PHYSICS 204B: METHODS OF MATHEMATICAL PHYSICS II

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Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Monday, January 6, 2020

"What did you do over break?" "Don't ask. No rest for the wicked."
—Mark Samuel Abbott and Nemanja Kaloper

The only outstanding logistical details here are that office hours will be posted later, and the TA is now Morgane König rather than Cameron Langer. All else is basically the same as last quarter.

Let's talk about Green's functions in more than one dimension. Our discussion will be somehwat sketchy, but we'll get a rough idea of the topic. A Green's function is the inverse of a differential operator, and it lives under integrals. In one dimension, we wrote that to solve

$$\mathcal{L}\phi = J, \tag{1.1}$$

we could construct G such that

$$\mathcal{L}G = \delta, \tag{1.2}$$

a function which gives a delta function upon being hit by a differential operator. We would like to solve the problem of finding the linear response of a field ϕ at a point $\mathbf{r}_1$ due to a source $J(\mathbf{r}_2)$ at a point $\mathbf{r}_2$.

Recall that the adjoint of an operator is given by

$$\langle\psi|\mathcal{L}\phi\rangle = \langle\mathcal{L}^\dagger\psi|\phi\rangle, \tag{1.3}$$

and an operator is self-adjoint if

$$\mathcal{L} = \mathcal{L}^\dagger. \tag{1.4}$$

That is, for an operator given by

$$\mathcal{L}\phi = \nabla \cdot (p\nabla\phi) + q\phi, \tag{1.5}$$

we must check that

$$\int \psi^*(\nabla \cdot (p\nabla\phi)) + \int q\psi^*\phi = \int \nabla \cdot (p\nabla\psi^*)\phi + \int q\psi^*\phi. \tag{1.6}$$

The q terms cancel, so we find that

$$\int [\psi^*\nabla \cdot (p\nabla\phi) - \nabla \cdot (p\nabla\psi^*)\phi] = 0, \tag{1.7}$$

and if we integrate by parts, then

$$\int_V dV\, \nabla \cdot (\psi^* p\nabla\phi - (\nabla\psi^*)p\phi) = 0. \tag{1.8}$$

By the divergence theorem,

$$\int_S d\mathbf{S} \cdot p[\psi^*\nabla\phi - (\nabla\psi^*)\phi] = 0. \tag{1.9}$$

Dirichlet or Neumann boundary conditions will guarantee self-adjointness. That is, if the functions vanish on S or their normal derivatives vanish on S, then $\mathcal{L}$ is self-adjoint. There are also mixed conditions we could impose, but Dirichlet and Neumann conditions are sufficient to make our operator Hermitian.

For a Hermitian operator, the corresponding Green's function obeys[1]

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.10}$$

For recall that

$$\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1)|G(\mathbf{r}, \mathbf{r}_2)\rangle = \langle G(\mathbf{r}, \mathbf{r}_1)|\mathcal{L}G(\mathbf{r}, \mathbf{r}_2)\rangle \tag{1.11}$$

by self-adjointness. By the definition of the Green's function and the inner product, we can replace $\mathcal{L}G$ by a delta function and get

$$\int \delta(\mathbf{r} - \mathbf{r}_1)G(\mathbf{r}, \mathbf{r}_2) = G(\mathbf{r}_1, \mathbf{r}_2) \tag{1.12}$$

on the LHS and

$$\int G^*(\mathbf{r}, \mathbf{r}_1)\delta(\mathbf{r}, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.13}$$

Hence

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.14}$$

Now let's construct the eigenfunction expansion of the Green's function. Consider

$$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.15}$$

We're keeping the $\mathbf{r}_1, \mathbf{r}_2$ dependence in G because there might be operators that are not translationally invariant. In other words, we can't assume that quantities depend only on $|\mathbf{r}_1 - \mathbf{r}_2|$. That is,

$$\mathcal{L}|_{\mathbf{r}} \neq \mathcal{L}|_{\mathbf{r}+\mathbf{a}}. \tag{1.16}$$

Suppose we construct the eigenfunctions ϕ_λ of the operator $\mathcal{L}$, such that

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda. \tag{1.17}$$

WLOG we may take them to be orthonormal,

$$\langle \phi_\lambda|\phi_\mu\rangle = \delta_{\lambda\mu}. \tag{1.18}$$

For now, we shall assert that they are a complete set– in general we will have to prove this. The expansion for the delta function is just the completeness relation:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \int_\lambda \phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1), \tag{1.19}$$

[1] This property is also responsible for Green's reciprocity theorem in electromagnetism, i.e. the statement that the potential energy of a charge distribution ρ_2 in a field produced by another distribution ρ_2 is equal to the energy of ρ_1 in the field produced by ρ_2. If you like, the theorem is a special case/application since the Laplacian operator is Hermitian.

since

$$f(\mathbf{r}) = \int_\lambda f_\lambda \phi_\lambda(\mathbf{r}) = \int_{\mathbf{r}_1} f(\mathbf{r}_1)\delta(\mathbf{r}-\mathbf{r}_1) = \int_\lambda \underbrace{\int_{\mathbf{r}_1} f(\mathbf{r}_1)\phi_\lambda^*(\mathbf{r}_1)}_{f_\lambda} \phi_\lambda(\mathbf{r}). \tag{1.20}$$

Now we can put our delta function decomposition back in: suppose that

$$G(\mathbf{r},\mathbf{r}_1) = \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}), \tag{1.21}$$

so that

$$\begin{aligned} \mathcal{L}G &= \mathcal{L}\int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1)\mathcal{L}\phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1)\lambda\phi_\lambda(\mathbf{r}), \end{aligned}$$

and this last expression must be equal to the expansion of the delta function:

$$0 = \int_\lambda [C_\lambda(\mathbf{r}_1)\lambda - \phi_\lambda^*(\mathbf{r}_1)]\phi_\lambda(\mathbf{r}). \tag{1.22}$$

Hence

$$c_\lambda(\mathbf{r}_1) = \frac{\phi^*(\mathbf{r}_1)}{\lambda}, \tag{1.23}$$

so

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda}. \tag{1.24}$$

This makes the hermiticity of G totally clear:

$$G(\mathbf{r}_1,\mathbf{r}_2) = G^*(\mathbf{r}_2,\mathbf{r}_1). \tag{1.25}$$

The only problem might be if we have a zero eigenvalue, in which case we have to be careful. Some people define a generalized Green's function by

$$(\mathcal{L}-Z)G = \delta, \tag{1.26}$$

so that our expression is just modified to

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda - Z}, \tag{1.27}$$

and we can study the $z \to 0$ limit.

Let's consider some examples. The Laplace equation is

$$\boldsymbol{\nabla}^2\phi = J, \tag{1.28}$$

and it is already in self-adjoint form,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}G) = \delta(\mathbf{r}_1-\mathbf{r}_2). \tag{1.29}$$

Suppose we had a solution

$$\int \boldsymbol{\nabla}^2\phi = J \tag{1.30}$$

such that $\int_V J = Q$ and a homogenous solution

$$\boldsymbol{\nabla}^2\chi = 0. \tag{1.31}$$

But now

$$\int_V dV\boldsymbol{\nabla}^2(\phi+\chi) = \int_V J = Q. \tag{1.32}$$

We can rewrite the first expression as a surface integral, $\int d\mathbf{S}\cdot\boldsymbol{\nabla}(\phi+\chi)$. Hence we find that since the integral of the $\boldsymbol{\nabla}\phi$ term is already Q, it must be that

$$\int d\mathbf{S}\cdot\boldsymbol{\nabla}\chi = 0, \tag{1.33}$$

and therefore by the uniqueness theorems, χ is at most a constant. This is an example of a gauge symmetry, actually, but we won't go too much into that. So given appropriate boundary conditions, solutions of the Laplace equation are unique up to a constant.

Now let us note the Laplace equation *is* translationally invariant, so we can write

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r})) = \delta(\mathbf{r}). \tag{1.34}$$

In fact, it is also manifestly spherically symmetric in this form. We've just chosen coordinates to put our charge at the origin. Let us integrate over a spherical region R. Then

$$1 = \int \delta(\mathbf{r}) = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r}) = 4\pi R^2 \frac{dG}{dr}. \tag{1.35}$$

We conclude that

$$G = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.36}$$

which is nothing more than the Coulomb potential for a unit charge.

If we instead enclosed the charge in a Faraday cage (setting the potential to zero somewhere), we would add a homogeneous solution to the Laplace equation to our Green's function in order to fit the new boundary conditions. Note that in 2 dimensions, we would just have $1 = 2\pi R \frac{dG}{dr}$, which gives our Green's function as a log of $|r|$ instead.

Lecture 2.

Wednesday, January 8, 2020

> *"If I had taken a potato instead of a box, I would be hard-pressed to do this. It would be much much harder. Except in two dimensions. Two-dimensional potatoes are very cool because Laplacians in 2 dimensions are special."*
>
> —Nemanja Kaloper

Let's play a bit more with Green's functions. Consider the Laplace equation. We can fit a solution using

$$\boldsymbol{\nabla}^2 (G + F) = \delta. \tag{2.1}$$

That is, we can construct a solution with a source using the Green's function

$$\boldsymbol{\nabla}^2 G = \delta \tag{2.2}$$

and add on homogeneous solutions F such that

$$\boldsymbol{\nabla}^2 F = 0 \tag{2.3}$$

to fit boundary conditions.

Example 2.4. Let us solve the following electrostatics problem. We take a charge and place it inside a grounded conductor shaped like a cube. Can we solve the Laplace equation in this setup?

The charge has spherical symmetry, but the cube does not. That means that the field lines near the sides of the cube will deform so the field becomes normal to the faces of the cube in the static, equilibrium configuration.

