OMIS 6000

Week 7:

- **Mixed Integer-Linear Programs (MILP)**

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Models with Multiple Variable Types

- In many managerial situations, you need to make several types of decisions. For example:
  1. Should you produce a particular product?
  2. If so, how much of that product should you produce?
- Deciding to produce a product typically incurs a ***fixed-cost***. Then, there is an additional per-unit (***variable***) cost for every unit that is produced.
- In this case, what two types of decision variables should we include in an optimization problem?

MILP Models

- Problems with multiple types of decision variables are called **Mixed Integer-Linear Programs** or **MILP** models. These problems have some combination of integer, binary, and continuous variables.
- **Theorem:** The computational complexity of MILP models are no harder to solve then an integer or binary programs.

Why?

MILP Models

Solving MILP models (Dakin's Algorithm):

- The **Simplex** method is first used to find an optimal solution to the **LP relaxation**.
- The **Branch-and-Bound** method is applied only to variables with integer restrictions:
  - If the **LP relaxation** assigns integer values to all integer decision variables, the solution is feasible and represents the best solution that can be obtained (how lucky are we!).
  - Solvers apply additional techniques that improve computational efficiency (choosing what decision variable to branch on, what branching value to use, etc.).

Mixed Integer-Linear Programs (MILPs)

In-house vs. Outsourcing

Condominium Construction

Production and Transportation

Tree Planting Program

In-house vs. Outsourcing Manufacturing

Gooseberry Electronics produces a variety of electrical components. Their manufacturing facility, in Waterloo, Ontario, focuses on producing two products: *a drone controller* and an *interface for smartphones*.

In-house vs. Outsourcing Manufacturing

Each product consists of 3 subassembly units: a base, a display, and a keypad. Both the *drone controller* and *smartphone interface* use the same base but different displays and keypads. Gooseberry forecasts that exactly 7000 *drone controllers* and 5000 *smartphone interfaces* will be needed to satisfy demand during the upcoming Christmas season. There are 500 hours of in-house manufacturing time available although Gooseberry does have the option of purchasing some, or all, of the subassembly units from outside suppliers. If Gooseberry does manufacture a subassembly unit in-house, it incurs a fixed setup cost as well as a per unit manufacturing cost. The following table (next slide) shows the setup cost, the manufacturing time per subassembly unit, and the cost to purchase each subassembly unit from an outside supplier.

In-house vs. Outsourcing Manufacturing

Subassembly	Setup Cost	Manufacturing Time (min per unit)	Manufacturing Cost (per unit)	Purchase Cost (per unit)
Base	\$1000	0.9	\$0.40	\$0.65
Drone Display	\$1200	2.2	\$2.90	\$3.45
Smartphone Display	\$1900	3.0	\$3.15	\$3.70
Drone Keypad	\$1500	0.8	\$0.30	\$0.50
Smartphone Keypad	\$1500	1.0	\$0.55	\$0.70

In-house vs. Outsourcing Manufacturing

How many of each subassembly unit should Gooseberry manufacture and/or purchase to minimize the total cost of supply?

In-house vs. Outsourcing Manufacturing

- What kinds of decisions must be made?
  1. Should Gooseberry manufacture the ***base***? If so, how many units should they produce?
  2. Should Gooseberry manufacture the ***display*** for the drone? If so, how many units should they produce?
  3. Should Gooseberry manufacture the ***display*** for the smartphone? If so, how many units should they produce?
  4. Should Gooseberry manufacture the ***keypad*** for the drone? If so, how many units should they produce?
  5. Should Gooseberry manufacture the ***keypad*** for the smartphone? If so, how many units should they produce?

In-house vs. Outsourcing Manufacturing

Define the objective

Define the decision variables

In-house vs. Outsourcing Manufacturing

Define the objective

Minimize the cost of supply

Define the decision variables

In-house vs. Outsourcing Manufacturing

Define the objective

Minimize the cost of supply

Define the decision variables

$$y_i = \begin{cases} 1, \textit{ if Gooseberry decides to manufacture subassembly unit } i \textit{ where } i = \{\textit{base, drone display, smartphone display, drone keypad, smartphone keypad}\} \\ 0, \textit{ otherwise} \end{cases}$$

In-house vs. Outsourcing Manufacturing

Define the objective

Minimize the cost of supply

Define the decision variables

$x_i =$ ***the amount that Gooseberry <u>manufactures</u> of subassembly unit i where $i =$ {base, drone display, smartphone display, drone keypad, smartphone keypad}***

In-house vs. Outsourcing Manufacturing

Define the objective

Minimize the cost of supply

Define the decision variables

$z_i =$ ***the amount that Gooseberry <u>purchases</u> of subassembly unit i where $i =$ {base, drone display, smartphone display, drone keypad, smartphone keypad}***

In-house vs. Outsourcing Manufacturing

Write the mathematical objective function

Minimize $Z =$

(What are the fixed costs?)

