

# PHYSICS 204B: METHODS OF MATHEMATICAL PHYSICS II

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Lecture 1.

## Monday, January 6, 2020

“What did you do over break?” “Don’t ask. No rest for the wicked.”  
—Mark Samuel Abbott and Nemanja Kaloper

The only outstanding logistical details here are that office hours will be posted later, and the TA is now Morgane König rather than Cameron Langer. All else is basically the same as last quarter.

Let's talk about Green's functions in more than one dimension. Our discussion will be somewhat sketchy, but we'll get a rough idea of the topic. A Green's function is the inverse of a differential operator, and it lives under integrals. In one dimension, we wrote that to solve

$$\mathcal{L}\phi = J, \quad (1.1)$$

we could construct  $G$  such that

$$\mathcal{L}G = \delta, \quad (1.2)$$

a function which gives a delta function upon being hit by a differential operator. We would like to solve the problem of finding the linear response of a field  $\phi$  at a point  $\mathbf{r}_1$  due to a source  $J(\mathbf{r}_2)$  at a point  $\mathbf{r}_2$ .

Recall that the adjoint of an operator is given by

$$\langle \psi | \mathcal{L}\phi \rangle = \langle \mathcal{L}^\dagger \psi | \phi \rangle, \quad (1.3)$$

and an operator is self-adjoint if

$$\mathcal{L} = \mathcal{L}^\dagger. \quad (1.4)$$

That is, for an operator given by

$$\mathcal{L}\phi = \nabla \cdot (p \nabla \phi) + q\phi, \quad (1.5)$$

we must check that

$$\int \psi^* (\nabla \cdot (p \nabla \phi)) + \int q \psi^* \phi = \int \nabla \cdot (p \nabla \psi^*) \phi + \int q \psi^* \phi. \quad (1.6)$$

The  $q$  terms cancel, so we find that

$$\int [\psi^* \nabla \cdot (p \nabla \phi) - \nabla \cdot (p \nabla \psi^*) \phi] = 0, \quad (1.7)$$

and if we integrate by parts, then

$$\int_V dV \nabla \cdot (\psi^* p \nabla \phi - (\nabla \psi^*) p \phi) = 0. \quad (1.8)$$

By the divergence theorem,

$$\int_S d\mathbf{S} \cdot p [\psi^* \nabla \phi - (\nabla \psi^*) \phi] = 0. \quad (1.9)$$

Dirichlet or Neumann boundary conditions will guarantee self-adjointness. That is, if the functions vanish on  $S$  or their normal derivatives vanish on  $S$ , then  $\mathcal{L}$  is self-adjoint. There are also mixed conditions we could impose, but Dirichlet and Neumann conditions are sufficient to make our operator Hermitian.

For a Hermitian operator, the corresponding Green's function obeys<sup>1</sup>

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \quad (1.10)$$

For recall that

$$\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) | G(\mathbf{r}, \mathbf{r}_2) \rangle = \langle G(\mathbf{r}, \mathbf{r}_1) | \mathcal{L}G(\mathbf{r}, \mathbf{r}_2) \rangle \quad (1.11)$$

by self-adjointness. By the definition of the Green's function and the inner product, we can replace  $\mathcal{L}G$  by a delta function and get

$$\int \delta(\mathbf{r} - \mathbf{r}_1) G(\mathbf{r}, \mathbf{r}_2) = G(\mathbf{r}_1, \mathbf{r}_2) \quad (1.12)$$

on the LHS and

$$\int G^*(\mathbf{r}, \mathbf{r}_1) \delta(\mathbf{r}, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \quad (1.13)$$

Hence

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \quad (1.14)$$

Now let's construct the eigenfunction expansion of the Green's function. Consider

$$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (1.15)$$

<sup>1</sup>This property is also responsible for Green's reciprocity theorem in electromagnetism, i.e. the statement that the potential energy of a charge distribution  $\rho_2$  in a field produced by another distribution  $\rho_2$  is equal to the energy of  $\rho_1$  in the field produced by  $\rho_2$ . If you like, the theorem is a special case/application since the Laplacian operator is Hermitian.

We're keeping the  $\mathbf{r}_1, \mathbf{r}_2$  dependence in  $G$  because there might be operators that are not translationally invariant. In other words, we can't assume that quantities depend only on  $|\mathbf{r}_1 - \mathbf{r}_2|$ . That is,

$$\mathcal{L}|\mathbf{r} \neq \mathcal{L}|\mathbf{r}+\mathbf{a}. \quad (1.16)$$

Suppose we construct the eigenfunctions  $\phi_\lambda$  of the operator  $\mathcal{L}$ , such that

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda. \quad (1.17)$$

WLOG we may take them to be orthonormal,

$$\langle \phi_\lambda | \phi_\mu \rangle = \delta_{\lambda\mu}. \quad (1.18)$$

For now, we shall assert that they are a complete set– in general we will have to prove this. The expansion for the delta function is just the completeness relation:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \int_\lambda \phi_\lambda^*(\mathbf{r}_2) \phi_\lambda(\mathbf{r}_1), \quad (1.19)$$

since

$$f(\mathbf{r}) = \int_\lambda f_\lambda \phi_\lambda(\mathbf{r}) = \int_{\mathbf{r}_1} f(\mathbf{r}_1) \delta(\mathbf{r} - \mathbf{r}_1) = \int_\lambda \underbrace{\int_{\mathbf{r}_1} f(\mathbf{r}_1) \phi_\lambda^*(\mathbf{r}_1)}_{f_\lambda} \phi_\lambda(\mathbf{r}). \quad (1.20)$$

Now we can put our delta function decomposition back in: suppose that

$$G(\mathbf{r}, \mathbf{r}_1) = \int_\lambda C_\lambda(\mathbf{r}_1) \phi_\lambda(\mathbf{r}), \quad (1.21)$$

so that

$$\begin{aligned} \mathcal{L}G &= \mathcal{L} \int_\lambda C_\lambda(\mathbf{r}_1) \phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1) \mathcal{L}\phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1) \lambda \phi_\lambda(\mathbf{r}), \end{aligned}$$

and this last expression must be equal to the expansion of the delta function:

$$0 = \int_\lambda [C_\lambda(\mathbf{r}_1) \lambda - \phi_\lambda^*(\mathbf{r}_1)] \phi_\lambda(\mathbf{r}). \quad (1.22)$$

Hence

$$c_\lambda(\mathbf{r}_1) = \frac{\phi_\lambda^*(\mathbf{r}_1)}{\lambda}, \quad (1.23)$$

so

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2) \phi_\lambda(\mathbf{r}_1)}{\lambda}. \quad (1.24)$$

This makes the hermiticity of  $G$  totally clear:

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \quad (1.25)$$

The only problem might be if we have a zero eigenvalue, in which case we have to be careful. Some people define a generalized Green's function by

$$(\mathcal{L} - Z)G = \delta, \quad (1.26)$$

so that our expression is just modified to

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2) \phi_\lambda(\mathbf{r}_1)}{\lambda - Z}, \quad (1.27)$$

and we can study the  $z \rightarrow 0$  limit.

Let's consider some examples. The Laplace equation is

$$\nabla^2 \phi = J, \quad (1.28)$$

and it is already in self-adjoint form,

$$\nabla \cdot (\nabla G) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (1.29)$$

Suppose we had a solution

$$\int \nabla^2 \phi = J \quad (1.30)$$

such that  $\int_V J = Q$  and a homogenous solution

$$\nabla^2 \chi = 0. \quad (1.31)$$

But now

$$\int_V dV \nabla^2 (\phi + \chi) = \int_V J = Q. \quad (1.32)$$

We can rewrite the first expression as a surface integral,  $\int d\mathbf{S} \cdot \nabla (\phi + \chi)$ . Hence we find that since the integral of the  $\nabla \phi$  term is already  $Q$ , it must be that

$$\int d\mathbf{S} \cdot \nabla \chi = 0, \quad (1.33)$$

and therefore by the uniqueness theorems,  $\chi$  is at most a constant. This is an example of a gauge symmetry, actually, but we won't go too much into that. So given appropriate boundary conditions, solutions of the Laplace equation are unique up to a constant.

Now let us note the Laplace equation *is* translationally invariant, so we can write

$$\nabla \cdot (\nabla G(\mathbf{r})) = \delta(\mathbf{r}). \quad (1.34)$$

In fact, it is also manifestly spherically symmetric in this form. We've just chosen coordinates to put our charge at the origin. Let us integrate over a spherical region  $R$ . Then

$$1 = \int_R \delta(\mathbf{r}) = \int_R \nabla \cdot (\nabla G(\mathbf{r})) = 4\pi R^2 \frac{dG}{dr}. \quad (1.35)$$

We conclude that

$$G = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (1.36)$$

which is nothing more than the Coulomb potential for a unit charge.

If we instead enclosed the charge in a Faraday cage (setting the potential to zero somewhere), we would add a homogeneous solution to the Laplace equation to our Green's function in order to fit the new boundary conditions. Note that in 2 dimensions, we would just have  $1 = 2\pi R \frac{dG}{dr}$ , which gives our Green's function as a log of  $|r|$  instead.

Lecture 2.

**Wednesday, January 8, 2020**

*"If I had taken a potato instead of a box, I would be hard-pressed to do this. It would be much much harder. Except in two dimensions. Two-dimensional potatoes are very cool because Laplacians in 2 dimensions are special."*

—Nemanja Kaloper

Let's play a bit more with Green's functions. Consider the Laplace equation. We can fit a solution using

$$\nabla^2 (G + F) = \delta. \quad (2.1)$$

That is, we can construct a solution with a source using the Green's function

$$\nabla^2 G = \delta \quad (2.2)$$

and add on homogeneous solutions  $F$  such that

$$\nabla^2 F = 0 \quad (2.3)$$

to fit boundary conditions.

**Example 2.4.** Let us solve the following electrostatics problem. We take a charge and place it inside a grounded conductor shaped like a cube. Can we solve the Laplace equation in this setup?

The charge has spherical symmetry, but the cube does not. That means that the field lines near the sides of the cube will deform so the field becomes normal to the faces of the cube in the static, equilibrium configuration.

Consider a single charge near a grounded plane. We can solve this by the method of images. Suppose the charge sits on the  $z$  axis, while the grounded plane is the  $xy$  plane. In particular, it sits at a point  $(0, 0, a)$  from the origin. We now consider a point a distance  $\rho$  from the  $z$  axis and at a height  $z$  along the  $z$  axis.

The method of images says that we can solve for the potential by considering any (fictitious) charge distribution which satisfies the same boundary conditions for the potential. That is, instead of just measuring the potential from the single charge, we can write

$$\frac{1}{\sqrt{\rho^2 + (z - a)^2}} \quad (2.5)$$

as the potential from the charge alone and suppose we add another image charge  $Q'$  on the other side of the grounded plane at a location  $(0, 0, -a')$ . Then the potential from this configuration of the charge and image charge is

$$\frac{1}{\sqrt{\rho^2 + (z - a)^2}} + \frac{Q'}{\sqrt{\rho^2 + (z + a')^2}}. \quad (2.6)$$

Now we consider the plane  $z = 0$  and see that

$$V(z = 0) = \frac{1}{\sqrt{\rho^2 + a^2}} + \frac{Q'}{\sqrt{\rho^2 + a'^2}}. \quad (2.7)$$

But this still seems to depend on  $\rho$ . Setting  $\rho = 0$  gives

$$\frac{1}{a} + \frac{Q'}{a'} = 0, \quad (2.8)$$

and taking a derivative gives

$$\frac{1}{a^3} + \frac{Q'}{a'^3} = 0, \quad (2.9)$$

so solving gives

$$a = a', \quad Q' = -1. \quad (2.10)$$

The potential on the real side of the grounded plane is just the dipole potential in the region we care about. The other charge is equal and opposite, and equally far away behind the plane.

What about two conducting planes? We can construct a dipole to set the potential on one plane, but then the other won't be grounded. We can add an image dipole outside the other plane (see image). Then we the second plane is okay but the first plane is now not grounded. So we go on adding dipoles in this way ad infinitum, and the overall potential is

$$\sum_{n=-\infty}^{\infty} \frac{1}{[\rho^2 + (z - a + n(2D))^2]^{-1/2}} - \frac{1}{[\rho^2 + (z + a + n(2D))^2]^{-1/2}}. \quad (2.11)$$

Expanded, this is

$$\sum_n \frac{1}{|\mathbf{r} - 2Dn\mathbf{k}|} - \frac{1}{|\mathbf{r} + 2Dn\mathbf{k}|}. \quad (2.12)$$

The sum looks like it might diverge, since each term goes as  $1/n$ , but in fact since we're dealing with dipoles, the  $1/n$  dependence will cancel and away from the corners, this will converge. These image charges help us to satisfy the boundary conditions on the two planes.

The generalization to three dimensions is to extend the periodicity into three dimensions. We get an infinite lattice of image dipoles which is at once simple and complicated.

Let's write a nice fact about multipoles:

$$\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \theta). \quad (2.13)$$

This simply says that the potential of a multipole drops off as  $1/r^{l+1}$ , and inside some spherical region it increases as  $r^l$ . Its angular dependence is given by the Legendre polynomials. This comes from the fact that

$$\sum x^n P_l(t) = \frac{1}{\sqrt{1-2xt+x^2}}, \quad (2.14)$$

the generating function for Legendre polynomials, with  $-1 < t < 1$  and  $0 < x < 1$ . Taking this denominator as  $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + 2r_1r_2 \cos \theta + r_2^2}$ , if we pull out  $r_1$  then we get the desired result with this formula.

One more example. Suppose we write the Schrödinger equation as

$$(\nabla^2 + k^2)\psi = 2mV\psi, \quad (2.15)$$

where  $k^2 = 2mE$ .

We may write the Green's function

$$(\nabla^2 + k^2)G = \delta, \quad (2.16)$$

satisfying

$$\int_{S^2} d\mathbf{S} \cdot \nabla G + k^2 \int_V dV G = 1, \quad (2.17)$$

where  $V$  is a sphere centered on the origin. In the limit of very small spheres, we shall argue that the second term vanishes. For the  $k^2$  term is constant, whereas as  $r$  grows small, the  $\nabla^2$  blows up as  $1/r^2$ . Near the origin, we can solve

$$\int d\mathbf{S} \cdot \nabla G = 1, \quad (2.18)$$

while away from the origin, our delta function is zero. We can exploit the spherical symmetry to rewrite the Laplacian as

$$\frac{1}{r^2} \partial_r (r^2 \partial_r G) + k^2 G = 0. \quad (2.19)$$

If we write  $G = \phi/r$  to account for the asymptotic behavior of the Green's function, then we find

$$\phi'' + k^2 \phi = 0. \quad (2.20)$$

This has solutions

$$\phi = e^{\pm ikr}. \quad (2.21)$$

The  $+$  solution is an outgoing wave, while the  $-$  solution is an incoming wave. Now

$$G = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}. \quad (2.22)$$

We can now treat the Schrödinger equation as an integral equation with a source. Thus

$$\psi(\mathbf{r}_1) = 2m \int d\mathbf{r}_2 G(\mathbf{r}_1, \mathbf{r}_2) V(\mathbf{r}_2) \psi(\mathbf{r}_2). \quad (2.23)$$

This is self-consistent, but is it useful? It will be if  $V$  is small. Thus we can expand  $\psi$  perturbatively in orders of  $V$ , considering each order of scattering and writing

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots \quad (2.24)$$

where each order is given by solving the integral equation with the previous order.

Last time, we rewrote the Schrödinger equation as an integral equation,

$$\psi(\mathbf{r}_1) = -\frac{2m}{4\pi} \int \frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|} V(\mathbf{r}_2) \psi(\mathbf{r}_2). \quad (3.1)$$

To solve this perturbatively, we require that the potential is small,

$$V = \lambda W \quad (3.2)$$

for some  $\lambda \ll 1$ , and moreover that these integrals converge.

We must make a guess for the initial wave as

$$\psi_0 = e^{-ik \cdot \mathbf{r}}, \quad (3.3)$$

which solves the homogeneous equation, and hence the first-order correction is

$$\psi_1 = -\frac{2m}{4\pi} \int d^3r_2 \frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|} V(\mathbf{r}_2) e^{-ik \cdot \mathbf{r}}. \quad (3.4)$$

This will give us the order  $\lambda$  correction.

**Fourier series** Consider a function  $f(x)$  defined on the interval  $[-\pi, \pi]$ . As long as a function satisfies certain “Dirichlet conditions,” it has a valid Fourier decomposition that converges in the mean. These conditions require that the function has only finitely many finite discontinuities. So any smooth function on the interval will work, but functions with a finite jump can still give us a valid Fourier decomposition. It turns out that at the point of the discontinuity, the Fourier series will have the mean value of the limits from the left and right.

Things that will break the decomposition include infinite discontinuities ( $1/x$ ) or that have infinitely many finite discontinuities. There are functions which are not Dirichlet which have Fourier decompositions, but the conditions are sufficient to guarantee the convergence of the Fourier decomposition.

The decomposition takes the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad (3.5)$$

with

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \quad (3.6)$$

so that  $a_0/2$  is the mean value of the function. The coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \quad (3.7)$$

This is nothing more than the decomposition of a function in an infinite-dimensional vector space. Thanks to the orthogonality of cosines and sines, we can treat this as a complete basis and compute the components, which are exactly the Fourier coefficients.

On our function space, there is an inner product defined by

$$f \cdot g = \int_{-\pi}^{\pi} f^*(x) g(x). \quad (3.8)$$

Notice that

$$0 = \int_{-\pi}^{\pi} \sin nx \cos mx dx, \quad (3.9)$$

since the trig addition formulas tell us we are integrating sines  $\sin(n+m)x$  and  $\sin(n-m)x$  over whole intervals.<sup>2</sup> If we have

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{n} \int_{-\pi}^{\pi} \sin nxd(\cos mx) = \frac{1}{n} (\sin nx \cos mx)|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \cos nxd(\sin mx). \quad (3.10)$$

<sup>2</sup>Equivalently the function is odd and integrated over a symmetric interval.

The boundary terms are zero since  $\sin n\pi = 0$ , and we get that<sup>3</sup>

$$\int \sin^2 nx dx = \int \cos^2 nx dx. \quad (3.11)$$

Since

$$1 = \sin^2 nx + \cos^2 nx \implies \underbrace{\int_{-\pi}^{\pi} dx}_{2\pi} = \int_{-\pi}^{\pi} dx (\sin^2 nx + \cos^2 nx) = 2 \int_{-\pi}^{\pi} dx \sin^2 nx, \quad (3.12)$$

we conclude that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx \sin^2 nx = 1. \quad (3.13)$$

We can also do this with complex exponentials pretty easily. We find that<sup>4</sup>

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx \sin nx \sin mx = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \cos nx \cos mx = \delta_{nm}. \quad (3.14)$$

Let  $f_N$  be the truncated Fourier series approximation of  $f$ ,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx + b_n \sin nx, \quad (3.15)$$

One may show that

$$0 \leq |f - f_N|^2 = \int f^2 + f_N^2 - 2ff_N, \quad (3.16)$$

which is just the Bessel's inequality. We have a term independent of the  $a_n$ s, one quadratic in them, and one linear in them.

It is sometimes convenient to use a more symmetric expansion for the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \quad (3.17)$$

This is perfectly equivalent to sines and cosines, as we know. The matching works out as

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n + ib_n), \quad c_{-n} = c_n^* = \frac{1}{2}(a_n - ib_n) \quad (3.18)$$

assuming that  $a, b$  are real.

These eigenfunctions satisfy the Sturm-Liouville problem

$$y'' = -\lambda y \quad (3.19)$$

where

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi). \quad (3.20)$$

The eigenvalues are  $\lambda = n^2$  and the sets of solutions are

$$y_n = \sin nx, \cos nx. \quad (3.21)$$

<sup>3</sup>We can also derive this by noting that the integral of  $\sin nx$  over  $-\pi$  to  $\pi$  (i.e. integer multiples of the period) is equal to the integral of  $\cos nx$  over the same interval, since a sine is just a phase-shifted cosine. Hence  $\int_{-\pi}^{\pi} \sin(nx) dx = \int_{-\pi}^{\pi} \cos(nx) dx$  and the same is true for their squares. Since they both complete the same number of full periods over the interval, their integrals must be the same. So we can immediately conclude that  $\int_{-\pi}^{\pi} \sin^2 x dx = \int_{-\pi}^{\pi} \cos^2 x dx$ .

<sup>4</sup>See the end of this section for a bit of extra detail.