Consider a single charge near a grounded plane. We can solve this by the method of images. Suppose the charge sits on the z axis, while the grounded plane is the xy plane. In particular, it sits at a point $(0, 0, a)$ from the origin. We now consider a point a distance ρ from the z axis and at a height z along the z axis.

The method of images says that we can solve for the potential by considering any (fictitious) charge distribution which satisfies the same boundary conditions for the potential. That is, instead of just measuring the potential from the single charge, we can write

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} \tag{2.5}$$

as the potential from the charge alone and suppose we add another image charge Q' on the other side of the grounded plane at a location $(0, 0, -a')$. Then the potential from this configuration of the charge and image charge is

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} + \frac{Q'}{\sqrt{\rho^2 + (z+a')^2}}. \tag{2.6}$$

Now we consider the plane $z = 0$ and see that

$$V(z=0) = \frac{1}{\sqrt{\rho^2 + a^2}} + \frac{Q'}{\sqrt{\rho^2 + a'^2}}. \tag{2.7}$$

But this still seems to depend on ρ. Setting $\rho = 0$ gives

$$\frac{1}{a} + \frac{Q'}{a'} = 0, \tag{2.8}$$

and taking a derivative gives

$$\frac{1}{a^3} + \frac{Q'}{a'^3} = 0, \tag{2.9}$$

so solving gives

$$a = a', \quad Q' = -1. \tag{2.10}$$

The potential on the real side of the grounded plane is just the dipole potential in the region we care about. The other charge is equal and opposite, and equally far away behind the plane.

What about two conducting planes? We can construct a dipole to set the potential on one plane, but then the other won't be grounded. We can add an image dipole outside the other plane (see image). Then we the second plane is okay but the first plane is now not grounded. So we go on adding dipoles in this way ad infinitum, and the overall potential is

$$\sum_{n=-\infty}^{\infty} \frac{1}{[\rho^2 + (z - a + n(2D))^2]^{-1/2}} - \frac{1}{[\rho^2 + (z + a + n(2D))^2]^{-1/2}}. \tag{2.11}$$

Expanded, this is

$$\sum_n \frac{1}{|\mathbf{r} - 2Dn\mathbf{k}|} - \frac{1}{|\mathbf{r} + 2Dn\mathbf{k}|}. \tag{2.12}$$

The sum looks like it might diverge, since each term goes as $1/n$, but in fact since we're dealing with dipoles, the $1/n$ dependence will cancel and away from the corners, this will converge. These image charges help us to satisfy the boundary conditions on the two planes.

The generalization to three dimensions is to extend the periodicity into three dimensions. We get an infinite lattice of image dipoles which is at once simple and complicated.

Let's write a nice fact about multipoles:

$$\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_<^{l+1}} P_l(\cos\theta). \tag{2.13}$$

This simply says that the potential of a multipole drops off as $1/r^{l+1}$, and inside some spherical region it increases as r^l. Its angular dependence is given by the Legendre polynomials. This comes from the fact that

$$\sum x^n P_l(t) = \frac{1}{\sqrt{1 - 2xt + x^2}}, \tag{2.14}$$

the generating function for Legendre polynomials, with $-1 < t < 1$ and $0 < x < 1$.Taking this denominator as $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + 2r_1 r_2 \cos\theta + r_2^2}$, if we pull out r_1 then we get the desired result with this formula.

One more example. Suppose we write the Schrödinger equation as

$$(\boldsymbol{\nabla}^2 + k^2)\psi = 2mV\psi, \tag{2.15}$$

where $k^2 = 2mE$.

We may write the Green's function

$$(\boldsymbol{\nabla}^2 + k^2)G = \delta, \tag{2.16}$$

satisfying

$$\int_{S^2} d\mathbf{S} \cdot \nabla G + k^2 \int_V dV\, G = 1, \tag{2.17}$$

where V is a sphere centered on the origin. In the limit of very small spheres, we shall argue that the second term vanishes. For the k^2 term is constant, whereas as r grows small, the ∇^2 blows up as $1/r^2$. Near the origin, we can solve

$$\int d\mathbf{S} \cdot \nabla G = 1, \tag{2.18}$$

while away from the origin, our delta function is zero. We can exploit the spherical symmetry to rewrite the lLplacian as

$$\frac{1}{r^2}\partial_r(r^2 \partial_r G) + k^2 G = 0. \tag{2.19}$$

If we write $G = \phi/r$ to account for the asymptotic behavior of the Green's function, then we find

$$\phi'' + k^2 \phi = 0. \tag{2.20}$$

This has solutions

$$\phi = e^{\pm ikr}. \tag{2.21}$$

The $+$ solution is an outgoing wave, while the $-$ solution is an incoming wave. Now

$$G = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{2.22}$$

We can now treat the Schrödinger equation as an integral equation with a source. Thus

$$\psi(\mathbf{r}_1) = 2m \int dr_2^3\, G(\mathbf{r}_1, \mathbf{r}_2) V(\mathbf{r}_2) \psi(\mathbf{r}_2). \tag{2.23}$$

This is self-consistent, but is it useful? It will be if V is small. Thus we can expand ψ perturbatively in orders of V, considering each order of scattering and writing

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots \tag{2.24}$$

where each order is given by solving the integral equation with the previous order.

Lecture 3.

Friday, January 10, 2020

"If wavelength is very small compared to the interstitial distance, this just goes through like hot knife through butter. That's why we exist."
—Nemanja Kaloper

Last time, we rewrote the Schrödinger equation as an integral equation,

$$\psi(\mathbf{r}_1) = -\frac{2m}{4\pi} \int \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|} V(\mathbf{r}_2) \psi(\mathbf{r}_2). \tag{3.1}$$

To solve this perturbatively, we require that the potential is small,

$$V = \lambda W \tag{3.2}$$

for some $\lambda \ll 1$, and moreover that these integrals converge.

We must make a guess for the initial wave as

$$\psi_0 = e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{3.3}$$

which solves the homogeneous equation, and hence the first-order correction is

$$\psi_1 = -\frac{2m}{4\pi} \int d^3 r_2 \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|} V(\mathbf{r}_2) e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{3.4}$$

This will give us the order λ correction.

Fourier series Consider a function $f(x)$ defined on the interval $[-\pi, \pi]$. As long as a function satisfies certain "Dirichlet conditions," it has a valid Fourier decomposition that converges in the mean. These conditions require that the function has only finitely many finite discontinuities. So any smooth function on the integral will work, but functions with a finite jump can still give us a valid Fourier decomposition. It turns out that at the point of the discontinuity, the Fourier series will have the mean value of the limits from the left and right.

Things that will break the decomposition include infinite discontinuities ($1/x$) or that have infinitely many finite discontinuities. There are functions which are not Dirichlet which have Fourier decompositions, but the conditions are sufficient to guarantee the convergence of the Fourier decomposition.

The decomposition takes the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{3.5}$$

with

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx \tag{3.6}$$

so that $a_0/2$ is the mean value of the function. The coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx) dx. \tag{3.7}$$

This is nothing more than the decomposition of a function in an infinite-dimensional vector space. Thanks to the orthogonality of cosines and sines, we can treat this as a complete basis and compute the components, which are exactly the Fourier coefficients.

On our function space, there is an inner product defined by

$$f \cdot g = \int_{-\pi}^{\pi} f^*(x) g(x). \tag{3.8}$$

Notice that

$$0 = \int_{-\pi}^{\pi} \sin nx \cos mx dx, \tag{3.9}$$

since the trig addition formulas tell us we are integrating sines $\sin(n+m)x$ and $\sin(n-m)x$ over whole intervals.[2] If we have

$$\int_{-\pi}^{\pi} \sin nx \sin nx dx = \frac{1}{n}\int_{-\pi}^{\pi} \sin nx d(\cos nx) = \frac{1}{n}(\sin nx \cos nx)|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \cos nx d(\sin nx). \tag{3.10}$$

The boundary terms are zero since $\sin n\pi = 0$, and we get that[3]

$$\int \sin^2 nx dx = \int \cos^2 nx dx. \tag{3.11}$$

Since

$$1 = \sin^2 nx + \cos^2 nx \implies \underbrace{\int_{-\pi}^{\pi} dx}_{2\pi} = \int_{-\pi}^{\pi} dx(\sin^2 nx + \cos^2 nx) = 2\int_{-\pi}^{\pi} dx \sin^2 nx, \tag{3.12}$$

we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx \sin^2 nx = 1. \tag{3.13}$$

We can also do this with complex exponentials pretty easily. We find that[4]

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx \sin nx \sin mx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos nx \cos mx = \delta_{nm}. \tag{3.14}$$

[2] Equivalently the function is odd and integrated over a symmetric interval.

[3] We can also derive this by noting that the integral of $\sin nx$ over $-\pi$ to π (i.e. integer multiples of the period) is equal to the integral of $\cos nx$ over the same interval, since a sine is just a phase-shifted cosine. Hence $\int_{-\pi}^{\pi} \sin(nx) dx = \int_{-\pi}^{\pi} \cos(nx) dx$ and the same is true for their squares. Since they both complete the same number of full periods over the interval, their integrals must be the same. So we can immediately conclude that $\int_{-\pi}^{\pi} \sin^2 x dx = \int_{-\pi}^{\pi} \cos^2 x dx$.

[4] See the end of this section for a bit of extra detail.

Let f_N be the truncated Fourier series approximation of f,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx, \tag{3.15}$$

One may show that

$$0 \le |f - f_N|^2 = \int f^2 + f_N^2 - 2 f f_N, \tag{3.16}$$

which is just the Bessel's inequality. We have a term independent of the a_ns, one quadratic in them, and one linear in them.

