PHYSICS 200C: ELECTROMAGNETISM II

IAN LIM
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Lecture 1.

Monday, March 30, 2020

We've completed our study of electrostatics, and now it's time to begin electrodynamics. We'll write the versions in materials to recap:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \tag{1.1a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.1b}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.1c}$$
$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.1d}$$

In statics, the total charge is

$$\rho = \rho_f - \nabla \cdot \mathbf{P}, \tag{1.2}$$

where $-\nabla \cdot \mathbf{P}$ is the polarization charge, and the total current is

$$\mathbf{J} = \mathbf{J}_f + \nabla \times \mathbf{M}, \tag{1.3}$$

where $\nabla \times \mathbf{M}$ is the magnetization current.[1]

In dynamics, there is an additional term in the total current. We need not only free current and magnetization current but also *polarization current*:

$$\mathbf{J} = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \tag{1.4}$$

We'll also define the auxiliary fields

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}, \tag{1.5}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu}\mathbf{B} \tag{1.6}$$

Good dielectrics have $\epsilon/\epsilon_0 \sim$ 4-10, while good magnets have permittivities of over 1000.

Maxwell introduced an important correction to Ampère's law. In statics,

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{1.7}$$

It follows that since the divergence of a curl vanishes,

$$\nabla \cdot \mathbf{J} = 0 \tag{1.8}$$

in statics. This makes sense, since it says that charge is not accumulating or depleting from anywhere. More generally, the continuity equation tells us that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0, \tag{1.9}$$

so since $\rho_f = \nabla \cdot \mathbf{D}$, we can write

$$\nabla \cdot \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \tag{1.10}$$

and now we have a proper divergence-free quantity to put on the RHS of Ampere's law:

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{1.11}$$

Example 1.12. We have an infinite straight wire with cross-sectional area πa^2, and the wire has a break. We wish to find the displacement current in the gap. A current $I(t) = I_0 \cos \omega t$ flows in the wire.

Since the charge can't jump the gap, we get a charge density σ building up on each of the faces of the wire. Namely,

$$\mathbf{E}_{\text{gap}} = \frac{\sigma}{\epsilon_0} \tag{1.13}$$

with

$$\sigma = \int \mathbf{J} dt = \int \frac{I(t)}{\pi a^2} dt, \tag{1.14}$$

and then the displacement current is

$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \epsilon_0 \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}_{\text{gap}}}{\partial t} = \frac{\partial \sigma}{\partial t} = \frac{I(t)}{\pi a^2}. \tag{1.15}$$

We conclude that

$$I_D = \pi a^2 \mathbf{J}_D = I(t), \tag{1.16}$$

so in fact the displacement current is equal to the current in the wire. Note we had to have a time dependence, or else our capacitor would charge up. At 20,000 V/cm, air breaks down and we get a spark.

[1] Last quarter we also considered fictitious magnetic charges.

Faraday's law is the other important time-dependent Maxwell equation,

$$\nabla \times vecE = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.17}$$

Changing magnetic fields can induce an electromotive force (EMF), such that

$$\epsilon = \oint_C \mathbf{E} \cdot d\mathbf{l}, \tag{1.18}$$

such that $\mathbf{E}$ is the electric field at a line element $d\mathbf{l}$ of the circuit. This minus sign is important enough to have its own name, Lenz's law. Faraday's observations said that

$$\epsilon = -K\frac{d\Phi}{dt}, \tag{1.19}$$

where $k = 1$ in SI units. For a moving circuit (say, a wire loop), we see that the flux can change either if $\mathbf{B}$ changes or if the motion changes the flux through the circuit,

$$\frac{d}{dt}\int_S \mathbf{B} \cdot \hat{\mathbf{n}} dA = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} dA + \oint (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l}, \tag{1.20}$$

Now the electromotive force is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -k\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} dA, \tag{1.21}$$

where

$$\mathbf{E}' = \mathbf{E} + k(v \times \mathbf{B}), \tag{1.22}$$

with $\mathbf{E}'$ in the (co-)moving frame moving at velocity $\mathbf{v}$, and $\mathbf{E}$ measured in the lab frame. A conduction electron is basically at rest in the frame of the circuit, so it experiences a purely electric force, $\mathbf{F} = q\mathbf{E}'$ and so in the lab frame, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

If the circuit is fixed in the lab frame, then

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} dA = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} dA, \tag{1.23}$$

so

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.24}$$

Lecture 2.

Wednesday, April 1, 2020

Note. Reading: Zangwill 10.5-10.10. The slides for the lecture should be posted on the previous night. Homeworks will be due at midnight/11:59 pm on Mondays rather than 10:30 am.

Quasistatic fields Quasistatics describes situations where fields and sources are changing slowly with respect to the relevant length scales. We'll see how to turn this into something formal shortly

Griffiths mentions the skin depth briefly in his discussion of dispersion relations, while Jackson talks about it at the end of magnetostatics. Many useful (and sometimes annoying) effects can be derived in the quasistatic approximation, and Zangwill spends the most time on this of the various graduate texts.

Well, we have our time-dependent Maxwell equations:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{2.1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{2.1d}$$

We're always looking for simplifications to solve these equations explicitly. A useful one is where we can drop the time derivatives, either the Faraday part of the electric field $\frac{\partial \mathbf{B}}{\partial t}$ or the displacement current $\frac{\partial \mathbf{D}}{\partial t}$ are negligible.

For *quasi-electrostatic fields*, we have

$$\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2.2}$$

and for *quasi-magnetostatic fields*, we have

$$\frac{\partial \mathbf{D}}{\partial t} = 0. \tag{2.3}$$

Quasi-electrostatic case Let's begin by considering a slowly varying charge distribution $\rho(\mathbf{r}, t)$. It has a characteristic length scale l and oscillation period T where $1/T \sim \omega$, a characteristic frequency of oscillation. We can decompose arbitrary oscillations in Fourier series, so we don't lose too much by working with sinusoids. That is, spatial derivatives give

$$\nabla \cdot e^{ix/l} \to \frac{1}{l} e^{ix/l}, \tag{2.4}$$

and time derivatives give

$$\frac{\partial}{\partial t}(e^{i\omega t}) \to \omega e^{i\omega t}. \tag{2.5}$$

These are the relevant length and time scales involved. Thus we can rewrite the continuity equation in terms of these length scales:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \implies \frac{\mathbf{J}}{l} + \omega\rho = 0. \tag{2.6}$$

Rewriting, we find that

$$J \sim \omega l \rho. \tag{2.7}$$

In particular, we can decompose the electric field in the following way, thinking of the Helmholtz theorem:

$$\mathbf{E} = \mathbf{E}_c + \mathbf{E}_f, \tag{2.8}$$

into a Coulomb part such that

$$\nabla \cdot \mathbf{E}_c = \frac{\rho}{\epsilon} \implies \frac{E_c}{l} \sim \rho/\epsilon \implies E_C \sim \frac{l}{\epsilon}\rho, \tag{2.9}$$

while the Faraday part goes as

$$\nabla \times \mathbf{E}_f \sim \frac{\partial \mathbf{B}}{\partial t} \implies \frac{E_f}{l} \sim \omega B \implies E_f \sim \omega B l. \tag{2.10}$$

Similarly we can estimate the magnetic field as

$$\nabla \times \mathbf{B} = \mu(\mathbf{J} + \epsilon\frac{\partial \mathbf{E}}{\partial t}) \implies B \sim \mu\omega\rho l^2. \tag{2.11}$$

If we compare the Faraday and Coulomb fields, we have

$$\frac{E_F}{E_C} \sim \mu\epsilon\omega^2 l^2 \sim \frac{\omega^2 l^2}{v_n^2}, \tag{2.12}$$

where v_n indicates the speed of light in the medium. We see that $E_F/E_C \ll 1$ when

$$\omega^2 \ll \frac{v_n^2}{l^2}, \tag{2.13}$$

i.e. the transit time for light l/v_n in the medium is much less than $1/\omega$.

Quasi-magnetosatics The quasi-magnetostatic case is similar, except here we instead need to consider a slowly varying current $\mathbf{J}(\mathbf{r}, t)$. We have the ampere part of the magnetic field,

$$\boldsymbol{\nabla} \times \mathbf{B}_A = \mu \mathbf{J} \implies B_A \sim \mu J l \tag{2.14}$$

and the Faraday law gives

$$\boldsymbol{\nabla} \times \mathbf{E}_F = \frac{\partial \mathbf{B}_A}{\partial t} \implies E_F \sim l\omega B_a \sim \mu\omega l^2 J. \tag{2.15}$$

The displacement current is then

$$\mathbf{J}_D = \epsilon \frac{\partial \mathbf{E}_F}{\partial t} \implies \tag{2.16}$$

What does slowly varying mean in materials? For materials that obey Ohm's law,

$$\mathbf{J}_f = \sigma \mathbf{E}, \tag{2.17}$$

where σ is the conductance (in units of $1/\Omega m$). we can plug into the continuity equation,

$$\boldsymbol{\nabla} \cdot \mathbf{J}_F + \frac{\partial \rho_f}{\partial t} = 0 \implies \boldsymbol{\nabla} \cdot (\sigma \mathbf{E}) + \frac{\partial \rho_f}{\partial t} = 0, \tag{2.18}$$

so

$$\frac{\partial \rho_f}{\partial t} + \frac{\sigma}{\epsilon}\rho_f = 0 \implies \rho_f(\mathbf{r}, t) = \rho_0(\mathbf{r}) e^{-t/\tau}, \tag{2.19}$$

which says that free charge decays with a characteristic *electric time constant*

$$\tau_E \equiv \epsilon/\sigma. \tag{2.20}$$

Highly conductive materials redistribute charge quickly.

There's also a corresponding magnetic time constant,

$$B_F \sim \mu l J_F \sim \mu\sigma\omega l^2 B_{\text{ext}} \tag{2.21}$$

where we consider an external magnetic field. Then

$$\frac{B_F}{B_{\text{ext}}} \sim \mu\sigma\omega l^2 = \omega\tau_M \implies \tau_M \equiv \mu\sigma l^2. \tag{2.22}$$

We see that τ_M scales directly with the conductance, while τ_E scales inversely with σ.

Poor conductors (lossy dielectrics) have $\sigma \sim 10$ S/m,[2] while good conductors have $\sigma \sim 10^8$ S/m.

Skin depth For quasistatic magnetic fields, Faraday's law rerwritten in terms of the vector potential is

$$\boldsymbol{\nabla} \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0. \tag{2.23}$$

if we write $\mathbf{E}$ in terms of the vector potential as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{2.24}$$

then Ampere's law becomes

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu \mathbf{J} = \mu\sigma \mathbf{E} \tag{2.25}$$

and so

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mu\sigma\left(-\frac{\partial \mathbf{A}}{\partial t}\right). \tag{2.26}$$

Now we use the double curl identity as find that

$$-\nabla^2 \mathbf{A} = -\mu\sigma\frac{\partial \mathbf{A}}{\partial t}. \tag{2.27}$$

We recover a diffusion equation for $\mathbf{A}$. If we consider a semi-infinite slab of material and a field

$$H_x(z, t) = h(z) e^{i\omega t}, \tag{2.28}$$

then

$$(\frac{d^2}{dz^2} + i\mu\sigma\omega) h(z) = 0 \; implies h(z) = H_0 e^{ikz} \tag{2.29}$$

[2] The unit of conductance is Siemens per meter, or sometimes mhOs per meter

which gives us a characteristic penetration depth $\kappa = ik$.

Lecture 3.

Friday, April 3, 2020

Today we'll go through chapter 15, which focuses on general fields and conservation laws. In dynamics, we still have

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{3.1}$$

which means we can define a time-dependent vector potential,

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t). \tag{3.2}$$

We can plug this into Faraday's law and see that

$$\boldsymbol{\nabla} \times \mathbf{E} = -\boldsymbol{\nabla} \times \left(\frac{\partial \mathbf{A}}{\partial t}\right), \tag{3.3}$$

so that

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{3.4}$$

This suggests that we can define a time-dependent scalar potential Φ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\Phi \tag{3.5}$$

or equivalently

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{3.6}$$

We'd now like to find the equations of motion for the new potentials. Plugging our expression for $\mathbf{E}$ into Gauss's law, we have

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}), \tag{3.7}$$

and if we plug the equations for $\mathbf{B}$ and $\mathbf{E}$ into Ampère's law, we can use the double-curl identity to find

$$\underbrace{\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}}_{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})} = -\mu_0 \mathbf{J} + \underbrace{\mu_0\epsilon_0 \frac{\partial}{\partial t}\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})}_{\text{displacement current}}. \tag{3.8}$$

We therefore have a pair of coupled inhomogenous equations for the potentials int erms fo the sources.

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}, \tag{3.9}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\left(\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{\partial}{\partial t}\boldsymbol{\nabla}\Phi\right) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) = -\mu_0 \mathbf{J}. \tag{3.10}$$

Now the potentials are only defined up to a gauge function $\Lambda(s^\alpha) = \Lambda(t, \mathbf{r})$, i.e.

