PHYSICS 204B: METHODS OF MATHEMATICAL PHYSICS II

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These notes were taken for Physics 204B, *Methods of Mathematical Physics II*, as taught by Nemanja Kaloper at the University of California, Davis in winter quarter 2020. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Monday, January 6, 2020

"What did you do over break?" "Don't ask. No rest for the wicked."
—Mark Samuel Abbott and Nemanja Kaloper

The only outstanding logistical details here are that office hours will be posted later, and the TA is now Morgane König rather than Cameron Langer. All else is basically the same as last quarter.

Let's talk about Green's functions in more than one dimension. Our discussion will be somehwat sketchy, but we'll get a rough idea of the topic. A Green's function is the inverse of a differential operator, and it lives under integrals. In one dimension, we wrote that to solve

$$\mathcal{L}\phi = J, \tag{1.1}$$

we could construct G such that

$$\mathcal{L}G = \delta, \tag{1.2}$$

a function which gives a delta function upon being hit by a differential operator. We would like to solve the problem of finding the linear response of a field ϕ at a point $\mathbf{r}_1$ due to a source $J(\mathbf{r}_2)$ at a point $\mathbf{r}_2$.

Recall that the adjoint of an operator is given by

$$\langle\psi|\mathcal{L}\phi\rangle = \langle\mathcal{L}^\dagger\psi|\phi\rangle, \tag{1.3}$$

and an operator is self-adjoint if

$$\mathcal{L} = \mathcal{L}^\dagger. \tag{1.4}$$

That is, for an operator given by

$$\mathcal{L}\phi = \nabla\cdot(p\nabla\phi) + q\phi, \tag{1.5}$$

we must check that

$$\int \psi^*(\nabla\cdot(p\nabla\phi)) + \int q\psi^*\phi = \int \nabla\cdot(p\nabla\phi^*)\phi + \int q\psi^*\phi. \tag{1.6}$$

The q terms cancel, so we find that

$$\int [\psi^*\nabla\cdot(p\nabla\phi) - \nabla\cdot(p\nabla\psi^*)\phi] = 0, \tag{1.7}$$

and if we integrate by parts, then

$$\int \nabla\cdot(\psi^* p\nabla\phi - (\nabla\psi^*)p\phi) = 0. \tag{1.8}$$

By the divergence theorem,

$$\int_S p[\psi^* \nabla\phi - (\nabla\psi^*)\phi] = 0. \tag{1.9}$$

Dirichlet or Neumann boundary conditions will guarantee self-adjointness. There are also mixed conditions, but these sorts of conditions are sufficient to make our operator Hermitian.

For a Hermitian operator, the corresponding Green's function obeys[1]

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.10}$$

For recall that

$$\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) | G(\mathbf{r}, \mathbf{r}_2) \rangle = \langle G(\mathbf{r}, \mathbf{r}_1) | \mathcal{L}G(\mathbf{r}, \mathbf{r}_2 \rangle \tag{1.11}$$

by self-adjointness. By the definition of the Green's function and the inner product, we can replace $\mathcal{L}G$ by a delta function and get

$$\int \delta(\mathbf{r} - \mathbf{r}_1) G(\mathbf{r}, \mathbf{r}_2) = G(\mathbf{r}_1, \mathbf{r}_2) \tag{1.12}$$

on the LHS and

$$\int G^*(\mathbf{r}, \mathbf{r}_1) \delta(\mathbf{r}, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.13}$$

Hence

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.14}$$

Now let's construct the eigenfunction expansion of the Green's function. Consider

$$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.15}$$

We're keeping the $\mathbf{r}_1, \mathbf{r}_2$ dependence in G because there might be operators that are not translationally invariant. In other words, we can't assume that quantities depend only on $|\mathbf{r}_1 - \mathbf{r}_2|$. That is,

$$\mathcal{L}|_{\mathbf{r}} \neq \mathcal{L}|_{\mathbf{r}+\mathbf{a}}. \tag{1.16}$$

Suppose we construct the eigenfunctions ϕ_λ of the operator $\mathcal{L}$, such that

