Quantitative Macroeconomics

Dynamic Programming

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Constrained optimization

A simple two-periods economy

$$\max_{c_1, c_2} U(c_1, c_2) = \frac{c_1^{1-\gamma_1}}{1 - -\gamma_1} + \beta \frac{c_2^{1-\gamma_2}}{1-\gamma_2} \tag{1}$$

s.t.

$$c_1 = y - s \tag{2}$$

$$c_2 = y + Rs \tag{3}$$

We take income (y) and gross interest rates (R) as given.

A simple two-periods economy

We can get rid of s by replacing equation (2) into equation (2) and the resulting equation into (1) and assume an interior solution. The first-order conditions with respect to c_1 and c_2 are:

$$c_1^{-\gamma_1} - \beta R c_2^{-\gamma_2} = 0 \tag{4}$$

$$c_1 + \frac{c_2}{R} - y - \frac{y}{R} = 0 \tag{5}$$

A simple two-periods economy with credit contraint

What if agentes are credit-contrained? We can set up the Lagrangian and use the Karush-Kuhn-Tucker method:

$$\max_{c_1, c_2} U(c_1, c_2) = \frac{c_1^{1-\gamma_1}}{1-\gamma_1} + \beta \frac{c_2^{1-\gamma_2}}{1-\gamma_2} \tag{6}$$

$$c_1 = y - s \tag{7}$$

$$c_2 = y + Rs \tag{8}$$

$$s \geq 0 \tag{9}$$

A simple two-periods economy with credit contraint

$$\mathcal{L} = \frac{c_1^{1-\gamma_1}}{1-\gamma_1} + \beta \frac{c_2^{1-\gamma_2}}{1-\gamma_2} + \lambda_1 (y - s - c_1) + \lambda_2 (y + Rs - c_2) + \lambda_3 s \tag{10}$$

$$c_1^{-\gamma_1} - \lambda_1 = 0 \tag{11}$$

$$\beta c_2^{-\gamma_2} - \lambda_2 = 0 \tag{12}$$

$$-\lambda_1 + R\lambda_2 + \lambda_3 = 0 \tag{13}$$

$$y - s - c_1 = 0 \tag{14}$$

From sequential problems to Bellman equations

Dynamic Progamming and the value Function

- Decision problem at certain point of **time**.
- It depends on in **initial conditions**.
- It also depends on **initial choices**.
- The **value of the remaining decision problem** arises from those items above.

Dynamic Progamming and the value Function

- We will break a **dynamic optimization problem** into **simple subproblems**, as prescribed by Bellman's **Principle of Optimality**.
- Continuous-time problem: a partial differential equation called **Hamilton-Jacobi-Bellman (HJB)** equation.

1) The problem must posses some objective (maximizing utility or profits, minimizing costs or loss, etc.)
2) We will keep tracking the evolution of decisions throughout the time.
3) The information of the current situation of the problem is called **state**. It is what is needed to make a correct decision. E.g. How much to consume today, given my wealth?

Dynamic Progamming and the value Function

3) The variable that are **chosen** at any given point in time are the **control variables**.

4) A choice regarding the the control variable(s) may be equivalent to choosing the next state, i.e., choosing how much to consume is equivalent of choosing next period's stock of capital.

5) **Policy function**: we will look for a rule that determines the control variables as a function of the state.

6) We will assume impatiance (discount factor) and some característics of functional forms.

7) Value function: it is the best possible value of the objective function when it is written as a function of the state.

The dynamic problem

Following Ljungqvist and Sargent (2012), chapter 3, let's work with the following **sequential** problem:

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r\left(x_t, u_t\right) \tag{17}$$

s.t.

$$x_{t+1} = g\left(x_t, u_t\right), \tag{18}$$

given $x_0 \in \mathbb{R}^n$.

- $\beta \in (0, 1)$: discount factor;
- $r\left(x_t, u_t\right)$: ((concave) objective function;
- $\{u_t\}_{t=0}^{\infty}$: sequence of controls;
- x_t: state.