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda \tag{3.11}$$

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \tag{3.12}$$

That is, the physics is all the same no matter what gauge we work in.

There are basically two gauge choices we often make in classical electrodynamics.[3] The first is Lorenz gauge,[4]

$$\boldsymbol{\nabla} \cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0. \tag{3.13}$$

[3] In QED we sometimes use Feynman/Landau gauge.

[4] In covariant notation it's $\partial_\mu A^\mu = 0$.

This uncouples the differential equations nicely to

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\epsilon_0 \tag{3.14}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{3.15}$$

Another gauge sometimes useful is the Coulomb gauge (also radiation gauge or transverse gauge),

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{3.16}$$

Then the Φ equation simplifies to Poisson's equation,

$$\nabla^2\Phi = -\rho/\epsilon_0, \tag{3.17}$$

but we lose covariance. The vector potential is still coupled,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}. \tag{3.18}$$

The $\frac{\partial\Phi}{\partial t}$ term cancels the longitudinal current, so what's left on the RHS in the "transverse" current. This is not manifestly covariant, since Poisson's equation has no time dependence in it. It says that there is no propagation time for the scalar potential; all the complicated time dependence is in the vector potential.

Conservation laws As we know, the continuity equation is a statement of conservation of charge. We define a total charge in a region

$$Q(t) = \int_V \rho(\mathbf{r}',t)d^3x', \tag{3.19}$$

and define the current through the bounding surface to be

$$\frac{dQ}{dt} = -\oint_S \mathbf{J}\cdot d\mathbf{a}, \tag{3.20}$$

so by the divergence theorem, we have

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{J}, \tag{3.21}$$

a local statement of conservation of charge. The divergence of $\mathbf{J}$ at a point tells us whether charge is building up or flowing away from a point.

Conservation of energy in electromagnetism also looks similar. The result is called Poynting's theorem. We begin by the following construction: dot the $\mathbf{E}$-field with Ampère's law and the $\mathbf{B}$-field with Faraday's law. Thus

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}). \tag{3.22}$$

We can simplify and rewrite as

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) = \mu_0\mathbf{J}\cdot\mathbf{E} + \frac{\partial}{\partial t}\left(\frac{B^2}{2} + \frac{E^2}{2c^2}\right). \tag{3.23}$$

and with a bit of rearrangement and a vector identity we can rewrite as

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right) + \frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) + \mathbf{E}\cdot\mathbf{J} = 0 \tag{3.24}$$

where we define the Poynting vector as

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \mathbf{E}\times\mathbf{H} \tag{3.25}$$

and the local energy density

$$u_{\text{em}} = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}. \tag{3.26}$$

This has the form of a conservation law,

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} + \mathbf{E}\cdot\mathbf{J} = 0, \tag{3.27}$$

where $\mathbf{E}\cdot\mathbf{J}$ is the change in mechanical energy density, i.e. the work done by the electric field. Written compactly we have

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{EM}}) = -\boldsymbol{\nabla}\cdot\mathbf{S}, \tag{3.28}$$

which is now in the form of a conservation law, with the Poynting vector representing the local energy flow.

Similarly we can find a conservation law for linear momentum by taking cross products rather than dot products in order to get a vectorial equation. Since momentum is a vector we will need something which has two indices in order to be left with a vector quantity when we take the divergence. That is,

$$\frac{\partial}{\partial t}\mathbf{P} = +\boldsymbol{\nabla}\cdot\mathbf{T}, \tag{3.29}$$

where $\mathbf{T}$ is the Maxwell stress tensor and $\mathbf{P}$ is the momentum density in the fields.

If we take cross products with Ampère's law and Faraday's law, we now have

$$\mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right) + \mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\right). \tag{3.30}$$

Rearranging, we have

$$\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{H}) + \mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \mathbf{J}\times\mathbf{B} = 0. \tag{3.31}$$

If we subtract $\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) = 0$ and also $\mathbf{E}[\boldsymbol{\nabla}\cdot(\epsilon\mathbf{E}) - \rho = 0]$ then we have

$$\frac{\partial}{\partial t}(\epsilon\mathbf{E}\times\mathbf{B}) + (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}) + \left\{\frac{1}{\mu}[\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B})] + \epsilon[\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})]\right\} = 0. \tag{3.32}$$

We recognize $\mathbf{f}_{\text{mech}} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$ as a force density from the Lorentz force. Moreover, if we define

$$\mathbf{g} \equiv \mathbf{D}\times\mathbf{B} = \epsilon\mathbf{E}\times\mathbf{B} = \epsilon\mu\mathbf{E}\times\mathbf{H}, \tag{3.33}$$

then some of these terms will clean up. By some vector identities, we may find that

$$(\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}[\boldsymbol{\nabla}\cdot\mathbf{B}])_i = -\partial_j\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right), \tag{3.34}$$

so then we have the appropriate conservation law,

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{em}} + \mathbf{p}_{\text{mech}})_i + \partial_j T_{ij} = 0, \tag{3.35}$$

where we've defined the Maxwell stress tensor

$$T_{ij} \equiv +\epsilon E_iE_j + \frac{1}{\mu}B_iB_j - u_{\text{em}}\delta_{ij}. \tag{3.36}$$

Finally, from conservation of momentum we can find a corresponding angular momentum. Since

$$\frac{\partial\mathbf{g}}{\partial t} - \boldsymbol{\nabla}\cdot\mathbf{T} = -\mathbf{f}_{\text{mech}}, \tag{3.37}$$

by taking the cross-product with $\mathbf{r}$ we have

$$\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{g}) - \mathbf{r}\times\boldsymbol{\nabla}\cdot\mathbf{T}) = -\mathbf{r}\times\mathbf{f}_{\text{mech}} \tag{3.38}$$

or equivalently

$$\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{g}) + \boldsymbol{\nabla}\cdot(\mathbf{T}\times\mathbf{r}) = -\mathbf{r}\times\mathbf{f}_{\text{mech}}, \tag{3.39}$$

which again looks like a conservation law with

$$\mathbf{M} \equiv \mathbf{T}\times\mathbf{r} \tag{3.40}$$

the angular momentum current density. Like the Maxwell stress tensor, it is also a rank two tensor. We therefore define a torque

$$\mathbf{N}_{\text{mech}} = \frac{d\mathbf{L}_{\text{mech}}}{dt} = -\int_V\left\{\frac{\partial}{\partial t}(\mathbf{r}\times\mathbf{g}) + \boldsymbol{\nabla}\cdot\mathbf{M}\right\}dv, \tag{3.41}$$

where

$$\mathbf{L}_{\text{em}} \equiv \mathbf{r} \times \mathbf{g} \tag{3.42}$$

defines an angular momentum carried by the fields.

Lecture 4.

Monday, April 6, 2020

Today we'll cover sections 16.1-16.4, on the wave equation. Zangwill's discussion is of waves in vacuum, but in general we can do this with non-conducting, linear, homogeneous, isotropic media. We can break this down:

- non-conducting: ϵ and μ are real (for conducting media, ϵ and μ have real and imaginary parts)
- linear: $\mathbf{D} = \epsilon \mathbf{E}, \mathbf{B} = \mu \mathbf{H}$ (there is a whole field of non-linear optics)
- homogenous: no spatial dependence (graded optics have many applications)
- isotropic: no preferred direction (anisotropic media also have interesting applications)

The fact that conducting media respond to cancel applied fields means that there is not just transmission but absorption; waves passing through are damped as the charge carriers in the medium reshuffle to counter the applied field.

Ferromagnets are an example of nonlinear media, as is the Kerr effect which changes the polarization of a wave. Nonlinearity usually kicks in when we hit physical limits on our ability to magnetize or polarize a substance, e.g. when the dipole moments in a ferromagnet are maximally aligned.

We sum this up as simple media, which vacuum certainly is, but in general there are many others that will satisfy these conditions. Consider a simple medium in the absence of sources, $\rho = 0, \mathbf{J} = 0$. The source-free Maxwell equations have a nice symmetry:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{4.1a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{4.1b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1c}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}. \tag{4.1d}$$

If we take the curl of Faraday's law, then

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\boldsymbol{\nabla} \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}). \tag{4.2}$$

Now we use the double curl identity to rewrite the LHS as

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \tag{4.3}$$

by Gauss's law, and the RHS by Ampère's law becomes

$$-\frac{\partial}{\partial t}\left(\mu\epsilon \frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{4.4}$$

which is a wave equation for each of the components of $\mathbf{E}$,

$$\nabla^2 \mathbf{E} + \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{4.5}$$

Jackson takes a different approach; he writes the Maxwell equations and goes to the potential right away, with

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}, \mathbf{B} = \boldsymbol{\nabla} \times vecA. \tag{4.6}$$

Inserting the potentials into Ampère's law, we have

$$\frac{1}{\mu}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \epsilon \frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\Phi - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right). \tag{4.7}$$

Expanding out as before gives us

$$\nabla^2 - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \mu\epsilon \frac{\partial \Phi}{\partial t}\right) = 0. \tag{4.8}$$

We can choose Coulomb gauge to write $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ and choose $\Phi = 0$. we get Poisson's equation for Φ and the wave equation for $\mathbf{A}$, i.e.

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{4.9}$$

and

$$\epsilon \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}) = 0, \tag{4.10}$$

so

$$\nabla^2 \Phi = 0. \tag{4.11}$$

The wave equation was well-known by the time it was discovered Maxwell's equations had wave solutions. What was different was that EM waves could propagate through vacuum, without any medium necessary.

Solutions t the wave equations can be written in terms of null coordinates. If we define

$$\xi = vt - \hat{\mathbf{n}} \cdot \mathbf{r} = \hat{n}^\mu s_\mu, \tag{4.12}$$

where $\hat{\mathbf{n}}$ indicates the direction of propagation, then we can consider harmonic solutions:

$$\mathbf{A}(\xi) = \mathrm{Re}\Big(\mathbf{A}_0 e^{-ik(vt - \hat{\mathbf{n}}\cdot\mathbf{r})}\Big), \tag{4.13}$$

such that

$$kv = \omega, \mathbf{k} = \sqrt{\mu\epsilon\omega^2}\hat{\mathbf{n}}, \tag{4.14}$$

or equivalently

$$\mathbf{A}(\mathbf{r}, t) = \mathrm{Re}\Big(\mathbf{A}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\Big) \tag{4.15}$$

The electric field is

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = i\omega \mathbf{A} = i\omega \mathbf{A}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}. \tag{4.16}$$

The magnetic field is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = i\mathbf{k} \times \mathbf{A} = i\mathbf{k} \times \mathbf{A}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}. \tag{4.17}$$

Note we could also pick the complementary null coordinate $\eta = vt + \hat{\mathbf{n}} \cdot \mathbf{r}$, which simply propagates in the $-\hat{\mathbf{n}}$ direction.

We could equivalently write the waves in terms of $e^{i\phi}$, where $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$, and this defines a phase velocity $v = \frac{d\mathbf{r}}{dt} = \frac{\omega}{k}$. The general solution need not be harmonic or periodic, but by a Fourier transform we can treat wave packets as a sum of Fourier modes.

From the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, we can see that[5]

$$\hat{\mathbf{n}} \cdot \frac{d\mathbf{A}}{d\xi} = 0, \tag{4.18}$$

so $\mathbf{A} \perp \hat{\mathbf{n}}$ (up to a constant). It follows that if we choose $\Phi = 0$, then

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -v \frac{\partial vecA}{\partial \xi}, \tag{4.19}$$

while

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = -\hat{\mathbf{n}} \times \frac{\partial \mathbf{A}}{\partial \xi} = \frac{1}{v}\hat{\mathbf{n}} \times \mathbf{E} \implies |\mathbf{B}| = \frac{1}{v}|\mathbf{E}|. \tag{4.20}$$

We see that $\hat{\mathbf{n}}, \mathbf{E}$, and $\mathbf{B}$ form a right-handed orthogonal set, whre $\mathbf{E}$ and $\mathbf{B}$ are perpendicular and in phase.

The energy density is

$$u = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{B} \cdot \mathbf{H} = \frac{\epsilon}{2}E^2 + \frac{1}{2\mu}B^2. \tag{4.21}$$

One can similarly analyze the Poynting vector and the angular momentum in the fields.

[5] I think one can do this from a chain rule manipulation.

A final approach to this is to simply assume the form of a harmonic solution to the Maxwell equations (in particular, a plane wave) and write down a wave equation.

If we study plane waves, we have solutions with $1/v = \sqrt{\mu\epsilon}$ and a wavevector $k = \sqrt{\mu\epsilon}\omega$, with a phase velocity $\omega/k = 1/\sqrt{\mu\epsilon} = c/n$, where n is the index of refraction. The index of refraction may be complex, which indicates absorption, and moreover it may be frequency-dependent.