Subassembly	Setup Cost	Manufacturing Time (min per unit)	Manufacturing Cost (per unit)	Purchase Cost (per unit)
Base	$1000	0.9	$0.40	$0.65
Drone Display	$1200	2.2	$2.90	$3.45
Smartphone Display	$1900	3.0	$3.15	$3.70
Drone Keypad	$1500	0.8	$0.30	$0.50
Smartphone Keypad	$1500	1.0	$0.55	$0.70

In-house vs. Outsourcing Manufacturing

Write the mathematical objective function

Minimize $Z = 1000y_1 + 1200y_2 + 1900y_3 + 1500y_4 + 1500y_5$

(What are the variable costs?)

Subassembly	Setup Cost	Manufacturing Time (min per unit)	Manufacturing Cost (per unit)	Purchase Cost (per unit)
Base	$1000	0.9	$0.40	$0.65
Drone Display	$1200	2.2	$2.90	$3.45
Smartphone Display	$1900	3.0	$3.15	$3.70
Drone Keypad	$1500	0.8	$0.30	$0.50
Smartphone Keypad	$1500	1.0	$0.55	$0.70

In-house vs. Outsourcing Manufacturing

Write the mathematical objective function

Minimize $$Z = 1000y_1 + 1200y_2 + 1900y_3 + 1500y_4 + 1500y_5 + 0.40x_1 + 2.90x_2 + 3.15x_3 + 0.30x_4 + 0.55x_5 + 0.65z_1 + 3.45z_2 + 3.70z_3 + 0.50z_4 + 0.70z_5$$

Subassembly	Setup Cost	Manufacturing Time (min per unit)	Manufacturing Cost (per unit)	Purchase Cost (per unit)
Base	$1000	0.9	$0.40	$0.65
Drone Display	$1200	2.2	$2.90	$3.45
Smartphone Display	$1900	3.0	$3.15	$3.70
Drone Keypad	$1500	0.8	$0.30	$0.50
Smartphone Keypad	$1500	1.0	$0.55	$0.70

In-house vs. Outsourcing Manufacturing

Formulating the constraints

There are four types of constraints:

1. Demand constraints
2. Capacity constraints
3. **Linking constraints**
4. Non-negativity and binary constraints

In-house vs. Outsourcing Manufacturing

Formulating the demand constraints

Gooseberry forecasts that exactly 7000 drone controllers and 5000 smartphone interfaces will be needed to satisfy demand during the upcoming Christmas season.

To *exactly* achieve these demand targets, remember that Gooseberry can either manufacture the subassembly units themselves or outsource this capability by purchasing them from its suppliers.

In-house vs. Outsourcing Manufacturing

Formulating the demand constraints

Gooseberry forecasts that exactly 7000 drone controllers and 5000 smartphone interfaces will be needed to satisfy demand during the upcoming Christmas season.

(total demand for bases)

(demand for drone displays)

(demand for smartphone displays)

(demand for drone keypads)

(demand for smartphone keypads)

In-house vs. Outsourcing Manufacturing

Formulating the demand constraints

Gooseberry forecasts that exactly 7000 drone controllers and 5000 smartphone interfaces will be needed to satisfy demand during the upcoming Christmas season.

$x_1 + z_1 = 12000$ **(total demand for bases)**

$x_2 + z_2 = 7000$ **(demand for drone displays)**

$x_3 + z_3 = 5000$ **(demand for smartphone displays)**

$x_4 + z_4 = 7000$ **(demand for drone keypads)**

$x_5 + z_5 = 5000$ **(demand for smartphone keypads)**

In-house vs. Outsourcing Manufacturing

Formulating the capacity constraints

There are 500 hours of in-house manufacturing time.

