PHYSICS 204B: METHODS OF MATHEMATICAL PHYSICS II

IAN LIM
LAST UPDATED JANUARY 16, 2020

These notes were taken for Physics 204B, *Methods of Mathematical Physics II*, as taught by Nemanja Kaloper at the University of California, Davis in winter quarter 2020. I live-TeXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.
Many thanks to Arun Debray for the LaTeX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

CONTENTS

Lecture 1.

Monday, January 6, 2020

"What did you do over break?" "Don't ask. No rest for the wicked."
—Mark Samuel Abbott and Nemanja Kaloper

The only outstanding logistical details here are that office hours will be posted later, and the TA is now Morgane König rather than Cameron Langer. All else is basically the same as last quarter.

Let's talk about Green's functions in more than one dimension. Our discussion will be somehwat sketchy, but we'll get a rough idea of the topic. A Green's function is the inverse of a differential operator, and it lives under integrals. In one dimension, we wrote that to solve

$$\mathcal{L}\phi = J, \tag{1.1}$$

we could construct G such that

$$\mathcal{L}G = \delta, \tag{1.2}$$

a function which gives a delta function upon being hit by a differential operator. We would like to solve the problem of finding the linear response of a field ϕ at a point $\mathbf{r}_1$ due to a source $J(\mathbf{r}_2)$ at a point $\mathbf{r}_2$.

Recall that the adjoint of an operator is given by

$$\langle \psi | \mathcal{L}\phi \rangle = \langle \mathcal{L}^\dagger \psi | \phi \rangle, \tag{1.3}$$

and an operator is self-adjoint if

$$\mathcal{L} = \mathcal{L}^\dagger. \tag{1.4}$$

That is, for an operator given by

$$\mathcal{L}\phi = \boldsymbol{\nabla} \cdot (p\boldsymbol{\nabla}\phi) + q\phi, \tag{1.5}$$

we must check that

$$\int \psi^*(\boldsymbol{\nabla} \cdot (p\boldsymbol{\nabla}\phi)) + \int q\psi^*\phi = \int \boldsymbol{\nabla} \cdot (p\boldsymbol{\nabla}\psi^*)\phi + \int q\psi^*\phi. \tag{1.6}$$

The q terms cancel, so we find that

$$\int [\psi^*\boldsymbol{\nabla} \cdot (p\boldsymbol{\nabla}\phi) - \boldsymbol{\nabla} \cdot (p\boldsymbol{\nabla}\psi^*)\phi] = 0, \tag{1.7}$$

and if we integrate by parts, then

$$\int_V dV\, \boldsymbol{\nabla} \cdot (\psi^* p \boldsymbol{\nabla} \phi - (\boldsymbol{\nabla}\psi^*) p \phi) = 0. \tag{1.8}$$

By the divergence theorem,

$$\int_S d\mathbf{S} \cdot p[\psi^* \boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\psi^*)\phi] = 0. \tag{1.9}$$

Dirichlet or Neumann boundary conditions will guarantee self-adjointness. That is, if the functions vanish on S or their normal derivatives vanish on S, then $\mathcal{L}$ is self-adjoint. There are also mixed conditions we could impose, but Dirichlet and Neumann conditions are sufficient to make our operator Hermitian.

For a Hermitian operator, the corresponding Green's function obeys[1]

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.10}$$

For recall that

$$\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) | G(\mathbf{r}, \mathbf{r}_2) \rangle = \langle G(\mathbf{r}, \mathbf{r}_1) | \mathcal{L}G(\mathbf{r}, \mathbf{r}_2) \rangle \tag{1.11}$$

by self-adjointness. By the definition of the Green's function and the inner product, we can replace $\mathcal{L}G$ by a delta function and get

$$\int \delta(\mathbf{r} - \mathbf{r}_1) G(\mathbf{r}, \mathbf{r}_2) = G(\mathbf{r}_1, \mathbf{r}_2) \tag{1.12}$$

on the LHS and

$$\int G^*(\mathbf{r}, \mathbf{r}_1) \delta(\mathbf{r}, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.13}$$

Hence

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.14}$$

Now let's construct the eigenfunction expansion of the Green's function. Consider

$$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.15}$$

We're keeping the $\mathbf{r}_1, \mathbf{r}_2$ dependence in G because there might be operators that are not translationally invariant. In other words, we can't assume that quantities depend only on $|\mathbf{r}_1 - \mathbf{r}_2|$. That is,

$$\mathcal{L}|_{\mathbf{r}} \neq \mathcal{L}|_{\mathbf{r}+\mathbf{a}}. \tag{1.16}$$

Suppose we construct the eigenfunctions ϕ_λ of the operator $\mathcal{L}$, such that