**Non-lectured: exponentials and integrals of  $\sin nx$ , two ways** We can compute the integral of  $\sin^2 nx$  by passing to the complex exponential form,

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}. \quad (3.22)$$

Then for  $n \neq m$ ,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \left( \frac{e^{inx} - e^{-inx}}{2i} \right) \left( \frac{e^{imx} - e^{-imx}}{2i} \right) \quad (3.23)$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \left( e^{i(n+m)x} + e^{-i(n+m)x} - e^{-ix(n-m)} - e^{ix(n-m)} \right) \quad (3.24)$$

$$= -\frac{1}{2\pi} \left[ \frac{e^{ix(n+m)}}{i(n+m)} - \frac{e^{-ix(n+m)}}{i(n+m)} + \frac{e^{-ix(n-m)}}{i(n-m)} - \frac{e^{ix(n-m)}}{i(n-m)} \right]_{-\pi}^{\pi} \quad (3.25)$$

However, notice that these exponentials are all at least  $2\pi$  periodic. That is,

$$e^{i(2\pi)(n+m)} = 1 \implies e^{i\pi(n+m)} = e^{-i\pi(n+m)} \quad (3.26)$$

for  $n, m \in \mathbb{Z}$ . That means that all these exponential terms vanish when  $n \neq m$ . If  $n = m$ , then the second line, Eqn. (3.24), becomes

$$-\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \left( e^{i(n+m)x} + e^{-i(n+m)x} - 2 \right) = -\frac{1}{2\pi} (-2\pi) = 1. \quad (3.27)$$

We conclude that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx dx = \delta_{nm}. \quad (3.28)$$

But wait, there's more! Here is a high-powered way to do this integral using contour integration. If you don't know contour integration, read ahead to when these notes cover them and come back later. Let's write this integral suggestively in terms of  $\theta$ ,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta. \quad (3.29)$$

Since  $\theta$  runs from  $-\pi$  to  $\pi$ , we can think of this integral as tracing a unit circle contour in the complex plane (cf. Arfken 11.8). That is, we shall make the change of variables

$$z = e^{i\theta} \quad (3.30)$$

so that

$$d\theta = -i \frac{dz}{z}, \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = -i \frac{z^n - z^{-n}}{2}. \quad (3.31)$$

Now our integral becomes

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta &= -\frac{i}{\pi} \oint_C \left( -i \frac{z^n - z^{-n}}{2} \right) \left( -i \frac{z^m - z^{-m}}{2} \right) \frac{dz}{z} \\ &= \frac{i}{4\pi} \oint (z^{n+m} - z^{n-m} - z^{-n+m} + z^{-n-m}) \frac{dz}{z}. \end{aligned}$$

We can evaluate this with the Cauchy residue theorem! Just look for poles inside the unit circle. Actually, this function only has poles at the origin. We may recall that anything that's not a simple pole in this sort of integral will have zero residue. For instance, suppose  $n = 1, m = 2$ . Then our integral would be

$$\frac{i}{4\pi} \oint (z^{1+2} - z^{1-2} - z^{-1+2} + z^{-1-2}) \frac{dz}{z}. \quad (3.32)$$

Just by power counting, none of these are simple poles. For the  $z^{n+m}$  term to give a simple pole, we would need  $n + m = 0$ , and for  $n, m \in \mathbb{Z}_+$  there are no solutions to this equation. The same is true for the  $z^{-n-m}$  term. Our only hope is for the middle terms to give us a simple pole, and this happens exactly when  $n = m$ . When  $n = m$ , we have

$$\frac{i}{4\pi} \oint (z^{2n} - 2 + z^{-2n}) \frac{dz}{z} = \frac{i}{4\pi} (2\pi i (-2)) = 1 \quad (3.33)$$

by the Cauchy residue theorem. As before, we conclude that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \sin m\theta d\theta = \delta_{nm}. \quad \boxtimes \quad (3.34)$$

Lecture 4.

**Monday, January 13, 2020**

*"If I attempted to do it [the completeness relation on Fourier modes] any other way, any self-respecting mathematician would decapitate me immediately. So I won't do it because I like my neck just the way it is."*

—Nemanja Kaloper

Suppose we have

$$\phi_n = \frac{e^{inx}}{\sqrt{2\pi}}, \quad (4.1)$$

such that the inner product is given by

$$\langle \phi_n | \phi_m \rangle = \int \phi_n^* \phi_m. \quad (4.2)$$

For a general function  $f$  let us now take its inner product with the basis vectors,

$$c_n = \langle \phi_n | f \rangle \quad (4.3)$$

and for a general set of coefficients  $\{\bar{c}_n\}$ , write the function (linear combination)

$$\bar{f}_N = \sum \bar{c}_n \phi_n. \quad (4.4)$$

We shall show that the Fourier decomposition of  $f$  converges to  $f$  faster than any other sequence  $\bar{f}_N$ . To do this, let us use the triangle inequality and write

$$0 \leq |f - \bar{f}_N|^2 = \langle f - \bar{f}_N | f - \bar{f}_N \rangle \quad (4.5)$$

$$= |f|^2 - \langle \bar{f}_N | f \rangle - \langle f | \bar{f}_N \rangle + |\bar{f}_N|^2 \quad (4.6)$$

$$= |f|^2 - 2\text{Re}\langle \bar{f}_N | f \rangle + |\bar{f}_N|^2. \quad (4.7)$$

We can compute these inner products explicitly.

$$\langle \bar{f}_N | \bar{f}_N \rangle = \sum_{n,m} \bar{c}_n^* \bar{c}_m \phi_n \phi_m = \sum_n \bar{c}_n^* \bar{c}_n = \sum_n |\bar{c}_n|^2. \quad (4.8)$$

The cross-terms are

$$\langle \bar{f}_N | f \rangle = \sum \bar{c}_n^* \langle \phi_n | f \rangle = \sum \bar{c}_n^* c_n. \quad (4.9)$$

Plugging back in, we have

$$0 \leq |f|^2 - \sum (\bar{c}_n^* c_n + \bar{c}_n c_n^*) + \sum \bar{c}_n^* \bar{c}_n. \quad (4.10)$$

This is guaranteed to be positive semi-definite by the positivity of the norm. This defines a distance  $D(f, \bar{f}_N)$  on the space of functions.

Notice there are really two sets of free parameters,  $c_n$  and  $\bar{c}_n$  or equivalently  $c_n$  and  $\bar{c}_n^*$ . If we take a derivative with respect to  $\bar{c}_n^*$  to extremize this, we get

$$\bar{c}_n = c_n, \bar{c}_n^* = c_n^*. \quad (4.11)$$

That is, we want to approximate our function as fast as we can, and the best way to do it is by picking the Fourier coefficients.<sup>5</sup> The derivative guarantees it. This is exactly the same as the least-squares fit, just by a different process.

The convergence of these sorts of series may be in question when the metric on the space is not positive definite (e.g. Minkowski signature). In that case, these approximations may converge to something but perhaps not uniquely, and we may require additional information to ensure uniqueness.

<sup>5</sup>If you like, this is a discrete version of a functional derivative where we just vary the components of each function.

Fourier series have some nice properties. Notice that since the Fourier basis elements  $\cos nx, \sin nx$  are periodic in  $2\pi$ , the entire Fourier series is also periodic in  $2\pi$ . Note that the function we are approximating need not itself have the same value at  $-\pi$  and  $\pi$ ; this just counts as a single discontinuity.

What if our function is periodic on a different interval, say  $2L$ ? That is,

$$f(x + 2L) = f(x). \quad (4.12)$$

Then define  $x = \alpha \bar{x}$  such that we can write

$$x + 2L = \alpha(\bar{x} + 2\pi), \quad (4.13)$$

since we know how to deal with functions which are  $2\pi$  periodic. We find immediately that

$$\alpha = L/\pi. \quad (4.14)$$

Hence our Fourier expansion can now be written as

$$f = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \quad (4.15)$$

All this corresponds to is changing the radius of the unit circle. The circle has a radius given by  $L$  instead of 1. Then the coefficients are just

$$a_0 = \frac{1}{L} \int_{-L}^L dx f(x) \quad (4.16)$$

and the other coefficients are<sup>6</sup>

$$a_n = \frac{1}{L} \int_{-L}^L dx f(x) \cos \frac{n\pi x}{L}, \quad b_n = \frac{1}{L} \int_{-L}^L dx f(x) \sin \frac{n\pi x}{L}. \quad (4.17)$$

Equivalently we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^L dx f(x) e^{-\frac{in\pi x}{L}}. \quad (4.18)$$

Let us define a function which is given on the fundamental domain as

$$f(x) = x, \quad -\pi < x < \pi. \quad (4.19)$$

and  $2\pi$  periodic, so e.g.  $f(x) = x - 2\pi, \pi < x < 3\pi$  and so on. This is a sawtooth wave. What is the Fourier decomposition of this wave? One can do the integrals in Eq. (4.17) to figure it out. One finds that all the cosine coefficients go away because this function is odd. It will only have sines:

$$x = f(x) = 2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right] = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}. \quad (4.20)$$

This does not converge absolutely, if one checks. The ratio test is indeterminate. This is like a harmonic series. But it should converge, and the reason it does is because it is an alternating series. Alternating series have nicer convergence properties in general.

We can also evaluate this at  $\pi/2$ . Remember this was the Fourier expansion of  $x$ . So then by plugging  $x = \pi/2$  into Eqn. (4.20), we get

$$\frac{\pi}{2} = 2 \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right], \quad (4.21)$$

which lets us sum a series we might not have expected.

We could actually do this for a triangle wave instead, defined as  $|x|$  on  $-\pi, \pi$ . If we wrote down the Fourier series we would find that

$$f(x) = \frac{\pi}{2} - \sum \frac{4}{\pi} \frac{\cos((2n+1)x)}{(2n+1)^2}. \quad (4.22)$$

This series is no longer alternating, but it actually converges faster, like  $\sum 1/n^2$ , since the function is not discontinuous like our sawtooth wave. The derivative is discontinuous, yes, but we still have nicer convergence. In general, the Fourier decompositions of functions which are smoother will converge faster.

<sup>6</sup>You can do this by just making a change of variables in the integral to get your limits of integration how you want them.

It's fairly clear that we can add two Fourier series to get a new one.. But in fact we can also integrate Fourier series by linearity (assuming the original sum converges). When we integrate Fourier series, our sines and cosines get factors of  $1/n$  in the series coefficients, i.e. for  $f(x)$  given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad (4.23)$$

the integral of  $f(x)$  is

$$\int_{-\pi}^x f(x') dx' = \frac{a_0}{2}(x + \pi) + \sum_{n=1}^{\infty} a_n \frac{\sin nx}{n} - b_n \frac{\cos nx}{n} + \text{constant boundary terms.} \quad (4.24)$$

These factors of  $1/n$  mean that the convergence properties of the integrated function are nicer than the original, i.e. it converges at least as fast as the expansion of  $f(x)$ .

Lecture 5.

**Wednesday, January 15, 2020**

*"I'm gonna skip this example of the full wave rectifier, blah blah blah. I never cared much for electronics. This is not a statement of preference but of personal incompetence."*

—Nemanja Kaloper

Today we will continue our discussion of Fourier series. In general the integral of a Fourier series is not itself a Fourier series, since it has a linear term. But its convergence properties are nicer than the original. What's happening is that we put frequencies in the denominator, which makes the different frequency modes decouple more.

Recall we derived the Fourier series of a sawtooth wave,

$$x = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}. \quad (5.1)$$

Can we take derivatives of this, as

$$1 = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos nx? \quad (5.2)$$

Alas, this is nonsense. Suppose we evaluated this at  $x = 0$ . Then we would find

$$1 = 2(1 - 1 + 1 - 1 + \dots) \quad (5.3)$$

which is clearly not convergent in the usual sense.<sup>7</sup>

However, if we take the Fourier expansion of  $|x|$ , we can get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad (5.4)$$

which converges faster than  $1/n$  (i.e. uniformly). Hence we can define the derivative such that

$$\frac{d}{dx}|x| = +\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases} \quad (5.5)$$

This is correct, as

$$\frac{d}{dx}|x| = 2\theta(x) - 1, \quad (5.6)$$

with  $\theta$  the Heaviside step function.

Suppose now we are handed a series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n}. \quad (5.7)$$

What function does this correspond to? Let's think qualitatively. The cosine is bounded by  $\pm 1$ , so this might converge. The point  $x = 0$  is dangerous since our series becomes a harmonic series, which is divergent (albeit only logarithmically). This is true at any integer multiple of  $2\pi$ , in fact.

<sup>7</sup>However, see [https://en.wikipedia.org/wiki/Grandi%27s\\_series](https://en.wikipedia.org/wiki/Grandi%27s_series). One can define a sensible Cesàro sum which actually is  $1/2$ , but it's not what we mean by convergence.

Let's make sense of this as follows. We shall write this as

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \operatorname{Re} \sum_{n=1}^{\infty} \frac{e^{inx}}{n}. \quad (5.8)$$

Now  $e^{inx}$  is a point on a unit circle in the complex plane, and we can further think of this as a limit

$$\operatorname{Re} \sum_{n=1}^{\infty} \frac{e^{inx}}{n} = \lim_{r \rightarrow 1} \operatorname{Re} \left( \sum_{n=1}^{\infty} \frac{(re^{ix})^n}{n} \right) = \left| \sum_{n=1}^{\infty} \frac{q^n}{n} \right| < \sum_{n=1}^{\infty} \frac{|q|^n}{n}. \quad (5.9)$$

If we take this limit where  $|q| = r < 1$  then this surely converges.

Notice that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n. \quad (5.10)$$

This is not quite what we want, since we're missing the  $n$  in the denominator. But we can do a trick.

$$\int \frac{dq}{1-q} = \sum_{n=0}^{\infty} \int q^n dq = \sum_{n=0}^{\infty} \frac{q^{n+1}}{n+1}. \quad (5.11)$$

If we shift the dummy variable (noting that  $n = 0$  just gives  $q$ ), we can write

$$-\ln(1-q) = \sum_{n=0}^{\infty} \frac{q^n}{n}. \quad (5.12)$$

It follows that our function was

$$\ln(2 - 2\cos x)^{1/2} = -\ln[2\sin(x/2)]. \quad (5.13)$$

This passes a reasonable sanity check, which is that when  $x$  is a multiple of  $2\pi$ , then this series is the log of zero, which is clearly divergent.<sup>8</sup>

Consider some square-wave potential which is  $2\pi$  periodic, such that

$$f(x) = \begin{cases} -\pi < x < 0 & 0, \\ 0 < x < \pi & h. \end{cases} \quad (5.14)$$

The Fourier transform is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n+1)x}{2n+1}. \quad (5.15)$$

Note that only odd sines appear, since the odd sines will have nonzero integral over the half-interval  $0, \pi$ , while the even ones vanish.

We can also switch to exponential notation,

$$f(x) = \sum c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x) e^{-inx}. \quad (5.16)$$

What's interesting now is to consider the finite truncation of the Fourier series,

$$f_N(x) = \sum_{n=-N}^N c_n e^{inx} = \sum_{n=-N}^N e^{inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) e^{-int}. \quad (5.17)$$

If this sum were infinite, we could not exchange the order of integration and summation with impunity. But for finite  $N$ , there are no problems with convergence. So we can do this freely and write

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \sum_{n=-N}^N e^{in(x-t)}. \quad (5.18)$$

This is in fact just a sum of cosines. For notice that if we define

$$q = e^{i(x-t)}, \quad (5.19)$$

<sup>8</sup>However, something a little weird happens when we take  $x$  to be e.g.  $-\pi$ . On the RHS we have a log of something negative, which is only well-defined in terms of the analytic continuation of  $\ln(z)$ . In fact, the log of a negative number has an imaginary part. The LHS looks like we're summing real numbers, so this is a bit unusual; it has to do again with the lack of uniform convergence.

then our sum is

$$S = \sum_{n=-N}^N q^n = \frac{1}{q^N} \sum_{n=0}^{2N} q^n \quad (5.20)$$

Note that the sum with  $N \rightarrow \infty$  is the geometric series  $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ . It follows that the finite sum can be written as the difference of two infinite sums,

$$\sum_{n=0}^{2N} q^n = \sum_{n=0}^{\infty} q^n - \sum_{n=2N+1}^{\infty} q^n = S_{\infty} - q^{2N} S_{\infty} = \frac{1 - q^{2N+1}}{1 - q}. \quad (5.21)$$

We get that this is

$$\frac{q^N - q^{-N}}{1 - q} = \frac{\sin((N + \frac{1}{2})(x - t))}{\sin \frac{x-t}{2}}. \quad (5.22)$$

Hence

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \left[ \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} \right], \quad (5.23)$$

which is the best approximation to our function in terms of a finite truncation. The goal is now to determine what properties  $f$  must have such that  $f_N$  actually converges nicely to  $f$ .

Note also that the term in the brackets is just a delta function. It is simply the sum  $\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-t)}$ , and in a sense this is just the statement of the completeness of complex exponentials.

Lecture 6.

**Friday, January 17, 2020**

*"If you're going to try to violate decoupling, don't try to do it with something as stupid as continuous functions." "Bastards." "That's correct."*

—Nemanja Kaloper and Mark Samuel Abbott

Last time, we found the explicit formula for the partial sums of a Fourier series to be

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \left[ \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} \right]. \quad (6.1)$$

Let us consider a Heaviside step function,

$$f(x) = \frac{h}{2} (2\Theta(x) - 1). \quad (6.2)$$

We found the Fourier series to be

$$f(x) = \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{2n+1}, \quad (6.3)$$

and we'd like to study what happens when the sum is finite rather than infinite. That is, let us plug in  $f(t)$ . Then

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt \left( \frac{h}{2} (2\Theta(t) - 1) \right) \left[ \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} \right] \quad (6.4)$$

$$= \frac{h}{4\pi} \left[ - \int_{-\pi}^0 dt \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} + \int_0^{\pi} dt \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} \right]. \quad (6.5)$$

Since this is a finite series and the function is piecewise continuous, we can now take the limit of the interpolating functions  $\lim_{N \rightarrow \infty} f_N(x)$  to get the infinite sum.

Note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dt \frac{\sin[(N + 1/2)(x - t)]}{\sin \frac{x-t}{2}} = 1. \quad (6.6)$$

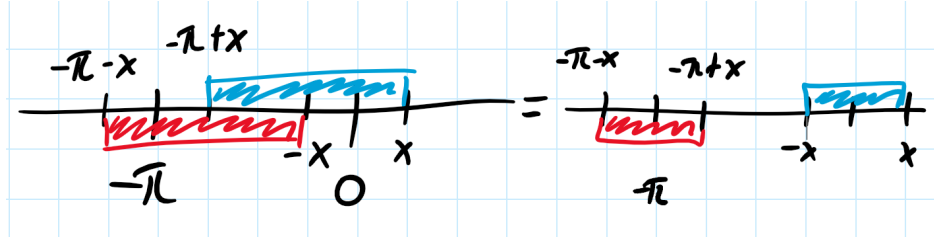


FIGURE 1. An illustration of how to rewrite the integration limits in Eqn. (6.9).

This is just the integral of (the limiting series of) a delta function. Why? This was just the sum

$$\sum_{n=-N}^N e^{in(x-t)} = \frac{1}{2\pi} \frac{\sin[(N+1/2)(x-t)]}{\sin[\frac{x-t}{2}]}, \quad (6.7)$$

and if we integrate the LHS from  $-\pi$  to  $\pi$ , then we have

$$\int_{-\pi}^{\pi} dt \sum_{n=-N}^N e^{in(x-t)} = \sum_{n=-N}^N e^{inx} \int_{-\pi}^{\pi} dt e^{-int} = \sum_{n=-N}^N e^{inx} (2\pi \delta_{n,0}), \quad (6.8)$$

so the other terms in the sum drop out.

We can rewrite the terms in our integrals (6.5) in terms of new variables  $-s = x - t$  in the first integral and  $s = x - t$  in the second integral to write

$$\begin{aligned} f_N(x) &= \frac{h}{4\pi} \left[ - \int_{-\pi}^0 dt \frac{\sin((N+1/2)(x-t))}{\sin[\frac{x-t}{2}]} + \int_0^{\pi} dt \frac{\sin((N+1/2)(x-t))}{\sin[\frac{x-t}{2}]} \right] \\ &= \frac{h}{\pi} \left[ - \int_{-\pi-x}^{-x} ds \frac{\sin((N+1/2)(-s))}{\sin[\frac{-s}{2}]} - \int_x^{-\pi+x} ds \frac{\sin((N+1/2)s)}{\sin[\frac{s}{2}]} \right] \\ &= \frac{h}{4\pi} \left[ - \int_{-\pi-x}^{-x} ds \frac{\sin((N+1/2)s)}{\sin[\frac{s}{2}]} + \int_{-\pi+x}^x ds \frac{\sin((N+1/2)s)}{\sin[\frac{s}{2}]} \right] \end{aligned} \quad (6.9)$$

Let's justify the signs here. There are two sign flips on the second term. One comes from  $ds = -dt$ , while the other comes from exchanging the limits of integration in the third line. There are no sign flips on the first term, since

$$\begin{aligned} \sin((N+1/2)(-s)) &= -\sin((N+1/2)s), \quad \sin(-s/2) = -\sin(s/2) \\ \implies \frac{\sin((N+1/2)(-s))}{\sin(-s/2)} &= \frac{\sin((N+1/2)s)}{\sin(s/2)}. \end{aligned}$$

We can now shuffle the limits of integration in Eqn. (6.9) a bit (see Fig. 1) to get

$$f_N(x) = \frac{h}{4\pi} \left[ + \int_{-x}^{+x} ds \frac{\sin((N+1/2)s)}{\sin[\frac{s}{2}]} - \int_{-\pi-x}^{-\pi+x} ds \frac{\sin((N+1/2)s)}{\sin[\frac{s}{2}]} \right]. \quad (6.10)$$

One way to do this formally is to recognize that the relative sign between these two integrals allows us to rewrite the definite integrals as differences of indefinite integrals (denoted  $\Phi$ ):

$$\begin{aligned} - \int_{-\pi-x}^{-x} f(x) dx + \int_{-\pi+x}^x f(x) dx &= -[\Phi(-x) - \Phi(-\pi-x)] + [\Phi(x) - \Phi(-\pi+x)] \\ &= (\Phi(x) - \Phi(-x)) - (\Phi(-\pi+x) - \Phi(-\pi-x)) \\ &= \int_{-x}^x f(x) dx - \int_{-\pi-x}^{-\pi+x} f(x) dx. \end{aligned}$$

Graphically or otherwise, we arrive at the result (6.10).

Now what happens as  $x \rightarrow 0$ ? Our integration measure  $ds$  gets vanishingly small, and if the integrand were finite on that range, the integral would vanish. Well, the second term, the integral on an interval around  $-\pi$ , is bounded, since  $\sin((N + 1/2)s)$  is bounded around  $s = -\pi$ , and the denominator

$$\lim_{s \rightarrow -\pi} \sin(s/2) = \sin(-\pi/2) = -1$$

is perfectly well-behaved. So indeed the second integral in (6.10) is zero. More precisely, it is proportional to the interval length  $2x$ , which goes to zero as  $x \rightarrow 0$ .

