PHYSICS 204B: METHODS OF MATHEMATICAL PHYSICS II

IAN LIM
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These notes were taken for Physics 204B, *Methods of Mathematical Physics II*, as taught by Nemanja Kaloper at the University of California, Davis in winter quarter 2020. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

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Lecture 1.

Monday, January 6, 2020

"What did you do over break?" "Don't ask. No rest for the wicked."
—Mark Samuel Abbott and Nemanja Kaloper

The only outstanding logistical details here are that office hours will be posted later, and the TA is now Morgane König rather than Cameron Langer. All else is basically the same as last quarter.

Let's talk about Green's functions in more than one dimension. Our discussion will be somehwat sketchy, but we'll get a rough idea of the topic. A Green's function is the inverse of a differential operator, and it lives under integrals. In one dimension, we wrote that to solve

$$\mathcal{L}\phi = J, \tag{1.1}$$

we could construct G such that

$$\mathcal{L}G = \delta, \tag{1.2}$$

a function which gives a delta function upon being hit by a differential operator. We would like to solve the problem of finding the linear response of a field ϕ at a point $\mathbf{r}_1$ due to a source $J(\mathbf{r}_2)$ at a point $\mathbf{r}_2$.

Recall that the adjoint of an operator is given by

$$\langle \psi | \mathcal{L}\phi \rangle = \langle \mathcal{L}^\dagger \psi | \phi \rangle, \tag{1.3}$$

and an operator is self-adjoint if

$$\mathcal{L} = \mathcal{L}^\dagger. \tag{1.4}$$

That is, for an operator given by

$$\mathcal{L}\phi = \boldsymbol{\nabla} \cdot (p \boldsymbol{\nabla} \phi) + q\phi, \tag{1.5}$$

we must check that

$$\int \psi^* (\boldsymbol{\nabla} \cdot (p \boldsymbol{\nabla} \phi)) + \int q\psi^* \phi = \int \boldsymbol{\nabla} \cdot (p \boldsymbol{\nabla} \psi^*)\phi + \int q\psi^* \phi. \tag{1.6}$$

The q terms cancel, so we find that

$$\int [\psi^* \boldsymbol{\nabla} \cdot (p \boldsymbol{\nabla} \phi) - \boldsymbol{\nabla} \cdot (p \boldsymbol{\nabla} \psi^*)\phi] = 0, \tag{1.7}$$

and if we integrate by parts, then

$$\int_V dV\, \boldsymbol{\nabla} \cdot (\psi^* p \boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\psi^*) p \phi) = 0. \tag{1.8}$$

By the divergence theorem,

$$\int_S d\mathbf{S} \cdot p[\psi^* \boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\psi^*)\phi] = 0. \tag{1.9}$$

Dirichlet or Neumann boundary conditions will guarantee self-adjointness. That is, if the functions vanish on S or their normal derivatives vanish on S, then $\mathcal{L}$ is self-adjoint. There are also mixed conditions we could impose, but Dirichlet and Neumann conditions are sufficient to make our operator Hermitian.

For a Hermitian operator, the corresponding Green's function obeys[1]

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.10}$$

For recall that

$$\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) | G(\mathbf{r}, \mathbf{r}_2)\rangle = \langle G(\mathbf{r}, \mathbf{r}_1) | \mathcal{L}G(\mathbf{r}, \mathbf{r}_2)\rangle \tag{1.11}$$

by self-adjointness. By the definition of the Green's function and the inner product, we can replace $\mathcal{L}G$ by a delta function and get

$$\int \delta(\mathbf{r} - \mathbf{r}_1) G(\mathbf{r}, \mathbf{r}_2) = G(\mathbf{r}_1, \mathbf{r}_2) \tag{1.12}$$

on the LHS and

$$\int G^*(\mathbf{r}, \mathbf{r}_1)\delta(\mathbf{r}, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.13}$$

