PHYSICS 200B: ELECTROMAGNETISM I

IAN LIM
LAST UPDATED FEBRUARY 27, 2020

These notes were taken for Physics 200B, *Electromagnetism I*, as taught by Rena Zieve at the University of California, Davis in winter quarter 2020. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Tuesday, January 7, 2020

Note. This course will have discussion sections! I will not be writing notes for those since they're like homework. Exams will be closed-book but with a formula sheet—two pages, front and back, typed okay. The final exam can be moved with signed consent of all students.

The textbook for this course is Zangwill *Modern Electrodynamics*. The plan is to cover chapters 1-13, mostly electrostatics and magnetostatics (time-independent stuff). Homeworks due Tuesdays apart from the very last one, which is due Thursday.

Most of Chapter 1 of Zangwill is mathematical preliminaries. Chapter 2 is largely a review of Maxwell's equations. There is one important auxiliary to the Maxwell equations, though, and it's the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{1.1}$$

This equation just says there's stuff (ρ) and the stuff can flow in a way defined by the vector field $\mathbf{j}$. The divergence $\nabla \cdot \mathbf{j}$ tells us how much charge is flowing out of a point, on average.

To review, the first two Maxwell's equations are the following:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.3}$$

The next one we might guess is

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{1.4}$$

that current flow sources magnetic fields. This equation can't be quite right, because if we take the divergence, the divergence of a curl always vanishes, so

$$0 = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0 \boldsymbol{\nabla} \cdot \mathbf{j}, \tag{1.5}$$

which by the continuity equation says that $\frac{\partial \rho}{\partial t} = 0$ everywhere. That seems nonphysical, so let us amend the equation by adding a $\mathbf{j}_D$ term, called the *displacement current*. That is,

$$0 = \mathbf{j_D} + \mu_0 \left(-\frac{\partial \rho}{\partial t} \right) = -\mu_0 \epsilon_0 \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_D. \tag{1.6}$$

This *defines* the displacement current as

$$\mu_0 \epsilon_0 \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_D, \tag{1.7}$$

and since we recall that $\mu_0 \epsilon_0 = \frac{1}{c^2}$, we can rewrite what we recognize as Ampère's law as

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{1.8}$$

The last Maxwell equation is Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.9}$$

These are Maxwell's equations in vacuum, and they can be modified to describe fields in materials with some little tweaks.

There's also a "glitch" in Zangwill's exposition. In the text, Zangwill mentions that Maxwell wrote down 12 equations prior to the development of vector notation (relating each of the components), but in fact he wrote down *eight*. Why eight? Each of the curl equations is a vector equation (three components), but each of the divergence equations is only one (a scalar equation). Hence $2 \times 3 + 2 \times 1 = 8$.

Altogether, we have the differential forms of Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.10}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.11}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.12}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{1.13}$$

We should also be familiar with the integral forms of the Maxwell equations,

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} Q_{\text{enc}} \tag{1.14}$$

$$\int \mathbf{B} \cdot d\mathbf{S} = 0 \tag{1.15}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} \tag{1.16}$$

$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 I_{\text{enc}} + \frac{1}{c^2} \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{S}. \tag{1.17}$$

If we have matter, then the laws need to be adapted because generically, charges and magnetic dipoles reorient themselves based on the applied fields. Materials have electric and magnetic polarizabilities. It's useful to define the total charge as a sum

$$\rho = \rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}, \tag{1.18}$$

where $\mathbf{P}$ is the *polarization vector*. That is, we can put some charge ρ_f on a material, but in general when there's a background applied field, the charges will redistribute themselves. We know this as the fact that charges in a dielectric generically rearrange themselves to oppose the applied field. Recall that $\mathbf{P}$ points from negative to positive charge. Thus a positive divergence of $\mathbf{P}$ acts like a local negative charge.

We can do the same for magnetism, though ti's a bit messier.

$$\mathbf{j} = \mathbf{j}_f + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M}, \tag{1.19}$$

where $\mathbf{M}$ is the *magnetization vector*, which we think of as little magnetic dipoles or eddy currents orienting themselves to minimize their energy. This one depends on the current due to free charge $\mathbf{j_f} = \frac{\partial \rho_f}{\partial t}$, the rate of change of the bound charge from polarization, and the curl of the magnetization.

This leads us to define auxiliary fields:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \tag{1.20}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B}. \tag{1.21}$$

For the electric field, the sign conventions (whether to add or subtract $\mathbf{p}$ make sense. It's very natural for the polarization pointing from $-$ to $+$ to add to an effective field $\mathbf{D}$. But for the magnetic field, it could go either way. Magnets near superconductors induce a magnetization that perfectly cancels the applied field, whereas in ferromagnets like iron, the magnetization is in the same direction as the applied field. We just have to pick a convention and stick to it.

The Maxwell equations in materials take on kind of a nice form:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \tag{1.22}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.23}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.24}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.25}$$

Helmholtz theorem The Helmholtz theorem says that if we specify the divergence and curl of a function everywhere, then the function is uniquely defined up to adding functions which have zero divergence and zero curl. Such functions are called *harmonic functions*, which we'll see more later, and whether or not we should add them corresponds to fitting boundary conditions.

We can actually write the electric field explicitly in terms of the divergence and curl:

$$\mathbf{E}(\mathbf{r}) = -\underbrace{\boldsymbol{\nabla} \int d^3r' \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}}_{\text{curl}=0} + \underbrace{\boldsymbol{\nabla} \times \int d^3r' \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}}_{\text{div}=0}. \tag{1.26}$$

The second term is actually zero if there is no time dependence, from Faraday's law. If we apply Gauss's law to rewrite $\boldsymbol{\nabla}' \cdot \mathbf{E}$,, we recognize what remains as

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = -\boldsymbol{\nabla} \int d^3r' \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.27}$$

We've recovered Coulomb's law, just written in a more vectorial language.

Let's consider the force on a charge distribution ρ^* due to the field from a distribution ρ. That is,

$$\mathbf{F}(\rho, \rho^*) = \int d^3r \rho^*(\mathbf{r}) \mathbf{E}(\mathbf{r}) \tag{1.28}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r' \frac{\rho^*(\mathbf{r})\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.29}$$

But we notice that we can exchange $\mathbf{r}$ and $\mathbf{r}'$ at the cost of picking up a sign flip:

$$\mathbf{F}(\rho, \rho^*) = -\mathbf{F}(\rho^*, \rho). \tag{1.30}$$

If we set $\rho = \rho^*$, then $\mathbf{F}(\rho, \rho) = 0$, which tells us that a charge distribution cannot exert a net force on itself.

Finally, let's introduce potentials. The electric field in electrostatics can be written as the gradient of the scalar potential,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \tag{1.31}$$

so that

$$\frac{\rho}{\epsilon_0} = \boldsymbol{\nabla} \cdot \mathbf{E} = -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = -\boldsymbol{\nabla}^2\phi. \tag{1.32}$$

The equation

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\epsilon_0} \tag{1.33}$$

is hopefully familiar to us as Poisson's equation, and the homogeneous case ($\rho = 0$) is

$$\boldsymbol{\nabla}^2\phi = 0, \tag{1.34}$$

which is Laplace's equation.

Lecture 2.

Thursday, January 9, 2020

Today we'll start our discussion with Gauss's law, moving rapidly into chapter 3 of Zangwill. Gauss's law lets us calculate electric fields rapidly for situations with high amounts of symmetry. Basically, we can solve systems with spherical symmetry, cylindrical symmetry, and translational plane symmetry. It doesn't get us too far but it's a lot better than Coulomb's law volume integrals.

As we know, Gauss's law (differential form) states

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

or in integral form

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V \boldsymbol{\nabla} \cdot \mathbf{E} d^3r = \int_V \frac{\rho}{\epsilon_0} d^3r = \frac{Q_{\text{enc}}}{\epsilon_0}. \tag{2.2}$$

The simplest example is for the point charge. For a single positive charge $q > 0$, we can reason that the field must point radially outwards and it is constant at surfaces of constant R, so that

$$\oint_S \mathbf{E} \cdot d\mathbf{Q} = |E| 4\pi R^2 = \frac{q}{\epsilon_0}, \tag{2.3}$$

which yields

$$\mathbf{E} = \frac{1}{4\pi R^2 \epsilon_0} \hat{\mathbf{r}}. \tag{2.4}$$

Gauss's law also gives us a nice result (sometimes known as Newton's shell theorem), which says that the electric field due to a spherically symmetric shell of charge *inside* that shell ($r < R$) is zero.

We can also do Gauss's law for an infinite plane by drawing a Gaussian pillbox, say, extending a height h above and below the plane. Rotational symmetry lets us reason that the field can only point in the normal direction to the plane, while reflection symmetry says its magnitude is the same equal distances above and below the plane. It follows that

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \sigma A, \tag{2.5}$$

where A is the area of the face of the Gaussian pillbox. Since the field is normal to the plane and the faces of the pillbox are oriented outwards, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2E(h)A, \tag{2.6}$$

so that

$$\mathbf{E}(h) = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z < 0. \end{cases} \tag{2.7}$$

The reason there is no h dependence is because the infinite plane is scale-invariant. If we rescale all the coordinates, the infinite plane still looks like an infinite plane. This result isn't necessarily useful to us because real life is filled with infinite planes of charge, or even because the problem is exactly solvable; rather, it's because any reasonably smooth (in the mathematical sense) surface looks locally flat, which means that near the surface, we have essentially the field from an infinite plane.

By the same Gaussian pillbox arguments, we find that $E_{\parallel}$ is continuous at surfaces of charge, while $E_{\perp}$ is discontinuous by $\frac{\sigma}{\epsilon_0}$. How should we calculate the force on a little surface of charge, given that the electric field is discontinuous above and below the field? Well, we can just take the average. This seems physically reasonable, but we can justify it. Remember the discontinuity comes from the charge at the surface itself, and as we showed last time, charges cannot exert forces on themselves. So if we average above and below the surface, we will basically average away the contribution of the charged surface itself and get the right answer.

We can also state Earnshaw's theorem– in a closed region with no charge, any extrema of the potential must be on the boundary. For suppose there was a local maximum of the potential in the interior. The gradient of the potential vanishes at that point, and a little bit away we can draw a Gaussian surface it is pointing towards that point everywhere. It follows that the **E**-field points away everywhere, so our integral $\oint_S \mathbf{E} \cdot d\mathbf{A} > 0$, which violates our assumption that there was no charge in the region.[1] This tells us that we cannot make an electrostatic cage; no charge distribution can hold itself in a static configuration under Coulomb interactions alone. Earnshaw's theorem has told us that any charge dropped in a region with no other charge will move to the edges of that region, since it cannot sit at an extremum.

Potential and potential energy For our purposes, we will follow Zangwill and denote potential energy by V and electrostatic potential by φ. We say the potential energy changes as

$$\delta V = -\mathbf{F} \cdot d\mathbf{s} \tag{2.8}$$

for small displacements, i.e.

$$\mathbf{F} = -\boldsymbol{\nabla} V. \tag{2.9}$$

In addition, when forces are due only to electric fields, then

$$\mathbf{F} = q\mathbf{E} = -q\boldsymbol{\nabla}\varphi \implies V = q\varphi. \tag{2.10}$$

Hence the electrostatic potential is the energy per charge.

Let's now prove Green's reciprocity relation for the potential energy. That is, suppose there are two charge distributions ρ_1, ρ_2. The energy of ρ_2 in the potential φ_1 created by ρ_1 is

$$V = \int d^3r\, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}), \tag{2.11}$$

such that

$$\delta V = \int d^3r\, [\rho_2(\mathbf{r} - \delta\mathbf{s}) - \rho_2(\mathbf{r})]\varphi_1(\mathbf{r}). \tag{2.12}$$

The minus sign comes from that active/passive transformation jazz. If the distribution is moved by $\delta\mathbf{s}$, then looking at the "same" point in space $\mathbf{r}$ for the new distribution is equivalent to looking at the original distribution at a point $\mathbf{r} - \delta\mathbf{s}$.

We can now Taylor expand as $\delta\mathbf{s}$ gets small, such that

$$\delta V \approx -\delta\mathbf{s} \cdot \int d^3r\, (\boldsymbol{\nabla}\rho_2(\mathbf{r}))\varphi_1(\mathbf{r}) = -\delta\mathbf{s} \cdot \int d^3r(\boldsymbol{\nabla}(\rho_2\varphi_1) - \rho_2\boldsymbol{\nabla}\varphi_1), \tag{2.13}$$

after a product rule manipulation (basically an integration by parts). Now this first term is a total derivative, so it vanishes as we take our integration region d^3r to be all space. What's left is

$$\delta V = -\delta\mathbf{s} \cdot \int d^3r\, \rho_2(\mathbf{r})\mathbf{E}_1(\mathbf{r}). \tag{2.14}$$

Let us rewrite this energy in a different way:

$$V = \int d^3r\, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r'\, \rho_2(\mathbf{r}) \frac{\rho_1(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.15}$$

[1] One can also argue this directly from the form of Laplace's equation.

But notice that this integral is manifestly symmetric in $\mathbf{r}$ and $\mathbf{r}'$ (i.e. $|\mathbf{r}-\mathbf{r}'| = |\mathbf{r}'-\mathbf{r}|$). This certainly converges, so we can switch the order of integration and do the d^3r integral first to find

$$V = \int d^3r\, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}) = \int d^3r'\, \rho_1(\mathbf{r}')\varphi_2(\mathbf{r}'). \tag{2.16}$$

This relation means that the energy in an electrostatic charge distribution doesn't depend on how we build it, only on the final geometry.

Consider the following example.

Example 2.17. Suppose we have a spherical region of radius R centered on the origin with no charge inside. As it turns out, the potential at the center is the average potential on the surface:[2]

$$\varphi(0) = \frac{1}{4\pi R^2}\int dS\varphi(\mathbf{r}). \tag{2.18}$$

Can we use Green's reciprocity relation to find this result? Take

$$\varphi_1 = \varphi, \quad \rho_1 = \rho; \quad \rho_2(\mathbf{r}) = \frac{q}{4\pi R^2}\delta(r-R). \tag{2.19}$$

Then

$$\varphi_2 = \begin{cases} \frac{q}{4\pi\epsilon_0 r} & r \geq R, \\ \frac{q}{4\pi\epsilon_0 R} & r \leq R. \end{cases} \tag{2.20}$$

It follows that

$$\int d^3r \frac{q}{4\pi R^2}\delta(r-R)\varphi(\mathbf{r}) = \int d^3r\rho(\mathbf{r})\varphi_2(\mathbf{r}) = \int_R^\infty r^2 dr \int d\Omega\rho(\mathbf{r})\frac{q}{4\pi\epsilon_0 r}. \tag{2.21}$$

But this last expression up the the factor of q is exactly the integral we would use to compute the potential at the origin. That is,

$$\int_R^\infty r^2 dr \int d\Omega\rho(\mathbf{r})\frac{q}{4\pi\epsilon_0 r} = q\varphi(\mathbf{r}=0). \tag{2.22}$$

Meanwhile, the LHS says that we just integrate

$$\int d^3r \frac{q}{4\pi R^2}\delta(r-R)\varphi(\mathbf{r}) = \frac{q}{4\pi R^2}\int dS\,\varphi(\mathbf{r}). \tag{2.23}$$

Cancelling the factors of q, we have exactly the desired result. The potential at the center of the sphere is equal to the average potential on its surface.

We can write the potential energy of a charge distribution as

$$U_E = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^N\sum_{i>j}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{2.24}$$

or for continuous distributions,

$$U_E = \frac{1}{8\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{2}\int d^3r\rho(\mathbf{r})\varphi(\mathbf{r}) \tag{2.25}$$

Note that for all-positive or all-negative distributions of charges, these formulae make it clear that the energy of assembling the distribution is positive-definite. The answer is somewhat less clear when we have a mix of charges. Let's manipulate this result to see what happens.

$$\begin{aligned} U_E &= \frac{1}{2}\int d^3r\rho(\mathbf{r})\varphi(\mathbf{r}) \\ &= \frac{1}{2}\int d^3r\epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}\varphi(\mathbf{r}) \\ &= \frac{\epsilon_0}{2}\int d^3r(\boldsymbol{\nabla}\cdot(\mathbf{E}\varphi) - \mathbf{E}\cdot(\boldsymbol{\nabla}\phi)) \\ &\to \frac{\epsilon_0}{2}\int d^3r|E|^2 \end{aligned}$$

[2] Again, this is a property of Laplace's equation. For a good reference on this, see Evans *Partial Differential Equations*, section 2.2.2.

where the last step is taking the limit as the region of integration becomes all space, and the total derivative term goes away.

So U_E for continuous distributions defined in this way is *positive-definite*. How does this square with our idea that we could just bring one positive and one negative charge together? Well, it takes energy to assemble the charges in the first place. In fact, one can check that the energy of a point charge is divergent since it's a finite amount of charge squished into a single point of space. Part of the energy in this expression (summing up the energy in the E-field) comes from actually assembling the charge distribution, and we'll get some practice with this on the homework.

Lecture 3.

Tuesday, January 14, 2020

Multipole expansion Today we're moving into Chapter 4 of Zangwill, on multipoles. Multipoles are a way of thinking about the field of a charge distribution far away from that distribution. For instance, if there is a net charge in that distribution, far away we have a field that decays as $1/r^2$, like the field of a point charge.

