

PHYSICS 200B: ELECTROMAGNETISM I

IAN LIM

LAST UPDATED JANUARY 31, 2020

These notes were taken for Physics 200B, *Electromagnetism I*, as taught by Rena Zieve at the University of California, Davis in winter quarter 2020. I live-TeXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.

Many thanks to Arun Debray for the L^AT_EX template for these lecture notes: as of the time of writing, you can find him at <https://web.ma.utexas.edu/users/a.debray/>.

CONTENTS

1.	Tuesday, January 7, 2020	1
2.	Thursday, January 9, 2020	4
3.	Tuesday, January 14, 2020	7
4.	Thursday, January 16, 2020	9
5.	Thursday, January 23, 2020	12
6.	Friday, January 24, 2020	15
7.	Tuesday, January 28, 2020	17
8.	Thursday, January 30, 2020	19

Lecture 1.

Tuesday, January 7, 2020

Note. This course will have discussion sections! I will not be writing notes for those since they're like homework. Exams will be closed-book but with a formula sheet—two pages, front and back, typed okay. The final exam can be moved with signed consent of all students.

The textbook for this course is Zangwill *Modern Electrodynamics*. The plan is to cover chapters 1-13, mostly electrostatics and magnetostatics (time-independent stuff). Homeworks due Tuesdays apart from the very last one, which is due Thursday.

Most of Chapter 1 of Zangwill is mathematical preliminaries. Chapter 2 is largely a review of Maxwell's equations. There is one important auxiliary to the Maxwell equations, though, and it's the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (1.1)$$

This equation just says there's stuff (ρ) and the stuff can flow in a way defined by the vector field \mathbf{j} . The divergence $\nabla \cdot \mathbf{j}$ tells us how much charge is flowing out of a point, on average.

To review, the first two Maxwell's equations are the following:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1.2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.3)$$

The next one we might guess is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (1.4)$$

that current flow sources magnetic fields. This equation can't be quite right, because if we take the divergence, the divergence of a curl always vanishes, so

$$0 = \nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{j}, \quad (1.5)$$

which by the continuity equation says that $\frac{\partial \rho}{\partial t} = 0$ everywhere. That seems nonphysical, so let us amend the equation by adding a \mathbf{j}_D term, called the *displacement current*. That is,

$$0 = \mathbf{j}_D + \mu_0 \left(-\frac{\partial \rho}{\partial t} \right) = -\mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_D. \quad (1.6)$$

This *defines* the displacement current as

$$\mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_D, \quad (1.7)$$

and since we recall that $\mu_0 \epsilon_0 = \frac{1}{c^2}$, we can rewrite what we recognize as Ampère's law as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (1.8)$$

The last Maxwell equation is Faraday's law,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.9)$$

These are Maxwell's equations in vacuum, and they can be modified to describe fields in materials with some little tweaks.

There's also a "glitch" in Zangwill's exposition. In the text, Zangwill mentions that Maxwell wrote down 12 equations prior to the development of vector notation (relating each of the components), but in fact he wrote down *eight*. Why eight? Each of the curl equations is a vector equation (three components), but each of the divergence equations is only one (a scalar equation). Hence $2 \times 3 + 2 \times 1 = 8$.

Altogether, we have the differential forms of Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1.10)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.11)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.12)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \quad (1.13)$$

We should also be familiar with the integral forms of the Maxwell equations,

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} Q_{\text{enc}} \quad (1.14)$$

$$\int \mathbf{B} \cdot d\mathbf{S} = 0 \quad (1.15)$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} \quad (1.16)$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \frac{1}{c^2} \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{S}. \quad (1.17)$$

If we have matter, then the laws need to be adapted because generically, charges and magnetic dipoles reorient themselves based on the applied fields. Materials have electric and magnetic polarizabilities. It's useful to define the total charge as a sum

$$\rho = \rho_f - \nabla \cdot \mathbf{P}, \quad (1.18)$$

where \mathbf{P} is the *polarization vector*. That is, we can put some charge ρ_f on a material, but in general when there's a background applied field, the charges will redistribute themselves. We know this as the fact that charges in a dielectric generically rearrange themselves to oppose the applied field. Recall that \mathbf{P} points from negative to positive charge. Thus a positive divergence of \mathbf{P} acts like a local negative charge.

We can do the same for magnetism, though it's a bit messier.

$$\mathbf{j} = \mathbf{j}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}, \quad (1.19)$$

where \mathbf{M} is the *magnetization vector*, which we think of as little magnetic dipoles or eddy currents orienting themselves to minimize their energy. This one depends on the current due to free charge $\mathbf{j}_f = \frac{\partial \rho_f}{\partial t}$, the rate of change of the bound charge from polarization, and the curl of the magnetization.

This leads us to define auxiliary fields:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \quad (1.20)$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} = \frac{1}{\mu} \mathbf{B}. \quad (1.21)$$

For the electric field, the sign conventions (whether to add or subtract \mathbf{p} make sense. It's very natural for the polarization pointing from $-$ to $+$ to add to an effective field \mathbf{D} . But for the magnetic field, it could go either way. Magnets near superconductors induce a magnetization that perfectly cancels the applied field, whereas in ferromagnets like iron, the magnetization is in the same direction as the applied field. We just have to pick a convention and stick to it.

The Maxwell equations in materials take on kind of a nice form:

$$\nabla \cdot \mathbf{D} = \rho_f \quad (1.22)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.23)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.24)$$

$$\nabla \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.25)$$

Helmholtz theorem The Helmholtz theorem says that if we specify the divergence and curl of a function everywhere, then the function is uniquely defined up to adding functions which have zero divergence and zero curl. Such functions are called *harmonic functions*, which we'll see more later, and whether or not we should add them corresponds to fitting boundary conditions.

We can actually write the electric field explicitly in terms of the divergence and curl:

$$\mathbf{E}(\mathbf{r}) = -\underbrace{\nabla \int d^3 r' \frac{\nabla' \cdot \mathbf{E}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}}_{\text{curl}=0} + \underbrace{\nabla \times \int d^3 r' \frac{\nabla' \times \mathbf{E}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}}_{\text{div}=0}. \quad (1.26)$$

The second term is actually zero if there is no time dependence, from Faraday's law. If we apply Gauss's law to rewrite $\nabla' \cdot \mathbf{E}$, we recognize what remains as

$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r}) = -\nabla \int d^3 r' \frac{\rho(\mathbf{r}')}{4\pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi \epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.27)$$

We've recovered Coulomb's law, just written in a more vectorial language.

Let's consider the force on a charge distribution ρ^* due to the field from a distribution ρ . That is,

$$\mathbf{F}(\rho, \rho^*) = \int d^3 r \rho^*(\mathbf{r}) \mathbf{E}(\mathbf{r}) \quad (1.28)$$

$$= \frac{1}{4\pi \epsilon_0} \int d^3 r \int d^3 r' \frac{\rho^*(\mathbf{r}) \rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.29)$$

But we notice that we can exchange \mathbf{r} and \mathbf{r}' at the cost of picking up a sign flip:

$$\mathbf{F}(\rho, \rho^*) = -\mathbf{F}(\rho^*, \rho). \quad (1.30)$$

If we set $\rho = \rho^*$, then $\mathbf{F}(\rho, \rho) = 0$, which tells us that a charge distribution cannot exert a net force on itself.

Finally, let's introduce potentials. The electric field in electrostatics can be written as the gradient of the scalar potential,

$$\mathbf{E} = -\nabla \phi, \quad (1.31)$$

so that

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi. \quad (1.32)$$

The equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad (1.33)$$

is hopefully familiar to us as Poisson's equation, and the homogeneous case ($\rho = 0$) is

$$\nabla^2 \phi = 0, \quad (1.34)$$

which is Laplace's equation.

Lecture 2.

Thursday, January 9, 2020

Today we'll start our discussion with Gauss's law, moving rapidly into chapter 3 of Zangwill. Gauss's law lets us calculate electric fields rapidly for situations with high amounts of symmetry. Basically, we can solve systems with spherical symmetry, cylindrical symmetry, and translational plane symmetry. It doesn't get us too far but it's a lot better than Coulomb's law volume integrals.

As we know, Gauss's law (differential form) states

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (2.1)$$

or in integral form

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{E} d^3r = \int_V \frac{\rho}{\epsilon_0} d^3r = \frac{Q_{\text{enc}}}{\epsilon_0}. \quad (2.2)$$

The simplest example is for the point charge. For a single positive charge $q > 0$, we can reason that the field must point radially outwards and it is constant at surfaces of constant R , so that

$$\oint_S \mathbf{E} \cdot d\mathbf{Q} = |E|4\pi R^2 = \frac{q}{\epsilon_0}, \quad (2.3)$$

which yields

$$\mathbf{E} = \frac{1}{4\pi R^2 \epsilon_0} \hat{\mathbf{r}}. \quad (2.4)$$

Gauss's law also gives us a nice result (sometimes known as Newton's shell theorem), which says that the electric field due to a spherically symmetric shell of charge *inside* that shell ($r < R$) is zero.

