Symmetries and classical fields

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Classical field theory is the branch of mathematical physics dealing with non-quantized fields as representations of physical objects. The first historical motivation for using fields was Maxwell's theory of electromagnetism, which led to the development of various relativistic theories in the field formalism. This seminar first introduces the mathematical notion of field, and how it can be connected to physics through Lagrangians. The consequences of field symmetries are investigated, beginning with Noether's theorem and leading to the idea of gauge fields and covariant derivatives as the best mathematical description of fundamental interactions. The relation of *classical* field theory with relativistic quantum mechanics and general relativity is discussed, although no previous knowledge of these domains is assumed.

INTRODUCTION

Physics is historically the study of material points and bodies with finite extension. Newtonian mechanics describes the world in terms of a countable set of degrees of freedom. This vision seemed to culminate with quantum mechanics, which even quantizes energy and other physical observables that are classically considered continuous. In opposition, Maxwell's theory of electromagnetism finds the origin of the Lorentz force in physical *fields*, defined everywhere and representing a continuous infinity of degrees of freedom. At the same time, when trying to quantize the motion of relativistic particles, there seems to be no choice but to describe their classical motion, not with discrete coordinates, but with fields.

Classical field theory, the study of non quantized fields on a manifold, thus naturally emerges from relativistic theories (electromagnetism, relativistic quantum mechanics and general relativity). A gauge-theoretic argument on the subject can be found in appendix D.

One of the main motivations for the study of *classical* fields (the only ones we discuss here) is that it is a necessary first step before building a quantum field theory. The quantization of classical field theories stands among the main problems of contemporary mathematical physics.

Section I is an initiation to the mathematics of manifolds, fiber bundles and fields. Most of the concepts are given as keys towards a further exploration of the subject, but are not needed in the subsequent sections. Section II introduces fields as tools for writing physical theories through the Lagrangian formalism. Noether's theorem for fields is derived, while conserved currents and charges are discussed. Section III discusses gauge theories and covariant derivatives in connection with general relativity and quantum field theory.

Throughout this seminar, we use **natural units** where $c = \hbar = 1$. Readers should be familiar with tensor calculus, Ricci notation, and Einstein's summation convention [1].

I. MANIFOLDS AND FIELDS

A. The tangent bundle

Morally, a **manifold** is a set of points with topological properties nice enough to represent the world we live in. A curve, or a napkin, are illustrative examples of manifolds. A topological manifold is a set in which continuous paths can be drawn, and a differentiable manifold is a set in which differentiable paths can be drawn. Formal definitions can be found in appendix A.

Let A be a point in a differentiable manifold M. To each differentiable path γ passing through A, we can associate a tangent vector at A, representing the "velocity" of a particle taking the path γ. The set of all possible tangent vectors at A forms a vector space called the tangent space at A, and noted $T_A M$. On a given manifold, all tangent spaces are isomorphic, and hence have a common dimension termed the dimension of the manifold itself.

For example, the tangent spaces to a curve are simply its tangent lines, which are 1-dimensional vector spaces. A curve is thus a 1-dimensional manifold. The tangent spaces to a napkin are its tangent planes, and thus a napkin is a 2-dimensional manifold. Note that a manifold needs not belong to a higher-dimensional space for its tangent spaces to be well defined.

Of course, tangent spaces are of primary importance. They allow us to use *linear algebra* on a manifold although no operation on manifold points is naturally defined (we cannot add, subtract or multiply two points, for instance). This raises another problem: we can only add vectors that belong to the same tangent space (vectors that are defined at the same point A). Yet, we want to compare vectors at A and vectors at B (to integrate a vector field over a path from A to B, for instance). As is explained in appendix A, this is made possible by using transition maps between coordinate charts. Considering two path-connected points A, B, there is a (possibly infinite) sequence of intersecting neibourghoods and transition maps that allow us to compare vectors between A and B.

This construction containing the manifold and all its tangent spaces is called the **tangent bundle**. By choosing a reference frame in each tangent space, we define local reference frames (called vielbeins) for the manifold. If M is the spacetime of general relativity, a local basis would be a *vierbein* (vier = 4) $(e_\mu)_{0\le\mu\le3}$.

B. Fields on a manifold

The tangent bundle is a special case of a more general construction. We can associate to each point $A \in M$ a set of some kind (called the fiber at A) such that all fibers are isomorphic in some sense. The collection consisting of M and all its fibers is a known as a **fiber bundle**. Choosing an element in each fiber in a continous way yields a **section** of the fiber bundle.

If the fibers are affine spaces, we get an affine bundle, if the fibers are principal homogeneous spaces, we get a principal bundle. If the fibers are vector spaces (as in the tangent bundle), then we get a **vector bundle**. Considering a vector bundle, we can construct linear forms, endomorphisms and more generally (p, q)-tensors on each of its fibers. The tensor algebras at each point are of course isomorphic since all fibers are isomorphic. We then define the tensor bundle as the fiber bundle having tensor algebras as fibers.

A *vector field* is a section of the tangent bundle, that is, a choice of vector in each tangent space. A *tensor field* is a section of the tensor bundle formed on the tangent bundle, that is, a choice of tensor on each tangent space.

The dual space to a tangent space T_AM is known as the cotangent space T^*_AM, and the cotangent spaces together form the cotangent bundle. A section of the cotangent bundle (a choice of linear form on each tangent space) is called a *differential 1-form*. From these, we generate the exterior algebra of differential forms through rules specified in appendix B. We can define local reference frames for 1-forms by choosing a reference frame in each cotangent space. If M is the spacetime of general relativity, a local basis of 1-forms would be written $(\mathrm{d}x^\mu)_{0\le\mu\le3}$.

