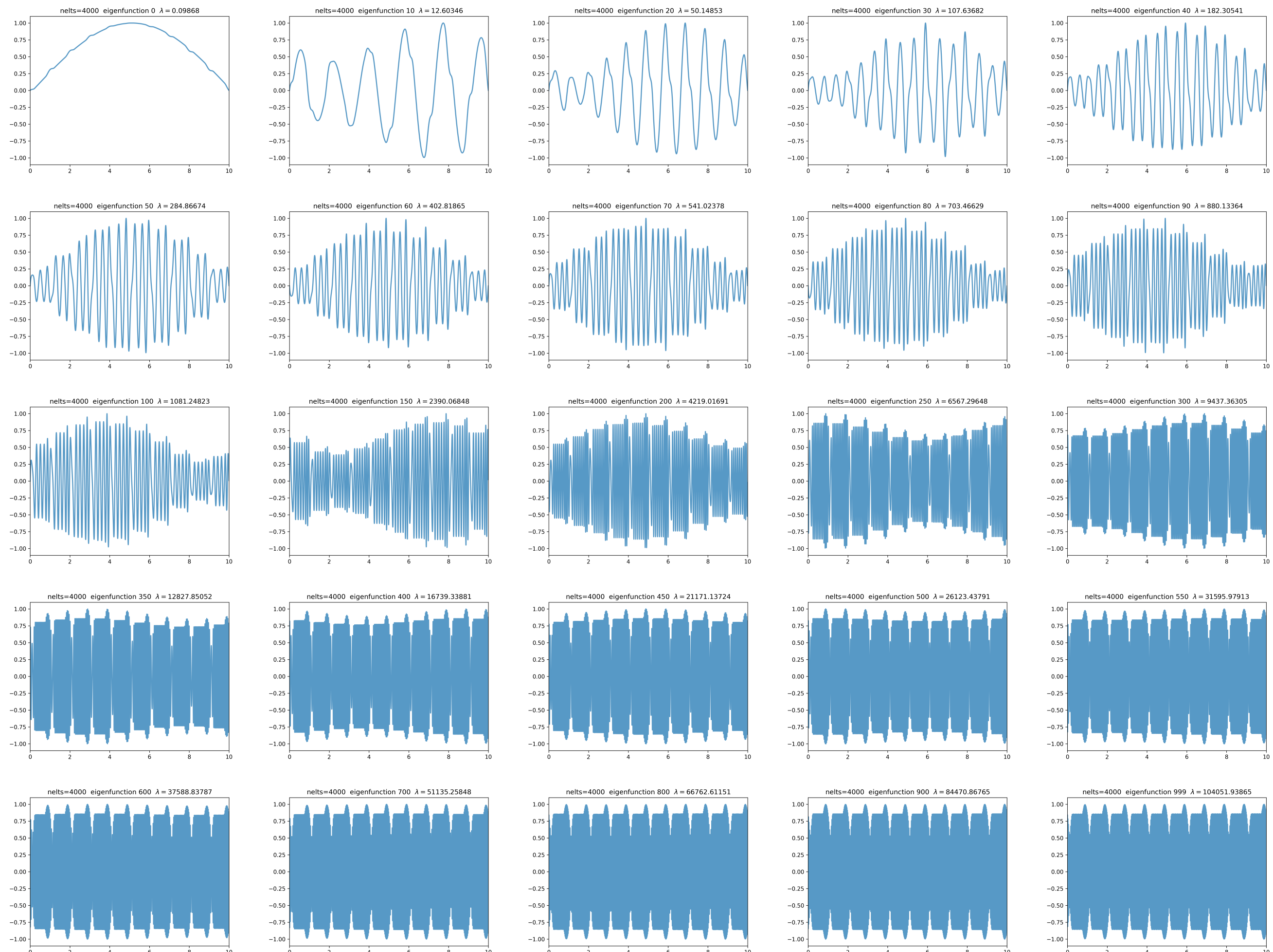

Figure 14: Over $[0, 10]$, dirichlet boundary. $d_{max} = 0$, i.e., no randomness and 1-periodic. eigen mode 0,10, 20,30,...

