Question 1

Each week, Pfizer vaccines for COVID-19 arrive by air to one of two airports in Toronto: Billy Bishop Toronto City Airport (100,000 doses) or Toronto Pearson Airport (250,000 doses). They are immediately transported to 7 hospital immunization clinics and 22 city run mass vaccination sites where the city can collectively administer exactly 50,000 vaccinations per day (7 days per week).

Airport \ Vaccination Sites	1-5	6-10	11-15	16-20	21-25	26-29
Billy Bishop Toronto City Airport	\$0.05	\$0.06	\$0.07	\$0.08	\$0.09	\$0.10
Toronto Pearson Airport	\$0.08	\$0.05	\$0.09	\$0.10	\$0.07	\$0.06

Table 1: Transportation costs per dose.

The cost per dose associated with transporting vaccines from each of the airports to each of the vaccination sites is given in Table 1. Each day, the seven hospital immunization clinics can administer four times as many vaccinations as compared to the city run vaccination clinics. To ensure a feasible transportation plan, the following restrictions must be adhered to:

1. The difference between the number of doses sent from either airport to sites 1-5 combined must be within 4,800 units of each other.
2. The number of doses sent from Toronto Pearson Airport to sites 21-25 combined must be less than or equal to eight times of the doses sent from Billy Bishop airport to sites 11-15 combined.
3. The number of doses sent from Billy Bishop airport to sites 26-29 combined must be greater than or equal to 80% of the doses sent from Toronto Pearson Airport to sites 16-20 combined.

Formulate and solve a linear program to determine how many doses of vaccine should be sent from the airports to each of the 29 vaccination locations to minimize transportation costs while adhering to all constraints. Then, answer the following 10 questions below

(a) How many vaccinations can a hospital administer per week? **28000**

(b) How many doses should be sent to each city run vaccination site? **7000**

(c) How many decision variables are in the formulation? **58**

(d) What type of constraint is the first restriction? **Absolute value constraint**

(e) Write down the constraint associated with ensuring 250,000 doses are transferred from Toronto Pearson Airport to the 29 vaccination sites. $\boldsymbol{\sum_{j=1}^{29} x_{2j} = 250000}$

(f) Write down the constraint associated with the third restriction. $\boldsymbol{\sum_{j=26}^{29} x_{1j} \geq 0.8 \sum_{j=16}^{20} x_{2j}}$

(g) What is the optimal transportation cost for the week? **\$24,828**

(h) How many doses are sent from Toronto Pearson Airport to vaccination site 3? **16,400**

(i) Are both of the ratio constraints binding or not binding? **Binding**

(j) What is preventing the solution from finding a lower cost transportation plan?

The ratio and absolute value constraints. First, they are binding constraints. Second, the constraints do not necessarily have to be binding as we require the demand constraints to be binding for feasibility. Third, they represent restrictions, that if changed, would result in a lower cost without changing the transportation network.

Linear Programming Formulation: Let x_{ij} represent the number of doses sent from the $i = 2$ airports to the $j = 1, \ldots, 29$ vaccination sites (the first seven are hospitals).

$$\text{minimize} \sum_{i=1}^{2} \sum_{j=1}^{7} c_j x_{ij} \qquad \text{subject to}$$

$$\sum_{j=1}^{29} x_{1j} = 100000 \tag{1}$$

$$\sum_{j=1}^{29} x_{2j} = 250000 \tag{2}$$

$$\sum_{i=1}^{2} x_{ij} = 7000 \qquad \text{for } j = 8, ..., 29 \tag{3}$$

$$\sum_{i=1}^{2} x_{ij} = 28000 \qquad \text{for } j = 1, ..., 7 \tag{4}$$

$$\sum_{j=1}^{5} x_{1j} - \sum_{j=1}^{5} x_{2j} \leq 4800 \tag{5}$$

$$\sum_{j=1}^{5} x_{2j} - \sum_{j=1}^{5} x_{1j} \leq 4800 \tag{6}$$

$$\sum_{j=21}^{25} x_{2j} \leq 8 \sum_{j=11}^{15} x_{1j} \tag{7}$$

$$\sum_{j=26}^{29} x_{1j} \geq 0.8 \sum_{j=16}^{20} x_{2j} \tag{8}$$

$$x_{ij} \geq 0, \qquad \text{for all } i, j \tag{9}$$

The objective is to minimize the cost of storing the vaccines. Constraints (1)-(2) represent the amount that can be transferred from the airports to all the vaccination sites. Constraints (3)-(4) is the amount we must transfer directly to the city run sites and hospitals from the airport to satisfy demand. Constraints (5)-(6) represent the absolute value constraints. Constraint (7)-(8) are the two ratio constraints. Finally, (9) represents the variable bounds.

