

# PHYSICS 230B: QUANTUM FIELD THEORY

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Lecture 1.

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The syllabus for this course is on Canvas.

**Massless spin 1 particles** What we're gonna start talking about is QFTs with massless spin 1 particles (i.e. photons). When we talk about particles, what we mean are states, i.e.

$$|p, s\rangle, \quad (1.1)$$

labeled by a four-momentum  $p$  and a spin state  $s$ . For massive spin 1 there are three spin states which can be distinguished by the helicity, i.e. the projection of the spin along the direction of the three-momentum, so our states can be labeled by  $s = 0, \pm 1$ .

In the massive case, we therefore have

$$|p, s\rangle_{s=0,\pm 1} \mapsto^\Lambda \sum_{s'} D_{s's}^{(1)}(W(\Lambda, p)) |\Lambda p, s'\rangle \quad (1.2)$$

where  $W(\Lambda, p)$  indicates a representation of the  $SO(3)$  symmetry.

In the massless case, it's a little different. Instead we have

$$|p, s\rangle_{s=\pm 1} \mapsto^\Lambda e^{-is\theta(W(\Lambda, p))} |\Lambda p, s\rangle. \quad (1.3)$$

The key difference is that we lose a polarization state in going from massive to massless.  $2 \neq 3$ . There is no rest frame for the photon, so we have to be careful in taking a massless limit or conversely turning on a mass.

How do we set up a QFT? We will need some fields, obviously. We can set up a Fock space with some creation and annihilation operators, so that given a vacuum state, we can define

$$|p_1, s_1, \dots, p_n, s_n\rangle = a_{s_1}^\dagger(p_1), \dots, a_{s_n}^\dagger(p_n) |0\rangle \quad (1.4)$$

where the creation and annihilation operators have commutator

$$[a_{s'}(p'), a_s^\dagger(p)] = \delta_{s's}(p'|p) = \delta_{s's} 2|\mathbf{p}| \cdot (2\pi)^3 \delta^3(\mathbf{p}' - \mathbf{p}). \quad (1.5)$$

We could define a vector field, since we want a nontrivial transformation under Lorentz. Thus

$$\hat{A}^\mu(x) = \int (dp) \sum_{s=\pm 1} \hat{a}_s(p) \epsilon_s^\mu(p) e^{-ip \cdot x} + \text{h.c.} \quad (1.6)$$

with  $dp = \frac{d^3p}{(2\pi)^3} \frac{1}{2|\mathbf{p}|}$ . That is, our vector field is an integral over  $d^4p$  with creation and annihilation operators summed over spin states, polarization vectors  $\epsilon_s^\mu(p)$  attached, and the corresponding exponentials.

Note that as a consequence of being massless, the field satisfies

$$\square A^\mu(x) = 0, \quad (1.7)$$

the massless Klein-Gordon equation. We can choose

$$p_\mu \epsilon_s^\mu(p) = 0 \iff \partial_\mu \hat{A}^\mu(x) = 0. \quad (1.8)$$

That gets rid of one linear combination, but we still seem to have too many degrees of freedom. In fact, we will show that  $A$  does not transform as an honest vector. That is, the equation

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(\Lambda) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x) \quad (1.9)$$

cannot be satisfied. This is nontrivial to show, as the left side is an infinite-dim unitary rep of the Lorentz group, whereas the right side is a finite-dim non-unitary rep of Lorentz.

However, what we notice is that the left side evaluates to

$$\hat{A}^\mu(x) = \int (dp) \sum_{s=\pm 1} \underbrace{\hat{U}^\dagger \hat{a}_s(p) \hat{U}}_{\sim e^{-is\theta} a_s} \epsilon_s^\mu(p) e^{-ip \cdot x} + \text{h.c.}, \quad (1.10)$$

and we know that the right side (by virtue of having Lorentz indices) must affect the polarization vector, i.e.

$$\Lambda^\mu_\nu \epsilon^\nu(\Lambda^{-1}p) \stackrel{?}{=} e^{-is\theta(W(\Lambda, \Lambda^{-1}p))} \epsilon_s^\mu(p). \quad (1.11)$$

For a general four-momentum we can certainly choose a frame where it takes the standard form

$$p^\mu \rightarrow n^\mu = (E, 0, 0, E), \quad (1.12)$$

such that for rotations about the  $x_3$  axis,

$$(e^{-i\theta J_3})^\mu_\nu \epsilon^\nu_\pm(n) = e^{-i\theta} \epsilon^\mu_\pm(n). \quad (1.13)$$

Here  $J_3$  is the generator of rotations around the  $x_3$  axis. Hence this becomes an eigenvalue equation,  $J^3 \epsilon_\pm(n) = \pm \epsilon_\pm(n)$ . We find that

$$\epsilon_\pm^\mu(n) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) + \alpha_\pm n^\mu. \quad (1.14)$$

There's an extra freedom in the  $\alpha$ s since  $n^\mu$  is a zero eigenvector of  $J_3$  and can therefore be added to  $\epsilon_\pm(n)$  with impunity.

Let us consider the action of the Little group, with the generators  $J_3, T^1, T^2$  (transcribe later). Thus

$$W(\theta, \beta_1, \beta_2)^\mu_\nu = \left[ e^{-i(\theta J^3 + \beta_1 T^1 + \beta_2 T^2)} \right]^\mu_\nu \quad (1.15)$$

defines the Wigner rotation. These are the operators which leave the form of the standard four-momentum unchanged.