It is sometimes convenient to use a more symmetric expansion for the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{3.17}$$

This is perfectly equivalent to sines and cosines, as we know. The matching works out as

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n + i b_n), \quad c_{-n} = c_n^* = \frac{1}{2}(a_n - i b_n) \tag{3.18}$$

assuming that a, b are real.

These eigenfunctions satisfy the Sturm-Liouville problem

$$y'' = -\lambda y \tag{3.19}$$

where

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi). \tag{3.20}$$

The eigenvalues are $\lambda = n^2$ and the sets of solutions are

$$y_n = \sin nx, \cos nx. \tag{3.21}$$

Non-lectured: exponentials and integrals of $\sin nx$, two ways We can compute the integral of $\sin^2 nx$ by passing to the complex exponential form,

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}. \tag{3.22}$$

Then for $n \neq m$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left(\frac{e^{inx} - e^{-inx}}{2i}\right)\left(\frac{e^{imx} - e^{-imx}}{2i}\right) \tag{3.23}$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \left(e^{i(n+m)x} + e^{-i(n+m)x} - e^{-ix(n-m)x} - e^{ix(n-m)x}\right) \tag{3.24}$$

$$= -\frac{1}{2\pi}\left[\frac{e^{ix(n+m)}}{i(n+m)} - \frac{e^{-ix(n+m)}}{i(n+m)} + \frac{e^{-ix(n-m)}}{i(n-m)} - \frac{e^{ix(n-m)}}{i(n-m)}\right]_{-\pi}^{\pi} \tag{3.25}$$

However, notice that these exponentials are all at least 2π periodic. That is,

$$e^{i(2\pi)(n+m)} = 1 \implies e^{i\pi(n+m)} = e^{-i\pi(n+m)} \tag{3.26}$$

for $n, m \in \mathbb{Z}$. That means that all these exponential terms vanish when $n \neq m$. If $n = m$, then the second line, Eqn. (3.24), becomes

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \left(e^{i(n+m)x} + e^{-i(n+m)x} - 2\right) = -\frac{1}{2\pi}(-2\pi) = 1. \tag{3.27}$$

We conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \delta_{nm}. \tag{3.28}$$

But wait, there's more! Here is a high-powered way to do this integral using contour integration. If you don't know contour integration, read ahead to when these notes cover them and come back later. Let's write this integral suggestively in terms of θ,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta. \tag{3.29}$$

Since θ runs from $-\pi$ to π, we can think of this integral as tracing a unit circle contour in the complex plane (cf. Arfken 11.8). That is, we shall make the change of variables

$$z = e^{i\theta} \tag{3.30}$$

so that

$$d\theta = -i\frac{dz}{z}, \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = -i\frac{z^n - z^{-n}}{2}. \tag{3.31}$$

Now our integral becomes

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta &= -\frac{i}{\pi}\oint_C \left(-i\frac{z^n - z^{-n}}{2}\right)\left(-i\frac{z^m - z^{-m}}{2}\right)\frac{dz}{z} \\ &= \frac{i}{4\pi}\oint (z^{n+m} - z^{n-m} - z^{-n+m} + z^{-n-m})\frac{dz}{z}.\end{aligned}$$

We can evaluate this with the Cauchy residue theorem! Just look for poles inside the unit circle. Actually, this function only has poles at the origin. We may recall that anything that's not a simple pole in this sort of integral will have zero residue. For instance, suppose $n = 1, m = 2$. Then our integral would be

$$\frac{i}{4\pi}\oint (z^{1+2} - z^{1-2} - z^{-1+2} + z^{-1-2})\frac{dz}{z}. \tag{3.32}$$

Just by power counting, none of these are simple poles. For the z^{n+m} term to give a simple pole, we would need $n + m = 0$, and for $n, m \in \mathbb{Z}_+$ there are no solutions to this equation. The same is true for the z^{-n-m} term. Our only hope is for the middle terms to give us a simple pole, and this happens exactly when $n = m$. When $n = m$, we have

$$\frac{i}{4\pi}\oint (z^{2n} - 2 + z^{-2n})\frac{dz}{z} = \frac{i}{4\pi}(2\pi i(-2)) = 1 \tag{3.33}$$

by the Cauchy residue theorem. As before, we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = \delta_{nm}. \quad \boxtimes \tag{3.34}$$

Lecture 4.

Monday, January 13, 2020

"If I attempted to do it [the completeness relation on Fourier modes] any other way, any self-respecting mathematician would decapitate me immediately. So I won't do it because I like my neck just the way it is."

—Nemanja Kaloper

Suppose we have

$$\phi_n = \frac{e^{inx}}{\sqrt{2\pi}}, \tag{4.1}$$

such that the inner product is given by

$$\langle \phi_n | \phi_m \rangle = \int \phi_n^* \phi_m. \tag{4.2}$$

For a general function f let us now take its inner product with the basis vectors,

$$c_n = \langle \phi_n | f \rangle \tag{4.3}$$

and for a general set of coefficients $\{\bar{c}_n\}$, write the function (linear combination)

$$\bar{f}_N = \sum \bar{c}_n \phi_n. \tag{4.4}$$

We shall show that the Fourier decomposition of f converges to f faster than any other sequence $\bar{f}_N$. To do this, let us use the triangle inequality and write

$$0 \leq |f - \bar{f}_N|^2 = \langle f - \bar{f}_N | f - \bar{f}_N \rangle \tag{4.5}$$

$$= |f|^2 - \langle \bar{f}_N | f \rangle - \langle f | \bar{f}_N \rangle + |\bar{f}_N|^2 \tag{4.6}$$

$$= |f|^2 - 2\text{Re}\langle \bar{f}_N | f \rangle + |\bar{f}_N|^2. \tag{4.7}$$

We can compute these inner products explicitly.

$$\langle \bar{f}_N | \bar{f}_N \rangle = \sum_{n,m} \bar{c}_n^* \bar{c}_m \phi_n \phi_m = \sum_n \bar{c}_n^* \bar{c}_n = \sum_n |\bar{c}_n|^2. \tag{4.8}$$

The cross-terms are

$$\langle \bar{f}_N | f \rangle = \sum \bar{c}_n^* \langle \phi_n | f \rangle = \sum \bar{c}_n^* c_n. \tag{4.9}$$

Plugging back in, we have

$$0 \leq |f|^2 - \sum (\bar{c}_n^* c_n + \bar{c}_n c_n^*) + \sum \bar{c}_n^* \bar{c}_n. \tag{4.10}$$

This is guaranteed to be positive semi-definite by the positivity of the norm. This defines a distance $D(f, \bar{f}_N)$ on the space of functions.

Notice there are really two sets of free parameters, c_n and $\bar{c}_n$ or equivalently c_n and $\bar{c}_n^*$. If we take a derivative with respect to $\bar{c}_n^*$ to extremize this, we get

$$\bar{c}_n = c_n, \bar{c}_n^* = c_n^*. \tag{4.11}$$

That is, we want to approximate our function as fast as we can, and the best way to do it is by picking the Fourier coefficients.[5] The derivative guarantees it. This is exactly the same as the least-squares fit, just by a different process.

The convergence of these sorts of series may be in question when the metric on the space is not positive definite (e.g. Minkowski signature). In that case, these approximations may converge to something but perhaps not uniquely, and we may require additional information to ensure uniqueness.

Fourier series have some nice properties. Notice that since the Fourier basis elements $\cos nx, \sin nx$ are periodic in 2π, the entire Fourier series is also periodic in 2π. Note that the function we are approximating need not itself have the same value at $-\pi$ and π; this just counts as a single discontinuity.

What if our function is periodic on a different interval, say $2L$? That is,

$$f(x + 2L) = f(x). \tag{4.12}$$

Then define $x = \alpha \bar{x}$ such that we can write

$$x + 2L = \alpha(\bar{x} + 2\pi), \tag{4.13}$$

since we know how to deal with functions which are 2π periodic. We find immediately that

$$\alpha = L/\pi. \tag{4.14}$$

Hence our Fourier expansion can now be written as

$$f = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \tag{4.15}$$

All this corresponds to is changing the radius of the unit circle. The circle has a radius given by L instead of 1. Then the coefficients are just

$$a_0 = \frac{1}{L} \int_{-L}^{L} dx\, f(x) \tag{4.16}$$

and the other coefficients are[6]

$$a_n = \frac{1}{L} \int_{-L}^{L} dx f(x) \cos \frac{n\pi x}{L}, \quad b_n = \frac{1}{L} \int_{-L}^{L} dx f(x) \sin \frac{n\pi x}{L}. \tag{4.17}$$

[5] If you like, this is a discrete version of a functional derivative where we just vary the components of each function.

[6] You can do this by just making a change of variables in the integral to get your limits of integration how you want them.

Equivalently we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-\frac{in\pi x}{L}}. \tag{4.18}$$

Let us define a function which is given on the fundamental domain as

$$f(x) = x, \quad -\pi < x < \pi. \tag{4.19}$$

and 2π periodic, so e.g. $f(x) = x - 2\pi, \pi < x < 3\pi$ and so on. This is a sawtooth wave. What is the Fourier decomposition of this wave? One can do the integrals in Eq. (4.17) to figure it out. One finds that all the cosine coefficients go away because this function is odd. It will only have sines:

$$x = f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}. \tag{4.20}$$

This does not converge absolutely, if one checks. The ratio test is indeterminate. This is like a harmonic series. But it should converge, and the reason it does is because it is an alternating series. Alternating series have nicer convergence properties in general.

We can also evaluate this at $\pi/2$. Remember this was the Fourier expansion of x. So then by plugging $x = \pi/2$ into Eqn. (4.20), we get

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right], \tag{4.21}$$

which lets us sum a series we might not have expected.