Question 2

The S&P500 is a stock market index of the 500 leading companies publicly traded in the U.S. stock market. As part of your role at a robo-advisor investment startup, your task is to design an investment portfolio based on 67 companies listed on the S&P500. The goal is to maximize the expected 1-year return (see the spreadsheet *sp500_data.xlsx*) subject to a total investment of $10 million. The portfolio must satisfy a number of restrictions to ensure it is sufficiently diversified.

- At most, $600,000 can be invested in any individual stock (for diversification purposes).
- No more than $500,000 can be invested in the Telecommunications sector.
- The amount invested in the Information Technology (IT) sector must be at least 75% the amount invested in the Telecommunications sector.
- The absolute difference between the total invested in the Consumer Discretionary sector and the Consumer Staples sector should not exceed $200,000 (complementary trends).
- At least $1 million must be invest in the Energy sector and at least $300,000 must be invested in companies headquartered in New York, New York.

Formulate and solve a linear program to design an optimal investment portfolio that maximizes the total expected return and satisfies all restrictions. Then, answer the following 10 questions below.

(a) How many companies are headquartered in New York City? **6**

(b) How many decision variables are there? **67**

(c) What does a decision variable represent? Given this definition, write down the constraint associated with the restriction that, at most, $600,000 can be invested in any individual stock. **A decision variable represents how much money is invested in company $\boldsymbol{i = 1, ..., 67}$. Then, $\boldsymbol{x_i \leq 600000 \quad \forall i}$**

(d) How much is invested in companies headquartered in NYC? **$600,000**

(e) What is the optimal expected return of the portfolio after 1 year? **$513,460**

(f) After seeing your report, a colleague has asked if it would be worth it to inquire whether the amount invested in the Energy sector could be reduced. If the shadow price associated with the Energy sector constraint is -1.31%, what would your answer be?

The shadow price of the Energy sector constraint is negative but it's actually -2.14% **(I used slightly different numbers so you would have to actually formulate and solve questions 2a-2d). Regardless, if either** -2.14% **or** -1.31% **is used, any *decrease* in the right-hand side of that constraint would *increase* the objective function value, which represents the expected return of the portfolio. Yes, <u>reduce it!</u>**

(g) It was just announced that the return for Coca-Cola Enterprises is actually 3.00%. Without re-solving the problem, if the allowable increase/decrease for this objective function coefficient is 2.4%/-Infinity, would you change your portfolio if you had this new return information?

The original return of Coca-Cola Enterprises is 2.02% **while the new return is** 3.00%. **Thus, the objective function coefficient has increased by** $3.00\% - 2.02\% = 0.98\%$. **This is within the allowable increase for the objective function coefficient** ($4.42\% - 2.02\% = 2.40\%$ **in the context of this problem and** $4.5\% - 2.02\% = 2.48\%$ **using the numbers in the sensitivity report). No, the optimal portfolio would <u>not change</u>.**

(h) Due to the announcement of a new version of a popular smartphone, one of your colleagues has proposed increasing the maximum amount invested in the Telecommunications sector to \$625,000. If the shadow price associated with the Telecommunications sector constraint is 9.85% and the allowable increase/decrease is $\$100,000/\$42,857.14$, is this a good decision? Without re-solving the problem, would the announcement change the composition of the optimal portfolio?

Yes, it is worth it. The fake shadow price used for this question (9.85%) and the actual shadow price of the constraint (9.78%) are both positive. Thus, increasing the right-hand-side of the constraint will increase the optimal expected return. The change is beyond the allowable increase of the Telecommunications constraint ($\$625,000 > \$100,000$); the optimal allocation of funds will change.

(i) The bank is running a special promotion and will let you borrow, at most, \$50,000. This must be paid back in exactly one one year but the 1.5% interest rate is extremely favorable. Assume that all borrowed money is used for investing and you receive the estimated returns in exactly one year. If the shadow price of the total investment constraint is 4.32% with an allowable increase/decrease of $\$75,000/\$525,000$, how much would you be willing to borrow, if anything?

The allowable increase is higher than the maximum amount we can borrow ($\$50,000$). If we borrow all of it, our increase in profit would be

$$4.36\% \times 50000 - 1.5\% \times 50000 > 0(\textbf{real})$$
$$4.32\% \times 50000 - 1.5\% \times 50000 > 0(\textbf{fake})$$

In both cases, the return on the investment is higher than the interest rate. Therefore, it makes sense to borrow from the bank and use the promotion.

(j) A company in New York City has attracted your attention by stating that it will revolutionize the Energy sector. However, with the help of a colleague based in New York City, you have estimated its yield to be 6%. Does it make sense to invest in this new opportunity?