We can also do Gauss's law for an infinite plane by drawing a Gaussian pillbox, say, extending a height h above and below the plane. Rotational symmetry lets us reason that the field can only point in the normal direction to the plane, while reflection symmetry says its magnitude is the same equal distances above and below the plane. It follows that

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \sigma A, \quad (2.5)$$

where A is the area of the face of the Gaussian pillbox. Since the field is normal to the plane and the faces of the pillbox are oriented outwards, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2E(h)A, \quad (2.6)$$

so that

$$\mathbf{E}(h) = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z < 0. \end{cases} \quad (2.7)$$

The reason there is no h dependence is because the infinite plane is scale-invariant. If we rescale all the coordinates, the infinite plane still looks like an infinite plane. This result isn't necessarily useful to us because real life is filled with infinite planes of charge, or even because the problem is exactly solvable; rather, it's because any reasonably smooth (in the mathematical sense) surface looks locally flat, which means that near the surface, we have essentially the field from an infinite plane.

By the same Gaussian pillbox arguments, we find that E_{\parallel} is continuous at surfaces of charge, while E_{\perp} is discontinuous by $\frac{\sigma}{\epsilon_0}$. How should we calculate the force on a little surface of charge, given that the electric field is discontinuous above and below the field? Well, we can just take the average. This seems physically reasonable, but we can justify it. Remember the discontinuity comes from the charge at the surface itself, and as we showed last time, charges cannot exert forces on themselves. So if we average above and below the surface, we will basically average away the contribution of the charged surface itself and get the right answer.

We can also state Earnshaw's theorem— in a closed region with no charge, any extrema of the potential must be on the boundary. For suppose there was a local maximum of the potential in the interior. The gradient of the potential vanishes at that point, and a little bit away we can draw a Gaussian surface it is pointing towards that point everywhere. It follows that the \mathbf{E} -field points away everywhere, so our integral $\oint_S \mathbf{E} \cdot d\mathbf{A} > 0$, which violates our assumption that there was no charge in the region.¹ This tells us that we cannot make an electrostatic cage; no charge distribution can hold itself in a static configuration under Coulomb interactions alone. Earnshaw's theorem has told us that any charge dropped in a region with no other charge will move to the edges of that region, since it cannot sit at an extremum.

Potential and potential energy For our purposes, we will follow Zangwill and denote potential energy by V and electrostatic potential by ϕ . We say the potential energy changes as

$$\delta V = -\mathbf{F} \cdot d\mathbf{s} \quad (2.8)$$

for small displacements, i.e.

$$\mathbf{F} = -\nabla V. \quad (2.9)$$

In addition, when forces are due only to electric fields, then

$$\mathbf{F} = q\mathbf{E} = -q\nabla\phi \implies V = q\phi. \quad (2.10)$$

Hence the electrostatic potential is the energy per charge.

Let's now prove Green's reciprocity relation for the potential energy. That is, suppose there are two charge distributions ρ_1, ρ_2 . The energy of ρ_2 in the potential ϕ_1 created by ρ_1 is

$$V = \int d^3r \rho_2(\mathbf{r})\phi_1(\mathbf{r}), \quad (2.11)$$

such that

$$\delta V = \int d^3r [\rho_2(\mathbf{r} - \delta\mathbf{s}) - \rho_2(\mathbf{r})]\phi_1(\mathbf{r}). \quad (2.12)$$

The minus sign comes from that active/passive transformation jazz. If the distribution is moved by $\delta\mathbf{s}$, then looking at the "same" point in space \mathbf{r} for the new distribution is equivalent to looking at the original distribution at a point $\mathbf{r} - \delta\mathbf{s}$.

We can now Taylor expand as $\delta\mathbf{s}$ gets small, such that

$$\delta V \approx -\delta\mathbf{s} \cdot \int d^3r (\nabla\rho_2(\mathbf{r}))\phi_1(\mathbf{r}) = -\delta\mathbf{s} \cdot \int d^3r (\nabla(\rho_2\phi_1) - \rho_2\nabla\phi_1), \quad (2.13)$$

after a product rule manipulation (basically an integration by parts). Now this first term is a total derivative, so it vanishes as we take our integration region d^3r to be all space. What's left is

$$\delta V = -\delta\mathbf{s} \cdot \int d^3r \rho_2(\mathbf{r})\mathbf{E}_1(\mathbf{r}). \quad (2.14)$$

Let us rewrite this energy in a different way:

$$V = \int d^3r \rho_2(\mathbf{r})\phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r' \rho_2(\mathbf{r}) \frac{\rho_1(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (2.15)$$

But notice that this integral is manifestly symmetric in \mathbf{r} and \mathbf{r}' (i.e. $|\mathbf{r} - \mathbf{r}'| = |\mathbf{r}' - \mathbf{r}|$). This certainly converges, so we can switch the order of integration and do the d^3r integral first to find

$$V = \int d^3r \rho_2(\mathbf{r})\phi_1(\mathbf{r}) = \int d^3r' \rho_1(\mathbf{r}')\phi_2(\mathbf{r}'). \quad (2.16)$$

¹One can also argue this directly from the form of Laplace's equation.

This relation means that the energy in an electrostatic charge distribution doesn't depend on how we build it, only on the final geometry.

Consider the following example.

Example 2.17. Suppose we have a spherical region of radius R centered on the origin with no charge inside. As it turns out, the potential at the center is the average potential on the surface:²

$$\varphi(0) = \frac{1}{4\pi R^2} \int dS \varphi(\mathbf{r}). \quad (2.18)$$

Can we use Green's reciprocity relation to find this result? Take

$$\varphi_1 = \varphi, \quad \rho_1 = \rho; \quad \rho_2(\mathbf{r}) = \frac{q}{4\pi R^2} \delta(r - R). \quad (2.19)$$

Then

$$\varphi_2 = \begin{cases} \frac{q}{4\pi\epsilon_0 r} & r \geq R, \\ \frac{q}{4\pi\epsilon_0 R} & r \leq R. \end{cases} \quad (2.20)$$

It follows that

$$\int d^3r \frac{q}{4\pi R^2} \delta(r - R) \varphi(\mathbf{r}) = \int d^3r \rho(\mathbf{r}) \varphi_2(\mathbf{r}) = \int_R^\infty r^2 dr \int d\Omega \rho(\mathbf{r}) \frac{q}{4\pi\epsilon_0 r}. \quad (2.21)$$

But this last expression up to the factor of q is exactly the integral we would use to compute the potential at the origin. That is,

$$\int_R^\infty r^2 dr \int d\Omega \rho(\mathbf{r}) \frac{q}{4\pi\epsilon_0 r} = q \varphi(\mathbf{r} = 0). \quad (2.22)$$

Meanwhile, the LHS says that we just integrate

$$\int d^3r \frac{q}{4\pi R^2} \delta(r - R) \varphi(\mathbf{r}) = \frac{q}{4\pi R^2} \int dS \varphi(\mathbf{r}). \quad (2.23)$$

Cancelling the factors of q , we have exactly the desired result. The potential at the center of the sphere is equal to the average potential on its surface.

We can write the potential energy of a charge distribution as

$$U_E = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^N \sum_{i>j}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (2.24)$$

or for continuous distributions,

$$U_E = \frac{1}{8\pi\epsilon_0} \int d^3r \int d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2} \int d^3r \rho(\mathbf{r}) \varphi(\mathbf{r}) \quad (2.25)$$

Note that for all-positive or all-negative distributions of charges, these formulae make it clear that the energy of assembling the distribution is positive-definite. The answer is somewhat less clear when we have a mix of charges. Let's manipulate this result to see what happens.

$$\begin{aligned} U_E &= \frac{1}{2} \int d^3r \rho(\mathbf{r}) \varphi(\mathbf{r}) \\ &= \frac{1}{2} \int d^3r \epsilon_0 \nabla \cdot \mathbf{E} \varphi(\mathbf{r}) \\ &= \frac{\epsilon_0}{2} \int d^3r (\nabla \cdot (\mathbf{E} \varphi) - \mathbf{E} \cdot (\nabla \varphi)) \\ &\rightarrow \frac{\epsilon_0}{2} \int d^3r |\mathbf{E}|^2 \end{aligned}$$

where the last step is taking the limit as the region of integration becomes all space, and the total derivative term goes away.

So U_E for continuous distributions defined in this way is *positive-definite*. How does this square with our idea that we could just bring one positive and one negative charge together? Well, it takes energy to assemble the charges in the first place. In fact, one can check that the energy of a point charge is

²Again, this is a property of Laplace's equation. For a good reference on this, see Evans *Partial Differential Equations*, section 2.2.2.

divergent since it's a finite amount of charge squished into a single point of space. Part of the energy in this expression (summing up the energy in the E-field) comes from actually assembling the charge distribution, and we'll get some practice with this on the homework.

Lecture 3.

Tuesday, January 14, 2020

Multipole expansion Today we're moving into Chapter 4 of Zangwill, on multipoles. Multipoles are a way of thinking about the field of a charge distribution far away from that distribution. For instance, if there is a net charge in that distribution, far away we have a field that decays as $1/r^2$, like the field of a point charge.