We can superpose waves (e.g. plane waves) and in geneal construct

$$\mathbf{E} = (\tilde{E}_1\hat{\mathbf{e}}_1 + \tilde{E}_2\hat{\mathbf{e}}_2)e^{i(\omega t + \mathbf{k}\cdot\mathbf{r})}, \tag{4.22}$$

where $\hat{\mathbf{n}}, \hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ form a right-handed set and $\tilde{E}_1, \tilde{E}_2$ are potnetially complex. That is, $\tilde{E}_{1/2} = E_{1/2}e^{i\phi_{1/2}}$.

If E_1 and E_2 have the same phase, we get linearly polarized light which oscillates in the plane. If $\phi_1 \neq \phi_2$, we get elliptic polarized light, where the E-field rotates around the axis of propagation. In particular when $|E_1| = |E_2|$ and $\phi_1 = \phi_2 + \pi/2$, we get *circularly polarized light*. We can consider special basis vectors

$$\hat{\mathbf{e}}_\pm = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2) \tag{4.23}$$

where the signs indicate the helicity (whether the field spirals in a right-handed or left-handed way. $+$ is counterclockwise, $-$ is clockwise.

Lecture 5.

Wednesday, April 8, 2020

Today we'll cover more realistic wave setups. We can write wave packets as integrals over Fourier modes,

$$\mathbf{E} = \mathrm{Re}\frac{1}{(2\pi)^3}\int d^3k\, \boldsymbol{\mathcal{E}}_\perp(\mathbf{k})e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} \tag{5.1}$$

and then the corresponding B-field is

$$c\mathbf{B} = \mathrm{Re}\frac{1}{(2\pi)^3}\int d^3k\, (\hat{\mathbf{k}} \times \boldsymbol{\mathcal{E}}_\perp(\mathbf{k}))e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \tag{5.2}$$

where $\mathbf{k}$ defines a wave vector (a direction of propagation).

Superposition and destructive interference allows us to constrain the transverse and longitudinal extent of the wavepacket. The $1/(2\pi)^3$ factor comes from taking the Fourier transform, working in radians; it is conventional. We're basically integrating over frequency components.

We can also write scalar wave packets in terms of the potentials rather than the fields. That is, we can write

$$U(\mathbf{r}, t) = \frac{1}{(2\pi)^3}\int d^3k\, \hat{U}(\mathbf{k})e^{-i(\omega(\mathbf{k})t - \mathbf{k}\cdot\mathbf{r})} \tag{5.3}$$

in terms of an amplitude u.

Suppose we take one dimension at a time,

$$u(x, t = 0; y, z) = \int_{-\infty}^{\infty} dk_x \hat{\mathbf{u}}(k_x)e^{ik_x x}. \tag{5.4}$$

We have to integrate over "negative frequencies" in order to treat waves as propagating forwards or backwards. We need to pick a distribution for $\hat{\mathbf{u}}(k_x)$. Let's posit a Gaussian in x:

$$\hat{\mathbf{u}}(k_x) = \frac{1}{\sqrt{\pi}\Delta k_x}\exp\left[-(k_x - k_{0x})^2/\Delta k_x^2\right]. \tag{5.5}$$

The characteristic spread in x is $\sigma = \Delta k_x$, and the beam is centered on k_{0x}. Why do we choose a Gaussian? Well, they are ubiquitous in nature and we have the mathematical apparatus to deal with them. If we insert this into our integral equation, we find that

$$u(x, t = 0) = \exp(ik_{0x}x)\exp(-x^2/(\Delta x)^2), \tag{5.6}$$

so we get a beam which travels with wave number k_{0x} and spatial width Δx such that[6]

$$\Delta x \Delta k_x \geq 1/2. \tag{5.7}$$

[6] I think this is saturated for the Gaussian beam.

The time evolution describes how the packet moves in space. We can talk of a "grup velocity," how fast the packet moves.

$$v_g = \nabla_k \omega(\mathbf{k})|_{\mathbf{k}=\mathbf{k}_0} = \frac{d\omega(\mathbf{k})}{d\mathbf{k})}|_{\mathbf{k}=\mathbf{k}_0}. \tag{5.8}$$

The frequency dependence defines a dispersion relations, so that e.g.

$$\omega = vk = \frac{c}{n}k \tag{5.9}$$

gives the group velocity as c/n, which depends on the index of refraction. Then

$$u(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3k \hat{\mathbf{u}}(\mathbf{k}) e^{i(\mathbf{k}-\mathbf{k}_0)(\mathbf{r}-v_g t)} e^{-i(\omega_0 t - \mathbf{k}_0\cdot\mathbf{r})}. \tag{5.10}$$

Fourier transforms allow us to go between time and frequency domains (the amplitude $u(\mathbf{r},t)$ and its Fourier transform $\hat{u}(\mathbf{r},\omega)$).

If we substitute

$$\hat{u}(\mathbf{r},\omega)e^{-i\omega t} \tag{5.11}$$

into the wave equation, we find that the wave equation becomes a condition on the Fourier transform,

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \implies \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\hat{u}(\mathbf{r},\omega) = 0. \tag{5.12}$$

We have transformed the wave equation into the *Helmholtz equation*, which is also well-studied.

Now let us further make the paraxial approximation, i.e. we suppose the rays are close enough to the axis that $\sin\theta \approx \tan\theta \approx \theta$. We start with the (source-free) Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0, \tag{5.13}$$

where ψ can be any component of E or B, where $k = 2\pi/\lambda$. We take

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r},t)e^{i(kz-\omega t)}, \tag{5.14}$$

where ψ_0 is the beam profile. Taking the Laplacian gives us

$$\nabla^2\psi = \frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2} + \left[ik\frac{\partial\psi_0}{\partial z} - k^2\psi_0 + \frac{\partial^2\psi_0}{\partial z^2} + ik\frac{\partial\psi_0}{\partial z}\right]. \tag{5.15}$$

Plugging back into the Helmholtz equation, the k^2 terms cancel. OUr paraxial apprixmation also says that the second derivative in z is small, so we have

$$\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2} + 2ik\frac{\partial\psi_0}{\partial z} = 0, \tag{5.16}$$

which we call the *paraxial wave equation*. In cylindrical coordinates we would instead have

$$\frac{\partial^2\psi_0}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi_)}{\partial\rho} + 2ik\frac{\partial\psi_0}{\partial z} = 0, \tag{5.17}$$

where we can assume azimuthal symmetry.

The solution to this equation has the form

$$\psi_0(\rho,z) = \frac{w_0}{w}\exp\left[-\frac{\rho^2}{w^2} - \frac{i\pi\rho^2}{\lambda R} + i\rho_0\right] \tag{5.18}$$

where this beam has a gaussian width profle in ρ and also oscillates in the radial direction, with

$$w(z) = w_0\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]^{1/2} \tag{5.19}$$

and

$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right], \tag{5.20}$$

which defines a radius of curvature of the wavefront as a function of z. That is, at $z = 0$ we have an "infinite" radius of curvature of the wavefront, and $w(z)$ defines a characteristic radius of the beam in the radial directions.

There's also an extent in z where the beam remains reasonably confined:

$$z_R = \frac{\pi \omega_0^2}{\lambda}, \tag{5.21}$$

which is the value of z for which the beam area doubles ($w(z_R) = \sqrt{2} w_0$).

This is actually just the lowest order mode solution. For other symmetries we can expand in Hermite or Laguerre polynomials, as needed.[7]

In rectangular symmetry we get Hermite polynomials

$$\psi_{0,mn}(z) = \frac{1}{w(z)} H_m \left[\frac{\sqrt{2}x}{w(z)} \right] H_n \left[\frac{\sqrt{2}y}{w(z)} \right] \dots \tag{5.22}$$

(I didn't finish writing it down) and for circular symmetry we would get Laguerre polynomials.

We can also discuss Fabry-Perot resonators, which are not covered in Zangwill. These are cavities which have two spherical mirrors set some distance from a center point. If we send light into the cavity (supposing the mirrors have some transmission), we can consider a symmetric resonator with equal focal lengths and find a nice expression for the waist of the beam. Moreover, we can work out what frequencies will constructively interfere in the cavity and find a characteristic frequency of the resonator.

There are a whole range of cavity setups we can construct based on the values of the focal lengths or equivalently the radii of curvature of the mirrors.

Lecture 6.

Friday, April 10, 2020

Today we'll discuss reflection and transmission of waves in simple matter (Zangwill 17.1-17.3).

Introduction As we've said, simple matter is matter that is linear, homogeneous, and isotropic. The polarization or magnetization responds linearly to applied fields, and the object is the same everywhere and in every direction. We also require that our media is nonconductive, so there is no attenuation (compare a Faraday cage). In conductive media, the E and B fields can get out of phase. We should also restrict to nondispersive media, which responds to all frequencies in the same way (as opposed to e.g. a prism of acrylic, where the index of refraction depends on the frequency of light passing through).

Reflection and transmission Strictly, we should work with wave packets and Gaussian optics. However, for our discussion we'll make the simplifying assumption of dealing with plane waves. Suppose we have two media separated at $z = 0$, defined by indices of reaction n and n'. We'll send in a wave with unit wavevector $\hat{\mathbf{k}}$ and two perpendicular directions $\hat{\mathbf{p}}, \hat{\mathbf{s}}$. Let's treat reflection in the xz-plane. We'll have $\hat{\mathbf{p}}$ parallel to the plane of incidence, and $\hat{\mathbf{s}}$ perpendicular.[8]

For $z < 0, n = \sqrt{\mu\epsilon}$, while $z > 0$ has an index of refraciton $n' = \sqrt{\mu'\epsilon'}$.

The incident electric field is

$$\mathbf{E}_{\text{inc}} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tag{6.1}$$

while the reflected field is

$$\mathbf{E}_{\text{ref}} = \mathbf{E}_0'' e^{i(\mathbf{k}''\cdot\mathbf{r} - \omega'' t)}, \tag{6.2}$$

and

$$\mathbf{E}_{\text{trans}} = \mathbf{E}_0' e^{i(\mathbf{k}'\cdot\mathbf{r} - \omega' t)}. \tag{6.3}$$

Note that $z > 0$ has just $\mathbf{E}_{\text{trans}}$, while $z < 0$ has the sum $\mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{ref}}$. At the interface $z = 0$, we apply matching conditions and find that the fields must have the same space and time dependence,

$$(\mathbf{k}\cdot\mathbf{r} - \omega t) = (\mathbf{k}'\cdot\mathbf{r} - \omega' t) = (\mathbf{k}''\cdot\mathbf{r} - \omega'' t). \tag{6.4}$$

[7] We shouldn't be surprised at finding the Hermite or Laguerre polynomials, since the Helmholtz equation is closely related to the Schrödinger equation.

[8] From *senkrect*, perepndicular in German

This gives us some important constraints.

- $\omega = \omega' = \omega''$ because the only time dependence is in ωt
- $k_x = k'_x = k''_x$ because the variables in $\mathbf{r}$ are independent
- $k_y = k'_y = k''_y = 0$
- $|k_z| = |k''_z|$ because $k = \omega/v = n\omega/c$

These give us the basic laws of optics:

- Plane of incidence (if the wave starts in the xz plane it stays in the plane)
- $\theta_{\text{inc}} = \theta_{\text{ref}}$
- Snell's law, $n \sin\theta_1 = n_2 \sin\theta_2$.

To get the transmission and reflection coefficients, we apply the boundary conditions.

- $\Delta E_{\tan} = 0 \implies (E_0 + E''_0)_{x,y} = (E'_0)_{x,y}$
- $\Delta D_{\text{norm}} = \sigma_f = 0 \implies \epsilon(E_0 + E''_0)_z = \epsilon'(E'_0)_z$
- $\Delta B_{\text{norm}} = 0 \implies (B_0 + B''_0)_z = (B'_0)_z$
- $\Delta H_{\tan} = \mathbf{K}_f = 0 \implies \frac{1}{\mu}(B_0 + B''_0)_{x,y} = \frac{1}{\mu'}(B'_0)_{x.y}$,

where tan is the tangential component and norm is the normal one.

Now we may choose a polarization. Let us define $\hat{\mathbf{s}} = \hat{\mathbf{y}}$ and $\hat{\mathbf{p}} \equiv \cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}}$. For the p-polarization, we choose $\mathbf{E}$ to lie along $\hat{\mathbf{p}}$ and $\mathbf{B}$ to lie along $\hat{\mathbf{s}}$. We could have also chosen the E-field to lie along $\hat{\mathbf{s}}$ instead, which is (quite reasonably) the s-polarization. We get four equations, though one is trivial. From the boundary conditions,

- $(E_0 + E''_0)\cos\theta = E'_0 \cos\theta'$ ($\hat{\mathbf{x}}$ component, the $\hat{\mathbf{y}}$ equation is trivial)
- $\epsilon(E_0 + E''_0)\sin\theta = \epsilon' E'_0 \sin\theta'$ (from the $\hat{\mathbf{z}}$ matching condition on D)
- $0 = 0$ (from the B_{norm} condition in this polarization)
- $\frac{1}{\mu}(\sqrt{\mu\epsilon}E_0 + \sqrt{\mu\epsilon}E''_0) = \frac{1}{\mu'}\sqrt{\mu'\epsilon'}E'_0$ (from the $\hat{\mathbf{y}}$ component of the $H_{\tan}$ equation).