*where 30000 minutes = 500 hours * 60 minutes / hours.*

In-house vs. Outsourcing Manufacturing

Formulating the capacity constraints

There are 500 hours of in-house manufacturing time.

$$\mathbf{0.9x_1 + 2.2x_2 + 3x_3 + 0.8x_4 + x_5 \leq 30000}$$

*where 30000 minutes = 500 hours * 60 minutes / hours.*

In-house vs. Outsourcing Manufacturing

Formulating the linking constraints

We need some way to link the decision of ***whether or not*** to manufacture the subassembly units, to the decision regarding ***how many*** subassembly units to manufacture.

That is, we must express the fact that if ***we decide to*** manufacture subassembly units, then we should produce some quantity. However, it should be no more than an upper limit. If we instead ***decide not to*** manufacture subassembly units, then we should produce zero units.

In-house vs. Outsourcing Manufacturing

Formulating the linking constraints

We need some way to link the decision of ***whether or not*** to manufacture the subassembly units, to the decision regarding ***how many*** subassembly units to manufacture.

The key to this relationship is to create a ***Big-M*** constraint:

$$\boldsymbol{x_i \leq M_i y_i}$$

where $\boldsymbol{M_i}$ is a sufficiently large number. Remember that $\boldsymbol{y_i}$ is a binary variable and $\boldsymbol{x_i}$ is a continuous variable.

- What happens if $\boldsymbol{y_i = 0}$? What about if $\boldsymbol{y_i = 1}$?

In-house vs. Outsourcing Manufacturing

Formulating the linking constraints

We need some way to link the decision of ***whether or not*** to manufacture the subassembly units, to the decision regarding ***how many*** subassembly units to manufacture.

The choice of which M_i to use depends on the problem. It should be chosen such that it is ***as small as possible,*** but the constraint should still yield the desired behavior.

M_i is called a hyperparameter and should be chosen so that it imposes an upper bound on the value of x_i.

In-house vs. Outsourcing Manufacturing

Formulating the linking constraints

Gooseberry forecasts that exactly 7000 drone controllers and 5000 smartphone interfaces will be needed to satisfy demand during the upcoming Christmas season.

M_i should impose an upper bound on the value of x_i:

$$x_1 \leq M_1 y_1$$
$$x_2 \leq M_2 y_2$$
$$x_3 \leq M_3 y_3$$
$$x_4 \leq M_4 y_4$$
$$x_5 \leq M_5 y_5$$

In-house vs. Outsourcing Manufacturing

Formulating the linking constraints

Gooseberry forecasts that exactly 7000 drone controllers and 5000 smartphone interfaces will be needed to satisfy demand during the upcoming Christmas season.

M_i should impose an upper bound on the value of x_i:

$$x_1 \leq 12000y_1$$
$$x_2 \leq 7000y_2$$
$$x_3 \leq 5000y_3$$
$$x_4 \leq 7000y_4$$
$$x_5 \leq 5000y_5$$

In-house vs. Outsourcing Manufacturing

Minimize $\boldsymbol{Z} = 1000y_1 + 1200y_2 + 1900y_3 + 1500y_4 + 1500y_5$
$+ 0.40x_1 + 2.90x_2 + 3.15x_3 + 0.30x_4 + 0.55x_5$
$+ 0.65z_1 + 3.45z_2 + 3.70z_3 + 0.50z_4 + 0.70z_5$

Demand Constraints

$x_1 + z_1 = 12000$
$x_2 + z_2 = 7000$
$x_3 + z_3 = 5000$
$x_4 + z_4 = 7000$
$x_5 + z_5 = 5000$

Linking Constraints

$x_1 \leq 12000y_1$
$x_2 \leq 7000y_2$
$x_3 \leq 5000y_3$
$x_4 \leq 7000y_4$
$x_5 \leq 5000y_5$

$0.9x_1 + 2.2x_2 + 3x_3 + 0.8x_4 + x_5 \leq 30000$ **(capacity constraint)**

Non-negativity and binary constraints:

$x_i, z_i \geq 0 \quad for\ i = 1, \ldots, 5$
$y_i \in \{0,1\}\ for\ i = 1, \ldots, 5$

In-house vs. Outsourcing: Python Solution

- Remember to define the decision variables differently depending if they are binary (`vtype=GRB.BINARY`) or continuous (`vtype=GRB.CONTINUOUS`).
- Notice that after paying the setup cost, we try to use as much capacity as we can.

What managerial intuition do you get from the Python solution?