The first term is different. We are taking an integral in a region around  $s = 0$ , but  $\sin(0)$  is zero, so the denominator will cause the integrand to diverge as  $s \rightarrow 0$ . The numerator  $\sin((n + 1/2)s)$  also goes to zero, so we get an indeterminate ratio which we can work out with L'Hôpital's rule or equivalently by Taylor expansion around  $s = 0$ . That is,

$$\lim_{x \rightarrow 0} \frac{h}{4\pi} \int_{-x}^x \frac{\sin((N + 1/2)s)}{\sin(s/2)} ds = \lim_{x \rightarrow 0} \frac{h}{4\pi} \int_{-x}^x ds \frac{(N + 1/2)s}{(s/2)} = \lim_{x \rightarrow 0} \frac{h}{2\pi} \int_{-x}^x ds (N + 1/2) \quad (6.11)$$

The factors of  $s$  have cancelled. Now we must be careful about the order of limits. If we hold  $N$  fixed and perform the  $ds$  integral, then we get

$$\lim_{x \rightarrow 0} f_N(x) = \lim_{x \rightarrow 0} \frac{h}{2\pi} (N + 1/2)(2x) = 0. \quad (6.12)$$

It follows that

$$f_N(0) = 0 \implies f(0) = \lim_{N \rightarrow \infty} f_N(0) = 0. \quad (6.13)$$

That is, the (shifted) step function takes on the value

$$f(0) = \Theta(0) - 1/2 = 0 \implies \Theta(0) = \frac{1}{2}, \quad (6.14)$$

the average of its value on either side of  $x = 0$ .

Moreover, if we look at the Fourier series approximation of the step function, we find that the approximation overshoots the step. This is called the *Gibbs phenomenon*. Curiously, the overshoot doesn't improve with more terms; in fact, it approaches a limiting value of about 18%. The reason for this is simply that we're trying to approximate a discontinuous function with a smooth one. We overshoot because the derivative cannot change too quickly. Next time, we will study the behavior of

$$\frac{h}{2\pi} \int_0^x \frac{\sin[(N + 1/2)s] \sin(s/2)}{d} ds, \quad (6.15)$$

and we can study the behavior of this integral in terms of a new variable

$$\zeta = (N + \frac{1}{2})s. \quad (6.16)$$

Then we have an integral

$$\frac{h}{2\pi} \int_0^{(N+1/2)x} \frac{\sin(\zeta)}{\sin(\zeta/(2N+1))} d\zeta \approx \frac{h(2N+1)}{2\pi} \frac{h}{2\pi} \int_0^{(N+1/2)x} \frac{\sin(\zeta)}{\zeta} d\zeta \quad (6.17)$$

and we can study this in certain limits of  $\zeta$ .

Lecture 7.

**Wednesday, January 22, 2020**

"Today I have a 'fold.' Combination of flu and cold. Keep your distance."

—Nemanja Kaloper

Let's recall our key equation for the truncated Fourier series,

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \left[ \frac{\sin((N + 1/2)(x - t))}{\sin \frac{x-t}{2}} \right]. \quad (7.1)$$

We shall start by writing a new integration variable

$$x = -t = -s \leftrightarrow s = t - x. \quad (7.2)$$



It follows that

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} ds f(x+s) \frac{\sin[(N+1/2)s]}{\sin(s/2)}. \quad (7.3)$$

We're integrating over complete intervals (note that while the numerator and denominator are  $4\pi$ -periodic, their quotient is actually  $2\pi$ -periodic).<sup>9</sup> So the shifting of the argument by  $x$  doesn't actually affect the value of the integral. That is,

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (7.4)$$

By evenness in  $t$ , we can write the integral as a symmetrized sum

$$f_N(x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} dt [f(x+t) + f(x-t)] \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (7.5)$$

Now we take the specific case of the step function

$$f(x) = 2\Theta(x) - 1, \quad (7.6)$$

such that for the first term,

$$x+t > 0, t > -x, \quad x+t < 0, t < -x. \quad (7.7)$$

Hence the first term splits into two integrals

$$+ \int_{-x}^{\pi} - \int_{-\pi}^{-x}. \quad (7.8)$$

Similarly, for the second term,

$$x-t > 0, t < x, \quad x-t < 0, t > x. \quad (7.9)$$

Hence the  $f(x-t)$  term also splits into

$$- \int_x^{\pi} + \int_{-\pi}^x. \quad (7.10)$$

we have an overall sum

$$+ \int_{-x}^{\pi} - \int_{-\pi}^{-x} - \int_x^{\pi} + \int_{-\pi}^x. \quad (7.11)$$

In the  $-\int_{-\pi}^{-x}$  integral, we can take  $t \rightarrow -t$  so that

$$- \int_{-\pi}^{-x} \rightarrow + \int_{+\pi}^{+x} \quad (7.12)$$

and similarly in the last integral we have

$$+ \int_{-\pi}^x \rightarrow - \int_{+\pi}^{-x}. \quad (7.13)$$

Hence we have

$$\int_{-x}^{\pi} + \int_{\pi}^x - \int_x^{\pi} - \int_{\pi}^{-x}. \quad (7.14)$$

If we now flip the limits of integration we see that terms double up, so this becomes

$$2 \left[ \int_{-x}^{\pi} + \int_{\pi}^x \right] = 2 \int_{-x}^x. \quad (7.15)$$

Hence our integral was really

$$\frac{1}{2\pi} \int_{-x}^x dt \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (7.16)$$

<sup>9</sup>The best way to see this is by using complex exponentials, i.e. by writing

$$\frac{\sin((N+1/2)x)}{\sin(x/2)} \stackrel{?}{=} \frac{\sin((N+1/2)(x+2\pi))}{\sin(1/2(x+2\pi))} = \frac{e^{i(N+1/2)(x+2\pi)} - e^{-i(N+1/2)(x+2\pi)}}{e^{i(1/2)(x+2\pi)} - e^{-i(1/2)(x+2\pi)}}.$$

Now the  $2\pi$  factors give  $e^{i(2N+1)\pi} = -1$  and  $e^{-i(2N+1)\pi} = -1$ , so we find that

$$\frac{e^{i(N+1/2)(x+2\pi)} - e^{-i(N+1/2)(x+2\pi)}}{e^{i(1/2)(x+2\pi)} - e^{-i(1/2)(x+2\pi)}} = \frac{-e^{i(N+1/2)x} + e^{-i(N+1/2)x}}{-e^{ix/2} + e^{-ix/2}} = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

One can also do this with trig addition formulae, which give the same result.

We can use the evenness of the integrand to get rid of a factor of 1/2 and write

$$f_N(x) = \frac{1}{\pi} \int_0^x dt \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (7.17)$$

For a fixed  $N$ , if we take  $x \rightarrow 0$ , every expression in the interpolating series is zero and it follows that  $f(0) = \lim_{N \rightarrow \infty} f_N(0) = 0$ .

Let us now consider positive  $x$  and define

$$\zeta = (N+1/2)t \quad (7.18)$$

so that

$$f_N = \frac{1}{\pi} \int_0^{(N+1/2)x} \frac{d\zeta}{N+1/2} \frac{\sin \zeta}{\sin\left(\frac{\zeta}{2N+1}\right)}. \quad (7.19)$$

In the limit as  $N \rightarrow \infty$ , we can approximate  $\sin(\zeta/(2N+1)) \approx \zeta$  and therefore

$$f_N \rightarrow \frac{2}{\pi} \int_0^\infty d\zeta \frac{\sin \zeta}{\zeta}. \quad (7.20)$$

How can we perform this? Let's introduce a regularizing function  $e^{-\lambda\zeta}$  and we'll take the limit  $\lambda \rightarrow 0$  later. This is Yukawa's trick for regularizing Coulombic integrals. That is, define

$$I(\lambda) = \int_0^\infty d\zeta e^{-\lambda\zeta} \frac{\sin \zeta}{\zeta}. \quad (7.21)$$

Now let us study the derivative with respect to  $\lambda$ , i.e.

$$I'(\lambda) = - \int_0^\infty d\zeta e^{-\lambda\zeta} \sin \zeta. \quad (7.22)$$

We could make some exponential substitution for  $\sin \zeta$  now to evaluate this. Even more simply, we could do integration by parts. Doing it twice gives the original expression back, which means we can evaluate this explicitly and solve the PDE for  $I$  in terms of  $\lambda$ . Taking the  $\lambda \rightarrow 0$  limit then yields our answer:

$$\int_0^\infty d\zeta \frac{\sin \zeta}{\zeta} = \frac{\pi}{2}. \quad (7.23)$$

Hence it follows that in the limit as  $N \rightarrow \infty$ , our function really does converge for  $x > 0$ :

$$\lim_{N \rightarrow \infty} f_N(x > 0) = \frac{2}{\pi} \left( \frac{\pi}{2} \right) = 1. \quad (7.24)$$

Finally, we must calculate the overshoot. We now take the limit as  $x \rightarrow 0$  for fixed  $N$  of

$$f_N = \frac{1}{\pi} \int_0^{(N+1/2)x} \frac{d\zeta}{N+1/2} \frac{\sin \zeta}{\sin\left(\frac{\zeta}{2N+1}\right)}. \quad (7.25)$$

We shall again take  $N$  to be large but not infinite, such that

$$f_N(x) = \frac{2}{\pi} \int_0^{(N+1/2)x} d\zeta \frac{\sin \zeta}{\zeta}. \quad (7.26)$$

At what point is this overshoot maximized? Well, it is maximized when the numerator  $\sin \zeta$  hits zero and starts to become negative, i.e. when the upper limit of integration is  $\zeta = \pi$ , i.e.

$$(N+1/2)x = \pi. \quad (7.27)$$

Hence

$$f_N = \frac{2}{\pi} \int_0^\pi \dots = \frac{2}{\pi} \int_0^{3\pi} - \int_\pi^{3\pi} = \frac{2}{\pi} \left[ \int_0^\infty - \int_\pi^{3\pi} - \int_{3\pi}^{5\pi} - \int_{5\pi}^{7\pi} - \dots \right]. \quad (7.28)$$

The integral  $\int_0^\infty$  was just  $\pi/2$ , as we calculated. We just increase the integration limits by a whole period and then subtract off what we added on. What about the first correction? We have

$$\int_\pi^{3\pi} d\zeta \frac{\sin \zeta}{\zeta}. \quad (7.29)$$

These integrals over periods  $\pi, 3\pi$  are small negative numbers, since the sin is negative from  $\pi$  to  $2\pi$  and positive from  $2\pi$  to  $3\pi$ , but the  $1/\zeta$  suppresses the positive contribution. We can't compute these exactly; at this point, we must resort to numerical methods. It turns out the sum of the corrections is very close to 0.18; this is the 18% overshoot of the Gibbs phenomenon.

Finally, let us prove the Fourier theorem. We have an integral

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)}, \quad (7.30)$$

which is manifestly  $2\pi$ -periodic. We may allow  $f$  to have finitely many finite discontinuities, e.g.

$$f(x_0^+) \neq f(x_0^-) \quad (7.31)$$

for some  $x_0$ . Now we split the integral into two, as

$$\frac{1}{2\pi} \left[ \int_{-\pi}^0 dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)} + \int_0^{\pi} dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)} \right]. \quad (7.32)$$

Hence if  $f$  has a discontinuity at  $x$ , then the left integral from  $t \in -\pi, 0$  picks up all the contributions up to the discontinuity at  $x$ . Similarly the right integral picks up all the contributions from  $x$  onwards. Let us add and subtract the left limit and rewrite this first term as

$$\frac{1}{2\pi} \int_{-\pi}^0 [f(x^-) + (f(x+t) - f(x^-))] \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (7.33)$$

This first term is just a constant integrated against the sin piece, so its integral over a half period is just  $\pi$ . That is, the  $f(x^-)$  integral becomes

$$\frac{1}{2} f(x^-). \quad (7.34)$$

The integral from 0 to  $\pi$  does something similar; it picks up the right limit,  $\frac{1}{2} f(x^+)$ . What about the rest of it? There's continuity argument to be made here, which we'll revisit next time. One can also do this with trig addition formulae, which give the same result.

Lecture 8.

**Friday, January 24, 2020**

*"When I was in kindergarten... you know, it's like that Opus Dei thing. You punish yourself to see how much you can take."*

—Nemanja Kaloper

Let's conclude our proof of the Fourier theorem. We wrote the partial sum

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)}, \quad (8.1)$$

and said that due to the  $2\pi$ -periodicity, we could simply shift the domain of integration in  $t$  as we like. If there's a single discontinuity at  $x$ , we can break up the domain of integration as

$$\frac{1}{2\pi} \left[ \int_{-\pi}^0 dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)} + \int_0^{\pi} dt f(x+t) \frac{\sin[(N+1/2)t]}{\sin(t/2)} \right]. \quad (8.2)$$

Consider just the first term, the integral from  $-\pi$  to 0. We can add and subtract the function's limit from the left,

$$f_N^- = \frac{1}{2\pi} \int_{-\pi}^0 [f(x^-) + (f(x+t) - f(x^-))] \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \quad (8.3)$$

We've just added zero. Hence the following integral can be evaluated as

$$\frac{1}{2\pi} \int_{-\pi}^0 [f(x^-)] \frac{\sin[(N+1/2)t]}{\sin(t/2)} = \frac{f(x^-)}{2\pi} (\pi) = \frac{f(x^-)}{2}. \quad (8.4)$$

We did the integral of the ratio of sines previously. On the interval  $-\pi$  to  $\pi$  it was 2, and by the evenness of the ratio of sines, the integral over the half-interval  $-\pi$  to 0 is just  $\pi$ .

Computing the integral  $f_N^+$  from 0 to  $\pi$ , we get something very similar—the right limit of the function,  $f(x^+)/2$ , and another integral.

We shall argue that the integrals cancel each other perfectly by the Riemann-Lebesgue lemma, which (roughly) states that the integral of fast functions which oscillate a lot about a point are zero except about some region where the oscillations are slow.

That is, we wish to compute

$$\frac{1}{2} \int_{-\pi}^0 dt \frac{f(x+t) - f(x^-)}{\sin(t/2)} \sin[(N+1/2)t]. \quad (8.5)$$

Notice that for  $t \rightarrow 0$  from the left, the numerator goes as  $f(x^-)t$  since  $f$  is continuous up to  $x$ . The denominator  $\sin(t/2)$  is a slow function; it is  $4\pi$  periodic and only goes to zero at  $t \rightarrow 0$ , where its leading order behavior is  $\sin(t/2) \sim t/2$ . In the limit as  $N$  grows very large, we need only worry about the integrand near  $t \rightarrow 0$ . That is, it goes as  $\sin((N+1/2)t) \sim (N+1/2)t + O(t^3)$ . Hence our slow function is approximately

$$\sigma(t) = \frac{f(x+t) - f(x^-)}{\sin(t/2)} \approx \frac{f'(x^-)t}{t/2} = 2f'(x^-), \quad (8.6)$$

so that

$$\lim_{N \rightarrow \infty} \int_a^b dt \sigma(t) \sin((N+1/2)t) = 0. \quad (8.7)$$

That is, this integral oscillates so fast and our slow function is well-behaved on the whole integral, so the whole integral vanishes. This is the conclusion of the Riemann-Lebesgue lemma.

Let's do a trick to see this works. Let  $\omega = N + 1/2$ , and define a new variable

$$t = t' + \frac{\pi}{N+1/2} = t' + \frac{\pi}{\omega} \quad (8.8)$$

so that

$$\int_{a-\pi/\omega}^{b-\pi/\omega} dt' \sigma(t' + \pi/\omega) \sin(\omega(t' + \pi/\omega)) = - \int_{a-\pi/\omega}^{b-\pi/\omega} dt' \sigma(t' + \pi/\omega) \sin(\omega t') \quad (8.9)$$

since the  $\sin$  is  $2\pi$ -periodic. This is clearly the same as our original integral, since we've just changed variables. Call the integral  $I$ . Then

$$2I = \int_a^b dt \sigma(t) \sin(\omega t) - \int_{a-\pi/\omega}^{b-\pi/\omega} dt \sigma(t + \pi/\omega) \sin(\omega t). \quad (8.10)$$

We see that as  $\omega \rightarrow \infty$ , these integrals perfectly cancel each other. More precisely, we get an integral

$$\int_a^{b-\pi/\omega} dt \sin(\omega t) [\sigma(t) - \sigma(t + \pi/\omega)] - \int_{a-\pi/\omega}^a dt \sigma(t + \pi/\omega) \sin \omega t + \int_{b-\pi/\omega}^b dt \sigma(t) \sin \omega t. \quad (8.11)$$

If we take the last two terms, we see they are both bounded. The last two are bounded by some multiple of  $\pi/\omega$ , since the interval from  $b - \pi/\omega$  to  $b$  becomes infinitesimal and the functions  $\sigma(t), \sin(\omega t)$  are well-behaved around  $b$ . A similar argument applies for the integral from  $a - \pi/\omega$  to  $a$ .

The first term is also bounded and vanishes since  $\sigma(t) - \sigma(t + \pi/\omega)$  is just a derivative  $\sigma'(t)$  times  $\pi/\omega$ , and  $\sigma'$  is well-behaved (nonsingular) over the interval. It follows that in the limit as  $\omega \rightarrow \infty$ , all these integrals go to zero, so that indeed  $I = 0$  and

$$\lim_{N \rightarrow \infty} f_N(x) = \frac{f(x^-) + f(x^+)}{2}, \quad (8.12)$$

and moreover we recover the completeness relation

$$\lim_{N \rightarrow \infty} \frac{\sin(N+1/2)(x-t)}{\sin(\frac{x-t}{2})} = \delta(x-t). \quad (8.13)$$

**Complex analysis** A complex number is a pair  $(x, y)$ , which we may write

$$z = x + iy, \quad x, y \in \mathbb{R} \quad (8.14)$$

in cartesian form. Complex numbers also have a polar form,

$$z = \rho e^{i\phi}, \quad \tan \phi = y/x, \quad \rho = \sqrt{x^2 + y^2}. \quad (8.15)$$

Complex functions on the complex plane may be multivalued. For instance, consider the function  $\ln z$ . If we write a number like  $A = Ae^{2\pi ni}$ ,  $n \in \mathbb{Z}$ , then

$$\log A = \log(Ae^{2\pi ni}) = \log A + 2\pi ni, \quad (8.16)$$

so the log is infinitely multivalued. We consider complex functions

$$f : \mathbb{C} \rightarrow \mathbb{C}, \quad (8.17)$$

mapping some  $z \in \mathbb{C}$  to  $w \in \mathbb{C}$ , and since the output is complex, we can write

$$f(z) = u(x, y) + iv(x, y) \quad (8.18)$$

where now  $u, v : \mathbb{C} \rightarrow \mathbb{R}$  are real-valued functions. Arbitrary complex functions are clearly as complicated as real functions, but there is a special subset of complex functions which are more constrained in interesting ways. These are the *analytic functions*.

Lecture 9.

**Monday, January 27, 2020**

*"Suppose you have a transparent beach ball. And you take one of these laser pointers that fools like to shine at airplanes and such things, and you shine it through your beach ball."*

—Nemanja Kaloper

Today's material is drawn from Arfken 6th edition, since the 7th edition has some sort of error we'll consider shortly. Consider a complex-valued function  $f(z) : \mathbb{C} \rightarrow \mathbb{C}$ . Note that complex functions need not be one-to-one, and in fact they can even be multivalued if we're not careful (cf. our discussion of the function  $\ln z, z \in \mathbb{C}$ ).

We can define a *Banach space* as a set such that for any two points  $z_1, z_2 \in \mathbb{C}$ , there exists a smooth function  $f : [0, 1] \rightarrow \mathbb{C}$  such that  $f(0) = z_1, f(1) = z_2$ .<sup>10</sup> This allows us to talk about derivatives of functions:

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \quad (9.1)$$

and the derivative  $\frac{df}{dz}$  through a point must be unique and independent of the path.

The existence and uniqueness of the derivative for complex functions is related to a set of constraints known as the Cauchy-Riemann conditions. That is,

$$\exists f'(z) \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad (9.2)$$

where our complex function is

$$f = u + iv, \quad u, v : \mathbb{C} \rightarrow \mathbb{R}, \quad (9.3)$$

and points in the domain are given by

$$z = x + iy, \quad x, y \in \mathbb{R}. \quad (9.4)$$

Let us consider two possible displacements,

$$\Delta z = \Delta x, \quad \Delta z = i\Delta y. \quad (9.5)$$

In the first case, the derivative takes the form

$$\frac{df}{dz} = \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (9.6)$$

in the limit as  $\Delta x \rightarrow 0$ . In the second case, we instead have

$$\frac{df}{dz} = \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \quad (9.7)$$

Now we equate real and imaginary parts and find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (9.8)$$

<sup>10</sup>This seems to be a statement of the vector space being complete, in the sense we can define limits.

Hence the existence of the derivative implies Cauchy-Riemann.

We shall now prove the converse, that Cauchy-Riemann implies the existence of the derivative. Suppose we have a path through a point  $z$ , and we are given the appropriate derivatives of  $u$  and  $v$  for some function  $f$ . Then

$$\Delta f = \left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left( \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right). \quad (9.9)$$

Hence

$$\frac{\Delta f}{\Delta z} = \frac{\left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left( \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right)}{\Delta x + i \Delta y}. \quad (9.10)$$

Let us now apply Cauchy-Riemann and eliminate the  $v$  derivatives. That is,

$$\frac{\Delta f}{\Delta z} = \frac{\left( \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left( -\frac{\partial u}{\partial y} \Delta x + \frac{\partial u}{\partial x} \Delta y \right)}{\Delta x + i \Delta y} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}. \quad (9.11)$$

It follows that all the path dependence in  $\Delta x$  and  $\Delta y$  drops out, so the derivative exists and moreover it is path-independent. We could also write

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad (9.12)$$

using Cauchy-Riemann.

Now consider the function

$$z^2 = (x + iy)^2 = \underbrace{x^2 - y^2}_u + i \underbrace{2xy}_v. \quad (9.13)$$

We see that

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}. \quad (9.14)$$

So this function is really analytic, as we might expect. In contrast,

$$|z| = \sqrt{x^2 + y^2} \quad (9.15)$$

is not analytic, since  $v = 0$  for this function but  $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \neq 0$  in general.