Suppose we have a charge distribution $\rho(\mathbf{r}')$ contained entirely within a sphere of radius R. That is, $\rho(\mathbf{r}') = 0$ for $|\mathbf{r}'| > R$. Then we can look at a point $\mathbf{r}$ outside the sphere with $|\mathbf{r}| > R$. Well, our potential is just given by

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.1}$$

and we'd like to expand the denominator when $|\mathbf{r}| \gg |\mathbf{r}'|$. Our expansion looks like

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} - \mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2}(\mathbf{r}'\cdot\boldsymbol{\nabla})^2\frac{1}{r} - \dots \tag{3.2}$$

If we're being careful we should really expand in r'/r as a small parameter. The derivatives are all with respect to unprimed coordinates, and then the $\mathbf{r}'$ determines which direction we're looking. This is much better in index notation. With the Einstein summation convention (i.e. sum over repeated indices), this becomes

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} - r_i'\boldsymbol{\nabla}_i\frac{1}{r} + \frac{1}{2}r_i'r_j'\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j\frac{1}{r} + \dots \tag{3.3}$$

If we plug this expansion back into the expression for the potential, we get

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\underbrace{\left[\int d^3r'\rho(\mathbf{r}')\right]}_{\text{monopole}=Q}\frac{1}{r} - \underbrace{\left[\int d^3r'\rho(\mathbf{r}')r_i'\right]}_{\text{dipole}=\mathbf{p}}\partial_i\frac{1}{r} + \underbrace{\left[\frac{1}{2}\int d^3r'\rho(\mathbf{r}')r_i'r_j'\right]}_{\text{quadrupole}=Q_{ij}}\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j\frac{1}{r} + \dots\right\} \tag{3.4}$$

A note on tensors. Tensors are like collections of numbers that have special properties under coordinate transformations. A rank 1 tensor is just a vector; we need only one number to label which component we are looking at. So if we write a vector $\mathbf{v} = (a, b, c)$ then we can also write its components v_i such that $v_1 = a, v_2 = b, v_3 = c$.

Similarly matrices are rank two tensors— we need two index labels to say which element we are looking at. That is, a 3×3 matrix requires two index labels to pick out an element, a row and a column. We can guess what a rank three tensor looks like, i.e. some sort of cube of numbers. After rank three it's best to just think of these in the abstract. However, not any collection of numbers is a tensor. Each row and column of a matrix must transform like a vector.

Note that many of the complications of working with tensors (e.g. where do you put the indices, up or down?) will drop out when we work in flat Cartesian coordinates.

Can we study the limiting behavior of this expansions? Sure, if we perform some dimensional analysis. We took our charge to be confined to a sphere of radius R; the monopole moment is independent of R, while the dipole and quadrupole components scale as

$$p_i \sim QR, \quad Q_{ij} \sim QR^2. \tag{3.5}$$

Derivatives with respect to the **r** unprimed coordinates act like $1/r$, so our approximation looks like

$$\varphi(r) \sim \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} - \frac{QR}{r^2} + \frac{QR^2}{r^3} + \ldots\right]. \tag{3.6}$$

If we work out the derivatives, we get

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{p_i r_i}{r^3} + Q_{ij}\frac{3r_i r_j - r^2\delta_{ij}}{r^5} + \ldots\right\} \tag{3.7}$$

Let us find the field of the dipole term far away.

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi_{\text{dipole}} \tag{3.8}$$

$$= -\frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right). \tag{3.9}$$

We will write this vector in terms of its components. Note that $\partial_i r = \partial_i\sqrt{x^2+y^2+z^2} = \frac{r_i}{\sqrt{x^2+y^2+z^2}} = r_i/r$. Then

$$\begin{aligned}
E_i &= -\frac{1}{4\pi\epsilon_0}\partial_i\left(\frac{p_j r_j}{r^3}\right) \\
&= -\frac{1}{4\pi\epsilon_0}\left[p_j\frac{\delta_{ij}}{r^3} - \frac{3p_j r_j}{r^4}\partial_i r\right] \\
&= -\frac{1}{4\pi\epsilon_0}\left[p_j\frac{\delta_{ij}}{r^3} - \frac{3p_j r_j r_i}{r^5}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}\cdot\mathbf{r}}{r^5}r_i - \frac{p_i}{r^3}\right].
\end{aligned}$$

Equating these as vectors gives

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\mathbf{p}\cdot\hat{\mathbf{r}}}{r^3}\hat{\mathbf{r}} - \frac{\mathbf{p}}{r^3}\right]. \tag{3.10}$$

One can model a physical dipole as two charges, one $+q$ and one $-q$, separated by a distance a and connected by a rigid "rod" of some sort. Then the dipole moment is $q\mathbf{a}$. One can imagine shrinking this dipole down by keeping the product qa fixed while taking $a \to 0$. In a sense, a dipole is like a derivative of a delta function.

In constant electric fields, there is no net force on a dipole since each of the charges experiences an equal and opposite force. (In general there will be a torque.) It's only when the field is changing that we experience a net force, so we may think of dipoles as sensitive to gradients in the field.

The quadrupole The quadrupole moment was defined as

$$Q_{ij} = \frac{1}{2}\int d^3r'(\rho(\mathbf{r}')r_i' r_j', \tag{3.11}$$

and from this expression we see that $Q_{ij} = Q_{ji}$, e.g. $Q_{12} = Q_{21}$, and so on. It is symmetric, and hence in 3 dimensions there are 6 independent components. If we choose our axes really well, we can diagonalize the quadrupole tensor and just compute three components.

We can also define the *traceless quadrupole tensor* as

$$\Theta_{ij} = 3Q_{ij} - Q_{kk}\delta_{ij}. \tag{3.12}$$

That is, if we take the trace of this, then we get

$$\Theta_{ii} = 3Q_{ii} - Q_{kk}\delta_{ii} = 3Q_{ii} - Q_{kk}(3) = 0, \tag{3.13}$$

since the trace of the Kronecker delta in 3 dimensions is 3. (More generally, it is the dimension of the space in d dimensions.) This will have *five* independent components, since we have imposed one algebraic constraint on the six components (that the trace vanishes).

Why is this useful? Well, we can rewrite the quadrupole potential as

$$\varphi_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}Q_{ij}\frac{3r_i r_j - \delta_{ij}r^2}{5} = \frac{1}{4\pi\epsilon_0}\Theta_{ij}\frac{r_i r_j}{r^5}. \tag{3.14}$$

If we ever need to go to higher orders, this trace-free condition becomes really useful, since we can take the trace with respect to any pair of indices, e.g. by setting $Q_{iij} = 0$ or $Q_{ijj} = 0$.

We can write the potential as

$$\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + Q_{ij} \frac{3\text{“}\cos\theta_i \cos\theta_j\text{”} - \delta_{ij}}{r^3}. \tag{3.15}$$

If we start writing out the expansion terms in spherical coordinates, we find that these are secretly (maybe not so secretly) Legendre and associated Legendre polynomials.

The reason for this is that our potential far away can be built out of solutions to Laplace's equation $\nabla^2 \varphi = 0$, since there is no charge there. In 204A, we considered separable solutions to Laplace's equation in spherical coordinates, which we wrote as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{A_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi). \tag{3.16}$$

It's perhaps not so surprising that the spherical harmonics arise like this. Moreover, if we think about how many A_{lm} we get for fixed l, there are $2l + 1$ choices for m. That is, the quadrupole moment, which corresponds to $l = 2$, really should have five independent components.[3]

For $l = 3$, we expect $2l + 1 = 7$ components. From the Cartesian perspective, there are a priori $3 \times 3 \times 3 = 27$ components. However, $Q_{ijk} = Q_{jik}$ and so on. Hence we get some different sets of components.

There's

$$Q_{111}, Q_{222}, Q_{333} \tag{3.17}$$

which are unaffected by the symmetry and all independent. There's also sets where all three indices are distinct, e.g.

$$Q_{123}, \tag{3.18}$$

and there are six of these (like Q_{213}), all related to this one by the symmetry. Then there are ones with two identical indices, like

$$Q_{122}, Q_{133}, Q_{112}, Q_{113}, Q_{233}, Q_{223} \tag{3.19}$$

Each of these is related to two other components by symmetry, e.g. $Q_{122} = Q_{212} = Q_{221}$, for a total of $3 \times 6 = 18$ components with two indices the same. Hence $3 + 6 + 18 = 27$.

All the other components are related to the ones we have listed by symmetry, so there are $3 + 1 + 6 = 10$ components. Moreover, if we impose three constraints on the traces

$$T_{iik} = 0, T_{iji} = 0, T_{ijj} = 0, \tag{3.20}$$

then we get $10 - 3 = 7$, which is exactly the right number of components as given by the spherical harmonic coefficients A_{2m}.

Lecture 4.

Thursday, January 16, 2020

It's very useful to note that we can expand angular dependence from the Legendre polynomials P_l in terms of spherical harmonics, i.e.

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=l}^{l} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi). \tag{4.1}$$

That is, we can express an arbitrary angular dependence in terms of spherical harmonics. For instance,

$$f(\theta, \phi; \theta', \phi') = \sum_{l'=0}^{\infty} \sum_{m=-l'}^{l'} A_{l'm}(\theta', \phi') Y_{l',m}(\theta, \phi), \tag{4.2}$$

[3] There's also a second solution to Laplace's equation scaling as r^l, but those solutions diverge as $r \to \infty$, so they represent interior rather than exterior solutions.

where we treat the $A_{l'm}$ as coefficients which depend on "fixing" some choice of θ', ϕ'. That is, $A_{l'm}$ is like picking a choice of coordinates, and in particular choosing f to be the Legendre polynomial $f(\theta, \phi; \theta', \phi') = P_l(\theta, \phi; \theta', \phi')$ tells us that

$$P_l(\theta, \phi; \theta', \phi') = \sum_{m=l}^{l} A_{lm}(\theta', \phi') Y_{lm}(\theta, \phi), \tag{4.3}$$

where

$$A_{lm}(\theta', \phi') = \frac{4\pi}{2l+1} Y_{lm}^*(\theta', \phi'). \tag{4.4}$$

That is, the only terms in the sum (4.2) we need are for $l' = l$.

The spherical harmonics Y_{lm} are polynomials in x, y, z of degree l. All terms in a given spherical harmonic either have even degree or odd degree. Moreover, we can make all terms have degree l by inserting $x^2 + y^2 + z^2$s, since this is equal to $r^2 = 1$ on the unit sphere. For instance,

$$P_2 = \frac{3}{2}z^2 - \frac{1}{2} = \frac{3}{2}z^2 - \frac{1}{2}(x^2 + y^2 + z^2) = z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2. \tag{4.5}$$

Hence we can always increase the power of a term by 2. Similarly Y_{3m} will contain terms like

$$x^3, y^3, z^3, xyx, xxy, yxx, \ldots,$$

and this sort of counting argument is exactly analogous to our multipole expansion and how elements of the tensor were related by symmetry. Then the $x^2 + y^2 + z^2 = 1$ constraint gives us our trace constraints.

We could also make a transformation $x \to x' \cos\alpha + y' \sin\alpha, y \to -x' \sin\alpha + y' \cos\alpha$, and we see that under such rotations, the orders of the terms in Y_{lm} don't change. It follows that rotations don't mix Y_{lm}s between different orders l.

Let's now make an argument from completeness using a 2D delta function defined on the surface of a sphere. We'll write it in terms of unit vectors rather than angles to avoid prefactors of the sphere area. Hence we shall assert that the delta function can be expanded in this way,

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm}(\hat{\mathbf{w}}) Y_{lm}(\hat{\mathbf{v}}) \tag{4.6}$$

where $Y_{lm}(\hat{\mathbf{v}})$ indicates the spherical harmonic as a function of the angular coordinates of $\hat{\mathbf{v}}$. The (as yet undetermined) coefficients c_{lm} depend on $\hat{\mathbf{w}}$, just as $A_{l'm}$ depended on θ', ϕ'.

Now we observe that by the orthonormality of the spherical harmonics,

$$\int d\Omega_v \sum_{l=0}^{\infty} \sum_{m} c_{lm} Y_{lm}(\hat{\mathbf{v}}) Y_{l'm'}^*(\hat{\mathbf{v}}) = c_{lm} \delta_{ll'} \delta_{mm'} = c_{l'm'}(\hat{\mathbf{w}}). \tag{4.7}$$

But this is equivalent to

$$\int d\Omega_v \left(\sum_{l=0}^{\infty} \sum_{m} c_{lm} Y_{lm}(\hat{\mathbf{v}}) \right) Y_{l'm'}^*(\hat{\mathbf{v}}) = \int d\Omega_v \delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) Y_{l'm'}^*(\hat{\mathbf{v}}) = Y_{l'm'}^*(\hat{\mathbf{w}}) \tag{4.8}$$

It follows that

$$c_{lm}(\hat{\mathbf{w}}) = Y_{lm}^*(\hat{\mathbf{w}}), \tag{4.9}$$

so

$$\boxed{\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{v}}) Y_{lm}^*(\hat{\mathbf{w}}).} \tag{4.10}$$

If we choose $\hat{\mathbf{w}} = \hat{\mathbf{z}}$, i.e. we fix $\hat{\mathbf{w}}$ to point along the z axis, then we have *cylindrical* symmetry, not just spherical. But only the $m = 0$ spherical harmonics have cylindrical symmetry, so in fact

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{z}}) = \sum_{l=0}^{\infty} Y_{l0}(\hat{\mathbf{v}}) Y_{l0}^*(\hat{\mathbf{z}}). \tag{4.11}$$

Moreover, note that $P_l(\cos\theta = 0) = 1$,

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l, \tag{4.12}$$

so it follows that

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{z}}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{v}} \cdot \hat{\mathbf{z}}). \tag{4.13}$$

It follows that since the LHS of this expression depends only on the difference $\hat{\mathbf{v}} - \hat{\mathbf{z}}$ and the RHS depends only on the dot product $\hat{\mathbf{v}} \cdot \hat{\mathbf{z}}$, this difference is independent of rotating our choice of coordinates, which means that in fact

$$\boxed{\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{v}} \cdot \hat{\mathbf{w}}).} \tag{4.14}$$

Equating the boxed equations and taking this order by order in l, we find precisely that

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi), \tag{4.15}$$

which is none other than Eqn. (4.1).

Back to the physics. Recall that Coulomb's law gives us scalar potentials as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{4.16}$$

There's a nice expansion we can use for this inverse separation:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'), \quad r' < r. \tag{4.17}$$

This is really a very sensible way to *define* the Legendre polynomials. The RHS is an expansion in increasing powers of $1/r$, and purely on dimensional grounds, the RHS must have dimensions of $1/\text{distance}$, which means that increasing powers of $1/r$ must be accompanied by powers of r'. That is, we need powers of r'/r. Plugging in our expansion (4.1) we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left(\frac{r'}{r}\right)^l Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'). \tag{4.18}$$

Last time, we saw that

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=l}^{l} A_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \implies A_{lm} = \int d^3r' \frac{4\pi}{2l+1} r'^l Y_{lm}^*(\theta', \phi') \rho(\mathbf{r}'). \tag{4.19}$$

The interior solution for Laplace's equation (i.e. the one that doesn't diverge as $r \to 0$) just replaces $\frac{r'^l}{r^{l+1}} \to \frac{r^l}{(r')^{l+1}}$, and then the A_{lm}s are modified.

Conductors For our purposes, we will use the terms conductors and metals interchangeably. The charge carriers in a conductor reorganize themselves to respond to external applied fields. Specifically, they configure themselves so that the electric field inside is zero (equivalently, this minimizes the energy in the field). Thus $\mathbf{E} = 0$ inside. On the surface, the electric field is perpendicular to the surface. If this weren't the case, the charge would reshuffle to cancel this.[4] To sum up,

$$\mathbf{E} = 0 \text{ inside}, \tag{4.20}$$

$$E_{\parallel} = 0 \text{ on surface}. \tag{4.21}$$

It follows from Gauss's law and the definition of potential that

$$\rho = 0, \quad \varphi = \text{constant inside}. \tag{4.22}$$

Let's work out the classic example of a neutral conducting sphere in a constant applied field E_0. See Fig. 1. The total field in the sphere is therefore the sum of the applied field and the induced field from the rearranged charges in the sphere.

That is, if the applied field is

$$\mathbf{E}_0 = E_0 \hat{\mathbf{z}}, \tag{4.23}$$

[4] At least in the absence of an external magnetic field.