Moreover, we can endow the manifold with a metric by imposing a metric tensor, *i.e.* a symmetric (0,2)-tensor field $g_{\mu\nu} = e_\mu \cdot e_\nu$. The manifold is said to be *Riemannian* if its metric is positive-definite at each point. It is *pseudo-Riemannian* if we only require it to be non-degenerate (that is, invertible). A Lorentzian manifold of dimension d is a pseudo-Riemannian manifold whose metric has signature $(1, d-1)$. The spacetime of general relativity is Lorentzian, with a metric of signature $(1, 3)$.

C. Index conventions

Lower-case Greek indices run over spacetime. The number and position of these indices refer to the type of the tensor field: a scalar field is written ϕ, a vector field ϕ^μ, a covector field ϕ_μ, an endomorphism field $\phi^\mu{}_\nu$, etc.

However, we also want to work with tensor fields of arbitrary rank, type and number. We thus write arbitrary fields ϕ^Σ, where Σ is a *multi-index*. For instance, if $\phi^\mu{}_\nu$ is a real (1,1)-tensor field, then $\Sigma = ({}^\mu{}_\nu)$, and hence $\phi_\Sigma = \phi_\mu{}^\nu$ and $\phi^\Sigma\psi_\Sigma = \phi^\mu{}_\nu\psi_\mu{}^\nu$. If we have two real fields ϕ^μ_1, ϕ^ν_2, then $\phi^\Sigma = (\phi^\mu_1, \phi^\nu_2)$, $\phi_\Sigma = (\phi_{1\mu}, \phi_{2\nu})$ and (assuming also two fields ψ^μ_1, ψ^ν_2) $\phi^\Sigma\psi_\Sigma = \phi^\mu_1\psi_{1\mu} + \phi^\nu_2\psi_{2\nu}$.

Complex fields are handled in the same manner: if ϕ^μ is a complex vector field, then $\phi^\Sigma = (\phi^\mu, \phi^{*\mu})$, $\phi_\Sigma = (\phi^*_\mu, \phi_\mu)$ and $\phi^\Sigma\psi_\Sigma = \phi^\mu\psi^*_\mu + \phi^{*\mu}\psi_\mu$. Multi-indices are upper-case Greek indices.

II. LAGRANGIANS AND SYMMETRIES

From now on, we model spacetime as a 4-dimensional Lorentzian vector space M (typically Minkowski space). All the results in this section are readily generalizable to a 4-dimensional Lorentzian *manifold* through a procedure outlined and justified throughout section III.

A **Lagrangian** is a function L of the system's coordinates and velocities, such that the action functional $S = \int \mathrm{d}t\, L$ is stationary for the physical trajectory. If there is a continuous infinity of coordinates, then the system is described by a field ϕ^Σ, and we had better define the Lagrangian density $\mathcal{L}(\phi^\Sigma, \partial_\mu\phi^\Sigma)$ such that $L(\phi^\Sigma, \dot{\phi}^\Sigma) = \int \mathrm{d}^3x\, \mathcal{L}(\phi^\Sigma, \partial_\mu\phi^\Sigma)$, and thus $S[\phi^\Sigma] = \int \mathrm{d}^4x\, \mathcal{L}(\phi^\Sigma, \partial_\mu\phi^\Sigma)$.

The Lagrangian density is now a function of the field ϕ^Σ and its derivatives $\partial_\mu\phi^\Sigma$, which are independent variables because they are taken at one given point. $\mathcal{L}$ should not depend on higher derivatives of ϕ^Σ because else we would need more boundary conditions to solve the equations of motion, and such conditions are considered unphysical. In practice, the Lagrangian L is never used, and we will simply call "Lagrangian" the Lagrangian density $\mathcal{L}$ (I deeply regret it, but this is standard practice).

Let us vary the action with respect to the field ϕ^Σ. We consider a transformation $\phi^\Sigma \mapsto \phi^\Sigma + \delta\phi^\Sigma$ where $\delta\phi^\Sigma$ is a "small" and "smally varying" field. Then, for *arbitrary* ϕ^Σ, $\delta\phi^\Sigma$ and integration domain D:

$$\begin{aligned}
\delta S &\overset{\text{def}}{=} S[\phi^\Sigma + \delta\phi^\Sigma] - S[\phi^\Sigma] \\
&= \int_D \mathrm{d}^4x \left[\mathcal{L}(\phi^\Sigma + \delta\phi^\Sigma, \partial_\mu\phi^\Sigma + \partial_\mu\delta\phi^\Sigma) - \mathcal{L}(\phi^\Sigma, \partial_\mu\phi^\Sigma)\right] \\
&= \int_D \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^\Sigma}\delta\phi^\Sigma + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\partial_\mu\delta\phi^\Sigma\right) \\
&= \int_D \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^\Sigma} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\right)\delta\phi^\Sigma + \int_D \mathrm{d}^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\delta\phi^\Sigma\right)
\end{aligned} \tag{1}$$

where we have integrated by part between the third and the fourth lines.

Now, a **continuous symmetry** is a group of transformations $\phi^\Sigma \mapsto \phi^\Sigma + \delta\phi^\Sigma$ such that $\delta S = \int_D \mathrm{d}^4x\, \partial_\mu K^\mu(\phi^\Sigma, \partial_\mu\phi^\Sigma)$ for some function $K^\mu(\phi^\Sigma, \partial_\mu\phi^\Sigma)$, and for arbitrary ϕ^Σ and D. A symmetry is thus a transformation that leaves the action invariant up to a total derivative, and total derivatives do not contribute to the equations of motion since their integral only depend on the boundary field values.

A. Symmetries and conserved currents

We will now use the very general result of eq. (1) to demonstrate two results. Please be very careful, the two proofs are analogous but different.