Condominium Construction

Condominium Construction

Diamante® is a condo builder that has developed The Florian, Domus, and The Royalton. They are currently working on a plan to build 4 low-rise apartment buildings and have considered 7 sites in Toronto. For each building, there can be anywhere between 4 and 12 floors with 13 or 26 units per floor. For each site $i = 1, \ldots, 7$, the cost of constructing an apartment complex consists of two components:

1) A fixed cost if an apartment is chosen to be built at site i.

2) A variable cost associated with the number of units built per floor (either 13 units or 26 units on each floor).

There are certain requirements (city ordinances and local mandates) that Diamante® must factor into their decisions.

Condominium Construction

Define the objective

Define the decision variables

Condominium Construction

Define the objective

Minimize the development costs

Define the decision variables

Condominium Construction

Define the objective

Minimize the development costs

Define the decision variables

$$z_i = \begin{cases} \mathbf{1}, \textit{if a condo is built on site i where i = 1, \ldots, 7} \\ \mathbf{0}, \ \textit{otherwise} \end{cases}$$

Condominium Construction

Define the objective

Minimize the development costs

Define the decision variables

$$x_i = \begin{cases} \textit{the number of } 13-\textit{unit floors at} \\ \textit{site } i \textit{ where } i = 1, \ldots, 7 \end{cases}$$

Condominium Construction

Define the objective

Minimize the development costs

Define the decision variables

$$y_i = \begin{cases} \textit{the number of 26-unit floors at} \\ \textit{site } i \textit{ where } i = 1, \ldots, 7 \end{cases}$$

Condominium Construction

Write the mathematical objective function

Minimize $\mathbf{Z} =$

Construction Site	Fixed Cost (C_i)	Variable Cost - 13 Units (A_i)	Variable Cost - 26 Units (B_i)
1	$3,600,139.42	$145,044.34	$1,388,290.16
2	$5,490,819.25	$535,503.09	$970,743.26
3	$4,881,866.51	$203,022.34	$658,878.96
4	$3,840,105.76	$229,679.10	$481,477.35
5	$4,059,055.94	$283,722.30	$696,580.10
6	$3,394,952.48	$919,371.26	$207,675.03
7	$5,082,906.08	$996,211.52	$94,350.22

Condominium Construction

Write the mathematical objective function

Minimize $$\mathbf{Z} = \sum_{i=1}^{7} C_i z_i$$

Construction Site	Fixed Cost (C_i)	Variable Cost - 13 Units (A_i)	Variable Cost - 26 Units (B_i)
1	$3,600,139.42	$145,044.34	$1,388,290.16
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6	$3,394,952.48	$919,371.26	$207,675.03
7	$5,082,906.08	$996,211.52	$94,350.22

Condominium Construction

Write the mathematical objective function

Minimize $$Z = \sum_{i=1}^{7} C_i z_i + \sum_{i=1}^{7} A_i x_i + \sum_{i=1}^{7} B_i y_i$$

Construction Site	Fixed Cost (C_i)	Variable Cost - 13 Units (A_i)	Variable Cost - 26 Units (B_i)
1	$3,600,139.42	$145,044.34	$1,388,290.16
2	$5,490,819.25	$535,503.09	$970,743.26
3	$4,881,866.51	$203,022.34	$658,878.96
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6	$3,394,952.48	$919,371.26	$207,675.03
7	$5,082,906.08	$996,211.52	$94,350.22

Condominium Construction

Formulating the constraints

For each site i, there can be anywhere between 4 and 12 floors with 13 or 26 units per floor provided a condo is built.

Condominium Construction

Formulating the constraints

For each site i, there can be anywhere between 4 and 12 floors with 13 or 26 units per floor provided a condo is built.

$$\boldsymbol{x_i + y_i \leq 12 z_i \ \forall i}$$

Condominium Construction

Formulating the constraints

For each site i, there can be anywhere between 4 and 12 floors with 13 or 26 units per floor provided a condo is built.

$$\boldsymbol{x_i + y_i \leq 12z_i \quad \forall i}$$
$$\boldsymbol{x_i + y_i \geq 4z_i \quad \forall i}$$

Condominium Construction

Formulating the constraints

Exactly 4 condominiums must be built.

Condominium Construction

Formulating the constraints

Exactly 4 condominiums must be built.

$$\sum_{i=1}^{7} z_i = 4$$

Condominium Construction

Formulating the constraints

There are also several planning requirements:

1. No less than 916 units can be built.
2. If an apartment is built on site 4, one must be built on site 7.
3. If an apartment is built on site 2, at least one must be built on sites 3 and 5.
4. If an apartment is built on site 1, one cannot be built on site 4.
5. No less than 33% of all floors in a building can have 13 units.