We could actually do this for a triangle wave instead, defined as $|x|$ on $-\pi, \pi$. If we wrote down the Fourier series we would find that

$$f(x) = \frac{\pi}{2} - \sum \frac{4}{\pi}\frac{\cos((2n+1)x)}{(2n+1)^2}. \tag{4.22}$$

This series is no longer alternating, but it actually converges faster, like $\sum 1/n^2$, since the function is not discontinuous like our sawtooth wave. The derivative is discontinuous, yes, but we still have nicer convergence. In general, the Fourier decompositions of functions which are smoother will converge faster.

It's fairly clear that we can add two Fourier series to get a new one.. But in fact we can also integrate Fourier series by linearity (assuming the original sum converges). When we integrate Fourier series, our sines and cosines get factors of $1/n$ in the series coefficients, i.e. for $f(x)$ given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{4.23}$$

the integral of $f(x)$ is

$$\int_{-\pi}^{x} f(x')dx' = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} a_n \frac{\sin nx}{n} - b_n \frac{\cos nx}{n} + \text{ constant boundary terms.} \tag{4.24}$$

These factors of $1/n$ mean that the convergence properties of the integrated function are nicer than the original, i.e. it converges at least as fast as the expansion of $f(x)$.

Lecture 5.

Wednesday, January 15, 2020

> *"I'm gonna skip this example of the full wave rectifier, blah blah blah. I never cared much for electronics. This is not a statement of preference but of personal incompetence."*
>
> —Nemanja Kaloper

Today we will continue our discussion of Fourier series. In general the integral of a Fourier series is not itself a Fourier series, since it has a linear term. But its convergence properties are nicer than the original. What's happening is that we put frequencies in the denominator, which makes the different frequency modes decouple more.

Recall we derived the Fourier series of a sawtooth wave,

$$x = 2\sum_{n=1}^{\infty}(-1)n+1\frac{\sin nx}{n}. \tag{5.1}$$

Can we take derivatives of this, as

$$1 = 2\sum(-1)^{n+1}\cos nx? \tag{5.2}$$

Alas, this is nonsense. Suppose we evaluated this at $x = 0$. Then we would find

$$1 = 2(1-1+1-1+\dots) \tag{5.3}$$

which is clearly not convergent in the usual sense.[7]

However, if we take the Fourier expansion of $|x|$, we can get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum\frac{\cos nx}{n^2}, \tag{5.4}$$

which converges faster than $1/n$ (i.e. uniformly). Hence we can define the derivative such that

$$\frac{d}{dx}|x| = +\frac{4}{\pi}\sum\frac{\sin nx}{n} = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases} \tag{5.5}$$

This is correct, as

$$\frac{d}{dx}|x| = 2\theta(x) - 1, \tag{5.6}$$

with θ the Heaviside step function.

Suppose now we are handed a series

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n}. \tag{5.7}$$

What function does this correspond to? Let's think qualitatively. The cosine is bounded by ± 1, so this might converge. The point $x = 0$ is dangerous since our series becomes a harmonic series, which is divergent (albeit only logarithmically). This is true at any integer multiple of 2π, in fact.

Let's make sense of this as follows. We shall write this as

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = \mathrm{Re}\sum\frac{e^{inx}}{n}. \tag{5.8}$$

Now e^{inx} is a point on a unit circle in the complex plane, and we can further think of this as a limit

$$\mathrm{Re}\sum\frac{e^{inx}}{n} = \lim_{r\to 1}\mathrm{Re}\left(\sum\frac{(re^{ix})^n}{n}\right) = |\sum\frac{q^n}{n}| < \sum\frac{|q|^n}{n}. \tag{5.9}$$

If we take this limit where $|q| = r < 1$ then this surely converges.

Notice that

$$\frac{1}{1-q} = \sum q^n. \tag{5.10}$$

This is not quite what we want, since we're missing the n in the denominator. But we can do a trick.

$$\int\frac{dq}{1-q} = \sum_{n=0}^{\infty}\int q^n dq = \sum_{n=0}^{\infty}\frac{q^{n+1}}{n+1}. \tag{5.11}$$

If we shift the dummy variable (noting that $n = 0$ just gives q), we can write

$$-\ln(1-q) = \sum_{n=0}^{\infty}\frac{q^n}{n}. \tag{5.12}$$

It follows that our function was

$$\ln(2-2\cos x)^{1/2} = -\ln[2\sin(x/2)]. \tag{5.13}$$

[7]However, see https://en.wikipedia.org/wiki/Grandi%27s_series. One can define a sensible Cesàro sum which actually is $1/2$, but it's not what we mean by convergence.

This passes a reasonable sanity check, which is that when x is a multiple of 2π, then this series is the log of zero, which is clearly divergent.[8]

Consider some square-wave potential which is 2π periodic, such that

$$f(x) = \begin{cases} -\pi < x < 0 & 0, \\ 0 < x < \pi & h. \end{cases} \tag{5.14}$$

The Fourier transform is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi} \sum \frac{\sin(2n+1)x}{2n+1}. \tag{5.15}$$

Note that only odd sines appear, since the odd sines will have nonzero integral over the half-interval $0, \pi$, while the even ones vanish.

We can also switch to exponential notation,

$$f(x) = \sum c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x) e^{-inx}. \tag{5.16}$$

What's interesting now is to consider the finite truncation of the Fourier series,

$$f_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} e^{inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) e^{-int}. \tag{5.17}$$

If this sum were infinite, we could not exchange the order of integration and summation with impunity. But for finite N, there are no problems with convergence. So we can do this freely and write

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \sum_{n=-N}^{N} e^{in(x-t)}. \tag{5.18}$$

This is in fact just a sum of cosines. For notice that if we define

$$q = e^{i(x-t)}, \tag{5.19}$$

then our sum is

$$S = \sum_{n=-N}^{N} q^n = \frac{1}{q^N} \sum_{n=0}^{2N} q^n \tag{5.20}$$

Note that the sum with $N \to \infty$ is the geometric series $\sum_{n=0}^{\infty} q = \frac{1}{1-q}$. It follows that the finite sum can be written as the difference of two infinite sums,

$$\sum_{n=0}^{2N} q^n = \sum_{n=0}^{\infty} q^n - \sum_{n=2N+1}^{\infty} q^n = S_\infty - q^{2N} S_\infty = \frac{1-q^{2N+1}}{1-q}. \tag{5.21}$$

We get that this is

$$\frac{q^N - q^{-N}}{1-q} = \frac{\sin(N+\frac{1}{2})(x-t))}{\sin\frac{x-t}{2}}. \tag{5.22}$$

Hence

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \left[\frac{\sin((N+1/2)(x-t))}{\sin\frac{x-t}{2}} \right], \tag{5.23}$$

which is the best approximation to our function in terms of a finite truncation. The goal is now to determine what properties f must have such that f_N actually converges nicely to f.

Note also that the term in the brackets is just a delta function. It is simply the sum $\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-t)}$, and in a sense this is just the statement of the completeness of complex exponentials.

[8] However, something a little weird happens when we take x to be e.g. $-\pi$. On the RHS we have a log of something negative, which is only well-defined in terms of the analytic continuation of $\ln(z)$. In fact, the log of a negative number has an imaginary part. The LHS looks like we're summing real numbers, so this is a bit unusual; it has to do again with the lack of uniform convergence.

Lecture 6.

Friday, January 17, 2020

"If you're going to try to violate decoupling, don't try to do it with something as stupid as continuous functions." "Bastards." "That's correct."
—Nemanja Kaloper and Mark Samuel Abbott

Last time, we found the explicit formula for the partial sums of a Fourier series to be

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, f(t)\left[\frac{\sin((N+1/2)(x-t))}{\sin\frac{x-t}{2}}\right]. \tag{6.1}$$

Let us consider a Heaviside step function,

$$f(x) = \frac{h}{2}(2\Theta(x)-1). \tag{6.2}$$

We found the Fourier series to be

$$f(x) = \frac{2h}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)x]}{2n+1}, \tag{6.3}$$

and we'd like to study what happens when the sum is finite rather than infinite. That is, let us plug in $f(t)$. Then

$$\begin{aligned} f_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\left(\frac{h}{2}(2\Theta(t)-1)\right)\left[\frac{\sin((N+1/2)(x-t))}{\sin\frac{x-t}{2}}\right] & (6.4)\\ &= \frac{h}{4\pi}\left[-\int_{-\pi}^{0} dt\frac{\sin((N+1/2)(x-t))}{\sin\left[\frac{x-t}{2}\right]} + \int_0^{\pi} dt\frac{\sin((N+1/2)(x-t))}{\sin\left[\frac{x-t}{2}\right]}\right]. & (6.5)\end{aligned}$$

Since this is a finite series and the function is piecewise continuous, we can now take the limit of the interpolating functions $\lim_{N\to\infty} f_N(x)$ to get the infinite sum.

Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dt\frac{\sin[(N+1/2)(x-t)]}{\sin\left[\frac{x-t}{2}\right]} = 1. \tag{6.6}$$

This is just the integral of (the limiting series of) a delta function. Why? This was just the sum

$$\sum_{n=-N}^{N} e^{in(x-t)} = \frac{1}{2\pi}\frac{\sin[(N+1/2)(x-t)]}{\sin\left[\frac{x-t}{2}\right]}, \tag{6.7}$$

and if we integrate the LHS from $-\pi$ to π, then we have

$$\int_{-\pi}^{\pi} dt \sum_{n=-N}^{N} e^{in(x-t)} = \sum_{n=-N}^{N} e^{inx}\int_{-\pi}^{\pi} dt\, e^{-int} = \sum_{n=-N}^{N} e^{inx}(2\pi\delta_{n,0}), \tag{6.8}$$

so the other terms in the sum drop out.