1. Euler-Lagrange equations. Let us assume the stationary action principle: if we choose $D = M$ (the whole spacetime) then there is a field ϕ^Σ such that $\delta S = 0$ (beware: the stationary action principle does not state that δS vanishes for any domain of integration, but only for $D = M$). This particular field ϕ^Σ is assumed to be the physical field, the one satisfying the equations of motion. For physical consistency, we also assume that $\delta\phi^\Sigma \to 0$ fast enough at infinity, which means the surface term vanishes in eq. (1), that is:

$$\int_M \mathrm{d}^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\delta\phi^\Sigma\right) = 0 \tag{2}$$

leading to

$$\int_M \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^\Sigma} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\right)\delta\phi^\Sigma = \delta S = 0 \tag{3}$$

This is true only for our particular choice of ϕ^Σ and $D = M$, but also for arbitrary $\delta\phi^\Sigma$, so:

$$\frac{\partial\mathcal{L}}{\partial\phi^\Sigma} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma} \qquad\qquad \text{Euler-Lagrange equations} \tag{4}$$

This result is now independent of the choice of an integration domain, since there is no integration anymore.

2. Noether's theorem. Let us now choose ϕ^Σ to be the physical field. We can use our first proof and insert the Euler-Lagrange equations in eq. (1), but this time the domain of integration remains arbitrary, so the surface term does not vanish:

$$\delta S = \int_D \mathrm{d}^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\delta\phi^\Sigma\right) \tag{5}$$

If we choose moreover our transformation $\phi^\Sigma \mapsto \phi^\Sigma + \delta\phi^\Sigma$ to be a symmetry, then $\delta\mathcal{L} = \partial_\mu K^\mu$, so:

$$0 = \int_D \mathrm{d}^4x\, \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\Sigma} \delta\phi^\Sigma - K^\mu \right)}_{\stackrel{\text{def}}{=} J^\mu} \tag{6}$$

This is true only for our particular choice of ϕ^Σ and $\delta\phi^\Sigma$, but also for arbitrary D, so:

$$\partial_\mu J^\mu = 0 \qquad \text{Continuity equation} \tag{7}$$

Therefore, *to each symmetry of the action, a conserved current (or Noether current) J^μ is associated*, and eq. (6) gives the explicit expression of the current. This is the (first) Noether theorem. Integrating the continuity equation (7) over space yields:

$$0 = \int \mathrm{d}^3x \left(\frac{\partial J^0}{\partial t} + \nabla \cdot \vec{J} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int \mathrm{d}^3x\, J^0}_{\stackrel{\text{def}}{=} Q} + \underbrace{\oint \mathrm{d}^2\vec{S} \cdot \vec{J}}_{\underset{\infty}{\to} 0} \tag{8}$$

because the current $\vec{J}$ is supposed to vanish at infinity. Then, Q is called the conserved charge (or Noether charge). Whereas Noether currents are locally conserved, Noether charges are only globally conserved.

B. The stress-energy tensor

Action symmetries are usually classified into *internal* symmetries acting on the fields (as we did in the previous section) and *external* symmetries acting on the coordinates. However, the action is a functional of the fields, not a function of the manifold coordinates. Therefore, any external symmetry can be recast as an internal one [2].

Translations are the simplest example of external symmetries. They are transformations of the form:

$$x'^\mu = x^\mu + a^\mu \tag{9}$$

This can be easily recast as a field transformation:

$$\phi'^\Sigma(x) = \phi^\Sigma(x - a) \tag{10}$$

Let us compute $\delta\phi^\Sigma$, the infinitesimal field transformation:

$$\delta\phi^\Sigma = \phi'^\Sigma(x) - \phi^\Sigma(x) = -a^\mu \partial_\mu \phi^\Sigma \tag{11}$$

The infinitesimal Lagrangian transformation is then:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^\Sigma} \delta\phi^\Sigma + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\Sigma} \partial_\mu \delta\phi^\Sigma = -a^\nu \left(\frac{\partial \mathcal{L}}{\partial \phi^\Sigma} \partial_\nu \phi^\Sigma + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\Sigma} \partial_\mu \partial_\nu \phi^\Sigma \right) = -a^\nu \partial_\nu \mathcal{L} \tag{12}$$

by applying the chain rule. Thus $\delta\mathcal{L}$ is a total derivative, which shows that *translations are symmetries of the action*, with $K^\mu = -a^\mu \mathcal{L}$. We can use eq. (6) to compute the associated Noether currents:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\Sigma} a^\nu \partial_\nu \phi^\Sigma - a^\nu \delta^\mu{}_\nu \mathcal{L} \tag{13}$$

Since a is an arbitrary vector in this expression, there are in fact four independent Noether currents, each associated to translational invariance in one of the four spacetime directions:

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\Sigma}\partial_\nu\phi^\Sigma - \delta^\mu{}_\nu\mathcal{L} \tag{14}$$

then we have $\partial_\mu T^\mu{}_\nu = 0$. T is called the **stress-energy tensor**. In particular, the Noether charge associated to time translation is:

$$\int \mathrm{d}^3x\, T^{00} = \int \mathrm{d}^3x\, \frac{\partial\mathcal{L}}{\partial\dot\phi^\Sigma}\dot\phi^\Sigma - \mathcal{L} = \int \mathrm{d}^3x\, \mathcal{H} \tag{15}$$

where we have recognized the Legendre transform of the Lagrangian, which is the Hamiltonian, whose integral yields the energy. Similarly, the three Noether charges associated to space translations yield the momentum vector. In fact, energy and linear momentum can be *defined* as the Noether charges associated with translational symmetries.

III. GAUGE THEORIES

A **gauge theory** is a general procedure that takes a Lagrangian invariant under some global continuous transformations, and changes it into a Lagrangian invariant under *local* continuous transformations. That is, gauge theories are a means to generalize a physical theory so that it satisfies stronger physical requirements, at the price of introducing new degrees of freedom in the form of *gauge fields*.