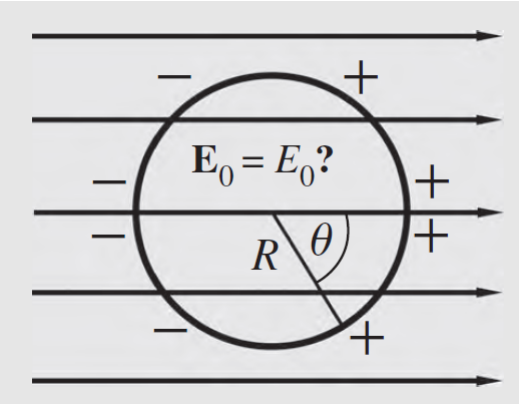


FIGURE 1. Zangwill Fig. 5.2.

then the self-field inside must be $-E_0\hat{\mathbf{z}}$. The potential from the induced charge is[5]

$$\varphi_{\text{self}}(r < R, \theta) = E_0 z = E_0 r\cos\theta. \tag{4.24}$$

A priori, the self-field outside is then

$$\varphi_{\text{self}}(r > R, \theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{A_0}{r} + \frac{A_1}{r^2}P_1(\cos\theta) + \frac{A_2}{r^3}P_2(\cos\theta) + \ldots\right] \tag{4.25}$$

We can now use matching conditions. The potential is continuous, so if we take $r = R$ and set φ_{self} inside equal to φ_{self} outside, then

$$E_0 R\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{A_1}{R^2}\cos\theta. \tag{4.26}$$

By the orthogonality of the Legendre polynomials, we can just match order by order in P_l, and if we solve for $A_1/4\pi\epsilon_0$, then

$$\varphi_{\text{self}}(r \geq R, \theta) = E_0 R^3\frac{\cos\theta}{r^2}. \tag{4.27}$$

This is *exactly* the field of a dipole. The quantity $\mathbf{p} = 4\pi R^3\epsilon_0\mathbf{E}$ is the dipole moment of this sphere, where $\alpha = 4\pi R^3$ is called the polarizability. In general, more complicated shapes will have a vanishing monopole moment but some some of higher multipole moments.

Suppose now we have some strangely-shaped conductor with a cavity machined out of it, and we place a small (say, positive) charge inside the cavity. Well, what happens to the charge in the conductor? Clearly the inside surface of the cavity must rearrange itself to cancel the field within the conductor. That takes some net negative charge on the inside surface. That is, negative charge accumulates to shield the interior. That charge had to come from somewhere, though. There must be a net positive charge on the exterior surface which rearranges itself in some energetically favorable way. The neat thing is that the solution on the exterior surface doesn't depend at all on the shape of the cavity, or where the charge is placed inside. This is what we mean by shielding. Our conductor hides the details of the interior and reveals only the net charge contained within.

Lecture 5.

Thursday, January 23, 2020

In full generality, we have an expression for the electrostatic potential near some charge distribution $\rho(\mathbf{r})$:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{5.1}$$

[5] This is clearly not constant! Nor should it be. The total potential is constant, and it is the sum of this induced potential and the potential sourcing the applied field.

However, this denominator is complicated; it depends both on $\mathbf{r}'$, which we integrate over, and also $\mathbf{r}$, which we don't. A very nice way to decompose this expression is in terms of Legendre polynomials, as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'). \tag{5.2}$$

If We're only interested in the potential on e.g. the z-axis, then this might be a good expansion to use. In that case, integrating over $\hat{\mathbf{r}}'$ is not too bad. But if we're interested in more general $\mathbf{r}$ (say, ranging over a sphere), we still want to separate the angles, and we can instead use the angle addition formula for the spherical harmonics, which says

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi'). \tag{5.3}$$

Capacitance Recall that the capacitance is defined as

$$C = \frac{Q}{V}, \tag{5.4}$$

where V is the potential relative to infinity. In a sense, the self-capacitance is the capacitance of the object with the other plate being a hollow sphere of infinite radius.

We're going to skip the details of the capacitance matrix and many-conductor systems; our main interest will be in systems with precisely two conductors, i.e. capacitors. Typically we put a $+Q$ charge on one and a $-Q$ charge on the other. These charges produce a potential difference V between the conductors, such that the ratio $C = Q/V$ is the capacitance.

The prototypical example is the parallel-plate capacitor, with two plates of area A and charge $+Q$ and $-Q$. So long as the plate separation d is much less than the lateral dimension $\sqrt{A}$, i.e. $\sqrt{A} \gg d$, we can approximate these as infinite plates. It follows that

$$\mathbf{E} = -\frac{Q}{\epsilon_0 A}\hat{\mathbf{z}}, \quad V = -\int_0^d E\, dz = \frac{Qd}{\epsilon_0 A}, \tag{5.5}$$

so that

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}. \tag{5.6}$$

Let's note that our plates aren't really infinite, so the fringing effects at the edges mean that the E-field is weaker at the edges. It follows that the potential difference is lower for the same charge, so the capacitance goes up. Apparently we have *underestimated* the capacitance.

Now we can calculate the energy in the capacitor as

$$U = \frac{1}{2}\int d^3r \rho(\mathbf{r}) V(\mathbf{r}) \tag{5.7}$$

Notice that if we have two conductors, this integral simplifies considerably. Suppose we have one conductor with a surface S_1 and a charge $+Q$ and another conductor with a surface S_2 and a charge $-Q$. Conductors are equipotentials, and it follows that since all the charge lies on the surface, the integral becomes

$$U = \frac{1}{2}\left[\int_{S_1} d^2S \sigma_+(\mathbf{r}) V_+ + \int_{S_2} d^2S \sigma_-(\mathbf{r}) V_-\right] = \frac{1}{2}[V_+ Q + V_-(-Q)] = \frac{1}{2}QV. \tag{5.8}$$

We can rewrite this as

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2, \tag{5.9}$$

and this has interesting consequences depending on whether we hold charge constant or voltage constant. That is, if we hold charge constant, then it makes sense that a higher-capacitance (in the case of parallel plates, smaller d) setup is energetically favored. But if we hold voltage constant, then the plates actually want to push apart (lower capacitance).

For our parallel plate capacitor, we can check that this works by integrating the energy in the E-field:

$$U_{\parallel} = \frac{1}{2}\epsilon_0 \int E^2 d^3r = \frac{\epsilon_0}{2}\frac{Q^2}{\epsilon_0^2 A^2} Ad = \frac{1}{2}QV. \tag{5.10}$$

Example 5.11 (The quantum dot)**.** We've just found that the self-energy of an object is given by

$$U = \frac{Q^2}{2C}. \tag{5.12}$$

For objects with large self-capacitances, adding a bit of charge doesn't affect the total energy much. But for very small objects on the nanoscale, this self-capacitance can be very small, so it may take a lot of energy to change the charge by a little bit. That is, U looks like a parabola centered at zero, and if we apply a "gate voltage," we can shift the zero over to e.g. $Q = 1/2$, so that whether we have 0 or 1 (electron) charges in the dot, both are equally favorable from an energy standpoint. Adding temperature into the picture can mess this up, though. Once the energy scale $k_B T$ becomes comparable to the energy cost of adding a charge, thermal effects will destroy this nice zero-or-one picture.

Dielectrics (Zangwill Ch. 6) Conductors are nice because their charge carriers redistribute to cancel applied electric fields. But insulators are more complicated because their cancellation is imperfect due to subtle material properties.

In materials, we can write the total charge ρ as a sum of two terms– a free (applied) charge and a polarization/bound charge due to the material response. That is,

$$\rho(\mathbf{r}) = \rho_f(\mathbf{r}) + \rho_P(\mathbf{r}). \tag{5.13}$$

The free charge ρ_f is charge we control by putting it on objects, while the polarization (bound) charge ρ_P is how the material responds to the applied charge/fields. For instance, in conductors, all the ρ_P (i.e. the induced charge) lies on the surface of the conductor.

If we consider a dielectric with no net charge, then

$$\int_V d^3r\, \rho_P(\mathbf{r}) + \int_S d^2S\, \sigma_P(\mathbf{r}) = 0. \tag{5.14}$$

That is, the sum of the polarization volume charge and the polarization surface charge is zero.

We can then motivate the polarization vector $\mathbf{P}(\mathbf{r})$ in the following way. A polarization vector should describe how charges redistribute in a material, such that

$$\mathbf{P}(\mathbf{r}) = 0 \text{ outside the material,} \tag{5.15}$$

$$\mathbf{P}(\mathbf{r}) \cdot \hat{\mathbf{n}} = \sigma_P(\mathbf{r}) \text{ on surface.} \tag{5.16}$$

We can plug this into our equation (5.14) to get

$$\int_V d^3r\, \rho_P(\mathbf{r}) + \int_V d^3r\, \boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{r}) = 0 \tag{5.17}$$

by the divergence theorem. Since these are both volume integrals, it follows that the integrand vanishes,

$$\boldsymbol{\nabla} \cdot \mathbf{P}(\mathbf{r}) = -\rho_P(\mathbf{r}) \text{ inside material.} \tag{5.18}$$

We can now integrate the polarization over the volume of the material. Note that $\boldsymbol{\nabla}_i r_j = \delta_{ij}$, so

$$\int_V d^3r\, P_j = \int d^3r\, P_i \boldsymbol{\nabla}_i r_j = \int_V d^3r\, \boldsymbol{\nabla}_i (r_j P_i) - \int_V d^3r\, r_j \boldsymbol{\nabla}_i P_i \tag{5.19}$$

by the chain rule. Then the first term is a divergence, so we can turn it into a surface integral, i.e.

$$\int_V d^3r\, P_j = \int_S d^2S\, r_j \sigma_P(\mathbf{r}) + \int_V d^3r\, r_j \rho_P(\mathbf{r}) \tag{5.20}$$

using the definitions of the polarization volume charge and surface charge. But now we recognize that these are moment integrals of charge distributions, which means that these are precisely in the form of a dipole moment. That is,

$$\int_V d^3r\, \mathbf{P} = \mathbf{p}, \tag{5.21}$$

the net dipole moment of the distribution. Hence we can think of polarization as dipole moment per unit volume.

Question: which quantity has more information, $\mathbf{P}(\mathbf{r})$ or $\rho(\mathbf{r})$? A priori, we might think that because of Eqn. (5.18), the charge density has less information than the polarization vector, since we've taken a derivative (which is a lossy operation). But if we take the integral $\int_V d^3r \mathbf{P} = \mathbf{p}$, we only specify one moment of the distribution.

As it turns out, the polarization has more information because it contains phase information. That is, the integral tells us one piece of information we can extract from $\mathbf{P}$, but there's in principle more we can do with $\mathbf{P}$.

Lecture 6.

Friday, January 24, 2020

Let's continue thinking about dielectrics. Let's assume the polarization $\mathbf{P}$ is known and see what we can do from there. For instance, what is the potential due only to the polarization charge?

$$\varphi_P(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{-\boldsymbol{\nabla}'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\int_S d\mathbf{S}'\cdot\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{6.1}$$

That is, we just integrate the influences of the volume charge $\rho_P = -\boldsymbol{\nabla}\cdot\mathbf{P}$ and the surface charge $\mathbf{P}\cdot\hat{\mathbf{n}}$. Now the electric field is given by

$$\mathbf{E}_P(\mathbf{r}) = -\boldsymbol{\nabla}\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{-\boldsymbol{\nabla}'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}') + \frac{1}{4\pi\epsilon_0}\int_S d\mathbf{S}'\cdot\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}'). \tag{6.2}$$

Now we'll make a simplifying assumption. If we not only know the polarization $\mathbf{P}$ but also know that $\mathbf{P}(\mathbf{r}')$ is uniform (within the sample), then $\boldsymbol{\nabla}'\cdot\mathbf{P}(\mathbf{r}') = 0$, so this first term goes away and we are left with just the surface term. By the divergence theorem, we can rewrite the flux integral as

$$\mathbf{E}_{P,i}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\boldsymbol{\nabla}'_j\left(\frac{\mathbf{P}_j(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(r_i - r'_i)\right) = \frac{P_j}{4\pi\epsilon_0}\int_V d^3r'\boldsymbol{\nabla}'_j\left(\frac{r_i - r'_i}{|\mathbf{r}-\mathbf{r}'|^3}\right). \tag{6.3}$$

However, notice that the final expression is symmetric in r and r' up to a minus sign, so

$$\boldsymbol{\nabla}'_j\left(\frac{r_i - r'_i}{|\mathbf{r}-\mathbf{r}'|^3}\right) = -\boldsymbol{\nabla}_j\left(\frac{r_i - r'_i}{|\mathbf{r}-\mathbf{r}'|^3}\right). \tag{6.4}$$

Hence

$$E_{P,i}(\mathbf{r}) = -\frac{P_j}{4\pi\epsilon_0}\boldsymbol{\nabla}_j\int d^3r'\,\frac{r_i - r'_i}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{6.5}$$

or in vector notation,

$$\mathbf{E}_P(\mathbf{r}) = -(\mathbf{P}\cdot\boldsymbol{\nabla})\boldsymbol{\epsilon}(\mathbf{r}), \tag{6.6}$$

where

$$\boldsymbol{\epsilon}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{6.7}$$

which is none other than the electric field of a uniform charge distribution

$$\rho_{\text{im}} = \begin{cases} 1 & \text{in } V, \\ 0 & \text{outside.} \end{cases} \tag{6.8}$$

We sometimes call this *Poisson's theorem.*

Example 6.9. Consider a uniformly polarized sphere. We can calculate its field the hard way (with spherical harmonics) or the easy way (with Poisson's theorem).

Let's do it the easy way. Our electric field from the constant density helper system is

$$\boldsymbol{\epsilon}(\mathbf{r}) = \begin{cases} \frac{1}{3\epsilon_0}\mathbf{r}, & r < R \\ \frac{1}{3\epsilon_0}\left(\frac{R}{r}\right)^3, & r > R. \end{cases} \tag{6.10}$$

Let's say that the polarization is constant in the $\hat{\mathbf{z}}$ direction. Then our theorem says that the field is

$$\mathbf{E}_P(\mathbf{r}) = \begin{cases} \frac{1}{3\epsilon_0}\mathbf{P}, & r < R \\ \frac{R^3}{3\epsilon_0}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{P})\hat{\mathbf{r}}-\mathbf{P}}{r^3}, & r > R. \end{cases} \tag{6.11}$$

That is, we get a dipole field outside and a uniform field inside, which is similar to what we saw for conductors, except that the interior has a nonzero field.

Alas, this result isn't as useful as we might hope, because uniform polarizations are hard to come by. In general systems are complicated and do not admit closed-form solutions for their charge distributions. Consider two dielectric spheres in a uniform electric field (say, in the $\hat{\mathbf{z}}$ direction) at some offset (say, in the $\hat{\mathbf{z}} + \hat{\mathbf{x}}$ direction). The dipole moment induced on one by the external field produces a dipole field that hits the other at some weird angle, and the other has to respond to the dipole field of the first.

Generally, we can write the polarization vector as

$$P_i = \epsilon_0 \chi_{ij} E_j + \epsilon_0 \chi^{(2)}_{ijk} E_j E_k + \dots, \tag{6.12}$$

where these χs are tensors indicating the response of the polarization to different electric fields in different directions. Real materials might have different responses to E-fields in the x versus y directions, for instance. We call this *anisotropy*.

However, having said that, we'll now run away from all these complications as fast as possible. We look at "simple" dielectrics with

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} = \epsilon_0(\kappa - 1)\mathbf{E}. \tag{6.13}$$

We call χ the *electric susceptibility*, ϵ the *permittivity*, and κ the *dielectric constant*. We need

$$\chi \geq 0, \epsilon \geq \epsilon_0, \kappa \geq 1, \tag{6.14}$$

where the bounds are saturated in vacuum. Note that χ and κ are dimensionless, while ϵ has the same units as ϵ_0.

Now we can write Gauss's law as

$$\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = \rho_f + \rho_P = \rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}. \tag{6.15}$$

If we define the auxiliary field (sometimes called the electric displacement) $\mathbf{D}$ as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{6.16}$$

then we get Gauss's law in materials,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f. \tag{6.17}$$

This field is nice because it depends only on the free charge, which we control. In particular, for electrostatic fields, it follows that

$$\boldsymbol{\nabla} \times \mathbf{D} = \boldsymbol{\nabla} \times \mathbf{P}. \tag{6.18}$$

We can also derive boundary conditions on the displacement by the same Gaussian pillbox arguments as before. At a surface, if there is no applied surface charge, then

$$\Delta D_\perp = 0, \tag{6.19}$$

$$\Delta D_\parallel = \Delta P_\parallel. \tag{6.20}$$

Example 6.21. Consider a point charge $+q$ embedded in a dielectric. Calculating the displacement around this charge is super easy, by the usual Gauss's law arguments. It is just

$$\mathbf{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{\mathbf{r}}, \tag{6.22}$$

with r the radial distance from the point charge. Now if we assume this is a linear dielectric, then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(\chi + 1)\mathbf{E} = \epsilon_0 \kappa \mathbf{E} = \epsilon \mathbf{E}. \tag{6.23}$$

It follows that

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{q/\kappa}{r^2} \hat{\mathbf{r}} \tag{6.24}$$

As a sanity check, we can recognize that since $\epsilon \geq \epsilon_0$ or equivalently $\kappa \geq 1$, the electric field is reduced from its vacuum value by a factor of κ. The charge is "screened" by the dielectric.