Suppose we have a charge distribution $\rho(\mathbf{r}')$ contained entirely within a sphere of radius R . That is, $\rho(\mathbf{r}') = 0$ for $|\mathbf{r}'| > R$. Then we can look at a point \mathbf{r} outside the sphere with $|\mathbf{r}| > R$. Well, our potential is just given by

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (3.1)$$

and we'd like to expand the denominator when $|\mathbf{r}| \gg |\mathbf{r}'|$. Our expansion looks like

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} - \mathbf{r}' \cdot \nabla \frac{1}{r} + \frac{1}{2} (\mathbf{r}' \cdot \nabla)^2 \frac{1}{r} - \dots \quad (3.2)$$

If we're being careful we should really expand in r'/r as a small parameter. The derivatives are all with respect to unprimed coordinates, and then the \mathbf{r}' determines which direction we're looking. This is much better in index notation. With the Einstein summation convention (i.e. sum over repeated indices), this becomes

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} - r'_i \nabla_i \frac{1}{r} + \frac{1}{2} r'_i r'_j \nabla_i \nabla_j \frac{1}{r} + \dots \quad (3.3)$$

If we plug this expansion back into the expression for the potential, we get

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \underbrace{\left[\int d^3r' \rho(\mathbf{r}') \right]}_{\text{monopole}=Q} \frac{1}{r} - \underbrace{\left[\int d^3r' \rho(\mathbf{r}') r'_i \right]}_{\text{dipole}=\mathbf{p}} \partial_i \frac{1}{r} + \underbrace{\left[\frac{1}{2} \int d^3r' \rho(\mathbf{r}') r'_i r'_j \right]}_{\text{quadrupole}=Q_{ij}} \nabla_i \nabla_j \frac{1}{r} + \dots \right\} \quad (3.4)$$

A note on tensors. Tensors are like collections of numbers that have special properties under coordinate transformations. A rank 1 tensor is just a vector; we need only one number to label which component we are looking at. So if we write a vector $\mathbf{v} = (a, b, c)$ then we can also write its components v_i such that $v_1 = a, v_2 = b, v_3 = c$.

Similarly matrices are rank two tensors—we need two index labels to say which element we are looking at. That is, a 3×3 matrix requires two index labels to pick out an element, a row and a column. We can guess what a rank three tensor looks like, i.e. some sort of cube of numbers. After rank three it's best to just think of these in the abstract. However, not any collection of numbers is a tensor. Each row and column of a matrix must transform like a vector.

Note that many of the complications of working with tensors (e.g. where do you put the indices, up or down?) will drop out when we work in flat Cartesian coordinates.

Can we study the limiting behavior of this expansions? Sure, if we perform some dimensional analysis. We took our charge to be confined to a sphere of radius R ; the monopole moment is independent of R , while the dipole and quadrupole components scale as

$$p_i \sim QR, \quad Q_{ij} \sim QR^2. \quad (3.5)$$

Derivatives with respect to the \mathbf{r} unprimed coordinates act like $1/r$, so our approximation looks like

$$\varphi(r) \sim \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} - \frac{QR}{r^2} + \frac{QR^2}{r^3} + \dots \right]. \quad (3.6)$$

If we work out the derivatives, we get

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q}{r} + \frac{p_i r_i}{r^3} + Q_{ij} \frac{3r_i r_j - r^2 \delta_{ij}}{r^5} + \dots \right\} \quad (3.7)$$

Let us find the field of the dipole term far away.

$$\mathbf{E} = -\nabla \varphi_{\text{dipole}} \quad (3.8)$$

$$= -\frac{1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right). \quad (3.9)$$

We will write this vector in terms of its components. Note that $\partial_i r = \partial_i \sqrt{x^2 + y^2 + z^2} = \frac{r_i}{\sqrt{x^2 + y^2 + z^2}} = r_i/r$.

Then

$$\begin{aligned} E_i &= -\frac{1}{4\pi\epsilon_0} \partial_i \left(\frac{p_j r_j}{r^3} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \left[p_j \frac{\delta_{ij}}{r^3} - \frac{3p_j r_j}{r^4} \partial_i r \right] \\ &= -\frac{1}{4\pi\epsilon_0} \left[p_j \frac{\delta_{ij}}{r^3} - \frac{3p_j r_j r_i}{r^5} \right] \\ &= \frac{1}{4\pi\epsilon_0} \left[\frac{\mathbf{p} \cdot \mathbf{r}}{r^5} r_i - \frac{p_i}{r^3} \right]. \end{aligned}$$

Equating these as vectors gives

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left[\frac{3\mathbf{p} \cdot \hat{\mathbf{r}}}{r^3} \hat{\mathbf{r}} - \frac{\mathbf{p}}{r^3} \right]. \quad (3.10)$$

One can model a physical dipole as two charges, one $+q$ and one $-q$, separated by a distance a and connected by a rigid “rod” of some sort. Then the dipole moment is qa . One can imagine shrinking this dipole down by keeping the product qa fixed while taking $a \rightarrow 0$. In a sense, a dipole is like a derivative of a delta function.

In constant electric fields, there is no net force on a dipole since each of the charges experiences an equal and opposite force. (In general there will be a torque.) It's only when the field is changing that we experience a net force, so we may think of dipoles as sensitive to gradients in the field.

The quadrupole The quadrupole moment was defined as

$$Q_{ij} = \frac{1}{2} \int d^3r' (\rho(\mathbf{r}') r'_i r'_j), \quad (3.11)$$

and from this expression we see that $Q_{ij} = Q_{ji}$, e.g. $Q_{12} = Q_{21}$, and so on. It is symmetric, and hence in 3 dimensions there are 6 independent components. If we choose our axes really well, we can diagonalize the quadrupole tensor and just compute three components.

We can also define the *traceless quadrupole tensor* as

$$\Theta_{ij} = 3Q_{ij} - Q_{kk} \delta_{ij}. \quad (3.12)$$

That is, if we take the trace of this, then we get

$$\Theta_{ii} = 3Q_{ii} - Q_{kk} \delta_{ii} = 3Q_{ii} - Q_{kk}(3) = 0, \quad (3.13)$$

since the trace of the Kronecker delta in 3 dimensions is 3. (More generally, it is the dimension of the space in d dimensions.) This will have *five* independent components, since we have imposed one algebraic constraint on the six components (that the trace vanishes).

Why is this useful? Well, we can rewrite the quadrupole potential as

$$\varphi_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} Q_{ij} \frac{3r_i r_j - \delta_{ij} r^2}{5} = \frac{1}{4\pi\epsilon_0} \Theta_{ij} \frac{r_i r_j}{r^5}. \quad (3.14)$$

If we ever need to go to higher orders, this trace-free condition becomes really useful, since we can take the trace with respect to any pair of indices, e.g. by setting $Q_{iij} = 0$ or $Q_{ijj} = 0$.

We can write the potential as

$$\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + Q_{ij} \frac{3'' \cos \theta_i \cos \theta_j'' - \delta_{ij}}{r^3}. \quad (3.15)$$

If we start writing out the expansion terms in spherical coordinates, we find that these are secretly (maybe not so secretly) Legendre and associated Legendre polynomials.

The reason for this is that our potential far away can be built out of solutions to Laplace's equation $\nabla^2 \varphi = 0$, since there is no charge there. In 204A, we considered separable solutions to Laplace's equation in spherical coordinates, which we wrote as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{A_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi). \quad (3.16)$$

It's perhaps not so surprising that the spherical harmonics arise like this. Moreover, if we think about how many A_{lm} we get for fixed l , there are $2l + 1$ choices for m . That is, the quadrupole moment, which corresponds to $l = 2$, really should have five independent components.³

For $l = 3$, we expect $2l + 1 = 7$ components. From the Cartesian perspective, there are a priori $3 \times 3 \times 3 = 27$ components. However, $Q_{ijk} = Q_{jik}$ and so on. Hence we get some different sets of components.

There's

$$Q_{111}, Q_{222}, Q_{333} \quad (3.17)$$

which are unaffected by the symmetry and all independent. There's also sets where all three indices are distinct, e.g.

$$Q_{123}, \quad (3.18)$$

and there are six of these (like Q_{213}), all related to this one by the symmetry. Then there are ones with two identical indices, like

$$Q_{122}, Q_{133}, Q_{112}, Q_{113}, Q_{233}, Q_{223} \quad (3.19)$$

Each of these is related to two other components by symmetry, e.g. $Q_{122} = Q_{212} = Q_{221}$, for a total of $3 \times 6 = 18$ components with two indices the same. Hence $3 + 6 + 18 = 27$.

All the other components are related to the ones we have listed by symmetry, so there are $3 + 1 + 6 = 10$ components. Moreover, if we impose three constraints on the traces

$$T_{iik} = 0, T_{iji} = 0, T_{ijj} = 0, \quad (3.20)$$

then we get $10 - 3 = 7$, which is exactly the right number of components as given by the spherical harmonic coefficients A_{2m} .