Condominium Construction

Formulating the constraints

No less than 916 units can be built.

Condominium Construction

Formulating the constraints

No less than 916 units can be built.

$$\sum_{i=1}^{7} 13x_i + \sum_{i=1}^{7} 26y_i \geq 916$$

Condominium Construction

Formulating the constraints

If an apartment is built on site 4, one must be built on site 7.

Condominium Construction

Formulating the constraints

If an apartment is built on site 4, one must be built on site 7.

$$z_4 \leq z_7$$

Condominium Construction

Formulating the constraints

If an apartment is built on site 2, at least one must be built on sites 3 and 5.

Condominium Construction

Formulating the constraints

If an apartment is built on site 2, at least one must be built on sites 3 and 5.

$$z_2 \leq z_3 + z_5$$

Condominium Construction

Formulating the constraints

If an apartment is built on site 1, one cannot be built on site 4.

Condominium Construction

Formulating the constraints

If an apartment is built on site 1, one cannot be built on site 4.

$$z_1 \leq 1 - z_4$$

Condominium Construction

Formulating the constraints

No less than 33% (or one-third) of all floors in a single building can have 13 units.

Condominium Construction

Formulating the constraints

No less than 33% (or one-third) of all floors in a single building can have 13 units.

$$x_i \geq \frac{1}{3}(x_i + y_i) \; \forall i$$

Condominium Construction

Minimize $\mathbf{Z} = \sum_{i=1}^{7} C_i z_i + \sum_{i=1}^{7} A_i x_i + \sum_{i=1}^{7} B_i y_i$

Subject to:

$x_i + y_i \leq 12 z_i \quad \forall i$		(Max unit construction constraint)
$x_i + y_i \geq 4 z_i \quad \forall i$		(Min unit construction constraint)
$\sum_{i=1}^{7} z_i = 4$		(Number of sites to build)
$\sum_{i=1}^{7} 13 x_i + \sum_{i=1}^{7} 26 y_i \geq 916$		(Number of units to build)
$z_4 \leq z_7$		(Planning requirement #1)
$z_2 \leq z_3 + z_5$		(Planning requirement #2)
$z_1 \leq 1 - z_4$		(Planning requirement #3)
$x_i \geq \frac{1}{3}(x_i + y_i) \ \forall i$		(Planning requirement #4)
$x_i, y_i \geq 0$	$for\ i = 1, .., 7$	(Non-negativity constraint)
$x_{i,}, y_{\mathrm{i}}\ is\ an\ integer$	$for\ i = 1, .., 7$	(Integrality constraint)
$z_i \in \{0,1\}$	$for\ i = 1, .., 7$	(Binary constraint)

Condominium Construction: Python Solution

- Remember to define the decision variables differently depending if they are binary (`vtype=GRB.BINARY`) or integral (`vtype=GRB.INTEGER`).
- Constraints can be associated with just the binary variables, just the integer variables, or both. However, there needs to be a class of constraints that link the decision variables.

What managerial intuition do you get from the Python solution?

Production and Transportation

Masters Best Friend Inc.® manufactures and distributes dog food to high-volume customers in various U.S. cities.

Production and Transportation

Because the company rents its production facilities, Masters Best Friend Inc.® can decide to operate at only *some* plants in any given month. The fixed monthly cost for operating any production facility is \$60,000. The monthly capacity of each plant is 2500 pounds and the production cost at any production facility is \$10.25 per pound. After the product is manufactured, it is shipped to customers at a cost of \$0.02 per pound per mile (I know, it's the US!). Note that customers, as you will see, are conveniently located in the same city as the production facilities.

Production and Transportation

Distance (miles) of Production Facilities to Customers:

From/To	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Boston	0	983	1815	1991	3036	1539	213	2664
Chicago	983	0	1205	1050	2112	1390	840	1729
Dallas	1815	1205	0	801	1425	1332	1604	1027
Denver	1991	1050	801	0	1174	2065	1780	836
LA	3036	2112	1425	1174	0	2757	2825	398
Miami	1539	1390	1332	2065	2757	0	1258	2359
NY	213	840	1604	1780	2825	1258	0	2442
Phoenix	2664	1729	1027	836	398	2359	2442	0

Production and Transportation

Distance (miles) of Production Facilities to Customers:

For simplicity, the distance between two cities will be defined as $\boldsymbol{t_{ij}}$ where $\boldsymbol{i,j}$ = {Boston, Chicago, Dallas, Denver, LA, Miami, NY, Phoenix} = {1,2,3,4,5,6,7,8}.

