# PHYSICS 200B: ELECTROMAGNETISM I

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These notes were taken for Physics 200B, *Electromagnetism I*, as taught by Rena Zieve at the University of California, Davis in winter quarter 2020. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to `itlim@ucdavis.edu`.

Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at `https://web.ma.utexas.edu/users/a.debray/`.

## Contents

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Lecture 1.

# Tuesday, January 7, 2020
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*Note.* This course will have discussion sections! I will not be writing notes for those since they're like homework. Exams will be closed-book but with a formula sheet—two pages, front and back, typed okay. The final exam can be moved with signed consent of all students.

The textbook for this course is Zangwill *Modern Electrodynamics*. The plan is to cover chapters 1-13, mostly electrostatics and magnetostatics (time-independent stuff). Homeworks due Tuesdays apart from the very last one, which is due Thursday.

Most of Chapter 1 of Zangwill is mathematical preliminaries. Chapter 2 is largely a review of Maxwell's equations. There is one important auxiliary to the Maxwell equations, though, and it's the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{1.1}$$

This equation just says there's stuff ($\rho$) and the stuff can flow in a way defined by the vector field $\mathbf{j}$. The divergence $\nabla \cdot \mathbf{j}$ tells us how much charge is flowing out of a point, on average.

To review, the first two Maxwell's equations are the following:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3}$$

The next one we might guess is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{1.4}$$

that current flow sources magnetic fields. This equation can't be quite right, because if we take the divergence, the divergence of a curl always vanishes, so

$$0 = \nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{j}, \tag{1.5}$$

which by the continuity equation says that $\frac{\partial \rho}{\partial t} = 0$ everywhere. That seems nonphysical, so let us amend the equation by adding a $\mathbf{j}_D$ term, called the *displacement current*. That is,

$$0 = \mathbf{j_D} + \mu_0\left(-\frac{\partial \rho}{\partial t}\right) = -\mu_0\epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_D. \tag{1.6}$$

This *defines* the displacement current as

$$\mu_0 \epsilon_0 \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_D, \tag{1.7}$$

and since we recall that $\mu_0 \epsilon_0 = \frac{1}{c^2}$, we can rewrite what we recognize as Ampère's law as

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{1.8}$$

The last Maxwell equation is Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.9}$$

These are Maxwell's equations in vacuum, and they can be modified to describe fields in materials with some little tweaks.

There's also a "glitch" in Zangwill's exposition. In the text, Zangwill mentions that Maxwell wrote down 12 equations prior to the development of vector notation (relating each of the components), but in fact he wrote down *eight*. Why eight? Each of the curl equations is a vector equation (three components), but each of the divergence equations is only one (a scalar equation). Hence $2 \times 3 + 2 \times 1 = 8$.

Altogether, we have the differential forms of Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.10}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.11}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.12}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{1.13}$$

We should also be familiar with the integral forms of the Maxwell equations,

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} Q_{\text{enc}} \tag{1.14}$$

$$\int \mathbf{B} \cdot d\mathbf{S} = 0 \tag{1.15}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} \tag{1.16}$$

$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 I_{\text{enc}} + \frac{1}{c^2} \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{S}. \tag{1.17}$$

If we have matter, then the laws need to be adapted because generically, charges and magnetic dipoles reorient themselves based on the applied fields. Materials have electric and magnetic polarizabilities. It's useful to define the total charge as a sum

$$\rho = \rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}, \tag{1.18}$$

where $\mathbf{P}$ is the *polarization vector*. That is, we can put some charge $\rho_f$ on a material, but in general when there's a background applied field, the charges will redistribute themselves. We know this as the fact that charges in a dielectric generically rearrange themselves to oppose the applied field. Recall that $\mathbf{P}$ points from negative to positive charge. Thus a positive divergence of $\mathbf{P}$ acts like a local negative charge.

We can do the same for magnetism, though ti's a bit messier.

$$\mathbf{j} = \mathbf{j}_f + \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{M}, \tag{1.19}$$

where $\mathbf{M}$ is the *magnetization vector*, which we think of as little magnetic dipoles or eddy currents orienting themselves to minimize their energy. This one depends on the current due to free charge $\mathbf{j_f} = \frac{\partial \rho_f}{\partial t}$, the rate of change of the bound charge from polarization, and the curl of the magnetization.