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda. \tag{1.17}$$

WLOG we may take them to be orthonormal,

$$\langle \phi_\lambda | \phi_\mu \rangle = \delta_{\lambda\mu}. \tag{1.18}$$

For now, we shall assert that they are a complete set– in general we will have to prove this. The expansion for the delta function is just the completeness relation:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \int_\lambda \phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1), \tag{1.19}$$

since

$$f(\mathbf{r}) = \int_\lambda f_\lambda \phi_\lambda(\mathbf{r}) = \int_{\mathbf{r}_1} f(\mathbf{r}_1)\delta(\mathbf{r} - \mathbf{r}_1) = \int_\lambda \underbrace{\int_{\mathbf{r}_1} f(\mathbf{r}_1)\phi_\lambda^*(\mathbf{r}_1)}_{f_\lambda} \phi_\lambda(\mathbf{r}). \tag{1.20}$$

Now we can put our delta function decomposition back in: suppose that

$$G(\mathbf{r}, \mathbf{r}_1) = \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}), \tag{1.21}$$

---

[1]This property is also responsible for Green's reciprocity theorem in electromagnetism, i.e. the statement that the potential energy of a charge distribution ρ_2 in a field produced by another distribution ρ_2 is equal to the energy of ρ_1 in the field produced by ρ_2. If you like, the theorem is a special case/application since the Laplacian operator is Hermitian.

so that

$$\mathcal{L}G = \mathcal{L} \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r})$$
$$= \int_\lambda C_\lambda(\mathbf{r}_1)\mathcal{L}\phi_\lambda(\mathbf{r})$$
$$= \int_\lambda C_\lambda(\mathbf{r}_1)\lambda\phi_\lambda(\mathbf{r}),$$

and this last expression must be equal to the expansion of the delta function:

$$0 = \int_\lambda [C_\lambda(\mathbf{r}_1)\lambda - \phi_\lambda^*(\mathbf{r}_1)]\phi_\lambda(\mathbf{r}). \tag{1.22}$$

Hence

$$c_\lambda(\mathbf{r}_1) = \frac{\phi^*(\mathbf{r}_1)}{\lambda}, \tag{1.23}$$

so

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda}. \tag{1.24}$$

This makes the hermiticity of G totally clear:

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.25}$$

The only problem might be if we have a zero eigenvalue, in which case we have to be careful. Some people define a generalized Green's function by

$$(\mathcal{L} - Z)G = \delta, \tag{1.26}$$

so that our expression is just modified to

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda - Z}, \tag{1.27}$$

and we can study the $z \to 0$ limit.

Let's consider some examples. The Laplace equation is

$$\nabla^2\phi = J, \tag{1.28}$$

and it is already in self-adjoint form,

$$\nabla \cdot (\nabla G) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.29}$$

Suppose we had a solution

$$\int \nabla^2\phi = J \tag{1.30}$$

such that $\int_V J = Q$ and a homogenous solution

$$\nabla^2\chi = 0. \tag{1.31}$$

But now

$$\int_V dV \nabla^2(\phi + \chi) = \int_V J = Q. \tag{1.32}$$

We can rewrite the first expression as a surface integral, $\int d\mathbf{S} \cdot \nabla(\phi + \chi)$. Hence we find that since the integral of the $\nabla\phi$ term is already Q, it must be that

$$\int d\mathbf{S} \cdot \nabla\chi = 0, \tag{1.33}$$

and therefore by the uniqueness theorems, χ is at most a constant. This is an example of a gauge symmetry, actually, but we won't go too much into that. So given appropriate boundary conditions, solutions of the Laplace equation are unique up to a constant.

Now let us note the Laplace equation *is* translationally invariant, so we can write

$$\nabla \cdot (\nabla G(\mathbf{r})) = \delta(\mathbf{r}). \tag{1.34}$$

In fact, it is also manifestly spherically symmetric in this form. We've just chosen coordinates to put our charge at the origin. Let us integrate over a spherical region R. Then

$$1 = \int \delta(\mathbf{r}) = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r}) = 4\pi R^2 \frac{dG}{dr}. \tag{1.35}$$

We conclude that

$$G = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.36}$$

which is nothing more than the Coulomb potential for a unit charge.

If we instead enclosed the charge in a Faraday cage (setting the potential to zero somewhere), we would add a homogeneous solution to the Laplace equation to our Green's function in order to fit the new boundary conditions. Note that in 2 dimensions, we would just have $1 = 2\pi R \frac{dG}{dr}$, which gives our Green's function as a log of $|r|$ instead.

Lecture 2.