We're left with three independent equations and three unknowns: θ', E'_0, E''_0. Snell's law relates θ' to θ in terms of the indices of refraction, so we can rewrite the second equation above in terms of $\sin\theta$. Solving the system of equations gives us an expression for the transmitted wave,

$$\frac{E'_0}{E_0} = \frac{2}{\frac{\mu n'}{\mu' n} + \frac{\cos\theta'}{\cos\theta}} \equiv t_{12,p} \tag{6.5}$$

when the wave has the p-polarization. One can similarly compute the transmission and reflection coefficients for the other polarization. These are called the Fresnel coefficients, after Augustine-Jean Fresnel, who originally worked out these coefficients. Fresnel is also known for the design known as the Fresnel lens, which allows focusing of light without having a (spherical) lens that gets very thick in the middle.

Note that we can also define

$$R = r_{12}^2, \quad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} t_{12}^2 \frac{\cos\theta_2}{\cos\theta_1}. \tag{6.6}$$

There are critical angles we might consider, where a ray going from a slow to fast medium can experience total internal reflection.

Lecture 7.

Monday, April 13, 2020

Today we'll move into sections 17.4-17.7, which is various topics in reflection and refraction for layered and conductive materials. Layered media are interesting because multiple reflections allow us to have interference effects. Suppose we have a single layer of material (say, glass) with index of refraction n and thickness d, surrounded by air (index of refracition $n_0 \approx 1$).

As we send in a beam at an angle θ_0, it is refracted to an angle θ in the interior and back to θ_0 as it leaves the second interface. We can calculate Fresnel coefficients for the air-glass interface and the glass-air interface, i.e. there is some r, t corresponding to entering the medium, and r', t' for leaving it.

In fact, it's not just that the ray is completely transmitted when it hits the glass-air side of the layer—part of it will be reflected back into the medium with an amplitude r'. These patterns of reflection give us interference patterns, an infinite series of reflections and transmissions:

$$E_T = E_I\left[tt' + tr'r't'e^{i\Delta\varphi} + t(r'r')^2t'e^{i2\Delta\varphi} + \dots\right] = \frac{tt'}{1 - r'r'e^{i\Delta\varphi}}E_I, \tag{7.1}$$

since this is a geometric series with ratio $r'^2e^{i\Delta\varphi}$. The phase difference $\Delta\varphi$ comes from the path length difference in the medium from multiple reflections. We might have seen this in thin lenses/optics, or in the interference patterns of soap bubbles and oil slicks.

Now, by analyzing a single interaction at the glass-air interface, we can relate some of the Fresnel coefficients and find the *Stokes relations*.

If we time-reverse the process, we find that

$$r^2 + tt' = 1 \text{ and } tr' + rt = 0, \tag{7.2}$$

so we learn that

$$r = r', \quad tt' = 1 - r^2 = 1 - R. \tag{7.3}$$

This allows us to write *Airy's formula* for the fraction of transmitted energy,

$$\left|\frac{E_T}{E_I}\right|^2 = \left[1 + \frac{4R}{(1-R)^2}\sin^2(\Delta\varphi/2)\right]^{-1} \tag{7.4}$$

where the phase shift is given explicitly by

$$\frac{\Delta\varphi}{2} = n\frac{\omega}{c}d\cos\theta \tag{7.5}$$

using Snell's law and a bit of geometry ($\Delta\varphi = 2kl - k_0a$ with $k = n\omega/c, k_0 = n_0\omega/c$). We see that when $\Delta\varphi/2$ is an integer multiple of π, we have maximum transmission ($|\mathcal{T}|^2 = 1$) and moreover we can make the transmission sharply peaked around these values by making R very close to 1.

Simple conducting media We previously required our media to be non-conducting. If we relax that constraint, we can still understand some aspects of the problem. We can take our media to still be linear and Ohmic:

$$\mathbf{D} = \epsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J}_f = \sigma\mathbf{E}. \tag{7.6}$$

Let's also assume that $\rho_f = 0$, there is no free charge density. Maxwell's equations take the form

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{H} = 0, \tag{7.7}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t}, \quad \boldsymbol{\nabla}\times\mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial\mathbf{E}}{\partial t}, \tag{7.8}$$

where we've chosen to write everything in terms of just the E and H fields. This allows us to set up a wave equation:

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0. \tag{7.9}$$

The first derivative term is new; it introduces dissipation into our wave equation. THe conducting medium drains energy as Joule heating:

$$\frac{dW}{dt} = \int_V d^3r\,\mathbf{J}_f\cdot\mathbf{E} = \sigma\int_V d^3r\,|\mathbf{E}|^2. \tag{7.10}$$

If we consider a plane wave of the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{7.11}$$

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)], \tag{7.12}$$

then the divergence equations (the Gauss laws/monopole law) tell us that

$$\mathbf{k}\cdot\mathbf{E} = 0, \quad \mathbf{k}\cdot\mathbf{B} = 0, \tag{7.13}$$

so the fields are still transverse. However, the curl equations give

$$\boldsymbol{\nabla}\times\mathbf{E} = \omega\mu\mathbf{H}, \quad \mathbf{k}\times\mathbf{H} = -\omega(\epsilon + i\frac{\sigma}{\omega})\mathbf{E}. \tag{7.14}$$

That is, we can treat our permittivity as complex, $\tilde{\epsilon}(\omega) = \epsilon + i\frac{\sigma}{\omega} = \epsilon' + i\epsilon''$.[9] Since we have a complex dielectric constant, we will have a complex and frequency-dependent index of refraction. That is, we expect absorption (dissipation) and also dispersion. We have

$$\mathbf{k} \cdot \mathbf{k} = \tilde{k}^2 = \tilde{\epsilon}(\omega)\mu\omega^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega = \tilde{n}^2(\omega)\frac{\omega^2}{c^2}, \tag{7.15}$$

where the index of refraction is complex as promised. Then

$$\mathbf{k} = \tilde{k}\hat{\mathbf{k}} = \tilde{n}\frac{\omega}{c}\hat{\mathbf{k}} = (n' + in'')\frac{\omega}{c}\hat{\mathbf{k}} \tag{7.16}$$

in Zangwill's notation, splitting the index of refraction into real and imaginary parts.

Note that we still have transverse plane waves,

$$\mathbf{k} = \tilde{n}\frac{\omega}{c}\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \cdot \mathbf{E} = 0, \tag{7.17}$$

and moreover we arrive at a complex impedance $\tilde{Z}(\omega)$ such that

$$\tilde{Z}(\omega)\mathbf{H} = \mathbf{k} \times \mathbf{E} \tag{7.18}$$

with $\tilde{Z}(\omega) = \sqrt{\mu/\tilde{\epsilon}(\omega)}$. The complex impedance means that waves in conductors (the **E** and **H** fields) are no longer in phase. The induced Faraday currents from the changing electric fields causes the fields to go out of phase.

Our wave takes the form

$$\mathbf{E} = \mathbf{E}_0 e^{-\frac{\omega}{c}n''\hat{\mathbf{k}}\cdot\mathbf{r}} \exp[i(\mathbf{k} \cdot \mathbf{r} - (?)] \tag{7.19}$$

which has manifestly an oscillating and decaying part. There's a skin depth to a material, $\delta(\omega) = \sqrt{2/\mu\sigma\omega}$, and If we consider the wave impedance, we can write it as

$$\tilde{Z} = \sqrt{\mu}\epsilon + i\sigma\omega \approx \sqrt{\frac{\mu\omega}{i\sigma}} = \sqrt{\frac{\mu\omega}{\sigma}}e^{i\pi/4} = \frac{1-i}{\sigma\delta} \tag{7.20}$$

under the assumption for a good conductor that $\sigma\omega/\mu \gg 1$. The physical electric field in a good conductor looks like a damped oscillator with an envelope $\exp(-z/\delta)$, where the skin depth δ defines a characteristic penetration length.

This leads us also to complex Fresnel coefficients in terms of the complex impedances. If we then compute the reflectivity, we find that

$$R(\omega) = |\tilde{r}(\omega)|^2 = \left|\frac{\tilde{n}_2 - n_1}{\tilde{n}_2 + n_1}\right|^2 \approx \frac{1 - n_1/n_2'}{1 + n_1/n_2'} = (?) \tag{7.21}$$

in terms of the conductivity.

Anisotropic media is also interesting and useful. The permittivity ϵ becomes a permittivity tensor ϵ_{ij} such that

$$\tilde{P}_i = \epsilon_0\chi_{ij}\tilde{E}_j, \tag{7.22}$$

since a crystal might have a different response to fields in different directions. A cubic crystal would be isotropic, the same in all directions, while crystals with preferred axes might produce more complicated polarizations and D-fields. We get two different phase speeds, leading to the phenomenon of birefringence.

Lecture 8.

Wednesday, April 15, 2020

Today we'll get started with Chapter 18, *Waves in Dispersive Media*. We'll spend most of our time on 18.4, transverse and longitudinal waves, studying the Lorentz-Drude model.

[9] It was kind of arbitrary to make the permittivity rather than the permeability complex. However, most of our materials in the lab are non-magnetic, so $\mu = \mu_0$ and we need not worry about magnetization effects.

Non-dispersive media obeys the Maxwell equations, as usual, and simple constitutive relations. For dispersive media, the medium response is not instantaneous, and the response does not vanish when the field is removed. The polarization and magnetization depend on the history of the fields:

$$\mathbf{P}(t) = \epsilon_0 \int_{-\infty}^{\infty} \mathbf{E}(t')G_e(t-t')dt' \tag{8.1}$$

$$\mathbf{M}(t) = \int_{-\infty}^{\infty} \mathbf{H}(t')G_M(t-t')dt' \tag{8.2}$$

Polarization and magnetization arise from microscopic phenomena– in the case of polarization, outer-shell electrons shift their positions in response to an applied field, while in the case of magnetization, microscopic magnetic dipole moments (due to spin or orbital effects) reorient to align with the field.

Here, the response functions G_e and G_M are Green's functions, specifically advanced or retarded propagators based on physical concerns about causality.

Let us now take Fourier transforms and go from the time domain tothe frequency domain:

$$\mathbf{P}(\omega) = \epsilon_0 \chi_e(\omega)\mathbf{E}(\omega), \tag{8.3}$$

$$\mathbf{M}(\omega) = \chi_M(\omega)\mathbf{H}(\omega). \tag{8.4}$$

We write the susceptibilities as Fourier transforms:

$$\chi(\omega) = \int_{-\infty}^{\infty} G(\tau)e^{i\omega\tau}d\tau \tag{8.5}$$

and

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega)e^{-i\omega\tau}d\omega. \tag{8.6}$$

For now, we can just say that our $\epsilon, \chi_e, \mu, \chi_m$ all become complex and frequency-dependent. The real part gives refraction, while the imaginary part is responsible for absorption.

For plane waves in dispersive media, we have

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{8.7}$$

$$\boldsymbol{\nabla} \times \mathbf{E} - i\omega\mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{H} + i\omega\mathbf{D} = \mathbf{J}_f. \tag{8.8}$$

As before we can write everything in terms of just two fields using the constitutive relations:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho_f/\tilde{\epsilon}, \quad \boldsymbol{\nabla} \cdot \mathbf{H} = 0, \tag{8.9}$$

$$\boldsymbol{\nabla} \times \mathbf{E} - i\omega\tilde{\mu}\mathbf{H} = 0, \quad \boldsymbol{\nabla} \times \mathbf{H} + i\omega\tilde{\epsilon}\mathbf{E} = \mathbf{J}. \tag{8.10}$$

Here, $\tilde{\epsilon}(\omega) = \epsilon_0[1+\chi_e(\omega)]$.

Taking curls of the curl equation and applying the double-curl identity gives us Helmholtz equations:

$$\nabla^2\mathbf{H} + \tilde{n}^2\frac{\omega^2}{c^2}\mathbf{H} = -\boldsymbol{\nabla} \times \mathbf{J}, \tag{8.11}$$

where the complex index of refraction is given in the most natural way as $\tilde{n} = \tilde{\mu}\tilde{\epsilon}/\mu_0\epsilon_0$.

We saw that metals have a complex index of refraction. Metals are dispersive, but not all dispersive materials are metals.[10] Given an index of refraction, we can determine a complex wave vector $\tilde{\mathbf{k}}$ where $\tilde{\mathbf{k}} \cdot \tilde{\mathbf{k}} = \tilde{n}^2\omega^2/c^2$. We can split up

$$\tilde{\mathbf{k}} = (q + i\kappa)\hat{\mathbf{k}}. \tag{8.12}$$

We can still define a unit vector $\hat{\mathbf{k}}$ which points in the direction of propagation. It's just that the overall wave vector is now complex.