Now

$$\Lambda^\mu_\nu \epsilon^\nu_\pm(\Lambda^{-1}p) = e^{-is\theta} \epsilon^\mu_\pm(p) + ip^\mu f_s, \quad (1.16)$$

where  $\theta, f_s$  are some functions of  $W(\Lambda, \Lambda^{-1}p)$ . We see that we've picked up an extra piece, the  $p^\mu$  term. Now  $T^{1,2} \cdot n = 0$ , since this is the definition of being in the Little group. But on the polarization vector, we have instead

$$T^{1,2} \cdot \epsilon(n) \propto n. \quad (1.17)$$

Thus when we write down the unitary transformation of our field  $A^\mu$ , we have

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(\Lambda) = \Lambda^\mu_\nu \hat{A}^\nu(\Lambda^{-1}x) + \partial^\mu \hat{\omega}(x). \quad (1.18)$$

There's some extra stuff here too,

$$\hat{\omega}(x) = \int (dp) \sum_{s=\pm} \left[ \hat{a}_s(p) \epsilon_s^\mu(p) e^{-ip \cdot x} f_s(W(\dots)) + \text{h.c.} \right], \quad (1.19)$$

which is some terrible stuff we don't want to deal with.

We recall (from Chapter 7) that there's a boost  $L(p)$  such that

$$\epsilon_{\pm}(p) = L(p) \cdot \epsilon_{\pm}(n), \quad (1.20)$$

with  $L(p) \cdot n = p$ .

For some standard  $n = (E, 0, 0, E)$ , we can boost along the 3-axis to turn  $E \rightarrow p^0$  and then rotate  $\mathbf{n} = (0, 0, p^0)$  to point in the direction of  $\mathbf{p}$ . Thus

$$\epsilon_{\pm}(n) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \quad (1.21)$$

such that  $\epsilon_{\pm}^0(p) = 0$  gives

$$\hat{A}^0(x) = 0, \quad (1.22)$$

which is clearly not Lorentz-invariant.

However, while  $A^\mu$  by itself is not good to work with, we can define

$$\hat{F}_{\mu\nu}(x) \equiv \partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x), \quad (1.23)$$

which defines an antisymmetric tensor observable. We might like to write down some sort of interacting Hamiltonian, which should have terms  $O(A^3)$ . For instance,

$$F^{\mu\nu} F_{\nu\rho} F^\rho{}_\mu \in \mathcal{H}_{\text{int}}. \quad (1.24)$$

What we'll see is that  $[A_\mu] = 1$ , which tells us that  $[F^3] = 6 > 4$ , i.e. in four spacetime dimensions this coupling is irrelevant. We know that light has nontrivial couplings to matter, so we'll have to figure out how this can be done.

**Gauge invariance** Gauge invariance comes into our theory through the “Stuckelberg trick.” Stuckelberg figured out how to go from a theory without gauge invariance to a theory with gauge invariance. We begin by declaring that a new field  $\hat{A}^\mu$  has the property

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(\Lambda) = \Lambda^\mu{}_\nu \hat{A}^\nu(\Lambda^{-1}x), \quad (1.25)$$

i.e. transforms as an honest vector, given that

$$A_\mu(x) \cong A_\mu(x) + \partial_\mu \omega(x). \quad (1.26)$$

That is, we say that adding  $\partial_\mu \omega(x)$  is a gauge redundancy, i.e. two fields differing only by a total derivative lead to the same physics.

Now, for some  $A_\mu$  our gauge transformations sweep out some gauge orbit which ends up foliating the space of  $A_\mu$ , and the configuration space is then the set of equivalence classes. This is known as the “geometrical viewpoint.”

Conversely, there is the gauge-fixed viewpoint in which we choose a representative  $A_\mu(x)$  from each orbit.

Notice also that our field strength tensor is gauge-invariant. In the free theory, if we only want to build the equations of motion out of  $F^{\mu\nu}$ , basically all we can write down is

$$\partial_\mu F^{\mu\nu} = 0. \quad (1.27)$$

This is both  $O(A_\mu)$  and quadratic in the derivatives,  $O(\partial^2)$ . And thus we may choose a gauge such that

$$\square A = 0 \quad (1.28)$$

$$A^0 = 0 \quad (1.29)$$

$$\nabla \cdot \mathbf{A} = 0. \quad (1.30)$$

This is *Coulomb gauge*.

Let  $A_\mu(x)$  be in a general gauge. Does there exist an  $\omega$  with

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \omega(x) \quad (1.31)$$

such that

$$0 \stackrel{?}{=} A'_0(x) = A_0(x) + \partial_0 \omega(x, t) \quad (1.32)$$

Sure, if we choose  $\omega(\mathbf{x}, t) = -\int^t dt' A_0(\mathbf{x}, t') + f(\mathbf{x})$ , where we can certainly add something that depends only on  $\mathbf{x}$ . Moreover we wish to impose

$$0 \stackrel{?}{=} \nabla \cdot \mathbf{A}'(\mathbf{x}, t) = \nabla \cdot \left[ A(\mathbf{x}, t) - \nabla \left( -\int_0^t dt' A_0(\mathbf{x}, t') \right) - \nabla f(\mathbf{x}) \right]. \quad (1.33)$$

In general this is not possible, since the first two terms here generically depend on time and the  $\nabla^2 f(\mathbf{x})$  term does not depend on time. However, we can do it for one particular time,  $t = 0$ , and set

$$\nabla \cdot \mathbf{A}'(\mathbf{x}, t = 0) = 0, \quad (1.34)$$

such that the equations of motion guarantee

$$\partial_t(\nabla \cdot \mathbf{A}) = 0. \quad (1.35)$$