Lecture 4.

Thursday, January 16, 2020

It's very useful to note that we can expand angular dependence from the Legendre polynomials P_l in terms of spherical harmonics, i.e.

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \quad (4.1)$$

That is, we can express an arbitrary angular dependence in terms of spherical harmonics. For instance,

$$f(\theta, \phi; \theta', \phi') = \sum_{l'=0}^{\infty} \sum_{m=-l'}^{l'} A_{l'm}(\theta', \phi') Y_{l',m}(\theta, \phi), \quad (4.2)$$

³There's also a second solution to Laplace's equation scaling as r^l , but those solutions diverge as $r \rightarrow \infty$, so they represent interior rather than exterior solutions.

where we treat the $A_{l'm}$ as coefficients which depend on “fixing” some choice of θ', ϕ' . That is, $A_{l'm}$ is like picking a choice of coordinates, and in particular choosing f to be the Legendre polynomial $f(\theta, \phi; \theta', \phi') = P_l(\theta, \phi; \theta', \phi')$ tells us that

$$P_l(\theta, \phi; \theta', \phi') = \sum_{m=l}^l A_{lm}(\theta', \phi') Y_{lm}(\theta, \phi), \quad (4.3)$$

where

$$A_{lm}(\theta', \phi') = \frac{4\pi}{2l+1} Y_{lm}^*(\theta', \phi'). \quad (4.4)$$

That is, the only terms in the sum (4.2) we need are for $l' = l$.

The spherical harmonics Y_{lm} are polynomials in x, y, z of degree l . All terms in a given spherical harmonic either have even degree or odd degree. Moreover, we can make all terms have degree l by inserting $x^2 + y^2 + z^2$ s, since this is equal to $r^2 = 1$ on the unit sphere. For instance,

$$P_2 = \frac{3}{2}z^2 - \frac{1}{2} = \frac{3}{2}z^2 - \frac{1}{2}(x^2 + y^2 + z^2) = z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2. \quad (4.5)$$

Hence we can always increase the power of a term by 2. Similarly Y_{3m} will contain terms like

$$x^3, y^3, z^3, xyx, xxy, yxx, \dots,$$

and this sort of counting argument is exactly analogous to our multipole expansion and how elements of the tensor were related by symmetry. Then the $x^2 + y^2 + z^2 = 1$ constraint gives us our trace constraints.

We could also make a transformation $x \rightarrow x' \cos \alpha + y' \sin \alpha, y \rightarrow -x' \sin \alpha + y' \cos \alpha$, and we see that under such rotations, the orders of the terms in Y_{lm} don't change. It follows that rotations don't mix Y_{lm} s between different orders l .

Let's now make an argument from completeness using a 2D delta function defined on the surface of a sphere. We'll write it in terms of unit vectors rather than angles to avoid prefactors of the sphere area. Hence we shall assert that the delta function can be expanded in this way,

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm}(\hat{\mathbf{w}}) Y_{lm}(\hat{\mathbf{v}}) \quad (4.6)$$

where $Y_{lm}(\hat{\mathbf{v}})$ indicates the spherical harmonic as a function of the angular coordinates of $\hat{\mathbf{v}}$. The (as yet undetermined) coefficients c_{lm} depend on $\hat{\mathbf{w}}$, just as $A_{l'm}$ depended on θ', ϕ' .

Now we observe that by the orthonormality of the spherical harmonics,

$$\int d\Omega_v \sum_{l=0}^{\infty} \sum_m c_{lm} Y_{lm}(\hat{\mathbf{v}}) Y_{l'm'}^*(\hat{\mathbf{v}}) = c_{lm} \delta_{ll'} \delta_{mm'} = c_{l'm'}(\hat{\mathbf{w}}). \quad (4.7)$$

But this is equivalent to

$$\int d\Omega_v \left(\sum_{l=0}^{\infty} \sum_m c_{lm} Y_{lm}(\hat{\mathbf{v}}) \right) Y_{l'm'}^*(\hat{\mathbf{v}}) = \int d\Omega_v \delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) Y_{l'm'}^*(\hat{\mathbf{v}}) Y_{l'm'}^*(\hat{\mathbf{v}}) = Y_{l'm'}^*(\hat{\mathbf{w}}) \quad (4.8)$$

It follows that

$$c_{lm}(\hat{\mathbf{w}}) = Y_{lm}^*(\hat{\mathbf{w}}), \quad (4.9)$$

so

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{v}}) Y_{lm}^*(\hat{\mathbf{w}}). \quad (4.10)$$

If we choose $\hat{\mathbf{w}} = \hat{\mathbf{z}}$, i.e. we fix $\hat{\mathbf{w}}$ to point along the z axis, then we have *cylindrical* symmetry, not just spherical. But only the $m = 0$ spherical harmonics have cylindrical symmetry, so in fact

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{z}}) = \sum_{l=0}^{\infty} Y_{l0}(\hat{\mathbf{v}}) Y_{l0}^*(\hat{\mathbf{z}}). \quad (4.11)$$

Moreover, note that $P_l(\cos \theta = 0) = 1$,

$$Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l, \quad (4.12)$$

so it follows that

$$\delta(\hat{\mathbf{v}} - \hat{\mathbf{z}}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{v}} \cdot \hat{\mathbf{z}}). \quad (4.13)$$

It follows that since the LHS of this expression depends only on the difference $\hat{\mathbf{v}} - \hat{\mathbf{z}}$ and the RHS depends only on the dot product $\hat{\mathbf{v}} \cdot \hat{\mathbf{z}}$, this difference is independent of rotating our choice of coordinates, which means that in fact

$$\boxed{\delta(\hat{\mathbf{v}} - \hat{\mathbf{w}}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{v}} \cdot \hat{\mathbf{w}}).} \quad (4.14)$$

Equating the boxed equations and taking this order by order in l , we find precisely that

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi), \quad (4.15)$$

which is none other than Eqn. (4.1).

Back to the physics. Recall that Coulomb's law gives us scalar potentials as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (4.16)$$

There's a nice expansion we can use for this inverse separation:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'), \quad r' < r. \quad (4.17)$$

This is really a very sensible way to *define* the Legendre polynomials. The RHS is an expansion in increasing powers of $1/r$, and purely on dimensional grounds, the RHS must have dimensions of $1/\text{distance}$, which means that increasing powers of $1/r$ must be accompanied by powers of r' . That is, we need powers of r'/r . Plugging in our expansion (4.1) we have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \left(\frac{r'}{r}\right)^l Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'). \quad (4.18)$$

Last time, we saw that

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l A_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \implies A_{lm} = \int d^3r' \frac{4\pi}{2l+1} r'^l Y_{lm}^*(\theta', \phi') \rho(\mathbf{r}'). \quad (4.19)$$

The interior solution for Laplace's equation (i.e. the one that doesn't diverge as $r \rightarrow 0$) just replaces $\frac{r'^l}{r^{l+1}} \rightarrow \frac{r^l}{(r')^{l+1}}$, and then the A_{lms} are modified.

Conductors For our purposes, we will use the terms conductors and metals interchangeably. The charge carriers in a conductor reorganize themselves to respond to external applied fields. Specifically, they configure themselves so that the electric field inside is zero (equivalently, this minimizes the energy in the field). Thus $\mathbf{E} = 0$ inside. On the surface, the electric field is perpendicular to the surface. If this weren't the case, the charge would reshuffle to cancel this.⁴ To sum up,

$$\mathbf{E} = 0 \text{ inside}, \quad (4.20)$$

$$E_{\parallel} = 0 \text{ on surface}. \quad (4.21)$$

It follows from Gauss's law and the definition of potential that

$$\rho = 0, \quad \varphi = \text{constant inside}. \quad (4.22)$$

Let's work out the classic example of a neutral conducting sphere in a constant applied field E_0 . See Fig. 1. The total field in the sphere is therefore the sum of the applied field and the induced field from the rearranged charges in the sphere.

That is, if the applied field is

$$\mathbf{E}_0 = E_0 \hat{\mathbf{z}}, \quad (4.23)$$

⁴At least in the absence of an external magnetic field.