Wednesday, January 8, 2020

"If I had taken a potato instead of a box, I would be hard-pressed to do this. It would be much much harder. Except in two dimensions. Two-dimensional potatoes are very cool because Laplacians in 2 dimensions are special."
—Nemanja Kaloper

Let's play a bit more with Green's functions. Consider the Laplace equation. We can fit a solution using

$$\boldsymbol{\nabla}^2(G + F) = \delta. \tag{2.1}$$

That is, we can construct a solution with a source using the Green's function

$$\boldsymbol{\nabla}^2 G = \delta \tag{2.2}$$

and add on homogeneous solutions F such that

$$\boldsymbol{\nabla}^2 F = 0 \tag{2.3}$$

to fit boundary conditions.

Example 2.4. Let us solve the following electrostatics problem. We take a charge and place it inside a grounded conductor shaped like a cube. Can we solve the Laplace equation in this setup?

The charge has spherical symmetry, but the cube does not. That means that the field lines near the sides of the cube will deform so the field becomes normal to the faces of the cube in the static, equilibrium configuration.

Consider a single charge near a grounded plane. We can solve this by the method of images. Suppose the charge sits on the z axis, while the grounded plane is the xy plane. In particular, it sits at a point $(0,0,a)$ from the origin. We now consider a point a distance ρ from the z axis and at a height z along the z axis.

The method of images says that we can solve for the potential by considering any (fictitious) charge distribution which satisfies the same boundary conditions for the potential. That is, instead of just measuring the potential from the single charge, we can write

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} \tag{2.5}$$

as the potential from the charge alone and suppose we add another image charge Q' on the other side of the grounded plane at a location $(0,0,-a')$. Then the potential from this configuration of the charge and image charge is

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} + \frac{Q'}{\sqrt{\rho^2 + (z+a')^2}}. \tag{2.6}$$

Now we consider the plane $z = 0$ and see that

$$V(z = 0) = \frac{1}{\sqrt{\rho^2 + a^2}} + \frac{Q'}{\sqrt{\rho^2 + a'^2}}. \tag{2.7}$$

But this still seems to depend on ρ. Setting $\rho = 0$ gives

$$\frac{1}{a} + \frac{Q'}{a'} = 0, \tag{2.8}$$

and taking a derivative gives

$$\frac{1}{a^3} + \frac{Q'}{a'^3} = 0, \tag{2.9}$$

so solving gives

$$a = a', \quad Q' = -1. \tag{2.10}$$

The potential on the real side of the grounded plane is just the dipole potential in the region we care about. The other charge is equal and opposite, and equally far away behind the plane.

What about two conducting planes? We can construct a dipole to set the potential on one plane, but then the other won't be grounded. We can add an image dipole outside the other plane (see image). Then we the second plane is okay but the first plane is now not grounded. So we go on adding dipoles in this way ad infinitum, and the overall potential is

$$\sum_{n=-\infty}^{\infty} \frac{1}{[\rho^2 + (z - a + n(2D))^2]^{-1/2}} - \frac{1}{[\rho^2 + (z + a + n(2D))^2]^{-1/2}}. \tag{2.11}$$

Expanded, this is

$$\sum_n \frac{1}{|\mathbf{r} - 2Dn\mathbf{k}|} - \frac{1}{|\mathbf{r} + 2Dn\mathbf{k}|}. \tag{2.12}$$

The sum looks like it might diverge, since each term goes as $1/n$, but in fact since we're dealing with dipoles, the $1/n$ dependence will cancel and away from the corners, this will converge. These image charges help us to satisfy the boundary conditions on the two planes.

The generalization to three dimensions is to extend the periodicity into three dimensions. We get an infinite lattice of image dipoles which is at once simple and complicated.

Let's write a nice fact about multipoles:

$$\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_<^{l+1}} P_l(\cos\theta). \tag{2.13}$$

This simply says that the potential of a multipole drops off as $1/r^{l+1}$, and inside some spherical region it increases as r^l. Its angular dependence is given by the Legendre polynomials. This comes from the fact that

$$\sum x^n P_l(t) = \frac{1}{\sqrt{1 - 2xt + x^2}}, \tag{2.14}$$

the generating function for Legendre polynomials, with $-1 < t < 1$ and $0 < x < 1$. Taking this denominator as $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + 2r_1 r_2 \cos\theta + r_2^2}$, if we pull out r_1 then we get the desired result with this formula.

One more example. Suppose we write the Schrödinger equation as

$$(\nabla^2 + k^2)\psi = 2mV\psi, \tag{2.15}$$

where $k^2 = 2mE$.