Hence

$$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{1.14}$$

Now let's construct the eigenfunction expansion of the Green's function. Consider

$$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.15}$$

We're keeping the $\mathbf{r}_1, \mathbf{r}_2$ dependence in G because there might be operators that are not translationally invariant. In other words, we can't assume that quantities depend only on $|\mathbf{r}_1 - \mathbf{r}_2|$. That is,

$$\mathcal{L}|_{\mathbf{r}} \neq \mathcal{L}|_{\mathbf{r}+\mathbf{a}}. \tag{1.16}$$

Suppose we construct the eigenfunctions ϕ_λ of the operator $\mathcal{L}$, such that

$$\mathcal{L}\phi_\lambda = \lambda\phi_\lambda. \tag{1.17}$$

WLOG we may take them to be orthonormal,

$$\langle \phi_\lambda | \phi_\mu \rangle = \delta_{\lambda\mu}. \tag{1.18}$$

For now, we shall assert that they are a complete set– in general we will have to prove this. The expansion for the delta function is just the completeness relation:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \int_\lambda \phi_\lambda^*(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1), \tag{1.19}$$

since

$$f(\mathbf{r}) = \int_\lambda f_\lambda \phi_\lambda(\mathbf{r}) = \int_{\mathbf{r}_1} f(\mathbf{r}_1)\delta(\mathbf{r} - \mathbf{r}_1) = \int_\lambda \underbrace{\int_{\mathbf{r}_1} f(\mathbf{r}_1)\phi_\lambda^*(\mathbf{r}_1)}_{f_\lambda} \phi_\lambda(\mathbf{r}). \tag{1.20}$$

Now we can put our delta function decomposition back in: suppose that

$$G(\mathbf{r}, \mathbf{r}_1) = \int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}), \tag{1.21}$$

[1] This property is also responsible for Green's reciprocity theorem in electromagnetism, i.e. the statement that the potential energy of a charge distribution ρ_2 in a field produced by another distribution ρ_2 is equal to the energy of ρ_1 in the field produced by ρ_2. If you like, the theorem is a special case/application since the Laplacian operator is Hermitian.

so that

$$\begin{aligned}\mathcal{L}G &= \mathcal{L}\int_\lambda C_\lambda(\mathbf{r}_1)\phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1)\mathcal{L}\phi_\lambda(\mathbf{r}) \\ &= \int_\lambda C_\lambda(\mathbf{r}_1)\lambda\phi_\lambda(\mathbf{r}),\end{aligned}$$

and this last expression must be equal to the expansion of the delta function:

$$0 = \int_\lambda [C_\lambda(\mathbf{r}_1)\lambda - \phi^*_\lambda(\mathbf{r}_1)]\phi_\lambda(\mathbf{r}). \tag{1.22}$$

Hence

$$c_\lambda(\mathbf{r}_1) = \frac{\phi^*(\mathbf{r}_1)}{\lambda}, \tag{1.23}$$

so

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int_\lambda \frac{\phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda}. \tag{1.24}$$

This makes the hermiticity of G totally clear:

$$G(\mathbf{r}_1,\mathbf{r}_2) = G^*(\mathbf{r}_2,\mathbf{r}_1). \tag{1.25}$$

The only problem might be if we have a zero eigenvalue, in which case we have to be careful. Some people define a generalized Green's function by

$$(\mathcal{L} - Z)G = \delta, \tag{1.26}$$

so that our expression is just modified to

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int_\lambda \frac{\phi^*_\lambda(\mathbf{r}_2)\phi_\lambda(\mathbf{r}_1)}{\lambda - Z}, \tag{1.27}$$

and we can study the $z \to 0$ limit.