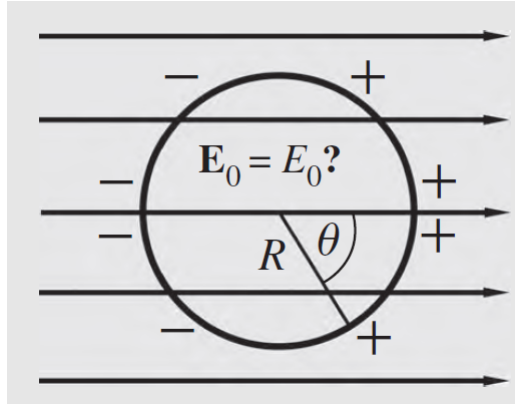


FIGURE 1. Zangwill Fig. 5.2.

then the self-field inside must be $-E_0\hat{z}$. The potential from the induced charge is⁵

$$\varphi_{\text{self}}(r < R, \theta) = E_0 z = E_0 r \cos \theta. \quad (4.24)$$

A priori, the self-field outside is then

$$\varphi_{\text{self}}(r > R, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{A_0}{r} + \frac{A_1}{r^2} P_1(\cos \theta) + \frac{A_2}{r^3} P_2(\cos \theta) + \dots \right] \quad (4.25)$$

We can now use matching conditions. The potential is continuous, so if we take $r = R$ and set φ_{self} inside equal to φ_{self} outside, then

$$E_0 R \cos \theta = \frac{1}{4\pi\epsilon_0} \frac{A_1}{R^2} \cos \theta. \quad (4.26)$$

By the orthogonality of the Legendre polynomials, we can just match order by order in P_l , and if we solve for $A_1/4\pi\epsilon_0$, then

$$\varphi_{\text{self}}(r \geq R, \theta) = E_0 R^3 \frac{\cos \theta}{r^2}. \quad (4.27)$$

This is *exactly* the field of a dipole. The quantity $\mathbf{p} = 4\pi R^3 \epsilon_0 \mathbf{E}$ is the dipole moment of this sphere, where $\alpha = 4\pi R^3$ is called the polarizability. In general, more complicated shapes will have a vanishing monopole moment but some of higher multipole moments.

Suppose now we have some strangely-shaped conductor with a cavity machined out of it, and we place a small (say, positive) charge inside the cavity. Well, what happens to the charge in the conductor? Clearly the inside surface of the cavity must rearrange itself to cancel the field within the conductor. That takes some net negative charge on the inside surface. That is, negative charge accumulates to shield the interior. That charge had to come from somewhere, though. There must be a net positive charge on the exterior surface which rearranges itself in some energetically favorable way. The neat thing is that the solution on the exterior surface doesn't depend at all on the shape of the cavity, or where the charge is placed inside. This is what we mean by shielding. Our conductor hides the details of the interior and reveals only the net charge contained within.

Lecture 5.

Thursday, January 23, 2020

In full generality, we have an expression for the electrostatic potential near some charge distribution $\rho(\mathbf{r})$:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (5.1)$$

⁵This is clearly not constant! Nor should it be. The total potential is constant, and it is the sum of this induced potential and the potential sourcing the applied field.

However, this denominator is complicated; it depends both on \mathbf{r}' , which we integrate over, and also \mathbf{r} , which we don't. A very nice way to decompose this expression is in terms of Legendre polynomials, as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'). \quad (5.2)$$

If we're only interested in the potential on e.g. the z -axis, then this might be a good expansion to use. In that case, integrating over $\hat{\mathbf{r}}'$ is not too bad. But if we're interested in more general \mathbf{r} (say, ranging over a sphere), we still want to separate the angles, and we can instead use the angle addition formula for the spherical harmonics, which says

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi'). \quad (5.3)$$

Capacitance Recall that the capacitance is defined as

$$C = \frac{Q}{V}, \quad (5.4)$$

where V is the potential relative to infinity. In a sense, the self-capacitance is the capacitance of the object with the other plate being a hollow sphere of infinite radius.

We're going to skip the details of the capacitance matrix and many-conductor systems; our main interest will be in systems with precisely two conductors, i.e. capacitors. Typically we put a $+Q$ charge on one and a $-Q$ charge on the other. These charges produce a potential difference V between the conductors, such that the ratio $C = Q/V$ is the capacitance.

The prototypical example is the parallel-plate capacitor, with two plates of area A and charge $+Q$ and $-Q$. So long as the plate separation d is much less than the lateral dimension \sqrt{A} , i.e. $\sqrt{A} \gg d$, we can approximate these as infinite plates. It follows that

$$\mathbf{E} = -\frac{Q}{\epsilon_0 A} \hat{\mathbf{z}}, \quad V = -\int_0^d E dz = \frac{Qd}{\epsilon_0 A}, \quad (5.5)$$

so that

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}. \quad (5.6)$$

Let's note that our plates aren't really infinite, so the fringing effects at the edges mean that the E -field is weaker at the edges. It follows that the potential difference is lower for the same charge, so the capacitance goes up. Apparently we have *underestimated* the capacitance.

Now we can calculate the energy in the capacitor as

$$U = \frac{1}{2} \int d^3r \rho(\mathbf{r}) V(\mathbf{r}) \quad (5.7)$$

Notice that if we have two conductors, this integral simplifies considerably. Suppose we have one conductor with a surface S_1 and a charge $+Q$ and another conductor with a surface S_2 and a charge $-Q$. Conductors are equipotentials, and it follows that since all the charge lies on the surface, the integral becomes

$$U = \frac{1}{2} \left[\int_{S_1} d^2S \sigma_+(\mathbf{r}) V_+ + \int_{S_2} d^2S \sigma_-(\mathbf{r}) V_- \right] = \frac{1}{2} [V_+ Q + V_- (-Q)] = \frac{1}{2} QV. \quad (5.8)$$

We can rewrite this as

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2, \quad (5.9)$$

and this has interesting consequences depending on whether we hold charge constant or voltage constant. That is, if we hold charge constant, then it makes sense that a higher-capacitance (in the case of parallel plates, smaller d) setup is energetically favored. But if we hold voltage constant, then the plates actually want to push apart (lower capacitance).

For our parallel plate capacitor, we can check that this works by integrating the energy in the E -field:

$$U_{\parallel} = \frac{1}{2} \epsilon_0 \int E^2 d^3r = \frac{\epsilon_0}{2} \frac{Q^2}{\epsilon_0^2 A^2} Ad = \frac{1}{2} QV. \quad (5.10)$$

Example 5.11 (The quantum dot). We've just found that the self-energy of an object is given by

$$U = \frac{Q^2}{2C}. \quad (5.12)$$

For objects with large self-capacitances, adding a bit of charge doesn't affect the total energy much. But for very small objects on the nanoscale, this self-capacitance can be very small, so it may take a lot of energy to change the charge by a little bit. That is, U looks like a parabola centered at zero, and if we apply a "gate voltage," we can shift the zero over to e.g. $Q = 1/2$, so that whether we have 0 or 1 (electron) charges in the dot, both are equally favorable from an energy standpoint. Adding temperature into the picture can mess this up, though. Once the energy scale $k_B T$ becomes comparable to the energy cost of adding a charge, thermal effects will destroy this nice zero-or-one picture.

Dielectrics (Zangwill Ch. 6) Conductors are nice because their charge carriers redistribute to cancel applied electric fields. But insulators are more complicated because their cancellation is imperfect due to subtle material properties.

In materials, we can write the total charge ρ as a sum of two terms— a free (applied) charge and a polarization/bound charge due to the material response. That is,

$$\rho(\mathbf{r}) = \rho_f(\mathbf{r}) + \rho_P(\mathbf{r}). \quad (5.13)$$

The free charge ρ_f is charge we control by putting it on objects, while the polarization (bound) charge ρ_P is how the material responds to the applied charge/fields. For instance, in conductors, all the ρ_P (i.e. the induced charge) lies on the surface of the conductor.

If we consider a dielectric with no net charge, then

$$\int_V d^3r \rho_P(\mathbf{r}) + \int_S d^2S \sigma_P(\mathbf{r}) = 0. \quad (5.14)$$

That is, the sum of the polarization volume charge and the polarization surface charge is zero.

We can then motivate the polarization vector $\mathbf{P}(\mathbf{r})$ in the following way. A polarization vector should describe how charges redistribute in a material, such that

$$\mathbf{P}(\mathbf{r}) = 0 \text{ outside the material,} \quad (5.15)$$

$$\mathbf{P}(\mathbf{r}) \cdot \hat{\mathbf{n}} = \sigma_P(\mathbf{r}) \text{ on surface.} \quad (5.16)$$

We can plug this into our equation (5.14) to get

$$\int_V d^3r \rho_P(\mathbf{r}) + \int_V d^3r \nabla \cdot \mathbf{P}(\mathbf{r}) = 0 \quad (5.17)$$

by the divergence theorem. Since these are both volume integrals, it follows that the integrand vanishes,

$$\nabla \cdot \mathbf{P}(\mathbf{r}) = -\rho_P(\mathbf{r}) \text{ inside material.} \quad (5.18)$$

We can now integrate the polarization over the volume of the material. Note that $\nabla_i r_j = \delta_{ij}$, so

$$\int_V d^3r P_j = \int_V d^3r P_i \nabla_i r_j = \int_V d^3r \nabla_i (r_j P_i) - \int_V d^3r r_j \nabla_i P_i \quad (5.19)$$

by the chain rule. Then the first term is a divergence, so we can turn it into a surface integral, i.e.

$$\int_V d^3r P_j = \int_S d^2S r_j \sigma_P(\mathbf{r}) + \int_V d^3r r_j \rho_P(\mathbf{r}) \quad (5.20)$$

using the definitions of the polarization volume charge and surface charge. But now we recognize that these are moment integrals of charge distributions, which means that these are precisely in the form of a dipole moment. That is,

$$\int_V d^3r \mathbf{P} = \mathbf{p}, \quad (5.21)$$

the net dipole moment of the distribution. Hence we can think of polarization as dipole moment per unit volume.