Since we have

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad (9.16)$$

we can now import all our results from regular calculus of real variables. The chain rule and the product rule all follow in the usual way.

Suppose moreover that the second derivatives of  $u$  and  $v$  exist. Consider deriving the first with respect to  $x$ , and the second with respect to  $y$ . That is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \partial y}. \quad (9.17)$$

By the equality of mixed partials, we learn that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}, \quad (9.18)$$

or equivalently

$$\Delta u = 0, \Delta v = 0, \quad (9.19)$$

since we can perform the same argument for  $v$  as well. This states exactly that  $u$  and  $v$  are *harmonic functions*; they satisfy the 2D Laplace equation.

Consider this in the form

$$0 = \delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y. \quad (9.20)$$

This defines a trajectory along an equipotential. Then

$$\left. \frac{\Delta y}{\Delta x} \right|_{u=\text{constant}} = -\frac{\partial_x u}{\partial_y u} = +\frac{\partial_y v}{\partial_x v}. \quad (9.21)$$

This tells us that in fact  $u$  and  $v$  are related in a special way—in fact, the equipotentials of  $u$  and  $v$  are orthogonal.

To understand the idea of a “point at infinity,” let us place a two-sphere on the complex plane, with the south pole at the origin of the complex plane. Suppose we trace a line from the north pole to a point on the sphere, and then extend that line until it intersects the complex plane. Clearly, this is a one-to-one mapping between the plane and the sphere. But where is the point at the north pole mapped to? Effectively, it is mapped to all the points on a circle of infinite radius.<sup>11</sup>

We can then define a Riemann sum on some path  $f(z)$  starting at  $z_0$  and ending at  $z$ . We can discretize the path as  $\{z_i\}$  and sample along the way as

$$\sum_i f(z_i) \Delta z. \quad (9.22)$$

If the limit  $\Delta z \rightarrow 0$  of this sum exists, then we say that

$$\lim_{\Delta z \rightarrow 0} \sum_i f(z_i) \Delta z \equiv \int_{z_0}^z f(z') dz'. \quad (9.23)$$

In fact, we can write

$$\int_C f(z) dz = \int (u + iv)(dx + idy) = \int (u dx - v dy) + i \int (u dy + v dx). \quad (9.24)$$

Lecture 10.

**Wednesday, January 29, 2020**

*“This is just his Majesty Sir Newton. Or Sir Issac, shall we say. You know, I think the purpose of old England was to give birth to Newton. Everything else we can discount.”*

—Nemanja Kaloper

Last time, we defined integrals of complex functions in terms of the limit of their Riemann (discretized) sums. In general, the integral between two endpoints will depend on the path taken to get there. But we know from classical mechanics that some forces are conservative, i.e. the integrals of certain functions are independent of path.

What is the condition for the integral of a function to be independent of path? This is the same as asking when any integral around a closed loop is zero, and the answer is provided by the following theorem.

**Theorem 10.1** (Cauchy’s integral theorem). *If a function  $f(z) : \mathbb{C} \rightarrow \mathbb{C}$  is analytic in and around a curve  $C$  in a region which is simply connected,<sup>12</sup> then*

$$\oint_C f(z) dz = 0. \quad (10.2)$$

Let us compute the integral

$$\oint_{|z|=1} dz z^n. \quad (10.3)$$

Since  $z = e^{i\phi}$  on this contour,  $dz = ie^{i\phi} d\phi$ , so

$$\oint_{|z|=1} dz z^n = i \int_0^{2\pi} d\phi e^{i\phi(n+1)} = \frac{i}{i(n+1)} e^{i\phi(n+1)} \Big|_0^{2\pi} = 2\pi i \delta_{n,-1}, \quad (10.4)$$

where we take  $n \in \mathbb{Z}$ .

One could imagine doing this integral around a square of corners  $a, b, c, d$ . Then

$$\oint dz z^n = \frac{z^{n+1}}{n+1} \Big|_{\text{endpoints}} = \frac{1}{n+1} (b^{n+1} - a^{n+1} + c^{n+1} - b^{n+1} + \dots) \quad (10.5)$$

where all the terms will cancel. Except in the case  $n = -1$ , where we get logs instead, i.e. a sum

$$\ln(b/a) + \ln(c/b) + \ln(d/c) + \ln(ae^{2\pi i}/d) = 2\pi i. \quad (10.6)$$

<sup>11</sup>This construction is better known as the Riemann sphere.

<sup>12</sup>Simply connected means informally that the region has no holes. Formally it means that the fundamental group  $\pi_1$  is trivial. That is, any closed curve is homotopic (can be continuously deformed) to a point.

So the case  $n = -1$  (which we'll later recognize as that of a simple pole) is special.

We also claimed last time that since  $f(z)$  can be written as real and imaginary parts and similarly  $z$  can be written in terms of real and imaginary parts, then

$$\int_C f(z) dz = \int (u + iv)(dx + idy) = \int (u dx - v dy) + i \int (u dy + v dx). \quad (10.7)$$

Recall from vector calculus that

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_A d\mathbf{S} (\nabla \times \mathbf{A}). \quad (10.8)$$

If we like, this is really a definition of the curl, as the circulation of a vector field around an infinitesimal loop. If we write this in terms of components, we have

$$\oint_C (V_x dx + V_y dy) = \int d\mathbf{S} \cdot (\partial_y V_x - \partial_x V_y) \hat{\mathbf{z}}. \quad (10.9)$$

Let's apply this to our integral. We find that

$$\oint_C dz f(z) = \int dA (\partial_y u + \partial_x v) + i \int dA (\partial_y v - \partial_x u) = 0 \quad (10.10)$$

by the Cauchy-Riemann conditions. However, this definition of the curl relies on not just the existence of the derivatives of  $A$  but their continuity. The full theorem is stronger. But let us take this as a first proof that if a function is analytic in a simply connected region, then its integral around any closed loop is zero.

Let us suppose now that we have a (planar) region that we break up with a fine mesh. Since the border of each mesh region in the interior is counted twice, but with opposite orientation, then

$$\sum_i \oint_{C_i} f(z) = \oint_C f(z). \quad (10.11)$$

We now pick a point  $z_i$  inside each cell  $C_i$  and define

$$\delta(z, z_i) = \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i). \quad (10.12)$$

Notice that for

$$|z - z_i| < \Delta, \quad |\delta(z, z_i)| < \epsilon. \quad (10.13)$$

That is, we can make the difference between the discretized derivative at  $z_i$  and the true derivative  $f'(z_i)$  arbitrarily small.

Then we rewrite  $f(z)$  in terms of  $\delta$ :

$$\oint f(z) = \sum \left[ \oint_{C_i} \delta(z - z_i) + \oint f'(z_i)(z - z_i) + \oint f(z_i) \right]. \quad (10.14)$$

But if we take the limit as the limit gets arbitrarily fine, this last term is just the integral of a constant over a closed contour. This is zero, since we are integrating  $f(z) = 1$  around a closed contour, and 1 is clearly analytic. The second term vanishes by the same argument, since  $z - z_i$  is also analytic. We're left with

$$\oint f(z) dz = \sum \oint_{C_i} \delta(z - z_i) dz < \epsilon A, \quad (10.15)$$

and since this can be made arbitrarily small, we find that the integral is zero.

Lecture 11.

**Friday, January 31, 2020**

*"We don't know what this does on the circle. Except we \*do\* know. We just haven't realized that we know."*

—Nemanja Kaloper



Last time, we discussed Cauchy's integral theorem. There's a way to deal with contour integrals near singular points. The contour integral over two contours that can be continuously deformed into one another without crossing singularities have the same value. This is precisely the Gauss's law statement that the flux through two surfaces that can be continuously deformed into each other is equal, provided that we have not enclosed any new charge.

Let's take a function

$$\frac{f(z)}{z - z_0}, \quad (11.1)$$

where  $z_0 \in \mathbb{C}$  is some point in the complex plane and  $f(z)$  is analytic in some domain of analyticity including  $z_0$ . Consider the integral on a contour within the domain of analyticity which contains  $z_0$ ,

$$\oint_C \frac{f(z)}{z - z_0}. \quad (11.2)$$

This integral is nonzero, since it includes a singular point.

How can we compute its value? Consider as our contour  $C_\rho$ , a circle centered on  $z_0$  of radius  $\rho$ . Each point on the circle is  $z = z_0 + \rho e^{i\phi}$ . Now we can compute the contour integral explicitly. By flux conservation (this is an isolated singularity), any other contour enclosing  $z_0$  has the same integral as over  $C_\rho$ . The integral is

$$\oint_{C_\rho} dz \frac{f(z)}{z - z_0} = i \int_0^{2\pi} d\phi \rho e^{i\phi} \frac{f(z_0 + \rho e^{i\phi})}{\rho e^{i\phi}} = i \int_0^{2\pi} d\phi f(z_0 + \rho e^{i\phi}). \quad (11.3)$$

If we now take the limit as  $\rho \rightarrow 0$ , we can use the fact that  $f$  is analytic and therefore continuous at  $z_0$  so that in fact

$$\oint_{C_\rho} dz \frac{f(z)}{z - z_0} = 2\pi i f(z_0). \quad (11.4)$$

We can combine this with our previous knowledge of integrals of analytic functions to say that the integral  $\oint_C dz \frac{f(z)}{z - z_0}$  is zero if  $z_0$  is outside the contour, and  $2\pi i f(z_0)$  if it is inside.

Moreover, let us write

$$f(z_0 + \delta z) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - (z_0 + \delta z)}. \quad (11.5)$$

Since

$$f(z_0) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z_0}, \quad (11.6)$$

it follows that expanding to first order and taking the limit  $\delta z \rightarrow 0$ , we get

$$f'(z_0) = \lim_{\delta z \rightarrow 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \frac{1}{2\pi i} \oint dz \frac{f(z)}{(z - z_0)^2}. \quad (11.7)$$

Equivalently we could just take a derivative with respect to  $z_0$  directly. In general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}. \quad (11.8)$$

This tells us that for analytic functions, it's not just the case that the first derivatives exist, but in fact all higher derivatives exist, for the integrand is perfectly well-behaved.

We shall now prove the converse of Cauchy's theorem, that the contour integral over any closed loop vanishing implies analyticity. This is known as *Morera's theorem*, and we'll do it in a way that corrects an error in the textbook.

**Theorem 11.9 (Morera).** *If a function  $f(z)$  is continuous in a simply connected region  $R$  and  $\oint_C f(z)dz = 0$  for every closed contour  $C$  within  $R$ , then  $f(z)$  is analytic throughout  $R$ .*

*Proof.* We are given that  $\oint f(z) = 0$  over any closed loop in  $R$ . it follows that for such a loop, we can take two points  $z_1, z_2$  on the loop and split it into two paths, such that the integral over each path is equal and opposite. Equivalently, the integral over any path depends only on the endpoints, so we can write

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} dz f(z) \quad (11.10)$$

in terms of an “indefinite integral”  $F$  which is single-valued throughout the region  $R$ . Now we write this expression in the following way:

$$\begin{aligned} \frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) &= \frac{\int_{z_1}^{z_2} dz f(z)}{z_2 - z_1} - f(z_1) \\ &= \frac{\int_{z_1}^{z_2} dz (f(z) - f(z_1))}{z_2 - z_1}. \end{aligned}$$

At this point, the book claims that because  $f$  is continuous we can replace it by its derivative. But that assumes that the derivative of  $f$  exists, which is what we’re trying to prove! The right way to say this is as follows. This expression is true for any  $z_1$  and  $z_2$ , so now continuity implies that  $f(z) - f(z_1) < \epsilon$  as  $z \rightarrow z_1$ , where  $\epsilon$  can be made arbitrarily small. Then

$$\lim_{z_2 \rightarrow z_1} \frac{\int_{z_1}^{z_2} dz (f(z) - f(z_1))}{z_2 - z_1} < \lim_{z_2 \rightarrow z_1} \frac{\int_{z_1}^{z_2} \epsilon dz}{z_2 - z_1} = \lim_{z_2 \rightarrow z_1} \epsilon \frac{z_2 - z_1}{z_2 - z_1} = \lim_{z_2 \rightarrow z_1} \epsilon = 0. \quad (11.11)$$

Hence

$$\lim_{z_2 \rightarrow z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = 0 \implies F'(z_1) = f(z_1). \quad (11.12)$$

Since  $f$  is well-defined (a fortiori, since it is continuous) in  $R$ , it follows that  $F$  is analytic everywhere in  $R$ . Since  $F$  is analytic, its derivative  $f(z)$  is also analytic.  $\square$

Consider the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (11.13)$$

on a circle of radius  $R$ . If  $|f| < M$  (the function is bounded on the circle), then we can prove that

$$|a_n| < \frac{M}{R^n}. \quad (11.14)$$

For notice that

$$a_n = \frac{1}{n!} f^{(n)}(z)|_{z=0} = \frac{1}{2\pi} \oint dz \frac{f(z)}{z^{n+1}}. \quad (11.15)$$

Then

$$|a_n| = \frac{1}{n!} |f^{(n)}(z)|_{z=0} = \frac{1}{2\pi} \left| \oint dz \frac{f(z)}{z^{n+1}} \right| < \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{M}{R^n} = \frac{M}{R^n}. \quad (11.16)$$

In fact, it follows that the only function which is analytic and bounded as  $R \rightarrow \infty$  is in fact the constant function.

Let us now prove the fundamental theorem of algebra, that a polynomial of order  $N$  has  $N$  complex roots counting degeneracy. Consider a polynomial  $P(z)$ , and suppose that  $P(z)$  has no root. Then the quantity  $1/P(z)$  is perfectly finite and bounded, which means that by our theorem,  $1/P(z)$  is actually constant, so it’s not a polynomial.

Thus it must have at least one root  $z_0$ , and we can define  $\hat{P} = \frac{P(z)}{z - z_0}$ . This process can terminate no sooner than after  $N$  steps, when what’s left really is constant.

The real power of analyticity is in the procedure of analytic continuation, however. Consider a point  $z_0$ , where the nearest singularity of a function  $f(z)$  lies at  $z_1$ . The value of the function  $f(z')$  at some  $z'$  nearby is then

$$f(z') = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z'} = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z_0} \sum \left( \frac{z' - z_0}{z - z_0} \right)^n = \sum \frac{(z' - z_0)^n}{2\pi i} \oint dz \frac{f(z)}{(z - z_0)^{n+1}}. \quad (11.17)$$

This is a Taylor series at  $z'$ , and now

$$f(z') = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z' - z_0)^n. \quad (11.18)$$

So in fact the function at a point and its derivatives gave us the function everywhere within a circle up to the closest singularity.

**Monday, February 3, 2020**

*"You know this from Lewis Carroll's Jabberwocky.<sup>13</sup> 'The time has come to talk of many things. Kings and cabbages and whether the sea is boiling hot and whether pigs have wings.' Singularities are the pigs with wings."*

—Nemanja Kaloper

*Note.* Homework submission deadline has been extended to 11 PM on Fridays. We'll get confirmation of this via email.

Last time, we began discussing analytic continuation. Analytic functions are highly constrained in their domain of analyticity. If we know the function and its derivatives at some point  $z_0$  and the nearest singularity is at  $z_1$ , then we can Taylor expand the function in a circular domain centered on  $z_0$  of radius  $|z_0 - z_1|$  given by

$$f(z') = \sum_{n=0}^{\infty} \frac{c_n}{n!} (z' - z_0)^n, \quad (12.1)$$

with the coefficients given by the contour integral

$$c_n = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}. \quad (12.2)$$

The intuition here is by analogy to charge. Potentials were expanded in multipole moments, and we shall see that complex functions have a similar expansion in terms of how singular they are. Suppose we have a complex function  $f$  with some singularity at  $z_1$  and  $z_2$ .

We can construct an annular region between  $z_1$  and  $z_2$ , and if we wish to find the value of some  $z'$  in the region, it is given by

$$f(z') = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z'}, \quad (12.3)$$

where the contour is taken to be  $C_1^{(-)} \cup C_2 \cup I^+ \cup I^-$ .

Now we can write this integral explicitly as

$$f(z) = \frac{1}{2\pi i} \left[ \oint_{C_2} \frac{f(z)}{z - z'} - \oint_{C_1} \frac{f(z)}{z - z'} \right] \quad (12.4)$$

Since

$$z - z' = (z - z_0) \left( 1 - \frac{z' - z_0}{z - z_0} \right), \quad (12.5)$$

we can rewrite

$$\frac{1}{z - z'} = \frac{1}{z - z_0} \sum_n \left( \frac{z' - z_0}{z - z_0} \right)^n \quad (12.6)$$

as before and therefore

$$f(z') = \frac{1}{2\pi i} \left( \sum_n (z' - z_0)^n \oint_{C_2} \frac{f(z)}{(z - z_0)^{n+1}} \right) - \frac{1}{2\pi i} \oint_{C_1} dz \frac{f(z)}{z - z'}. \quad (12.7)$$

The second integral would just be zero by Cauchy's theorem if there were no central region of non-analyticity. However, we can handle this also.

$$z - z' = z - z_0 - (z' - z_0). \quad (12.8)$$

Now we can make a similar expansion as

$$\left[ -\frac{z - z_0}{z' - z_0} + 1 \right] (z' - z_0), \quad (12.9)$$

<sup>13</sup>Editor's note: The poem was actually *The Walrus and the Carpenter*, which appeared in *Through the Looking-Glass*.

and this first term can be expanded in the same way since now  $|z - z_0| < |z' - z_0|$ . Hence the second term becomes

$$\frac{1}{2\pi i} \sum_{n=0}^{\infty} (z' - z_0)^{-(n+1)} \oint dz f(z) (z - z_0)^n. \quad (12.10)$$

These are equivalent to the interior and exterior expansions of the electromagnetic potential in spherical coordinates.

We can change the dummy variable and rewrite the exterior expansion as

$$\frac{1}{2\pi i} \sum_{n=-1}^{\infty} (z' - z_0)^n \oint dz f(z) (z - z_0)^{-(n+1)}. \quad (12.11)$$

It follows that we can write  $f(z')$  as a single sum

$$f(z') = \sum_{n=-\infty}^{\infty} c_n (z' - z_0)^n, \quad c_n \equiv \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}. \quad (12.12)$$

If  $f$  is analytic at  $z_0$ , then for all  $n = -1, -2, -3, \dots$  we get back the ordinary Taylor expansion because  $1/(z - z_0)^{n+1}$  will just be a positive power of  $z - z_0$ . Conversely if  $f$  has some sort of a singular point at  $z_0$ , we will pick up nontrivial  $c_n$  for negative  $n$ . These integrals for negative  $n$  are all zero iff the function is analytic at  $z_0$ .

If there are only finitely many  $c_n$  for negative  $n$  which are nonzero, then we call the function *meromorphic*. These sorts of singularities are called poles. Functions which moreover have at worst first-order poles ( $c_{-1} \neq 0$ ) are called holomorphic. If a function has infinitely many terms in its pole expansion we have instead an essential singularity.

**Branch points** We've seen that functions like  $\ln z$  are multivalent; they have multiple consistent values for the same complex input.

Consider first the function  $z^2$ . If we define a ray  $z = \rho e^{i\phi}$ , our function maps this ray at angle  $\phi$  to a new ray at angle  $2\phi$ . It follows that the image is  $2\phi = 2\pi \sim 0$  for  $\phi = \pi$ , and as we sweep out our ray from 0 to  $2\pi$ , the image of this ray will go around twice.

But now we have a difficulty when we want to define the inverse. That is, if we write

$$\sqrt{z} = \sqrt{\rho} e^{i\phi/2}, \quad (12.13)$$

then our function is only single-valued on some special region. There's a special line, the *branch cut*, which we are forbidden to cross. Branch cuts provide us with memory of which branch we are on, which gives us power to compute difficult integrals.<sup>14</sup>

Lecture 13.

Wednesday, February 5, 2020

*"Now the little beam-me-up-Scotty elevator is only this segment here to here. The only way to take the elevator is to close the loop around one only."*

—Nemanja Kaloper

Last time, we introduced branch points and branch cuts. Branch cuts tell us where we cross from one copy of the complex plane (Riemann sheet) to another.

A natural question to ask is where we should put our branch cut. In fact, it doesn't matter so long as our branch cut connects branch points. Notice for instance that  $\sqrt{z}$  has a branch point at 0, and it is also singular at the point at infinity. The placement of the branch cut is equivalent to us picking the fundamental domain of the Fourier series, as  $0, 2\pi$  or  $-\pi, \pi$ .

**Example 13.1.** Consider the function

$$f(z) = \sqrt{z^2 - 1} = \sqrt{z+1} \sqrt{z-1}. \quad (13.2)$$

<sup>14</sup>This is how many of the integrals in Gradshteyn and Ryzhik are done.

If we use  $z = 1/w$  and consider the behavior around  $w = 0$ , we find that

$$\sqrt{\frac{1}{w^2} - 1} = \frac{1}{w} \sqrt{1 - w} \sqrt{1 + w}. \quad (13.3)$$

It now makes sense to study this around  $w = 0$ . Indeed,  $w = 0$  is a pole with residue 1. However, it is not a branch point.

Let's go back to the original function. If we encircle the branch point at  $z = 1$  by a contour of radius  $\rho$ , we can parametrize this as  $z = 1 + \rho e^{i\phi}$  and then

$$f(z) = \sqrt{2 + \rho e^{i\phi}} \sqrt{\rho e^{i\phi}}. \quad (13.4)$$

We see that the second factor will pick up a nontrivial phase factor as we run from 0 to  $2\pi$ , whereas the second one is some number whose phase never exceeds  $\pi/2$ .<sup>15</sup>

The same reasoning applies to the branch point at  $z = -1$ . However, if we encircle both branch points, we pick up a phase factor of  $-1$  from each branch point, so the function is indeed single-valued so long as we don't cross the branch cut.