This leads us to define auxiliary fields:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E} \tag{1.20}$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} = \frac{1}{\mu} \mathbf{B}. \tag{1.21}$$

For the electric field, the sign conventions (whether to add or subtract $\mathbf{p}$ make sense. It's very natural for the polarization pointing from $-$ to $+$ to add to an effective field $\mathbf{D}$. But for the magnetic field, it could go either way. Magnets near superconductors induce a magnetization that perfectly cancels the applied field, whereas in ferromagnets like iron, the magnetization is in the same direction as the applied field. We just have to pick a convention and stick to it.

The Maxwell equations in materials take on kind of a nice form:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1.22}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.23}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.24}$$

$$\nabla \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.25}$$

**Helmholtz theorem** The Helmholtz theorem says that if we specify the divergence and curl of a function everywhere, then the function is uniquely defined up to adding functions which have zero divergence and zero curl. Such functions are called *harmonic functions*, which we'll see more later, and whether or not we should add them corresponds to fitting boundary conditions.

We can actually write the electric field explicitly in terms of the divergence and curl:

$$\mathbf{E}(\mathbf{r}) = - \underbrace{\nabla \int d^3 r' \frac{\nabla' \cdot \mathbf{E}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}}_{\text{curl}=0} + \underbrace{\nabla \times \int d^3 r' \frac{\nabla' \times \mathbf{E}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|}}_{\text{div}=0}. \tag{1.26}$$

The second term is actually zero if there is no time dependence, from Faraday's law. If we apply Gauss's law to rewrite $\nabla' \cdot \mathbf{E}$,, we recognize what remains as

$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r}) = -\nabla \int d^3 r' \frac{\rho(\mathbf{r}')}{4\pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi \epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{1.27}$$

We've recovered Coulomb's law, just written in a more vectorial language.

Let's consider the force on a charge distribution $\rho^*$ due to the field from a distribution $\rho$. That is,

$$\mathbf{F}(\rho, \rho^*) = \int d^3 r \rho^*(\mathbf{r}) \mathbf{E}(\mathbf{r}) \tag{1.28}$$

$$= \frac{1}{4\pi \epsilon_0} \int d^3 r \int d^3 r' \frac{\rho^*(\mathbf{r})\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{1.29}$$

But we notice that we can exchange $\mathbf{r}$ and $\mathbf{r}'$ at the cost of picking up a sign flip:

$$\mathbf{F}(\rho, \rho^*) = -\mathbf{F}(\rho^*, \rho). \tag{1.30}$$

If we set $\rho = \rho^*$, then $\mathbf{F}(\rho, \rho) = 0$, which tells us that a charge distribution cannot exert a net force on itself.

Finally, let's introduce potentials. The electric field in electrostatics can be written as the gradient of the scalar potential,

$$\mathbf{E} = -\nabla \phi, \tag{1.31}$$

so that

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi. \tag{1.32}$$

The equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{1.33}$$

is hopefully familiar to us as Poisson's equation, and the homogeneous case ($\rho = 0$) is

$$\nabla^2 \phi = 0, \tag{1.34}$$

which is Laplace's equation.

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Lecture 2.

# Thursday, January 9, 2020

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Today we'll start our discussion with Gauss's law, moving rapidly into chapter 3 of Zangwill. Gauss's law lets us calculate electric fields rapidly for situations with high amounts of symmetry. Basically, we can solve systems with spherical symmetry, cylindrical symmetry, and translational plane symmetry. It doesn't get us too far but it's a lot better than Coulomb's law volume integrals.

As we know, Gauss's law (differential form) states

$$\mathbf{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

or in integral form

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V \mathbf{\nabla} \cdot \mathbf{E} d^3r = \int_V \frac{\rho}{\epsilon_0} d^3r = \frac{Q_{\text{enc}}}{\epsilon_0}. \tag{2.2}$$

The simplest example is for the point charge. For a single positive charge $q > 0$, we can reason that the field must point radially outwards and it is constant at surfaces of constant $R$, so that

$$\oint_S \mathbf{E} \cdot d\mathbf{Q} = |E|4\pi R^2 = \frac{q}{\epsilon_0}, \tag{2.3}$$

which yields

$$\mathbf{E} = \frac{1}{4\pi R^2 \epsilon_0} \hat{\mathbf{r}}. \tag{2.4}$$

Gauss's law also gives us a nice result (sometimes known as Newton's shell theorem), which says that the electric field due to a spherically symmetric shell of charge *inside* that shell ($r < R$) is zero.