We need to compute the reduced cost of the new variable corresponding to this novel investment option. Notice that this variable will be added to the New York City constraint (shadow price of zero), the total amount constraint (shadow price of $4.32\%/4.36\%$), and the Energy sector constraint (shadow price of $-1.31\%/-2.14\%$):

$$\text{reduced cost} = 6 - (1 \times 0 + 1 \times 4.32\% - 1 \times 1.31\%) = 2.99\%(\textbf{real})$$
$$\text{reduced cost} = 6 - (1 \times 0 + 1 \times 4.36\% - 1 \times 2.14\%) = 3.78\%(\textbf{fake})$$

Since the reduced cost is positive, we will buy some stock in the company.

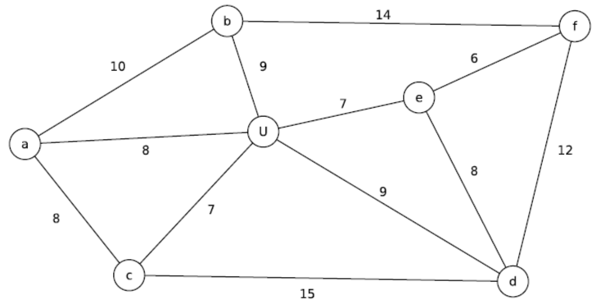
Linear Programming Formulation: Let x_i represent a decision variable that indicates how much money is invested in company i, for $i = 1, 2, \ldots, 67$. Let r_i be the return of stock i. Also, let T, I, D, S, and E be the set of indices corresponding to companies in the Telecommunications, IT, Consumer Discretionary, Consumer Staple, and Energy sectors, respectively. Finally, let NY be the indices of companies headquartered in New York city.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{67} (r_i\, x_i)/100 && \text{subject to} \\
& \sum_{i=1}^{67} x_i = 10000000 && \text{(Budget)} \\
& \sum_{i \in T} x_i \le 500000 && \text{(Telecommunications Restriction)} \\
& \sum_{i \in I} x_i \ge 0.75 \sum_{i \in T} x_i && \text{(IT and Telecom Restriction)} \\
& \sum_{i \in D} x_i - \sum_{i \in S} x_i \le 200000 && \text{(Discretionary/Staple 1)} \\
& \sum_{i \in S} x_i - \sum_{i \in D} x_i \le 200000 && \text{(Discretionary/Staple 2)} \\
& \sum_{i \in E} x_i \ge 1000000 && \text{(Energy Sector)} \\
& \sum_{i \in NY} x_i \ge 300000 && \text{(NYC)} \\
& x_i \le 600000, \quad \text{for } i = 1, 2, \ldots, 67 && \text{(Limit per stock)} \\
& x_i \ge 0 \;\; \text{for } i = 1, 2, \ldots, 67 && \text{(Non-negativity)}
\end{aligned}
$$

The objective is to maximize the expected return on the investments. The constraints represent the total budget and the various restrictions as outlined in the question.

Question 3

York University is planning to introduce a shuttle service between the Keele (U), Markham (f), and Glendon (a) campuses as well as a few other other stops. The shuttle can only travel directly from one stop to another if there is an arc between them in the network (see diagram below). All connections are two-way streets and arc labels represent the distance between stops (km).



Transportation analysts are considering 58 routes (see file *Routes.csv* for the list of routes and their maintenance cost per day) where the shuttle begins at the Keele campus (U), stops at several locations, and then returns back to the Keele campus (U). The following two factors must also be considered.

1. If the Glendon campus (a) is visited by more than one shuttle service, where the shuttles are serving different routes, an extra \$350 per day is incurred. Note that this cost is incurred for each *pair* of shuttles that visit the Glendon campus. For instance, if shuttle 1 and 2 visit the Glendon campus, an extra \$350 is incurred. If, instead, shuttles 1, 2, and 3 visit the campus, then there are three pairs (1 and 2, 1 and 3, 2 and 3). Thus, the total cost is $3 \times \$350 = \$1050$.

2. The student association will subsidize the project. That is, they will pay \$50 per day for each stop that is served by at least three routes (this does not include the Keele stop).

Formulate and solve a *linear* programming model with only binary decision variables to select a minimum-cost set of routes such that all stops are served by at least one route and the two restrictions discussed above are adhered to. Then, answer the following 10 questions:

(a) What route, among the 58 that are under consideration, has the largest cost? **35 or 51**

(b) If the cost per km is \$2.00, what is the smallest cost of a route that begins at the Keele campus (U), then visits the Glendon campus (a), the Markham campus (f) and then returns to the Keele campus (in that order)? Note that other stops may also be served in this route. **\$90**

(c) If no subsidy were to be provided by the student association, how many routes would be selected? **1 (there exists a single route that can serve all stops).**

(d) Write down the term in the objective associated with the student association subsidy.

$$-\sum_{j=1}^{6} 50z_i$$

(e) Write down the nonlinear term, associated with the Glendon campus (a) being visited by more than one shuttle service (i.e., the first restriction), *before* it is linearized.