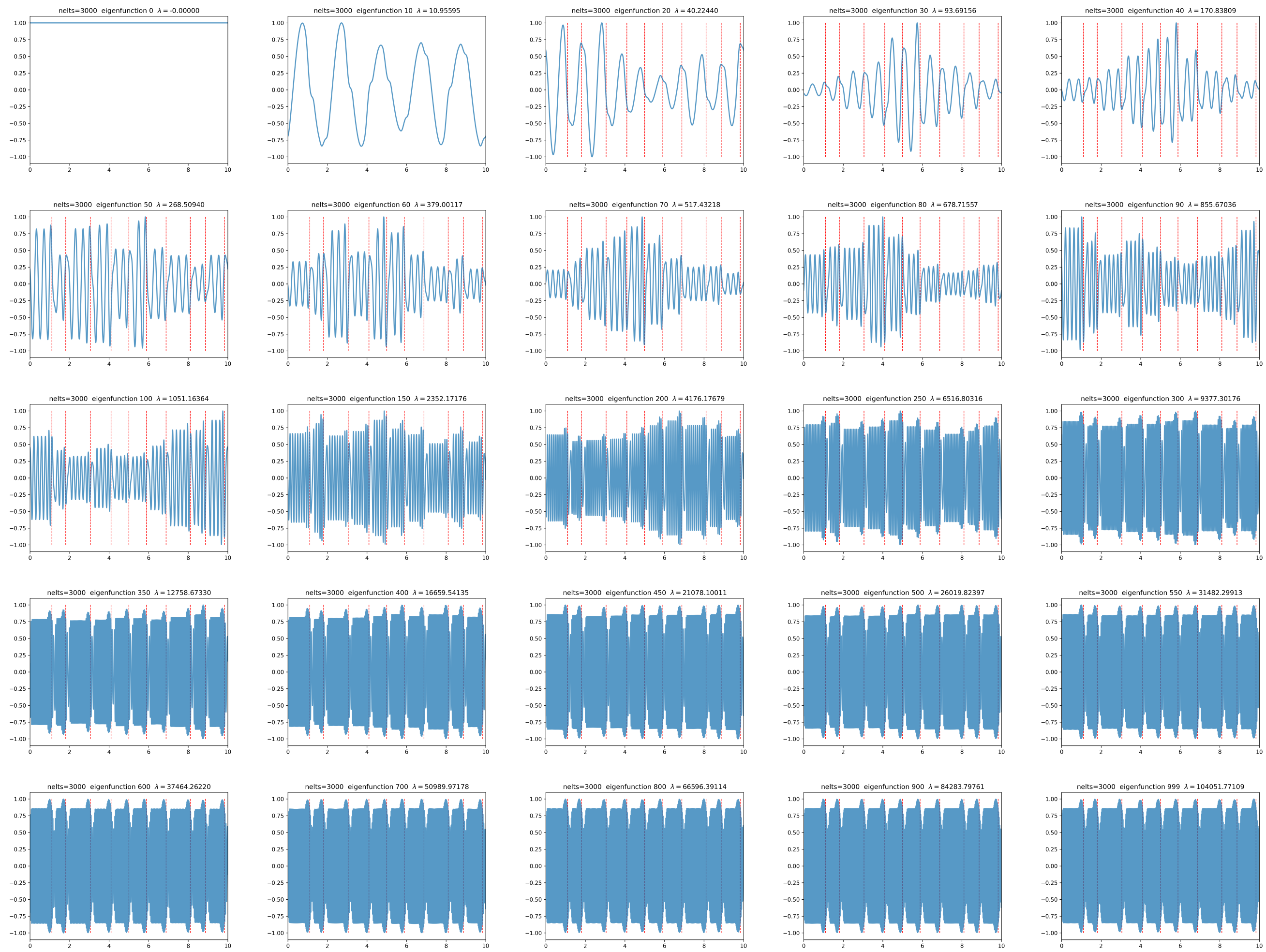

Figure 15: Over $[0, 10]$, periodic boundary. $d_{max} = 0.2$. red dash is the center of f, i.e., $n + d_n(\omega)$.