From/To	**Boston**	**Chicago**	**Dallas**	**Denver**	**LA**	**Miami**	**NY**	**Phoenix**
Boston	0	983	1815	1991	3036	1539	213	2664
Chicago	983	0	1205	1050	2112	1390	840	1729
Dallas	1815	1205	0	801	1425	1332	1604	1027
Denver	1991	1050	801	0	1174	2065	1780	836
LA	3036	2112	1425	1174	0	2757	2825	398
Miami	1539	1390	1332	2065	2757	0	1258	2359
NY	213	840	1604	1780	2825	1258	0	2442
Phoenix	2664	1729	1027	836	398	2359	2442	0

Production and Transportation

Customers submit order sizes and prices. For example, the customer in Boston requires an order of 1430 pounds next month and is willing to pay $75,740 for it. Masters Best Friend can decide to fill the *entire order* or not. If not, the customer will take its business to another company. For the upcoming month, Masters Best Friend Inc.® wants to maximize its profit. To do this, it must decide which production facilities to operate and which customers to serve from which operating plant.

Quantities and Prices:

	Quantity	Price
Boston	1430	$75,740
Chicago	870	$44,370
Dallas	770	$46,320
Denver	1140	$87,780
LA	700	$43,850
Miami	830	$21,000
NY	1230	$74,850
Phoenix	1070	$83,980

Production and Transportation

- What kinds of decisions must be made?
  1. Whether or not we should open a production facility?
  2. Whether or not we want to satisfy a customer's order?
  3. How many units should we ship from a production facility to a customer provided that the production facility is open, and we have decided to satisfy the customer’s order?

What is our objective and what kinds of decision variables do we need?

Production and Transportation

Define the objective

Maximize the monthly profit

Define the decision variables

$y_i =$

- ***1, if Masters Best Friend opens production facility*** i ***where*** $i =$ ***{Boston, Chicago, Dallas, Denver, LA, Miami, NY, Phoenix} = {1,2,3,4,5,6,7,8}***
- ***0, otherwise***

Production and Transportation

Define the objective

Maximize the monthly profit

Define the decision variables

$$z_j = \begin{cases} \textit{1, if Masters Best Friend chooses to serve customer } j \textit{ where } j = \{\textit{Boston, Chicago, Dallas, Denver, LA, Miami, NY, Phoenix}\} = \{1,2,3,4,5,6,7,8\} \\ \textit{0, otherwise} \end{cases}$$

Production and Transportation

Define the objective

Maximize the monthly profit

Define the decision variables

$x_{ij}=$ ***the amount of dog food (pounds) shipped from a production facility in city i to a customer in city j where $i, j =$ {Boston, Chicago, Dallas, Denver, LA, Miami, NY, Phoenix} = {1,2,3,4,5,6,7,8}***

Production and Transportation

Write the mathematical objective function

Maximize Z = Profit

= Revenues - Costs

Customers have both order sizes and prices. Masters Best Friend can decide to fill this *entire order* or not. If not, the customer will take its business to another company.

Note: The fixed monthly cost for operating any production facility is $60,000. The monthly capacity of each plant is 2500 pounds and the production cost at any production facility is $10.25 per pound. After the product is manufactured, it is shipped to customers at a cost of $0.02 per pound per mile.

Production and Transportation

Write the mathematical objective function

$$\text{Revenues} = 75740z_1 + 44370z_2 + 46320z_3 + 87780z_4 + 43850z_5 + 21000z_6 + 74850z_7 + 83980z_8$$

	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Quantity	1430	870	770	1140	700	830	1230	1070
Price	\$75,740	\$44,370	\$46,320	\$87,780	\$43,850	\$21,000	\$74,850	\$83,980

Production and Transportation

Write the mathematical objective function

Costs =

(cost of using facility *i*)

(cost of production)

(cost of transportation)

The fixed monthly cost for operating any production facility is $60,000. The capacity of each plant is 2500 pounds per month, and the production cost at any production facility is $10.25 per pound. After the product is manufactured, it is shipped to customers at a rate of $0.02 per pound per mile.