To study gauge theories, we will use the complex Klein-Gordon field, a classical scalar field governed by the Klein-Gordon Lagrangian:

$$\mathcal{L} = \partial^\mu\phi\partial_\mu\phi^* - m^2\phi\phi^* \tag{16}$$

Originally, the Klein-Gordon field was thought of as a relativistic wavefunction, which is strange since we only deal here with *classical* fields. *De facto*, interpreting the Klein-Gordon field as a wavefunction leads for example to the Klein paradox, and to negative energy solutions that are difficult to understand. On the contrary, if we see this field as classical, then we must still quantize it by replacing $\phi(x)$ by an operator field $\hat\phi(x)$. This is the object of *quantum field theory*, which solves the Klein paradox and reinterprets the negative energy solutions as antiparticles with positive energy. The relativistic nature of the theory seems to forbid the description of quantum systems by wavefunctions (even the Hamiltonian is not a covariant quantity!) and replaces them with *quantum fields* whose physical interpretation is unfortunately not so straightforward. The Klein-Gordon classical field $\phi(x)$ only provides a classical approximation to the associated quantum field $\hat\phi(x)$. Because the Klein-Gordon equation is obtained by a first "naive" quantization procedure (described in appendix C), quantum field theory is said to perform a "second quantization".

An introductory example set in the context of Newtonian mechanics can be found in appendix D, where Bour's formula and inertial forces are derived by gauging Galilean dynamics.

A. The U(1)-invariance of the Klein-Gordon equation

First, we want to show that the Klein-Gordon Lagrangian is invariant under *global* U(1) transformations, that is under the mapping:

$$\phi(x) \mapsto \mathrm{e}^{+i\alpha}\phi(x) \quad ; \quad \phi^*(x) \mapsto \mathrm{e}^{-i\alpha}\phi^*(x) \qquad \text{where } \alpha \text{ is some real constant.} \tag{17}$$

The proof is straightforward: since α is a constant, the exponential factors can be pulled out of the derivatives, and thus cancel each other. This invariance leads to a Noether current that can be computed with the infinitesimal field variation $\delta\phi = i\alpha\phi$:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi + c.c. = i\alpha(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \tag{18}$$

Now, imagine we want the Klein-Gordon Lagrangian to be invariant under *local* U(1) transformation, that is under the mapping:

$$\phi(x) \mapsto \mathrm{e}^{+iq\alpha(x)}\phi(x) \quad ; \quad \phi^*(x) \mapsto \mathrm{e}^{-iq\alpha(x)}\phi^*(x) \qquad \text{where } \alpha(x) \text{ is some real field and } q \text{ some real constant.} \tag{19}$$

The mass term is still invariant, but the kinetic term becomes (to first order in α):

$$\partial^\mu\phi\partial_\mu\phi^* \mapsto \partial^\mu\phi\partial_\mu\phi^* + iq\phi\partial^\mu\alpha\partial_\mu\phi^* - iq\phi^*\partial^\mu\alpha\partial_\mu\phi \tag{20}$$

Thus, to ensure local invariance of $\mathcal{L}$, we modify the Lagrangian by adding a new field $A_\mu(x)$ which transforms under U(1) such that $A_\mu \mapsto A_\mu + \partial_\mu\alpha$:

$$\mathcal{L} \overset{\text{def}}{=} (\partial^\mu\phi - iqA^\mu\phi)(\partial_\mu\phi^* + iqA_\mu\phi^*) - m^2\phi\phi^* \tag{21}$$

This is the new, gauge invariant Lagrangian. Hence, the so-called **gauge field** A_μ absorbs the unwanted terms that appear during a U(1) local transformation (called a gauge transformation in this context). Since the necessity of this procedure came from the derivative, the Lagrangian can be cleverly rewritten in terms of the **covariant derivative**:

$$D_\mu \overset{\text{def}}{=} \partial_\mu - iqA_\mu \tag{22}$$

$$\mathcal{L} = D^\mu\phi(D_\mu\phi)^* - m^2\phi\phi^* \tag{23}$$

In fact, all equations of a gauge theory can be found by formally replacing all partial derivatives with the corresponding covariant derivatives.

But this Lagrangian is still incomplete. Indeed, we need to add a kinetic term describing the dynamics of the gauge field, which is indeed a new degree of freedom. This new term must not contain any other field than the gauge field itself (else it would be an interaction term) and it must be gauge invariant. The only possible field respecting these constraints is the **field strength tensor**:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{24}$$

which must be twice contracted to yield a scalar, leading to the following Lagrangian:

$$\mathcal{L} = D^\mu\phi(D_\mu\phi)^* - m^2\phi\phi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{25}$$

Let us compute the associated Euler-Lagrange equations:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = -\partial_\mu F^{\mu\nu} \tag{26a}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = -iq(\phi(D^\nu\phi)^* - \phi^* D^\nu\phi) = -J^\nu \tag{26b}$$

hence $\partial_\mu F^{\mu\nu} = J^\nu$ where J^ν is the Noether current associated with gauge invariance.

But A_ν, as a covector field, is also a differential 1-form (see appendix B) called the **connection form**, and the field strength tensor is nothing but its exterior derivative, called the **curvature form** in this context:

$$A = A_\nu\,\mathrm{d}x^\nu \tag{27}$$

$$F = \mathrm{d}A = (\partial_\mu A_\nu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \sum_{\mu>\nu} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{28}$$

$$\mathrm{d}F = \mathrm{d}^2A = 0 \quad \Rightarrow \quad (\partial_\sigma F_{\mu\nu})\,\mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0 \tag{29}$$

This last equation means that the $(\partial_\sigma F_{\mu\nu})$ are zero under the condition that σ, μ, ν have different values. This can be rewritten $\varepsilon^{\rho\sigma\mu\nu}\partial_\sigma F_{\mu\nu} = 0$. Together, the equations

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{30a}$$
$$\varepsilon^{\rho\sigma\mu\nu}\partial_\sigma F_{\mu\nu} = 0 \tag{30b}$$

are Maxwell's equations, and the Noether current J^μ appears as the electric 4-current! Equation (25) is therefore described as the *Lagrangian of scalar electrodynamics*. A_μ is the 4-potential, and represents the photon field in the particle-theoretic point of view. The electromagnetic interaction is said to be *mediated* by photons.