**Example 13.5.** Let us study  $e^z = e^x e^{iy}$ . For  $y = 0$ , we have the real exponential  $e^x$ . The real line is mapped to the half-line under the exponential map, i.e.  $(-\infty, \infty) \mapsto [0, \infty)$ . If we look at the image of a line  $x + y_0$  for a fixed  $y_0 \neq 0$  and  $x \in \mathbb{R}$ , we see that  $y_0$  becomes the phase of the ray.

We can keep doing this until  $y_0 = 2\pi$ , at which point our line is mapped back to the original ray  $[0, \infty)$ . We see that in fact infinitely many points are mapped to the same ray, i.e. we get strips of width  $2\pi$  in the domain all mapped to the complex plane in the image.

We are also led to the idea of analytic continuation, i.e. riding the flux lines of a harmonic function.

**Example 13.6.** Consider the series

$$\sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (13.7)$$

and

$$\sum_{n=0}^{\infty} i^{n-1} (z-i)^n. \quad (13.8)$$

Notice the first one is simply

$$\sum (1-z)^n = \frac{1}{1-1(-z)} = \frac{1}{z}. \quad (13.9)$$

In fact, the other is similar; it is just  $1/z$  expanded around the point  $z = i$ .

Lecture 14.

**Friday, February 7, 2020**

*"These days you see a lot of crap on the internet asking for the abandonment of all rules because we had certain expectations and hopes that did not come true. Just because certain of these confusions have failed to materialize doesn't mean you should adopt perpetuum mobile and hundred-headed dragons and so on.*

*...unless you see one. Which I would love because it would keep me in business for another few decades."*

—Nemanja Kaloper

We had two expansions of  $1/z$ , as

$$\frac{1}{z} = \begin{cases} \sum (-1)^n (z-1)^n \\ \sum i^{n-1} (z-i)^n \end{cases} \quad (14.1)$$

<sup>15</sup>There's a book by E. Cartan on spin, which is related to these ideas. Spin  $1/2$  objects must undergo a rotation by  $4\pi$  to come back to the identity.

The radius of convergence for each series is 1. How could we prove that these series are one and the same? We need to look at the overlap and show that the series agree on an open set within the overlap region. Let us parametrize

$$z = \left(\alpha + \frac{1}{2}\right)(1 + i). \quad (14.2)$$

For  $\alpha$  real, we're just on the diagonal line between the axes. If we go a little bit off ( $\alpha$  complex), then we can explore the overlap region more. That is, our first expansion becomes

$$\sum (-1)^n \left( \left(\alpha + \frac{1}{2}\right)(1 + i) - 1 \right)^n = (-1)^n \left( \alpha + \frac{1}{2} - \frac{1}{1+i} \right)^n (1 + i)^n. \quad (14.3)$$

Now we use the binomial expansion to write

$$\sum_{n=0}^{\infty} (1 + i) \sum_{j=0}^n \binom{n}{j} \alpha^j \left( \frac{1}{1+i} - \frac{1}{2} \right)^{n-j} (-1)^j. \quad (14.4)$$

We can now change the order of summation as

$$\sum_{j=0}^{\infty} \sum_{n=j}^{\infty} (1 + i) \binom{n}{j} \alpha^j \left( \frac{1}{1+i} - \frac{1}{2} \right)^{n-j} (-1)^j. \quad (14.5)$$

If we now shift the dummy index  $n$  to start at 0 then we get

$$\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (1 + i) \binom{n+j}{j} \alpha^j \left( \frac{1}{1+i} - \frac{1}{2} \right)^n (-1)^j. \quad (14.6)$$

As it turns out, these binomial coefficients correspond to an expansion of some function. Recall that the ordinary binomial coefficients come from expanding  $(1 - x)^\alpha = 1 + \alpha + \frac{\alpha(\alpha-1)}{2!} + \dots$ . In our case,

$$\sum_{n=j}^{\infty} \binom{n}{j} x^{n-j} = \frac{1}{(1-x)^{j+1}}. \quad (14.7)$$

It follows that our series is really a single sum

$$\sum_{j=0}^{\infty} \frac{(-1)^j 2^{j+1}}{1+i} \alpha^j. \quad (14.8)$$

If we repeat the process (exercise, or in the textbook) for the second series, we find that these series have the same representation in terms of  $\alpha$  over some range of  $\alpha$  (in fact,  $|\alpha| < 1/2$ ). This is an example of analytic continuation—we've extended the region of validity of our expansion in order to show that two a priori different series in fact represent the same analytic function everywhere.

Notice that in principle a function  $f$  depends on both  $x$  and  $y$ , but analyticity gave us stricter conditions on their real and imaginary parts. We said that for a general function  $\phi$  with  $\Delta\phi = 0$ ,

$$\partial_z \partial_{z^*} \phi = 0 \implies \phi = \phi_1(z) + \phi_2(z^*), \quad (14.9)$$

a sum of a holomorphic and antiholomorphic part. One can show that for a function,

$$\frac{\partial}{\partial z^*} f = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z^*}. \quad (14.10)$$

Since  $z = x + iy, z^* = x - iy$ , we can write

$$x = \frac{z + z^*}{2}, \quad y = \frac{z - z^*}{2i} \quad (14.11)$$

such that we can recover the Cauchy-Riemann conditions from the requirement that our function is purely holomorphic,  $\frac{\partial f}{\partial z^*} = 0$ .

**Residue theorem** Suppose we have a complex function with a Laurent series,

$$f = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \quad (14.12)$$

We can now integrate over closed contours  $C$  to get

$$\oint_C f dz = \sum_{n=-\infty}^{\infty} a_n \oint_C (z - z_0)^n. \quad (14.13)$$

If  $z_0$  is at worst an isolated singularity, then we can rewrite this contour integral in terms of the contour around a small circle enclosing only  $z_0$ , as

$$\sum a_n \oint_{C_r} (z - z_0)^n = 2\pi i a_{-1}. \quad (14.14)$$

Hence integrating over the contour around some point  $z_0$  which is an isolated singularity only picks out the  $a_{-1}$  coefficient.

If there are multiple isolated singularities then

$$\oint f(z) dz = \sum_i \oint_{C_i} f(z) dz = 2\pi i \sum_i a_{-1}^{(i)}. \quad (14.15)$$

Hence we can compute integrals that look arbitrarily bad by taking the expansions of the function around the singular points and adding up the  $a_{-1}$  coefficients of the expansions at each point.

The coefficient of the  $1/z$  term is called the residue. We say that

$$\oint_C f(z) dz = 2\pi i \sum_{z_i \in \mathbb{C}} \text{Res} f(z_i). \quad (14.16)$$

What is the integral

$$\oint_{|z|=4} \frac{1}{(z-1)(z-2)(z-3)}? \quad (14.17)$$

We claim it is zero. Why? The reason is that this function goes to zero “real fast” as the radius of the circle goes to infinity. For the integral of the charges enclosed must be the negative of the integral of the charges outside, since we can flip the orientation of the curve and exchange inside and outside.<sup>16</sup>

Lecture 15.

**Monday, February 10, 2020**

*“What’s your favorite Fourier series?” “They’re all the same.”*

—Mark Samuel Abbott and Nemanja Kaloper

*Note.* As a logistical point, there will be no Green’s functions on the midterm. The focus will be on Fourier series and complex analysis.

We have found a wonderful generalization of the Cauchy integral theorem; it is the Cauchy residue theorem, which says we can compute the contour integral bounding some region by summing up the *residues* of the poles contained within. That is,

$$\oint_C f dz = 2\pi i \sum_{z_i \in \mathbb{C}} \text{Res}(z_i). \quad (15.1)$$

We can study the point at infinity with an inversion map,  $w = 1/z$ .<sup>17</sup>

It follows that if we reverse the direction of the contour and traverse it clockwise, we capture all the singularities outside the contour. Thus if we sum over all the residues in the entire complex plane,

$$\sum_{z_i} \text{Res}(z_i) = 0. \quad (15.2)$$

<sup>16</sup>This is a good check on some of the homework problems. The sum of all the residues in the entire complex plane must be zero.

<sup>17</sup>Further reading on conformal transformations can be found in W.K. Tung, and also Lie Groups and Algebras by Howard Georgi.

Let us also note that

$$\oint dz f(z) = - \oint \frac{dw}{w^2} g(w). \quad (15.3)$$

The residue is the  $a_{-1}$  coefficient of  $g(w)/w^2$ , or equivalently the  $a_1$  coefficient of  $g(w)$ .

Notice that for a function with a simple pole,

$$f = \frac{a_{-1}}{z - z_i} + f_A, \quad (15.4)$$

we can easily compute the residue by

$$(z - z_i)f|_{z \rightarrow z_i} = a_{-1} + (z - z_i)f_A|_{z \rightarrow z_i} = a_{-1}. \quad (15.5)$$

The residue of a pole of order  $n$  is a little more complicated, but not much. If we have

$$f(z) = \frac{a_{-n}}{(z - z_i)^n} + \dots + \frac{a_{-1}}{z - z_i} + f_A. \quad (15.6)$$

Then

$$(z - z_i)^n f = a_{-n} + a_{-(n-1)}(z - z_i) + \dots + a_{-1}(z - z_i)^{n-1} + f_A(z - z_i)^n. \quad (15.7)$$

Now we wish to extract the  $a_{-1}$  coefficient. But that's no problem. We take  $n - 1$  derivatives of this expression and then set  $z \rightarrow z_i$ . All the terms  $a_{-n}$  to  $a_{-2}$  will be killed off by the derivatives, while all the terms in  $f_A$  vanish when we take  $z \rightarrow z_i$ . It follows that

$$a_{-1} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_i)^n f] \quad (15.8)$$

Now the time has come to deal with singularities on the contour itself. We'll do this by using a technique from real analysis. Consider first the integral

$$\int_{-b}^a \frac{dx}{x}, \quad (15.9)$$

with  $a, b > 0$ . This is clearly ill-defined since  $1/x$  diverges as  $x \rightarrow 0$ . But we can consider a limit. Suppose we take the integral from  $-b$  to  $-\epsilon^{(-)}$  and  $\epsilon^{(+)}$  to  $a$ . That is,

$$\int_{-b}^a \frac{dx}{x} = \int_{-b}^{-\epsilon^{(-)}} \frac{dx}{x} + \int_{\epsilon^{(+)}}^a \frac{dx}{x} = \ln \left( -\frac{\epsilon^{(-)} a}{-b \epsilon^{(+)}} \right). \quad (15.10)$$

We see that the value depends on the cutoffs  $\epsilon^\pm$ . If we set  $\epsilon^{(-)} = \epsilon^{(+)}$ , then this integral has a sensible sort of limit, which we call the Cauchy principal value:

$$\mathcal{P} \int_{-b}^a \frac{dx}{x} = \ln \left( -\frac{\epsilon a}{-b \epsilon} \right) = \ln(a/b). \quad (15.11)$$

Consider now the integral

$$\int_0^\infty dx \frac{\sin x}{x}. \quad (15.12)$$

Near  $x = 0$  this is regular since  $\sin x \sim x$  for small  $x$ , and out at infinity, the oscillations of the sines make this function converge faster than  $1/x$ . It's like the alternating harmonic series.

We can also write this as an integral over complex exponentials:

$$\begin{aligned} \int_0^\infty dx \frac{e^{ix} - e^{-ix}}{2ix} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2i} \int_\epsilon^\infty dx \frac{e^{ix}}{x} - \frac{1}{2i} \int_\epsilon^\infty dx \frac{e^{-ix}}{x} \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2i} \int_\epsilon^\infty dx \frac{e^{ix}}{x} + \frac{1}{2i} \int_{-\infty}^{-\epsilon} dx \frac{e^{+ix}}{x} \\ &= \mathcal{P} \int_{-\infty}^\infty dx \frac{e^{ix}}{x}. \end{aligned}$$

If we extend this to an integral in the complex plane we can *close the contour* and evaluate this by our usual Cauchy tricks. But the singularity lies on the contour. Should we include it or exclude it?

Well, if we deform the contour as a small semicircle of radius  $\epsilon$  about the singularity, either we enclose the contour and get an integral  $\int_\pi^0 i \epsilon d\phi e^{i\phi} \frac{a_{-1}}{\epsilon e^{i\phi}}$  or we miss the singularity but we pick up extra distance.



Either way the integral has value  $\pi ia_{-1}$ . Well, if we deform the contour as a small semicircle of radius  $\epsilon$  about the singularity, either we enclose the contour and get an integral  $\int_{\pi}^0 i\epsilon d\phi e^{i\phi} \frac{a_{-1}}{\epsilon e^{i\phi}}$  or we miss the singularity but we pick up extra distance. Either way the integral has value  $\pi ia_{-1}$ .

Lecture 16.

**Wednesday, February 12, 2020**

Let us continue our discussion of the principal value of the contour integral running over a singularity on the contour. The principal value can be written in two ways. One way, it is equal to the integral with the contour deformed to miss the pole as

$$\mathcal{P} = 2\pi i \sum \text{Res}f(z_i) - \int_{\pi}^0 dz f(z) \quad (16.1)$$

and the other way it includes it as

$$\mathcal{P} = 2\pi i \sum \text{Res}f(z_i) + 2\pi i \text{Res}(z_0) - \int_{\pi}^{2\pi} dz f(z). \quad (16.2)$$

The first integral around the semicircle is half of the integral around it,

$$\mathcal{P} = 2\pi i \sum \text{Res}f(z_i) + i\pi \text{Res}(z_0), \quad (16.3)$$

while the second integral has the correct sign

$$\mathcal{P} = 2\pi i \sum \text{Res}f(z_i) + 2\pi i \text{Res}(z_0) - i\pi \text{Res}(z_0) = 2\pi i \sum \text{Res}f(z_i) + i\pi \text{Res}(z_0). \quad (16.4)$$

We find that in both cases, the principal value of the integral is  $2\pi i$  times the sum of the residues inside plus half the residue of the pole on the contour.<sup>18</sup>

**Mittag-Leffler theorem** Suppose we have a meromorphic function (isolated singularities) where all poles are at worst first-order. Let us also suppose there are no poles at infinity.

We are given a set of singularities—their locations and their residues. WLOG we can take  $f(0) \neq 0$ , i.e. the function at the origin is nonzero. Then we claim that the function with these singularities and the given residues is

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left( \frac{1}{z - z_n} + \frac{1}{z_n} \right). \quad (16.5)$$

This is just a way of summing up the Laurent series about each pole.

Consider the integral

$$\oint_{C_R} dw \frac{f(w)}{w(w-z)} = 2\pi i \left[ \sum \frac{b_n}{z_n(z_n - z)} + \frac{f(0)}{-z} + \frac{f(z)}{z} \right]. \quad (16.6)$$

That is, when  $z \neq z_i$  for all  $i$ , we pick up the residues  $b_n$  at  $w = z_n$  and we pick up poles at  $w = 0$  and  $w = z$ . We can write the LHS as

$$\oint id\phi Re^{i\phi} \frac{f(Re^{i\phi})}{Re^{i\phi}(Re^{i\phi} - z)}. \quad (16.7)$$

Let us take the limit as the radius of the circular contour grows arbitrarily large. By assumption there are no poles at infinity, so the integral of the LHS is in fact zero and the sum runs over all poles, i.e.

$$\lim_{R \rightarrow \infty} \oint id\phi Re^{i\phi} \frac{f(Re^{i\phi})}{Re^{i\phi}(Re^{i\phi} - z)} \rightarrow 0. \quad (16.8)$$

Solving for  $f(z)$ , we find that

$$f(z) = f(0) + \sum \frac{b_n z}{z_n(z_n - z)}, \quad (16.9)$$

which is just our equation (16.5).<sup>19</sup>

<sup>18</sup>This can be generalized to higher-order poles, with adjustments. Higher-order poles go around a bit too fast for a single semicircle to work, but the argument will work.

<sup>19</sup>The theorem can be modified for higher-order poles. There, we just need  $(z_n - z)^n$  in the denominator.

**Example 16.10.** Consider the function  $f(z) = \tan z$  for complex  $z$ . This function has simple poles whenever  $\cos z = 0$ , i.e. for  $z_k = \frac{\pi}{2} + k\pi$ . We can write down the residues at these poles by l'Hôpital:

$$\lim_{z \rightarrow z_k} \frac{(z - z_k) \sin z}{\cos z} = -1. \quad (16.11)$$

Hence the residue is  $-1$  for all poles and we have

$$\tan z = \sum_{k=1}^{\infty} (-1) \left[ \frac{1}{z - (\pi/2 + k\pi)} + \frac{1}{\pi/2 + k\pi} \right] + \sum_{k=-1}^{-\infty} (-1) \left[ \frac{1}{z - (\pi/2 + k\pi)} + \frac{1}{\pi/2 + k\pi} \right]. \quad (16.12)$$

If we combine the sums, we can rewrite this as

$$\tan z = \sum_{k=1}^{\infty} (-1) \left[ \frac{2z}{z^2 - (2k+1)(\pi/2)^2} \right]. \quad (16.13)$$

This is effectively the potential we would get from a series of charges at odd multiples of  $\pi/2$ .

Let

$$f(z) = (z - z_0)^n g(z), \quad (16.14)$$

where  $g$  is analytic. Then we can compute the logarithmic derivative,

$$\frac{f'}{f} = \frac{n}{z - z_0} + \frac{g'}{g}. \quad (16.15)$$

If we now take the contour integral of both sides around a contour enclosing  $z_0$ , then we get

$$\oint \frac{f'}{f} = \oint \frac{n}{z - z_0} + \frac{g'}{g} = 2\pi i n. \quad (16.16)$$

If  $n$  is positive then we have a zero of  $f(z)$ ; if  $n$  is negative we have a pole. More generally if we have  $N$  zeroes and  $P$  poles then

$$\oint \frac{f'}{f} = 2\pi i (N - P). \quad (16.17)$$

This is basically counting the sum of winding numbers around poles/zeroes.

An immediate corollary is Rouché's theorem, which says that if  $|g| < |f|$  on some domain, then  $f + g$  has no more zeroes than the original function  $f$ . We can prove this

$$\oint \frac{f'}{f} = 2\pi i N_f, \quad \oint \frac{f' + g'}{f + g} = 2\pi i N_{f+g}. \quad (16.18)$$

We have

$$N_f = \Delta \arg f, \quad (16.19)$$

while

$$N_{f+g} = \Delta \arg(f + g) = \Delta \arg f(1 + g/f) = \Delta \arg f + \Delta \arg(1 + g/f). \quad (16.20)$$

Here,  $\arg$  pulls out the phase (argument) of the complex number. The first term is the same as  $N_f$ , while the second is zero. It follows that

$$1 + \frac{g}{f} < 1 + \epsilon e^{i\phi}. \quad (16.21)$$

Correction to something from last time. Last time, we talked about the generalization of the Mittag-Leffler theorem. The correction is this—for higher-order poles, we need to consider the behavior of the function at infinity, i.e. ensure that  $f(z)/z \rightarrow 0$  as  $z \rightarrow \infty$ . If there are higher-order poles at infinity we can write

$$\oint_{C_R} dw \frac{f(w)}{(w-z)^{n+1}}. \quad (17.1)$$

Suppose we have a function  $f(z)$  which admits a Laurent expansion

$$f(z) = \underbrace{\sum_{n=-N}^{-1} a_n(z-z_k)^n}_{S(z)} + f_A(z) \quad (17.2)$$

about a singular point  $z_k$ . All the poles can be denoted as a singular part  $S(z)$ . Now if we take the sum

$$f(z) = \sum_{n=1}^{\infty} [S_n(z) - S_n(z_0)] + f(z_0). \quad (17.3)$$

That is, we sum up all the singular parts and subtract off the  $1/z_n$  pieces, and add on the analytic part. Mittag-Leffler says that

$$f(z) = \sum_{n=1}^{\infty} [S_n(z)] + Q, \quad (17.4)$$

where  $Q$  is some constant offset. This is true for  $z$  or  $z_0$ , so when we extract all the singular pieces, what remains is the same no matter where we expand. In fact, this is just a complex analysis statement that the electrostatic potential (which satisfies Laplace's equation) is determined by the boundary conditions, possibly up to a constant. We'll get the proof by an email announcement later.

**Trigonometric integrals by contour integration** Suppose we have some trigonometric integral

$$\int_0^{2\pi} f(\sin \phi, \cos \phi) d\phi. \quad (17.5)$$

Well, we can easily rewrite

$$\sin \phi = \frac{1}{2i}(z - z^*) = \frac{1}{2i}(z + z^{-1}) \quad (17.6)$$

when  $z$  lies on the unit circle,  $|z| = 1$ , and similarly

$$\cos \phi = \frac{1}{2}(z + z^{-1}) \quad (17.7)$$

where we have taken

$$z = e^{i\phi}, \quad d\phi = \frac{dz}{iz}. \quad (17.8)$$

It follows that our integral is completely equivalent to the contour integral

$$\oint_{|z|=1} dz f\left(\frac{1}{2i}(z - z^{-1}), \frac{1}{2}(z + z^{-1})\right) = 2\pi i \sum \text{Res} f \quad (17.9)$$

for the residues inside the unit circle. If a pole lies on the unit circle, we use the principal value.

**Example 17.10.** Consider the integral

$$\int_0^{2\pi} d\phi \frac{1}{1 + a \cos \phi} \quad (17.11)$$

for  $a$  a real number. By our change of variables this is just

$$\oint \frac{dz}{iz} \frac{1}{1 + a(z + z^{-1})/2} = \oint \frac{dz}{i} \frac{1}{z + \frac{a}{2}(z^2 + 1)}. \quad (17.12)$$

Factoring, we find that this is

$$-\frac{2i}{a} \oint \frac{dz}{(z - z_+)(z - z_-)} \quad (17.13)$$

with

$$z_{\pm} = -\frac{1 \pm \sqrt{1-a^2}}{a}. \quad (17.14)$$

If  $a < 1$  then both these roots are real, and moreover one lies outside the unit circle while the other lies inside. If  $a > 1$  then the roots are complex and may both lie inside the unit circle.