We can also do Gauss's law for an infinite plane by drawing a Gaussian pillbox, say, extending a height $h$ above and below the plane. Rotational symmetry lets us reason that the field can only point in the normal direction to the plane, while reflection symmetry says its magnitude is the same equal distances above and below the plane. It follows that

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \sigma A, \tag{2.5}$$

where $A$ is the area of the face of the Gaussian pillbox. Since the field is normal to the plane and the faces of the pillbox are oriented outwards, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2E(h)A, \tag{2.6}$$

so that

$$\mathbf{E}(h) = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{z}} & z < 0. \end{cases} \tag{2.7}$$

The reason there is no $h$ dependence is because the infinite plane is scale-invariant. If we rescale all the coordinates, the infinite plane still looks like an infinite plane. This result isn't necessarily useful to us because real life is filled with infinite planes of charge, or even because the problem is exactly solvable; rather, it's because any reasonably smooth (in the mathematical sense) surface looks locally flat, which means that near the surface, we have essentially the field from an infinite plane.

By the same Gaussian pillbox arguments, we find that $E_\parallel$ is continuous at surfaces of charge, while $E_\perp$ is discontinuous by $\frac{\sigma}{\epsilon_0}$. How should we calculate the force on a little surface of charge, given that the electric field is discontinuous above and below the field? Well, we can just take the average. This seems physically reasonable, but we can justify it. Remember the discontinuity comes from the charge at the surface itself, and as we showed last time, charges cannot exert forces on themselves. So if we average above and below the surface, we will basically average away the contribution of the charged surface itself and get the right answer.

We can also state Earnshaw's theorem– in a closed region with no charge, any extrema of the potential must be on the boundary. For suppose there was a local maximum of the potential in the interior. The gradient of the potential vanishes at that point, and a little bit away we can draw a Gaussian surface it is

pointing towards that point everywhere. It follows that the **E**-field points away everywhere, so our integral $\oint_S \mathbf{E} \cdot d\mathbf{A} > 0$, which violates our assumption that there was no charge in the region.[1] This tells us that we cannot make an electrostatic cage; no charge distribution can hold itself in a static configuration under Coulomb interactions alone. Earnshaw's theorem has told us that any charge dropped in a region with no other charge will move to the edges of that region, since it cannot sit at an extremum.

**Potential and potential energy** For our purposes, we will follow Zangwill and denote potential energy by $V$ and electrostatic potential by $\varphi$. We say the potential energy changes as

$$\delta V = -\mathbf{F} \cdot d\mathbf{s} \tag{2.8}$$

for small displacements, i.e.

$$\mathbf{F} = -\boldsymbol{\nabla} V. \tag{2.9}$$

In addition, when forces are due only to electric fields, then

$$\mathbf{F} = q\mathbf{E} = -q\boldsymbol{\nabla}\varphi \implies V = q\varphi. \tag{2.10}$$

Hence the electrostatic potential is the energy per charge.

Let's now prove Green's reciprocity relation for the potential energy. That is, suppose there are two charge distributions $\rho_1, \rho_2$. The energy of $\rho_2$ in the potential $\varphi_1$ created by $\rho_1$ is

$$V = \int d^3r \, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}), \tag{2.11}$$

such that

$$\delta V = \int d^3r \, [\rho_2(\mathbf{r} - \delta\mathbf{s}) - \rho_2(\mathbf{r})]\varphi_1(\mathbf{r}). \tag{2.12}$$

The minus sign comes from that active/passive transformation jazz. If the distribution is moved by $\delta\mathbf{s}$, then looking at the "same" point in space $\mathbf{r}$ for the new distribution is equivalent to looking at the original distribution at a point $\mathbf{r} - \delta\mathbf{s}$.

We can now Taylor expand as $\delta\mathbf{s}$ gets small, such that

$$\delta V \approx -\delta\mathbf{s} \cdot \int d^3r \, (\boldsymbol{\nabla}\rho_2(\mathbf{r}))\varphi_1(\mathbf{r}) = -\delta\mathbf{s} \cdot \int d^3r (\boldsymbol{\nabla}(\rho_2\varphi_1) - \rho_2\boldsymbol{\nabla}\varphi_1), \tag{2.13}$$

after a product rule manipulation (basically an integration by parts). Now this first term is a total derivative, so it vanishes as we take our integration region $d^3r$ to be all space. What's left is

$$\delta V = -\delta\mathbf{s} \cdot \int d^3r \, \rho_2(\mathbf{r})\mathbf{E}_1(\mathbf{r}). \tag{2.14}$$

