Question 1 [8 points]

GadgetMarket Inc., a well-known retailer in consumer electronics, specializes in two flagship products: the TechFit Smartwatch and the PowerSound Earbuds. As fall and winter approach, GadgetMarket Inc. anticipates varying demand due to seasonality, with special discounts planned for the week of Black Friday (week 12) and the week following Christmas (Boxing Day), i.e., week 16. They are looking to maximize revenue by dynamically adjusting prices for both products each week from September 1 to December 31. GadgetMarket Inc., has decided on the following weekly price dynamics.

Table 1: Price Requirements by Week

Weeks	Price Constraints
1-4	Static pricing for each product throughout this period.
5-8	Static pricing for each product during this period, with prices at least \$10 lower than in Weeks 1-4.
9-11	Static pricing for each product during this period, with prices at least \$20 higher than in Weeks 1-4.
12	Prices for each product should be the lowest of any week, by at least \$5.
13-15	Static pricing for each product during this period, with prices between those in Weeks 1-4 and Weeks 5-8.
16	Prices for each product should be at least \$4 higher than in Black Friday week and at least \$6 lower than any other week.
17	Prices for each product should be the highest of any week, by at least \$15.

Each product has a linear relationship between price and demand, and also accounts for the effect of the other product's price due to their complementarity. For instance, a change in the price of the TechFit Smartwatch may impact not only its own demand, but also the demand for the PowerSound Earbuds. The price-demand relationships for each product are given in *price_response.csv*.

(a) First, consider the TechFit Smartwatch with the price response functions provided in Table 2 for the first two weeks (assume no cross-elasticity). Using the KKT conditions, derive the optimal prices, assuming they are non-negative but otherwise have no restrictions.

Table 2: Price Response Functions for TechFit Smartwatch With No Cross-Elasticity.

Week	Intercept	Own-Price Coefficient
1	1000	-5
2	950	-4.5

(b) Consider again the TechFit Smartwatch with the same price response functions as in the previous question for the first two weeks (assume no cross-elasticity). Using the KKT conditions, derive the optimal prices assuming they are non-negative but must remain the same across both weeks.

(c) What do you observe about the optimal prices derived using the KKT conditions with and without the equality constraint? Based on our class discussion from the *Variable Pricing with Diversion* example, why does this occur? What is the benefit of dynamic pricing?

(d) Now consider both products. Using the price response functions in the *price_response.csv* file for the **first two weeks only**, determine the optimal prices using the projected gradient descent algorithm. For each product, assume static pricing across both weeks. Initialize all prices at zero, with a step size of 0.001 and a stopping criterion of 10^{-6}. What are the optimal prices?

(e) For the full model, why is this optimization problem considered to be a nonlinear program? Discuss why no linear reformulations of the problem are possible.

(f) Implement and solve the full model across all 17 weeks using Gurobi. Assume that dynamic pricing is allowed. What is the optimal revenue over the 17-week period?

(g) Generate a plot showing the price dynamics for each product over the 17-week period on the same graph. Briefly comment on whether you think GadgetMarket Inc. and its customers would find these price trends favorable. As a customer, is there anything you would find concerning?

(h) Benchmarking is useful for contextualization. Using Gurobi, what would the optimal revenue be for a dynamic pricing strategy without any price constraints except to ensure prices and demand are non-negative? Alternatively, what would the optimal revenue be if prices were constrained to be the same for all 17 weeks?

(i) Compare the benefits and drawbacks of (i) the static pricing model, (ii) the unconstrained dynamic pricing model, and (iii) the constrained dynamic pricing model.

Question 2 [8 points]

The city of Los Angeles recently enacted a new hotel worker protection ordinance, impacting workload limits for hotel attendants in the region. Based on this ordinance, the Beverley Hills Four Seasons is updating its scheduling process to avoid incurring penalties. The new regulations stipulate that:

1. All attendants receive a full 8 hours pay even if they work less then this. They are also allowed to work a maximum of 2 hours of overtime (in hourly increments) at 1.5 times their hourly wage.
2. An attendant may clean between 2 and 4 floors per day. Attendants who clean more than two floors (i.e., three of four) receive an additional \$75 per floor.
3. Each attendant can clean 3500 square feet of rooms per day. If they exceed this value, they double their regular hourly wage (this does not include overtime hours).

Data on rooms, including square footage, floor location, and the cleaning time (hours), has been provided in the file *hotels.csv*. Suppose there are 8 attendants working today, each earning an hourly wage of \$25. Formulate and solve a binary program to minimize the hotel's staffing cost while ensuring all rooms are cleaned and the new ordinance requirements are accounted for.

(a) Can you explain how the costs associated with the first two regulations can be correctly tracked using only binary variables? That is, describe how the overtime constraints should work.

(b) You will also need a binary variable, f_{ik} to capture whether attendant i has been assigned to floor k. This helps to determine how many floors attendant i has been assigned to. Write down the constraint such that if attendant i is assigned to at least one room on floor k, then $f_{ik} = 1$.

(c) What type of constraint is the third regulation and why?

(d) Considering the cost of violating the regulations, in what order do you think penalties would be incurred if they become necessary? How would this ordering change if attendants received double their regular hourly wage (instead of time and a half) for overtime hours worked?

(e) Use Gurobi to solve the binary program. What is the optimal cost, and how many total overtime hours and floor violations (in excess of two) occur across all attendants?

(f) You will probably notice that Gurobi does not immediately provide the optimal solution when solving the model. Instead, it goes through numerous branch-and-bound iterations. While there are ways to enhance performance, one approach is to solve an approximation of the problem by relaxing the solution. Using Gurobi's model.relax() procedure, what happens when you do this?

(g) Instead of using model.relax(), manually create a linear relaxation by converting the decision variables from binary to continuous with the appropriate bounds. What is the optimal cost? Compare this to the optimal solution of the binary program. What do your findings imply about using the solution of relaxed model as an approximation to the binary program?

(h) In the previous two questions, you explored two different methods for relaxing an integer (binary) programming model. Explain why these approaches yield different results?

(i) Now assume that attendants receive 2× their regular hourly wage (instead of time and a half) for the number of overtime hours worked. What is the optimal cost, and how many total overtime hours and floor violations (in excess of two) occur across all attendants? Compare this result to the optimal solution of the binary program in part (e). Was your intuition in part (d) correct?