Let's consider some examples. The Laplace equation is

$$\boldsymbol{\nabla}^2\phi = J, \tag{1.28}$$

and it is already in self-adjoint form,

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}G) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{1.29}$$

Suppose we had a solution

$$\int \boldsymbol{\nabla}^2\phi = J \tag{1.30}$$

such that $\int_V J = Q$ and a homogenous solution

$$\boldsymbol{\nabla}^2\chi = 0. \tag{1.31}$$

But now

$$\int_V dV\boldsymbol{\nabla}^2(\phi+\chi) = \int_V J = Q. \tag{1.32}$$

We can rewrite the first expression as a surface integral, $\int d\mathbf{S}\cdot\boldsymbol{\nabla}(\phi+\chi)$. Hence we find that since the integral of the $\boldsymbol{\nabla}\phi$ term is already Q, it must be that

$$\int d\mathbf{S}\cdot\boldsymbol{\nabla}\chi = 0, \tag{1.33}$$

and therefore by the uniqueness theorems, χ is at most a constant. This is an example of a gauge symmetry, actually, but we won't go too much into that. So given appropriate boundary conditions, solutions of the Laplace equation are unique up to a constant.

Now let us note the Laplace equation *is* translationally invariant, so we can write

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}G(\mathbf{r})) = \delta(\mathbf{r}). \tag{1.34}$$

In fact, it is also manifestly spherically symmetric in this form. We've just chosen coordinates to put our charge at the origin. Let us integrate over a spherical region R. Then

$$1 = \int \delta(\mathbf{r}) = \int_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} G(\mathbf{r}) = 4\pi R^2 \frac{dG}{dr}. \tag{1.35}$$

We conclude that

$$G = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.36}$$

which is nothing more than the Coulomb potential for a unit charge.

If we instead enclosed the charge in a Faraday cage (setting the potential to zero somewhere), we would add a homogeneous solution to the Laplace equation to our Green's function in order to fit the new boundary conditions. Note that in 2 dimensions, we would just have $1 = 2\pi R \frac{dG}{dr}$, which gives our Green's function as a log of $|r|$ instead.

Lecture 2.

Wednesday, January 8, 2020

> *"If I had taken a potato instead of a box, I would be hard pressed to do this. It would be much much harder. Except in two dimensions. Two-dimensional potatoes are very cool because Laplacians in 2 dimensions are special."*
>
> –Nemanja Kaloper

Let's play a bit more with Green's functions. Consider the Laplace equation. We can fit a solution using

$$\boldsymbol{\nabla}^2 (G + F) = \delta. \tag{2.1}$$

That is, we can construct a solution with a source using the Green's function

$$\boldsymbol{\nabla}^2 G = \delta \tag{2.2}$$

and add on homogeneous solutions F such that

$$\boldsymbol{\nabla}^2 F = 0 \tag{2.3}$$

to fit boundary conditions.

Example 2.4. Let us solve the following electrostatics problem. We take a charge and place it inside a grounded conductor shaped like a cube. Can we solve the Laplace equation in this setup?

The charge has spherical symmetry, but the cube does not. That means that the field lines near the sides of the cube will deform so the field becomes normal to the faces of the cube in the static, equilibrium configuration.

Consider a single charge near a grounded plane. We can solve this by the method of images. Suppose the charge sits on the z axis, while the grounded plane is the xy plane. In particular, it sits at a point $(0, 0, a)$ from the origin. We now consider a point a distance ρ from the z axis and at a height z along the z axis.

The method of images says that we can solve for the potential by considering any (fictitious) charge distribution which satisfies the same boundary conditions for the potential. That is, instead of just measuring the potential from the single charge, we can write

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} \tag{2.5}$$

as the potential from the charge alone and suppose we add another image charge Q' on the other side of the grounded plane at a location $(0, 0, -a')$. Then the potential from this configuration of the charge and image charge is

$$\frac{1}{\sqrt{\rho^2 + (z-a)^2}} + \frac{Q'}{\sqrt{\rho^2 + (z+a')^2}}. \tag{2.6}$$

Now we consider the plane $z = 0$ and see that

$$V(z=0) = \frac{1}{\sqrt{\rho^2 + a^2}} + \frac{Q'}{\sqrt{\rho^2 + a'^2}}. \tag{2.7}$$

But this still seems to depend on ρ. Setting $\rho = 0$ gives

$$\frac{1}{a} + \frac{Q'}{a'} = 0, \tag{2.8}$$

and taking a derivative gives

$$\frac{1}{a^3} + \frac{Q'}{a'^3} = 0, \tag{2.9}$$

so solving gives

$$a = a', \quad Q' = -1. \tag{2.10}$$

The potential on the real side of the grounded plane is just the dipole potential in the region we care about. The other charge is equal and opposite, and equally far away behind the plane.