We can also compute the polarization charge, since we know that $\mathbf{P} = \epsilon_0(\kappa - 1)\mathbf{E}$. Then

$$\rho_P = -\boldsymbol{\nabla} \cdot \mathbf{P} = -\epsilon_0(\kappa - 1)\boldsymbol{\nabla} \cdot \mathbf{E} = -\epsilon_0(\kappa - 1)\rho/\epsilon_0 = (1 - \kappa)(\rho_f + \rho_p). \tag{6.25}$$

It follows that

$$\rho_P = \frac{(1 - \kappa)\rho_f}{\kappa} = \left(\frac{1}{\kappa} - 1\right)\rho_f. \tag{6.26}$$

Example 6.27. What if we have charge $+1$ in a sphere of radius R with one dielectric constant κ_1 inside a bigger dielectric of dielectric constant κ_2? Most of the analysis is the same, but now

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q/\kappa_1}{r^2}\hat{\mathbf{r}} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q/\kappa_2}{r^2}\hat{\mathbf{r}}, & r > R. \end{cases} \tag{6.28}$$

How can the field outside not depend on the first dielectric? The answer is similar to what happened for conductors, except the screening is imperfect. The electric field is discontinuous at $r = R$ because there is an induced surface charge density there due to the polarizations from the dielectrics.

In general the game plan will be to calculate $\mathbf{D}$ using our old Gauss's law tricks and then convert to $\mathbf{E}$-fields given the linearity of the medium, and then find surface charge densities.

Lecture 7.

Tuesday, January 28, 2020

Last time, we discussed the field from a charge at the boundary between two dielectrics. At the boundary, $E_\perp$ is discontinuous because of the surface charge. But $\mathbf{D}$ seems to be continuous since it depends only on the free charge. What's happening? Why does E have a curl if D is continuous? Really, what's happening is that D also knows about the polarization:

$$\boldsymbol{\nabla} \times \mathbf{D} = \boldsymbol{\nabla} \times \mathbf{P}. \tag{7.1}$$

So E can be composed of a curl part and a divergence part, and the polarization itself is sourcing a curl at the boundary.

Poisson and Laplace equations Let us leave behind the world of matter for a bit and explore Laplace's equation. To begin our discussion, suppose we have two solutions of Poisson's equation,

$$\nabla^2\varphi_1 + \nabla^2\varphi_2 = -\frac{\rho(\mathbf{r})}{\epsilon}. \tag{7.2}$$

When are solutions unique? Define

$$\Phi = \varphi_1 - \varphi_2. \tag{7.3}$$

By linearity, it follows that

$$\nabla^2\Phi = 0. \tag{7.4}$$

In general, note that for scalar functions f, g, we have

$$\int_S d\mathbf{S}\cdot f\boldsymbol{\nabla}g = \int_V d^3r\,\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g) = \int_V d^3r(f\nabla^2 g + \boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g). \tag{7.5}$$

If we take $f = g = \Phi$, then

$$\int_S d\mathbf{S}\cdot\Phi\boldsymbol{\nabla}\Phi = \int_V d^3r\left(\Phi\underbrace{\nabla^2\Phi}_{=0} + |\boldsymbol{\nabla}\Phi|^2\right) \geq 0. \tag{7.6}$$

This first term is zero by assumption, while the second term is positive semi-definite and equal to zero only for $\boldsymbol{\nabla}\Phi = 0$.

When does the LHS vanish? Either Φ vanishes on the boundary or $\boldsymbol{\nabla}\Phi\cdot\hat{\mathbf{n}}$ vanishes on the boundary, i.e. the normal derivative is zero. Either way, we get $\boldsymbol{\nabla}\Phi = 0$, which forces $\Phi =$ constant.

If $\Phi = 0$ on S, then these are the Dirichlet boundary conditions, and in fact $\varphi_1 = \varphi_2 =$some known values on S. Alternately, if $\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\Phi = 0$, then that's equivalent to specifying the normal derivative $\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi$ on the boundary. These are *Neumann boundary conditions*, and the solutions can vary by a constant. We could also specify the value on part of the boundary and a normal derivative elsewhere; these are mixed boundary conditions.[6]

[6]There's also a fourth class of boundary conditions, the Cauchy boundary conditions. Cauchy BCs are needed for hyperbolic equations, e.g. the wave equation in (x,t). For hyperbolic equations, one must effectively specify both the position of the wave (function value) as well as its velocity (normal derivative) on an initial value surface (basically a time slice, or what's naturally called in GR a Cauchy surface). For elliptic equations like Laplace and Poisson, this is generally too restrictive, since either the function value *or* the normal derivative on a closed surface is sufficient to uniquely determine the solution.

This tells us when solutions are unique, though it does not prove existence. In general we need only specify φ or $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\varphi$ on the boundary; if we try to fix both, the problem will usually be overdetermined unless we're very lucky.

Let us make a guess when we try to solve Laplace's equation. We will make the following ansatz:

$$\varphi(u, v, w) = A(u)B(v)C(w) \tag{7.7}$$

in terms of some functions A, B, C. If we make this guess and plug it into the differential equation, then our PDE separates into ODEs for each variable, linked by separation constants. While this seems like a really simple guess, Zangwill says that this works in 13 different coordinate systems.[7]

Zangwill also claims to write down the general solution to Laplace's equation, but it's not true. *Do not believe Zangwill Eqn. 7.14.* When the equation separates, we get two linearly independent solutions for A, two for B, and two for C (the equation is second-order). That is, we have

$$A, \tilde{A}; B, \tilde{B}; C, \tilde{C}. \tag{7.8}$$

In fact, each ODE might give us a family of eigenfunctions (think of sines in a rectangular waveguide) indexed by some eigenvalues. So we really have

$$\{A_\alpha, \tilde{A}_\alpha\}; \{B_\beta, \tilde{B}_\beta\}; \{C_\gamma, \tilde{C}_\gamma\}, \tag{7.9}$$

and to construct the most general solution, we need 8 terms:

$$\varphi = \sum_{\alpha,\beta,\gamma} \left[a^{(1)}_{\alpha\beta\gamma} A_\alpha(u) B_\beta(v) C_\gamma(w) + a^{(2)}_{\alpha\beta\gamma} A_\alpha(u) B_\beta(v) \tilde{C}_\gamma(w) + a^{(3)}_{\alpha\beta\gamma} A_\alpha(u) \tilde{B}_\beta(v) C_\gamma(w) + \ldots \right]. \tag{7.10}$$

This is precisely equivalent to the construction of entangled states in quantum mechanics. That is, the most general solution isn't just a product of individual solutions but a sum of products. It is a tensor product structure, not a direct product. Contrast Zangwill 7.14, which claims the most general solution is

$$\sum_{\alpha\beta\gamma} (a_\alpha A_\alpha(u) + \tilde{a}_\alpha \tilde{A}_\alpha(u))(b_\beta B_\beta(v) + \tilde{b}_\beta \tilde{B}_\beta(V))(c_\gamma C_\gamma(w) + \tilde{c}_\gamma \tilde{C}_\gamma(w)). \tag{7.11}$$

This is just a product of solutions, but there aren't enough parameters for the solution space. There are really only four independent parameters here, since we can pull out an $a_\alpha b_\beta c_\gamma$ and treat this as a single number.

Now, in the theory of Sturm-Liouville problems, ODEs satisfying self-adjointness provide us with a complete orthonormal set of eigenfunctions. Completeness says that

$$\sum_k \psi_k(v) \psi_k^*(v') = \delta(v - v'), \tag{7.12}$$

where k indexes over eigenfunctions. It follows that we have a basis, so that

$$F(v) = \int \delta(v - v') F(v') dv' = \sum_k \underbrace{\int dv' \psi_k^*(v') F(v')}_{F_k} \psi_k(v) = \sum_k F_k \psi_k(v). \tag{7.13}$$

That is, we can decompose $F(v)$ into its components. Note that if $F(v) = \psi_k'(v)$, then we immediately have $F_k = \int dv' \psi_k^*(v') \psi_{k'}(v') = \delta_{kk'}$.

In Cartesian coordinates, separation of variables is very nice. We have

$$\nabla^2 \varphi = 0, \quad \varphi(x, y, z) = X(x)Y(y)Z(z). \tag{7.14}$$

If we plug in and divide by the original function φ, we get

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{7.15}$$

Since the LHS depends only on x and the RHS depends only on Y and Z, we can write

$$X'' = \alpha^2 X \implies X(x) = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0. \end{cases} \tag{7.16}$$

As we expected, there are two linearly independent solutions (1 and x for $\alpha = 0$ and $e^{\pm x}$ for $\alpha \neq 0$.

[7] I haven't checked.

For instance, we could set up a box (say, a cube of side length a) where the boundary condition is $V_1(x,y)$ at $z=0$ and $V=0$ on the sides. It follows that the X and Y dependence is sines which are periodic in a, namely

$$V(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{\text{mn}} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\frac{\sinh\gamma_{mn}(a-z)}{\sinh\gamma_{mn}a}, \tag{7.17}$$

such that

$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2. \tag{7.18}$$

This now satisfies $V(z=a)=0$, and the coefficients V_{mn} are given by taking the Fourier components of the boundary condition $V_1(x,y)$, i.e.

$$V_{mn}^{(1)} = \frac{4}{a^2}\int_0^a dx\int_0^a dy\, V_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}. \tag{7.19}$$

What if another wall had a nontrivial boundary condition, e.g. $V_2(y,z)$ at $x=a$? Fortunately, Poisson's equation is linear and homogeneous so we can just add solutions. That is, we can add on a solution

$$V_2(x,t,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{\text{mn}}^{(2)} \sin\frac{m\pi y}{a}\sin\frac{n\pi z}{a}\frac{\sinh\gamma_{mn}(x)}{\sinh\gamma_{mn}a}. \tag{7.20}$$

Example 7.21. Let's consider the solution to Laplace's equation in a Faraday cage. Suppose we have two rows of infinite wires. The wires are separated within the rows by a distance a and the two rows are separated by a distance d (so one is at $z=0$ and one is at $z=d$). Say the row extends in the x direction and the wires extend in the y direction. Each wire has linear charge density λ.

The charge density from the lower row of wires is

$$\sigma(x) = \lambda\sum_{p=-\infty}^{\infty}\delta(x-pa). \tag{7.22}$$

Because the charge distribution is symmetric (even) about the origin, we can write our charge distribution in terms of cosines,

$$\sigma(x) = A_0 + \sum_{m=1}^{\infty} A_m\cos\frac{2\pi mx}{a}. \tag{7.23}$$

This will turn out to be Neumann boundary conditions, since we're specifying a charge distribution on the boundary, which fixes the perpendicular component of the E-field at the boundary and therefore the normal derivative of V.

The coefficients are given by

$$A_0 = \frac{1}{a}\int_{-a/2}^{a/2} dx\,\sigma(x) = \frac{\lambda}{a}, \quad A_m = \frac{2}{a}\int_{-a/2}^{a/2} dx\,\sigma(x)\cos\frac{2\pi mx}{a} = \frac{2\lambda}{a}. \tag{7.24}$$

Since our period is a, we only ever see the delta function at the origin. We Since our charge density has a period a, we only ever see the delta function at the origin. We'll continue this next time.

Lecture 8.

Thursday, January 30, 2020

Let's finish our discussion from last time of the Faraday cage. The Fourier decomposition of the charge distribution is

$$\sigma(x, z=0) = \frac{\lambda}{a} + \sum_{m=1}^{\infty}\frac{2\lambda}{a}\cos\frac{2\pi mx}{a}, \tag{8.1}$$

and we see that cosines in x are part of the separable solutions to Laplace's equation in Cartesian coordinates. The wires extend in the y direction, so we want a constant solution in y, and it just remains to fit the z-dependence.

Note that σ specifies a normal derivative of φ (it is a discontinuity in E), so our general solution could look like

$$\varphi(x,y,z) = (ax + b|z| + cx|z| + d) + \sum_{\gamma} c_\gamma\cos(\gamma x)e^{-\gamma|z|}. \tag{8.2}$$

But by the symmetry in x, the terms linear in x go away and the constant d can be set to zero WLOG. There's no $e^{+\gamma|z|}$ terms since this would diverge as $|z| \to \infty$. Thus we're left with

$$\varphi(x,y,z) = b|z| + \sum_\gamma c_\gamma \cos(\gamma x) e^{-\gamma|z|}. \tag{8.3}$$

If we now take the normal derivative at $z \to 0$, we get

$$\frac{\partial\varphi}{\partial z}|_{z\to 0^+} = b + \sum_\gamma c_\gamma(\cos(\gamma x))(-\gamma)e^{-\gamma(0)} \tag{8.4}$$

and

$$\frac{\partial\varphi}{\partial z}|_{z\to 0^-} = -b + \sum_\gamma c_\gamma(\cos(\gamma x))(\gamma)e^{\gamma(0)}. \tag{8.5}$$

It follows that the electric fields are

$$E_\perp(z\to 0^+) = -\frac{\partial\varphi}{\partial z}|_{z\to 0^+} = -b + \sum_\gamma c_\gamma(\cos(\gamma x))\gamma, \tag{8.6}$$

$$E_\perp(z\to 0^-) = -\frac{\partial\varphi}{\partial z}|_{z\to 0^-} = +b - \sum_\gamma c_\gamma(\cos(\gamma x))\gamma. \tag{8.7}$$

If we now take their difference, we have

$$\frac{\sigma}{\epsilon_0} = E_\perp(z\to 0^+) - E_\perp(z\to 0^-) = -2b + 2\sum_\gamma c_\gamma(\cos\gamma x)\gamma. \tag{8.8}$$

By matching coefficients with Eq. (8.1), we find that

$$-2b = \frac{\lambda}{\epsilon_0 a}, \quad \gamma = \frac{2\pi}{a}m, \quad c_\gamma = \frac{\lambda}{a\gamma\epsilon_0} = \frac{\lambda}{\epsilon_0 2\pi m} \tag{8.9}$$

Plugging back into our solution, we have

$$\boxed{\varphi(x,y,z) = -\frac{\lambda}{2a\epsilon_0}|z| + \frac{\lambda}{2\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\cos\frac{2\pi m}{a}e^{-\frac{2\pi m}{a}|z|}.} \tag{8.10}$$

This diverges far away due to the $|z|$ term, but that actually makes sense because far from the cage (at large z, we see the field of an infinite plane of charge, i.e. a constant field and a potential changing linearly with distance.

That's the solution for a single wire plane. If we add two such planes separated by a distance d, then we have

$$\varphi(x,y,z) = -\frac{\lambda}{2a\epsilon)}\overbrace{(|z| + |d-z|)}^{=d \text{ "inside," } 0<z<d} + \sum\ldots \tag{8.11}$$

and the exponential terms both decay inside exponentially fast.

We can discuss separable solutions to Laplace's equation in some generality. For Cartesian coordinates, there are three separation constants α, β, γ with $\alpha^2 + \beta^2 + \gamma^2 = 0$, It follows that most of our solutions will have oscillation in at least one direction and decay (or growth) in another. For spherical solutions, there are technically two independent solutions for the θ dependence, the associated Legendre polynomials P_l^m and the other kind Q_l^m. The second kind are divergent at $\theta = \pi$ and sometimes at $\theta = 0$. Cylindrical solutions have many different options; the radial dependence can be growing exponentially, decaying exponentially, growing as a log, or oscillating/decaying as cylindrical Bessel functions. We won't really discuss this in any large degree of detail, but it's good to be aware that such solutions exist.[8]

Example 8.12. Consider a ring of charge Q and radius R. If we consider this in a spherical expansion, we can write

$$\varphi(r,\theta,\phi) = \begin{cases} \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & r > R, \\ \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & r < R. \end{cases} \tag{8.13}$$

[8] Arfken has a very detailed discussion of Bessel functions and Legendre polynomials

But since these solutions must agree at $r = R$, we can actually write

$$\varphi(r,\theta,\phi) = \begin{cases} \sum_{l=0}^{\infty} c_l \frac{R^{l+1}}{r^{l+1}} P_l(\cos\theta) & r > R, \\ \sum_{l=0}^{\infty} c_l \frac{r^l}{R^l} P_l(\cos\theta) & r < R. \end{cases} \tag{8.14}$$

Moreover, we can explicitly calculate

$$\varphi(r,0,0) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{r^2+R^2}} = \frac{Q}{4\pi\epsilon_0 R}\sum_{l=0}^{\infty}\Big(\frac{r}{R}\Big)^l P_l(0) \tag{8.15}$$

for $r < R$ and the expansion of $\frac{1}{\sqrt{1-2tz+t^2}}$. Now we can just evaluate our general expansion for $r < R$ at $\theta = \pi/2$ using the fact that $P_l(\cos(\theta = \pi/2)) = 1$. Matching the r dependence, we find that

$$c_l = \frac{Q}{4\pi\epsilon_0 R} P_l(0). \tag{8.16}$$

It's a bit like analytic continuation—if we know a function on some line, we can actually extend it consistently elsewhere. Now that we have the coefficients c_l, we have the solution for φ.

Example 8.17. Consider a solid grounded conductor with a "bite" taken out of it. Suppose we can analyze this system in cylindrical coordinates with translational symmetry along the z-axis. Then our solution (in almost full generality: see our discussion from last time) is

$$\varphi(\rho,\phi,z) = (A_0 + B_0\ln\rho)(C_0\phi + D_0) + \sum_{\alpha\neq 0}\left[A_\alpha\rho_\alpha^\alpha\rho^{-\alpha}\right]\left[C_\alpha e^{i\alpha\phi} + D_\alpha e^{-i\alpha\phi}\right]. \tag{8.18}$$

This will be the general solution so long as one of $A_\alpha, B_\alpha, C_\alpha, D_\alpha$ is zero for each α. We'd like to avoid singularities as $\rho \to 0$. We can now write

$$\varphi(\rho\to 0,\phi,z) \propto \rho^{\pi/\beta}\sin\frac{\pi\phi}{\beta} \tag{8.19}$$

as the leading dependence. That is, we've thrown away the $\rho^{-\alpha}$ and $\ln\rho$ dependence, and we also got rid of C_0 in order for our potential to be single-valued.