For the  $a < 1$  case only  $z_-$  lies inside the circle, and we get

$$2\pi i \left( -\frac{2i}{a} \right) \frac{1}{z_- - z_+} = \frac{2\pi}{\sqrt{1-a^2}}. \quad (17.15)$$

**Example 17.16.** Consider the integral

$$\int_{-\infty}^{\infty} dx f(x). \quad (17.17)$$

Suppose that  $f(x)$  is a real-valued function on the real line, and moreover it can be extended into the complex plane with finitely many singularities in the upper half-plane.

We also need  $f$  to go to zero sufficiently quickly, i.e.

$$f(z) \rightarrow \frac{1}{|x|^{1+\epsilon}} \quad (17.18)$$

as  $x \rightarrow \infty$ . Moreover let us say that

$$zf(z) \rightarrow 0 \quad (17.19)$$

as  $|z| \rightarrow \infty$ .

We can compute this real integral by *closing the contour*, i.e. by integrating along the real axis from  $-R$  to  $R$  and then along a semicircular path in the complex plane of radius  $R$ . That is, our integral becomes

$$\oint f(z)dz = \lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx + i \int_0^{\pi} d\phi R e^{i\phi} f(R e^{i\phi}) = \int_{-\infty}^{\infty} f(x)dx. \quad (17.20)$$

where the second integral has vanished in the limit as  $R \rightarrow \infty$ .

Note that we could have closed the contour in the lower half-plane, up to a minus sign.

**Example 17.21.** Consider

$$\int_0^{\infty} \frac{dx}{x^2 + 1}. \quad (17.22)$$

This is an even function, so we can rewrite it as

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)}. \quad (17.23)$$

This goes to zero faster than  $1/x$ , so we can close the contour either way. If we close it in the upper half-plane we get the pole at  $+i$ , and if we close it in the lower half-plane we get the pole at  $-i$  (and we pay a minus sign for running counterclockwise). That is,

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)} = \frac{1}{2} (2\pi i) \frac{1}{2i} = \pi/2. \quad (17.24)$$

**Example 17.25.** Consider

$$\int_{-\infty}^{\infty} dx f(x) e^{iax}, \quad (17.26)$$

with  $a > 0$ . We shall require that  $f(x) \rightarrow 0$  and in fact  $f(z) \rightarrow 0$  as  $z \rightarrow +\infty + i\epsilon$ .

If we close the contour in the upper half-plane, we have

$$\int_{-\infty}^{\infty} dx f(x) e^{iax} + i \int_0^{\pi} d\phi R e^{i\phi} f(R e^{i\phi}) e^{ia(R \cos \phi + iR \sin \phi)}. \quad (17.27)$$

Let us also bound it:

$$\left| \int_0^{\pi} d\phi R e^{i\phi} f(R e^{i\phi}) e^{ia(R \cos \phi + iR \sin \phi)} \right| < \int_0^{\pi} d\phi R |f(R e^{i\phi})| e^{-aR \sin \phi}. \quad (17.28)$$

The sine is even about  $\pi/2$ , so we can change the interval and notice that from  $[0, \pi/2]$ ,  $\sin \phi > \frac{2\phi}{\pi}$ . Hence

$$\begin{aligned} \int_0^\pi d\phi R \left| f(Re^{i\phi}) \right| e^{-aR \sin \phi} &= 2R \int_0^{\pi/2} d\phi \left| f(Re^{i\phi}) \right| e^{-aR \sin \phi} \\ &< 2R\epsilon \int_0^{\pi/2} d\phi e^{-aR \frac{2\phi}{\pi}} \\ &= 2R\epsilon \frac{\pi}{2aR} e^{-aR \frac{2\phi}{\pi}} \Big|_0^{\pi/2} \\ &= \frac{\pi}{a} \epsilon (1 - e^{-aR}) \rightarrow 0. \end{aligned}$$

It follows that the integral along the semicircle is in fact zero, so we can safely close the contour in the upper half-plane.

Fair game for the midterm is up to today's lecture, not including integrals that oscillate at infinity.

Lecture 18.

**Friday, February 21, 2020**

*"I don't cross the branch cuts. Otherwise Scotty beams me up to the next branch. Which is like a parallel universe, and I'm not supposed to go there."*

—Nemanja Kaloper

We did oscillatory integrals last time, integrals like

$$\int_{-\infty}^{\infty} dx f(x) e^{i\alpha x}. \quad (18.1)$$

**Example 18.2.** Let's consider the integral

$$\int_{-\infty}^{\infty} dx \frac{\cos x}{1+x^2}. \quad (18.3)$$

This integral goes to as  $1/x^2$  on the real axis, but the cosine will introduce trouble when we close the contour. Since  $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ , we cannot directly close the contour in either the upper or lower half-plane. One of these exponential factors will diverge there.

But we can write this as

$$\frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\cos x}{1+x^2} = \frac{1}{4} \int_{-\infty}^{\infty} dx \left[ \frac{e^{ix}}{1+x^2} + \frac{e^{-ix}}{1+x^2} \right] = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \frac{e^{ix}}{1+x^2} \right]. \quad (18.4)$$

We've flipped the sign on the second term under  $x \rightarrow -x$ , which flips the limits of integration but also introduces a minus sign from the  $dx$ . Hence we can now safely close the contour in the upper half-plane.

Our integral is<sup>20</sup>

$$\frac{1}{2} \oint dz \frac{e^{iz}}{(z-i)(z+i)} = \pi i \text{Res} \frac{e^{-1}}{2i} = \frac{\pi}{2e}. \quad (18.5)$$

**Example 18.6.** Consider the integral

$$\frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\sin x}{x}. \quad (18.7)$$

By the same trick, we write

$$\frac{1}{4i} \int_{-\infty}^{\infty} dx \left[ \frac{e^{ix}}{x} - \frac{e^{-ix}}{x} \right] = \frac{1}{4i} \int_{-\infty}^{\infty} dx \left[ \frac{e^{ix}}{x} + \frac{e^{+ix}}{x} \right] = \frac{1}{2i} \mathcal{P} \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}. \quad (18.8)$$

Now we compute the principal value of the pole on the contour. It is just half of the residue for a simple pole, and so we find that

$$\frac{1}{2i} \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} = \frac{\pi i}{2i} = \frac{\pi}{2}. \quad (18.9)$$

<sup>20</sup>We could have done this by flipping the first factor instead to  $e^{-ix}$  and picked up the pole at  $-i$ , but the price we pay is running the contour in the clockwise direction.

**Example 18.10.** Here's another integral.

$$\int_0^\infty dx \frac{1}{x^3 + 1}, \quad (18.11)$$

but this integral is now of indefinite parity. It does however drop very fast as  $x \rightarrow \infty$ . Is there another way we can close the contour to perform this integral?

Note that there is a branch point at the origin. If we sweep out an arc from 0 to  $2\pi/3$ , we see that

$$z^3 = x^3 e^{3i\phi} \big|_{\phi=2\pi/3} = x^3. \quad (18.12)$$

So we can integrate

$$\oint dz \frac{1}{1+z^3} = \int_0^\infty \frac{dx}{1+x^3} + \int_0^{2\pi/3} \frac{id\phi R e^{i\phi}}{1+R^3 e^{3i\phi}} - \int_0^R \frac{ie^{2\pi i/3}}{1+x^3}. \quad (18.13)$$

Hence the integral around the closed contour is

$$\left[1 - e^{2\pi i/3}\right] I, \quad (18.14)$$

where  $I$  is our original integral. Conversely we can compute this contour integral using the residue theorem. If we do the residue calculation, we find that

$$\int_0^\infty dx \frac{1}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}. \quad (18.15)$$

We can also use branch cuts to compute some integrals that a priori look pretty nasty. Consider the following.

**Example 18.16.** We wish to integrate

$$\int_0^\infty dx \frac{\ln x}{x^3 + 1}. \quad (18.17)$$

Let us use almost the same contour, except we will excise the singularity at  $z = 0$ . That is,

$$\oint dz \frac{\ln z}{1+z^3} = \int_0^\infty dx \frac{\ln x}{1+x^3} + \int R \frac{\ln(R e^{i\phi})}{1+R^3 e^{3i\phi}} + e^{2\pi i/3} \int_R^{0^+} dx \frac{\ln x + \frac{2\pi i}{3}}{1+x^3} + i \int_{2\pi/3}^0 d\phi \rho e^{i\phi} \frac{\ln(\rho e^{i\phi})}{1+\rho^3 e^{3i\phi}}. \quad (18.18)$$

The superscript  $+$  indicates that we're taking a principal value. The second term is zero because it vanishes as  $R \ln R / R^2$  as  $R \rightarrow \infty$ , while the third one is a bit modified from the integral we want. The fourth term goes to zero as  $\rho \ln \rho$ , so it also goes to zero.

Hence we have the contour integral being

$$(1 - e^{2\pi i/3})I - e^{2\pi i/3} \frac{2\pi i}{3} \int_0^\infty \frac{dx}{1+x^3}. \quad (18.19)$$

This last integral came from the fact that the log was sensitive to the phase, but we already computed it as  $2\pi/3\sqrt{3}$ . Finally, we compute the integral using the residue theorem with the pole again at  $e^{i\pi/3}$  and find that

$$I = -\frac{2\pi^2}{27}. \quad (18.20)$$

Next time, we'll compute this with a different branch cut.

Lecture 19.

**Monday, February 24, 2020**

*"There are no dumb questions, there can only be dumb answers."*

—Nemanja Kaloper

**Example 19.1.** Let us try to evaluate

$$\int_0^\infty dx \frac{x^p}{1+x^2}, \quad 0 < p < 1. \quad (19.2)$$

The integrand has a branch point at  $x = 0$ , so we can choose an contour to perform the integral. The value of the contour integral is

$$\oint = I + \int_{C_R} -e^{2\pi ip} I + \int_{C_\rho}, \quad (19.3)$$

where  $C_R$  indicates the circle at infinity and  $C_\rho$  indicates the small semicircle about the origin. The  $C_R$  integral is zero since this integral goes as  $R^{p-1}$  as  $R \rightarrow \infty$ , while the  $c_\rho$  integral is zero since this goes as  $\rho^{p+1}$  as  $\rho \rightarrow 0$ .

Thus

$$\oint = (1 - e^{2\pi ip})I \quad (19.4)$$

and by the residue theorem, the contour integral is

$$\oint = 2\pi i \left[ \frac{i^p}{2i} + \frac{(-i)^p}{-2i} \right]. \quad (19.5)$$

Solving, we find that

$$I = \frac{\pi}{2 \cos(\frac{p\pi}{2})}. \quad (19.6)$$

Notice this integral has singularities at odd integer  $p$ . This is very similar to the gamma function, which has an integral representation  $\Gamma(x) = \int_0^\infty dt t^{x-1} e^{-t}$  and diverges for negative integer  $x$ .

We can use branch cuts to compute certain integrals. We previously computed

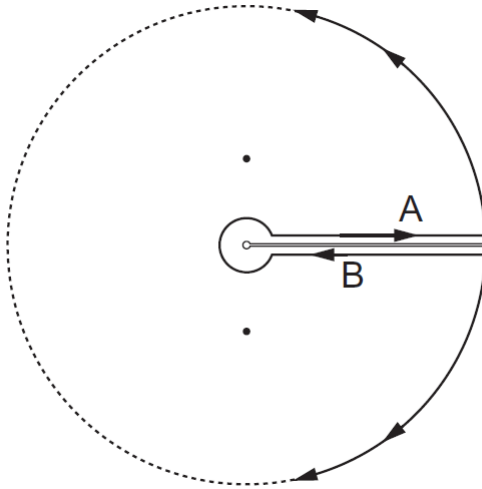
$$\int_0^\infty dx \frac{1}{x^3 + 1} \quad (19.7)$$

and also

$$\int_0^\infty dx \frac{\ln x}{1 + x^3} \quad (19.8)$$

over a very similar contour.

**Example 19.9.** Let us now show that we can use the second integral with the branch cut to work out the first one. That is, we shall choose the contour



and see what this does for us.

Note the poles marked here are for the previous example. Here, we have poles at  $e^{\pi i/3}, e^{\pi i}, e^{5\pi i/3}$  and we will need to evaluate the three residues. Now as before we get the integral over the upper segment A and the lower segment B;  $\ln x dx$  goes to zero quickly about the small circular region near the origin, and this integral goes away as  $\ln x/x^2$  as  $x \rightarrow \infty$ . Hence our integral is

$$\oint = \int_0^\infty dx \frac{\ln x}{1 + x^3} - \int_0^\infty dx \frac{\ln x + 2\pi i}{1 + x^3} = -2\pi i I, \quad (19.10)$$

where  $I$  is the integral we wanted. Since the contour integral is  $2\pi i \times$  the residues, we find that

$$-I = \text{Res}_1 + \text{Res}_2 + \text{Res}_3. \quad (19.11)$$

If we work it out, we find that

$$I = \frac{2\pi}{3\sqrt{3}}, \quad (19.12)$$

just as before.

**Example 19.13.** Consider

$$\int_0^\infty dx \frac{x}{\sinh x}. \quad (19.14)$$

The integrand is even, so we can extend it as

$$\frac{1}{2} \int_{-\infty}^\infty dx \frac{x}{\sinh x}. \quad (19.15)$$

As  $|x|$  grows large, the exponentials force this to converge. But how should we close the contour? The Jordan lemma doesn't apply. However, notice the following.

$$\sinh iy = \frac{1}{2}(e^{iy} - e^{-iy}) = i \sin y. \quad (19.16)$$

Recall that  $\sinh(x)$  follows the addition formula

$$\sinh(x + i\pi) = \sinh x \cosh(i\pi) + \cosh x \sinh(i\pi), \quad (19.17)$$

so because  $\sinh i\pi = i \sin \pi = 0$  and  $\cosh i\pi = \cos \pi = -1$ , we find that

$$\sinh(x + i\pi) = -\sinh x. \quad (19.18)$$

We see that  $\sinh$  is  $2\pi$ -periodic in the imaginary direction. Therefore this suggests that we take a rectangular contour of height  $i\pi$  and length  $2L$ . The sides are

$$\int_0^\pi idy \frac{\pm L + iy}{\sinh(\pm L + iy)}, \quad (19.19)$$

which goes away as  $L \rightarrow \infty$ , so our integral is

$$\oint = \int_{-\infty}^\infty + \int_{-\infty}^\infty dx \frac{x + i\pi}{\sinh x}. \quad (19.20)$$

If we treat the  $\frac{i\pi}{\sinh x}$  as a principal value, we find that its principal value is zero. Hence

$$\oint = 2 \int_{-\infty}^\infty dx \frac{x}{\sinh x}, \quad (19.21)$$

and all we need to do is add the residues. This has a pole at  $z = \pi i$ , so when we take the residue we find that it is

$$\lim_{z \rightarrow \pi i} (z - \pi i) \frac{z}{\sinh z} = -\pi i \quad (19.22)$$

and so

$$I = \frac{\pi^2}{4}. \quad (19.23)$$

Lecture 20.

**Wednesday, February 26, 2020**

Today we'll discuss using contour integrals to compute infinite sums. We'd like to sum a series like

$$\sum_n a_n, \quad (20.1)$$

and we want to write it in terms of a contour integral. Consider the function  $\cot z = \frac{\cos z}{\sin z}$ . The function  $\sin z$  has the product expansion

$$\sin z = z \prod_{n=1}^\infty \left(1 - \frac{z}{n\pi}\right) \left(1 + \frac{z}{n\pi}\right), \quad (20.2)$$



so we have (nondegenerate) zeroes at integer  $\pi$  and therefore  $\cot z$  has simple poles at integer  $\pi$ .<sup>21</sup> Thus the residue at such poles is

$$\lim_{z \rightarrow n\pi} (z - n\pi) \cot z = \frac{(z - n\pi) \cos z}{\sin z}. \quad (20.3)$$

We can set a normalization so that  $\cot \pi z$  has poles at integer  $n$ . Now consider the function

$$f(z) \cot(\pi z), \quad (20.4)$$

where  $f(z)$  does not have any poles at real integer values. Suppose that  $f(z)$  has some poles in the complex plane, however, at  $\{z_i\}$ . Then if we take the contour integral over a circle of radius  $R$  centered on the origin and take  $R \rightarrow \infty$ , then

$$\oint f(z) \cot(\pi z) dz = 2\pi i \sum_{n=-\infty}^{\infty} f(n) + 2\pi i \sum_{z_i} \text{Res} f(z) \cot \pi z. \quad (20.5)$$

If  $f$  decays sufficiently fast as  $z \rightarrow \infty$ , then the contour integral over the circle at infinity is zero, and we conclude that

$$\sum_{n=-\infty}^{\infty} f(n) = - \sum_{z_i} \text{Res} f(z) \cot \pi z. \quad (20.6)$$

**Example 20.7.** Suppose we wish to sum the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}. \quad (20.8)$$

Notice that the summand is even in  $n$ , so

$$S = \sum_{n=-1}^{\infty} \frac{1}{n^2 + a^2}. \quad (20.9)$$

We can easily add on the  $n = 0$  term, which is just  $1/a^2$ . Thus

$$2S + \frac{1}{a} = \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}, \quad (20.10)$$

which is related to the contour integral

$$\oint_{C_R} dz \frac{1}{z^2 + a^2} \cot(\pi z). \quad (20.11)$$

This has poles everywhere we want, but it also has poles at  $z = \pm ia$ . So the value of this integral is

$$2\pi i \left( \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} + 2 \frac{\cot(\pi ia)}{2ia} \right) = 0, \quad (20.12)$$

and so

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a). \quad (20.13)$$

**Example 20.14.** Consider the sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right). \quad (20.15)$$

There's a somewhat pedestrian way to sum this series. If we take the partial sums  $\sum_{n=1}^N$  then these sums are

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N+1} \right) = 1 - \frac{1}{N+1}, \quad (20.16)$$

and as  $N \rightarrow \infty$  we see the sum is just 1.

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<sup>21</sup>The degeneracy of a zero is easily counted by taking derivatives and evaluating the function at that zero until the net result is nonzero. For instance,  $f(x) = (x-1)^2$  has a double zero at  $x = 1$ . But  $f(1) = 0$  and  $f'(1) = 2(x-1)|_{x=1} = 0$ ; only  $f''(1) = 2 \neq 0$ .

Let's do this a tricky way. By shifting  $n \rightarrow -n$  and changing the index, we can see that

$$\sum_{n=-2}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad (20.17)$$

our original sum. We can almost extend this to the full infinite sum  $\sum_{n=-\infty}^{\infty}$ , but the  $n = 0$  and  $n = -1$  terms are no good. We can write

$$2S = \sum'_{-\infty} \frac{1}{n(n+1)}, \quad (20.18)$$

where the prime indicates we must omit  $n = 0$  and  $n = -1$ . These are poles, which suggests that we have picked up double-poles at  $n = 0, n = -1$ . That is,

$$0 = \oint dz \frac{1}{z(z+1)} \cot \pi z. \quad (20.19)$$

The residue at  $z = 0$  is the usual limit

$$\lim_{z \rightarrow 0} z^2 \cot \pi z \quad (20.20)$$

and  $z = -1$  is similar. Hence

$$\sum'_{-\infty} \frac{1}{n(n+1)} = -\text{Res} \left( \frac{1}{z(z+1)} \cot(\pi z) \right)_{z=0} - \text{Res} \left( \frac{1}{z(z+1)} \cot(\pi z) \right)_{z=-1}. \quad (20.21)$$

This is precisely analogous to having dipoles rather than monopoles at some integer locations on the real line.

**Schwartz reflection principle** The Schwartz reflection principle tells us how to take complex conjugates of various functions. We have seen previously that  $f^*(z) = f(z^*)$ , but in fact there are conditions. Consider

$$[(z - x_0)^n]^* = (z^* - x_0)^n, \quad (20.22)$$

where  $x_0 \in \mathbb{R}$ . When we take linear combinations  $\sum a_n (z - z_0)^n$ , we might have to complex conjugate the  $a_n$ s as well. The Schwartz reflection principle provides sufficient conditions so this does not happen.

Consider  $f(z)$  analytic. Hence we can Taylor expand  $f$  about a point  $x_0$  on the real line,

$$f(z) = \sum \frac{f^{(n)}(x_0)}{n!} (z - x_0)^n, \quad (20.23)$$

where  $z_0$  is real. Moreover, suppose that for real arguments  $x_0 \in \mathbb{R}$ ,  $f^*(x_0) = f(x_0)$ , i.e.  $f$  is real-valued for real arguments. Hence all the  $f^{(n)}(x_0)$  are also real. This must be the case since the  $(z - x_0)^n$  factors are all linearly independent. Hence

$$f^*(z) = \sum \frac{f^{(n)}(x_0)}{n!} (z^* - x_0)^n = f(z^*). \quad (20.24)$$

Lecture 21.

**Friday, February 28, 2020**

**Conformal mappings** Analytic functions naturally induce conformal (locally angle-preserving) mappings on the complex plane. To see this, consider a function  $f(z)$  analytic in some domain. Consider two smooth curves through a point  $z_0$ . The mapping  $f$  preserves angles between the curves. For suppose we had  $z_1(\lambda), z_2(\lambda)$ . The derivative of  $f$  at  $z_0$  is unique and independent of path, i.e.

$$f'(z_0) = \frac{\Delta w}{\Delta z}, \quad (21.1)$$

where  $\Delta w$  is the displacement in the image due to a small shift  $\Delta z$  in the domain. Then

$$\Delta w_i = f'(z_0) \Delta z_i, \quad (21.2)$$

where  $\Delta w_i, \Delta z_i, f'(z_0)$  are complex numbers. Hence we can expand them in polar form as

$$R_i e^{i\phi_i} = \rho e^{i\psi} \hat{\rho}_i e^{i\bar{\phi}_i}, \quad (21.3)$$

so that

$$\phi_2 = \psi + \bar{\phi}_2 \quad (21.4)$$

and similarly

$$\phi_1 = \psi + \bar{\phi}_1. \quad (21.5)$$

This tells us that the images of the original tangent vectors are shifted by the same angle  $\psi$ , so the angle between them has been preserved.<sup>22</sup>

Let us study some properties of conformal mappings. We can consider the inversion map,

$$w = \frac{1}{z}. \quad (21.6)$$

This maps points  $z = \rho e^{i\phi}$  to points  $w = \frac{1}{\rho} e^{-i\phi}$ . There are also special conformal transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad (21.7)$$

which form a group (they are an inversion followed by a translation and another inversion).