We can rewrite the terms in our integrals (6.5) in terms of new variables $-s = x-t$ in the first integral and $s = x-t$ in the second integral to write

$$\begin{aligned} f_N(x) &= \frac{h}{4\pi}\left[-\int_{-\pi}^{0} dt\frac{\sin((N+1/2)(x-t))}{\sin\left[\frac{x-t}{2}\right]} + \int_0^{\pi} dt\frac{\sin((N+1/2)(x-t))}{\sin\left[\frac{x-t}{2}\right]}\right]\\ &= \frac{h}{\pi}\left[-\int_{-\pi-x}^{-x} ds\frac{\sin((N+1/2)(-s))}{\sin\left[\frac{-s}{2}\right]} - \int_{x}^{-\pi+x} ds\frac{\sin((N+1/2)s)}{\sin\left[\frac{s}{2}\right]}\right]\\ &= \frac{h}{4\pi}\left[-\int_{-\pi-x}^{-x} ds\frac{\sin((N+1/2)s)}{\sin\left[\frac{s}{2}\right]} + \int_{-\pi+x}^{x} ds\frac{\sin((N+1/2)s)}{\sin\left[\frac{s}{2}\right]}\right] & (6.9)\end{aligned}$$

Let's justify the signs here. There are two sign flips on the second term. One comes from $ds = -dt$, while

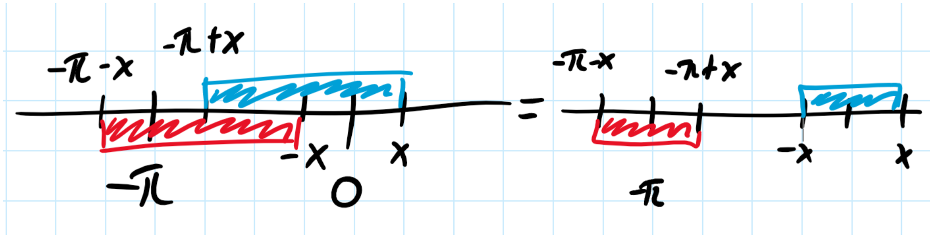

FIGURE 1. An illustration of how to rewrite the integration limits in Eqn. (6.9).

the other comes from exchanging the limits of integration in the third line. There are no sign flips on the first term, since

$$\sin((N+1/2)(-s)) = -\sin((N+1/2)s), \quad \sin(-s/2) = -\sin(s/2)$$
$$\implies \frac{\sin((N+1/2)(-s)}{\sin(-s/2)} = \frac{\sin((N+1/2)s)}{\sin(s/2)}.$$

We can now shuffle the limits of integration in Eqn. (6.9) a bit (see Fig. 1) to get

$$f_N(x) = \frac{h}{4\pi}\left[+\int_{-x}^{+x} ds \frac{\sin((N+1/2)s)}{\sin\left[\frac{s}{2}\right]} - \int_{-\pi-x}^{-\pi+x} ds \frac{\sin((N+1/2)s)}{\sin\left[\frac{s}{2}\right]}\right]. \tag{6.10}$$

One way to do this formally is to recognize that the relative sign between these two integrals allows us to rewrite the definite integrals as differences of indefinite integrals (denoted Φ):

$$\begin{aligned}
-\int_{-\pi-x}^{-x} f(x)dx + \int_{-\pi+x}^{x} f(x)dx &= -[\Phi(-x) - \Phi(-\pi-x)] + [\Phi(x) - \Phi(-\pi+x)] \\
&= (\Phi(x) - \Phi(-x)) - (\Phi(-\pi+x) - \Phi(-\pi-x)) \\
&= \int_{-x}^{x} f(x)dx - \int_{-\pi-x}^{-\pi+x} f(x)dx.
\end{aligned}$$

Graphically or otherwise, we arrive at the result (6.10).

Now what happens as $x \to 0$? Our integration measure ds gets vanishingly small, and if the integrand were finite on that range, the integral would vanish. Well, the second term, the integral on an interval around $-\pi$, is bounded, since $\sin((N+1/2)s)$ is bounded around $s = -\pi$, and the denominator

$$\lim_{s\to-\pi} \sin(s/2) = \sin(-\pi/2) = -1$$

is perfectly well-behaved. So indeed the second integral in (6.10) is zero. More precisely, it is proportional to the interval length $2x$, which goes to zero as $x \to 0$.

The first term is different. We are taking an integral in a region around $s = 0$, but $\sin(0)$ is zero, so the denominator will cause the integrand to diverge as $s \to 0$. The numerator $\sin((n+1/2)s)$ also goes to zero, so we get an indeterminate ratio which we can work out with L'Hôpital's rule or equivalently by Taylor expansion around $s = 0$. That is,

$$\lim_{x\to0} \frac{h}{4\pi} \int_{-x}^{x} \frac{\sin((N+1/2)s)}{\sin(s/2)} ds = \lim_{x\to0} \frac{h}{4\pi} \int_{-x}^{x} ds \frac{(N+1/2)s}{(s/2)} = \lim_{x\to0} \frac{h}{2\pi} \int_{-x}^{x} ds (N+1/2) \tag{6.11}$$

The factors of s have cancelled. Now we must be careful about the order of limits. If we hold N fixed and perform the ds integral, then we get

$$\lim_{x\to0} f_N(x) = \lim_{x\to0} \frac{h}{2\pi}(N+1/2)(2x) = 0. \tag{6.12}$$

It follows that

$$f_N(0) = 0 \implies f(0) = \lim_{N\to\infty} f_N(0) = 0. \tag{6.13}$$

That is, the (shifted) step function takes on the value

$$f(0) = \Theta(0) - 1/2 = 0 \implies \Theta(0) = \frac{1}{2}, \tag{6.14}$$

the average of its value on either side of $x = 0$.

Moreover, if we look at the Fourier series approximation of the step function, we find that the approximation overshoots the step. This is called the *Gibbs phenomenon*. Curiously, the overshoot doesn't improve with more terms; in fact, it approaches a limiting value of about 18%. The reason for this is simply that we're trying to approximate a discontinuous function with a smooth one. We overshoot because the derivative cannot change too quickly. Next time, we will study the behavior of

$$\frac{h}{2\pi}\int_0^x \frac{\sin[(N+1/2)s]\sin(s/2)}{d} x, \tag{6.15}$$

and we can study the behavior of this integral in terms of a new variable

$$\zeta = (N + \frac{1}{2})s. \tag{6.16}$$

Then we have an integral

$$\frac{h}{2\pi}\int_0^{(N+1/2)x} \frac{\sin(\zeta)}{\sin(\zeta/(2N+1))} d\zeta \approx \frac{h(2N+1)}{2\pi}\frac{h}{2\pi}\int_0^{(N+1/2)x}\frac{\sin(\zeta)}{\zeta} d\zeta \tag{6.17}$$

and we can study this in certain limits of ζ.

Lecture 7.

Wednesday, January 22, 2020

"Today I have a 'fold.' Combination of flu and cold. Keep your distance."
—Nemanja Kaloper

Let's recall our key equation for the truncated Fourier series,

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, f(t)\left[\frac{\sin((N+1/2)(x-t))}{\sin\frac{x-t}{2}}\right]. \tag{7.1}$$

We shall start by writing a new integration variable

$$x = -t = -s \leftrightarrow s = t - x. \tag{7.2}$$

It follows that

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} ds f(x+s)\frac{\sin[(N+1/2)s]}{\sin(s/2)}. \tag{7.3}$$

We're integrating over complete intervals (note that while the numerator and denominator are 4π-periodic, their quotient is actually 2π-periodic).[9] So the shifting of the argument by x doesn't actually affect the value of the integral. That is,

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{7.4}$$

By evenness in t, we can write the integral as a symmetrized sum

$$f_N(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi} dt[f(x+t) + f(x-t)]\frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{7.5}$$

Now we take the specific case of the step function

$$f(x) = 2\Theta(x) - 1, \tag{7.6}$$

[9] The best way to see this is by using complex exponentials, i.e. by writing

$$\frac{\sin((N+1/2)x)}{\sin(x/2)} \stackrel{?}{=} \frac{\sin((N+1/2)(x+2\pi))}{\sin(1/2(x+2\pi))} = \frac{e^{i(N+1/2)(x+2\pi)} - e^{-i(N+1/2)(x+2\pi)}}{e^{i(1/2)(x+2\pi)} - e^{-i(1/2)(x+2\pi)}}.$$

Now the 2π factors give $e^{i(2N+1)\pi} = -1$ and $e^{-i(2N+1)\pi} = -1$, so we find that

$$\frac{e^{i(N+1/2)(x+2\pi)} - e^{-i(N+1/2)(x+2\pi)}}{e^{i(1/2)(x+2\pi)} - e^{-i(1/2)(x+2\pi)}} = \frac{-e^{i(N+1/2)x} + e^{-i(N+1/2)x}}{-e^{ix/2} + e^{-ix/2}} = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

One can also do this with trig addition formulae, which give the same result.

such that for the first term,

$$x+t>0, t>-x, \quad x+t<0, t<-x. \tag{7.7}$$

Hence the first term splits into two integrals

$$+\int_{-x}^{\pi} - \int_{-\pi}^{-x}. \tag{7.8}$$

Similarly, for the second term,

$$x-t>0, t<x, \quad x-t<0, t>x. \tag{7.9}$$

Hence the $f(x-t)$ term also splits into

$$-\int_{x}^{\pi} + \int_{-\pi}^{x}. \tag{7.10}$$

we have an overall sum

$$+\int_{-x}^{\pi} - \int_{-\pi}^{-x} - \int_{x}^{\pi} + \int_{-\pi}^{x}. \tag{7.11}$$

In the $-\int_{-\pi}^{-x}$ integral, we can take $t \to -t$ so that

$$-\int_{-\pi}^{-x} \to + \int_{+\pi}^{+x} \tag{7.12}$$

and similarly in the last integral we have

$$+\int_{-\pi}^{x} \to - \int_{+\pi}^{-x}. \tag{7.13}$$

Hence we have

$$\int_{-x}^{\pi} + \int_{\pi}^{x} - \int_{x}^{\pi} - \int_{\pi}^{-x}. \tag{7.14}$$

If we now flip the limits of integration we see that terms double up, so this becomes

$$2\left[\int_{-x}^{\pi} + \int_{\pi}^{x}\right] = 2\int_{-x}^{x}. \tag{7.15}$$

Hence our integral was really

$$\frac{1}{2\pi}\int_{-x}^{x} dt \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{7.16}$$

We can use the evenness of the integrand to ge rid of a factor of $1/2$ and write

$$f_N(x) = \frac{1}{\pi}\int_{0}^{x} dt \frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{7.17}$$

For a fixed N, if we take $x \to 0$, every expression in the interpolating series is zero and it follows that $f(0) = \lim_{N\to\infty} f_N(0) = 0$.