If we look at $\mathbf{E} = -\boldsymbol{\nabla}\varphi$, we notice that

$$\frac{\partial\varphi}{\partial\rho} \propto \frac{\pi}{\beta}\rho^{\pi/\beta-1}\sin\frac{\pi\phi}{\beta}, \tag{8.20}$$

and we can now see that if $\beta > \pi$ (i.e. we have a spike, not a bite) then the corner will have very large electric fields at small ρ.

Example 8.21. Finally, let us consider two cylindrical rods aligned along the z-axis, separated by a little gap. We hold the rod at positive z at a potential V_R and the rod at negativ z at a potential V_L. What is the potential inside the gap? There is no ϕ dependence, so $\alpha = 0$ (we don't get the power-law decay/growth in ρ). Our boundary conditions are given in ϕ, z, so we will take the separation constant $k^2 < 0$ to expand in z, and define $k = i\kappa$ as is conventional.

This means we're expanding in Bessel functions I and K, and since K diverges as $\rho \to 0$, we will be using the functions I. THus

$$\varphi(\rho,\phi,z) = \frac{1}{2\pi}\int_0^\infty I_0(\kappa\rho)(E(\kappa)e^{i\kappa z} + F(\kappa)e^{-i\kappa z})d\kappa + \text{constant} \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\kappa)I_0(|\kappa|\rho)e^{i\kappa z}d\kappa + \text{constant}. \tag{8.22}$$

That is, we extended the limits of integration to $-\infty, \infty$ and now we recognize this integral as a Fourier transform. We can recover $A(\kappa)I_0(|\kappa|\rho)$ by taking an inverse Fourier transform.

Lecture 9.

Tuesday, February 4, 2020

Midterm logistics—we'll have the midterm Friday, during the regular section time. There will be 3-4 questions, of which we will choose a few and solve them in class. The full exam will then become a take-home which we'll work on over the weekend. Content runs through the Laplace chapter and not the Poisson chapter. In-class and take-home portions will be equally weighted (50/50).

Poisson's equation Laplace's equation is a special case of Poisson's equation. That is, Poisson's equation reads

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} \tag{9.1}$$

for vacuum or electric fields in (not-dielectric) matter, while

$$\nabla^2\varphi = -\frac{\rho_f}{\epsilon} \tag{9.2}$$

in matter, where ϵ is the permittivity. This latter form is sometimes nice when we know the free charge.

A useful technique for solving boundary value problems in electrostatics is the *method of images*. We'll start with the classic example, a charge above a grounded plane.

Example 9.3. Suppose we have some charge q a distance $z_0 > 0$ above a grounded conducting plane, $\varphi = 0$. The method of images says that we can solve potential boundary value problems by matching the potential with an equivalent fictitious charge distribution.

Notice that we can produce this same potential in the $z > 0$ half-space if we place an "image charge" at $-z_0$. The potential from this distribution is

$$\varphi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}}\right]. \tag{9.4}$$

The first term is the Poisson solution corresponding to our real charge, while the second term solves Poisson's equation in the upper half-space, so this will overall satisfy Poisson's equation for the *real* charge distribution in the space we care about.

One can easily check that on the plane $z = 0$, we have $\varphi(x,y,0) = 0$, so the boundary condition is satisfied.

Notice that the potential from the plane below it ($z < 0$) is its potential above, up to a $z \to -z$. That is,

$$\varphi_{\text{plane}}(x,y,z<0) = -\frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+(-z+z_0)^2}}. \tag{9.5}$$

But that tells us that

$$\varphi_{\text{plane}} + \varphi_{\text{charge}} = 0 \text{ for } z < 0, \tag{9.6}$$

which we expected. The conductor has screened out the influence of the charge in the lower half-space.

We can calculate the energy of this configuration by taking[9]

$$q\int_{z_0}^{\infty} dz\, E(z) = \int_{z_0}^{\infty} dz \frac{q^2}{4\pi\epsilon_0(2z)^2} = \frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{4z_0}\right). \tag{9.7}$$

Equivalently this is half of the interaction energy of the charge and the image charge; we can see this by integrating over the energy in the field of two real charges and then arguing by symmetry that we only have field in the half-space, so the actual energy is half the energy of the real dipole.

We can now calculate the charge distribution by looking at the normal derivative of potential at the conductor. Here,

$$-\frac{\partial\varphi}{\partial z}|_{z=0^+} = \frac{\sigma}{\epsilon_0}. \tag{9.8}$$

We have

$$\sigma/\epsilon_0 = \frac{q}{4\pi}\left[-\frac{z_0}{(\rho^2+z_0^2)^{3/2}} - \frac{z_0}{(\rho^2+z_0^2)^{3/2}}\right], \tag{9.9}$$

[9] I did this a little differently from lecture. This is the field of two point charges separated by a distance $2z$.

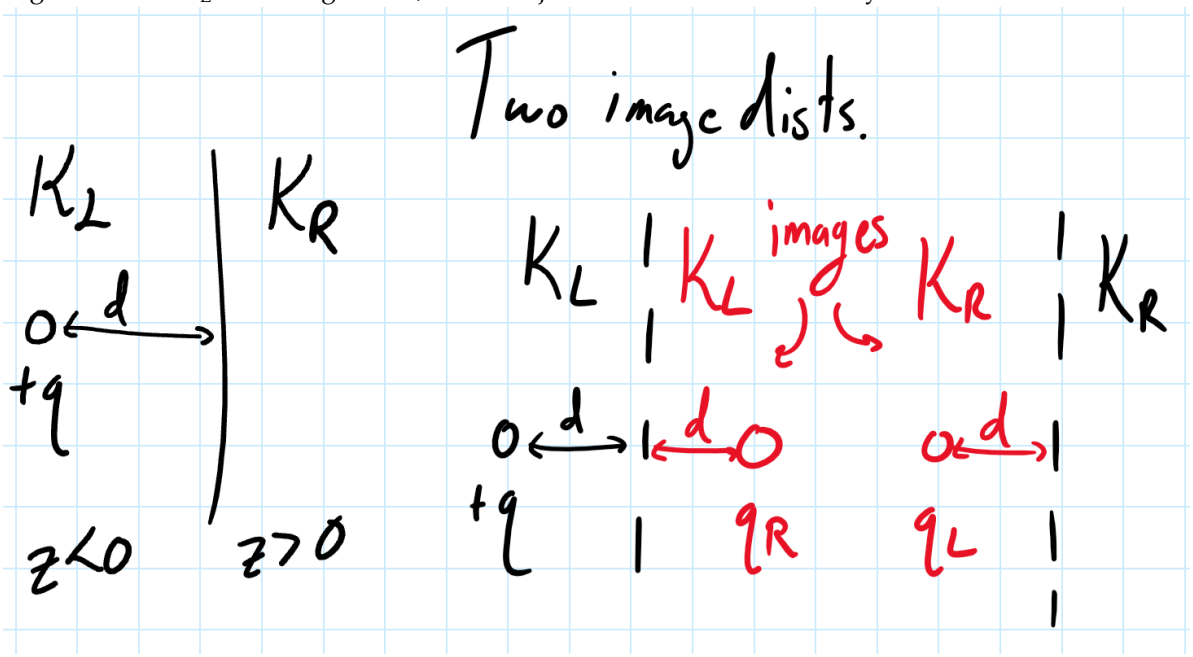
so then the total induced charge is

$$\begin{aligned} Q &= \int_0^\infty (2\pi\rho d\rho)\sigma(\rho,\phi) \\ &= \int_0^\infty (2\pi\rho d\rho)\frac{q}{4\pi}\left[-\frac{z_0}{(\rho^2+z_0^2)^{3/2}} - \frac{z_0}{(\rho^2+z_0^2)^{3/2}}\right] \\ &= -q\int_0^\infty d\rho \frac{\rho z_0}{(\rho^2+z_0^2)^{3/2}} \\ &= \frac{qz_0}{(\rho^2+z_0^2)^{1/2}}|_0^\infty \\ &= -q. \end{aligned}$$

We learn that the induced charge was exactly equal and opposite to the real charge, as we might have guessed.

Example 9.10. Let us now revisit a situation we've considered before. We have a charge $+q$ embedded in some dielectric of dielectric constant κ_L which fills the region $z<0$, a distance d from the interface with a different dielectric κ_R which fills $z>0$.

Emboldened by our success with the conducting plane, we might think that perhaps an image charge approach could work for the interface between dielectrics as well. That is, let's cook up a scenario where there's some image charge q_R in a dielectric κ_L on the right. It doesn't matter that the image charge configuration has κ_L on the right now, since we just want to fit the boundary conditions at $z=0$.



It follows that the potentials in each region are

$$\varphi_L(x,y,z) = \frac{1}{4\pi\epsilon_L}\left[\frac{q}{\sqrt{x^2+y^2+(z+d)^2}} + \frac{q_R}{\sqrt{x^2+y^2+(z-d)^2}}\right], \tag{9.11}$$

$$\varphi_R(x,y,z) = \frac{1}{4\pi\epsilon_R}\frac{q_L}{\sqrt{x^2+y^2+(z+d)^2}}. \tag{9.12}$$

The potential is continuous,

$$\varphi_L(x,y,0) = \varphi_R(x,y,0), \tag{9.13}$$

while its normal derivative is discontinuous,

$$\kappa_L\epsilon_0\frac{\partial\varphi_L}{\partial z}|_z = 0 = \kappa_R\frac{\partial\varphi_R}{\partial z}|_{z=0}. \tag{9.14}$$

Continuity tells us that

$$\frac{q+q_R}{\kappa_L} = \frac{q_L}{\kappa_R}, \tag{9.15}$$

and the normal derivative gives the condition that

$$-qd + q_R d = -q_L d. \tag{9.16}$$

That is,

$$q = q_R + q_L. \tag{9.17}$$

We now have two equations relating q, q_R, and q_L, which we can easily solve

$$q_L = \frac{2\kappa_R}{\kappa_R+\kappa_L} q, \quad q_R = \frac{\kappa_L - \kappa_R}{\kappa_L+\kappa_R} q. \tag{9.18}$$

We can then check that for $\kappa_L = \kappa_R$, we have $q_R = 0$ and $q_L = q$. This says there's no image charge on the right (there's no boundary) and our left image charge is just the original charge.

Like in Gauss's law, image charges only really work when we have a large amount of symmetry.

Lecture 10.

Thursday, February 6, 2020

Today, we'll discuss Green's functions. The principle behind Green's functions is this. Consider an inhomogenous differential equation like the Poisson equation,

$$\nabla^2 \varphi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{10.1}$$

A Green's function is the inverse of the differential operator. It is the solution to the inhomogenous equation with a delta function source, i.e. for $\rho(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$,

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\frac{1}{\epsilon_0}\delta(\mathbf{r}-\mathbf{r}'). \tag{10.2}$$

In fact, we know what potential corresponds to a point charge:

$$G_0(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{10.3}$$

Strictly, Zangwill calls this the free-space Green's function because for any solution of Poisson's equation, we can simply add another solution of Laplace's equation (the homogeneous equation). In general the exact solution will have to be fixed by boundary conditions. Trivially, we can write that

$$\int d^3r'\, \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') = \rho(\mathbf{r}). \tag{10.4}$$

If we multiply both sides of Eq. (10.2) by $\rho(\mathbf{r}')$ and perform the $\mathbf{r}'$ integral, we get

$$\nabla^2 \int d^3r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') = -\frac{1}{\epsilon_0}\int d^3r'\, \rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{10.5}$$

We conclude that in fact

$$\varphi(\mathbf{r}) = \int d^3r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'). \tag{10.6}$$

That is, the Green's function lets us build the solution to the inhomogeneous equation by integrating over (appropriately weighted) point charge potentials.

When we come to boundary value problems, there's an added complication. The potential of a point charge in a grounded box (and therefore its field) looks quite different than a potential in empty space. Let's start our discussion as follows:

$$\int d\mathbf{S}'\cdot f\boldsymbol{\nabla}' g = \int d^3r' \boldsymbol{\nabla}'\cdot(f\boldsymbol{\nabla}' g) = \int d^3r' \Big[\boldsymbol{\nabla}' f\cdot\boldsymbol{\nabla}' g + f\nabla'^2 g\Big]. \tag{10.7}$$

This is just a vector calculus manipulation. If we swap f and g and subtract from our original equation, we get

$$\int d\mathbf{S}'\cdot\big[f\boldsymbol{\nabla}' g - g\boldsymbol{\nabla}' f\big] = \int d^3r' \Big[f\nabla'^2 g - g\nabla'^2 f\Big]. \tag{10.8}$$

Let us now take

$$f(\mathbf{r}') = \varphi(\mathbf{r}'), \quad g(\mathbf{r}') = G(\mathbf{r}, \mathbf{r}'). \tag{10.9}$$

Then

$$\int d\mathbf{S}' \cdot \left[\varphi \boldsymbol{\nabla}' G - G \boldsymbol{\nabla}' \varphi\right] = \int d^3 r' \left[\varphi \boldsymbol{\nabla}'^2 G - G \boldsymbol{\nabla}'^2 \varphi\right] = -\frac{1}{\epsilon_0}\varphi(\mathbf{r}) + \frac{1}{\epsilon_0}\int d^3 r' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}'). \tag{10.10}$$

Hence we get back our inhomogenous solution, the $d^3 r'$ integral, plus a homogenous solution given by the $d\mathbf{S}'$ integral. This surface term is very nice—if φ is prescribed on the boundary (Dirichlet boundary conditions) then we can evaluate the first term in

$$\int d\mathbf{S}' \cdot \left[\varphi \boldsymbol{\nabla}' G - G \boldsymbol{\nabla}' \varphi\right], \tag{10.11}$$

and moreover if $G = 0$ on the boundary then we just have to compute the first integral. Conversely if we had Neumann boundary conditions, then we should set $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}' G = 0$ so that the normal derivative of G vanishes and we need only compute the second term.

Example 10.12 ("Splitting")**.** Let's consider Zangwill's third example. We wish to construct $G_D(\mathbf{r}, \mathbf{r}')$, the Green's function for a particular set of Dirichlet boundary conditions, $G_D(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r}$ on some surface. We now remark that the Dirichlet Green's function can be written as a sum

$$G_D(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') + \Lambda(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \Lambda(\mathbf{r}, \mathbf{r}'), \tag{10.13}$$

where Λ solves the homogeneous equation (the Laplace equation).

For instance, consider a point charge in a cylinder of length L and radius R. The process of constructing the Green's function is as follows. Consider the potential sourced by the point charge in empty space. We can solve for the potential on the cylinder, and then solve Laplace's equation for some Λ with *minus* that potential as the boundary conditions. If we add the two solutions, we get the field of a point charge in a grounded cylinder which appears in our ordinary $\int d^3 r' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')$ integral, and the boundary conditions will be fit by the surface term.

For Λ, we can build a solution in the cylinder as

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[e^{im\phi} I_m\left(\frac{n\pi}{L}\rho\right)\sin\frac{n\pi z}{L}\right], \tag{10.14}$$

corresponding to the $m \neq 0$ case and $k^2 < 0$ (so we have oscillations in the z-direction) and

$$\sum_{m=-\infty}^{\infty}\sum_{k}\left[e^{im\phi} J_m(k\rho)\left(A_{mk}e^{kz} + B_{mk}e^{-kz}\right)\right], \tag{10.15}$$

where the ks are now given by a different sort of discretizing condition such that $J_m(kR) = 0$.[10]

What's the use of the Green's function? If we have a single distribution given the boundary conditions, it might be better to just solve the problem once using our regular tricks. But if we have a set of similar problems with the same boundary conditions, it might be better to solve the Green's function once and then reuse it to generate solutions.

In 1D, Green's functions can be solved by direct integration:

$$\frac{\partial^2 G(x, x')}{\partial x^2} = \delta(x - x') \to \frac{\partial G(x, x')}{\partial x} = \Theta(x - x') + C \to G(x, x') = \frac{1}{2}|x - x'| + Cx + D. \tag{10.16}$$

In 3D, we can instead write $\delta(\mathbf{r} - \mathbf{r}')$ as the product of 1D delta functions. For instance, we might write

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(\rho - \rho')\delta(z - z')\frac{\delta(\phi - \phi')}{\rho}. \tag{10.17}$$

This is dimensionally correct; the $1/\rho$ comes from the fact the integration measure is $\rho d\rho d\phi dz$ in cylindrical coordinates.

[10] We don't need to treat the $m = 0$ case separtely, since the ρ dependence only changes when $k = 0$. While the $m = 0$ angular dependence would be $C_0\phi + D_0$, we can't have a linear piece in ϕ on the grounds of periodicity, so the constant D_0 can be absorbed into the overall constant.