In the absence of sources, the Helmholtz equation admits plane wave solutions,

$$\mathbf{H}(\mathbf{r}, \omega) = \mathbf{H}_0(\omega)e^{i\mathbf{k}\cdot\mathbf{r}} = \mathbf{H}_0(\omega)e^{(ik-\kappa)\hat{\mathbf{k}}\cdot\mathbf{r}}. \tag{8.13}$$

Now computing the intensity from the Poynting vector, $\mathbf{E} \times \mathbf{H}^*$, we get

$$S \propto e^{-2\kappa\hat{\mathbf{k}}\cdot\mathbf{r}}, \tag{8.14}$$

[10] For instance, acrylic certainly isn't a metal, but most cheap prisms are made of it.

so we see that

$$2\kappa = 2\frac{\mathrm{Im}(\tilde{n})}{c}\omega. \tag{8.15}$$

This tells us what it means for a material to be transparent– absorption is small, i.e. for a given frequency, $\kappa/q = \mathrm{Im}(\tilde{n})/\mathrm{Re}(\tilde{n}) \ll 1$.

Lorentz-Drude model In this model, the electrons in the medium are harmonically bound to the ions, which are forced to occupy lattice sites. An electric field displaces these electrons. They have equations of motion

$$\frac{d^2\mathbf{r}}{dt^2} + \gamma\frac{d\mathbf{r}}{dt} + \omega_r^2\mathbf{r} = \frac{q_e}{m}\mathbf{E}. \tag{8.16}$$

That is, the acceleration is equal to an electric force per mass minus a linear restoring force proportional to $\mathbf{r}$ and a damping (frictional) force proportional to $\frac{d\mathbf{r}}{dt}$. Here, ω_r is a resonant frequency, q_e is the electron charge, m is the electron mass, and γ is a damping coefficient.

The polarization is the dipole moment $q_e\mathbf{r}$ times the electron density n_e:

$$\mathbf{P} = n_e q_e \mathbf{r}. \tag{8.17}$$

Let us define a plasma frequency

$$\omega_p \equiv \sqrt{\frac{n_e q_e^2}{\epsilon_0 m}}. \tag{8.18}$$

The plasma frequency is the highest frequency which the electrons can respond to. Then

$$\frac{d^2\mathbf{r}}{dt^2} + \gamma\frac{d\mathbf{r}}{dt} + \omega_r^2\mathbf{r} = \epsilon_0\omega_p^2\mathbf{E}. \tag{8.19}$$

If we take the Fourier transform we get

$$(\omega_r^2 - \omega^2 - i\gamma\omega)\mathbf{P} = \epsilon\omega_p^2\mathbf{E}, \tag{8.20}$$

so we have a susceptibility

$$\chi_e(\omega) = \frac{\omega_p^2}{\omega_r^2 - \omega^2 - i\gamma\omega} \tag{8.21}$$

with poles at

$$\omega = -i\gamma/2 \pm (?) \tag{8.22}$$

The imaginary part of the susceptibility comes from damping, while the real part is related to the index of refraction and reflects electron oscillations.

Let us ignore magnetic effects and set $\mu(\omega) = \mu_0$. If we consider underdamped oscillators $\gamma/\omega_r \ll 1$, then

$$\mathrm{Re}(\mu\epsilon) = \frac{1}{\omega^2}(q^2 - \kappa^2) \tag{8.23}$$

and

$$\mathrm{Im}(\mu\epsilon) = 2\frac{q\kappa}{\omega^2}. \tag{8.24}$$

Lecture 9.

Friday, April 17, 2020

Today we'll finish Chapter 18, sections 18.6-18.8. We considered the Lorentz-Drude model to understand how polarization electrons are bound to nuclei and how they respond to polarizing electric fields.

Last time we had a susceptibility

$$\chi_e(\omega) = \frac{\omega_p^2}{\omega_r^2 - \omega^2 - i\gamma\omega}, \tag{9.1}$$

and when the resonant frequency goes to zero, $\omega_r \to 0$, we have unbound electrons (basically conductors):

$$\chi_e \to \frac{-\omega_p^2}{\omega(\omega + i\gamma)}. \tag{9.2}$$

Now since the bound current is $\mathbf{J}_b = \frac{\partial\mathbf{P}}{\partial t}$.

For non-magnetic material we can take the Fourier transform, using the fact that the polarization is proportional to the E-field:

$$\tilde{\mathbf{J}}_b = -i\omega\tilde{\mathbf{P}} = -i\omega\epsilon_0\chi_e\tilde{\mathbf{E}}. \tag{9.3}$$

We can also consider electrons to be free (Ohm's law), so

$$\tilde{\mathbf{J}} = \sigma(\omega)\tilde{\mathbf{E}} \tag{9.4}$$

where we have a complex frequency-dependent conductivity

$$\sigma(\omega) = -i\omega\epsilon_0\chi_e(\omega) = \frac{i\epsilon_0\omega_p^2}{\omega + i\gamma}. \tag{9.5}$$

In the low-frequency limit this reduces to

$$\sigma \to \frac{\epsilon_0\omega_p^2}{\gamma} = \sigma_0. \tag{9.6}$$

In the free electron theory of metals,

$$\sigma_0 = \frac{n_e q_e^2 \tau}{m_e} \tag{9.7}$$

where $\tau \simeq 1/\tau$. Now we get a complex index of refraction

$$n = \left[1 - \frac{\omega_p^2(\omega^2 - i\omega\gamma)}{\omega^4 + \omega^2\gamma^2}\right]^{1/2} \tag{9.8}$$

with $\omega \ll \gamma, \mathrm{Re}(n^2 \approx 1 - (\omega_p/\gamma)^2$.

We have a complex index of refraction

$$n = \sqrt{1 + \chi_e} = \sqrt{1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}} \tag{9.9}$$

in this limit. For a typical metal, $\omega_p\tau \gg 1$. At high frequencies $\omega > \omega_p$, the index of refraction is real; for $\omega < \omega_p$, the index of refraction is imaginary, and the crossover is $\omega = \omega_p$. Note that as $\omega/\omega_p \to \infty$, we have $\chi_e \to 0$ and $n \to 1.0$ (real). In the other limit, $\omega \to 0$, we get instead

$$n(\omega) = \sqrt{\frac{\omega_p^2}{2\omega\gamma}}(1 + i). \tag{9.10}$$

Notice that at long wavelengths (low frequencies), the real and imaginary parts of n have the same magnitude, and the absorption (proportional to the imaginary part) is the largest. At short wavelengths, the real part goes to 1 and the imaginary part goes to zero.

Wave packets in dispersive media We've considered plane waves. What if we now think about wave packets? We can use some of our mathematical tools from plane waves to understand wave packets. We write

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\,\frac{1}{(2\pi)^3}\int d^3k\, \boldsymbol{\mathcal{E}}_\perp(\mathbf{k})e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}. \tag{9.11}$$

We've seen that in vacuum or nondispersive media, a wave packet spreads in the transverse directions. We get free-space diffraction. In dispersive media, packets will also spread longitudinally in the direction of propagation because the speed of light is now frequency-dependent:

$$\mathbf{k}(\omega) = k(\omega)\hat{\mathbf{z}} = n(\omega)\frac{\omega}{c}\hat{\mathbf{z}} \tag{9.12}$$

and the field is

$$E(z, t) = \int_0^\infty A(\omega)e^{i(k(\omega)z - \omega t)}. \tag{9.13}$$

We still get the diffraction in transverse directions, but we want to understand what happens for the longitudinal one.

Our packet is centered in frequency space around ω_0, so if we define $\delta\omega = \omega - \omega_0$, then we can expand $k(\omega)$ in a Taylor series about ω_0 as

$$k(\omega) = k(\omega_0) + \delta\omega \frac{dk}{d\omega}|_{\omega_0} + \frac{1}{2}(\delta\omega)^2 \frac{d^2k}{d\omega^2}|_{\omega=0} + \dots \tag{9.14}$$

where the wavepacket becomes

$$E(z,t) = A(z,t)e^{i(k_0 z - \omega_0 t)} \tag{9.15}$$

with the envelope function being

$$A(z,t) = \int_{-\infty}^{\infty} d\delta\omega A(\omega_0 + \delta\omega) \exp\left(i(\delta\omega k_0' + \frac{1}{2}(\delta\omega)^2 k_0'' + \dots)z - i\delta\omega t \right). \tag{9.16}$$

The task is to solve for the envelope function.

Well, we can make a group velocity approximation: recall that

$$v_g = \frac{\partial\omega}{\partial\mathbf{k}}|_{\mathbf{k}=\mathbf{k}_0} = \frac{\partial\omega}{\partial k}|_{k=k_0}\hat{\mathbf{k}}. \tag{9.17}$$

That is, the overall packet moves with some group velocity, as compared to the phase velocity $v_p = \omega/k$. Without dispersion,

$$k = n\omega/c \implies \omega = kc/n, \frac{\partial\omega}{\partial k} = c/n, \tag{9.18}$$

so in nondispersive media, $v_p = v_g$.

On the other hand, for media with normal dispersion, $\frac{d\chi}{d\omega} > 0$ and so the index of refraction increases as ω increase (higher frequencies are slower), while for anomalous dispersion, $\frac{d\chi}{d\omega} < 0$, so higher (bluer) frequencies move faster.

To make any more progress we must constrain $A(z,t)$ somehow. Jackson limits his discussion to Gaussian wavepacket, Brau limits to the first two terms of the Taylor expansion, and Zangwill limits to cases where k_0'' is small. The Gaussian approximation is a little more constraining—it's a special case of the second derivative being small.

In this limit,

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)A(z,t) = 0. \tag{9.19}$$

Thus

$$E(z,t) = A(z - v_g t)e^{ik_0 z - \omega_0 t)}. \tag{9.20}$$

In Drude matter, above the plasma frequency we have

$$ck = \sqrt{\omega^2 - \omega_p^2} \tag{9.21}$$

and so the group velocity is

$$v_g = c^2/v_p = c\sqrt{1 - \omega_p^2/\omega^2} < c. \tag{9.22}$$

We can also write this in terms of the index of refraction,

$$v_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial k}\left(\frac{k}{n}\right) = \frac{c}{n} - \frac{ck}{n^2}\frac{\partial n}{\partial\omega}v_g. \tag{9.23}$$

In Lorentz matter, we have

$$v_g = \frac{c}{n + \omega\frac{dn}{d\omega}}. \tag{9.24}$$

This allows us to have either normal dispersion or anomalous dispersion:

- normal dispersion ($dn/d\omega > 0$) $\implies v_g < v_p$,
- anomalous dispersion ($dn/d\omega < 0$) $\implies v_g > v_p$. Anomalous dispersion can produce group velocities which exceed c; this isn't a problem since the information only travels with the phase velocity.

There are some consequences of causality, known as the Kramers–Krönig relations. The response functions G must be real and causal:

$$\mathrm{Re}(\chi(\omega_0)) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(\chi(\omega_0))}{\omega - \omega_0} d\omega \tag{9.25}$$

and a similar expression for the imaginary part holds.

Lecture 10.

Monday, April 20, 2020

We're moving on to waveguides today, roughly sections 19.1-4 in Zangwill. The purpose of a waveguide is to guide EM waves (clearly), ideally with low losses. This has applications to power transmission, communications, and signal propagation. There are conducting waveguides (transmission lines) as well as dielectric guides (fiber optics).

Transmission lines have multiple conductors (typically two, as in coaxial cables) to carry electrical power or signal with minimal distortion, while wave guides are a single conducting tube. We'll focus on transmission lines today.

Let's recall that fields at a conductive boundary satisfy

$$\nabla \cdot \mathbf{D} = \rho_f \implies \hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f \tag{10.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \implies \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{10.2}$$

$$\nabla \cdot \mathbf{B} = 0 \implies \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{10.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \implies \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f. \tag{10.4}$$

For a perfect conductor, the conductance becomes infinity, $\sigma \to \infty$, $R = 1/\sigma \to 0$ and the skin depth goes to zero, $\delta = \sqrt{2/\mu\sigma\omega} \to 0$. It follows that the fields inside the perfect conductor are exactly zero,

$$\mathbf{E}_C = \mathbf{H}_C = 0. \tag{10.5}$$

Outside, let μ, ϵ be constant and real. That is, we wish for our medium to be non-absorbing, so any power losses come from interaction with the conductor.

By the boundary conditions,

$$\Delta E_{\tan} = 0 \implies E_{2,\tan} = 0 \Delta B_{\text{norm}} = 0 \qquad \implies B_{2,\text{norm}} = 0 \Delta D_{\text{norm}} = \sigma \implies D_{2,\text{norm}} = \sigma \tag{10.6}$$

$$\Delta H_{\tan} = \mathbf{K} \implies \hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{K}. \tag{10.7}$$

We notice that $\mathbf{E} \times \mathbf{H}$ is therefore tangent to the boundary, and so $\mathbf{S} \cdot \hat{\mathbf{n}} = 0$. That is, there are no power losses to the boundaries.