Let us now consider positive x and define

$$\zeta = (N+1/2)t \tag{7.18}$$

so that

$$f_N = \frac{1}{\pi}\int_{0}^{(N+1/2)x} \frac{d\zeta}{N+1/2} \frac{\sin\zeta}{\sin\left(\frac{\zeta}{2N+1}\right)}. \tag{7.19}$$

In the limit as $N \to \infty$, we can approximate $\sin(\zeta/(2N+1)) \approx \zeta$ and therefore

$$f_N \to \frac{2}{\pi}\int_{0}^{\infty} d\zeta \frac{\sin\zeta}{\zeta}. \tag{7.20}$$

How can we perform this? Let's introduce a regularizing function $e^{-\lambda\zeta}$ and we'll take the limit $\lambda \to 0$ later. This is Yukawa's trick for regularizing Coulombic integrals. That is, define

$$I(\lambda) = \int_{0}^{\infty} d\zeta\, e^{-\lambda\zeta} \frac{\sin\zeta}{\zeta}. \tag{7.21}$$

Now let us study the derivative with respect to λ, i.e.

$$I'(\lambda) = -\int_0^\infty d\zeta\, e^{-\lambda\zeta}\sin\zeta. \tag{7.22}$$

We could make some exponential substitution for $\sin\zeta$ now to evaluate this. Even more simply, we could do integration by parts. Doing it twice gives the original expression back, which means we can evaluate this explicitly and solve the PDE for I in terms of λ. Taking the $\lambda \to 0$ limit then yields our answer:

$$\int_0^\infty d\zeta \frac{\sin\zeta}{\zeta} = \frac{\pi}{2}. \tag{7.23}$$

Hence it follows that in the limit as $N \to \infty$, our function really does converge for $x > 0$:

$$\lim_{N\to\infty} f_N(x > 0) = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1. \tag{7.24}$$

Finally, we must calculate the overshoot. We now take the limit as $x \to 0$ for fixed N of

$$f_N = \frac{1}{\pi}\int_0^{(N+1/2)x} \frac{d\zeta}{N+1/2}\frac{\sin\zeta}{\sin\left(\frac{\zeta}{2N+1}\right)}. \tag{7.25}$$

We shall again take N to be large but not infinite, such that

$$f_N(x) = \frac{2}{\pi}\int_0^{(N+1/2)x} d\zeta\frac{\sin\zeta}{\zeta}. \tag{7.26}$$

At what point is this overshoot maximized? Well, it is maximized when the numerator $\sin\zeta$ hits zero and starts to become negative, i.e. when the upper limit of integration is $\zeta = \pi$, i.e.

$$(N+1/2)x = \pi. \tag{7.27}$$

Hence

$$f_N = \frac{2}{\pi}\int_0^\pi \cdots = \frac{2}{\pi}\int_0^{3\pi} - \int_\pi^{3\pi} = \frac{2}{\pi}\left[\int_0^\infty - \int_\pi^{3\pi} - \int_{3\pi}^{5\pi} - \int_{5\pi}^{7\pi} - \dots\right]. \tag{7.28}$$

The integral $\int_0^\infty$ was just $\pi/2$, as we calculated. We just increase the integration limits by a whole period and then subtract off what we added on. What about the first correction? We have

$$\int_\pi^{3\pi} d\zeta\frac{\sin\zeta}{\zeta}. \tag{7.29}$$

These integrals over periods $\pi, 3\pi$ are small negative numbers, since the sin is negative from π to 2π and positive from 2π to 3π, but the $1/\zeta$ suppresses the positive contribution. We can't compute these exactly; at this point, we must resort to numerical methods. It turns out the sum of the corrections is very close to 0.18; this is the 18% overshoot of the Gibbs phenomenon.

Finally, let us prove the Fourier theorem. We have an integral

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}, \tag{7.30}$$

which is manifestly 2π-periodic. We may allow f to have finitely many finite discontinuities, e.g.

$$f(x_0^+) \neq f(x_0^-) \tag{7.31}$$

for some x_0. Now we split the integral into two, as

$$\frac{1}{2\pi}\left[\int_{-\pi}^0 dt\, f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)} + \int_0^\pi dt\, f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}\right]. \tag{7.32}$$

Hence if f has a discontinuity at x, then the left integral from $t \in -\pi, 0$ picks up all the contributions up to the discontinuity at x. Similarly the right integral picks up all the contributions from x onwards. Let us add and subtract the left limit and rewrite this first term as

$$\frac{1}{2\pi}\int_{-\pi}^0 \left[f(x^-) + \left(f(x+t) - f(x^-)\right)\right]\frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{7.33}$$

This first term is just a constant integrated against the sin piece, so its integral over a half period is just π. That is, the $f(x^-)$ integral becomes

$$\frac{1}{2}f(x^-). \tag{7.34}$$

The integral from 0 to π does something similar; it picks up the right limit, $\frac{1}{2}f(x^+)$. What about the rest of it? There's continuity argument to be made here, which we'll revisit next time. One can also do this with trig addition formulae, which give the same result.

Lecture 8.

Friday, January 24, 2020

"When I was in kindergarten... you know, it's like that Opus Dei thing. You punish yourself to see how much you can take."

—Nemanja Kaloper

Let's conclude our proof of the Fourier theorem. We wrote the partial sum

$$f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}, \tag{8.1}$$

and said that due to the 2π-periodicity, we could simply shift the domain of integration in t as we like. If there's a single discontinuity at x, we can break up the domain of integration as

$$\frac{1}{2\pi}\left[\int_{-\pi}^{0} dt\, f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)} + \int_{0}^{\pi} dt\, f(x+t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}\right]. \tag{8.2}$$

Consider just the first term, the integral from $-\pi$ to 0. We can add and subtract the function's limit from the left,

$$f_N^- = \frac{1}{2\pi}\int_{-\pi}^{0}\left[f(x^-) + \left(f(x+t) - f(x^-)\right)\right]\frac{\sin[(N+1/2)t]}{\sin(t/2)}. \tag{8.3}$$

We've just added zero. Hence the following integral can be evaluated as

$$\frac{1}{2\pi}\int_{-\pi}^{0}\left[f(x^-)\right]\frac{\sin[(N+1/2)t]}{\sin(t/2)} = \frac{f(x^-)}{2\pi}(\pi) = \frac{f(x^-)}{2}. \tag{8.4}$$

We did the integral of the ratio of sines previously. On the interval $-\pi$ to π it was 2, and by the evenness of the ratio of sines, the integral over the half-interval $-\pi$ to 0 is just π.

Computing the integral $f_N^+$from 0 to π, we get something very similar—the right limit of the function, $f(x^+)/2$, and another integral.

We shall argue that the integrals cancel each other perfectly by the Riemann-Lebesgue lemma, which (roughly) states that the integral of fast functions which oscillate a lot about a point are zero except about some region where the oscillations are slow.

That is, we wish to compute

$$\frac{1}{2}\int_{-\pi}^{0} dt\frac{f(x+t) - f(x^-)}{\sin(t/2)}\sin[(N+1/2)t]. \tag{8.5}$$

Notice that for $t \to 0$ from the left, the numerator goes as $f(x^-)t$ since f is continuous up to x. The denominator $\sin(t/2)$ is a slow function; it is 4π periodic and only goes to zero at $t \to 0$, where its leading order behavior is $\sin(t/2) \sim t/2$. In the limit as N grows very large, we need only worry about the integrand near $t \to 0$. That is, it goes as $\sin((N+1/2)t) \sim (N+1/2)t + O(t^3)$. Hence our slow function is approximately

$$\sigma(t) = \frac{f(x+t) - f(x^-)}{\sin(t/2)} \approx \frac{f'(x^-)t}{t/2} = 2f'(x^-), \tag{8.6}$$

so that

$$\lim_{N\to\infty}\int_a^b dt\,\sigma(t)\sin((N+1/2)t) = 0. \tag{8.7}$$

That is, this integral oscillates so fast and our slow function is well-behaved on the whole integral, so the whole integral vanishes. This is the conclusion of the Riemann-Lebesgue lemma.