Hence, just by imposing some local invariance on the free Lagrangian, we have derived Maxwell's electromagnetism. In the context of high-energy physics, gauge theories seem to be the correct way to obtain all fundamental interactions. This is developed in the next subsection.

B. Yang-Mills theories

Although we used the Klein-Gordon field in the last subsection, most particles of interest in high-energy are fermions (electrons, quarks...), and are thus described by spinors satisfying the Dirac equation:

$$\mathcal{L}(\psi, \partial_\mu \psi) = i\bar{\psi}\,\partial\!\!\!/\,\psi - m\bar{\psi}\psi \tag{31}$$

where ψ is a 4D vector (a Dirac spinor, in fact, but let us not dwell on that), $\partial\!\!\!/ \overset{\text{def}}{=} \gamma^\mu \partial_\mu$ is the Feynman-slashed derivative with the four Dirac matrices γ^μ, $\bar{\psi} \overset{\text{def}}{=} \psi^\dagger \gamma^0$ is the Dirac conjugate, and m is the mass again.

The Dirac Lagrangian is globally U(1)-invariant too, and it can be made locally U(1) invariant by following the same steps as in the previous section. This leads to Maxwell's electromagnetism in the case of spin 1/2 particles. Once quantized, the corresponding theory is known as **quantum electrodynamics**.

The procedure can be generalized to SU(n) invariance in the following way. Let us consider $n \geq 2$ free Dirac fields with the same mass m:

$$\mathcal{L}(\psi^a, \partial_\mu \psi^a) = i\bar{\psi}_1\,\partial\!\!\!/\,\psi_1 - m\bar{\psi}_1\psi_1 + i\bar{\psi}_2\,\partial\!\!\!/\,\psi_2 - m\bar{\psi}_2\psi_2 + \cdots = i\bar{\psi}_a\,\partial\!\!\!/\,\psi^a - m\bar{\psi}_a\psi^a \tag{32}$$

where a is an index running over the various fields ($a = 1, \ldots, n$). Then, $(\psi^a)_{1\leq a\leq n}$ is a vector in n dimensions. We can apply a SU(n) transformation to this vector, and the Lagrangian (32) is invariant under such a global transformation.

Again, we can impose $\mathcal{L}$ to be locally invariant under SU(n) by introducing a suitable covariant derivative, resulting in a **Yang-Mills theory**. However, because the group SU(n) is not Abelian, several gauge fields are needed to define the covariant derivative [3]. Indeed, the gauge fields appear in order to compensate terms coming from the derivative of the gauge transformation. Any local transformation $U(x) \in \mathrm{SU}(n)$ can be written in the form $U(x) = \mathrm{e}^{iH(x)}$, where $H(x)$ is a hermitic operator. Because it can be used to generate an element of SU(n), H is said to belong to the **Lie algebra** $\mathfrak{su}(n)$ associated with SU(n). The Lie algebra is a vector space of dimension $n^2 - 1$, which means the Lie group SU(n) admits $n^2 - 1$ independent generators $T_\mathfrak{a} \in \mathfrak{su}(n)$ (such that $H(x) = \lambda^\mathfrak{a}(x) T_\mathfrak{a}$). Each generator $T_\mathfrak{a}$ yields a term in the derivation of U, and thus must be compensated by a gauge field. Abelian groups like U(1) only have one degree of freedom, *i.e.* only one generator.

Because of non-Abelianity, eq. (24) is modified by additional terms containing several gauge fields (details can be found in appendix E). Physically, this introduces interactions between the mediating bosons.

For instance, local SU(3) invariance introduces $3^2 - 1 = 8$ gauge fields, identified as the eight gluons (each carrying a different *colour charge*), which mediate the so-called *strong interaction* responsible for the formation of *hadrons* from quarks. Unlike photons, gluons interact with other gluons, a consequence of the non-Abelianity of Yang-Mills theories. The three fermionic fields ψ_1, ψ_2, ψ_3, upon which the gauge group acts, are the same quark with different colour charges (usually called red, blue and green). It works since changing the colour of a quark does not change its mass. This theory is called **chromodynamics**.

The SU(2) Yang-Mills theory contains 3 gauge fields but does not represent any known physical interaction. However, the locally U(1) $\times$ SU(2)-invariant Dirac Lagrangian yields 4 gauge fields (1 for U(1) and 3 for SU(2)) whose linear combinations are the photon, the Z, the W^+ and the W^- bosons (the gauge fields and the mediating bosons are here two different bases of the same state space). This is the Glashow-Weinberg-Salam **electroweak theory**.

The quantization of these classical theories can be performed by Feynman calculus in the perturbative regime, when the interaction constant is small. However, finding a general quantization procedure for Yang-Mills theories remains an open problem in mathematical physics.

C. The general gauging procedure

Gauge theories can be summarized into an (almost) simple recipe:

1. Take a free Lagrangian possessing some global symmetry.
2. Identify the gauge fields (connection form) as the generators of the Lie algebra associated with the symmetry group.
3. Replace all derivatives with covariant derivatives formed by adding some kind of product between the gauge field and the field under study.
4. Compute the field strength tensor (curvature form) associated with the gauge fields.
5. Constrain the gauge fields by adding a kinetic term formed from the field strength tensor (to ensure its gauge-invariance).