We might then get rid of two delta functions via Fourier (or other integral) transforms to write

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} = \frac{1}{\pi}\int_0^\infty dk \cos k(z-z'), \tag{10.18}$$

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}. \tag{10.19}$$

Putting it back together, it follows that

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{2\pi^2}\frac{\delta(\phi-\phi')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos k(z-z'). \tag{10.20}$$

We can expand the Green's function in the same basis, as

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos k(z-z')G_m(\rho,\rho';k), \tag{10.21}$$

where we get some coefficients $G_m(\rho,\rho';k)$ which depend on the discrete values of m as well as the continuous variable k.

If you buy that we can expand the Green's function in this way, then we can explicitly compute the Laplacian of G and find the coefficients G_m by comparison to the expansion of the delta function in this basis (orthogonality, if you like). That is,

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\frac{1}{\epsilon_0}(\mathbf{r}-\mathbf{r}') \implies \nabla^2 G_m(\rho,\rho';k) = -\frac{\delta(\rho-\rho')}{\epsilon_0\rho}. \tag{10.22}$$

This is now an equation we can solve for G_m; it is a 1D Green's function problem. If we compute the Laplacian in cylindrical coordinates, we have

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dG_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)G_m = -\frac{1}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}. \tag{10.23}$$

Notice what happens. For $\rho<\rho'$ we have Laplace's equation since the RHS vanished. For $\rho>\rho'$ we also have Laplace's equation. What we have to do is match the solutions at the boundaries, and that's where we'll pick up on Tuesday.

Lecture 11.

Tuesday, February 11, 2020

Last time, we started thinking about constructing the Green's function for a cylinder. We found that if we expanded the Green's function in a Fourier basis as

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos k(z-z')G_m(\rho,\rho';k), \tag{11.1}$$

and via Laplace's (Poisson's) equation, the coefficients G_m satisfy

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dG_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)G_m = -\frac{1}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}, \tag{11.2}$$

which is Bessel's equation away from $\rho=\rho'$ but with negative k.

One can then use the solutions to Bessel's equation to construct the Green's function in the regions $\rho<\rho'$ and $\rho>\rho'$. That is,

$$G_m(\rho,\rho';k) = \begin{cases} a_m I_m(k\rho), & \rho<\rho' \\ b_m K_m(k\rho), & \rho>\rho'. \end{cases} \tag{11.3}$$

That is, we consider ρ' fixed and we want the solution for a delta-function charged cylinder. In fact, we write[11]

$$G_m(\rho,\rho';k) = \begin{cases} a_m I_m(k\rho)K_m(k\rho'), & \rho < \rho' \\ a_m K_m(k\rho)I_m(k\rho'), & \rho > \rho' \end{cases} \tag{11.4}$$

by the matching condition of continuity at $\rho = \rho'$.

There's another matching condition, the discontinuity in the derivative about $\rho = \rho'$.[12] That is,

$$\int_{\rho'-\Delta}^{\rho'+\Delta} d\rho \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G_m}{\partial\rho}\right) - (k^2 + \frac{m^2}{\rho^2})G_m\right] = \int -\frac{1}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}d\rho. \tag{11.5}$$

The term proportional to G_m is regular over the interval so it goes away; the other gives a derivative

$$\frac{\partial G_m}{\partial\rho}|\rho \to \rho'^{+} - \frac{\partial G_m}{\partial\rho}|_{\rho\to\rho'^{-}} = -\frac{1}{\epsilon_0\rho}. \tag{11.6}$$

Now we explicitly compute the derivative.

$$\frac{\partial G_m}{\partial\rho}|_{\rho\to\rho'^{-}} = a_m k K_m(k\rho')I'_m(k\rho'), \quad \frac{\partial G_m}{\partial\rho}|_{\rho\to\rho'^{+}} = a_m k K'_m(k\rho')I_m(k\rho'), \tag{11.7}$$

so we find that

$$a_m k\left[K'_m(\rho')I_m(k\rho') - I'_m(k\rho')K_m(k\rho')\right] = -\frac{1}{\epsilon_0\rho'}. \tag{11.8}$$

In fact, the LHS is just the Wronskian of the solutions to the Bessel equation. It is equal to

$$a_m k\left[K'_m(\rho')I_m(k\rho') - I'_m(k\rho')K_m(k\rho')\right] = a_m k\left(-\frac{1}{k\rho'}\right), \tag{11.9}$$

so we find that in fact

$$a_m = \frac{1}{\epsilon_0}. \tag{11.10}$$

Zangwill and Jackson both try to condense the notation by using $\rho_<$ and $\rho_>$ using the symmetry as

$$G_m(\rho,\rho';k) = \begin{cases} \frac{1}{\epsilon_0}I_m(k\rho)K_m(k\rho'), & \rho < \rho' \\ \frac{1}{\epsilon_0}K_m(k\rho)I_m(k\rho'), & \rho > \rho' \end{cases} = \frac{1}{\epsilon_0}I_m(k\rho_<)K_m(k\rho_>). \tag{11.11}$$

We've solved the Neumann boundary condition and found an expression which solves Laplace's equation away from the cylinder and has the right discontinuity on the wall ($\rho = \rho'$). What about the Dirichlet condition? We can add on a piece (another Bessel solution in ρ) to make this vanish at the wall $\rho = R$, i.e.

$$G_m(\rho,\rho';k) = \begin{cases} \frac{1}{\epsilon_0}I_m(k\rho)\left[K_m(k\rho') - I_m(k\rho')\frac{K_m(kR)}{I_m(kR)}\right], & \rho < \rho' \\ \frac{1}{\epsilon_0}K_m(k\rho)\left[I_m(k\rho') - I_m(k\rho)\frac{K_m(kR)}{I_m(kR)}\right], & \rho > \rho'. \end{cases} \tag{11.12}$$

Eigenfunction expansion We can do an eigenfunction expansion of the following (Sturm-Liouville) problem:

$$-\nabla^2\psi_n = \lambda_n\psi_n. \tag{11.13}$$

This is a Schrödinger equation, as we know. And we will set $\psi = 0$ on a boundary, specifically a cylinder. That is, we have a particle in a box. Suppose that we have some set of eigenfunctions $\{\psi_n\}$, suitably normalized (e.g. by the usual L^2 norm), and moreover the set is complete,

$$\sum_n \psi_n(\mathbf{r})\psi_n^*(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \tag{11.14}$$

[11] Equivalently this comes from the fact that the Green's function for a self-adjoint operator is symmetric in its arguments, cf. Arfken 8.2.

[12] This is very much like the delta-function potential in the Schrödinger equation.

We see that we've got a delta function; can we turn the LHS into something that looks like a Laplacian? We write down

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon_0} \sum_n \frac{\psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')}{\lambda_n}. \tag{11.15}$$

We check that this is right by taking the Laplacian explicitly:

$$\nabla^2 G_D = \frac{1}{\epsilon_0} \sum_n (-\psi_n(\mathbf{r}))\psi_n^*(\mathbf{r}') = -\frac{1}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}'). \tag{11.16}$$

So this works! But the hard part is loaded into actually finding the eigenfunctions ψ_n.

Let's see what happens in cylindrical coordinates. We make the separable ansatz

$$\psi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z), \tag{11.17}$$

and we write

$$\nabla^2 \psi + \lambda \psi = 0. \tag{11.18}$$

Separation of variables yields

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \underbrace{-\lambda - \frac{1}{Z}\frac{d^2Z}{dz^2}}_{\equiv -k^2}. \tag{11.19}$$

We pick our first separation constant to be $-k^2$ and find that

$$(\lambda - k^2)Z = -\frac{d^2Z}{dz^2} \implies Z(z) = e^{\pm i\sqrt{\lambda - k^2}z}. \tag{11.20}$$

Now we have

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k^2, \tag{11.21}$$

and then we can multiply through by ρ^2 and set

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \implies \Phi(\phi)e^{\pm im\phi}. \tag{11.22}$$

Finally, the last equation from separation of variables turns out to be Bessel's equation, so they have solutions

$$R(\rho) = J_m(k\rho), N_m(k\rho). \tag{11.23}$$

This time we want the oscillatory solutions in ρ since we are in a confined region that does not extend to infinity. In fact we should throw away the N_m, since these diverge as $\rho \to 0$. The allowed values of k come from the boundary condition on the side of the cylinder. Moreover we can choose

$$Z(z) = \sin(\sqrt{\lambda - k^2}z), \tag{11.24}$$

which has the property that it will vanish on the circular faces of the cylinder (say, of height L) provided that $\sqrt{\lambda - k^2} = \frac{n\pi}{L}$. We find that

$$\psi_{mkn} = A_{mkn} \sin\frac{n\pi z}{L} e^{im\phi} J_m(k\rho), \tag{11.25}$$

and to summarize, the condition of m being discrete comes from periodicity in ϕ; k being discrete comes from setting the J_ms to be zero on the side of the cylinder; and n comes from setting $Z(z)$ to be zero at $z = 0$ and $z = L$. The eigenvalue λ is then a function of k and n, as

$$\lambda = \frac{n^2\pi^2}{L} + k^2. \tag{11.26}$$

We'll mostly skip Chapter 9 of Zangwill, which deals with steady-state currents without discussing magnetic fields. Next time we'll consider the Aharanov-Bohm effect, which is a fun and cool thing not in the book.

Let's just start thinking about steady-state charge and current distributions. In this limit, Maxwell's equations read

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \boldsymbol{\nabla} \times \mathbf{E} = 0, \tag{11.27}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{11.28}$$

Since there is no time dependence, we can still use our tricks from potential theory and so on. The two equations dealing with $\mathbf{E}$ are totally unchanged.

We'll consider a version of Ohm's law which is as follows:

$$\mathbf{j} = \sigma \mathbf{E} = \frac{1}{\rho}\mathbf{E}, \tag{11.29}$$

where σ is conductivity and ρ is resistivity.

Lecture 12.

Thursday, February 13, 2020

Magnetostatics Magnetostatics is the area of electromagnetism dealing with steady-state currents (i.e. currents that do not change in time). The two magnetism Maxwell's equations then read

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{12.1}$$

Away from currents, we actually have

$$\boldsymbol{\nabla} \times \mathbf{B} = 0 \tag{12.2}$$

and therefore we can define a magnetic scalar potential,

$$\mathbf{B} = -\boldsymbol{\nabla}\chi \text{ with } \nabla^2 \chi = 0. \tag{12.3}$$

That is, for magnetostatics, one can define a scalar potential which also solves Laplace's equation away from current (density).

However, in regions where there is current we cannot quite do this. Instead, we can apply the Helmholtz theorem and write

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \underbrace{\left(\frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right)}_{\mathbf{A}(\mathbf{r})}. \tag{12.4}$$

One can equivalently move the curl inside (acting on unprimed coordinates) and write

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \tag{12.5}$$

which is simply the Biot-Savart law. When our current is confined to wires of constant current I, we may write

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int d\mathbf{l} \times \frac{(\mathbf{r} - \mathbf{l})}{|\mathbf{r} - \mathbf{l}|^3}, \tag{12.6}$$

We can turn Ampère's law into its integral form by taking the surface integral on both sides, i.e.

$$\int d\mathbf{S} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0 \int d\mathbf{S} \cdot \mathbf{j} \implies \oint_C d\mathbf{l} \cdot \mathbf{B} = \mu_0 I. \tag{12.7}$$

Like Gauss's law, Ampère's law admits only a few geometries we can solve exactly. One we can solve is an infinite solenoid, i.e. a cylinder made of wire coils which extends to infinity. If we take an Ampèrian loop outside, we know that the enclosed current is zero. We can argue that the field is actually constant outside along the axis of the cylinder. Inside, Ampère's law says that

$$B_z L = \mu_0 I_{\text{enc}} = \mu_0 n I L, \tag{12.8}$$

so we get a constant magnetic field strength inside,

$$B_z = \mu_0 n I, \tag{12.9}$$

where n is the number of loops per unit length and I is the current running in a single loop.

The magnetic vector potential is only defined up to the addition of any curl-free function; this is the idea of gauge freedom. Just as we could add a constant to the electrostatic scalar potential, we can add functions satisfying $\boldsymbol{\nabla} \times \tilde{\mathbf{A}} = 0$ to the vector potential. For instance, $\boldsymbol{\nabla}\chi$ can be added for any scalar χ.

Here is an explicit construction of the vector potential:

$$A_x = \int dz B_y, \quad A_y = -\int dz B_x, \quad A_z = A_z(z) \text{ (i.e. no dependence on } x \text{ or } y). \tag{12.10}$$

We can check that this will give the magnetic field:

$$\boldsymbol{\nabla} \times \mathbf{A} = \left(B_x, B_y, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \tag{12.11}$$

These first two terms look good; the last one is

$$\int dz \left(-\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \right) = \int dz \frac{\partial B_z}{\partial z} = B_z, \tag{12.12}$$

since the two derivatives in the first equation are part of the divergence of $\mathbf{B}$, and we know that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$.

Suppose we now turn time-dependence on. Then

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \mathbf{A}}{\partial t} \implies \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{12.13}$$

It follows that we can write a modified scalar potential such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\varphi. \tag{12.14}$$

If we changed $\mathbf{A}$ by a gauge transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$, then we would also need to modify the scalar potential as $\varphi \to \varphi - \frac{\partial \chi}{\partial t}$.

One of the nicest gauges[13] is Coulomb gauge,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{12.15}$$

We can always choose this gauge: suppose for some $\mathbf{A}$ we had $\boldsymbol{\nabla} \cdot \mathbf{A} \neq 0$. Then we can just solve the Poisson equation

$$\boldsymbol{\nabla}^2 \chi = \boldsymbol{\nabla} \cdot \mathbf{A} \tag{12.16}$$

and subtract off $\boldsymbol{\nabla}\chi$.

Eherenberg-Siday effect (Aharanov-Bohm effect) Historically, Aharanov and Bohm wrote a paper in 1959 on the idea of an electron outside a solenoid being sensitive to the vector potential, even when the magnetic field in a region is zero. In fact, Eherenber and Siday wrote about this same effect in 1949, but their paper was largely forgotten about until after Aharanov and Bohm's work.

Consider an infinite solenoid. If we take the integral of the vector potential on a loop around a solenoid, then

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot d\mathbf{S}. \tag{12.17}$$

That is, if we were sensitive to the integral of the vector potential, we could detect whether the solenoid was on. Classically the field is all that matters, so there can be no effect. But quantum mechanically, the story changes. The Hamiltonian for an electron in electric and magnetic fields is

$$\hat{H}\psi = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \psi + e\varphi\psi. \tag{12.18}$$

We'll take the electric field to be zero and $\varphi = 0$ identically. Then

$$\hat{H}\psi = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - e\mathbf{A})^2 \psi. \tag{12.19}$$

Let us now make a change of variables

$$\psi = e^{ig(\mathbf{r})}\tilde{\psi}, \text{ with } g(\mathbf{r}) = \frac{e}{\hbar}\int_0^{\mathbf{r}} A(\mathbf{r}') \cdot d\mathbf{r}'. \tag{12.20}$$

[13] In non-relativistic electrodynamics, anyway. For a more covariant version we might choose Loren(t)z gauge, $\partial_\mu A^\mu = 0$.

This integral is well-defined so long as $\nabla \times \mathbf{A} = 0$, i.e. in the region where the magnetic field is zero.

Then

$$\nabla\psi = i\nabla g\psi + e^{ig(\mathbf{r})}\nabla\tilde{\psi} = \frac{ie}{\hbar}\mathbf{A}\psi + e^{ig}\nabla\tilde{\psi}. \tag{12.21}$$

But notice that

$$(-i\hbar\nabla - e\mathbf{A})\psi = \frac{\hbar}{i}e^{ig}\nabla\tilde{\psi}, \tag{12.22}$$

and acting on this with $(-i\hbar\nabla - e\mathbf{A})$ again does much the same thing. That is,

$$\hat{H}\psi = \frac{1}{2m}(-i\hbar\nabla - e\mathbf{A})(e^{ig}\nabla\tilde{\psi}) = \frac{1}{2m}(-\hbar^2)e^{ig}\nabla^2\tilde{\psi}. \tag{12.23}$$

Now Schrödinger's equation tells us that $\hat{H}\psi = i\hbar\frac{\partial}{\partial t}$, so

$$\hat{H} = -\frac{\hbar^2}{2m}e^{ig}\nabla^2\tilde{\psi} = i\hbar e^{ig}\frac{\partial\tilde{\psi}}{\partial t}. \tag{12.24}$$

We conclude that

$$-\frac{\hbar^2}{2m}\nabla^2\tilde{\psi} = i\hbar\frac{\partial\tilde{\psi}}{\partial t}, \tag{12.25}$$

which is the free-particle Schrödinger equation.

This says that $\tilde{\psi}$ obeys the free-particle equation, while the actual wavefunction picks up a phase shift. Practically speaking, we could imagine sending some electrons around the left side of a solenoid and some around the right side and looking for interference due to the different phases when they meet up. Around a loop, we pick up a phase

$$\frac{e}{\hbar}\oint \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}', \tag{12.26}$$

and moreover this phase is unaffected by gauge transformations since

$$\oint(\mathbf{A} + \nabla\chi) = \oint \mathbf{A}. \tag{12.27}$$

Lecture 13.