Question: which quantity has more information, $\mathbf{P}(\mathbf{r})$ or $\rho(\mathbf{r})$? A priori, we might think that because of Eqn. (5.18), the charge density has less information than the polarization vector, since we've taken a derivative (which is a lossy operation). But if we take the integral $\int_V d^3r \mathbf{P} = \mathbf{p}$, we only specify one moment of the distribution.

As it turns out, the polarization has more information because it contains phase information. That is, the integral tells us one piece of information we can extract from \mathbf{P} , but there's in principle more we can do with \mathbf{P} .

Lecture 6.

Friday, January 24, 2020

Let's continue thinking about dielectrics. Let's assume the polarization \mathbf{P} is known and see what we can do from there. For instance, what is the potential due only to the polarization charge?

$$\varphi_P(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi\epsilon_0} \int_S d\mathbf{S}' \cdot \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (6.1)$$

That is, we just integrate the influences of the volume charge $\rho_P = -\nabla \cdot \mathbf{P}$ and the surface charge $\mathbf{P} \cdot \hat{\mathbf{n}}$. Now the electric field is given by

$$\mathbf{E}_P(\mathbf{r}) = -\nabla \varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{-\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') + \frac{1}{4\pi\epsilon_0} \int_S d\mathbf{S}' \cdot \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}'). \quad (6.2)$$

Now we'll make a simplifying assumption. If we not only know the polarization \mathbf{P} but also know that $\mathbf{P}(\mathbf{r}')$ is uniform (within the sample), then $\nabla' \cdot \mathbf{P}(\mathbf{r}') = 0$, so this first term goes away and we are left with just the surface term. By the divergence theorem, we can rewrite the flux integral as

$$\mathbf{E}_{P,i}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V d^3r' \nabla'_j \left(\frac{\mathbf{P}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (r_i - r'_i) \right) = \frac{P_j}{4\pi\epsilon_0} \int_V d^3r' \nabla'_j \left(\frac{r_i - r'_i}{|\mathbf{r} - \mathbf{r}'|^3} \right). \quad (6.3)$$

However, notice that the final expression is symmetric in r and r' up to a minus sign, so

$$\nabla'_j \left(\frac{r_i - r'_i}{|\mathbf{r} - \mathbf{r}'|^3} \right) = -\nabla_j \left(\frac{r_i - r'_i}{|\mathbf{r} - \mathbf{r}'|^3} \right). \quad (6.4)$$

Hence

$$E_{P,i}(\mathbf{r}) = -\frac{P_j}{4\pi\epsilon_0} \nabla_j \int d^3r' \frac{r_i - r'_i}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (6.5)$$

or in vector notation,

$$\mathbf{E}_P(\mathbf{r}) = -(\mathbf{P} \cdot \nabla) \boldsymbol{\epsilon}(\mathbf{r}), \quad (6.6)$$

where

$$\boldsymbol{\epsilon}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V d^3r' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (6.7)$$

which is none other than the electric field of a uniform charge distribution

$$\rho_{\text{im}} = \begin{cases} 1 & \text{in } V, \\ 0 & \text{outside.} \end{cases} \quad (6.8)$$

We sometimes call this *Poisson's theorem*.

Example 6.9. Consider a uniformly polarized sphere. We can calculate its field the hard way (with spherical harmonics) or the easy way (with Poisson's theorem).

Let's do it the easy way. Our electric field from the constant density helper system is

$$\boldsymbol{\epsilon}(\mathbf{r}) = \begin{cases} \frac{1}{3\epsilon_0} \mathbf{r}, & r < R \\ \frac{1}{3\epsilon_0} \left(\frac{R}{r} \right)^3 \mathbf{r}, & r > R. \end{cases} \quad (6.10)$$

Let's say that the polarization is constant in the $\hat{\mathbf{z}}$ direction. Then our theorem says that the field is

$$\mathbf{E}_P(\mathbf{r}) = \begin{cases} \frac{1}{3\epsilon_0} \mathbf{P}, & r < R \\ \frac{R^3}{3\epsilon_0} \frac{3(\hat{\mathbf{r}} \cdot \mathbf{P})\hat{\mathbf{r}} - \mathbf{P}}{r^3}, & r > R. \end{cases} \quad (6.11)$$

That is, we get a dipole field outside and a uniform field inside, which is similar to what we saw for conductors, except that the interior has a nonzero field.

Alas, this result isn't as useful as we might hope, because uniform polarizations are hard to come by. In general systems are complicated and do not admit closed-form solutions for their charge distributions. Consider two dielectric spheres in a uniform electric field (say, in the \hat{z} direction) at some offset (say, in the $\hat{z} + \hat{x}$ direction). The dipole moment induced on one by the external field produces a dipole field that hits the other at some weird angle, and the other has to respond to the dipole field of the first.

Generally, we can write the polarization vector as

$$P_i = \epsilon_0 \chi_{ij} E_j + \epsilon_0 \chi_{ijk}^{(2)} E_j E_k + \dots, \quad (6.12)$$

where these χ s are tensors indicating the response of the polarization to different electric fields in different directions. Real materials might have different responses to E -fields in the x versus y directions, for instance. We call this *anisotropy*.

However, having said that, we'll now run away from all these complications as fast as possible. We look at "simple" dielectrics with

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} = (\epsilon - \epsilon_0) \mathbf{E} = \epsilon_0 (\kappa - 1) \mathbf{E}. \quad (6.13)$$

We call χ the *electric susceptibility*, ϵ the *permittivity*, and κ the *dielectric constant*. We need

$$\chi \geq 0, \epsilon \geq \epsilon_0, \kappa \geq 1, \quad (6.14)$$

where the bounds are saturated in vacuum. Note that χ and κ are dimensionless, while ϵ has the same units as ϵ_0 .

Now we can write Gauss's law as

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_f + \rho_P = \rho_f - \nabla \cdot \mathbf{P}. \quad (6.15)$$

If we define the auxiliary field (sometimes called the electric displacement) \mathbf{D} as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (6.16)$$

then we get Gauss's law in materials,

$$\nabla \cdot \mathbf{D} = \rho_f. \quad (6.17)$$

This field is nice because it depends only on the free charge, which we control. In particular, for electrostatic fields, it follows that

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}. \quad (6.18)$$

We can also derive boundary conditions on the displacement by the same Gaussian pillbox arguments as before. At a surface, if there is no applied surface charge, then

$$\Delta D_{\perp} = 0, \quad (6.19)$$

$$\Delta D_{\parallel} = \Delta P_{\parallel}. \quad (6.20)$$

Example 6.21. Consider a point charge $+q$ embedded in a dielectric. Calculating the displacement around this charge is super easy, by the usual Gauss's law arguments. It is just

$$\mathbf{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{\mathbf{r}}, \quad (6.22)$$

with r the radial distance from the point charge. Now if we assume this is a linear dielectric, then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 (\chi + 1) \mathbf{E} = \epsilon_0 \kappa \mathbf{E} = \epsilon \mathbf{E}. \quad (6.23)$$

It follows that

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{q/\kappa}{r^2} \hat{\mathbf{r}} \quad (6.24)$$

As a sanity check, we can recognize that since $\epsilon \geq \epsilon_0$ or equivalently $\kappa \geq 1$, the electric field is reduced from its vacuum value by a factor of κ . The charge is "screened" by the dielectric.

We can also compute the polarization charge, since we know that $\mathbf{P} = \epsilon_0 (\kappa - 1) \mathbf{E}$. Then

$$\rho_P = -\nabla \cdot \mathbf{P} = -\epsilon_0 (\kappa - 1) \nabla \cdot \mathbf{E} = -\epsilon_0 (\kappa - 1) \rho / \epsilon_0 = (1 - \kappa) (\rho_f + \rho_P). \quad (6.25)$$

It follows that

$$\rho_P = \frac{(1 - \kappa) \rho_f}{\kappa} = \left(\frac{1}{\kappa} - 1 \right) \rho_f. \quad (6.26)$$

Example 6.27. What if we have charge $+1$ in a sphere of radius R with one dielectric constant κ_1 inside a bigger dielectric of dielectric constant κ_2 ? Most of the analysis is the same, but now

$$\mathbf{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q/\kappa_1}{r^2} \hat{\mathbf{r}} & r < R \\ \frac{1}{4\pi\epsilon_0} \frac{q/\kappa_2}{r^2} \hat{\mathbf{r}}, & r > R. \end{cases} \quad (6.28)$$

How can the field outside not depend on the first dielectric? The answer is similar to what happened for conductors, except the screening is imperfect. The electric field is discontinuous at $r = R$ because there is an induced surface charge density there due to the polarizations from the dielectrics.

In general the game plan will be to calculate \mathbf{D} using our old Gauss's law tricks and then convert to \mathbf{E} -fields given the linearity of the medium, and then find surface charge densities.

Lecture 7.