$\boldsymbol{350x_ix_j}$ for all $\boldsymbol{i, j \in A}$ where $\boldsymbol{i > j}$.

(f) Write down constraints that linearize the model.

$$y_{ij} \leq x_i, y_{ij} \leq x_j, \text{ and } y_{ij} \geq x_i + x_j - 1, \quad \forall i, j \in A, i > j$$

(g) How many decision variables are needed in the linear program?

Depending on the constraints included in the model, at least 844 (58 + 6 + $\sum_{n=1}^{40} n$), most likely, 1664 (58 + 6 + 40 × 40), and at most 3428 (58 + 6 + 58 × 58).

(h) What is minimum cost of serving all stops and satisfying the restrictions? **108**

(i) In the optimal solution, how many routes are selected? **3**

(j) In the optimal solution, how many routes stop at the Glendon campus? **1**

Linear Programming Formulation: Define A, B, C, D, E and F as sets associated with routes that serve stops a, b, c, d, e and f, respectively. Let x_i be a binary variable that equals one if route $i = 1, ..., 58$ is selected and zero otherwise. Let y_{ij} be a binary variable if route i and route j both visit the Glendon campus, i.e., i and j are both in set A for $i \neq j$, and zero otherwise. We also define z_k to be a binary variable that equals one if at least three routes serve stop $k = 1, ..., 6$ and zero otherwise. Finally, let W_i be the cost of selecting route $i = 1, ..., 58$.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{58} W_i x_i + \sum_{i,j,\in A, i>j} 350 y_{ij} - \sum_{j=1}^{6} 50 z_i && \text{subject to} \\
& \sum_{i \in A} x_i \geq 1 + 2z_1 && \text{(stop a)} \\
& \sum_{i \in B} x_i \geq 1 + 2z_2 && \text{(stop b)} \\
& \sum_{i \in C} x_i \geq 1 + 2z_3 && \text{(stop c)} \\
& \sum_{i \in D} x_i \geq 1 + 2z_4 && \text{(stop d)} \\
& \sum_{i \in E} x_i \geq 1 + 2z_5 && \text{(stop e)} \\
& \sum_{i \in F} x_i \geq 1 + 2z_6 && \text{(stop f)} \\
& y_{ij} \leq x_i && \forall i, j \in A, i > j \\
& y_{ij} \leq x_j && \forall i, j \in A, i > j \\
& y_{ij} \geq x_i + x_j - 1 && \forall i, j \in A, i > j \\
& y_{ij} = 0 && \forall i, j \in A, i \leq j \\
& x_i \in \{0, 1\} && \text{for } i = 1, \ldots, 58 \\
& y_{ij} \in \{0, 1\} && \text{for } i \neq j, i, j \in A \\
& z_k \in \{0, 1\} && \text{for } k = 1, \ldots, 6
\end{aligned}
$$

Note that, in order to ensure we don't double count pairs of shuttles, we constrain $y_{ij} = 0$ for all $i, j \in A$ where $i \leq j$ (*note:* there are multiple correct ways to do this). This has two effects:

1. For all $i \leq j$, it ensures $y_{ij} = 0$, and as a result, only one of x_i and x_j can equal to one.
2. For all $i > j$, $y_{ij} \geq 0$ which means that both x_i and x_j can equal to one.

Nevertheless, even if you iterate over the set A incorrectly, you will still get the same answer although the optimal objective function value may differ depending on whether the objective function is correct.

Question 4

OptiDiet, a prominent corporate wellness consultancy, has been commissioned to develop an extensive diet program for a large multinational corporation. The initiative aims to offer a nutritious, budget-friendly, and varied diet plan, addressing the diverse dietary preferences and requirements of the corporation's employees. OptiDiet has access to a database of 120 distinct food items, each characterized by its nutritional value and cost. The consultancy also tracks 60 different nutrient categories to ensure a comprehensive nutritional profile. Their objective is to devise a low-cost diet plan that fulfills the nutritional needs of the corporations employees while adhering to the following constraints:

1. **Nutritional Balance:** Each nutrient category has minimum and maximum weekly consumption requirements (refer to *nutrient_requirements.csv*). These constraints are designed to ensure that the selection and quantity of foods provided over the week maintains nutritional balance and prevent excessive intake of any individual nutrient. The file *nutrient_content.csv* contains the per gram contribution of each food item to the various nutrient requirements.

