PHYSICS 230B: QUANTUM FIELD THEORY

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Lecture 1.

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The syllabus for this course is on Canvas.

Massless spin 1 particles What we're gonna start talking about is QFTs with massless spin 1 particles (i.e. photons). When we talk about particles, what we mean are states, i.e.

$$|p, s\rangle, \tag{1.1}$$

labeled by a four-momentum p and a spin state s. For massive spin 1 there are three spin states which can be distinguished by the helicity, i.e. the projection of the spin along the direction of the three-momentum, so our states can be labeled by $s = 0, \pm 1$.

In the massive case, we therefore have

$$|p, s\rangle_{s=0,\pm 1} \mapsto^{\Lambda} \sum_{s'} D^{(1)}_{s's}(W(\Lambda, p))|\Lambda p, s'\rangle \tag{1.2}$$

where $W(\Lambda, p)$ indicates a representaiton of the $SO(3)$ symmetry.

In the massless case, it's a little different. Instead we have

$$|p, s\rangle_{s=\pm 1} \mapsto^{\Lambda} e^{-is\theta(W(\Lambda,p))}|\Lambda p, s\rangle. \tag{1.3}$$

The key difference is that we lose a polarization state in going from massive to massless. $2 \neq 3$. There is no rest frame for the photon, so we have to be careful in taking a massless limit or conversely turning on a mass.

How do we set up a QFT? We will need some fields, obviously. We can set up a Fock space with some creation and annihilation operators, so that given a vacuum state, we can define

$$|p_1, s_1, \ldots, p_n, s_n\rangle = a^\dagger_{s_1}(p_1), \ldots, a^\dagger_{s_n}(p_n)|0\rangle \tag{1.4}$$

where the creation and annihilation operators have commutator

$$[a_{s'}(p'), a^\dagger_s(p)] = \delta_{s's}(p'|p) = \delta_{s's} 2|\mathbf{p}| \cdot (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}). \tag{1.5}$$

We could define a vector field, since we want a nontrivial transformation under Lorentz. Thus

$$\hat{A}^\mu(x) = \int (dp) \sum_{s=\pm 1} \hat{a}_s(p)\epsilon^\mu_s(p) e^{-ip\cdot x} + \text{h.c.} \tag{1.6}$$

with $dp = \frac{d^3p}{(2\pi)^3}\frac{1}{2|\mathbf{p}|}$. That is, our vector field is an integral over d^4p with creation and annihilation operators summed over spin spates, polarization vectors $\epsilon_s^\mu(p)$ attached, and the corresponding exponentials.

Note that as a consequence of being massless, the field satisfies

$$\Box A^\mu(x) = 0, \tag{1.7}$$

the massless Klein-Gordon equation. We can choose

$$p_\mu \epsilon_s^\mu(p) = 0 \iff \partial_\mu \hat{A}^\mu(x) = 0. \tag{1.8}$$

That gets rid of one linear combination, but we still seem to have too many degrees of freedom. In fact, we will show that A does not transform as an honest vector. That is, the equation

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x)\hat{U}(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{1.9}$$

cannot be satisfied. This is nontrivial to show, as the left side is an infinite-dim unitary rep of the Lorentz group, whereas the right side is a finite-dim non-unitary rep of Lorentz.

However, what we notice is that the left side evaluates to

$$\hat{A}^\mu(x) = \int (dp) \sum_{s=\pm 1} \underbrace{\hat{U}^\dagger \hat{a}_s(p) U}_{\sim e^{-is\theta} a_s} \epsilon_s^\mu(p) e^{-ip\cdot x} + \text{h.c.}\ , \tag{1.10}$$

and we know that the right side (by virtue of having Lorentz indices) must affect the polarization vector, i.e.

$$\Lambda^\mu{}_\nu \epsilon^\nu(\Lambda^1 p) \stackrel{?}{=} e^{-is\theta(W(\Lambda,\Lambda^{-1}p))}\epsilon_s^\mu(p). \tag{1.11}$$

For a general four-momentum we can certainly choose a frame where it takes the statndard form

$$p^\mu \to n^\mu = (E, 0, 0, E), \tag{1.12}$$

such that for rotations about the x_3 axis,

$$(e^{-i\theta J_3})^\mu{}_\nu \epsilon_\pm^\nu(n) = e^{-i\theta}\epsilon_\pm^\mu(n). \tag{1.13}$$

Here J_3 is the generator of rotations around the x_3 axis. Hence this becomes an eigenvalue equation, $J^3\epsilon_\pm(n) = \pm\epsilon_\pm(n)$. We find that

$$\epsilon_\pm^\mu(n) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) + \alpha_\pm n^\mu. \tag{1.14}$$

There's an extra freedom in the αs since n^μ is a zero eigenvector of J_3 and can therefore be added to $\epsilon_\pm(n)$ with impunity.

Let us consider the action of the Little group, with the generators J_3, T^1, T^2 (transcribe later). Thus

$$W(\theta, \beta_1, \beta_2)^\mu{}_\nu = \left[e^{-i(\theta J^3 + \beta_1 T^1 + \beta_2 T^2)}\right]^\mu{}_\nu \tag{1.15}$$

defines the Wigner rotation. These are the operators which leave the form of the standard four-momentum unchanged.