We may write the Green's function

$$(\nabla^2 + k^2)G = \delta, \tag{2.16}$$

satisfying

$$\int_{S^2} d\mathbf{S} \cdot \nabla G + k^2 \int_V dV\, G = 1, \tag{2.17}$$

where V is a sphere centered on the origin. In the limit of very small spheres, we shall argue that the second term vanishes. For the k^2 term is constant, whereas as r grows small, the ∇^2 blows up as $1/r^2$. Near the origin, we can solve

$$\int d\mathbf{S} \cdot \nabla G = 1, \tag{2.18}$$

while away from the origin, our delta function is zero. We can exploit the spherical symmetry to rewrite the lLplacian as

$$\frac{1}{r^2}\partial_r(r^2\partial_r G) + k^2 G = 0. \tag{2.19}$$

If we write $G = \phi/r$ to account for the asymptotic behavior of the Green's function, then we find

$$\phi'' + k^2\phi = 0. \tag{2.20}$$

This has solutions

$$\phi = e^{\pm ikr}. \tag{2.21}$$

The $+$ solution is an outgoing wave, while the $-$ solution is an incoming wave. Now

$$G = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{2.22}$$

We can now treat the Schrödinger equation as an integral equation with a source. Thus

$$\psi(\mathbf{r}_1) = 2m \int dr_2^3 \, G(\mathbf{r}_1, \mathbf{r}_2)V(\mathbf{r}_2)\psi(\mathbf{r}_2). \tag{2.23}$$

This is self-consistent, but is it useful? It will be if V is small. Thus we can expand ψ perturbatively in orders of V, considering each order of scattering and writing

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots \tag{2.24}$$

where each order is given by solving the integral equation with the previous order.

┌─ Lecture 3. ───

Friday, January 10, 2020

└──

> *"If wavelength is very small compared to the interstitial distance, this just goes through like hot knife through butter. That's why we exist."*
> —Nemanja Kaloper

Last time, we rewrote the Schrödinger equation as an integral equation,

$$\psi(\mathbf{r}_1) = -\frac{2m}{4\pi}\int \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}V(\mathbf{r}_2)\psi(\mathbf{r}_2). \tag{3.1}$$

To solve this perturbatively, we require that the potential is small,

$$V = \lambda W \tag{3.2}$$

for some $\lambda \ll 1$, and moreover that these integrals converge.

We must make a guess for the initial wave as

$$\psi_0 = e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{3.3}$$

which solves the homogeneous equation, and hence the first-order correction is

$$\psi_1 = -\frac{2m}{4\pi}\int d^3r_2 \frac{e^{ik|\mathbf{r}_1 - \mathbf{r}_2|}}{|\mathbf{r}_1 - \mathbf{r}_2|}V(\mathbf{r}_2)e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{3.4}$$

This will give us the order λ correction.

Fourier series Consider a function $f(x)$ defined on the interval $[-\pi, \pi]$. As long as a function satisfies certain "Dirichlet conditions," it has a valid Fourier decomposition that converges in the mean. These conditions require that the function has only finitely many finite discontinuities. So any smooth function on the integral will work, but functions with a finite jump can still give us a valid Fourier decomposition. It turns out that at the point of the discontinuity, the Fourier series will have the mean value of the limits from the left and right.

Things that will break the decomposition include infinite discontinuities ($1/x$) or that have infinitely many finite discontinuities. There are functions which are not Dirichlet which have Fourier decompositions, but the conditions are sufficient to guarantee the convergence of the Fourier decomposition.

The decomposition takes the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{3.5}$$

with

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \tag{3.6}$$

so that $a_0/2$ is the mean value of the function. The coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx. \tag{3.7}$$

This is nothing more than the decomposition of a function in an infinite-dimensional vector space. Thanks to the orthogonality of cosines and sines, we can treat this as a complete basis and compute the components, which are exactly the Fourier coefficients.

On our function space, there is an inner product defined by

$$f \cdot g = \int_{-\pi}^{\pi} f^*(x)g(x). \tag{3.8}$$

Notice that

$$0 = \int_{-\pi}^{\pi} \sin nx \cos mx dx, \tag{3.9}$$

since the trig addition formulas tell us we are integrating sines $\sin(n + m)x$ and $\sin(n - m)x$ over whole intervals.[2] If we have

$$\int_{-\pi}^{\pi} \sin nx \sin nx dx = \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, d(\cos nx) = \frac{1}{n}(\sin nx \cos nx)|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} \cos nx \, d(\sin nx). \tag{3.10}$$

The boundary terms are zero since $\sin n\pi = 0$, and we get that[3]

$$\int \sin^2 nx dx = \int \cos^2 nx dx. \tag{3.11}$$

Since

$$1 = \sin^2 nx + \cos^2 nx \implies \underbrace{\int_{-\pi}^{\pi} dx}_{2\pi} = \int_{-\pi}^{\pi} dx(\sin^2 nx + \cos^2 nx) = 2 \int_{-\pi}^{\pi} dx \sin^2 nx, \tag{3.12}$$

we conclude that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx \sin^2 nx = 1. \tag{3.13}$$

We can also do this with complex exponentials pretty easily. We find that[4]

$$\frac{1}{\pi} \int_{-\pi}^{\pi} dx \sin nx \sin mx = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \cos nx \cos mx = \delta_{nm}. \tag{3.14}$$

---

[2]Equivalently the function is odd and integrated over a symmetric interval.