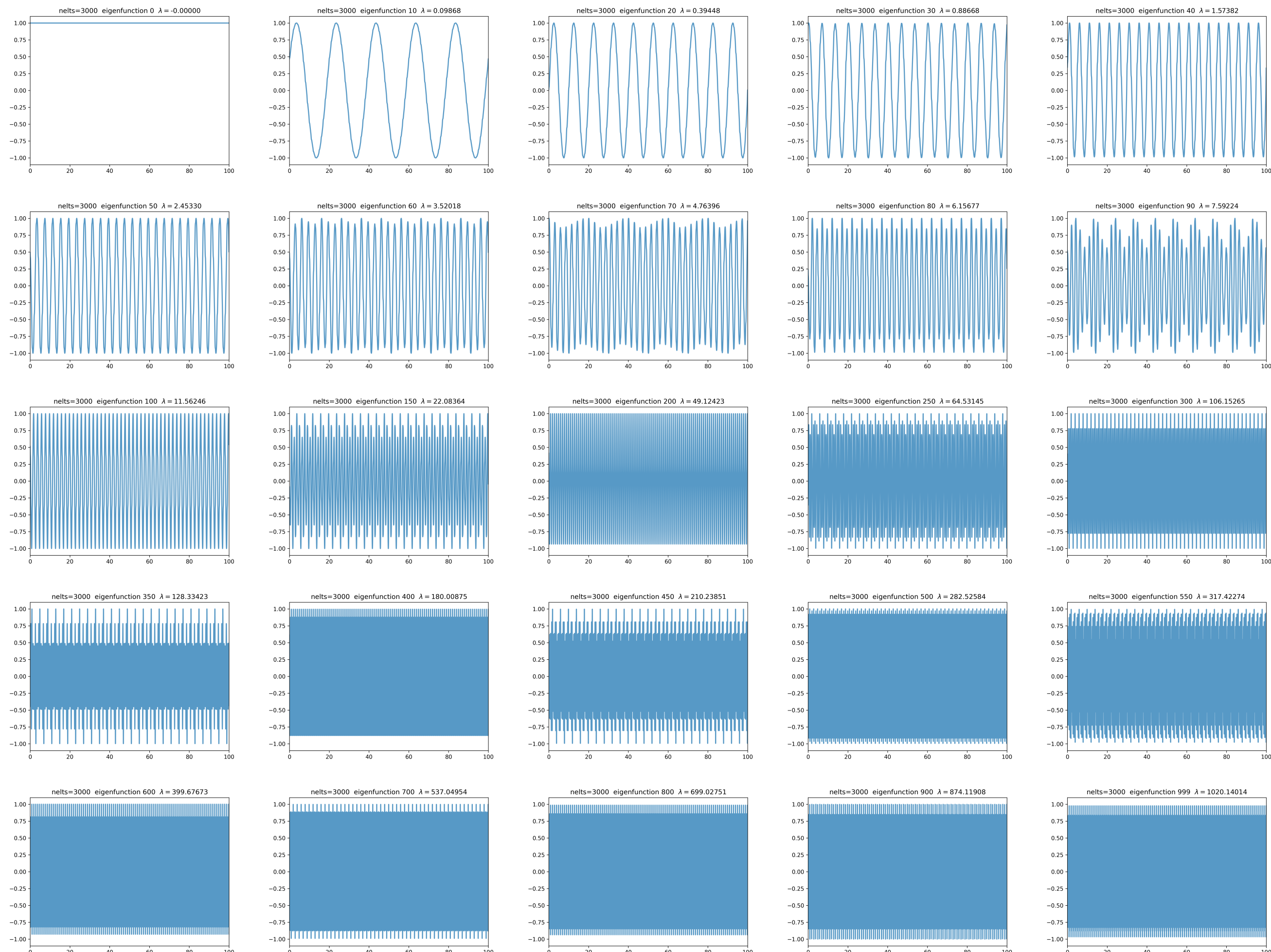

Figure 16: Over $[0, 100]$, periodic boundary. $d_{max} = 0$, i.e., no randomness and 1-periodic.