2. **Dietary Preferences:** These constraints determine the total quantity (grams) of each food item that can be incorporated into the diet to accommodate everyone (All), Vegetarian, Vegan, Kosher, and Halal options. Details regarding the demand for each food category and their dietary classifications can be found in the *food_preferences.csv* and *food_categories.csv* files.

3. **Variety:** Make sure that the proportion of each food item used per week is less than 3%.

Formulate a linear program to determine a cost-effective production strategy (refer to the cost information in *food_categories.csv*) that determines how much of each food (in grams) to include in the diet plan over the upcoming week and such that all of the above requirements are met.

(a) What is the minimum number and type of the decision variables used to model this problem? **(120, continuous)**

(b) How many constraints are in the linear program not including non-negativity? What Gurobi method can be used to determine this? **(245, numConstrs)**

(c) Suppose that V_K represents the minimum amount of food item K that must be incorporated into the diet plan where $K \in$ {Vegetarian, Vegan, Kosher, Halal, All}. Write down the constraint that ensures the weekly diet plan contains at least V_K grams for every K. $\boldsymbol{\sum_{i \in K} x_i \geq V_K}$

(d) Write down the *Variety* constraint(s). $\boldsymbol{x_i \leq 0.03 \sum_{k=1}^{120} x_k \qquad i}$

(e) Formulate and solve the linear program. What is the optimal food production cost? **$54,312**

(f) In the optimal solution, what proportion of grams are from Halal and Kosher foods? **36%**

(g) Suppose we omitted the *Variety* constraint(s). How many fewer food items would be produced? In addition, how much higher or lower is the production cost as compared to the original model?

$38 - 12 = 26$. The cost is approximately $10,000 lower or $43,539.

(h) What does your analysis from (g) suggest about the significance of the *Variety* constraint(s) in encouraging OptiDiet to create a diverse range of meals? Why is this the case?

Very important. The reason is that we must meet the nutritional requirements of the program. We can do this in two ways. We can either choose a few low-cost items and exploit their nutritious properties to provide the necessary nutritional requirements. Alternatively, we can choose a breadth of items.

(i) Unfortunately, it turns out that the right-hand-side value for the dietary preference constraints were underestimated by $10,000$ grams each. If we were to increase the right-hand-side values of any of these constraints by this amount, how would it impact the objective function?

Nothing would happen. Their shadow prices are all zero and the allowable increase is larger than $10,000$ for every constraint. Thus, nothing would change.

(j) Currently, the first food item is not being included in the diet plan. How much less costly would it need to be for it to be included in the optimal solution?

Look at the reduced cost. It is $\$0.0396$ which means that if the coefficient were less costly by at least $\$0.0396$, the first food item would be included in the diet plan.

Let x_i be the quantity (in grams) of the i^{th} food item included in the diet plan, for $i = 1, 2, \ldots, 120$. Note that x_i is a continuous and non-negative variable. Then, the objective is to

$$\text{Minimize} \quad Z = \sum_{i=1}^{120} c_i x_i$$

where c_i is the cost per gram of the i^{th} food item. The nutrient constraints are

$$\sum_{i=1}^{120} n_{ij} x_i \geq R_j^{\min}, \quad \forall j$$
$$\sum_{i=1}^{120} n_{ij} x_i \leq R_j^{\max}, \quad \forall j$$

where n_{ij} is the amount of nutrient j in one gram of food item i, and $R_j^{\min}, R_j^{\max}$ are the minimum and maximum required amounts of nutrient j, respectively (all data from files). Further, we have

$$\sum_{i \in K} x_i \geq V_K \qquad \forall K \in \{\text{Vegetarian, Vegan, Kosher, Halal, All}\}$$

where V_K is the minimum total quantity of food items in set K. Finally, the variety constraint is

$$x_i \leq 0.03 \sum_{k=1}^{120} x_k \qquad \forall i$$

All decision variables must non-negative and continuous,

$$x_i \geq 0, \quad \forall i.$$

Question 5

In the non-profit industry, large funding organizations provide resources to multiple nonprofits in the form of grants, with the expectation of achieving certain outputs. For instance, an output could be the number of mosquito nets distributed to a particular demographic. Both funders and nonprofits aim to achieve good outputs, however, they also have different objectives.