In reality we don't have perfect conductors Instead,

$$\left\langle \frac{\partial P}{\partial V} \right\rangle = \frac{1}{2}\mathbf{J} \cdot \mathbf{E}^*. \tag{10.8}$$

If the conductor is ohmic, then $\mathbf{J} = \sigma \mathbf{E}_c$ and so

$$-\left\langle \frac{\partial P}{\partial V} \right\rangle = \frac{1}{2\sigma}[\mathbf{J} \cdot \mathbf{J}^*]. \tag{10.9}$$

(?) (slide 4) For fields in the non-conducting medium, we can make the ansatz that the fields are harmonic, $\mathbf{E}, \mathbf{H} \sim e^{-i\omega t}$. Suppose the medium otherwise has no charge or current, $\rho = \mathbf{J} = 0$. Then Maxwell's equations give us

$$\nabla \cdot \mathbf{E} = 0 \tag{10.10}$$

$$\nabla \times \mathbf{E} = i\omega\mu \mathbf{H} \tag{10.11}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{10.12}$$

$$\nabla \times \mathbf{H} = -i\epsilon\omega \mathbf{E} \tag{10.13}$$

Using our usual double-curl tricks and the Gauss law, we find a Helmholtz equation

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = 0. \tag{10.14}$$

We might also define $k_0^2 = \mu\epsilon\omega^2/c^2$.

Consider an infinite pipe made of a perfect conductor extending in the *uvz* direction. Inside,

$$\mathbf{E}, \mathbf{H} \sim e^{\pm ikz - i\omega t} \tag{10.15}$$

which allows the wave to travel in the $\pm\hat{\mathbf{z}}$ direction. It cannot travel in the transverse directions because there's a piece of metal in the way.

Now if the waves have this sort of dependence, then

$$\frac{\partial^2}{\partial z^2} = -k^2 \implies \nabla^2 = \nabla_\perp^2 + \partial_z^2 = \nabla_\perp^2 - k^2. \tag{10.16}$$

Then the Helmholtz equation in terms of k_0 becomes

$$(\nabla_\perp^2 - k^2 + k_0^2)\mathbf{E} = 0, \tag{10.17}$$

and we can define

$$\gamma^2 \equiv k_0^2 - k^2. \tag{10.18}$$

Thus our Helmholtz equation becomes

$$\left(\nabla_\perp^2 + \gamma^2\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = 0. \tag{10.19}$$

We see that the field separates into axial and transverse components,

$$\mathbf{E} = \mathbf{E}_\perp + \hat{\mathbf{z}}E_z. \tag{10.20}$$

But each of the components must satisfy the Helmholtz equation, i.e.

$$(\nabla_\perp^2 + \gamma^2)E_z = 0, (\nabla_\perp^2 + \gamma^2)E_\perp = 0. \tag{10.21}$$

If we consider Faraday's law we have

$$\boldsymbol{\nabla} \times_\perp \times \mathbf{E}_\perp = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right) \times \left(\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y\right) = C\hat{\mathbf{z}} = i\mu\omega H_z\hat{\mathbf{z}}. \tag{10.22}$$

Taking a double curl gives us some nice results. We find that

$$\gamma^2\mathbf{E}_\perp = \nabla_\perp\left(\frac{\partial E_z}{\partial z}\right) - i\mu\omega\hat{\mathbf{z}} \times \nabla_\perp H_z, \tag{10.23}$$

and a similar equation for $\mathbf{H}_\perp$. This tells us that the fields are entirely determined by the axial components.

Now we can make some different choices.

- Transverse Magnetic (TM). $H_z = 0, \gamma \neq 0$. In that case,

  $$(\nabla_\perp^2 + \gamma^2)E_z = 0. \tag{10.24}$$

  We have Dirichlet boundary conditions where $\hat{\mathbf{n}} \times \mathbf{E}|_S = 0 \implies E_z|_S = 0$, which tells us that the field at the surface is zero.
- Transverse Electric (TE). $E_z = 0$. This leads to $\hat{\mathbf{n}} \cdot \mathbf{H}|_S = 0 \implies \hat{\mathbf{n}} \cdot \mathbf{H}_\perp|S = 0$. These are Neumann boundary conditions:

  $$(\nabla_\perp^2 + \gamma^2)H_z = 0, \quad \hat{\mathbf{n}} \cdot \nabla_\perp H_z|S = \frac{\partial H_z}{\partial n}|_S = 0. \tag{10.25}$$

- Transverse Electric/Magnetic (TEM). $H_z = E_z = 0$. These are only allowed in coaxial setups (transmission lines) because otherwise we could not fit the boundary conditions at the origin.

In waveguides, our TM waves have

- $H_z = 0$ everywhere by definition
- $E_z = \psi(x, y)e^{i(kz-\omega t)} \implies (\nabla_\perp^2 + \gamma^2)\psi = 0$. Rectangular pipes will give us sines and cosines; cylindrical pipes will give Bessel functions. This is precisely analogous to the free particle in QM (the Helmholtz equation is like the Schrödinger equation)

- $\mathbf{E}_\perp = \pm \frac{ik}{\gamma^2}(\nabla_\perp E_z)$
- (another condition for H?)

There are some conditions for TE waves. There are usually no TEM waves in a waveguide.

Lecture 11.

Wednesday, April 22, 2020

Today we'll continue our discussion of waveguides. As we said last time, one can propagate TM and TE waves down a conducting wave guide, but for TEM waves we need transmission lines– the latter is only allowed in co-axial setups. In TE waves, the magnetic field points a little bit along the axis of propagation ($H_z \neq 0$); in TM waves, it is instead the electric field that points a bit along the z-axis.

In the Tm case, $E_z|_S = 0$ on the boundary will give us sines, and in the TE case the derivative of H_z being zero will give us cosines.

In general, $\mathbf{E}_\perp, \mathbf{H}_\perp$, and $\hat{\mathbf{z}}$ form a RH orthoganl set, with

$$|\mathbf{E}_\perp| = Z|\mathbf{H}_\perp|. \tag{11.1}$$

$E_\perp$ and $H_\perp$ have no phase difference, but either $E_\perp$ and E_z or $H_\perp$ and H_z are 90° out of phase.

If we define $\gamma^2 = k_0^2 - k^2 = \mu\epsilon\omega^2$, solving for k^2 gives

$$k_\lambda^2 \equiv \mu\epsilon\omega^2 - \gamma_\lambda^2, \tag{11.2}$$

where we take ω to be given and γ determined. Now this leads us to write a cutoff frequency

$$\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\epsilon}} \tag{11.3}$$

such that above ω_λ, the wavenumber

$$k_\lambda = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_\lambda^2} \tag{11.4}$$

becomes imaginary. That is, $\omega > \omega_\lambda$ gives k_λ real and propagating waves, whereas for $\omega < \omega_\lambda$ we have cut-off modes and decaying exponential solutions.

In general we want to design our waveguide so that only one mode propagates. Consider a specific geometry, a rectangular waveguide. It has width a in the x-direction and height b in the y direction. It's standard to take $a = 2b$. Well, by separation of variables we can write down the solutions to the Helmholtz equation.

$$\psi_{nm} = E_0 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right), n, m = 1, 2, 3, \ldots \quad \text{TM waves} \tag{11.5}$$

$$\psi_{nm} = H_0 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right), n, m = 0, 1, 2, \ldots \quad \text{TE waves} \tag{11.6}$$

Note that our convention for the lower indices is switched from Zangwill. The ψs here are proportional to E_z^{nm} and H_z^{nm} respectively.

In either case, we can calculate the corresponding γs:

$$\gamma_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2. \tag{11.7}$$

THese are the eigenvalues, with

$$\omega_{nm} = \frac{1}{\sqrt{\mu\epsilon}}\gamma_{nm}. \tag{11.8}$$

We can see that the cutoff wavelength is $\lambda = 2\pi/\gamma_{nm}$.

In the standard geometry, $a = 2b$, we have

$$\gamma_{nm} = \frac{\pi}{a}\sqrt{n^2 + 4m^2}, \tag{11.9}$$

which gives us the lowest modes as

$$\gamma_{10} = \pi/a, \gamma_{01} = \gamma - 20 = 2\pi/a. \tag{11.10}$$

These must be TE modes, since one of their indices is zero. THe next ones up are

$$\gamma_{11} = \sqrt{5}\pi/a, \gamma_{21}\sqrt{8}\pi/a, \tag{11.11}$$

which can be TE or TM.

The corresponding cutoff frequencies are

$$\omega_{nm} = \frac{c}{n}\gamma_{nm}, \quad k_{nm} = \frac{c}{n}\sqrt{\omega^2 - \omega - nm^2}. \tag{11.12}$$

We notice that the usual situation gives us

$$\omega_{10} < \omega < \omega_{01} \text{ or } \omega_{20}, \tag{11.13}$$

then we can maximize the bandwidth for a desired frequency range.

The math is easiest for rectangular waveguides, but we can understand circular pipes as well; these give us Bessel functions in the radial direction instead.

Transmission lines with a coaxial geometry allow us to set up a potential difference between the two conductors, so TEM waves can propagate. Consider a coaxial cable with inner radius a and outer raduis b. We can set up a solution for the potential

$$\Phi = -\frac{\lambda}{2\pi\epsilon}\ln s, \tag{11.14}$$

then

$$E_s = -\frac{\partial\Phi}{\partial s} \tag{11.15}$$

and

$$\mathbf{E} = (A/s)\hat{\mathbf{s}}, \mathbf{B} = \frac{A}{cs}\hat{\boldsymbol{\phi}}. \tag{11.16}$$

Note that dielectric waveguides use total internal reflection instead of metallic reflectivity to guide the wave. If the core has a higher index of refraction, $n_1 > n_2$, then total internal reflection is possible. The index of refraction can either change by step (two separate materials) or in a continuous, graded way. It depends on what frequency response we want from our fiber optic.

In closed resonant cavities rather than waveguides, we again solve Laplace's equation and fit boundary conditions. We can do this for parallelepiped cavities, closed tubes, and also spherical cavities.

Lecture 12.

Friday, April 24, 2020

Today we'll begin our discussion of radiation and retardation. In general, Maxwell's equations are time-dependent.

Radiation is interesting to us because it allows us to have a nonzero power density at infinity. Accelerating charges allow us to have fields that do not fall off fast enough and extend to infinity.

Retardation is simply the phenomenon that signals take time to propagate in space, so there are no instantaneous responses to changing charges or currents. That is, the fields reflect the behavior of the sources at an earlier retarded time.

If we take the curl of Faraday's law, we get wave equations

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho + \mu_0\frac{\partial\mathbf{J}}{\partial t} \tag{12.1}$$

and

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu_0\boldsymbol{\nabla}\times\mathbf{J}. \tag{12.2}$$

These are inhomogeneous wave equations, so they are harder to solve. We can consider fields produced by dynamically polarized/magnetized matter, $\rho = -\boldsymbol{\nabla}\cdot\mathbf{P}, \mathbf{J} = \frac{\partial\mathbf{P}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}$. Now for these wave equations we can similarly define

$$\varphi_L = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{\mathrm{e}}, \quad \mathbf{A} = \frac{1}{c^2}\frac{\partial\boldsymbol{\pi}_e}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{\pi}_{\mathrm{m}} \tag{12.3}$$

Consider the fields from a point charge moving at constant velocity. That is,

$$\mathbf{v} = v\hat{\mathbf{z}} \tag{12.4}$$

$$\rho(\mathbf{r}, t) = q\delta(x)\delta(y)\delta(z - vt) \tag{12.5}$$

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{v}\rho(\mathbf{r}, t). \tag{12.6}$$

At a time t, the particle is at a point $vt\hat{\mathbf{z}}$, and we can study the field at some point $\mathbf{r} = (x, y, z)$. We can also define $\mathbf{R}$, which is the vector from the charge at a time t to the observation point (i.e. $vt\hat{\mathbf{z}} + \mathbf{R} = \mathbf{r}$). The potentials are now

$$\varphi(x, y, z - vt), \quad A(x, y, z - vt)\hat{\mathbf{z}}, \tag{12.7}$$

where we notice that since the current is in the $\hat{\mathbf{z}}$ direction, so is the vector potential.

If we let $\xi = z - vt$ and $\beta = v/c$, then Poisson's equation tells us

$$\nabla \cdot (\nabla\varphi) - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\rho/\epsilon_0 \implies \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + (1 - \beta^2)\frac{\partial^2\varphi}{\partial \xi^2} = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(\xi). \tag{12.8}$$

Now if we define $\gamma = (1 - \beta^2)^{-1/2}$ and $z' = \gamma\xi = \gamma(z - vt)$, then Poisson's equation becomes

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z'^2} = -\frac{q}{\epsilon_0}\delta(x)\delta(y)\delta(\xi). \tag{12.9}$$

Lecture 13.