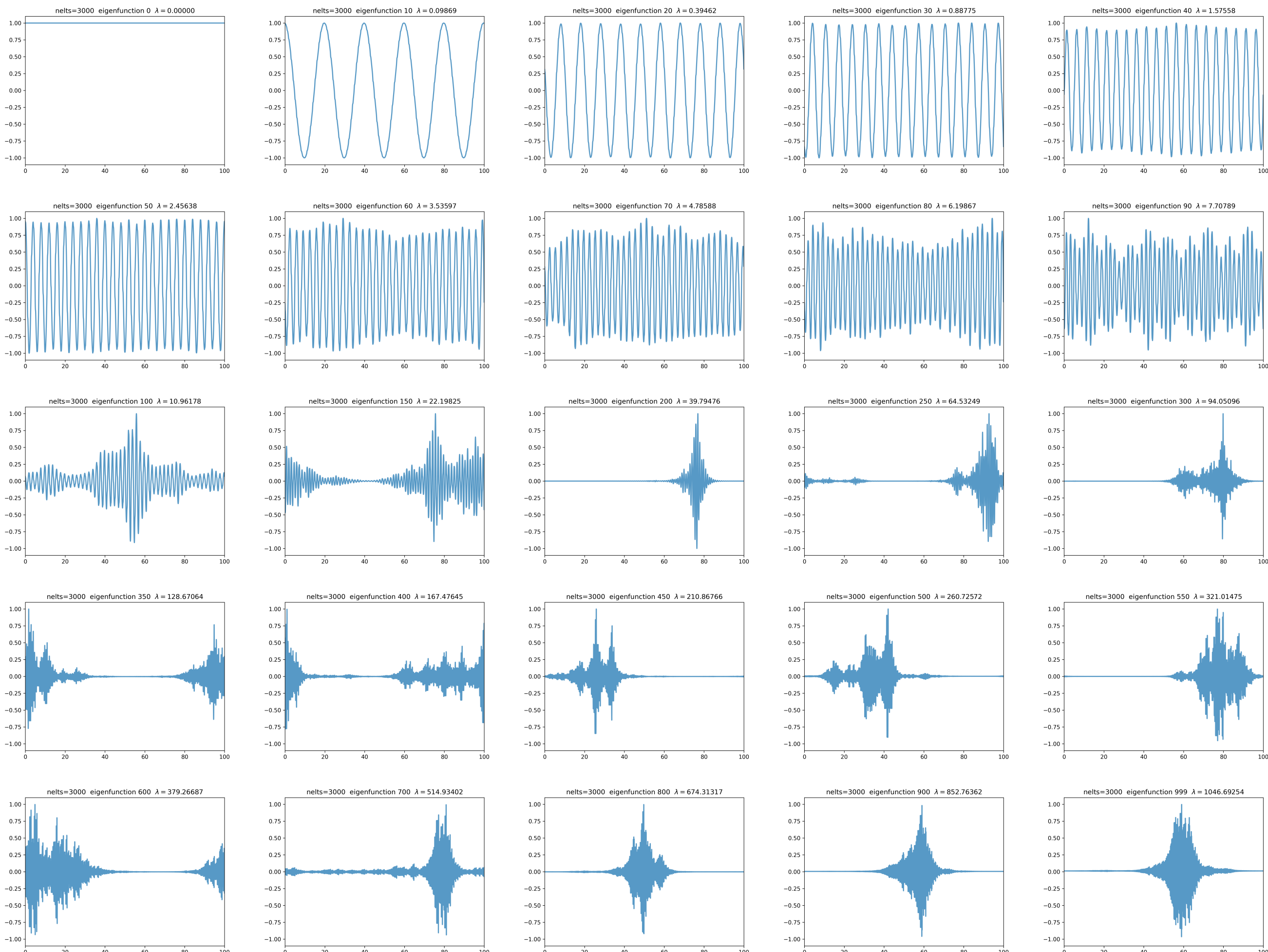

Figure 17: Over $[0, 100]$, periodic boundary. $d_{max} = 0.2$.