Let us rewrite this energy in a different way:

$$V = \int d^3r \, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r' \, \rho_2(\mathbf{r}) \frac{\rho_1(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.15}$$

But notice that this integral is manifestly symmetric in $\mathbf{r}$ and $\mathbf{r}'$ (i.e. $|\mathbf{r} - \mathbf{r}'| = |\mathbf{r}' - \mathbf{r}|$). This certainly converges, so we can switch the order of integration and do the $d^3r$ integral first to find

$$V = \int d^3r \, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}) = \int d^3r' \, \rho_1(\mathbf{r}')\varphi_2(\mathbf{r}'). \tag{2.16}$$

This relation means that the energy in an electrostatic charge distribution doesn't depend on how we build it, only on the final geometry.

Consider the following example.

**Example 2.17.** Suppose we have a spherical region of radius $R$ centered on the origin with no charge inside. As it turns out, the potential at the center is the average potential on the surface:[2]

$$\varphi(0) = \frac{1}{4\pi R^2} \int dS \varphi(\mathbf{r}). \tag{2.18}$$

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[1]One can also argue this directly from the form of Laplace's equation.

[2]Again, this is a property of Laplace's equation. For a good reference on this, see Evans *Partial Differential Equations*, section 2.2.2.

Can we use Green's reciprocity relation to find this result? Take

$$\varphi_1 = \varphi, \quad \rho_1 = \rho; \quad \rho_2(\mathbf{r}) = \frac{q}{4\pi R^2}\delta(r - R). \tag{2.19}$$

Then

$$\varphi_2 = \begin{cases} \frac{q}{4\pi\epsilon_0 r} & r \geq R, \\ \frac{q}{4\pi\epsilon_0 R} & r \leq R. \end{cases} \tag{2.20}$$

It follows that

$$\int d^3r \frac{q}{4\pi R^2}\delta(r - R)\varphi(\mathbf{r}) = \int d^3r \rho(\mathbf{r})\varphi_2(\mathbf{r}) = \int_R^\infty r^2 dr \int d\Omega \rho(\mathbf{r})\frac{q}{4\pi\epsilon_0 r}. \tag{2.21}$$

But this last expression up the the factor of $q$ is exactly the integral we would use to compute the potential at the origin. That is,

$$\int_R^\infty r^2 dr \int d\Omega \rho(\mathbf{r})\frac{q}{4\pi\epsilon_0 r} = q\varphi(\mathbf{r} = 0). \tag{2.22}$$

Meanwhile, the LHS says that we just integrate

$$\int d^3r \frac{q}{4\pi R^2}\delta(r - R)\varphi(\mathbf{r}) = \frac{q}{4\pi R^2}\int dS\, \varphi(\mathbf{r}). \tag{2.23}$$

Cancelling the factors of $q$, we have exactly the desired result. The potential at the center of the sphere is equal to the average potential on its surface.

We can write the potential energy of a charge distribution as

$$U_E = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^N \sum_{i>j}^N \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{2.24}$$

or for continuous distributions,

$$U_E = \frac{1}{8\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2}\int d^3r \rho(\mathbf{r})\varphi(\mathbf{r}) \tag{2.25}$$

Note that for all-positive or all-negative distributions of charges, these formulae make it clear that the energy of assembling the distribution is positive-definite. The answer is somewhat less clear when we have a mix of charges. Let's manipulate this result to see what happens.

$$\begin{aligned} U_E &= \frac{1}{2}\int d^3r \rho(\mathbf{r})\varphi(\mathbf{r}) \\ &= \frac{1}{2}\int d^3r \epsilon_0 \nabla \cdot \mathbf{E}\varphi(\mathbf{r}) \\ &= \frac{\epsilon_0}{2}\int d^3r (\nabla \cdot (\mathbf{E}\varphi) - \mathbf{E} \cdot (\nabla\phi)) \\ &\to \frac{\epsilon_0}{2}\int d^3r |E|^2 \end{aligned}$$

where the last step is taking the limit as the region of integration becomes all space, and the total derivative term goes away.

So $U_E$ for continuous distributions defined in this way is *positive-definite*. How does this square with our idea that we could just bring one positive and one negative charge together? Well, it takes energy to assemble the charges in the first place. In fact, one can check that the energy of a point charge is divergent since it's a finite amount of charge squished into a single point of space. Part of the energy in this expression (summing up the energy in the E-field) comes from actually assembling the charge distribution, and we'll get some practice with this on the homework.