Let's do a trick to see this works. Let $\omega = N + 1/2$, and define a new variable

$$t = t' + \frac{\pi}{N+1/2} = t' + \frac{\pi}{\omega} \tag{8.8}$$

so that

$$\int_{a-\pi/\omega}^{b-\pi/\omega} dt'\, \sigma(t' + \pi/\omega) \sin(\omega(t' + \pi/\omega)) = -\int_{a-\pi/\omega}^{b-\pi/\omega} dt'\, \sigma(t' + \pi/\omega) \sin(\omega t') \tag{8.9}$$

since the sin is 2π-periodic. This is clearly the same as our original integral, since we've just changed variables. Call the integral I. Then

$$2I = \int_a^b dt\, \sigma(t) \sin(\omega t) - \int_{a-\pi/\omega}^{b-\pi/\omega} dt\, \sigma(t + \pi/\omega) \sin(\omega t). \tag{8.10}$$

We see that as $\omega \to \infty$, these integrals perfectly cancel each other. More precisely, we get an integral

$$\int_a^{b-\pi/\omega} dt \sin(\omega t)[\sigma(t) - \sigma(t + \pi/\omega)] - \int_{a-\pi/\omega}^{a} dt\, \sigma(t + \pi/\omega) \sin \omega t + \int_{b-\pi/\omega}^{b} dt \sigma(t) \sin \omega t. \tag{8.11}$$

If we take the last two terms, we see they are both bounded. The last two are bounded by some multiple of π/ω, since the interval from $b - \pi/\omega$ to b becomes infinitesimal and the functions $\sigma(t), \sin(\omega t)$ are well-behaved around b. A similar argument applies for the integral from $a - \pi/\omega$ to a.

The first term is also bounded and vanishes since $\sigma(t) - \sigma(t + \pi/\omega)$ is just a derivative $\sigma'(t)$ times π/ω, and σ' is well-behaved (nonsingular) over the interval. It follows that in the limit as $\omega \to \infty$, all these integrals go to zero, so that indeed $I = 0$ and

$$\lim_{N\to\infty} f_N(x) = \frac{f(x^-) + f(x^+)}{2}, \tag{8.12}$$

and moreover we recover the completeness relation

$$\lim_{N\to\infty} \frac{\sin(N+1/2)(x-t)}{\sin(\frac{x-t}{2})} = \delta(x-t). \tag{8.13}$$

Complex analysis A complex number is a pair (x, y), which we may write

$$z = x + iy, \quad x, y \in \mathbb{R} \tag{8.14}$$

in cartesian form. Complex numbers also have a polar form,

$$z = \rho e^{i\phi}, \quad \tan\phi = y/x, \quad \rho = \sqrt{x^2 + y^2}. \tag{8.15}$$

Complex functions on the complex plane may be multivalued. For instance, consider the function $\ln z$. If we write a number like $A = Ae^{2\pi n i}$, $n \in \mathbb{Z}$, then

$$\log A = \log(Ae^{2\pi n i}) = \log A + 2\pi n i, \tag{8.16}$$

so the log is infinitely multivalued. We consider complex functions

$$f : \mathbb{C} \to \mathbb{C}, \tag{8.17}$$

mapping some $z \in \mathbb{C}$ to $w \in \mathbb{C}$, and since the output is complex, we can write

$$f(z) = u(x, y) + iv(x, y) \tag{8.18}$$

where now $u, v : \mathbb{C} \to \mathbb{R}$ are real-valued functions. Arbitrary complex functions are clearly as complicated as real functions, but there is a special subset of complex functions which are more constrained in interesting ways. These are the *analytic functions.*

Lecture 9.

Monday, January 27, 2020

"Suppose you have a transparent beach ball. And you take one of these laser pointers that fools like to shine at airplanes and such things, and you shine it through your beach ball."
—Nemanja Kaloper

Today's material is drawn from Arfken 6th edition, since the 7th edition has some sort of error we'll consider shortly. Consider a complex-valued function $f(z) : \mathbb{C} \to \mathbb{C}$. Note that complex functions need not be one-to-one, and in fact they can even be multivalued if we're not careful (cf. our discussion of the function $\ln z, z \in \mathbb{C}$).

We can define a *Banach space* as a set such that for any two points $z_1, z_2 \in \mathbb{C}$, there exists a smooth function $f : [0,1] \to \mathbb{C}$ such that $f(0) = z_1, f(1) = z_2$.[10] This allows us to talk abut derivatives of functions:

$$f'(z) = \lim_{\Delta z \to)} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \tag{9.1}$$

and the derivative $\frac{df}{dz}$ through a point must be unique and independent of the path.

The existence and uniqueness of the derivative for complex functions is related to a set of constraints known as the Cauchy-Riemann conditions. That is,

$$\exists f'(z) \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}, \tag{9.2}$$

where our complex function is

$$f = u + iv, \quad u, v : \mathbb{C} \to \mathbb{R}, \tag{9.3}$$

and points in the domain are given by

$$z = x + iy, \quad x, y \in \mathbb{R}. \tag{9.4}$$

Let us consider two possible displacements,

$$\Delta z = \Delta x, \quad \Delta z = i\Delta y. \tag{9.5}$$

In the first case, the derivative takes the form

$$\frac{df}{dz} = \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \to \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{9.6}$$

in the limit as $\Delta x \to 0$. In the second case, we instead have

$$\frac{df}{dz} = \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{9.7}$$

Now we equate real and imaginary parts and find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{9.8}$$

Hence the existence of the derivative implies Cauchy-Riemann.

We shall now prove the converse, that Cauchy-Riemann implies the existence of the derivative. Suppose we have a path through a point z, and we are given the appropriate derivatives of u and v for some function f. Then

$$\Delta f = \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right). \tag{9.9}$$

Hence

$$\frac{\Delta f}{\Delta z} = \frac{\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)}{\Delta x + i\Delta y}. \tag{9.10}$$

[10] This seems to be a statement of the vector space being complete, in the sense we can define limits.

Let us now apply Cauchy-Riemann and eliminate the v derivatives. That is,

$$\frac{\Delta f}{\Delta z} = \frac{\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(-\frac{\partial u}{\partial y}\Delta x + \frac{\partial u}{\partial x}\Delta y\right)}{\Delta x + i\Delta y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}. \tag{9.11}$$

It follows that all the path dependence in Δx and Δy drops out, so the derivative exists and moreover it is path-independent. We could also write

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{9.12}$$

using Cauchy-Riemann.

Now consider the function

$$z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + i\underbrace{2xy}_{v}. \tag{9.13}$$

We see that

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}. \tag{9.14}$$

So this function is really analytic, as we might expect. In contrast,

$$|z| = \sqrt{x^2+y^2} \tag{9.15}$$

is not analytic, since $v = 0$ for this function but $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \neq 0$ in general.

Since we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial X}, \tag{9.16}$$

we can now import all our results from regular calculus of real variables. The chain rule and the product rule all follow in the usual way.

Suppose moreover that the second derivatives of u and v exist. Consider deriving the first with respect to x, and the second with respect to y. That is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \partial y}. \tag{9.17}$$

By the equality of mixed partials, we learn that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}, \tag{9.18}$$

or equivalently

$$\Delta u = 0, \Delta v = 0, \tag{9.19}$$

since we can perform the same argument for v as well. This states exactly that u and v are *harmonic functions*; they satisfy the 2D Laplace equation.

Consider this in the form

$$0 = \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y. \tag{9.20}$$

This defines a trajectory along an equipotential. Then

$$\left.\frac{\Delta y}{\Delta x}\right|_{\text{u=constant}} = -\frac{\partial_x u}{\partial_y u} = +\frac{\partial_y v}{\partial_x v}. \tag{9.21}$$

This tells us that in fact u and v are related in a special way—in fact, the equipotentials of u and v are orthogonal.

To understand the idea of a "point at infinity," let us place a two-sphere on the complex plane, with the south pole at the origin of the complex plane. Suppose we trace a line from the north pole to a point on the sphere, and then extend that line until it intersects the complex plane. Clearly, this is a one-to-one mapping between the plane and the sphere. But where is the point at the north pole mapped to? Effectively, it is mapped to all the points on a circle of infinite radius.[11]

[11] This construction is better known as the Riemann sphere.

We can then define a Riemann sum on some path $f(z)$ starting at z_0 and ending at z. We can discretize the path as $\{z_i\}$ and sample along the way as

$$\sum_i f(z_i)\Delta z. \tag{9.22}$$

If the limit $\Delta z \to 0$ of this sum exists, then we say that

$$\lim_{\Delta z\to 0} \sum_i f(z_i)\Delta z \equiv \int_{z_0}^{z} f(z')dz'. \tag{9.23}$$

In fact, we can write

$$\int_C f(z)dz = \int (u+iv)(dx+idy) = \int (udx - vdy) + i\int (udy + vdx). \tag{9.24}$$

Lecture 10.

Wednesday, January 29, 2020

"This is just his Majesty Sir Newton. Or Sir Issac, shall we say. You know, I think the purpose of old England was to give birth to Newton. Everything else we can discount."
—Nemanja Kaloper

Last time, we defined integrals of complex functions in terms of the limit of their Riemann (discretized) sums. In general, the integral between two endpoints will depend on the path taken to get there. But we know from classical mechanics that some forces are conservative, i.e. the integrals of certain functions are independent of path.

What is the condition for the integral of a function to be independent of path? This is the same as asking when any integral around a closed loop is zero, and the answer is provided by the following theorem.

Theorem 10.1 (Cauchy's integral theorem)**.** *If a function $f(z): \mathbb{C} \to \mathbb{C}$ is analytic in and around a curve C in a region which is simply connected,*[12] *then*

$$\oint_C f(z)dz = 0. \tag{10.2}$$

Let us compute the integral

$$\oint_{|z|=1} dz\, z^n. \tag{10.3}$$

Since $z = e^{i\phi}$ on this contour, $dz = ie^{i\phi}d\phi$, so

$$\oint_{|z|=1} dz\, z^n = i\int_0^{2\pi} d\phi e^{i\phi(n+1)} = \frac{i}{i(n+1)} e^{i\phi(n+1)}|_0^{2\pi} = 2\pi i \delta_{n,-1}, \tag{10.4}$$

where we take $n \in \mathbb{Z}$.