Consider a large funder (e.g., Bill and Melinda Gates Foundation) and N nonprofits all vying to be allocated money. Let $a_i \geq 0$ be the amount of money given to nonprofit $i = 1, ..., N$. Given an allocation a_i, nonprofit i allocates their efforts e_i so as to maximize the utility function

$$u_i(e_i \,|\, a_i) = \alpha_i a_i - \frac{1}{2}e_i^2 + 2\sqrt{e_i \beta_i a_i},$$

where $\alpha_i \geq 0$ indicates the nonprofit's preference for how they value allocation versus demonstrating output and $\beta_i \geq 0$ represents the efficiency of nonprofit i reflecting how effectively they convert allocations into outputs; these values are provided in *non_profits.csv*. Lastly, the term $2\sqrt{e_i \beta_i a_i}$ represents the nonprofit's *output function*, which funders exclusively use when making allocation decisions.

Imagine you are a fund manager working for the Bill and Melinda Gates Foundation with a budget of \$50 million. Your responsibility is to determine an allocation policy that maximizes the total output by determining how much money to invest into each of the N nonprofits. To do this, please follow the steps below to formulate and solve a nonlinear optimization problem.

(a) To determine the effort level e_i^* for nonprofit i that maximizes its utility function given an allocation a_i, why do you only need the *stationarity* requirement from the KKT conditions?

There are no constraints in this problem except for non-negativity. However, if you observe that the optimal effort level is negative, you adjust the effort level to zero.

(b) Using the utility function provided above, show that the closed-form expression for the optimal effort e_i^* of nonprofit i, provided they are assigned allocation a_i, is equal to $e_i^* = (\beta_i a_i)^{\frac{1}{3}}$.

Take the derivative of the utility function $u_i(e_i \,|\, a_i)$, as given above, with respect to e_i. Then, set this derivative to zero, and solve for e_i.

(c) Given that $e_i^* = (\beta_i a_i)^{\frac{1}{3}}$, what is the expression for nonprofit i's optimal *output function*?

$\boldsymbol{2\beta_i^{2/3} a_i^{2/3}}$

(d) Due to the strong nonlinear nature of the optimal *output function* (i.e., a_i is being raised to a fractional exponent), this problem requires the implementation of a specific function within Gurobi, namely addGenConstrPow(). How do we assign the decision variable x_i to approximate the term with a fractional exponent using the addGenConstrPow() function with 1500 pieces.

model.addGenConstrPow(a[i], x[i], 2.0/3.0, "Allocation", "FuncPieces=1500")

(e) From the perspective of representatives of the Bill and Melinda Gates Foundation, formulate an optimization model to determine the allocation strategy that maximizes output using the addGenConstrPow() function discussed above. How many decision variables and constraints are needed to faithfully model the problem? **(180*2 = 360, 1)**

(f) What output value can the funding organization expect in the optimal allocation strategy? **\$22,356,203**

(g) How much of the budget is allocated in an optimal solution? Why is this the case?

All of it. We are maximizing a non-negative utility function with no other constraints except for the budget restriction.

(h) How many of the 180 nonprofits receive no funding at all? **Only 2.**

(i) As noted in (h), the optimal solution results in few nonprofits receiving no funding at all. This happens naturally within the problem, and there is no need to introduce specific constraints to increase funding diversity. What property of the problem leads to this outcome?

The objective function is the sum of concave increasing functions which means that there is a diminishing marginal utility to donating to any single nonprofit. Thus, as the allocation to a nonprofit increases, the additional benefit/output gained by the funder decreases. The result is that the optimal allocation strategy naturally spreads the available resources across a large number of nonprofits.

Let $a_i \geq 0$ be the amount allocation to nonprofit i for $i = 1, ..., N$ where $N = 180$.

$$\text{Maximize} \quad Z = \sum_{i=1}^{N} 2\beta_i^{2/3} a_i^{2/3} \text{ subject to } \sum_{i=1}^{N} a_i \leq 50,000,000$$

Question 6

Consider an Intensive Care Unit (ICU) in a large hospital comprising of 26 nurses. The staff is categorized into three groups: Senior Registered Nurses (SRN), Registered Nurses (RN), and Nurses in-training (NIT). The ICU operates 24/7, necessitating continuous staffing in two 12-hour shifts per day. A day shift is from 7 am to 7 pm and a night shift from 7 pm to 7 am. Therefore, there are $j = 1, \ldots, 14$ possible shifts per week, with odd numbers designating day shifts and even numbers designating night shifts: $j = 1$ denotes the Monday day shift, $j = 2$ denotes the Monday night shift, $\ldots$, $j = 13$ denotes the Sunday day shift, $j = 14$ denotes the Sunday night shift.

The scheduling costs for each nurse varies between weekdays and weekends (see *nurse_shift_costs.csv*). For the upcoming week, you are tasked with creating and solving a binary program that allocates **nurses** to **shifts** in a manner that **minimizes** overall costs and adheres to the following constraints:

- Each shift must be staffed with at least 6 nurses.
- Each nurse can work between 36 and 60 hours per week.
- Each of the 14 weekly shifts must include at least one senior registered nurse (SRN).
- Nurses cannot be scheduled for back-to-back shifts to ensure adequate rest periods.