Production and Transportation

Write the mathematical objective function

Costs = $\mathbf{60000} \sum_{i=1}^{8} y_i$ **(cost of using facility *i*)**

(cost of production)

(cost of transportation)

The fixed monthly cost for operating any production facility is $60,000. The capacity of each plant is 2500 pounds per month, and the production cost at any production facility is $10.25 per pound. After the product is manufactured, it is shipped to customers at a rate of $0.02 per pound per mile.

Production and Transportation

Write the mathematical objective function

$$\text{Costs} = 60000\sum_{i=1}^{8} y_i \quad \text{(cost of using facility } i\text{)}$$

$$+10.25\sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij} \quad \text{(cost of production)}$$

(cost of transportation)

The fixed monthly cost for operating any production facility is $60,000. The capacity of each plant is 2500 pounds per month, and the production cost at any production facility is $10.25 per pound. After the product is manufactured, it is shipped to customers at a rate of $0.02 per pound per mile.

Production and Transportation

Write the mathematical objective function

Costs = $\mathbf{60000 \sum_{i=1}^{8} y_i}$ **(cost of using facility *i*)**

$\mathbf{+10.25 \sum_{i=1}^{8} \sum_{j=1}^{8} x_{ij}}$ **(cost of production)**

$\mathbf{+0.02 \sum_{i=1}^{8} \sum_{j=1}^{8} t_{ij}\, x_{ij}}$ **(cost of transportation)**

The fixed monthly cost for operating any production facility is $60,000. The capacity of each plant is 2500 pounds per month, and the production cost at any production facility is $10.25 per pound. After the product is manufactured, it is shipped to customers at a rate of $0.02 per pound per mile.

Production and Transportation

Formulating the constraints

There are three types of constraints:

1. Resource constraints
2. Demand constraints
3. Non-negativity and binary constraints

Production and Transportation

Formulating the resource constraints

The monthly capacity of each plant is 2500 pounds.

$$\sum_{j=1}^{8} x_{ij} \leq 2500 \ \ for\ every\ production\ facility\ i$$

Is this correct?

Production and Transportation

Formulating the resource constraints

The monthly capacity of each plant is 2500 pounds.

$$\sum_{j=1}^{8} x_{ij} \leq 2500 \;\; for\ every\ production\ facility\ i$$

Is this correct?

Production and Transportation

Formulating the resource constraints

The monthly capacity of each plant is 2500 pounds.

$$\sum_{j=1}^{8} x_{ij} \leq 2500y_i \;\; for\ every\ production\ facility\ i$$

The monthly capacity of a plant is 2500 pounds provided we decide that the plant is operational.

Production and Transportation

Formulating the demand constraints

	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Quantity	1430	870	770	1140	700	830	1230	1070
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8

*For simplicity, lets define the order quantity for customer **j** as Q_j. The table above lists the corresponding values for each Q_j.*

Production and Transportation

Formulating the demand constraints

	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Quantity	1430	870	770	1140	700	830	1230	1070
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8

If we decide to satisfy a customer, we must fill their entire order.

$$\sum_{i=1}^{8} x_{ij} = Q_j \ \ for\ every\ customer\ j$$

Is this correct?

Production and Transportation

Formulating the demand constraints

	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Quantity	1430	870	770	1140	700	830	1230	1070
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8

If we decide to satisfy a customer, we must fill their entire order.

$$\cancel{\sum_{i=1}^{8} x_{ij} = Q_j \text{ for every customer } j}$$

Is this correct?

Production and Transportation

Formulating the demand constraints

	Boston	Chicago	Dallas	Denver	LA	Miami	NY	Phoenix
Quantity	1430	870	770	1140	700	830	1230	1070
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8

$$\sum_{i=1}^{8} x_{ij} = Q_j z_j \ \ \textit{for every customer } j$$

We must satisfy a customer's entire order provided we decide that we want to serve that customer.

Production and Transportation

Maximize $Z = 75740z_1 + 44370z_2 + 46320z_3$
$+ 87780z_4 + 43850z_5 + 21000z_6 + \ 74850z_7$
$+83980z_8 - 60000\sum_{i=i}^{8} y_i$
$-10.25\sum_{i=i}^{8}\sum_{j=1}^{8} x_{ij} - 0.02\ \sum_{i=i}^{8}\sum_{j=1}^{8} t_{ij}\, x_{ij}$

Subject to:

$\sum_{j=1}^{8} x_{ij} \leq 2500y_i\ \ for\ i = 1, \dots, 8$ (Resource constraints)