Tuesday, February 18, 2020

Today we'll begin our dive into chapters 10-13, exploring magnetostatics. We'll skip most of the vector multipoles in spherical coordinates since they're a bit more specialized. There will be no class on Thursday, March 5, since Professor Zieve is away.

Previously, we discussed the E-field matching conditions at surfaces,

$$\Delta E_\perp = \frac{\sigma}{\epsilon_0}, \quad \Delta\mathbf{E}_\parallel = 0. \tag{13.1}$$

For the magnetic field, there are also matching conditions, but they are a little different. We have instead

$$\hat{\mathbf{n}}\times(\mathbf{B}_1 - \mathbf{B}_2) = \mu_0\mathbf{K}, \quad \hat{\mathbf{n}}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0. \tag{13.2}$$

where $\mathbf{K}$ is the surface current density. That is, it is the component of the $\mathbf{B}$-field parallel to the surface (but perpendicular to the current) that jumps discontinuously, while the perpendicular component has no discontinuity ($\Delta B_\perp = 0$).

Recall that in electrostatics, Gauss's law in vacuum and Faraday's law (in steady state) gave us

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \nabla\times\mathbf{E} = 0, \mathbf{E} = -\nabla\varphi \implies \nabla^2\varphi = 0. \tag{13.3}$$

For magnetostatics, things are different. We have now

$$\nabla\cdot\mathbf{B} = 0, \nabla\times\mathbf{B} = \mu_0\mathbf{j} \implies B = -\nabla\chi, \nabla^2\chi = 0, \tag{13.4}$$

i.e. we can still define a scalar potential satisfying Laplace's equation when there are no volume current densities but χ is *not continuous* at surface currents; it is only defined piecewise.

Example 13.5. Let's consider the case of two concentric circular rings in the xy-plane. A current I flows counterclockwise in the inner ring of radius a, while that same current I flows clockwise in the outer ring of radius b. We can then import our results from Laplace solutions in electrostatics and write down the form of the potential χ. This setup has azimuthal symmetry, so let us expand in Legendre polynomials. The magnetostatic scalar potential is then

$$\chi(r,\theta,\phi) = \begin{cases} \sum_{l=0}^{\infty} A_l \left(\frac{r}{a}\right)^l P_l(\cos\theta) & r < a \\ \sum_{l=0}^{\infty} \left[B_l \left(\frac{r}{b}\right)^l + C_l \left(\frac{a}{r}\right)^{l+1}\right] P_l(\cos\theta) & a < r < b \\ \sum_{l=0}^{\infty} D_l \left(\frac{b}{r}\right)^{l+1} P_l(\cos\theta) & r > b \end{cases} \tag{13.6}$$

where we've just pulled out some factors in anticipation of the coefficients A_l, B_l, C_l, D_l.[14] We cannot say that that χ is continuous at $r = a$ and $r = b$, but we have continuity of the perpendicular component,

$$\Delta(\boldsymbol{\nabla}\chi)_\perp = 0 \text{ at } r = a, r = b. \tag{13.7}$$

At $r = a$ this becomes

$$\sum_{l=0}^{\infty} l A_l \frac{1}{a} P_l(\cos\theta) = \sum_{l=0}^{\infty} \left(B_l l \frac{a^{l-1}}{b^l} - (l+1) C_l \frac{a^{l+1}}{a^{l+2}} \right) P_l(\cos\theta) \tag{13.8}$$

$$\implies l A_l = l B_l \left(\frac{a}{b}\right)^l - (l+1) C_l. \tag{13.9}$$

by orthogonality of the Legendre polynomials. Similarly at $r = b$ we will find that

$$l B_l - (l+1) C_l \left(\frac{a}{b}\right)^{l+1} = (-l+1) D_l \left(\frac{a}{b}\right)^{l+1}. \tag{13.10}$$

This gives us two constraints on the coefficients.

From the perpendicular jump condition $\hat{\mathbf{n}} \times (\mathbf{B}_1 - \mathbf{B}_2) = \mu_0 \mathbf{K}$ we get a jump in the perpendicular derivative of χ, i.e. $\frac{1}{r}\frac{\partial \chi}{\partial \theta}$. Taking the derivatives of the Legendre polynomials isn't too bad; we just get associated Legendre polynomials. Where things get bad is when we start to do the matching. In general we'll get things like

$$\sum_{l=0}^{\infty} A_l \frac{1}{a} P_l^1(\cos\theta) - \sum_{l=0}^{\infty} \left[B_l \frac{1}{a} \left(\frac{a}{b}\right)^l + C_l \right] P_l^1(\cos\theta) \tag{13.11}$$

and we could in principle expand the current density

$$I \frac{\delta(r-a)}{r} \delta(\theta - \pi/2) \tag{13.12}$$

in terms of Legendre polynomials using

$$\delta(\theta - \pi/2) = \sum_{l=0}^{\infty} \alpha_l P_l(\cos\theta) \tag{13.13}$$

where

$$\alpha_m = \frac{2m+1}{2} \int_0^\pi d\theta \sin\theta \, \delta(\theta - \pi/2) P_m(\cos\theta) = \frac{2m+1}{2} P_m(0). \tag{13.14}$$

Thus

$$\delta(\theta - \pi/2) = \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(0) P_l(\cos\theta). \tag{13.15}$$

Now we could match up all the coefficients using this expansion of this current density in terms of Legendre polynomials. We won't actually solve through for the coefficients in this way. Instead, we'll use Zangwill's

[14] Adding a length scale also has the benefit that all the sets of coefficients will have the same units. If we wrote $A_l r^l$, we would have to remember that the A_ls all have different units like $[\chi]/[r]^l$.

approach of "solve it on the axis" and extend our solution off the axis by uniqueness of the expansion coefficients.

For the field on the axis of a ring, we can just do the Biot-Savart calculation. Normally we have to crunch through computing the integral, but by symmetry, any components of the field which lie in the plane of the ring cancel; the only nonzero component is the one along the axis of the ring, $\hat{\mathbf{z}}$. That is,

$$\mathbf{B}(z) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{l})}{|\mathbf{r}-\mathbf{r}|^3} \tag{13.16}$$

$$= \frac{\mu_0 I}{4\pi}(2\pi a)\frac{1}{z^2+a^2}\frac{a}{\sqrt{z^2+a^2}}\hat{\mathbf{z}} \tag{13.17}$$

$$= \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}\hat{\mathbf{z}}. \tag{13.18}$$

This is the field. We wanted the potential, so let us integrate

$$\chi(z) = \int_z^\infty \mathbf{B}\cdot d\mathbf{z} = \frac{\mu_0 I}{2}\left(\frac{-z}{\sqrt{a^2+z^2}}\right) \tag{13.19}$$

which we can do by a trig substitution or equivalently Mathematica.

In our case,

$$\chi(z) = \frac{\mu_0 I z}{2}\left(\frac{1}{\sqrt{b^2+z^2}} - \frac{1}{\sqrt{a^2+z^2}}\right). \tag{13.20}$$

If we use our generating function tricks and expand in Legendre polynomials, we will have

$$\chi(z) = \frac{\mu_0 I}{2}\sum_{l=0}^{\infty}\left[\left(\frac{z}{b}\right)^{l+1} - \left(\frac{a}{z}\right)^{l}\right]P_l(0) \tag{13.21}$$

$$= \frac{\mu_0 I}{2}\left[\sum_{l=1}^{\infty}\left(\frac{z}{b}\right)^{l}P_{l-1}(0) - 1 - \sum_{l=0}^{\infty}\left(\frac{a}{z}\right)^{l+1}P_{l+1}(0)\right]. \tag{13.22}$$

Now we can equate the z-dependence with the coefficients in Eq. (13.6) and solve for e.g. the B_l and C_l coefficients, and then use our matching conditions to get the A_ls and D_ls.

This might be kind of painful to actually solve, but the upshot is that we can solve this analytically. Moreover, $C_0 = 0$, which we can see from the matching condition at $r = a$ setting $l = 0$. A bit more matching shows that in fact $D_0 = 0$ as well, and therefore the leading-order behavior as $r \to \infty$ is $(1/r)^{l+1}|_{l=1} = 1/r^2$, which looks very much like the potential from an *electric* dipole. Notice there is no monopole term here. We shall see that this is more than just an analogy, and that in fact a full multipole expansion will also apply for the magnetic *vector* potential.

Lecture 14.

Thursday, February 20. 2020

In the electrostatic case, we constructed a multipole expansion of the scalar potential V by expanding $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$. Here, we have a vector potential defined (in Coulomb gauge) by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{14.1}$$

Notice that for an individual component of the vector potential, the expansion looks the same as for the scalar potential (up to a prefactor). That is, since

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} - \mathbf{r}'\cdot\boldsymbol{\nabla}\frac{1}{r} + \frac{1}{2!}(\mathbf{r}'\cdot\boldsymbol{\nabla})^2\frac{1}{r} - \dots \tag{14.2}$$

where we've expanded in powers of $\mathbf{r}'$. The minus signs are correct here—we'll pick up some more from taking derivatives of $1/r$. Thus for an individual component $A_k(\mathbf{r})$, we have

$$A_k(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int d^3r'\, j_k(\mathbf{r}') + \frac{\mathbf{r}}{r^3}\cdot\int d^3r'\, j_k(\mathbf{r}')\mathbf{r}' + \dots\right] \tag{14.3}$$

which has our monopole moment and dipole moment terms written out explicitly. Note that in the magnetic case, there's no equivalent to the discrete sums we had in electrostatics. There, we considered point charges, but there are no monopoles in our theory of magnetism. Indeed, the $1/r$ term should vanish since we're integrating the current flowing around closed loops, i.e. the integral of the vector pointing around these closed loops is zero.

We can be slightly more formal. Let us write[15]

$$\boldsymbol{\nabla}' \cdot (\mathbf{j} r_k') = r_k' \boldsymbol{\nabla}' \cdot \mathbf{j}(\mathbf{r}') + \mathbf{j}(\mathbf{r}') \cdot \underbrace{\boldsymbol{\nabla}' r_k'}_{\hat{\mathbf{r}}_k'} = j_k(\mathbf{r}'), \tag{14.4}$$

where the first term vanishes by the continuity equation and the second one is just the kth component of $\mathbf{j}(\mathbf{r}')$. Hence we can write

$$\int d^3r' j_k(\mathbf{r}') = \int d^3r' \, \boldsymbol{\nabla}' \cdot (\mathbf{j} r_k') = \oint d\mathbf{A} \cdot \mathbf{j} r_k', \tag{14.5}$$

and this is zero when we demand that the currents are constrained to a finite volume (no currents go off to infinity). Note that the $1/r$ term is zero *because we are in Coulomb gauge*.

Now for the dipole we can write

$$A_k^{\text{dipole}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{r_l}{r^3} T_{kl}, \tag{14.6}$$

and we can play the same trick to rewrite the dipole moment. Let's write the volume integral as the divergence of something. That is, we will try

$$\boldsymbol{\nabla} \cdot (\mathbf{j} r_k r_l) = r_k r_l \boldsymbol{\nabla} \cdot \mathbf{j} + r_k \mathbf{j} \cdot \hat{\mathbf{r}}_l + r_l \mathbf{j} \cdot \hat{\mathbf{r}}_k = r_k j_l + r_l j_k. \tag{14.7}$$

The first term goes away since $\boldsymbol{\nabla} \cdot \mathbf{j}$ is zero. The other terms are proportional to components of j. This is nicely symmetric in k and l, as it should be, but we just want the j_k term (or equivalently the r_l term). However, we may write

$$\epsilon_{lki} (\mathbf{r} \times \mathbf{j})_i = r_l j_k - r_k j_l, \tag{14.8}$$

the form of a cross product,[16] and therefore

$$2 r_l j_k = \boldsymbol{\nabla} \cdot (\mathbf{j} r_k r_l) + \epsilon_{lki} (\mathbf{r} \times \mathbf{j})_i. \tag{14.9}$$

This first term is the divergence term which goes away when we change to a surface integral. The second one therefore gives us our dipole tensor. It follows that

$$T_{kl} = \frac{1}{2} \epsilon_{lki} \int d^3r' (\mathbf{r}' \times \mathbf{j}(\mathbf{r}'))_i \equiv \epsilon_{lki} m_i, \tag{14.10}$$

where m_i is the magnetic dipole.

We see now that T_{kl} is a completely antisymmetric rank 2 tensor, i.e. while a priori it could have had 9 different components, it turns out to be proportional to the Levi-Civita symbol. Hence all the diagonal elements vanish and there are only three independent components which we can associate to a vector $\mathbf{m}$.

It follows that the dipole vector potential is

$$A_k(\mathbf{r}) = \frac{r_l}{r^3} \frac{\mu_0}{4\pi} \epsilon_{lki} m_i = \frac{\mu_0}{4\pi} \frac{(\mathbf{m} \times \mathbf{r})_k}{r^3}. \tag{14.11}$$

As a single vector equation,

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}.} \tag{14.12}$$

We can now calculate the vector potential at the center of a ring of uniform current, say of radius a. It is proportional to

$$\frac{I \oint \mathbf{r}' \times d\mathbf{r}'}{2} = \frac{2\pi a^2 I}{2} = \pi a^2 I. \tag{14.13}$$

That is, it is the current in the loop times the area of the loop.[17]

---

[15] One way to make this guess is to suppose that the thing we take the divergence of should have an extra factor of r' in it.

[16] I usually write this as $(\mathbf{r} \times \mathbf{j})_i = \epsilon_{ikl} r_k j_l$, but one can multiply both sides by another epsilon and then use the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ to get it in this form.

[17] A factor of $2\pi a$ comes from the circumference, while another factor of a comes from the fact that $|\mathbf{r}'| = a$ on the circle.

Higher dipoles At this point, Zangwill makes a weird change of notation. Instead of writing $\mathbf{r}, \mathbf{r}'$ variables, he changes $\mathbf{r}'$ to $\mathbf{s}$ and also changes his derivative notation.[18]

$$\begin{aligned}\mathbf{r}' &\to \mathbf{s}\\ \nabla_i &\to \nabla_i\\ \nabla'_i &\to \partial_i\end{aligned}$$

The general multipole term in $\mathbf{A}(\mathbf{r})$ is

$$\frac{\mu_0}{4\pi}\frac{(-1)^n}{n!}\left[\int d^3s\, j_k(\mathbf{r}) s_{l_1}s_{l_2}\dots s_{l_n}\right]\nabla_{l_1}\nabla_{l_2}\dots\nabla_{l_n}\frac{1}{r}. \tag{14.14}$$

We know what to do now. We write our divergence term

$$\begin{aligned}\partial_p(j_p s_k s_k s_{l_1}\dots s_{l_n}) &= (\partial_p j_p)s_k s_{l_1}\dots s_{l_n} + j_p\delta_{pk}s_{l_1}\dots s_{l_n} + j_p s_k\delta_{pl_1}s_{l_2}\dots s_{l_n}+\dots\\ &= (\partial_p j_p)s_k s_{l_1}\dots s_{l_n} + j_k s_{l_1}\dots s_{l_n} + j_{l_1}s_k s_{l_2}\dots s_{l_n}+\dots\end{aligned}$$

But remember these are eventually multplied by $\nabla_{l_1}\nabla_{l_2}\dots\nabla_{l_n}$, which is completely symmetric in the l_i indices. That means that all the n terms proportional to j_{l_i} contribute in the same way to the final expression. The first term proportional to $\partial_p j_p$ is zero by conservation of charge. We can moreover turn the extra terms not proportional to j_k into terms that are proportional to j_k using cross products,

$$\epsilon_{ikl_1}(\mathbf{s}\times\mathbf{j})_i = s_k j_{l_1} - s_{l_1}j_k. \tag{14.15}$$

Hence our divergence turns into

$$\partial_p(j_p s_k s_k s_{l_1}\dots s_{l_n}) = (n+1)j_k s_{l_1}\dots s_{l_n} + n\epsilon_{ikl_1}(\mathbf{s}\times\mathbf{j})_i s_{l_2}\dots s_{l_n}. \tag{14.16}$$

That is,

$$\int d^3s\, j_k(\mathbf{s})(\mathbf{s}\cdot\boldsymbol{\nabla})^n = \frac{n}{n+1}\epsilon_{kil_1}\int d^3s(\mathbf{s}\times\mathbf{j})_i(\mathbf{s}\cdot\boldsymbol{\nabla})^{n-1}\boldsymbol{\nabla}_{l_1} \tag{14.17}$$

We find that the general multipole moment is[19]

$$A_k(\mathbf{r}) = \frac{\mu_0}{4\pi}\epsilon_{kil}\sum_{n=1}^{\infty}(-1)^n m^{(n)}_{ip_1\dots p_{n-1}}\nabla_{p_1}\dots\nabla_{p_n}\nabla_l\frac{1}{r}, \tag{14.18}$$

where

$$m^{(n)}_{ip_1\dots p_{n-1}} = \frac{n}{(n+1)!}\int d^3s\,(\mathbf{s}\times\mathbf{j})_i s_{p_1}\dots s_{p_{n-1}}. \tag{14.19}$$

More on dipoles After considering the higher multipoles, the dipole moment looks really simple. SInce

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3},$$

we can compute the magnetic field as

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0}{4\pi}\left[m\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{r^3}\right) - (\mathbf{m}\cdot\boldsymbol{\nabla})\left(\frac{\mathbf{r}}{r^3}\right)\right]. \tag{14.20}$$

This first one is a delta function—it's $\nabla^2 1/r$, telling us that there is a divergent vector potential at the location of the dipole. The second one turns out to be

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}\right), \tag{14.21}$$

which looks exactly like the electric field from an electric dipole.