Additionally, there is an incremental overtime cost (see *nurse_shift_costs.csv*) that is incurred for **each scheduled shift** once the total number of working hours for any nurse exceeds 36. Specifically, for every shift that a nurse works in excess of 36, an additional overtime cost of *Cost_Overtime* is incurred.

(a) There are three classes of binary variables to include in the model. Explain what each represents.

The first is x_{ns}, which is a binary variable that equals one if nurse n is assigned to shift s and zero otherwise. Then, y_n and z_n indicate whether nurse n has worked a fourth or fifth shift, respectively (or not), because nurses cannot work more than five shifts per week as indicated in the question. Finally, note that there are other correct formulations that I also accepted for this problem.

(b) What are the three types of costs that must be included in the objective function?

Cost of weekday shifts, weekend shifts, and overtime.

(c) How many decision variables are necessary to include in the model ? $\mathbf{26 + 26 + 26 \times 14 = 416}$

(d) Write down the constraints that ensure each nurse works between 36 and 60 hours per week. $\boldsymbol{36 \leq 12 \sum_{s=1}^{14} x_{ns} \leq 60, \quad \forall n \in \{1, \ldots, 26\}}$

(e) Write down the following constraint(s): *Nurses cannot be scheduled for back-to-back shifts.*

$$\boldsymbol{x_{ns} + x_{n(s+1)} \leq 1, \qquad \forall n, s}$$

(f) To track the total number of overtime shifts and the associated cost in the objective function, we must track how many overtime shifts each nurse works. Write down these constraints.

$$\boldsymbol{\sum_{s=1}^{14} x_{ns} \leq 3 + y_n + z_n, \quad \forall n}$$

(g) What is the optimal objective function with the minimum cost? **\$28,862**

(h) In the optimal schedule, how many overtime shifts are needed? **6**

(i) Given that the task is to create a scheduling model for the upcoming week only, which specific types of back-to-back shift constraints can potentially be violated if this type of scheduling approach becomes standard and is to be utilized at the start of each week?

Sunday night of the current week and Monday morning of the next week.

(j) Surprisingly, quantitative approaches like this are seldom implemented in practical settings. Can you think of a barrier that would prevent the adoption of such a tool in a real-world application?

One such barrier includes dynamic preference management. Each week, nurses may have different preferences as to which shifts they will work (e.g., vacation). Further, management may have different requirements based on the patients in the ICU.

Decision Variables

Let x_{ns} be a binary variable that equals one if nurse n is assigned to shift s for $n = 1, ..., 26$ and $s = 1, ..., 14$ and zero otherwise. We also define y_n and z_n to indicate whether nurse n has worked a fourth or fifth shift, respectively. The objective is to minimize the total scheduling costs:

$$\text{Minimize} \sum_{n=1}^{26} \sum_{s \in \text{weekday\_shifts}} \text{Cost\_Weekday}_n x_{ns} + \sum_{n=1}^{26} \sum_{s \in \text{weekend\_shifts}} \text{Cost\_Weekend}_n x_{ns}$$
$$+ \sum_{n=1}^{26} \text{Cost\_Overtime}_n (y_n + z_n)$$

Staffing Constraints

Each shift must be staffed with at least 6 nurses. In addition, each of the 14 weekly shifts must include at least one Senior Registered Nurse (SRN) where srn_n is a binary indicator that equals one if a nurse is an SRN; it can be defined using the data set that was provided.

$$\sum_{n=1}^{26} x_{ns} \geq 6, \quad \forall s \in \{1, \ldots, 14\}$$

$$\sum_{n=1}^{26} x_{ns} \times \text{srn}_n \geq 1, \quad \forall s \in \{1, \ldots, 14\}$$

Working Hours Constraints

Each nurse works between 36 and 60 hours per week (or between 3 and 5 shifts):

$$36 \leq 12 \sum_{s=1}^{14} x_{ns} \leq 60, \quad \forall n \in \{1, \ldots, 26\}$$

Back-to-Back Shifts Constraints

Nurses cannot be scheduled for back-to-back shifts:

$$x_{ns} + x_{n(s+1)} \leq 1, \quad \forall n \in \{1, \ldots, 26\}, s \in \{1, \ldots, 13\}$$

Overtime Constraints

We need to keep track of the overtime shifts that are worked

$$\sum_{s=1}^{14} x_{ns} \leq 3 + y_n + z_n, \quad \forall n \in \{1, \ldots, 26\}.$$

Question 7

Sunnyshore Bay is a charming, privately-owned waterpark located near the serene Blue Lake, operating from May to September. Given the seasonal nature of the business, several significant expenses must be covered at the start of the waterpark season, ahead of achieving a stable cash flow. Over the four months of the season, Sunnyshore Bay expects the following revenues and expenses.