One could imagine doing this integral around a square of corners a, b, c, d. Then

$$\oint dz\, z^n = \frac{z^{n+1}}{n+1}|_{\text{endpoints}} = \frac{1}{n+1}(b^{n+1} - a^{n+1} + c^{n+1} - b^{n+1} + \dots) \tag{10.5}$$

where all the terms will cancel. Except in the case $n = -1$, where we get logs instead, i.e. a sum

$$\ln(b/a) + \ln(c/b) + \ln(d/c) + \ln(ae^{2\pi i}/d) = 2\pi i. \tag{10.6}$$

So the case $n = -1$ (which we'll later recognize as that of a simple pole) is special.

We also claimed last time that since $f(z)$ can be written as real and imaginary parts and similarly z can be written in terms of real and imaginary parts, then

$$\int_C f(z)dz = \int (u+iv)(dx+idy) = \int (udx - vdy) + i\int (udy + vdx). \tag{10.7}$$

Recall from vector calculus that

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_A d\mathbf{S}(\boldsymbol{\nabla}\times\mathbf{A}). \tag{10.8}$$

[12] Simply connected means informally that the region has no holes. Formally it means that the fundamental group π_1 is trivial. That is, any closed curve is homotopic (can be continuously deformed) to a point.

If we like, this is really a definition of the curl, as the circulation of a vector field around an infinitesimal loop. If we write this in terms of components, we have

$$\oint_C (V_x dx + V_y dx) = \int d\mathbf{S} \cdot (\partial_y V_x - \partial_x V_y)\hat{\mathbf{z}}. \tag{10.9}$$

Let's apply this to our integral. We find that

$$\oint_C dz\, f(z) = \int dA(\partial_y u + \partial_x v) + i \oint dA(\partial_y v - \partial_x u) = 0 \tag{10.10}$$

by the Cauchy-Riemann conditions. However, this definition of the curl relies on not just the existence of the derivatives of A but their continuity. The full theorem is stronger. But let us take this as a first proof that if a function is analytic in a simply connected region, then its integral around any closed loop is zero.

Let us suppose now that we have a (planar) region that we break up with a fine mesh. Since the border of each mesh region in the interior is counted twice, but with opposite orientation, then

$$\sum_i \oint_{C_i} f(z) = \oint_C f(z). \tag{10.11}$$

We now pick a point z_i inside each cell C_i and define

$$\delta(z, z_i) = \frac{f(z) - f(z_i)}{z - z_i} - f'(z_i). \tag{10.12}$$

Notice that for

$$|z - z_i| < \Delta, \quad |\delta(z, z_i)| < \epsilon. \tag{10.13}$$

That is, we can make the difference between the discretized derivative at z_i and the true derivative $f'(z_i)$ arbitrarily small.

Then we rewrite $f(z)$ in terms of δ:

$$\oint f(z) = \sum \left[\oint_{C_i} \delta(z - z_i) + \oint f'(z_i)(z - z_i) + \oint f(z_i) \right]. \tag{10.14}$$

But if we take the limit as the limit gets arbitrarily fine, this last term is just the integral of a constant over a closed contour. This is zero, since we are integrating $f(z) = 1$ around a closed contour, and 1 is clearly analytic. The second term vanishes by the same argument, since $z - z_i$ is also analytic. We're left with

$$\oint f(z)dz = \sum \oint_{C_i} \delta(z - z_i) dz < \epsilon A, \tag{10.15}$$

and since this can be made arbitrarily small, we find that the integral is zero.

Lecture 11.

Friday, January 31, 2020

*"We don't know what this does on the circle. Except we *do* know. We just haven't realized that we know."*

—Nemanja Kaloper

Last time, we discussed Cauchy's integral theorem. There's a way to deal with contour integrals near singular points. The contour integral over two contours that can be continuously deformed into one another without crossing singularities have the same value. This is precisely the Gauss's law statement that the flux through two surfaces that can be continuously deformed into each other is equal, provided that we have not enclosed any new charge.

Let's take a function

$$\frac{f(z)}{z - z_0}, \tag{11.1}$$

where $z_0 \in \mathbb{C}$ is some point in the complex plane and $f(z)$ is analytic in some domain of analyticity including z_0. Consider the integral on a contour within the domain of analyticity which contains z_0,

$$\oint_C \frac{f(z)}{z - z_0}. \tag{11.2}$$

This integral is nonzero, since it includes a singular point.

How can we compute its value? Consider as our contour C_ρ, a circle centered on z_0 of radius ρ. Each point on the circle is $z = z_0 + \rho e^{i\phi}$. Now we can compute the contour integral explicitly. By flux conservation (this is an isolated singularity), any other contour enclosing z_0 has the same integral as over C_ρ. THe integral is

$$\oint_{C_\rho} dz \frac{f(z)}{z - z_0} = i \int_0^{2\pi} d\phi \rho e^{i\phi} \frac{f(z_0 + \rho e^{i\phi})}{\rho e^{i\phi}} = i \int_0^{2\pi} d\phi f(z_0 + \rho e^{i\phi}). \tag{11.3}$$

If we now take the limit as $\rho \to 0$, we can use the fact that f is analytic and therefore continuous at z_0 so that in fact

$$\oint_{C_\rho} dz \frac{f(z)}{z - z_0} = 2\pi i f(z_0). \tag{11.4}$$

We can combine this with our previous knowledge of integrals of analytic functions to say that the integral $\oint_C dz \frac{f(z)}{z-z_0}$ is zero if z_0 is outside the contour, and $2\pi i f(z_0)$ if it is inside.

Moreover, let us write

$$f(z_0 + \delta z) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - (z_0 + \delta z)}. \tag{11.5}$$

Since

$$f(z_0) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z_0}, \tag{11.6}$$

it follows that expanding to first order and taking the limit $\delta z \to 0$, we get

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \frac{1}{2\pi i} \oint dz \frac{f(z)}{(z - z_0)^2}. \tag{11.7}$$

Equivalently we could just take a derivative with respect to z_0 directly. In general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z - z_0)^{n+1}}. \tag{11.8}$$

This tells us that for analytic functions, it's not just the case that the first derivatives exist, but in fact all higher derivatives exist, for the integrand is perfectly well-behaved.

We shall now prove the converse of Cauchy's theorem, that the contour integral over any closed loop vanishing implies analyticity. This is known as Morera's theorem, and we'll do it in a way that improves upon the text.

Suppose that $\oint f(z) = 0$ over any closed loop in a region. it follows that for such a loop, we can take two points z_1, z_2 on the loop and split it into two paths, such that the integral over each path is equal and opposite:

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} dz f(z). \tag{11.9}$$

Now we write it in the following way:

$$\begin{aligned} \frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) &= \frac{\int_{z_1}^{z_2} dz f(z)}{z_2 - z_1} - f(z_1) \\ &= \frac{\int_{z_1}^{z_2} dz (f(z) - f(z_1))}{z_2 - z_1}. \end{aligned}$$

At this point, the book claims that because f is continuous we can replace it by its derivative. But that assumes that the derivative of f exists, which is what we're trying to prove! The right way to say this is that $f(z) - f(z_1) < \epsilon$ as $z \to z_1$, so then

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} dz (f(z) - f(z_1))}{z_2 - z_1} < \epsilon \frac{z_2 - z_1}{z_2 - z_1} = \epsilon. \tag{11.10}$$

Hence

$$\lim \frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = 0 \implies F'(z_1) = f(z_1). \tag{11.11}$$

Now we know that $f'(z_1) = F''(z_1)$, and all higher derivatives exist.

Consider the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{11.12}$$

on a circle of radius R. If $|f| < M$ (the function is bounded on the circle), then we can prove that

$$|a_n| < \frac{M}{R^n}. \tag{11.13}$$

For notice that

$$a_n = \frac{1}{n!} f^{(n)}(z)|_{z=0} = \frac{1}{2\pi} \oint dz \frac{f(z)}{z^{n+1}}. \tag{11.14}$$

Then

$$|a_n| = \frac{1}{n!} |f^{(n)}(z)|_{z=0} = \frac{1}{2\pi} |\oint dz \frac{f(z)}{z^{n+1}}| < \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{M}{R^n} = \frac{M}{R^n}. \tag{11.15}$$

In fact, it follows that the only function which is analytic and bounded as $R \to \infty$ is in fact the constant function.

Let us now prove the fundamental theorem of algebra, that a polynomial of order N has N complex roots counting degeneracy. Consider a polynomial $P(z)$, and suppose that $P(z)$ has no root. Then the quantity $1/P(z)$ is perfectly finite and bounded, which means that by our theorem, $1/P(z)$ is actually constant, so it's not a polynomial.

Thus it must have at least one root z_0, and we can define $\hat{P} = \frac{P(z)}{z-z_0}$. This process can terminate no sooner than after N steps, when what's left really is constant.

The real power of analyticity is in the procedure of analytic continuation, however. Consider a point z_0, where the nearest singularity of a function $f(z)$ lies at z_1. The value of the function $f(z')$ at soe z' nearby is then

$$f(z') = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z'} = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z - z_0} \sum \left(\frac{z' - z_0}{z - z_0} \right)^n = \sum \frac{(z' - z_0)}{2\pi i} \oint dz \frac{f(z)}{(z - z_0)^{n+1}}. \tag{11.16}$$

This is a Taylor series at z', and now

$$f(z') = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z' - z_0)^n. \tag{11.17}$$

So in fact the function at a point and its derivatives gave us the function everywhere within a circle up to the closest singularity.