Appendix E gives general expressions for each of the steps not previously discussed.

D. General relativity as a gauge theory

Special relativity is invariant under *global* coordinate changes, while *all* coordinate changes are unphysical. In other words: we have a theory on flat spacetime, how do we change our derivatives to take *curvature* into account? Let us compute the differential of a vector $\mathbf{v} = v^\mu \mathbf{e}_\mu$.

$$\begin{aligned} \mathrm{d}\mathbf{v} &= \mathrm{d}(v^\mu)\,\mathbf{e}_\mu + v^\mu\,\mathrm{d}(\mathbf{e}_\mu) = \partial_\nu v^\mu\,\mathrm{d}x^\nu\,\mathbf{e}_\mu + v^\sigma \partial_\nu \mathbf{e}_\sigma\,\mathrm{d}x^\nu \\ &= (\partial_\nu v^\mu + v^\sigma \Gamma^\mu_{\sigma\nu})\,\mathrm{d}x^\nu\,\mathbf{e}_\mu \end{aligned} \tag{33}$$

where $\Gamma^\mu_{\sigma\nu}\mathbf{e}_\mu \overset{\text{def}}{=} \partial_\nu \mathbf{e}_\sigma$ defines the *Christoffel symbols of the second kind*. Together, they form a connection on the spacetime manifold. They can be computed from the metric $g_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) \tag{34}$$

So because we want to use any coordinate frame, and not only inertial frames, we must use the following covariant derivative:

$$D_\nu v^\mu = \partial_\nu v^\mu + v^\sigma \Gamma^\mu_{\sigma\nu} \tag{35}$$

From there, we can compute the curvature form, known as the Riemann tensor in this context:

$$R^\alpha{}_{\beta\gamma\delta} \overset{\text{def}}{=} \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\sigma\delta}\Gamma^\sigma_{\beta\gamma} \tag{36}$$

The dynamics of spacetime is thus given by an equation formed from the Riemann tensor. For brevity, we introduce the Ricci tensor and the Ricci scalar:

$$R_{\mu\nu} \overset{\text{def}}{=} R^\alpha{}_{\mu\alpha\nu} \quad ; \quad R \overset{\text{def}}{=} R^\mu{}_\mu \tag{37}$$

The most general equation formed from the metric, its first derivatives, and linear in its second derivatives, is then the **Einstein field equation**:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{38}$$

which is the fundamental equation of general relativity. The cosmological constant Λ has been the subject of much discussion since it governs the expansion of the universe.

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Appendix A: Mathematical complement on differentiable manifolds

A **topological space** is a set in which the neighbourhoods of all points have been specified. Equivalently, a topological space is a set in which we know what are the open and closed subsets. This defines a notion of "closeness" between the points of the set, but not necessarily a metric (metric spaces are topological spaces, but the converse is not true). A T_2-separated, or **Hausdorff** space is a topological space in which two distinct points necessarily have disjoint neighbourhoods (all metric spaces are Hausdorff, but the converse is not true). This guarantees the unicity of limits: in a non-Hausdorff space, a sequence can converge towards two distinct points at the same time.

A **homeomorphism** ψ between two sets U, V is a continuous bijection $\psi : U \to V$ whose reciprocal bijection $\psi^{-1} : V \to U$ is also continuous.

A **topological manifold** is a Hausdorff space such that every point has a neighbourhood (called a Euclidean neighbourhood) that is homeomorphic to an open subset of $\mathbb{R}^n$ (a topological manifold is thus said to be **locally Euclidean**). These homeomorphisms $\psi_U : U \to \psi_U(U) \subset \mathbb{R}^n$ define **coordinate charts** that can be used to parameterize the points in Euclidean neighbourhoods. A set of Euclidean neighbourhoods that cover the entire manifold, together with their corresponding coordinate charts, is termed an **atlas**. Given two Euclidean neighbourhoods U, V with non empty intersection and their coordinate charts ψ_U, ψ_V, there is a **transition map** f that takes the coordinate of a point $x \in U \cap V$ and maps it to its coordinate in the other chart $f : \psi_U(x) \mapsto \psi_V(x)$. f is then a homeomorphism. These transition functions allow us to connect the different Euclidean neighbourhoods, so their properties are of primary importance if we want to do geometry on the manifold.

A **differentiable atlas** is an atlas in which all transition maps are differentiable. A **differentiable manifold** is now a topological manifold endowed with a differentiable atlas. A $\mathcal{C}^k$-atlas is an atlas in which all transition maps are $\mathcal{C}^k$, and a smooth atlas is an atlas in which all transition maps are $\mathcal{C}^\infty$. Notions of $\mathcal{C}^k$-manifold and smooth manifold naturally arise from these definitions. All the manifolds we work with here are smooth.

Appendix B: Differential forms

A **differential 1-form** is a covector field. We can define local reference frames for 1-forms by choosing a reference frame in each cotangent space (the dual space to the tangent space). If M is the spacetime of general relativity, then we can define a local basis of 1-forms $(\mathrm{d}x^\mu)_{0\leq\mu\leq 3}$. A general 1-form is written:

$$A = A_\mu(x)\,\mathrm{d}x^\mu \tag{B1}$$

where x is a point on M (remember A is a covector *field*).

We can form the **exterior product** of two differential 1-forms to obtain a differential 2-form:

$$A \wedge B = A_\mu B_\nu\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \sum_{\mu<\nu}(A_\mu B_\nu - A_\nu B_\mu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = (A_\mu B_\nu - A_\nu B_\mu)\,\mathrm{d}x^{|\mu\nu|} \tag{B2}$$

because the exterior product is associative and antisymmetric on 1-forms. $\mathrm{d}x^{\mu\nu}$ is short for $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, and the bars $|\mu\nu|$ denote summation over $\mu < \nu$. In general, a p-form is then a sum of exterior products of p 1-forms. If we note d the dimension of the fiber, then differential forms only exist if $p \leq d$ because an exterior product of $(d+1)$ 1-forms is necessarily 0 since the exterior product is antisymmetric, and there are only d independent 1-forms.