[3]We can also derive this by noting that the integral of $\sin nx$ over $-\pi$ to π (i.e. integer multiples of the period) is equal to the integral of $\cos nx$ over the same interval, since a sine is just a phase-shifted cosine. Hence $\int_{-\pi}^{\pi} \sin(nx)dx = \int_{-\pi}^{\pi} \cos(nx)dx$ and the same is true for their squares. Since they both complete the same number of full periods over the interval, their integrals must be the same. So we can immediately conclude that $\int_{-\pi}^{\pi} \sin^2 xdx = \int_{-\pi}^{\pi} \cos^2 xdx$.

[4]See the end of this section for a bit of extra detail.

Let f_N be the truncated Fourier series approximation of f,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx, \tag{3.15}$$

One may show that

$$0 \leq |f - f_N|^2 = \int f^2 + f_N^2 - 2ff_N, \tag{3.16}$$

which is just the Bessel's inequality. We have a term independent of the a_ns, one quadratic in them, and one linear in them.

It is sometimes convenient to use a more symmetric expansion for the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{3.17}$$

This is perfectly equivalent to sines and cosines, as we know. The matching works out as

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{1}{2}(a_n + ib_n), \quad c_{-n} = c_n^* = \frac{1}{2}(a_n - ib_n) \tag{3.18}$$

assuming that a, b are real.

These eigenfunctions satisfy the Sturm-Liouville problem

$$y'' = -\lambda y \tag{3.19}$$

where

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi). \tag{3.20}$$

The eigenvalues are $\lambda = n^2$ and the sets of solutions are

$$y_n = \sin nx, \cos nx. \tag{3.21}$$

Non-lectured: exponentials and integrals of $\sin nx$, two ways We can compute the integral of $\sin^2 nx$ by passing to the complex exponential form,

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}. \tag{3.22}$$

Then for $n \neq m$,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \left(\frac{e^{imx} - e^{-imx}}{2i} \right) \tag{3.23}$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \left(e^{i(n+m)x} + e^{-i(n+m)x} - e^{-ix(n-m)x} - e^{ix(n-m)x} \right) \tag{3.24}$$

$$= -\frac{1}{2\pi} \left[\frac{e^{ix(n+m)}}{i(n+m)} - \frac{e^{-ix(n+m)}}{i(n+m)} + \frac{e^{-ix(n-m)}}{i(n-m)} - \frac{e^{ix(n-m)}}{i(n-m)} \right]_{-\pi}^{\pi} \tag{3.25}$$

However, notice that these exponentials are all at least 2π periodic. That is,

$$e^{i(2\pi)(n+m)} = 1 \implies e^{i\pi(n+m)} = e^{-i\pi(n+m)} \tag{3.26}$$

for $n, m \in \mathbb{Z}$. That means that all these exponential terms vanish when $n \neq m$. If $n = m$, then the second line, Eqn. (3.24), becomes

$$-\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \left(e^{i(n+m)x} + e^{-i(n+m)x} - 2 \right) = -\frac{1}{2\pi}(-2\pi) = 1. \tag{3.27}$$

We conclude that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \delta_{nm}. \tag{3.28}$$

But wait, there's more! Here is a high-powered way to do this integral using contour integration. If you don't know contour integration, read ahead to when these notes cover them and come back later. Let's write this integral suggestively in terms of θ,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \sin m\theta\, d\theta. \tag{3.29}$$

Since θ runs from $-\pi$ to π, we can think of this integral as tracing a unit circle contour in the complex plane (cf. Arfken 11.8). That is, we shall make the change of variables

$$z = e^{i\theta} \tag{3.30}$$

so that

$$d\theta = -i\frac{dz}{z}, \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = -i\frac{z^n - z^{-n}}{2}. \tag{3.31}$$

Now our integral becomes

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = -\frac{i}{\pi}\oint_C \left(-i\frac{z^n - z^{-n}}{2}\right)\left(-i\frac{z^m - z^{-m}}{2}\right)\frac{dz}{z}$$

$$= \frac{i}{4\pi}\oint (z^{n+m} - z^{n-m} - z^{-n+m} + z^{-n-m})\frac{dz}{z}.$$

We can evaluate this with the Cauchy residue theorem! Just look for poles inside the unit circle. Actually, this function only has poles at the origin. We may recall that anything that's not a simple pole in this sort of integral will have zero residue. For instance, suppose $n = 1, m = 2$. Then our integral would be

$$\frac{i}{4\pi}\oint (z^{1+2} - z^{1-2} - z^{-1+2} + z^{-1-2})\frac{dz}{z}. \tag{3.32}$$