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda. \tag{1.17}$$

WLOG we may take them to be orthonormal,

$$\langle \phi_\lambda | \phi_\mu \rangle = \delta_{\lambda\mu}. \tag{1.18}$$

For now, we shall assert that they are a complete set– in general we will have to prove this. The expansion for the delta function is just the completeness relation:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \int_\lambda \phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1), \tag{1.19}$$

since

$$f(\mathbf{r}) = \int_\lambda f_\lambda \phi_\lambda(\mathbf{r}) = \int_{\mathbf{r}_1} f(\mathbf{r}_1)\delta(\mathbf{r} - \mathbf{r}_1) = \int_\lambda \underbrace{\int_{\mathbf{r}_1} f(\mathbf{r}_1)\phi^*_\lambda(\mathbf{r}_1)}_{f_\lambda} \phi_\lambda(\mathbf{r}). \tag{1.20}$$

Now we can put our delta function decomposition back in: suppose that

$$G(\mathbf{r}, \mathbf{r}_1) = \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}), \tag{1.21}$$

so that

$$\begin{aligned}
\mathcal{L}G &= \mathcal{L} \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}) \\
&= \int_\lambda C_\lambda(\mathbf{r}_1)\mathcal{L}\phi_\lambda(\mathbf{r}) \\
&= \int_\lambda C_\lambda(\mathbf{r}_1)\lambda\phi_\lambda(\mathbf{r}),
\end{aligned}$$

[1] This property is also responsible for Green's reciprocity theorem in electromagnetism, i.e. the statement that the potential energy of a charge distribution ρ_2 in a field produced by another distribution ρ_2 is equal to the energy of ρ_1 in the field produced by ρ_2. If you like, the theorem is a special case/application since the Laplacian operator is Hermitian.

and this last expression must be equal to the expansion of the delta function:

$$0 = \int_\lambda [C_\lambda(\mathbf{r}_1)\lambda - \phi^*_\lambda(\mathbf{r}_1)]\phi_\lambda(\mathbf{r}). \tag{1.22}$$

Hence

$$c_\lambda(\mathbf{r}_1) = \frac{\phi^*(\mathbf{r}_1)}{\lambda}, \tag{1.23}$$

so

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda}. \tag{1.24}$$

This makes the hermiticity of G totally clear:

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.25}$$

The only problem might be if we have a zero eigenvalue, in which case we have to be careful. Some people define a generalized Green's function by

$$(\mathcal{L} - Z)G = \delta, \tag{1.26}$$

so that our expression is just modified to

$$G(\mathbf{r}_1, \mathbf{r}_2) = \int_\lambda \frac{\phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda - Z}, \tag{1.27}$$

and we can study the $z \to 0$ limit.

Let's consider some examples. The Laplace equation is

$$\boldsymbol{\nabla}^2 \phi = J, \tag{1.28}$$

and it is of self-adjoint form,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.29}$$

Suppose we had a solution

$$\int \boldsymbol{\nabla}^2 \phi = J \tag{1.30}$$

such that $\int_V J = Q$ and a homogenous solution

$$\boldsymbol{\nabla}^2 \chi = 0. \tag{1.31}$$

But now

$$\int_V dV \boldsymbol{\nabla}^2(\phi + \chi) = \int_V J = Q. \tag{1.32}$$

We can rewrite the first epxression as a surface integral, $\int d\mathbf{S} \cdot \boldsymbol{\nabla}(\phi + \chi)$. Hence we find that since the integral of the $\boldsymbol{\nabla}\phi$ term is already Q, it must be that

$$\int d\mathbf{S} \cdot \boldsymbol{\nabla}\chi = 0, \tag{1.33}$$

and therefore by the uniqueness theorems, χ is at most a constant. This is an example of a gauge symmetry, actually, but we won't go too much into that. So given appropriate boundary conditions, solutions of the Laplace equation are unique up to a constant.

Now let us note the Laplace equation *is* translationally invariant, so we can write

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r})) = \delta(\mathbf{r}). \tag{1.34}$$

In fact, it is also manifestly spherically symmetric in this form. We've just chosen coordinates to put our charge at the origin. Let us integrate over a spherical region R. Then

$$1 = \int \delta(\mathbf{r}) = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r}) = 4\pi R^2 \frac{dG}{dr}. \tag{1.35}$$

We conclude that

$$G = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.36}$$

which is nothing more than the Coulomb potential for a unit charge.

If we instead enclosed the charge in a Faraday cage (setting the potential to zero somewhere), we would add a homogeneous solution to the Laplace equation to our Green's function in order to fit the new boundary conditions. Note that in 2 dimensions, we would just have $1 = 2\pi R \frac{dG}{dr}$, which gives our Green's function as a log of $|r|$ instead.