What about two conducting planes? We can construct a dipole to set the potential on one plane, but then the other won't be grounded. We can add an image dipole outside the other plane (see image). Then we the second plane is okay but the first plane is now not grounded. So we go on adding dipoles in this way ad infinitum, and the overall potential is

$$\sum_{n=-\infty}^{\infty} \frac{1}{[\rho^2 + (z - a + n(2D))^2]^{-1/2}} - \frac{1}{[\rho^2 + (z + a + n(2D))^2]^{-1/2}}. \tag{2.11}$$

Expanded, this is

$$\sum_n \frac{1}{|\mathbf{r} - 2Dn\mathbf{k}|} - \frac{1}{|\mathbf{r} + 2Dn\mathbf{k}|}. \tag{2.12}$$

The sum looks like it might diverge, since each term goes as $1/n$, but in fact since we're dealing with dipoles, the $1/n$ dependence will cancel and away from the corners, this will converge. These image charges help us to satisfy the boundary conditions on the two planes.

The generalization to three dimensions is to extend the periodicity into three dimensions. We get an infinite lattice of image dipoles which is at once simple and complicated.

Let's write a nice fact about multipoles:

$$\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_<^{l+1}} P_l(\cos\theta). \tag{2.13}$$

This simply says that the potential of a multipole drops off as $1/r^{l+1}$, and inside some spherical region it increases as r^l. Its angular dependence is given by the Legendre polynomials. This comes from the fact that

$$\sum x^n P_l(t) = \frac{1}{\sqrt{1 - 2xt + x^2}}, \tag{2.14}$$

the generating function for Legendre polynomials, with $-1 < t < 1$ and $0 < x < 1$.Taking this denominator as $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + 2r_1 r_2 \cos\theta + r_2^2}$, if we pull out r_1 then we get the desired result with this formula.

One more example. Suppose we write the Schrödinger equation as

$$(\boldsymbol{\nabla}^2 + k^2)\psi = 2mV\psi, \tag{2.15}$$

where $k^2 = 2mE$.

We may write the Green's function

$$(\boldsymbol{\nabla}^2 + k^2)G = \delta, \tag{2.16}$$

satisfying

$$\int_{S^2} d\mathbf{S} \cdot \boldsymbol{\nabla} G + k^2 \int_V dV\, G = 1, \tag{2.17}$$

where V is a sphere centered on the origin. In the limit of very small spheres, we shall argue that the second term vanishes. For the k^2 term is constant, whereas as r grows small, the $\boldsymbol{\nabla}^2$ blows up as $1/r^2$. Near the origin, we can solve

$$\int d\mathbf{S} \cdot \boldsymbol{\nabla} G = 1, \tag{2.18}$$

while away from the origin, our delta function is zero. We can exploit the spherical symmetry to rewrite the laplacian as

$$\frac{1}{r^2}\partial_r(r^2\partial_r G) + k^2 G = 0. \tag{2.19}$$

If we write $G = \phi/r$ to account for the asymptotic behavior of the Green's function, then we find

$$\phi'' + k^2\phi = 0. \tag{2.20}$$

This has solutions

$$\phi = e^{\pm ikr}. \tag{2.21}$$

The $+$ solution is an outgoing wave, while the $-$ solution is an incoming wave. Now

$$G = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}. \tag{2.22}$$

We can now treat the Schrödinger equation as an integral equation with a source. Thus

$$\psi(\mathbf{r}_1) = 2m\int dr_2^3\, G(\mathbf{r}_1,\mathbf{r}_2)V(\mathbf{r}_2)\psi(\mathbf{r}_2). \tag{2.23}$$

This is self-consistent, but is it useful? It will be if V is small. Thus we can expand ψ perturbatively in orders of V, considering each order of scattering and writing

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots \tag{2.24}$$

where each order is given by solving the integral equation with the previous order.