The Bellman Equation

The **value function** is given by

$$V(x_t) = \max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \tag{19}$$

Let's work with the value function as follows:

$$\begin{aligned}
V(x_t) &= \max_{x_{t+1}} r(x_t, u_t) + \sum_{t=1}^{\infty} \beta^t r(x_{t+1}, u_{t+1}) \\
V(x_t) &= \max_{x_{t+1}} r(x_t, u_t) + \sum_{k=0}^{\infty} \beta^{k+1} (x_{t+k+1}, u_{t+k+1}) \\
V(x_t) &= \max_{x_{t+1}} r(x_t, u_t) + \beta \sum_{k=0}^{\infty} \beta^k r(x_{t+k+1}, u_{t+k+1}) \\
V(x_t) &= \max_{x_{t+1}} r(x_t, u_t) + \beta V(x_{t+1}).
\end{aligned} \tag{20}$$

The Bellman Equation

We are looking for a **policy function**, $h(\cdot)$, such that we can map the state x_t into the control u_t. This is given by iterating the following functions:

$$\begin{aligned} u_t &= h\left(x_t\right) \\ x_{t+1} &= g\left(x_t, u_t\right). \end{aligned} \tag{21}$$

The muffins-eating problem

The muffins-eating problem

The fundamental problem of economics: how to share a muffin?

- Suppose you are deciding how to consume a muffin over some (finite) periods of time $t = 1, 2, \ldots, T$.
- We can write the problem as follows:

$$\max_{c_t} \sum_{t=0}^{T} \beta u(c_t) \tag{22}$$

s.t.

$$x_{t+1} = x_t - c_t \tag{23}$$

The muffins-eating problem

The **value function** is given by

$$V(x_t) = \max_{x_{t+1}} \sum_{t=0}^{T} \beta u\left(x_t - x_{t+1}\right), \tag{24}$$

and the Bellaman equation can be written as follows:

$$V(x_t) = \max_{x_{t+1}} u\left(x_t - x_{t+1}\right) + \beta V(x_{t+1}), \tag{25}$$

Let us define $u(c_t) = \ln(c_t)$ and $T = 2$. What is the optimal bite in each period?

The muffins-eating problem

Let us solve the problem at time $t = 2 = T$. By definition, $x_3 = 0$. Therefore, we have that

$$V(x_2) = \ln(x_2), \tag{26}$$

At $t = 1$ we have

$$V(x_1) = \max_{x_2} \ln(x_1 - x_2) + \beta \ln(x_2), \tag{27}$$

The muffins-eating problem

First-order condition:

$$\frac{\partial V(x_1)}{\partial x_2} = 0 \iff -\frac{1}{x_1 - x_2} + \frac{\beta}{x_2} = 0. \tag{28}$$

Therefore,

$$x_2^* = \frac{\beta x_1}{1+\beta} \tag{29}$$

Replacing the result above into $V(x_1)$ yields:

$$V^*(x_1) = \ln\left(\frac{x_1}{1+\beta}\right) + \beta \ln\left(\frac{\beta x_1}{1+\beta}\right), \tag{30}$$

The muffins-eating problem

Now, at $t = 0$, we have that

$$V(x_0) = \max_{x_1} \ln(x_0 - x_1) + \beta V(x_1), \tag{31}$$

which can be rewritten as follows:

$$V(x_0) = \max_{x_1} \ln(x_0 - x_1) + \beta \left(\ln\left(\frac{x_1}{1+\beta} \right) + \beta \ln\left(\frac{\beta x_1}{1+\beta} \right) \right), \tag{32}$$

First-order condition:

$$\frac{\partial V(x_0)}{\partial x_1} = 0 \iff -\frac{1}{x_0 - x_1} + \frac{\beta}{x_1} + \frac{\beta^2}{x_1} = 0. \tag{33}$$

The muffins-eating problem

Policy function for any *given* x_0:

$$x_1^* = \frac{(\beta + \beta^2)x_0}{1 + \beta + \beta^2} \tag{33}$$

$$x_2^* = \frac{\beta x_1}{1 + \beta} \tag{29}$$

Deterministic Growth Model

The Economy

Consider the following model:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c^{1-\gamma}}{1-\gamma} \tag{34}$$

s.t.

$$k_{t+1} = k_t^{\theta} - c_t + (1-\delta)k_t \tag{35}$$

Note that $\gamma = 1 \implies u(c) = \ln(c)$.