	May	June	July	August
Revenues	\$180,000	\$260,000	\$420,000	\$580,000
Expenses	\$300,000	\$400,000	\$350,000	\$200,000

Sunnyshore Bay starts the season with an initial cash balance of \$140,000. Additionally, the company can opt to borrow money from a local bank at the following rates and term structures:

	1-month	2-months	3-months
Interest Rate	1.75%	2.25%	2.75%

Money is borrowed at the end of a month and repaid, with interest, at the end of the month in which the obligation is due. For example, if the company borrows \$10,000 at a 3-month rate in May, they must pay back \$10,275 at the end of August (the rate is not annualized). Sunnyshore Bay is not allowed to borrow money that cannot be paid back in this period (e.g., money cannot be borrowed at the 3-month rate in July). In addition, the following financial restrictions must be satisfied.

- The company must maintain a cash balance of at least \$25,000 in May, \$20,000 in June, \$35,000 in July, and \$18,000 in August, which is verified at the end of each month.
- The total amount borrowed in a month, i.e. taking into account any term/rate structures for that month, cannot exceed \$250,000 in May, \$150,000 in June, and \$350,000 in July.
- The cash balance at the end of July (after accounting for all loans) must be at least 65% of the combined total cash balances from May and June.

Formulate and solve a linear program that minimizes the total amount that Sunnyshore Bay has to repay to the bank over the summer months. Answer the following questions along the way:

(a) How many different investments can be made over the 4-month period? **6**

(b) Write down the cash balance constraint for money on-hand at the end of June.

$$\boldsymbol{z_2 = z_1 + b_{23} + b_{24} - 1.0175 b_{12} - 140}$$

(c) Write down the linear ratio constraint associated with the cash balance at the end of July. $\boldsymbol{z_3 \geq 0.65(z_1 + z_2)}$

(d) What is the total amount that Sunnyshore Bay has to repay to the bank over the entire season? **\$142,904.73**

(e) How much money does Sunnyshore Bay withdraw in May from all loans? **\$5000**

(f) What is the cash balance at the end of August? **\$327,095.27**

(g) Due to potential unexpected repairs, one of the managers has suggested increasing the minimum cash balance for June to \$27,500. How much will now have to be repaid if this change is approved?

This will increase the right-hand side of the financial restriction constraint in June by \$7,500 to \$27,500 (before it was \$20,000). Since this is within the allowable increase (\$8,846) and the shadow price of this constraint is 1.017585, the total is

$$\$142,904.73 + \$7,500 \times 1.01785 = \$150,536.62$$

The objective function value increases (an extra \$7631.89 in interest must be paid) because we will end up borrowing more money to ensure we have the cash balance.

Linear Program: Let b_{ij} be the amount to borrow from month i to be paid back in month j, for $i, j = 1, 2, 3, 4$. In addition, let z_i be cash balance at the end of month i, for $i = 1, 2, 3, 4$. Then,

$$
\begin{array}{lll}
\text{Minimize} & 1.0175(b_{12} + b_{23} + b_{34}) + 1.0225(b_{13} + b_{24}) + 1.0275b_{14} & \\
& z_1 = 140 + 180 + b_{12} + b_{13} + b_{14} - 300 & \text{(Balance - May)} \\
& z_2 = z_1 + 260 + b_{23} + b_{24} - 1.0175b_{12} - 400 & \text{(Balance - June)} \\
& z_3 = z_2 + 420 + b_{34} - 1.0175b_{23} - 1.0225b_{13} - 350 & \text{(Balance - July)} \\
& z_4 = z_3 + 580 - 1.0175b_{34} - 1.0225b_{24} - 1.0275b_{14} - 200 & \text{(Balance - August)} \\
& z_1 \geq 25 & \text{(Restriction A - May)} \\
& z_2 \geq 20 & \text{(Restriction A - June)} \\
& z_3 \geq 35 & \text{(Restriction A - July)} \\
& z_4 \geq 18 & \text{(Restriction A - August)} \\
& b_{12} + b_{13} + b_{14} \leq 250 & \text{(Restriction B - May)} \\
& b_{23} + b_{24} \leq 150 & \text{(Restriction B - June)} \\
& b_{34} \leq 350 & \text{(Restriction B - July)} \\
& z_3 \geq 0.65(z_1 + z_2) & \text{(Restriction C)} \\
& z_i \geq 0, b_{ij} \geq 0 & \forall i, j.
\end{array}
$$