[18] One of these should *really* be a covariant derivative. I feel uncomfortable writing ∇_i for a partial derivative.

[19] There also exists a spherical multipole expansion for vector fields in terms of Legendre polynomials. They exist and they're in Zangwill and we're not really going to do anything with them in this course.

Lecture 15.

Tuesday, February 25, 2020

Last time, we found the magnetic field of a dipole,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\Big[\mathbf{m}(\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{r^3} - (\mathbf{m}\cdot\boldsymbol{\nabla})\frac{\mathbf{r}}{r^3}\Big]. \tag{15.1}$$

The first term is secretly a delta-function (it is just $\nabla^2 1/r$), while the second term gives us our familiar dipole field away from $r = 0$,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}\right]. \tag{15.2}$$

In fact, we need to make a correction to this, which we'll see shortly.

Thinking of point dipoles can be useful; fundamental particles are often modeled as point charges which nevertheless have an intrinsic quantum mechanical spin and therefore a magnetic moment.

A point dipole is the limit of a ring of charge shrinking to zero radius while increasing the current. That is, for a physical loop we have $\mathbf{m} = \pi r^2 I$, and by increasing I while reducing r, we can keep $\mathbf{m}$ constant. Now let us take the volume integral over a sphere of radius R:

$$\int_V d^3r\, \mathbf{B}(\mathbf{r}) = \int_V d^3r \frac{\mu_0}{4\pi}\int_V d^3r' \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{15.3}$$

$$= -\frac{\mu_0}{4\pi}\int d^3r'\, \mathbf{j}(\mathbf{r}')\times\int_V d^3r \underbrace{\frac{(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3}}_{\frac{4\pi}{3}\frac{(r')^3}{r'^2}\hat{\mathbf{r}}' = \frac{4\pi}{3}\mathbf{r}'} \tag{15.4}$$

$$= -\frac{\mu_0}{3}\int_V d^3r' \mathbf{j}(\mathbf{r}')\times\mathbf{r}', \tag{15.5}$$

where we have identified the d^3r integral as the electric field from a constant charge density $4\pi\epsilon_0$ in a spherical volume V at a point r'. Hence we get

$$\int_V d^3r\, \mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{3}\int_V d^3r'\, \mathbf{j}(\mathbf{r}')\times\mathbf{r}' = \frac{2\mu_0}{\mathbf{m}} \tag{15.6}$$

Notice this is totally independent of V so long as the current source $\mathbf{j}$ is totally contained, in particular it works as $V \to 0$. Hence our dipole field is in fact

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3}\right] + \frac{2\mu_0}{3}\mathbf{m}\delta(\mathbf{r}), \tag{15.7}$$

where we have accounted for the delta function at the origin.

The Lorentz force law for current densities is

$$\mathbf{F} = \int d^3r'\, \mathbf{j}(\mathbf{r}')\times\mathbf{B}(\mathbf{r}'), \tag{15.8}$$

which is the generalization of our old $q\mathbf{v}\times\mathbf{B}$. For a current loop, the force on such a loop is

$$\mathbf{F} = \int d^3r'\, \mathbf{j}(\mathbf{r}')\times\big[\mathbf{B}(\mathbf{r}) + (\mathbf{r}'-\mathbf{r})\cdot\boldsymbol{\nabla})\mathbf{B}(\mathbf{r}) + \ldots\big], \tag{15.9}$$

where we have expanded $\mathbf{B}(\mathbf{r}')$ in a multipole expansion. We claim that the $\mathbf{j}(\mathbf{r}')\times\mathbf{B}(\mathbf{r})$ term goes away, since the integral of $\mathbf{j}$ around closed loops is zero. The term proportional to $\mathbf{r}$ goes away for the same reason. What remains is

$$F_p = \int d^3r(\mathbf{j}(\mathbf{r}')\times(r'_m\nabla_m)\mathbf{B}(\mathbf{r}))_p$$

$$= \int d^3r'\, j_k(\mathbf{r}')r'_m\nabla_m B_l(\mathbf{r})\epsilon_{klp}.$$

We recall that the magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2}\int d^3r'\, \mathbf{r}'\times\mathbf{j}(\mathbf{r}'), \tag{15.10}$$

so component-wise we have

$$\int d^3r'\, j_k r'_m = -\frac{1}{2}\epsilon_{kmi}\int d^3r'(\mathbf{r}'\times\mathbf{j})_i = -\epsilon_{kmi}m_i. \tag{15.11}$$

Hence

$$F_p = (-\epsilon_{kmi}m_i)\nabla_m B_l(\mathbf{r})\epsilon_{klp}, \tag{15.12}$$

and we can contract the epsilons as

$$-\epsilon_{klp}\epsilon_{kmi} = \delta_{li}\delta_{pm} - \delta_{lm}\delta_{pi}. \tag{15.13}$$

The force therefore becomes

$$F_p = \nabla_p B_l(\mathbf{r})m_l - \underbrace{\nabla_l B_l}_{=0} m_p \tag{15.14}$$

and we see that

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{B}(\mathbf{r})\cdot\mathbf{m}). \tag{15.15}$$

That is, dipoles feel no net force in a constant field, but they are sensitive to gradients in the field. This leads us naturally to the energy of a dipole in a field as $-\mathbf{B}\cdot\mathbf{m}$, i.e. dipoles tend to align with the field.

We can also consider the force between two dipoles. Since the magnetic field from the dipole drops off as $1/r^3$, the interaction between dipoles goes as $\boldsymbol{\nabla}(1/r^3)\sim 1/r^4$. Note that van der Waals interactions are also dipole interactions, but we might have heard from an undergraduate class that those interactions vary as $1/r^6$. What's going on? It turns out that the dipoles in such an interaction are not fixed in their strength; they are *induced* dipoles, so $\mathbf{m}$ depends on r. The induced dipole gets stronger as the applied dipole gets closer.

We can also think about torques,

$$\mathbf{N} = \int d^3r'\,\mathbf{r}'\times\left[\mathbf{j}(\mathbf{r}')\times\mathbf{B}(\mathbf{r}')\right] \sim \mathbf{m}\times\mathbf{B}. \tag{15.16}$$

Let's figure out the equations of motion for the magnetic moment. Let

$$\mathbf{m} = \gamma\mathbf{J}, \tag{15.17}$$

i.e. the magnetic moment comes from the angular momentum $\mathbf{J}$. The torque is the change of angular momentum with respect to time,

$$\mathbf{N} = \frac{d\mathbf{J}}{dt}. \tag{15.18}$$

Putting it together,

$$\frac{d\mathbf{m}}{dt} = \gamma\mathbf{N} = \gamma\mathbf{m}\times\mathbf{B}. \tag{15.19}$$

If we dot both sides with $\mathbf{m}$, we see that

$$0 = \mathbf{m}\cdot\frac{d\mathbf{m}}{dt} = \frac{1}{2}\frac{d(m^2)}{dt}. \tag{15.20}$$

If we dot with $\mathbf{B}$ instead, we get

$$\mathbf{B}\cdot\frac{d\mathbf{m}}{dt} = 0. \tag{15.21}$$

These two equations tell us that the magnitude of m doesn't change, but the angle $\mathbf{B}\cdot\mathbf{m}$ also doesn't change. We sometimes call γB the *Larmor frequency*. We know what's happening now. The magnetic moment rotates around the axis of $\mathbf{B}$ at a frequency γB like a gyroscope.

Let us try to consider the energy in magnetic interactions. For a loop of wire, we have

$$\delta W = -\sum_i q_i\mathbf{E}\cdot\mathbf{v}_i\delta_t, \tag{15.22}$$

i.e. we have some charges which experience a force in a field, and we multiply that force by the little distance they travel in a time interval δt. Now

$$\delta W = -\sum_i q_i \mathbf{E} \cdot \mathbf{v}_i \delta_t \tag{15.23}$$

$$= -\int d^3 r \mathbf{j}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) \delta t \tag{15.24}$$

$$= -I \oint C d\mathbf{l} \cdot \mathbf{E} \delta t \tag{15.25}$$

$$= -I \int_S d\mathbf{S} (\boldsymbol{\nabla} \times \mathbf{E}) \delta t \tag{15.26}$$

$$= I \int_S d\mathbf{S} \left(\frac{\partial \mathbf{B}}{\partial t} \right) \delta t \tag{15.27}$$

$$= I \frac{\partial \Phi}{\partial t} \delta t = I \delta \Phi, \tag{15.28}$$

with Φ the magnetic flux. That is, there is no work done if the flux is not changing. Next time, we'll build this to energy in a field distribution.

Lecture 16.

Thursday, February 27, 2020

Last time, we argued that changing magnetic fields (and fluxes) can do work on the charges in a wire, as

$$\delta W = I \delta \Phi. \tag{16.1}$$

If we consider the total work required for some final current, suppose that

$$I(t) = \frac{t}{t_f} I_f, \quad \Phi(t) = \frac{t}{t_f} \Phi_f. \tag{16.2}$$

Then

$$\delta \Phi = \delta t \frac{\Phi_f}{t_f}, \tag{16.3}$$

so the work is

$$\int_0^{\Phi_f} I d\Phi = \int_0^{t_f} \frac{t}{t_f} I_f \frac{\Phi_f}{t_f} dt = \frac{1}{2} \frac{t_f^2}{t_f^2} I_f \Phi_f = \frac{1}{2} I_f \Phi_f. \tag{16.4}$$

It follows that the stored energy in some current loop is

$$U = \frac{1}{2} I \Phi, \tag{16.5}$$

and if we define a geometrical constant

$$L = \frac{\Phi}{I}, \tag{16.6}$$

we see that the energy is

$$\frac{1}{2} L I^2 = \frac{1}{2} \frac{\Phi^2}{L}. \tag{16.7}$$

This L is the self-inductance; it says that the magnetic flux in a loop is directly proportional to the current running in that loop.

Let's now write

$$I\Phi = \int d^3 r \, \mathbf{j}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \tag{16.8}$$

so that

$$\begin{aligned}\delta W = I\delta\Phi &= I\delta(\int_S d\mathbf{S}\cdot\mathbf{B})\\ &= I\delta\int_S d\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\\ &= \int d^3r\,\mathbf{j}\cdot\delta\mathbf{A}.\end{aligned}$$

The total work is

$$W = \frac{1}{2}\int d^3r\,\mathbf{j}\cdot\mathbf{A}, \tag{16.9}$$

where the factor of $1/2$ comes from the fact that both $\mathbf{j}$ and $\mathbf{A}$ are changing with time, so this expression is secretly quadratic with time. Integrating over time to get rid of the δ provides us with the needed $1/2$.

Now the total work is

$$W \equiv U_B = \frac{1}{2}\int d^3r\,\mathbf{j}\cdot\mathbf{A} = \frac{1}{2}\int d^3r\frac{\boldsymbol{\nabla}\times\mathbf{B}}{\mu_0}\cdot\mathbf{A} = \frac{1}{2\mu_0}\int d^3r[\mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{A} - \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B})] = \frac{1}{2\mu_0}\int d^3rB^2. \tag{16.10}$$

Note that this work is the energy stored in the magnetic field. It is independent of any effects related to resistance; it is purely the work required to get the charges moving and produce a magnetic field.

Another complication is that in the electric case, we thought about keeping either potential constant or the charge constant (by electrically isolating an object). In the magnetic case, we can certainly run a constant current through the loop by hooking up a power supply, but there is no analogy for isolating the current loop. In general other forces can change the current running in the loop even if it is totally isolated.

Consider a square loop in the xy plane of side length L with a current I running through it. The lower left corner sits at the origin. We apply a magnetic field

$$\mathbf{B}_\odot = (B_0 + ax)\hat{\mathbf{z}}. \tag{16.11}$$

Then the force is

$$\mathbf{F} = \int d^3r\,\mathbf{j}\times\mathbf{B} = (-\hat{\mathbf{x}})(B_0)IL + \hat{\mathbf{x}}(B_0 + aL)IL = \hat{\mathbf{x}}IaL^2 \tag{16.12}$$

As it turns out, the constant B-field part washes out of the calculation. We see that the work done on the loop at an instant is

$$\delta W = IaL^2(v\delta t). \tag{16.13}$$

But magnetic forces aren't supposed to be doing work, yet our loop is clearly accelerating. What's going on? The catch is that the current in the wire must not be constant, i.e. I is changing with time. As the loop moves, it sees a changing magnetic flux, so

$$\delta\Phi = (av\delta t)L^2 \tag{16.14}$$

and therefore an EMF is produced,

$$\epsilon = -\frac{\delta\Phi}{\delta t} = -avL^2. \tag{16.15}$$

Hence the energy dissipated in an instant is

$$I\epsilon = -avL^2I. \tag{16.16}$$

The kinetic energy we pick up from the acceleration is precisely equal to the energy lost by the charges moving in the loop, so no work has been done.[20] It's this sort of work term that's hiding in our calculation of parallel currents attracting. If those currents were part of isolated wire loops, then we cannot assume the currents are constant. If there is a battery powering the currents, then there's an extra energy to keep track of.

[20] Griffiths has some very nice examples of places where magnetic fields appear to be doing work, and where there are always corrections like this that ensure that once all the energy is accounted for, no work is done by the magnetic force.

Magnetic fields in matter Just as we did in the electric case, we can talk about magnetic fields in matter. We can define *free current densities* $\mathbf{j}_f$ which we apply, and magnetization ("bound") currents $\mathbf{j}_m$, which describe the material response to the applied field. We produce $\mathbf{j}_f$ with an applied field $\mathbf{B}_{\text{ext}}$ and the sample responds with a self-field $\mathbf{B}_{\text{self}}$. The total current is the sum

$$\mathbf{j} = \mathbf{j}_f + \mathbf{j}_m, \tag{16.17}$$

and the total field is similarly the sum

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{self}}. \tag{16.18}$$

Self-consistency is the name of the game here.

There are two possibilities for where the magnetization current comes from: spin effects and orbital effects. For spins, we can write a total magnetic moment as

$$\mathbf{M}_S = \sum_{i=1}^{N} \mathbf{m}_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{16.19}$$

where some tiny dipoles $\mathbf{m}_i$ live on sites $\mathbf{r}_i$ in the material. We can think of these dipoles like current loops, which each are associated to a little current

$$\mathbf{j}_i = \boldsymbol{\nabla} \times [\mathbf{m}_i \delta(\mathbf{r} - \mathbf{r}_i)] \tag{16.20}$$

and therefore collectively produce an overall current:

$$\mathbf{j}_S = \sum \mathbf{j}_i = \sum \boldsymbol{\nabla} \times [\mathbf{m}_i \delta(\mathbf{r} - \mathbf{r}_i)] \boldsymbol{\nabla} \times \mathbf{M}_S. \tag{16.21}$$

which is called the "spin magnetization current density." At a surface, there is generally a surface magnetization current

$$\mathbf{K}_S = \mathbf{M}_S \times \hat{\mathbf{n}}. \tag{16.22}$$

If the magnetic moments $\mathbf{m}_i$ are not uniform, then we can have a net magnetization current density within the material.

The other source of magnetization current is orbital effects. Let us define a magnetization

$$\mathbf{M}_O = \begin{cases} 0 & \text{outside sample} \\ \mathbf{j}_O = \boldsymbol{\nabla} \times \mathbf{M}_O & \text{inside sample} \end{cases} \tag{16.23}$$

such that

$$\mathbf{K}_O = \mathbf{M}_O \times \hat{\mathbf{n}} \text{ at surface.} \tag{16.24}$$

Now if we slice through our material, then it should be the case that the surface current density we've defined is equivalent to the loop integral of the internal current density at the surface before we sliced the material. That is,

$$\int_S d\mathbf{S} \cdot \mathbf{j}_O(\mathbf{r}) + \oint_C d\mathbf{l} \cdot (\mathbf{K}_O(\mathbf{r}) \times \hat{\mathbf{n}}) = \oint d\mathbf{l} \cdot \mathbf{M}_O - \oint d\mathbf{l} \cdot \mathbf{M}_O = 0. \tag{16.25}$$

Now we avoid further complications by defining an overall magnetization

$$\mathbf{M} = \mathbf{M}_S + \mathbf{M}_O \tag{16.26}$$

and a total magnetic moment $\mathbf{m}$ which *includes* the orbital part, so that

$$\mathbf{m} = \frac{1}{2} \int d^3 r (\mathbf{r} \times \text{"}\mathbf{j}\text{"}) \tag{16.27}$$

where "$\mathbf{j}$" indicates a sum of $\mathbf{j}$ and $\mathbf{K}$, any internal and surface currents.[21] Equivalently

$$\mathbf{m} = \int_V d^3 \mathbf{M}, \tag{16.28}$$

the integral of the overall magnetization.

[21] The units don't quite match, hence the quotation marks, but this can be made sensible.