Tuesday, January 28, 2020

Last time, we discussed the field from a charge at the boundary between two dielectrics. At the boundary, E_\perp is discontinuous because of the surface charge. But \mathbf{D} seems to be continuous since it depends only on the free charge. What's happening? Why does E have a curl if D is continuous? Really, what's happening is that D also knows about the polarization:

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}. \quad (7.1)$$

So E can be composed of a curl part and a divergence part, and the polarization itself is sourcing a curl at the boundary.

Poisson and Laplace equations Let us leave behind the world of matter for a bit and explore Laplace's equation. To begin our discussion, suppose we have two solutions of Poisson's equation,

$$\nabla^2 \varphi_1 + \nabla^2 \varphi_2 = -\frac{\rho(\mathbf{r})}{\epsilon}. \quad (7.2)$$

When are solutions unique? Define

$$\Phi = \varphi_1 - \varphi_2. \quad (7.3)$$

By linearity, it follows that

$$\nabla^2 \Phi = 0. \quad (7.4)$$

In general, note that for scalar functions f, g , we have

$$\int_S d\mathbf{S} \cdot f \nabla g = \int_V d^3r \nabla \cdot (f \nabla g) = \int_V d^3r (f \nabla^2 g + \nabla f \cdot \nabla g). \quad (7.5)$$

If we take $f = g = \Phi$, then

$$\int_S d\mathbf{S} \cdot \Phi \nabla \Phi = \int_V d^3r \left(\underbrace{\Phi \nabla^2 \Phi}_{=0} + |\nabla \Phi|^2 \right) \geq 0. \quad (7.6)$$

This first term is zero by assumption, while the second term is positive semi-definite and equal to zero only for $\nabla \Phi = 0$.

When does the LHS vanish? Either Φ vanishes on the boundary or $\nabla \Phi \cdot \hat{\mathbf{n}}$ vanishes on the boundary, i.e. the normal derivative is zero. Either way, we get $\nabla \Phi = 0$, which forces $\Phi = \text{constant}$.

If $\Phi = 0$ on S , then these are the Dirichlet boundary conditions, and in fact $\varphi_1 = \varphi_2 = \text{some known values on } S$. Alternately, if $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$, then that's equivalent to specifying the normal derivative $\hat{\mathbf{n}} \cdot \nabla \varphi$ on the boundary. These are *Neumann boundary conditions*, and the solutions can vary by a constant. We could also specify the value on part of the boundary and a normal derivative elsewhere; these are mixed or Cauchy boundary conditions.

This tells us when solutions are unique, though it does not prove existence. In general we need only specify φ or $\hat{\mathbf{n}} \cdot \nabla \varphi$ on the boundary; if we try to fix both, the problem will usually be overdetermined unless we're very lucky.

Let us make a guess when we try to solve Laplace's equation. We will make the following ansatz:

$$\varphi(u, v, w) = A(u)B(v)C(w) \quad (7.7)$$

in terms of some functions A, B, C . If we make this guess and plug it into the differential equation, then our PDE separates into ODEs for each variable, linked by separation constants. While this seems like a really simple guess, Zangwill says that this works in 13 different coordinate systems.⁶

Zangwill also claims to write down the general solution to Laplace's equation, but it's not true. *Do not believe Zangwill Eqn. 7.14*. When the equation separates, we get two linearly independent solutions for A , two for B , and two for C (the equation is second-order). That is, we have

$$A, \tilde{A}; B, \tilde{B}; C, \tilde{C}. \quad (7.8)$$

In fact, each ODE might give us a family of eigenfunctions (think of sines in a rectangular waveguide) indexed by some eigenvalues. So we really have

$$\{A_\alpha, \tilde{A}_\alpha\}; \{B_\beta, \tilde{B}_\beta\}; \{C_\gamma, \tilde{C}_\gamma\}, \quad (7.9)$$

and to construct the most general solution, we need 8 terms:

$$\varphi = \sum_{\alpha, \beta, \gamma} \left[a_{\alpha\beta\gamma}^{(1)} A_\alpha(u) B_\beta(v) C_\gamma(w) + a_{\alpha\beta\gamma}^{(2)} A_\alpha(u) B_\beta(v) \tilde{C}_\gamma(w) + a_{\alpha\beta\gamma}^{(3)} A_\alpha(u) \tilde{B}_\beta(v) C_\gamma(w) + \dots \right]. \quad (7.10)$$

This is precisely equivalent to the construction of entangled states in quantum mechanics. That is, the most general solution isn't just a product of individual solutions but a sum of products. It is a tensor product structure, not a direct product. Contrast Zangwill 7.14, which claims the most general solution is

$$\sum_{\alpha\beta\gamma} (a_\alpha A_\alpha(u) + \tilde{a}_\alpha \tilde{A}_\alpha(u)) (b_\beta B_\beta(v) + \tilde{b}_\beta \tilde{B}_\beta(v)) (c_\gamma C_\gamma(w) + \tilde{c}_\gamma \tilde{C}_\gamma(w)). \quad (7.11)$$

This is just a product of solutions, but there aren't enough parameters for the solution space. There are really only four independent parameters here, since we can pull out an $a_\alpha b_\beta c_\gamma$ and treat this as a single number.

Now, in the theory of Sturm-Liouville problems, ODEs satisfying self-adjointness provide us with a complete orthonormal set of eigenfunctions. Completeness says that

$$\sum_k \psi_k(v) \psi_k^*(v') = \delta(v - v'), \quad (7.12)$$

where k indexes over eigenfunctions. It follows that we have a basis, so that

$$F(v) = \int \delta(v - v') F(v') dv' = \sum_k \underbrace{\int dv' \psi_k^*(v') F(v')}_{F_k} \psi_k(v) = \sum_k F_k \psi_k(v). \quad (7.13)$$

That is, we can decompose $F(v)$ into its components. Note that if $F(v) = \psi'_k(v)$, then we immediately have $F_k = \int dv' \psi_k^*(v') \psi'_{k'}(v') = \delta_{kk'}$.

In Cartesian coordinates, separation of variables is very nice. We have

$$\nabla^2 \varphi = 0, \quad \varphi(x, y, z) = X(x)Y(y)Z(z). \quad (7.14)$$

If we plug in and divide by the original function φ , we get

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z}. \quad (7.15)$$

Since the LHS depends only on x and the RHS depends only on Y and Z , we can write

$$X'' = \alpha^2 X \implies X(x) = \begin{cases} A_0 + B_0 x & \alpha = 0 \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0. \end{cases} \quad (7.16)$$

As we expected, there are two linearly independent solutions (1 and x for $\alpha = 0$ and $e^{\pm x}$ for $\alpha \neq 0$).

For instance, we could set up a box (say, a cube of side length a) where the boundary condition is $V_1(x, y)$ at $z = 0$ and $V = 0$ on the sides. It follows that the X and Y dependence is sines which are periodic in a , namely

$$V(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \frac{\sinh \gamma_{mn}(a-z)}{\sinh \gamma_{mn}a}, \quad (7.17)$$

⁶I haven't checked.

such that

$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2. \quad (7.18)$$

This now satisfies $V(z = a) = 0$, and the coefficients V_{mn} are given by taking the Fourier components of the boundary condition $V_1(x, y)$, i.e.

$$V_{mn}^{(1)} = \frac{4}{a^2} \int_0^a dx \int_0^a dy V_1(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}. \quad (7.19)$$

What if another wall had a nontrivial boundary condition, e.g. $V_2(y, z)$ at $x = a$? Fortunately, Poisson's equation is linear and homogeneous so we can just add solutions. That is, we can add on a solution

$$V_2(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}^{(2)} \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{a} \frac{\sinh \gamma_{mn}(x)}{\sinh \gamma_{mn}a}. \quad (7.20)$$

Example 7.21. Let's consider the solution to Laplace's equation in a Faraday cage. Suppose we have two rows of infinite wires. The wires are separated within the rows by a distance a and the two rows are separated by a distance d (so one is at $z = 0$ and one is at $z = d$). Say the row extends in the x direction and the wires extend in the y direction. Each wire has linear charge density λ .

The charge density from the lower row of wires is

$$\sigma(x) = \lambda \sum_{p=-\infty}^{\infty} \delta(x - pa). \quad (7.22)$$

Because the charge distribution is symmetric (even) about the origin, we can write our charge distribution in terms of cosines,

$$\sigma(x) = A_0 + \sum_{m=1}^{\infty} A_m \cos \frac{2\pi mx}{a}. \quad (7.23)$$

This will turn out to be Neumann boundary conditions, since we're specifying a charge distribution on the boundary, which fixes the perpendicular component of the E -field at the boundary and therefore the normal derivative of V .

The coefficients are given by

$$A_0 = \frac{1}{a} \int_{-a/2}^{a/2} dx \sigma(x) = \frac{\lambda}{a}, \quad A_m = \frac{2}{a} \int_{-a/2}^{a/2} dx \sigma(x) \cos \frac{2\pi mx}{a} = \frac{2\lambda}{a}. \quad (7.24)$$

Since our period is a , we only ever see the delta function at the origin. We Since our charge density has a period a , we only ever see the delta function at the origin. We'll continue this next time.