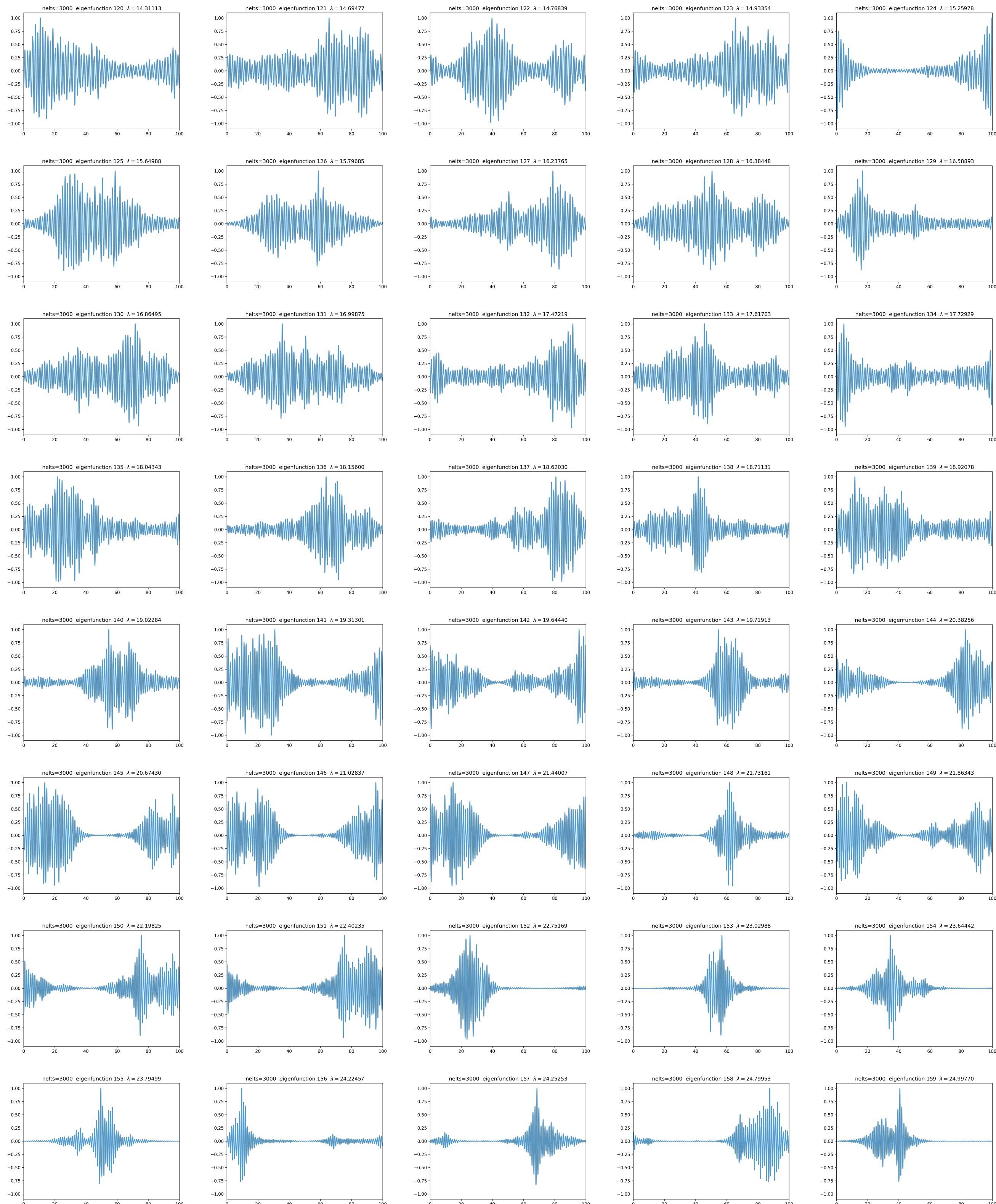

Figure 18: Over $[0, 100]$, periodic boundary. $d_{max} = 0.2$. modes 120 to 160, transition to localization

2.3 Localization length and Lyapunov exponent?

References

[1] DUERINCKX, MITIA and GLORIA, ANTOINE. Large-scale dispersive estimates for acoustic operators: homogenization meets localization. *Preprint* (2023)

[2] Bordenave, Charles, Arnab Sen, and Balint Virag. Mean quantum percolation. *Journal of the European Mathematical Society* 19.12 (2017): 3679-3707.

[3] Kuchment, Peter. An overview of periodic elliptic operators. *Bulletin of the American Mathematical Society* 53.3 (2016): 343-414.

[4] Sims, Robert, and GÃŒnter Stolz. Localization in One Dimensional Random Media: A Scattering Theoretic Approach *Communications in Mathematical Physics* 213 (2000): 575-597.

[5] Figotin, Alexander, and Abel Klein Localization of classical waves I: Acoustic waves. *Communications in Mathematical Physics* 180.2 (1996): 439-482.

[6] Kuchment, Peter An overview of periodic elliptic operators *Bulletin of the American Mathematical Society* no.3 (2016): 343-414.

[7] Filonov, N Second-order elliptic equation of divergence form having a compactly supported solution *Journal of Mathematical Sciences* 106 (2001): 3078–3086.

[8] Armstrong, Scott and Kuusi, Tuomo and Smart, Charles Large-Scale Analyticity and Unique Continuation for Periodic Elliptic Equations *Communications on Pure and Applied Mathematics* 76 (2023): 73–113