Just by power counting, none of these are simple poles. For the z^{n+m} term to give a simple pole, we would need $n + m = 0$, and for $n, m \in \mathbb{Z}_+$ there are no solutions to this equation. The same is true for the z^{-n-m} term. Our only hope is for the middle terms to give us a simple pole, and this happens exactly when $n = m$. When $n = m$, we have

$$\frac{i}{4\pi}\oint (z^{2n} - 2 + z^{-2n})\frac{dz}{z} = \frac{i}{4\pi}(2\pi i(-2)) = 1 \tag{3.33}$$

by the Cauchy residue theorem. As before, we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = \delta_{nm}. \quad \boxtimes \tag{3.34}$$

Lecture 4.

Monday, January 13, 2020

> *"If I attempted to do it [the completeness relation on Fourier modes] any other way, any self-respecting mathematician would decapitate me immediately. So I won't do it because I like my neck just the way it is."*
> —Nemanja Kaloper

Suppose we have

$$\phi_n = \frac{e^{inx}}{\sqrt{2\pi}}, \tag{4.1}$$

such that the inner product is given by

$$\langle \phi_n | \phi_m \rangle = \int \phi_n^* \phi_m. \tag{4.2}$$

For a general function f let us now take its inner product with the basis vectors,

$$c_n = \langle \phi_n | f \rangle \tag{4.3}$$

and for a general set of coefficients $\{\bar{c}_n\}$, write the function (linear combination)

$$\bar{f}_N = \sum \bar{c}_n \phi_n. \tag{4.4}$$

We shall show that the Fourier decomposition of f converges to f faster than any other sequence $\bar{f}_N$. To do this, let us use the triangle inequality and write

$$0 \le |f - \bar{f}_N|^2 = \langle f - \bar{f}_N | f - \bar{f}_N \rangle \tag{4.5}$$

$$= |f|^2 - \langle \bar{f}_N | f \rangle - \langle f | \bar{f}_N \rangle + |\bar{f}_N|^2 \tag{4.6}$$

$$= |f|^2 - 2\text{Re}\langle \bar{f}_N | f \rangle + |\bar{f}_N|^2. \tag{4.7}$$

We can compute these inner products explicitly.

$$\langle \bar{f}_N | \bar{f}_N \rangle = \sum_{n,m} \bar{c}_n^* \bar{c}_m \phi_n \phi_m = \sum_n \bar{c}_n^* \bar{c}_n = \sum_n |\bar{c}_n|^2. \tag{4.8}$$

The cross-terms are

$$\langle \bar{f}_N | f \rangle = \sum \bar{c}_n^* \langle \phi_n | f \rangle = \sum \bar{c}_n^* c_n. \tag{4.9}$$

Plugging back in, we have

$$0 \le |f|^2 - \sum(\bar{c}_n^* c_n + \bar{c}_n c_n^*) + \sum \bar{c}_n^* \bar{c}_n. \tag{4.10}$$

This is guaranteed to be positive semi-definite by the positivity of the norm. This defines a distance $D(f, \bar{f}_N)$ on the space of functions.

Notice there are really two sets of free parameters, c_n and $\bar{c}_n$ or equivalently c_n and $\bar{c}_n^*$. If we take a derivative with respect to $\bar{c}_n^*$ to extremize this, we get

$$\bar{c}_n = c_n, \bar{c}_n^* = c_n^*. \tag{4.11}$$

That is, we want to approximate our function as fast as we can, and the best way to do it is by picking the Fourier coefficients.[5] The derivative guarantees it. This is exactly the same as the least-squares fit, just by a different process.

The convergence of these sorts of series may be in question when the metric on the space is not positive definite (e.g. Minkowski signature). In that case, these approximations may converge to something but perhaps not uniquely, and we may require additional information to ensure uniqueness.

Fourier series have some nice properties. Notice that since the Fourier basis elements $\cos nx, \sin nx$ are periodic in 2π, the entire Fourier series is also periodic in 2π. Note that the function we are approximating need not itself have the same value at $-\pi$ and π; this just counts as a single discontinuity.

