PHYSICS 204A: MATH METHODS

IAN LIM
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These notes were taken for Physics 204A, *Math Methods*, as taught by Nemanja Kaloper at the University of California, Davis in fall quarter 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itlim@ucdavis.edu.
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Lecture 1.

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Reading assignment: read Ch. 2 and 3 of the course text (Arfken/Weber). This is basic linear algebra and vector analysis.

The purpose of this course is to learn formal aspects of quantum mechanics. We'll focus on doing analysis in Hilbert space. It's a remarkable fact about the natural world that most of our physical world is well-approximated by linear systems.

In the simplest form, we may think of vectors as arrays of numbers,

$$(v_1, v_2, \ldots). \tag{1.1}$$

But we can also think of some real function $f(x) : \mathbb{R} \to \mathbb{R}$ as a collection of numbers too, just by taking its values at arbitrarily close points.

Definition 1.2. A *linear vector space* over a field F, denoted $L(F)$, is a set $\{|v\rangle\}$ with an addition operation $+$ such that

- for $|v\rangle, |u\rangle \in L, |u\rangle + |v\rangle = |w\rangle \in L$ (closure)
- for $c \in F$, $c|v\rangle \in L$ (scalar multiplication).

These axioms directly imply that any linear combination of vectors in L is also in the vector space:

$$c_v|v\rangle + c_u|u\rangle \in L. \tag{1.3}$$

This leads us naturally to the notion of *linear (in)dependence*. Suppose we take some vectors $|v_k\rangle \in L$ and make a linear combo,

$$\sum_k c_k|v_k\rangle. \tag{1.4}$$

Definition 1.5. A set of vectors $\{|v_k\rangle\}$ is *linearly dependent* if there exists some $\{c_k\}$ not all zero such that

$$\sum_k c_k|v_k\rangle = |0\rangle, \tag{1.6}$$

and such that $|0\rangle \notin \{|v_k\rangle\}$.

We need this last condition because otherwise we could simply take the coefficient of the $|0\rangle$ vector to be 1 and then arrive at a trivial solution.

Definition 1.7. If a set of vectors is not linearly dependent, it is *linearly independent*.

The next question we might ask is as follows: what is the size of the biggest set of linearly independent vectors we can construct for a given vector space?

Definition 1.8. The maximum number of linearly independent vectors associated to a given vector space is called the *dimension*.

Example 1.9. We may consider an infinitely differentiable (C^∞) function. It has a Taylor expansion

$$f(x) = \sum \frac{f^{(n)}(0}{n!} x^n, \tag{1.10}$$

which we may think of as an expansion in the basis $(1, x, x^2, \ldots)$.

So this is a vector space with countably infinite dimension. But we can have uncountably infinite-dimensional spaces too, e.g. the space of Fourier-transformable functions in a basis $e^{ikx}, k \in \mathbb{R}$. These factors are not just linearly independent; introducing an appropriate inner product, they are orthogonal.

It follows that for a vector space L_D of dimension D, any set with more than D vectors must be linearly dependent, i.e. $\exists \bar{c}_v, \bar{c}_k$ not all zero such that

$$\bar{c}_v |v\rangle + \sum_k \bar{c}_k |v_k\rangle = 0. \tag{1.11}$$

Moreover $\bar{c}_v \neq 0$ or else the original set $|v_k\rangle$ would be linearly dependent. Hence we can divide through, define $\hat{c}_k = \bar{c}_k / \bar{c}_v$, and write

$$|v\rangle = \sum_k \hat{c}_k |v_k\rangle. \tag{1.12}$$

That is, we have *decomposed* a general vector $|v\rangle$ in terms of its components $\hat{c}_k$ with respect to a basis $|v_k\rangle$.

We might now be interested in adding more structure to our vector space. Consider L_D with $|v\rangle, |u\rangle$.

Definition 1.13. We define an *inner product* by

$$\langle v|u\rangle : (|v\rangle, |u\rangle) \to F \tag{1.14}$$

as a map from the input vectors to the field over which the vector space is defined, with the following properties:

- $\langle u|(\lambda|v_1\rangle + \mu|v_2\rangle) = \lambda\langle u|v_1\rangle + \mu\langle u|v_2\rangle$ (linearity)
- $\langle v|u\rangle = \langle u|v\rangle^*$
- $\langle v|v\rangle \geq 0$, with equality only for the zero vector (positive semi-definite).

If we choose a basis $\{|v_k\rangle\}$, then if our vectors $|u\rangle, |v\rangle$ have some expansion in this basis then by linearity that

$$\langle u|v\rangle = \sum_k \hat{c}_k^v \langle u|v_k\rangle, \tag{1.15}$$

and we can expand each of these inner products as

$$\langle u|v_k\rangle = (\langle v_k|u\rangle)^* = \sum_n c_n^{u*} \langle v_n|v_k\rangle. \tag{1.16}$$

It follows that we can write a general inner product as

$$\langle u|v\rangle = \sum_{n,k} = c_k^v c_n^{u*} \langle v_n|v_k\rangle. \tag{1.17}$$

Moreover if we could choose a nice basis which had a special property of *orthogonality* or better yet *orthonormality*, we could reduce this to a single sum

$$\langle u|v\rangle = \sum_k c_k^v c_k^* \tag{1.18}$$

in terms of the components alone.