The inversion map takes rays to rays (e.g.  $z = \rho e^{i\theta_0}$ , but they bring points from far distances in close to the origin, and vice versa. Inversion maps also take circles of radius  $R$  centered on the origin (say, run counterclockwise) to circles of  $1/R$  run clockwise.

What is the image of a circle that goes through the origin? Suppose we have the circle

$$\left| z - \frac{i}{2} \right| = \frac{1}{2}. \quad (21.8)$$

What is its inversion? We claim it is actually a straight line at  $w = -i$ . How can we see this? Well,

$$\left| \frac{1}{w} - \frac{i}{2} \right|^2 = \left( \frac{1}{w} - \frac{i}{2} \right) \left( \frac{1}{w^*} - \frac{i}{2} \right) = \frac{1}{4}. \quad (21.9)$$

We can show that in fact the imaginary part of  $w$  does not change, so the image is in fact a straight line.<sup>23</sup>

**Fourier transforms** Recall that sufficiently nice functions (Dirichlet conditions, etc.) have exponential fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}} \quad (21.10)$$

where

$$c_n = \frac{1}{2L} \int_{-L}^L dx' e^{-\frac{i n \pi x'}{L}} f(x'). \quad (21.11)$$

Let us take the Fourier coefficients to be bounded such that

$$\sum |c_n|^2 < \infty \quad (21.12)$$

and take the domain  $[L, L]$  to infinity. Hence

$$\int_{-\infty}^{\infty} dx |f|^2 < \infty \quad (21.13)$$

at a minimum. For instance, functions like  $x^2$  will not have Fourier transforms over the whole real line. Now

$$f(x) = \sum_{n=-\infty}^{\infty} e^{i n \pi x / L} \frac{1}{2L} \int_{-L}^L dx' e^{-i n \pi x' / L} f(x'). \quad (21.14)$$

Notice that the steps between Fourier indices  $n$  is

$$\Delta \frac{n \pi}{L} = \frac{\pi}{L} \Delta n. \quad (21.15)$$

<sup>22</sup>There's a lot of power in conformal mappings. We can take unusual shapes in 2D and map them into much simpler boundaries that we can explicitly integrate. This is part of the magic of doing conformal field theory in 2 dimensions, is that analyticity provides a powerful constraint in 2D.

<sup>23</sup>Here,

Suppose we define

$$dp = \frac{\pi}{L}, \quad p = \frac{n\pi}{L}. \quad (21.16)$$

Then our Fourier transform becomes

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \Delta n e^{ipx} \frac{1}{2L} \int_{-L}^L dx e^{-ipx'} f(x') \\ &= \sum_{n=-\infty}^{\infty} \Delta p e^{ipx} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ipx'} f(x'). \end{aligned}$$

This sum is precisely the definition of the Riemann-Stieltjes integral, and if we replace  $\Delta p$  with a  $dp$  (as  $L \rightarrow \infty$ , we see that

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} e^{ipx} \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}} e^{-ipx'} f(x'). \quad (21.17)$$

Incidentally, we see that

$$\int_{-\infty}^{\infty} dp e^{ip(x-x')} = 2\pi \delta(x-x'). \quad (21.18)$$

This is none other than the completeness relation on exponentials. We can define the Fourier transform

$$g(p) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}} e^{-ipx'} f(x') \quad (21.19)$$

and the inverse transform

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} e^{ipx} g(p). \quad (21.20)$$

The Fourier transform is a continuous unitary transform which applies to functions which are square-integrable, the generalization of the Dirichlet condition on the infinite interval. Moreover we see that these integrals are now of the form we previously considered, of an oscillating factor times some other term which we can evaluate as contour integrals by the Jordan lemma.

**Non-lectured: inverse map of circle** For the circle we discussed,

$$z = (i + e^{i\theta})/2 = (e^{i\pi/2} + e^{i\theta})/2 \quad (21.21)$$

so

$$\begin{aligned} w = 1/z &= \frac{2}{e^{i\pi/2} + e^{i\theta}} = 2 \frac{e^{-i\pi/2} + e^{-i\theta}}{2 + e^{i\pi/2}e^{-i\theta} + e^{-i\pi/2}e^{i\theta}} \\ &= \frac{e^{i\pi/2} + e^{-i\theta}}{1 + 1 \cos(\theta - \pi/2)} \\ &= \frac{-i(1 + e^{-i(\theta - \pi/2)})}{1 + \cos(\theta - \pi/2)} \\ &= -i \frac{1 + \cos(\theta - \pi/2) - i \sin(\theta - \pi/2)}{1 + \cos(\theta - \pi/2)}. \end{aligned}$$

We can see that the imaginary part is actually just  $-i$ , while the real part is

$$- \frac{\sin(\theta - \pi/2)}{1 + \cos(\theta - \pi/2)} \quad (21.22)$$

and this sweeps out the whole real line since  $\cos(\theta - \pi/2)$  goes through  $-1$ .

**Non-lectured: analytic function has zero antiholomorphic derivative** Suppose we have a function  $f(z)$  which is analytic. Then its derivative with respect to  $z^*$  is zero, i.e.  $\frac{df}{dz^*} = 0$ .

*Proof.* Recall that

$$z = x + iy, \quad z^* = x - iy. \quad (21.23)$$

Thus

$$x = \frac{z + z^*}{2}, \quad y = \frac{z - z^*}{2i}. \quad (21.24)$$

We can write  $f = u + iv$ , so then by the chain rule, the derivative  $\frac{df}{dz^*}$  is the following:

$$\begin{aligned} \frac{df}{dz^*} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z^*} \\ &= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (1/2) + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) (-1/2i) \\ &= \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\ &= 0 \text{ by Cauchy-Riemann.} \end{aligned} \quad (21.25)$$

☒

Lecture 22.

**Monday, March 2, 2020**

*“Any of you who take the bold and foolish step of getting lost in theory will become familiar with these things [Fourier-transformed Gaussians].”*

—Nemanja Kaloper

Last time, we defined the Fourier transform

$$g(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx} \quad (22.1)$$

and the inverse transform<sup>24</sup>

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} g(k) e^{ikx}, \quad (22.2)$$

such that the completeness relations

$$\int_{-\infty}^{\infty} dx e^{ik(x-x')} = 2\pi \delta(x - x'), \quad (22.3)$$

$$\int_{-\infty}^{\infty} dx e^{i(k-q)x} = 2\pi \delta(k - q) \quad (22.4)$$

hold. This uses the convention that  $e^{i\omega t}$  is the time dependence of a positive-frequency wave moving forward in time.

Suppose  $f(x)$  has definite parity, e.g.  $f$  is even,  $f(-x) = f(x)$ . Then we can just do the integral over the half-line and write

$$\begin{aligned} g(k) &= \int_0^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx} + \int_0^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{+ikx} \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx f(x) \cos(kx). \end{aligned}$$

This is the *even Fourier transform*. By an equivalent argument, if our function is instead odd, then

$$\tilde{g}(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx f(x) \sin(kx), \quad (22.5)$$

<sup>24</sup>It's a matter of convention whether we decide to put the  $\sqrt{2\pi}$ s in both the normal and inverse transform, or whether we put e.g. the  $1/2\pi$  with the inverse transform. In QFT I usually write  $d^d k / (2\pi)^d$  and put the  $2\pi$ s with the inverse transform when I take  $dk$  integrals, where  $d$  is the dimension we're working in (often in e.g.  $d = 4$ ) but there's not a lot of difference.

where the tilde indicates we have dropped a factor of  $i$  in going from exponentials to a sine.

**Example 22.6.** Take the function  $f(x) = e^{-\alpha|x|}$ , with  $\alpha > 0$ . Then

$$\begin{aligned} g(k) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty dk e^{-\alpha x} e^{-ikx} + \frac{1}{\sqrt{2\pi}} \int_0^\infty dx e^{-\alpha x} e^{+ikx} \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\alpha + ik} + \frac{1}{\alpha - ik} \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + k^2}. \end{aligned}$$

We can easily take the Fourier transform of a delta function as well,  $\delta(x)$ . Well,

$$g^\delta(x) = \frac{1}{\sqrt{2\pi}}. \quad (22.7)$$

The fact that the Fourier transform is constant in momentum space tells us that if we want to build a completely localized wavepacket in position space, we need to use components of all frequencies in momentum space.

**Example 22.8.** Now let us show that the inverse transform gives back the original function using  $f(x) = e^{-\alpha|x|}$ . We write the transformed function as

$$\sqrt{\frac{2}{\pi}} \frac{k}{x^2 + k^2}, \quad (22.9)$$

and now

$$\frac{k}{\pi} \int_{-\infty}^\infty dx \frac{e^{ik'x}}{x^2 + k^2} \quad (22.10)$$

is our original function. But which way we close the contour clearly depends on the sign of  $k$ . For  $k > 0$  we should close it in the lower half-plane, so

$$\frac{k}{\pi} \int_{-\infty}^\infty dx \frac{e^{-ikx}}{(x - ik)(x + ik)} = \frac{k}{\pi} \left( 2\pi i \frac{e^{-ik(-ik)}}{(-2ik)} \right) = e^{-k'k}. \quad (22.11)$$

**Example 22.12.** Let's take the Fourier transform of a Gaussian. That is,

$$f(x) = e^{-\alpha x^2} \quad (22.13)$$

so that the transform is

$$\begin{aligned} g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dx e^{-\alpha(x^2 + ik/\alpha)} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dx e^{-\alpha([x + ik/2\alpha]^2 + k^2/4\alpha^2)}. \end{aligned}$$

The  $k^2/4\alpha^2$  term is constant– it does not depend on  $x$ . The rest is a Gaussian, but with a complex shift:

$$g(k) = \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}} \int_{-\infty}^\infty dx e^{-\alpha[x + ik/2\alpha]^2}. \quad (22.14)$$

Now we can make a change of variables to write

$$g(x) = \frac{e^{-k^2/4\alpha}}{\sqrt{2\pi}} \int_{-\infty + ik/2\alpha}^{+\infty + ik/2\alpha} d\hat{x} e^{-\alpha\hat{x}^2}. \quad (22.15)$$

This isn't quite our original Gaussian integral because it is not over the real line yet, and we'd very much like to use the result  $\int_{-\infty}^\infty e^{-\alpha x^2} = \sqrt{\pi/\alpha}$ .

We can relate this to the integral over the real line, however, by taking a rectangular contour of length  $2L$  and height  $ik/2\alpha$ . The sides are exponentially suppressed as we take  $L \rightarrow \infty$ , and there are no poles in the rectangular contour, so the integral over the line  $x + ik/2\alpha$  is equal to the integral over the real line,  $x \in \mathbb{R}$ .

It follows that this integral really is our Gaussian integral, so the result is

$$g(k) = \frac{e^{-k^2/4\alpha}}{\sqrt{2\alpha}}. \quad (22.16)$$

As it turns out, the Fourier transform is another Gaussian, but with a different width of  $\sqrt{2\alpha}$  rather than  $1/\sqrt{\alpha}$ .

**Example 22.17.** Suppose we have a sine wave which is only on for some finite pulse,

$$f(x) = \begin{cases} \sin(\omega_0 t) & |t| < \frac{N\pi}{\omega_0}, \\ 0 & |t| > \frac{N\pi}{\omega_0}. \end{cases} \quad (22.18)$$

This is an odd function, so we can compute

$$\sqrt{\frac{2}{\pi}} \int_0^{N\pi/\omega_0} dt \sin(\omega_0 t) \sin(\omega t). \quad (22.19)$$

Now we can use trig addition formulae to turn the product of sines into regular cosines:

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]. \quad (22.20)$$

If we put the pieces together and do the integral, we get

$$\frac{1}{\sqrt{2\pi}} \left( \frac{\sin[(\omega_0 - \omega)(N\pi/\omega_0)]}{\omega_0 - \omega} - \frac{\sin[(\omega_0 + \omega)(N\pi/\omega_0)]}{\omega_0 + \omega} \right) \quad (22.21)$$

This function has a resonance at  $\omega = \omega_0$ . The next peak is when  $(\omega_0 - \omega)(N\pi/\omega_0) = \pi/2$ , so the distance to the next peak is  $\delta\omega = \omega_0/N$ . It follows that  $\Delta\omega\Delta t = \pi$ , i.e. the spacing between peaks in frequency space and the spacing between peaks in time is constant.

Lecture 23.

**Wednesday, March 4, 2020**

*"I like all things if they're valuable, okay? I collect them."*

—Nemanja Kaloper

Building off last time, we can define 3-dimensional equivalents of the Fourier transform,

$$f(\mathbf{x}) = \int \frac{d^d \mathbf{k}}{(2\pi)^{d/2}} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad (23.1)$$

and

$$g(\mathbf{k}) = \int \frac{d^d \mathbf{x}}{(2\pi)^{d/2}} f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (23.2)$$

**Example 23.3.** Consider the Yukawa potential,

$$f(r) = \frac{e^{-\alpha r}}{r} = \frac{e^{-r/m}}{r}, \quad (23.4)$$

where  $\alpha = m > 0$ . We may do this in order to ensure convergence. Consider now its Fourier transform,

$$\int d^3 x \frac{e^{-\alpha|\mathbf{x}|}}{|\mathbf{x}|} e^{-i|\mathbf{k}||\mathbf{x}| \cos \theta}. \quad (23.5)$$

Since we integrate over all space and all angles, WLOG we can take  $\mathbf{k}$  to point along the  $\hat{\mathbf{z}}$  direction in order to perform the integral. Notice that this integral retains an azimuthal symmetry, so it's most natural to transform to spherical coordinates. This integral becomes

$$2\pi \int_0^\infty dr r^2 \int_0^\pi d\theta \sin \theta \frac{e^{-\alpha|\mathbf{x}|}}{|\mathbf{x}|} e^{-i|\mathbf{k}||\mathbf{x}| \cos \theta}. \quad (23.6)$$

The book resorts to expanding in spherical harmonics, but there's really no need to do it. We have  $d \sin \theta = d(-\cos \theta)$  so the angular integral is easy to do. Our integral simplifies to

$$\begin{aligned} g(\mathbf{k}) &= 2\pi \int_0^\infty dr r e^{-\alpha r} \int_{-1}^1 d(\cos \theta) e^{-ikr(\cos \theta)} \\ &= \frac{2\pi i}{k} \int_0^\infty dr e^{-\alpha r} (e^{-ikr} - e^{+ikr}) \\ &= \frac{2\pi i}{k} \int_0^\infty dr (e^{-(\alpha+ik)r} - e^{-(\alpha-ik)r}). \end{aligned}$$

This last expression is similar to what we computed last time. The  $e^{\pm ikr}$  phase doesn't affect convergence, since  $e^{-\alpha r} |e^{\pm ikr}|$  converges absolutely. Thus

$$g(k) = \frac{2\pi i}{k} \left[ \frac{1}{\alpha + ik} - \frac{1}{\alpha - ik} \right] = \frac{2\pi i}{k} \frac{-2ik}{\alpha^2 + k^2} = \frac{4\pi}{\alpha^2 + k^2}. \quad (23.7)$$

The Fourier transform of the Yukawa potential has turned out to be a Lorentzian in  $k$ -space.<sup>25</sup> However, note that if the mass is zero, we lose the factor  $e^{-\alpha|x|}$  which gave us convergence. What happens now? Our integral becomes

$$\frac{2\pi i}{k} \int_0^\infty dr (e^{-ikr} - e^{+ikr}) = -\frac{2\pi i}{k} \left( \int_0^L + \int_0^{-L} \right) dr e^{ikr}. \quad (23.8)$$

The lower limit gives us the correct answer, but we have to understand the behavior at infinity. Inserting the regularizing factor  $e^{-\alpha|x|}$  allows us to actually cut off the behavior at infinity (at large distances  $x$ ) and then take the  $\alpha \rightarrow 0$  limit.

Hence we can say that (restoring factors of  $2\pi$ )

$$g(k) = \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{\alpha^2 + k^2} \quad (23.9)$$

and in the  $\alpha \rightarrow 0$  limit,

$$g(k) \rightarrow \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{k^2}. \quad (23.10)$$

Let us take the inverse transform and confirm that it all works out. That is, we shall integrate

$$\begin{aligned} \int d^3x \frac{e^{-i|\mathbf{k}||\mathbf{x}|\cos \theta}}{|\mathbf{x}|^2} &= 2\pi \int_0^\infty dr \int_{-1}^1 d(\cos \theta) e^{-ikr(\cos \theta)} \\ &= \frac{2\pi i}{k} \int_0^\infty dr \frac{1}{r} (e^{-ikr} - e^{+ikr}) \\ &= -\frac{2\pi i}{k} \int_0^\infty \frac{dr}{r} (e^{+ikr} - e^{-ikr}). \end{aligned}$$

Now we can flip the sign in the exponent of the second term and get

$$-\frac{2\pi i}{k} \left[ \int_0^\infty \frac{dr}{r} e^{ikr} - \int_0^{-\infty} \frac{dr}{r} e^{+ikr} \right], \quad (23.11)$$

which we recognize as a single principal value integral,

$$-\frac{2\pi i}{k} \int_{-\infty}^\infty \frac{dr}{r} e^{ikr}. \quad (23.12)$$

Moreover the combination  $dr/r$  is scale-invariant, as are the limits of integration, so we can reparametrize to  $\int_{-\infty}^\infty \frac{dz}{z} e^{-iz}$ . Performing the integral gives us our final answer,

$$\frac{4\pi^2}{k}, \quad (23.13)$$

which upon restoring  $2\pi s$  gives us our original Coulombic potential as promised.

<sup>25</sup>See <http://mathworld.wolfram.com/LorentzianFunction.html>.



Let's mention some properties of the Fourier transform. If we have a function  $f(\mathbf{x})$ , let us denote its Fourier transform as

$$[f(\mathbf{x})]^T = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (23.14)$$

Then the Fourier transform of the shifted function is

$$[f(\mathbf{r} - \mathbf{R})]^T = e^{i\mathbf{k}\cdot\mathbf{R}} g(\mathbf{k}). \quad (23.15)$$

Similarly the Fourier transform of the rescaled argument is

$$[f(\alpha\mathbf{r})]^T = \frac{1}{\alpha^3} g\left(\frac{\mathbf{k}}{\alpha}\right), \quad (23.16)$$

where the overall scaling factor comes from the integration measure  $d^3x$ . Inversion gives

$$[f(-\mathbf{r})]^T = g(-\mathbf{k}), \quad (23.17)$$

setting  $\alpha = -1$ . Note there's no overall minus sign because both the measure and the limits of integration flip. Complex conjugation and inversion gives

$$[f^*(-\mathbf{r})]^T = g^*(\mathbf{k}), \quad (23.18)$$

while taking a derivative brings down an  $ik$ :

$$[\nabla f(\mathbf{r})]^T = -i\mathbf{k}g(\mathbf{k}), \quad (23.19)$$

and doing the Laplacian gives

$$[\nabla^2 f(\mathbf{x})]^T = -k^2 g(\mathbf{k}). \quad (23.20)$$

Lecture 24.

**Friday, March 6, 2020**

*"The number of beans in the universe is conserved. So if you count the number of beans in one basis and the number of beans in another basis, you better get the same number of beans."*<sup>26</sup>

—Nemanja Kaloper

Today we'll consider some applications of Fourier transforms.

**Example 24.1.** Consider the wave equation with unit velocity,

$$\partial_x^2 y = \partial_t^2 y. \quad (24.2)$$

Let us consider a solution with initial conditions

$$y(x, 0) = f(x), \quad (24.3)$$

$$\partial_t y(x, 0) = 0. \quad (24.4)$$

This is the propagation of the vibration of a plucked string. It must be infinite, or else we would be using a Fourier series. Note that if the string were half-infinite we might instead use a Laplace transform.

We write  $y$  in terms of its Fourier transform  $Y$ , as

$$y = \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi}} Y(\alpha, t) e^{-i\alpha x} \quad (24.5)$$

and the inverse transform

$$Y = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} y e^{+i\alpha x}. \quad (24.6)$$

<sup>26</sup>Editor's note—when you're talking about particles, the concept of particle is actually not well-defined if your spacetime is not stationary. The notion of how to distinguish positive and negative frequency modes is only well-defined when the part of the spacetime you're looking at is stationary, i.e. it has a well-defined time translation symmetry. In spacetimes which are stationary at early times and late times but dynamical in between, one can define a change of basis such that the vacuum at early times is seen to be full of thermal radiation at late times. In fact, the non-conservation of "beans" in non-stationary spacetimes is exactly the phenomenon of Hawking radiation.

Then the relevant derivatives are

$$\partial_t^2 y = \int \frac{d\alpha}{\sqrt{2\pi}} (\partial_t^2 Y) e^{-i\alpha x}, \quad \partial_x^2 y = \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi}} Y(-\alpha^2) e^{-i\alpha x}. \quad (24.7)$$

Our PDE says that these are equal, so this reduces to a second-order ODE,

$$\partial_t^2 Y = -\alpha^2 Y \implies Y = A e^{i\alpha t} + B e^{-i\alpha t}. \quad (24.8)$$

The initial velocity condition tells us that

$$\partial_t y(x, 0) = 0 \implies \partial_t Y(\alpha, 0) = 0. \quad (24.9)$$

The initial “wavefunction” condition says that

$$Y(\alpha, 0) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} y(x, 0) e^{i\alpha x} = F(\alpha), \quad (24.10)$$

where  $F$  is the Fourier transform of the initial condition. Applying initial conditions, we have

$$A + B = F(\alpha) \quad (24.11)$$

for the position condition and

$$A - B = 0 \quad (24.12)$$

for the velocity condition. Thus

$$A = B = \frac{1}{2} F(\alpha), \quad (24.13)$$

so

$$Y(\alpha, t) = \frac{F(\alpha)}{2} [e^{i\alpha t} + e^{-i\alpha t}]. \quad (24.14)$$

Then the solution to our differential equation in real space is

$$\begin{aligned} y &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{2\pi}} \left( F(\alpha) e^{i\alpha(t-x)} + F(\alpha) e^{-i\alpha(t+x)} \right) \\ &= \frac{1}{2} [f(t-x) + f(t+x)], \end{aligned}$$

which is the d’Alembert solution to the wave equation we derived before.