What if our function is periodic on a different interval, say $2L$? That is,

$$f(x + 2L) = f(x). \tag{4.12}$$

Then define $x = \alpha \bar{x}$ such that we can write

$$x + 2L = \alpha(\bar{x} + 2\pi), \tag{4.13}$$

since we know how to deal with functions which are 2π periodic. We find immediately that

$$\alpha = L/\pi. \tag{4.14}$$

Hence our Fourier expansion can now be written as

$$f = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \tag{4.15}$$

All this corresponds to is changing the radius of the unit circle. The circle has a radius given by L instead of 1. Then the coefficients are just

$$a_0 = \frac{1}{L} \int_{-L}^{L} dx\, f(x) \tag{4.16}$$

and the other coefficients are[6]

$$a_n = \frac{1}{L} \int_{-L}^{L} dx f(x) \cos\frac{n\pi x}{L}, \quad b_n = \frac{1}{L} \int_{-L}^{L} dx f(x) \sin\frac{n\pi x}{L}. \tag{4.17}$$

---

[5]If you like, this is a discrete version of a functional derivative where we just vary the components of each function.

[6]You can do this by just making a change of variables in the integral to get your limits of integration how you want them.

Equivalently we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} dx f(x) e^{-\frac{in\pi x}{L}}. \tag{4.18}$$

Let us define a function which is given on the fundamental domain as

$$f(x) = x, \quad -\pi < x < \pi. \tag{4.19}$$

and 2π periodic, so e.g. $f(x) = x - 2\pi, \pi < x < 3\pi$ and so on. This is a sawtooth wave. What is the Fourier decomposition of this wave? One can do the integrals in Eq. (4.17) to figure it out. One finds that all the cosine coefficients go away because this function is odd. It will only have sines:

$$x = f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right] = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}. \tag{4.20}$$

This does not converge absolutely, if one checks. The ratio test is indeterminate. This is like a harmonic series. But it should converge, and the reason it does is because it is an alternating series. Alternating series have nicer convergence properties in general.

We can also evaluate this at $\pi/2$. Remember this was the Fourier expansion of x. So then by plugging $x = \pi/2$ into Eqn. (4.20), we get

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right], \tag{4.21}$$

which lets us sum a series we might not have expected.

We could actually do this for a triangle wave instead, defined as $|x|$ on $-\pi, \pi$. If we wrote down the Fourier series we would find that

$$f(x) = \frac{\pi}{2} - \sum \frac{4}{\pi} \frac{\cos((2n+1)x)}{(2n+1)^2}. \tag{4.22}$$

This series is no longer alternating, but it actually converges faster, like $\sum 1/n^2$, since the function is not discontinuous like our sawtooth wave. The derivative is discontinuous, yes, but we still have nicer convergence. In general, the Fourier decompositions of functions which are smoother will converge faster.

It's fairly clear that we can add two Fourier series to get a new one.. But in fact we can also integrate Fourier series by linearity (assuming the original sum converges). When we integrate Fourier series, our sines and cosines get factors of $1/n$ in the series coefficients, i.e. for $f(x)$ given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{4.23}$$

the integral of $f(x)$ is

$$\int_{-\pi}^{x} f(x')dx' = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} a_n \frac{\sin nx}{n} - b_n \frac{\cos nx}{n} + \text{ constant boundary terms}. \tag{4.24}$$

These factors of $1/n$ mean that the convergence properties of the integrated function are nicer than the original, i.e. it converges at least as fast as the expansion of $f(x)$.

Lecture 5.

Wednesday, January 15, 2020

> *"I'm gonna skip this example of the full wave rectifier, blah blah blah. I never cared much for electronics. This is not a statement of preference but of personal incompetence."*
> —Nemanja Kaloper

Today we will continue our discussion of Fourier series. In general the integral of a Fourier series is not itself a Fourier series, since it has a linear term. But its convergence properties are nicer than the original. What's happening is that we put frequencies in the denominator, which makes the different frequency modes decouple more.

Recall we derived the Fourier series of a sawtooth wave,

$$x = 2 \sum_{n=1}^{\infty} (-1)n + 1 \frac{\sin nx}{n}. \tag{5.1}$$

Can we take derivatives of this, as

$$1 = 2 \sum (-1)^{n+1} \cos nx? \tag{5.2}$$

Alas, this is nonsense. Suppose we evaluated this at $x = 0$. Then we would find

$$1 = 2(1 - 1 + 1 - 1 + \dots) \tag{5.3}$$

which is clearly not convergent in the usual sense.[7]

However, if we take the Fourier expansion of $|x|$, we can get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum \frac{\cos nx}{n^2}, \tag{5.4}$$

which converges faster than $1/n$ (i.e. uniformly). Hence we can define the derivative such that

$$\frac{d}{dx}|x| = +\frac{4}{\pi} \sum \frac{\sin nx}{n} = \begin{cases} 1 & x > 0, \\ -1 & x < 0. \end{cases} \tag{5.5}$$

This is correct, as

$$\frac{d}{dx}|x| = 2\theta(x) - 1, \tag{5.6}$$

with θ the Heaviside step function.