Lecture 8.

Thursday, January 30, 2020

Let's finish our discussion from last time of the Faraday cage. The Fourier decomposition of the charge distribution is

$$\sigma(x, z = 0) = \frac{\lambda}{a} + \sum_{m=1}^{\infty} \frac{2\lambda}{a} \cos \frac{2\pi mx}{a}, \quad (8.1)$$

and we see that cosines in x are part of the separable solutions to Laplace's equation in Cartesian coordinates. The wires extend in the y direction, so we want a constant solution in y , and it just remains to fit the z -dependence.

Note that σ specifies a normal derivative of φ (it is a discontinuity in E), so our general solution could look like

$$\varphi(x, y, z) = (ax + b|z| + cx|z| + d) + \sum_{\gamma} c_{\gamma} \cos(\gamma x) e^{-\gamma|z|}. \quad (8.2)$$

But by the symmetry in x , the terms linear in x go away and the constant d can be set to zero WLOG. There's no $e^{+\gamma|z|}$ terms since this would diverge as $|z| \rightarrow \infty$. Thus we're left with

$$\varphi(x, y, z) = b|z| + \sum_{\gamma} c_{\gamma} \cos(\gamma x) e^{-\gamma|z|}. \quad (8.3)$$

If we now take the normal derivative at $z \rightarrow 0$, we get

$$\frac{\partial \varphi}{\partial z} \Big|_{z \rightarrow 0^+} = b + \sum_{\gamma} c_{\gamma} (\cos(\gamma x)) (-\gamma) e^{-\gamma(0)} \quad (8.4)$$

and

$$\frac{\partial \varphi}{\partial z} \Big|_{z \rightarrow 0^-} = -b + \sum_{\gamma} c_{\gamma} (\cos(\gamma x)) (\gamma) e^{\gamma(0)}. \quad (8.5)$$

It follows that the electric fields are

$$E_{\perp}(z \rightarrow 0^+) = -\frac{\partial \varphi}{\partial z} \Big|_{z \rightarrow 0^+} = -b + \sum_{\gamma} c_{\gamma} (\cos(\gamma x)) \gamma, \quad (8.6)$$

$$E_{\perp}(z \rightarrow 0^-) = -\frac{\partial \varphi}{\partial z} \Big|_{z \rightarrow 0^-} = +b - \sum_{\gamma} c_{\gamma} (\cos(\gamma x)) \gamma. \quad (8.7)$$

If we now take their difference, we have

$$\frac{\sigma}{\epsilon_0} = E_{\perp}(z \rightarrow 0^+) - E_{\perp}(z \rightarrow 0^-) = -2b + 2 \sum_{\gamma} c_{\gamma} (\cos \gamma x) \gamma. \quad (8.8)$$

By matching coefficients with Eq. (8.1), we find that

$$-2b = \frac{\lambda}{\epsilon_0 a}, \quad \gamma = \frac{2\pi}{a} m, \quad c_{\gamma} = \frac{\lambda}{a \gamma \epsilon_0} = \frac{\lambda}{\epsilon_0 2\pi m} \quad (8.9)$$

Plugging back into our solution, we have

$$\varphi(x, y, z) = -\frac{\lambda}{2a\epsilon_0} |z| + \frac{\lambda}{2\pi\epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \cos \frac{2\pi m}{a} e^{-\frac{2\pi m}{a} |z|}. \quad (8.10)$$

This diverges far away due to the $|z|$ term, but that actually makes sense because far from the cage (at large z , we see the field of an infinite plane of charge, i.e. a constant field and a potential changing linearly with distance.

That's the solution for a single wire plane. If we add two such planes separated by a distance d , then we have

$$\varphi(x, y, z) = -\frac{\lambda}{2a\epsilon} \overbrace{(|z| + |d - z|)}^{=d \text{ "inside," } 0 < z < d} + \sum \dots \quad (8.11)$$

and the exponential terms both decay inside exponentially fast.

We can discuss separable solutions to Laplace's equation in some generality. For Cartesian coordinates, there are three separation constants α, β, γ with $\alpha^2 + \beta^2 + \gamma^2 = 0$. It follows that most of our solutions will have oscillation in at least one direction and decay (or growth) in another. For spherical solutions, there are technically two independent solutions for the θ dependence, the associated Legendre polynomials P_l^m and the other kind Q_l^m . The second kind are divergent at $\theta = \pi$ and sometimes at $\theta = 0$. Cylindrical solutions have many different options; the radial dependence can be growing exponentially, decaying exponentially, growing as a log, or oscillating/decaying as cylindrical Bessel functions. We won't really discuss this in any large degree of detail, but it's good to be aware that such solutions exist.⁷

Example 8.12. Consider a ring of charge Q and radius R . If we consider this in a spherical expansion, we can write

$$\varphi(r, \theta, \phi) = \begin{cases} \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & r > R, \\ \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & r < R. \end{cases} \quad (8.13)$$

But since these solutions must agree at $r = R$, we can actually write

$$\varphi(r, \theta, \phi) = \begin{cases} \sum_{l=0}^{\infty} c_l \frac{R^{l+1}}{r^{l+1}} P_l(\cos \theta) & r > R, \\ \sum_{l=0}^{\infty} c_l \frac{r^l}{R^l} P_l(\cos \theta) & r < R. \end{cases} \quad (8.14)$$

⁷Arfken has a very detailed discussion of Bessel functions and Legendre polynomials

Moreover, we can explicitly calculate

$$\varphi(r, 0, 0) = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{r^2 + R^2}} = \frac{Q}{4\pi\epsilon_0 R} \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^l P_l(0) \quad (8.15)$$

for $r < R$ and the expansion of $\frac{1}{\sqrt{1-2tz+t^2}}$. Now we can just evaluate our general expansion for $r < R$ at $\theta = \pi/2$ using the fact that $P_l(\cos(\theta = \pi/2)) = 1$. Matching the r dependence, we find that

$$c_l = \frac{Q}{4\pi\epsilon_0 R} P_l(0). \quad (8.16)$$

It's a bit like analytic continuation—if we know a function on some line, we can actually extend it consistently elsewhere. Now that we have the coefficients c_l , we have the solution for φ .

Example 8.17. Consider a solid grounded conductor with a “bite” taken out of it. Suppose we can analyze this system in cylindrical coordinates with translational symmetry along the z -axis. Then our solution (in almost full generality: see our discussion from last time) is

$$\varphi(\rho, \phi, z) = (A_0 + B_0 \ln \rho)(C_0 \phi + D_0) + \sum_{\alpha \neq 0} [A_\alpha \rho^\alpha \rho^{-\alpha}] [C_\alpha e^{i\alpha\phi} + D_\alpha e^{-i\alpha\phi}]. \quad (8.18)$$

This will be the general solution so long as one of $A_\alpha, B_\alpha, C_\alpha, D_\alpha$ is zero for each α . We'd like to avoid singularities as $\rho \rightarrow 0$. We can now write

$$\varphi(\rho \rightarrow 0, \phi, z) \propto \rho^{\pi/\beta} \sin \frac{\pi\phi}{\beta} \quad (8.19)$$

as the leading dependence. That is, we've thrown away the $\rho^{-\alpha}$ and $\ln \rho$ dependence, and we also got rid of C_0 in order for our potential to be single-valued.

If we look at $\mathbf{E} = -\nabla \varphi$, we notice that

$$\frac{\partial \varphi}{\partial \rho} \propto \frac{\pi}{\beta} \rho^{\pi/\beta-1} \sin \frac{\pi\phi}{\beta}, \quad (8.20)$$

and we can now see that if $\beta > \pi$ (i.e. we have a spike, not a bite) then the corner will have very large electric fields at small ρ .

Example 8.21. Finally, let us consider two cylindrical rods aligned along the z -axis, separated by a little gap. We hold the rod at positive z at a potential V_R and the rod at negative z at a potential V_L . What is the potential inside the gap? There is no ϕ dependence, so $\alpha = 0$ (we don't get the power-law decay/growth in ρ). Our boundary conditions are given in ϕ, z , so we will take the separation constant $k^2 < 0$ to expand in z , and define $k = i\kappa$ as is conventional.

This means we're expanding in Bessel functions I and K , and since K diverges as $\rho \rightarrow 0$, we will be using the functions I . Thus

$$\varphi(\rho, \phi, z) = \frac{1}{2\pi} \int_0^\infty I_0(\kappa\rho) (E(\kappa)e^{i\kappa z} + F(\kappa)e^{-i\kappa z}) d\kappa + \text{constant} = \frac{1}{2\pi} \int_{-\infty}^\infty A(\kappa) I_0(|\kappa|\rho) e^{i\kappa z} d\kappa + \text{constant}. \quad (8.22)$$

That is, we extended the limits of integration to $-\infty, \infty$ and now we recognize this integral as a Fourier transform. We can recover $A(\kappa) I_0(|\kappa|\rho)$ by taking an inverse Fourier transform.