We can also take the **exterior derivative** of a p-form A:

$$\mathrm{d}A = \partial_\mu A_{\nu_1\cdots\nu_p}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \tag{B3}$$

So the exterior derivative also raises the degree of the differential form. Naturally, the 1-form $\mathrm{d}x^\mu$ is the exterior derivative of the 0-form (scalar field) x^μ.

As a straightforward consequence of Schwartz's theorem, the exterior derivative is nilpotent: $\mathrm{d}^2 A = 0$ for any differential form A.

Although it is more practical to define p-forms iteratively from 1-forms and the exterior product, they do have an independent mathematical characterization as antisymmetric $(0, p)$-tensor fields.

Appendix C: Derivation of the Klein-Gordon Lagrangian

We are looking for a relativistic wave equation, so we begin with the relativistic mass-shell formula:

$$E^2 = p^2 + m^2 \tag{C1}$$

where E is the energy, p the 3-momentum and m the mass. Then we apply the quantization rule $\hat{P}^\mu = i\partial^\mu$, yielding:

$$-(\partial_0)^2 = -\nabla^2 + m^2 \tag{C2}$$

which, when applied to a scalar field ϕ, is the Klein-Gordon equation:

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi = 0 \tag{C3}$$

Now, if we take:

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi^* - m^2 \phi \phi^* \tag{C4}$$

we obtain:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*} = \partial^\mu \partial_\mu \phi \tag{C5}$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \tag{C6}$$

and the Euler-Lagrange equation correctly reproduces the Klein-Gordon equation.

Appendix D: Covariant Newtonian dynamics

This appendix aims to provide a simple example of gauge theory in the context of classical mechanics. Here, the manifold under study is 1-dimensional (it only represents time), and to each point on this manifold we associate a vector space representing the Galilean reference frame in which we study the dynamics of our system (so we have a vector bundle). The fields on the time manifolds are the functions of time. Therefore, even though we are dealing with discrete coordinates, Newtonian dynamics is a *1-dimensional field theory* on the time manifold, where the coordinate vector $\mathbf{q}(t)$ of the point particle under study is our field. In this theory, we have the following free equation of motion:

$$\frac{\mathrm{d}^2 \mathbf{q}}{\mathrm{d}t^2} = 0 \tag{D1}$$

(For simplicity, the mass constant is absorbed in the coordinate vector.) This leads to the following Lagrangian:

$$L_{\text{inertial}}(\mathbf{q}) = \frac{1}{2}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)^2 \tag{D2}$$

This Lagrangian is of course invariant under a global Special Euclidean transformation, *i.e.* it does not change if at each point in time we rotate (or translate) the Galilean reference frame. A SE(3) infinitesimal transformation is written:

$$\mathbf{q} \mapsto \mathbf{q} + \boldsymbol{\alpha} \times \mathbf{q} + \boldsymbol{\beta} \tag{D3}$$

for some infinitesimal vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$, respectively representing a rotation angle and a linear displacement. Using SE(3) as our gauge group, we can try to impose local invariance on the Newton Lagrangian. *Local* must be understood here in the context of the 1D time manifold, hence a local SE(3) transformation consists in making the transformation parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ depend on time. From the physical point of view, it means ours frames would not be Galilean anymore, because the frame at some time t_1 could be rotated with respect to the frame at t_0, so the reference frame must loose its "Galileanity" at some point between t_0 and t_1.

Under a gauge transformation, additional terms come out of the time derivative now that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ depend on time:

$$L'_{\text{inertial}}(\mathbf{q}) = \frac{\dot{\mathbf{q}}^2}{2} + (\dot{\boldsymbol{\alpha}} \times \mathbf{q}) \cdot \dot{\mathbf{q}} + \dot{\boldsymbol{\beta}} \cdot \dot{\mathbf{q}} \tag{D4}$$

We introduce gauge fields $\boldsymbol{\omega}(t), \mathbf{v}(t)$ transforming as $\boldsymbol{\omega}' = \boldsymbol{\omega} - \dot{\boldsymbol{\alpha}}$ and $\mathbf{v}' = \mathbf{v} - \dot{\boldsymbol{\beta}}$.

$$L_{\text{covariant}}(\mathbf{q}) = \frac{\dot{\mathbf{q}}^2}{2} + (\boldsymbol{\omega} \times \mathbf{q}) \cdot \dot{\mathbf{q}} + \mathbf{v} \cdot \dot{\mathbf{q}} \tag{D5}$$

This can be rewritten using the covariant derivative:

$$\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} \stackrel{\text{def}}{=} \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} + \boldsymbol{\omega} \times \mathbf{q} + \mathbf{v} \tag{D6}$$

$$L_{\text{covariant}}(\mathbf{q}) = \frac{1}{2}\left(\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}\right)^2 \tag{D7}$$

which, when we add an external potential $V(t)$, leads to **Newton's Covariant Second Law**, valid in any reference frame:

$$\frac{\mathrm{D}^2\mathbf{q}}{\mathrm{D}t^2} = \mathbf{F} \tag{D8}$$

where $\mathbf{F} = -\boldsymbol{\nabla} V$ is the external force. Developing the covariant derivative yields a kinetic term (of course) along with the inertial forces that appear as a consequence of non-Galileanity.

The Lagrangian should be adequately modified to account for the gauge fields' dynamics:

$$L_{\text{gauge}}(\mathbf{q}) = \frac{1}{2}\left(\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}\right)^2 - \mathbf{L} \cdot \boldsymbol{\omega} - \mathbf{P} \cdot \mathbf{v} \tag{D9}$$

where $\mathbf{L}, \mathbf{P}$ are some constant vectors, leading to the following equations of motion:

$$\mathbf{q} \times \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} = \mathbf{L} \qquad ; \qquad \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t} = \mathbf{P} \tag{D10}$$

which express the conservation of angular and linear momenta in non Galilean reference frames.