Wednesday, April 29, 2020

Today we'll talk about antenna theory. In Lorenz gauge, we've seen that

$$(\nabla^2 + k^2)\mathbf{A} = \mu_0\mathbf{J}, \tag{13.1}$$

with a wave number $k = \omega/c$. In 4-vector notation, $\Box A = -\mu_0 j$. If we suppose that A oscillates in time like a sinusoid with frequency ω then our wave equation becomes

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{A} = (\nabla^2 + \omega^2/c^2)\mathbf{A} = -\mu_0\mathbf{J}. \tag{13.2}$$

At this point, Zangwill resorts to a Green's function approach to solve the Helmholtz equation with source:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{13.3}$$

with the fields

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{13.4}$$

$$\mathbf{E} = \frac{i}{k}\nabla \times c\mathbf{B}, \tag{13.5}$$

where the latter comes from $\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$. This is subject to the Sommerfeld radiation condition, which is that sources only radiate waves; they do not absorb them. Zangwill is actually more formal about this than Jackson.

Now in general we can expand the separation $|\mathbf{r} - \mathbf{r}'|$ in orders of $\hat{\mathbf{n}} = \mathbf{r}/r$:

$$|\mathbf{r} - \mathbf{r}'| = r\left[1 - \frac{\hat{\mathbf{n}} \cdot \mathbf{r}'}{r} + \frac{r'^2}{2r^2} - \frac{1}{8}\left(\frac{2\hat{\mathbf{n}} \cdot \mathbf{r}'}{r}\right)^2 + \dots\right] \tag{13.6}$$

where this expansion occurs in the complex exponential.

Now we have a constraint $r \gg d^2/8\lambda$, known as the Fraunhofer limit. This tells us that we can neglect quadratic corrections in $\mathbf{r}'$:

$$|\mathbf{r} - \mathbf{r}'| \to r - \hat{\mathbf{n}} \cdot \mathbf{r}'. \tag{13.7}$$

The denominator of our solution for $\mathbf{A}$ can be approximated as r if $r \gg d$, the characteristic size of the source.

Jackson identifies three zones:

$$d \ll r \ll \lambda \equiv \text{near (static) zone} \tag{13.8}$$
$$d \ll r \sim \lambda \equiv \text{intermediate (induction) zone} \tag{13.9}$$
$$d \ll \lambda \ll r \equiv \text{far (radiation) zone} \tag{13.10}$$

Zangwill uses time scales instead, $r \ll c\tau, r \sim c\tau, r \gg c\tau$. We can consider the radiation zone, where $r \gg d$, in which case our solution for $\mathbf{A}$ simplifies to

$$\mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r})e^{-ik\hat{\mathbf{n}}\cdot\mathbf{r}'}d^3r', \tag{13.11}$$

which is an outgoing spherical wave. If we compute the curls, we have

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A} \simeq ik\hat{\mathbf{n}}\times\mathbf{A} = ik\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \hat{\mathbf{n}}\times\mathbf{J}(\mathbf{r})e^{ik\hat{\mathbf{n}}\cdot\mathbf{r}'}d^3r' \tag{13.12}$$

and

$$\mathbf{E}(\mathbf{r}) = c\mathbf{B}\times\hat{\mathbf{n}} = ick(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}} \tag{13.13}$$

The radiation fields are mutually orthogonal and transverse to $\mathbf{r}$ ($\hat{\mathbf{n}}$). They also satisfy $E = cB \propto r^{-1}$, which is slower than the usual $1/r^2$ falloff.

We can expand

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(ik)^n}{n!}\int \mathbf{J}(\mathbf{r}')(\hat{\mathbf{n}}\cdot\mathbf{r}')^n d^3r' \tag{13.14}$$

and come up with some multipole expansion.

To get electric dipole radiation, we take the zeroth order term,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')d^3r', \tag{13.15}$$

which we can readily compute. If we integrate by parts then

$$\int \mathbf{J}(\mathbf{r}')d^3r' = -\int \mathbf{r}'(\boldsymbol{\nabla}'\cdot\mathbf{J})d^3r' = -i\omega\int \mathbf{r}'\rho(\mathbf{r}')d^3r' = \dot{\mathbf{p}}, \tag{13.16}$$

where we've assumed the charge density is oscillating,

$$\boldsymbol{\nabla}\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} = i\omega\rho. \tag{13.17}$$

We notice that this defines a time derivative of an electric dipole moment, so this is dipole radiation.

Lecture 14.

Friday, May 1, 2020

Today we'll continue with antennas. We can compute the time-average power from oscillating radiative sources. The power per unit surface area for an electric dipole is

$$\frac{dP}{d\Omega} = \frac{p_0^2\omega^4}{32\pi^2\epsilon_0 c^3}\sin^2\theta, \tag{14.1}$$

where $p(\mathbf{t}) = \mathbf{p}_0 e^{i\omega t}$ is the dipole moment as a function of time. If we perform the angular integral over $\int \sin^2\theta d\Omega$, we find that

$$P_{\text{tot}}^{\text{ED}} = \frac{p^2\omega^4}{12\pi\epsilon_0 c^3} = \frac{|\ddot{p}_0|^2}{6\pi\epsilon_0 c^3}. \tag{14.2}$$

The reason we might want to write this as a time derivative is to make contact with expressions like the Larmor formula, which relate the acceleration to the power radiated.

One can write similar formulae for the magnetic dipole and the electric quadrupole.

$$\frac{dP^{(\text{MD})}}{d\Omega} = \frac{\mu_0 m_0^2\omega^4}{32\pi 2c^3} \tag{14.3}$$

and

$$P_{\text{tot}} = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}. \tag{14.4}$$

For the quadrupole,

$$\frac{dP^{(\text{MD})}}{d\Omega} = \frac{\mu_0 \dddot{Q}_{zz}^2}{512\pi^2 c^3} \cos^2\theta \sin^2\theta \tag{14.5}$$

with the total power

$$P = \frac{\mu_0 (3/2) \dddot{Q}_{zz}^2}{1420\pi c^3}. \tag{14.6}$$

For thin wire antennas, we can write the current as

$$I(z,t) = I_0 \sin k(d - |z|) e^{i\omega t}, \tag{14.7}$$

a linear sinusoidal center-fed antenna. Then

$$\mathbf{j}(\mathbf{r},t) = I_0 \sin k(d-|z|) e^{i\omega t} \delta(x)\delta(y)\hat{\mathbf{z}}. \tag{14.8}$$

We can then calculate a time-averaged power radiated by taking again the lowest-order nonvanishing term in our radiation approximation. We get

$$\frac{dP}{d\Omega} = \frac{\mu_0 c I_0^2}{8\pi^2} \left[\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta} \right]^2. \tag{14.9}$$

With $kd \gg 1$ (at wavelengths much smaller than the size of the antenna) this simplifies to

$$\frac{dP}{d\Omega} = \frac{\mu_0 c I_0^2}{32\pi^2} (?) \tag{14.10}$$

Fun facts about light and radiation.

- The visible spectrum is defined by the frequencies of light we can see.
- X-ray is separated from ultraviolet by its capacity to ionize materials.
- Radio is separated from infrared by Joule heating (infrared warms things up)
- Gamma rays overlap with X-rays in their frequency band because they're distinguished by their sources (gamma from nuclear, X-rays from overall atomic/electronic emissions)

If we have a single charged particle moving as

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}(t)\delta[\mathbf{r} - \mathbf{r}_0(t)] \tag{14.11}$$

where $\mathbf{v} = \dot{\mathbf{r}}_0$, then we can calculate the radiation from the charge. The power per area is

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{(?)\pi^2 c^3} \sin^2\theta. \tag{14.12}$$

Lecture 15.

Monday, May 4, 2020

Visible light has a wavelength of 700-400 nm (7000-4000 Angstroms), whereas the gas molecules in the atmosphere have a size on the order of a few angstroms (0-5). Hence the light has a wavelength much larger than the size of the particles in the atmosphere.

In the long wavelength limit, $\lambda \gg d$, we need only consider the lowest-order multipole in the expansion. Scattering is a process of sending in an incident wave, letting it interact with some object, and having that object re-emit the wave.

We'll consider the surrounding medium to be vacuum, ϵ_0, μ_0.

Consider an incident wave

$$\mathbf{E}_{\text{inc}} = \boldsymbol{\epsilon} - 0 E_0 e^{ik\hat{\mathbf{n}}_0 \cdot \mathbf{r}} e^{-i\omega t}, \tag{15.1}$$

and

$$\mathbf{H}_{\text{inc}} = \hat{\mathbf{n}}_0 \times \mathbf{E}_{\text{inc}} / Z_0, \tag{15.2}$$

where $Z_0 \equiv \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Note Jackson drops the time dependence (like taking a time average). The scattered fields are a spherical wave with a multipole expansion:

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}} - \hat{\mathbf{n}} \times \mathbf{m}/c], \tag{15.3}$$

where these are the electric and magnetic dipole terms.

Note that molecules and atoms naturally have a polarizability from charge distribution when exposed to electric fields, and a magnetizability due to spin effects in magnetic fields.

The differential scattering cross-section can be computed here by computing the Poynting vector, noting that

$$\frac{d\sigma}{d\Omega} = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{\text{sc}} \rangle}{| \langle \mathbf{S}_{\text{inc}} \rangle |} \tag{15.4}$$

and for pure dipole radiation we get

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} = |\hat{\boldsymbol{\epsilon}}^* \cdot \mathbf{p} + (\hat{\mathbf{n}} \times \hat{\boldsymbol{\epsilon}}^*) \cdot \mathbf{m}/c|^2. \tag{15.5}$$

For instance, we can compute scattering off a dielectric sphere. This only produces an electric dipole term. We know the dipole moment of the sphere:

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r - a}{\epsilon_r + 2} \right) a^3 \mathbf{E}_{\text{inc}} \tag{15.6}$$

and if we compute the cross section in directions parallel and perpendicular to the scattering plane, we get

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{1}{4} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \cos^2\theta$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{4} k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2$$

The scattering is proportional both to the size of the object (measured by a) and the frequency (measured by k).

Note that scattering generically polarizes light. When the light source is directly overhead, then 90 degrees from that source gives the maximally polarized light (i.e. near the horizon).

Scattering is proportional to k, and higher frequencies like blue are scattered more. But there's also an effect of the sun's spectrum (the sun emits less violet light in general). This tells us why the sky is blue on Earth. On the other hand, at sunset the blue light is scattered the most, and so as the light passes through more of the atmosphere, the red light (being scattered the least) is what makes it to our eyes.

Why does the sky on Mars look pink sometimes? It turns out this isn't due to scattering from gas in the atmosphere but rather from dust storms, which kick up iron compounds into the air. Are all skies blue? Basically, yes. This is impacted somewhat by the spectrum of the star, but higher frequencies are always scattered more.

In a different regime $\lambda \sim d$ we can consider Mie scattering, where the outgoing waves take the form of Bessel functions.

For a conducting sphere, we can also study the relevant scattering. Conducting spheres can have induced electric and magnetic fields:

$$\mathbf{p} = 4\pi epsilon_0 a^3 \mathbf{E}_{\text{inc}}$$

$$\mathbf{m} = -2\pi a^3 \mathbf{H}_{\text{inc}}.$$

We can do the same calculation for the electric and magnetic dipole terms in the differential cross-section.

Incidentally radar has a maximal differential cross-section peaked at -180 degrees, i.e. scattering is maximal directly back at the source.

Lecture 16.

Wednesday, May 6, 2020

Today we'll move onto Rayleigh scattering. In Thomson scattering, we had an EM wave incident on a charged particle. The **E** and **B** fields exert force on the particle, which sets the particle into motion. There's energy flux in the direction of the wave (given by the Poynting vector). A periodic wave creates periodic motion; when the particle accelerates, this produces radiation.