Suppose now we are handed a series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n}. \tag{5.7}$$

What function does this correspond to? Let's think qualitatively. The cosine is bounded by ± 1, so this might converge. The point $x = 0$ is dangerous since our series becomes a harmonic series, which is divergent (albeit only logarithmically). This is true at any integer multiple of 2π, in fact.

Let's make sense of this as follows. We shall write this as

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \operatorname{Re} \sum \frac{e^{inx}}{n}. \tag{5.8}$$

Now e^{inx} is a point on a unit circle in the complex plane, and we can further think of this as a limit

$$\operatorname{Re} \sum \frac{e^{inx}}{n} = \lim_{r \to 1} \operatorname{Re} \left(\sum \frac{(re^{ix})^n}{n} \right) = |\sum \frac{q^n}{n}| < \sum \frac{|q|^n}{n}. \tag{5.9}$$

If we take this limit where $|q| = r < 1$ then this surely converges.

Notice that

$$\frac{1}{1-q} = \sum q^n. \tag{5.10}$$

This is not quite what we want, since we're missing the n in the denominator. But we can do a trick.

$$\int \frac{dq}{1-q} = \sum_{n=0}^{\infty} \int q^n dq = \sum_{n=0}^{\infty} \frac{q^{n+1}}{n+1}. \tag{5.11}$$

If we shift the dummy variable (noting that $n = 0$ just gives q), we can write

$$1 - \ln(1-q) = \sum_{n=0}^{\infty} \frac{q^n}{n}. \tag{5.12}$$

It follows that our function was

$$\ln(2 - 2\cos x)^{1/2} = -\ln[2\sin(x/2)]. \tag{5.13}$$

---

[7]However, see https://en.wikipedia.org/wiki/Grandi%27s_series. One can define a sensible Cesàro sum which actually is $1/2$, but it's not what we mean by convergence.

This passes a reasonable sanity check, which is that when x is a multiple of 2π, then this series is the log of zero, which is clearly divergent.[8]

Consider some square-wave potential which is 2π periodic, such that

$$f(x) = \begin{cases} -\pi < x < 0 & 0, \\ 0 < x < \pi & h. \end{cases} \tag{5.14}$$

The Fourier transform is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi} \sum \frac{\sin(2n+1)x}{2n+1}. \tag{5.15}$$

Note that only odd sines appear, since the odd sines will have nonzero integral over the half-interval $0, \pi$, while the even ones vanish.

We can also switch to exponential notation,

$$f(x) = \sum c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x) e^{-inx}. \tag{5.16}$$

What's interesting now is to consider the finite truncation of the Fourier series,

$$f_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} e^{inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) e^{-int}. \tag{5.17}$$

If this sum were infinite, we could not exchange the order of integration and summation with impunity. But for finite N, there are no problems with convergence. So we can do this freely and write

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \sum_{n=-N}^{N} e^{in(x-t)}. \tag{5.18}$$

This is in fact just a sum of cosines. For notice that if we define

$$q = e^{i(x-t)}, \tag{5.19}$$

then our sum is

$$S = \sum_{n=-N}^{N} q^n = \frac{1}{q^N} \sum_{n=0}^{2N} q^n \tag{5.20}$$

Note that the sum with $N \to \infty$ is the geometric series $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$. It follows that the finite sum can be written as the difference of two infinite sums,

$$\sum_{n=0}^{2N} q^n = \sum_{n=0}^{\infty} q^n - \sum_{n=2N+1}^{\infty} q^n = S_\infty - q^{2N} S_\infty = \frac{1-q^{2N}}{1-q}. \tag{5.21}$$

We get that this is

$$\frac{q^N - q^{-N}}{1-q} = \frac{\sin(N+\frac{1}{2})(x-t)}{\sin\frac{x-t}{2}}. \tag{5.22}$$

Hence

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) \left[\frac{\sin((N+1/2)(x-t))}{\sin\frac{x-t}{2}} \right], \tag{5.23}$$

which is the best approximation to our function in terms of a finite truncation. The goal is now to determine what properties f must have such that f_N actually converges nicely to f.

Note also that the term in the brackets is just a delta function. It is simply the sum $\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-t)}$, and in a sense this is just the statement of the completeness of complex exponentials.

---

[8]However, something a little weird happens when we take x to be e.g. $-\pi$. On the RHS we have a log of something negative, which is only well-defined in terms of the analytic continuation of $\ln(z)$. In fact, the log of a negative number has an imaginary part. The LHS looks like we're summing real numbers, so this is a bit unusual; it has to do again with the lack of uniform convergence.