Thus, applying a gauge theory to Newtonian dynamics yields several results in a very elegant way:

1. the covariant derivative is a general case of Bour's formula for changing the derivation frame;

2. the coupling of the gauge fields to the field $\mathbf{q}$ is just the inertial forces.

Of course, this is a non-relativistic theory. In a relativistic theory, time cannot be separated from space, and thus we cannot use discrete coordinates (that is, a 1D field theory) as we did here, but we must use a 4D field theory where space and time are on the same footing. This explains why field theory is the natural setting of relativistic theories.

Appendix E: Details on the general gauging procedure

Let (M, E) be a vector bundle of base manifold M and fiber E. A symmetry group G acts on E, and its generators $T_\mathfrak{a}$ form a basis of the Lie algebra $\mathfrak{g}$. Greek indices run over M, Latin indices run over E and fraktur indices run over $\mathfrak{g}$. The Lie algebra is endowed with a Lie bracket characterized by its structure constants $f_{\mathfrak{bc}}{}^{\mathfrak{a}}$ such that $[T_\mathfrak{b}, T_\mathfrak{c}] = f_{\mathfrak{bc}}{}^{\mathfrak{a}} T_\mathfrak{a}$.

Each generator introduces a gauge field $B^\mathfrak{a}_\mu$, and the exterior derivative reads:

$$D_\mu \overset{\text{def}}{=} \partial_\mu + B^\mathfrak{a}_\mu T_\mathfrak{a} \tag{E1}$$

The generators can be represented by matrices $(T_\mathfrak{a})^a{}_b$ acting on the fibers. We can introduce the **connection** ω, a **Lie algebra-valued 1-form** which can be represented by a matrix of differential 1-forms defined by

$$\omega^a{}_b \overset{\text{def}}{=} \omega^a{}_{b\mu} \, \mathrm{d}x^\mu \overset{\text{def}}{=} B^\mathfrak{a}_\mu T_\mathfrak{a}{}^a{}_b \, \mathrm{d}x^\mu \tag{E2}$$

We can now apply the covariant derivative to a vector in E:

$$D_\mu v^a = \partial_\mu v^a + B^\mathfrak{a}_\mu T_\mathfrak{a}{}^a{}_b v^b = \partial_\mu v^a + \omega^a{}_{b\mu} v^b \tag{E3}$$

The exterior derivative also needs to be modified into an **exterior covariant derivative**. Let $(A^a{}_b)_{1 \leq a,b \leq \dim E}$ be a Lie algebra-valued differential form, then:

$$DA^a{}_b \overset{\text{def}}{=} \mathrm{d}A^a{}_b + \omega^a{}_c \wedge A^c{}_b \tag{E4}$$

is the exterior covariant derivative of A. This equation is more compactly written $DA = \mathrm{d}A + \omega \wedge A$. The **curvature** $\Omega \overset{\text{def}}{=} D\omega$ is a Lie algebra-valued 2-form. Using previous expressions, we get:

$$\begin{aligned}
\Omega^a{}_b &= \mathrm{d}\omega^a{}_b + \left(\omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}\right) \mathrm{d}x^{|\mu\nu|} \\
&= \mathrm{d}\omega^a{}_b + \left(B^\mathfrak{a}_\mu T_\mathfrak{a}{}^a{}_c B^\mathfrak{b}_\nu T_\mathfrak{b}{}^c{}_b - B^\mathfrak{b}_\nu T_\mathfrak{b}{}^a{}_c B^\mathfrak{a}_\mu T_\mathfrak{a}{}^c{}_b\right) \mathrm{d}x^{|\mu\nu|} \\
&= \mathrm{d}\omega^a{}_b + B^\mathfrak{a}_\mu B^\mathfrak{b}_\nu (T_\mathfrak{a}{}^a{}_c T_\mathfrak{b}{}^c{}_b - T_\mathfrak{b}{}^a{}_c T_\mathfrak{a}{}^c{}_b) \, \mathrm{d}x^{|\mu\nu|} \\
&= \left(\partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + B^\mathfrak{a}_\mu B^\mathfrak{b}_\nu [T_\mathfrak{a}, T_\mathfrak{b}]^a{}_b\right) \mathrm{d}x^{|\mu\nu|} \\
&= \underbrace{\left(\partial_\mu B^\mathfrak{a}_\nu - \partial_\nu B^\mathfrak{a}_\mu + B^\mathfrak{b}_\mu B^\mathfrak{c}_\nu f_{\mathfrak{bc}}{}^\mathfrak{a}\right) T_\mathfrak{a}{}^a{}_b}_{\overset{\text{def}}{=} \Omega^a{}_{b\mu\nu}} \mathrm{d}x^{|\mu\nu|}
\end{aligned} \tag{E5}$$

where the **field strength tensor** $\Omega^a{}_{b\mu\nu}$ is obtained from the components of Ω.

Finally, let us recall the second Bianchi identity, which is sometimes useful:

$$D\Omega = 0 \tag{E6}$$

[1] L. Delafosse, Tensor calculus for physicists, Strasbourg Students Physical Letters **1**, 0201, hal-04722859v1.

[2] M. Bañados and I. Reyes, A short review on noether's theorems, gauge symmetries and boundary terms, International Journal of Modern Physics D **25**, 1630021 (2016).

[3] D. J. Griffiths, *Introduction to Elementary Particles* (Wiley, 2008).

[4] J. Bennett, A pedagogical review of gravity as a gauge theory (2021), arXiv.2104.02627.