Now

$$\Lambda^\mu{}_\nu \epsilon_\pm^\nu(\Lambda^{-1}p) = e^{-is\theta}\epsilon_\pm^\mu(p) + ip^\mu f_s, \tag{1.16}$$

where θ, f_s are some functions of $W(\Lambda, \Lambda^{-1}p)$. We see that we've picked up an extra piece, the p^μ term. Now $T^{1,2}\cdot n = 0$, since this is the definition of being in the Little group. But on the polarization vector, we have instead

$$T^{1,2}\cdot \epsilon(n) \propto n. \tag{1.17}$$

Thus when we write down the unitary transformation of our field A^μ, we have

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x)\hat{U}(x) = \Lambda^\mu{}_\nu \hat{A}^\nu(\Lambda^{-1}x) + \partial^\mu \hat{\omega}(x). \tag{1.18}$$

There's some extra stuff here too,

$$\hat{\omega}(x) = \int (dp) \sum_{s=\pm} \left[\hat{a}_s(p)\epsilon_s^\mu(p) e^{-ip\cdot x} f_s(W(\ldots)) + \text{h.c.}\right], \tag{1.19}$$

which is some terrible stuff we don't want to deal with.

We recall (from Chapter 7) that there's a boost $L(p)$ such that

$$\epsilon_{\pm}(p) = L(p) \cdot \epsilon_{\pm}(n), \tag{1.20}$$

with $L(p) \cdot n = p$.

For some standard $n = (E, 0, 0, E)$, we can can boost along the 3-axis to turn $E \to p^0$ and then rotate $\mathbf{n} = (0, 0, p^0)$ to point in the direction of $\mathbf{p}$. Thus

$$\epsilon_{\pm}(n) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \tag{1.21}$$

such that $\epsilon^0_{\pm}(p) = 0$ gives

$$\hat{A}^0(x) = 0, \tag{1.22}$$

which is clearly not Lorentz-invariant.

However, while A^μ by itself is not good to work with, we can define

$$\hat{F}_{\mu\nu}(x) \equiv \partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x), \tag{1.23}$$

which defines an antisymmetric tensor observable. We might like to write down some sort of interacting Hamiltonian, which should have terms $O(A^3)$. For instance,

$$F^{\mu\nu} F_{\nu\rho} F^{\rho}{}_{\mu} \in \mathcal{H}_{\text{int}}. \tag{1.24}$$

What we'll see is that $[A_\mu] = 1$, which tells us that $[F^3] = 6 > 4$, i.e. in four spacetime dimensions this coupling is irrelevant. We know that light has nontrivial couplings to matter, so we'll have to figure out how this can be done.

Gauge invariance Gauge invariance comes into our theory through the "Stuckelberg trick." Stuckelberg figured out how to go from a theory without gauge invariance to a theory with gauge invariance. We begin by declaring that a new field $\hat{A}^\mu$ has the property

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(x) = \Lambda^\mu{}_\nu \hat{A}^\nu(\Lambda^{-1} x), \tag{1.25}$$

i.e. transforms as an honest vector, given that

$$A_\mu(x) \cong A_\mu(x) + \partial_\mu \omega(x). \tag{1.26}$$

That is, we say that adding $\partial_\mu \omega(x)$ is a gauge redundancy, i.e. two fields differing only by a total derivative lead to the same physics.

Now, for some A_μ our gauge transformations sweep out some gauge orbit which ends up foliating the space of A_μ, and the configuration space is then the set of equivalence classes. This is known as the "geometrical viewpoint."

Conversely, there is the gauge-fixed viewpoint in which we choose a representative $A_\mu(x)$ from each orbit.

Notice also that our field strength tensor is gauge-invariant. In the free theory, if we only want to build the equations of motion out of $F^{\mu\nu}$, basically all we can write down is

$$\partial_\mu F^{\mu\nu} = 0. \tag{1.27}$$

This is both $O(A_\mu)$ and quadratic in the derivatives, $O(\partial^2)$. And thus we may choose a gauge such that

$$\Box A = 0 \tag{1.28}$$

$$A^0 = 0 \tag{1.29}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{1.30}$$

This is *Coulomb gauge*.

Let $A_\mu(x)$ be in a general gauge. Does there exist an ω with

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x) \tag{1.31}$$

such that

$$0 \stackrel{?}{=} A'_0(x) = A_0(x) + \partial_0 \omega(x, t) \tag{1.32}$$

Sure, if we choose $\omega(\mathbf{x},t) = -\int^t dt' A_0(\mathbf{x},t') + f(\mathbf{x})$, where we can certainly add something that depends only on $\mathbf{x}$. Moreover we wish to impose

$$0 \stackrel{?}{=} \boldsymbol{\nabla} \cdot \mathbf{A}'(\mathbf{x},t) = \boldsymbol{\nabla} \cdot \left[A(\mathbf{x},t) - \boldsymbol{\nabla}\left(-\int_0^t dt' A_0(\mathbf{x},t') \right) - \boldsymbol{\nabla} f(\mathbf{x}) \right]. \tag{1.33}$$

In general this is not possible, since the first two terms here generically depend on time and the $\nabla^2 f(\mathbf{x})$ term does not depend on time. However, we can do it for one particular time, $t = 0$, and set

$$\boldsymbol{\nabla} \cdot \mathbf{A}'(\mathbf{x}, t = 0) = 0, \tag{1.34}$$

such that the equations of motion guarantee

$$\partial_t (\boldsymbol{\nabla} \cdot \mathbf{A}) = 0. \tag{1.35}$$