Bellamn Equation

$$V(k) = \max_{0 \leqslant c \leqslant k^{\alpha}+(1-\delta)k} \frac{c^{1-\gamma}}{1-\gamma} + \beta V\left(k'\right) \tag{36}$$

$$c = k^{\alpha} + (1-\delta)k - k' \tag{37}$$

$$V(k) = \max_{(1-\delta)k \leqslant k' \leqslant k^{\alpha}+(1-\delta)k} \frac{\left(k^{\alpha} + (1-\delta)k - k'\right)^{1-\gamma}}{1-\gamma} + \beta V\left(k'\right) \tag{38}$$

Algorithm

From the **contraction mapping theorem** we can compute to the bellman equation by applying the following algorithm:

1) Define the grid ($\Xi = \{x_1, \dots, x_{n_x}\}$) for the state variable(s).
2) Set the initial guess for $V_0(x)$.
3) Define a stopping criterion $d > 0$.
4) $\forall x_i \in \Xi$ compute $V(x') = \max\{V(x_{t+1}, x_{t+1}, \dots, x_{n_x})\}$.
5) Find the x_i that maximizes $V(x')$.
6) Calculate the distance between $V(x)$ and $V(x')$. If $d > \left|V^{n+1} - V^n\right|$ back to point 4. If $V^{n+1} = V^n$, then we have reached a **fixed point** and the problem is solved.

Stochastic Growth Model

Real Business Cycles

Let's work with the model by Brock and Mirman (1972). The household's problem is defined by

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c^{1-\gamma}}{1-\gamma} \tag{39}$$

s.t.

$$k_{t+1} = z_t k_t^{\alpha} - c_t + (1-\delta) k_t \tag{40}$$

and

$$z_{t+1} = \rho z_t + \epsilon_{t+1} \tag{41}$$

The idea is similar to the deterministic case. But now we have **two state** variables: (the stock of) **capital** and **productivity**. Moreover, we need to consider the possible values for the exogenous productivity shock. We will approximate it follwoing Tauchen (1986).

The Life Cycle Model

The Life Cycle Model - constant discounting

Let's assume that an individual lives T periods and is born with no assets.

$$\max_{\{c_t, a_{t+1}\}_1^T} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{42}$$

s.t.

$$c_t + a_{t+1} = Ra_t + w_t \tag{43}$$

$$a_{T+1} = 0 \tag{44}$$

$$a_1 = 0 \tag{45}$$

The Life Cycle Model - constant discounting

$$\mathcal{L} = \sum_{t-1}^{T} \beta^{t-1} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t \left(Ra_t + w_t - c_t - a_{t+1} \right) \tag{46}$$

F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \iff \beta^{t-1} c_t^{-\sigma} = \lambda_t \tag{47}$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \iff \lambda_t = \lambda_{t+1} R \tag{48}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \iff Ra_t + w_t = c_t + a_{t+1} \tag{49}$$

The Life Cycle Model - constant discounting

Combining equation (47) with equation (48) yields the following Euler equation:

$$c_{t+1} = (\beta R)^{\frac{1}{\sigma}} c_t \tag{50}$$

The Life Cycle Model - deterministic discounting

If we define

$$\beta(t) = 95 - 1, \tag{51}$$

the previous Euler equation becomes:

$$c_t^{-\sigma} = \frac{\beta_{t+1}}{\beta_t} R c_{t+1}^{-\sigma}. \tag{52}$$

References

Brock, William, and Leonard Mirman. 1972. "Optimal Economic Growth and Uncertainty: The Discounted Case." *Journal of Economic Theory* 4 (3): 479–513.

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Tauchen, George. 1986. "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions." *Economics Letters* 20 (2): 177–81.