**Example 24.15.** Let us find the Coulombic potential, i.e. the Green’s function for the Laplacian operator such that

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}'). \quad (24.16)$$

Well,  $G$  can be written in terms of its Fourier transform as

$$G = \int \frac{d^3 k}{(2\pi)^{3/2}} g(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{r}} \quad (24.17)$$

and similarly

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int d^3 k e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')} = \int \frac{d^3 k}{(2\pi)^{3/2}} \left( \frac{e^{i\mathbf{k} \cdot \mathbf{r}'}}{(2\pi)^{3/2}} \right) e^{-i\mathbf{k} \cdot \mathbf{r}} \quad (24.18)$$

Comparing left and right sides,

$$-k^2 g(\mathbf{k}, \mathbf{r}') = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k} \cdot \mathbf{r}'}. \quad (24.19)$$

Thus

$$G = -\frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}}{k^2} = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \quad (24.20)$$

**Example 24.21.** Let us consider the forced harmonic oscillator,

$$\ddot{y} + \omega_0 y = f(t). \quad (24.22)$$

We can solve for the Green's function:

$$\ddot{G} + \omega_0^2 G = -\delta(t - t'). \quad (24.23)$$

We're interested in the retarded Green's function, i.e. the Green's function which vanishes for  $t < t'$  (so that influences propagate to the future). We write the Green's function in terms of its Fourier transform,

$$G = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} g(\omega) e^{-i\omega t} \quad (24.24)$$

and the delta function is

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}. \quad (24.25)$$

Plugging in, we have

$$(-\omega^2 + \omega_0^2)g + \omega_0^2 g = -\frac{1}{\sqrt{2\pi}} e^{i\omega t'}, \quad (24.26)$$

so

$$g = -\frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t'}}{(\omega^2 - \omega_0^2)}. \quad (24.27)$$

Now we take the inverse transform to write

$$G = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t')}}{(\omega - \omega_0)(\omega + \omega_0)} \quad (24.28)$$

By now, it should be natural that we write this as a contour integral and decide how to close the contour by the Jordan lemma. Which way we close the contour depends on the relative signs of  $t$  and  $t'$ : for  $t < t'$ , we must close the contour in the upper half-plane, while for  $t > t'$  we close in the lower half-plane.

But there's an important condition. We can't just blithely compute the principal value of the poles at  $\pm\omega_0$ , since we require that this is the *retarded Green's function*. In fact, we must push the poles down off the contour to  $\pm\omega_0 - i\epsilon$  so that

$$G = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-t')}}{(\omega - (\omega_0 - i\epsilon))(\omega - (-\omega_0 - i\epsilon))}. \quad (24.29)$$

Thus we have used an  $i\epsilon$ -prescription (i.e. pushed the poles off the contour) and then let  $\epsilon \rightarrow 0$ .<sup>27</sup> The value of this contour integral gives us

$$G(t, t') = \begin{cases} 0, & t < t' \\ \frac{1}{i} \left[ \frac{e^{-i\omega_0(t-t')}}{2\omega_0} - \frac{e^{+i\omega_0(t-t')}}{2\omega_0} \right], & t > t'. \end{cases} \quad (24.30)$$

We can moreover combine factors and write

$$G(t, t') = \begin{cases} 0, & t < t' \\ -\frac{1}{\omega_0} \sin(\omega_0(t - t')), & t > t'. \end{cases} \quad (24.31)$$

This is exactly the answer we derived previously by solving the differential equation and fitting the boundary conditions, except we previously set  $\omega_0$  to 1.

<sup>27</sup>There are other choices for how to push the poles. There's also the advanced propagator, which results in retrocausal effects (effects propagate to the past). And there's a time-ordered propagator, where one pole is pushed up and one is pushed down. Hence positive frequency moves forward in time and negative frequency moves backwards in time. The negative frequency modes are to be interpreted as antiparticles.

**Convolution** For two functions  $f$  and  $g$ , we can define their convolution (or “folding,” or *Faltung*) as

$$f * g = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} g(y) f(x-y). \quad (24.32)$$

This operation is commutative, as we can see by a change of variables. One nice property of the convolution is that the Fourier transform of the convolution is the product of the Fourier transforms:

$$(f * g)^T = F(k)G(k). \quad (24.33)$$

We can prove this:

$$f * g = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} g(y) f(x-y) \quad (24.34)$$

$$= \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} G(k) e^{-ikx} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} F(q) e^{-iq(x-y)}. \quad (24.35)$$

So long as these transforms exist and converge, we can exchange the order of integration. This becomes

$$f * g = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dk dq G(k) F(q) e^{-iqx} \int_{-\infty}^{\infty} dy e^{i(q-k)y}, \quad (24.36)$$

where the  $dy$  integral is the delta function. Its value is  $2\pi\delta(k-q)$ , so now we have

$$f * g = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} G(k) F(k) e^{-ikx}, \quad (24.37)$$

which shows that  $f * g$  is the inverse Fourier transform of  $G(k)F(k)$ . Hence  $(f * g)^T = F(k)G(k)$ .  $\square$

There's one slight complication we'll deal with later, which is that sometimes the Fourier transform of the convolution exists when the individual Fourier transforms do not.

Lecture 25.

**Monday, March 9, 2020**

Today we'll continue our discussion of convolutions (*Faltung*). The convolution was defined as

$$f * g = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} g(y) f(x-y) = \int \frac{dk}{\sqrt{2\pi}} G(k) F(k) e^{-ikx}, \quad (25.1)$$

where  $F$  and  $G$  are the Fourier transforms of  $f$  and  $g$  respectively. In particular, if we take  $x = 0$  then we get the result known as the Parseval relation,

$$\int dy g(y) f(-y) = \int dk G(k) F(k). \quad (25.2)$$

This almost looks like an overlap integral, but we can rewrite a bit. Let us define a function  $\hat{f}(y)$  such that

$$\hat{f}^*(-y) = f(y). \quad (25.3)$$

Then its Fourier transform  $\hat{F}$  is just the complex conjugate of  $F$ , i.e.

$$F(k) = [f(y)]^T = [\hat{f}^*(-y)]^T = \hat{F}^*(k). \quad (25.4)$$

Thus

$$\langle \hat{f} | g \rangle = \int dy \hat{f}^*(y) g(y) = \int dk \hat{F}^*(k) G(k) = \langle \hat{F} | G \rangle. \quad (25.5)$$

We find that the Fourier transform is unitary; it preserves the inner product on the space of square integrable functions. That is,

$$\langle f | g \rangle = \langle F | G \rangle = \langle Uf | Ug \rangle \quad (25.6)$$

for  $F = Uf$ , where we've written the Fourier transform as a (linear) operator  $U$  on functions (vectors in a Hilbert space). In other words, going from coordinate (position) space to momentum space is just a unitary transformation.

Now suppose we have an equation we wish to solve,

$$\mathcal{L}y = J \quad (25.7)$$

for  $\mathcal{L}$  some differential operator. Then

$$y(x) = \int_{-\infty}^{\infty} J(y)G(x-y)dy = \int dk G^T(k)J^T(k)e^{-ikx}. \quad (25.8)$$

That is, we can determine the Green's function for an operator (given appropriate boundary conditions) and then Fourier transform the convolution integral required to solve the inhomogeneous equation.

For instance, we can write an electrostatic potential as

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{\rho^T(k)}{k^2} e^{-i\mathbf{k} \cdot \mathbf{r}}. \quad (25.9)$$

This gives us the electrostatic potential in terms of the Fourier transform of the charge distribution.

We can define multiple convolutions as well.

$$\begin{aligned} (f * (g * h))(x) &= \int \frac{dy}{\sqrt{2\pi}} (g * h)(y) f(x-y) \\ &= \int \frac{dy}{\sqrt{2\pi}} f(x-y) \int \frac{dz}{\sqrt{2\pi}} h(z) g(y-z) \\ &= \int \frac{d\tilde{y}}{\sqrt{2\pi}} \frac{dz}{\sqrt{2\pi}} f(\tilde{y}) g(\tilde{y} - z - x) h(z) \end{aligned}$$

with  $\tilde{y} = x - y$ . Equivalently we could use  $\tilde{z} = y - z$  and write this as

$$\int \frac{d\tilde{y}}{\sqrt{2\pi}} \frac{d\tilde{z}}{\sqrt{2\pi}} f(\tilde{y}) g(\tilde{z}) h(x - \tilde{y} - \tilde{z}). \quad (25.10)$$

It follows that convolution is associative, and moreover one can show that  $f * g = g * f$ , so convolution gives a commutative algebra.

For the electrostatic potential

$$\psi(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (25.11)$$

the interaction energy of two charge distributions is

$$\begin{aligned} W &= \int d^3r \rho(\mathbf{r}) \psi(\mathbf{r}) \\ &= \frac{1}{4\pi} \int d^3r d^3r' \rho(\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}'). \end{aligned}$$

Now we recognize this as a convolution integral and write this in terms of the Fourier transforms to find that

$$W = 4\pi \int d^3k d^3k' \rho^T(\mathbf{k}) \frac{1}{k^2} \rho^T(-\mathbf{k}'). \quad (25.12)$$

In a sense, the relation

$$[g * f]^T = F(k)G(k) \quad (25.13)$$

defines the Fourier transform. Conversely

$$[F * G]^T = fg. \quad (25.14)$$

That is, a convolution in one space becomes a product in the Fourier-transformed space. Notice that the converse is not exactly true:

$$[fg]^T = F * G \text{ if } F \text{ and } G \text{ exist.} \quad (25.15)$$

Sometimes  $F$  does not exist, but  $f$  may have a Taylor expansion and we can write the Fourier transform of a product order-by-order for some  $\sum a_n [x^n g]^T$ . Moreover we know that multiplying by  $x$  corresponds to taking  $i \frac{\partial}{\partial k}$  derivatives of the Fourier transform. That is, if  $f$  has a Taylor series, then

$$[f(x)g]^T = \sum a_k \left( i \frac{\partial}{\partial k} \right)^k g^T(k) = f \left( i \frac{\partial}{\partial k} \right) g^T(k). \quad (25.16)$$

We can thereby make sense of expressions like

$$\frac{1}{z-D} = \frac{1}{z} \frac{1}{1-\frac{1}{z}D} = \frac{1}{z} \sum \left( \frac{1}{z} D \right)^n, \quad (25.17)$$

and we recognize that the inverse of a derivative is of course an integral, so an integral can in a sense be written as an infinite sum of derivatives.

Our aim in much of this is often to solve the Schrödinger equation,

$$\left[ \frac{\hat{p}^2}{2m} + V(x) \right] |\psi\rangle = E |\psi\rangle. \quad (25.18)$$

Notice we can rewrite as

$$\left[ \frac{p^2}{2m} + V(i\nabla_{\mathbf{p}}) \right] \psi^T(\mathbf{p}) = E \psi^T(p). \quad (25.19)$$

In general this might be harder to solve than the original equation, depending on how nasty  $V(x)$  is. But for the harmonic oscillator (rescaled so  $m = 1$  and  $\omega = 1$ ) takes a very simple form:

$$\left[ \frac{p^2}{2} + \frac{x^2}{2} \right] \psi = E \psi, \quad (25.20)$$

which is now manifestly symmetric in  $x$  and  $p$  so that the equation in Fourier space looks exactly the same as it did in position space. We work in coordinate space because things appear to be localized in position space, but from a theoretical/calculational perspective either is perfectly good. which is now manifestly symmetric in  $x$  and  $p$  so that the equation in Fourier space looks exactly the same as it did in position space. We work in coordinate space because things appear to be localized in position space, but from a theoretical/calculational perspective either is perfectly good.

Lecture 26.

**Wednesday, March 11, 2020**

We have options for the final. -no final -regular 2 hour exam plus 30 mins for formatting and submission -“long homework” (48 hours)

**Legendre polynomials, revisited** Let us recall that Legendre polynomials are a complete set of functions which solve the differential equation

$$(1-x^2)P''(x) + 2xP'(x) + \lambda P(x) = 0. \quad (26.1)$$

For generic  $\lambda$ , power series (Frobenius) expansions do not terminate, except for special values of  $\lambda$  such that

$$\lambda = l(l+1), \quad l = 0, 1, 2, \dots \quad (26.2)$$

and we label the solutions as  $P_l(x)$ . We also saw the Legendre polynomials when we were solving differential equations with spherical symmetry. The Legendre polynomials emerge when we write the Laplacian in spherical coordinates and attempt to solve the Poisson/Laplace equations.

Consider the equation

$$\Delta\psi = q\delta(\mathbf{r}-\mathbf{r}'). \quad (26.3)$$

This has the Coulomb form of the potential,

$$\psi(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2 - rr' \cos \theta + r'^2}}. \quad (26.4)$$

If the point of interest  $\mathbf{r}$  is much farther away from the origin than the point charge,  $|\mathbf{r}| > |\mathbf{r}'|$ , then we can write

$$\psi(\mathbf{r}) = \frac{1}{r} \frac{1}{1-2tx+t^2} = \frac{1}{r} \sum_{n=0}^{\infty} P_n(x) t^n, \quad (26.5)$$

where we have defined

$$t = \frac{r'}{r} < 1, \quad x = \cos \theta. \quad (26.6)$$

This is an exterior spherical expansion. Notice that the identity

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} P_n(x)t^n \quad (26.7)$$

therefore tells us all the Legendre polynomials—we need only to expand in powers of  $t$  and collect their coefficients in  $x$ . We call this a *generating function* for the Legendre polynomials. There exist also associated Legendre polynomials, which are related to the usual ones by derivatives.<sup>28</sup> One can also analytically continue the argument of the Legendre polynomials, depending on what one is interested in. There's a generating function for Bessel functions, too:

$$e^{-\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x)z^n. \quad (26.8)$$

Generating functions are useful because they reveal certain commonalities between all the special functions in a family, e.g. special values. For instance, what is  $P_n(x = \pm 1)$ ? Well,

$$\sum_{n=0}^{\infty} P_n(x = \pm 1)t^n = \frac{1}{\sqrt{1-2(\pm 1)t+t^2}} = \frac{1}{1 \mp t}. \quad (26.9)$$

We can Taylor expand now.

$$\frac{1}{1-t} = \sum t^n, \quad \frac{1}{1+t} = \sum (-1)^n t^n, \quad (26.10)$$

and that immediately gives us

$$P_n(x = \pm 1) = (\pm 1)^n. \quad (26.11)$$

We can calculate the value at  $x = 0$  too:

$$\sum_{n=0}^{\infty} P_n(x = 0)t^n = \frac{1}{\sqrt{1+t^2}}. \quad (26.12)$$

One finds that

$$P_n(0) = \begin{cases} 0 & n = 2l + 1 \\ \frac{1}{(2n)!} D^{2n}(1+t^2)^{-1} & \text{otherwise} \end{cases} \quad (26.13)$$

since there are no odd powers of  $t$  in the Taylor expansion. This derivative turns out to be  $\binom{-1/2}{n} = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ . Equivalently one may write

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-z)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-2xt+t^2)^n \quad (26.14)$$

in a binomial expansion.

Now, we claimed that the generating function somehow knows about all the properties of the Legendre polynomials. That suggests it should have a link to the differential equation. Well, suppose we derive the sum expression by  $t$ . We get

$$\sum_{n=0}^{\infty} n P_n(x) t^{n-1} = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n, \quad (26.15)$$

and the derivative of the square root expression is

$$\frac{d}{dt} \frac{1}{\sqrt{1-2xt+t^2}} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{\sqrt{1-2xt+t^2}(1-2x+t^2)}. \quad (26.16)$$

It follows that

$$(1-2xt+t^2) \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n = (x-t) \sum_{n=0}^{\infty} P_n(x) t^n, \quad (26.17)$$

---

<sup>28</sup>There's a paper in Reviews of Modern Physics by Bander and Itzykson on properties of the Legendre polynomials and spherical harmonics.

where we have recognized and expanded the square root using the generating function. If we count order by order in  $t$ , we recover a recursion relation on the  $P_n$ :

$$(2n+1)xP_n(x) = (n+1)P_n(x) + nP_{n-1}(x). \quad (26.18)$$

This is not the only recursion relation we can find. We can also find one with derivatives with respect to  $x$ , i.e.

$$\sum P'_n(x)t^n = \frac{d}{dx} \left[ \frac{1}{\sqrt{1-2xt+t^2}} \right]. \quad (26.19)$$

Taking the derivative and comparing by orders of  $t$  again yields

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n, \quad (26.20)$$

a recursion relation between derivatives of Legendre polynomials. This may not look pretty, but in fact it's a direct consequence of symmetry. The relations come from representations of Lie algebras.

Lecture 27.

### Friday, March 13, 2020

Derivatives with respect to  $t$  gave us the recursion relation

$$(2n+1)xP_n(x) = (n+1)P_n(x) + nP_{n-1}(x), \quad (27.1)$$

while derivatives with respect to  $x$  give

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \quad (27.2)$$

If one takes a derivative with respect to  $x$  of the first one and multiplies by two, then the first becomes

$$2(2n+1)xP'_n + 2(2n+1)P_n = 2(n+1)P'_{n+1} + 2nP'_{n-1}. \quad (27.3)$$

Multiplying the latter by  $2n+1$  gives

$$(2n+1)P'_{n+1}(x) + (2n+1)P'_{n-1}(x) = 2(2n+1)xP'_n(x) + (2n+1)P_n(x). \quad (27.4)$$

Taking their difference now gives

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n \quad (27.5)$$

Now we have

$$P'_{n+1} = xP'_n + (n+1)P_n \quad (27.6)$$

and

$$P'_{n-1} = xP'_n - nP_n. \quad (27.7)$$

By shifting the index on the second equation we have

$$P'_n = xP'_{n+1} - (n+1)P_{n+1}. \quad (27.8)$$

Taking derivatives of each now tells us that

$$\begin{aligned} P''_n &= xP''_{n+1} - nP'_{n+1} \\ &= x[xP''_n + (n+2)P'_n] - n[xP'_n + (n+1)P_n], \end{aligned}$$

which is now a single equation in  $P_n$  at the same order  $n$ . Now

$$(1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0, \quad (27.9)$$

which we recognize as the Legendre equation. Moreover, note that the relation

$$(2n+1)xP_n(x) = (n+1)P_n(x) + nP_{n-1}(x),$$

allows us to figure out the Gram-Schmidt procedure. If we pick  $P_0 = 1$  to be normalized then  $P_1 = x$  and now we can read off  $P_2 = 3x^2/2$ .<sup>29</sup>

To wrap up, let us derive the Rodriguez formula for the Legendre polynomials. Let's state it first:

$$P_n = \frac{1}{2^n n!} D^n (x^2 - 1)^n. \quad (27.10)$$

<sup>29</sup>It may be possible to recover the inner product on this set of functions from the recursion relations, but it's easier to do once we have the differential equation and put it into self-adjoint form.



How do we show this works? Well, we can first use a binomial expansion:

$$\begin{aligned}
 P_n &= \frac{1}{2^n n!} D^n (x^2 - 1)^n \\
 &= \frac{1}{2^n n!} \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!k!} D^n x^{2n-2k} \\
 &= \frac{1}{2^n n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{(n-k)!k!} D^n x^{2n-2k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2n-2k)!}{2^n (n-k)!k!(n-2k)!} x^{n-2k}
 \end{aligned}$$

We notice that  $k$  when  $k > n/2$ , the derivatives are more than the power of  $x$ , so we can restrict the sum to  $\lfloor n/2 \rfloor$ .

If we expand the generating function for the Legendre polynomials and collect terms by orders in  $t$ , we find that the Legendre polynomials precisely agree with this closed-form expression for  $P_n$ . Moreover, by taking the Legendre equation and putting it in self-adjoint form,

$$\left[ (1-x^2)P_n' \right]' - n(n+1)P_n = 0, \quad (27.11)$$

we can then show that

$$\int_{-1}^1 \left[ (1-x^2)(P_m P_n' - P_n P_m') \right]' = [n(n+1) - m(m+1)] \int_{-1}^1 P_n P_m = 0 \quad (27.12)$$

unless  $n = m$ .

A more elegant relation using the generating function is to square the generating function and take its integral,

$$\int_{-1}^1 \left( \frac{1}{\sqrt{1-2xt+t^2}} \right)^2 = \sum_{n,k} \int_{-1}^1 dx P_k(x) P_n(x) t^{n+k} = \sum_{n=0}^{\infty} \int_{-1}^1 P_n^2 t^{2n}. \quad (27.13)$$

The integral of the squared generating function is just a log; it is

$$\frac{1}{t} \ln \frac{1+t}{1-t} = 2 \sum_n \frac{t^{2n}}{2n+1}, \quad (27.14)$$

which tells us that

$$\int_{-1}^1 P_n(x)^2 dx = \frac{2}{2n+1}. \quad (27.15)$$

We conclude that

$$\int_{-1}^1 P_n P_m = \delta_{nm} \frac{2}{2n+1}. \quad (27.16)$$

We can easily normalize each of these now that we know the norm by writing

$$\phi(x) = \sqrt{\frac{2n+1}{2}} P_n(x), \quad (27.17)$$

and we can use these like Fourier modes to expand functions as

$$f = \sum_n f_n \phi_n \quad (27.18)$$

with the coefficients given by

$$f_n = \langle \phi_n | f \rangle = \int_{-1}^1 dx \phi_n(x) f(x). \quad (27.19)$$