Consider a linearly polarized monochromatic plane wave with E-field

$$\mathbf{E}(\mathbf{r}, t) = E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\boldsymbol{\epsilon}}. \tag{16.1}$$

By our usual Fourier arguments, we can just understand a plane wave and then the general case follows from assembling a wave packet.

$$\mathbf{F} = q\mathbf{E} = m\ddot{x}, \tag{16.2}$$

so

$$\ddot{x} = -\omega^2 \chi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{16.3}$$

In the non-relativistic limit, we can approximate the magnetic force as being zero– since it is proportional to the velocity, if the particle never moves too fast then it's basically just the electric force. Plugging in the acceleration into the Larmor formula, we find

$$\frac{dP}{d\Omega} = \frac{q^2\ddot{x}^2}{16\pi^2\epsilon_0 c^3}\sin^2\theta \tag{16.4}$$

or

$$\frac{d\langle P\rangle}{d\Omega} = \frac{q^2\omega^4 x_0^2}{32\pi^2\epsilon_0 c^3}\sin^2\theta, \tag{16.5}$$

where θ is the angle relative to the axis of the particle motion. We can also write

$$\ddot{x}^2 = \frac{q^2}{m^2}E^2 \implies \left\langle \ddot{x}^2 \right\rangle = \frac{1}{2}\frac{q^2}{m^2}E_0^2, \tag{16.6}$$

so in terms of the charge and the field strength,

$$\begin{aligned} \frac{d\langle P\rangle}{d\Omega} &= \left(\frac{q^4}{16\pi^2\epsilon_0^2 m^2 c^4}\right)\frac{\epsilon_0 c E_0^2}{2}\sin^2\theta \\ &= \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 \frac{\epsilon_0 c E_0^2}{2}\sin^2\theta, \end{aligned}$$

where $\frac{q^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron. Incidentally, this classical radius of the electron comes from a calculation where we can imagine a spherical shell of charge and calculate the energy stored in the electric field. If we set that energy equal to the mass energy of the electron, we can calculate what the radius would be.

We can write a time-averaged Poynting vector for the process,

$$\langle S\rangle = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{2}c\epsilon_0 E_0^2\hat{\mathbf{z}}. \tag{16.7}$$

As a remark, charged particles generically emit radiation when they accelerate; this usually happens either because of ionizing or otherwise interacting with materials. In the case we're describing now we have an electron interacting with a background plane wave.

We can also compute a scattering cross-section, which represents an equivalent area of the incident wave front:

$$\sigma = \frac{\text{total re-radiated power}}{\langle S\rangle}, \tag{16.8}$$

how much power goes out normalized to the incident power flux. In this case, note that

$$\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\langle S\rangle} = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2\sin^2\theta. \tag{16.9}$$

Performing the $d\Omega$ integral we have $\int \sin^2\theta d\Omega = 8\pi/3$, and so the total scattering cross-section is

$$\sigma = \int_0^\pi \frac{d\sigma}{d\Omega} 2\pi \sin\theta d\theta = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2. \tag{16.10}$$

We've accounted for polarized light. What if the light is randomly polarized? In that case, we can average over possible polarizations. Note that that when $\hbar\omega \sim mc^2$, we need quantum corrections to model a photon scattering off an electron. The cross-section is given by the Klein-Nishima formula, and the scattering is primarily forward, as opposed to the classical formula, which is more symmetric in $\cos\theta$.

We can (re)conisder dielectric spheres. Using the equation of motion for the induced dipole moment, $\ddot{\mathbf{p}} = q\ddot{x}\hat{\mathbf{x}}$, and applying the Larmor formula, we can find a cross-section for the dielectric sphere as well (slide 8).

Lecture 17.

Friday, May 8, 2020

Today we'll discuss diffraction in various limits. We probably recall that $m\lambda = D\sin\theta$ gives the maxima in single slit diffraction, where D is the distance from the single slit to the screen. In the double slit case, we get a similar pattern. Here, maxima occur at $m\lambda = d\sin\theta$, with d the slit separation.

For circular apertures, we get a similar pattern but the first minimum occurs at $1.22\lambda = D\sin\theta$, where this value has to do with the first minimum of the Bessel function.

Long wavelengths correspond to a scattering limit; short wavelengths correspond to a diffraction limit ($\lambda/d \ll 1$). In scattering, we need to consider how the medium/target responds to the wave and re-radiates power; in diffraction, we just need to know how waves bend around obstacles.

In scalar diffraction theory, we get a wave equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0. \tag{17.1}$$

By taking the Fourier transform we get a Helmholtz equation

$$\nabla^2\tilde{A} + k^2\tilde{A} = 0 \tag{17.2}$$

and we can try to solve this with Green's function methods.

The Green's function for the 3D Helmholtz equation is (cf. Zangwill Eq. 10.79)

$$G(\mathbf{r}',\mathbf{r}) = \frac{e^{\pm k|\mathbf{r}'-\mathbf{r}|}}{4\pi|\mathbf{r}'-\mathbf{r}|} \tag{17.3}$$

which solves

$$\nabla^2 G(\mathbf{r}',\mathbf{r}) + k^2 G(\mathbf{r}',\mathbf{r}) = -\delta(\mathbf{r}'-\mathbf{r}). \tag{17.4}$$

Green's theorem(/identity) tells us we can write

$$-a(\mathbf{r}) = \int_V \left[a(\mathbf{r}')\nabla'^2 G(\mathbf{r}',\mathbf{r}) - G(\mathbf{r}',\mathbf{r})\nabla'^2 a(\mathbf{r}')\right]dV' = \oint_S \left[a(\mathbf{r}')\nabla' G(\mathbf{r}',\mathbf{r}) - G(\mathbf{r}',\mathbf{r})\nabla' a(\mathbf{r}')\right]dS' \tag{17.5}$$

so we can rewrite

$$a_k(\mathbf{r}) = -\int_{S_0}\left[a(\mathbf{r}')\nabla' G(\mathbf{r}',\mathbf{r}) - G(\mathbf{r}',\mathbf{r})\nabla' a(\mathbf{r}')\right]dS' \tag{17.6}$$

and now by specifying boundary conditions on the surface S_0 we can determine the solution a.

There are different limits we might take here, the near-field and far-field limits. Near-field is Fresnel diffraction, while far-field is Fraunhofer diffraction. Close to the aperture, edge effects are more prominent as the wave bends around the edges of the aperture; farther away, we recover the single-slit pattern.

Let's consider far-field diffraction, $\lambda \ll \mathbf{r}' \ll r$, where $\mathbf{r}'$ ranges over the aperture and r is the location of the screen, which must be much larger than the size of the aperture. In this limit, we can write $\mathbf{r}' - \mathbf{r} \approx -\mathbf{r}$. Since we are working in a $\lambda \ll \mathbf{r}'$ limit, we also know that $k|\mathbf{r}'| \gg 1$ since $k = 2\pi/\lambda$.

Now

$$a_D(\mathbf{r}) = \frac{-i}{2\pi}\int_{S_0} \frac{k(\mathbf{r}'-\mathbf{r})\cdot\hat{\mathbf{n}}'}{|\mathbf{r}'-\mathbf{r}|} a(\mathbf{r}') e(?) \tag{17.7}$$

For a circular aperture we have

$$a_D(\mathbf{r}) = ia_0\mathbf{k}\cdot\hat{\mathbf{n}}'\frac{e^{ikr}}{2\pi r}\int_0^R\int_0^{2\pi} e^{-i\Delta\mathbf{k}\cdot\mathbf{r}'}d\phi' r' dr' \tag{17.8}$$

where $e^{-i\Delta\mathbf{k}\cdot\mathbf{r}'} = e^{i\Delta k r'\cos\phi'}$, with Δk indicating the difference between the incident wavevector and the diffracted wavevector,

$$\Delta k = \mathbf{k} - \mathbf{k}_0. \tag{17.9}$$

Performing the integral gives us a Bessel function,

$$\int_0^{2\pi} e^{-i\Delta k\cos\phi'}d\phi' = 2\pi J_0(\Delta k r') \tag{17.10}$$

and integrating the r' integral gives

$$\int_0^R J_0(\Delta k r')r'dr' = \frac{R}{\Delta k}J_1(\Delta k R). \tag{17.11}$$

We find that

$$a(\mathbf{r}) = ia_0(\mathbf{k}\cdot\hat{\mathbf{n}}')\frac{J_1(\Delta k R)}{\Delta k R}\frac{R^2 e^{ikr}}{r}. \tag{17.12}$$

This is our amplitude function, and by squaring it to get the intensity we see that minima correspond to zeroes of the Bessel function.

Fresnel diffraction is the other limit, where $\lambda \ll \mathbf{r} \ll \mathbf{r}'$. Then we can expand

$$|\mathbf{r} - \mathbf{r}'|^2 = r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}' \tag{17.13}$$

and $k|\mathbf{r}' - \mathbf{r}|$ has an expansion in r/r'.

$$kr = k\mathbf{r}'\cdot\hat{\mathbf{r}} + \frac{kr'^2}{2r} - \frac{k(\mathbf{r}\cdot\hat{\mathbf{r}}')^2}{2r} + \dots \tag{17.14}$$

Assuming Dirichlet boundary conditions we can perform a Kirchhoff integral,

$$a(x, y0 = -ia_0\frac{kL}{2\pi}\int dx'dy'\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|^2} \tag{17.15}$$

and find the expression for Fresnel diffraction.

Lecture 18.

Monday, May 11, 2020

In special relativity, there are two basic postulates.

- The laws of physics are the same for all observers in an inertial reference frame (i.e. one where Newton's first law holds– no fictitious forces)
- The speed of light in vacuum has the same value c for all observers.

Historically, these were motivated by an evolving understanding of light as having particle properties (Newton's corpuscle theory) and wave-like properties (as advocated by Fresnel and others). The discovery of wave properties of light like interference and diffraction led physicists of the day to try to figure out whether there was a medium of propagation (the "luminiferous aether"), since all other known waves required a medium.

In 1864 Maxwell's equations presented a unified theory of electricity and magnetism, and the wave solutions suggested that light was an electromagnetic wave. Michelson and Morley's experiments testing the dependence of the velocity of light on the direction of motion contradicted the presence of an aether. Fitzgerald and Lorentz posited a transformation which mixed time and space in moving reference frames, but its implications were not fully understood.

Einstein concluded that Galilean relativity was incorrect, and that the (Fitzgerald-)Lorentz transformation was the correct way to describe physics in moving reference frames.

In the Galilean model, the only thing that changes is a spatial coordinate. That is,

$$\begin{aligned} t &= t' \\ x &= x' \\ y &= y' \\ z &= z' + ut', \end{aligned}$$

where the primed frame moves with velocity u in the $+z$-direction. Velocities add quite simply.

On the other hand, the Lorentz transformation says that the time and space coordinates are not independent when we change frames.

$$ct = \gamma\Big(ct' + \frac{u}{c}z'\Big) = \gamma(ct' + \beta z') \tag{18.1}$$

$$x = x' \tag{18.2}$$

$$y = y' \tag{18.3}$$

$$z = \gamma(z' + ut') = \gamma(z' + \beta ct'). \tag{18.4}$$

The forward transformation is

$$\begin{aligned} ct' &= \gamma(ct - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta ct), \end{aligned}$$

where

$$\beta \equiv u/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{18.5}$$

Sometimes it's convenient to define a boost parameter

$$\xi = \tanh^{-1}\beta \tag{18.6}$$

so that

$$\begin{aligned} \beta &\equiv \tanh\xi \\ \gamma &\equiv \cosh\xi \\ \gamma\beta &= \sin\xi. \end{aligned}$$

That is, if we let $\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$, then

$$ct' = ct\cosh\xi - z\sinh\xi, \tag{18.7}$$

$$z' = -ct\sin\xi + z\cosh\xi. \tag{18.8}$$

In matrix notation,

$$\begin{pmatrix} ct' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi, \end{pmatrix}\begin{pmatrix} ct \\ z \end{pmatrix} \tag{18.9}$$

which looks a lot like a rotation but with a hyperbolic signature (note the signs).

Lecture 19.

Wednesday, May 13, 2020

Four-vectors are pretty useful. We have contravariant vectors (index up) and covariant vectors (index down), and they transform differently under coordinate transformations.

Metrics $g_{\mu\nu}$ (and the inverse metric $g^{\mu\nu}$) tell us how to raise and lower indices. Equivalently they give us an inner product on the space, such that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{19.1}$$

Lecture 20.

Wednesday, June 3, 2020

Today we'll talk about some practicalities of charged particle detection. In the days of Rutherford, scattered particles were detected by using a phosphor (luminescent) screen and a manual "detector" (a graduate student). These days, we use electronic detectors, but the principle is the same.

Charged particles passing through a medium mostly see the electron cloud. Atoms are about 1 Angstrom in scale, and the nucleus itself is on the order of femtometers. Electrons are bound to nuclei with about 1 eV (e.g. the ground state energy of the innermost electron in hydrogen is 13.7 eV). But charged particles passing through have much higher energies on the order of GeV.

It follows that charged particles can easily ionize materials; ionizing radiation is similar. In detectors, the electrons can be ionized and collected as a signal; we apply an overall electric field to prevent the electrons from recombining with the nuclei and instead drift them over to a collector where we can measure a signature.

Energetic charged particles have a characteristic lifetime based on their energy, and therefore an average distance traveled $c\tau$ before they decay. For a muon this is about 658 meters, i.e. the product of the particle lifetime (in its rest frame) and the speed of light. In practice there is also a γ boost from the fact these particles are moving relativistically, and can often travel much farther in our frame because of time dilation effects.

Consider a collision between an energetic heavy charged particle of mass M, energy $E = \gamma Mc^2$, momentum $p = \gamma\beta Mc$, with a free electron at rest.