$\sum_{i=1}^{8} x_{ij} = Q_j z_j \quad for\ j = 1, \dots, 8$ (Demand constraints)

$y_i \in \{0,1\} \quad for\ i = 1, \ldots, 8$ (Binary constraints)

$z_j \in \{0,1\} \quad for\ j = 1, \ldots, 8$ (Binary constraints)

$x_{ij} \geq 0 \quad for\ i, j = 1, \ldots, 8$ (Non-negativity constraints)

Production and Transportation Problem: Python Solution

- The linking constraints affect many decision variables at once via the summation constraint.
- Managerially, an interesting observation is that in the optimal solution, not all production facilities are used even though there are customers in the same locations as the facilities.
  - This is due to the high fixed cost associated with opening facilities for production and is not always true.

What managerial intuition do you get from the Python solution?

Tree Planting

Tree Planting

The 50 million tree program is a tree planting charity whose mandate is to increase forest cover in Ontario. As of 2024, more than 30 million trees have been planted. Last year, the organization made a big push to acquire funding so that they could plant 10 million trees in 2025. There are a total of 36 potential planting locations in Ontario, but it remains to determine which sites should be chosen. For each site $i = 1, \ldots, 36$ there is a cost c_i for each tree that is planted where

$$c_i = 0.05 + \frac{1}{20}(i \; mod \; 10)$$

Unfortunately, due to the Conservation Authorities Act of Ontario, many intricate laws must be adhered to.

Tree Planting

Implications of the relevant laws:

- At least 13 planting locations must be chosen in Ontario and for any location that is selected, between 33,111 and 668,457 trees can be planted.
- At most one planting location can be chosen amongst sites 1, 10, and 20 and no more than 4 planting locations must be chosen amongst sites 2, 4, 6, 8, 12, 14, and 16.
- If planting location 30 is chosen, then the sites 31 and 32 cannot be chosen and if planting location 21 is chosen then the sites 22 and 23 must be chosen.
- The sum of all trees planted at sites 1-18 must equal the sum of all trees planted at sites 19-36.

Tree Planting

Define the objective

Define the decision variables

Tree Planting

Define the objective

Minimize the cost of planting trees

Define the decision variables

Tree Planting

Define the objective

Minimize the cost of planting trees

Define the decision variables

$$y_i = \begin{cases} 1, & \textit{if location i is chosen } i = 1, \ldots, 36 \\ 0, & \textit{otherwise} \end{cases}$$

Tree Planting

Define the objective

Minimize the cost of planting trees

Define the decision variables

$$x_i = \begin{cases} \textbf{\textit{the number of trees planted at}} \\ \textbf{\textit{at location i where }} i = 1, \dots, 36 \end{cases}$$

Tree Planting

Write the mathematical objective function

For each site $i = 1, \ldots, 36$ *define* c_i *to be the per tree cost of planting trees at that site. Then,*

$$\boldsymbol{Minimize} \qquad \mathbf{Z} =$$

Tree Planting

Write the mathematical objective function

For each site $i = 1, \ldots, 36$ *define* c_i *to be the per tree cost of planting trees at that site. Then,*

$$\boldsymbol{Minimize} \qquad \mathbf{Z} = \sum_{i=1}^{36} c_i x_i$$

Tree Planting

Formulating the constraints

At least 13 planting locations must be chosen in Ontario.

Tree Planting

Formulating the constraints

At least 13 planting locations must be chosen in Ontario.

$$\sum_{i=1}^{36} y_i \geq 13$$

Tree Planting

Formulating the constraints

For any location that is selected, between 33,111 and 668,457 trees can be planted.

Tree Planting

Formulating the constraints

For any location that is selected, between 33,111 and 668,457 trees can be planted.

$$x_i \geq 33111 y_i \quad \forall i$$
$$x_i \leq 668457 y_i \quad \forall i$$

Tree Planting

Formulating the constraints

At most one planting location can be chosen amongst sites 1, 10, and 20 and no more than 4 planting locations must be chosen amongst sites 2, 4, 6, 8, 12, 14, and 16.

Tree Planting

Formulating the constraints

At most one planting location can be chosen amongst sites 1, 10, and 20 and no more than 4 planting locations must be chosen amongst sites 2, 4, 6, 8, 12, 14, and 16.

$$y_1 + y_{10} + y_{20} \leq 1$$

Tree Planting

Formulating the constraints

At most one planting location can be chosen amongst sites 1, 10, and 20 and no more than 4 planting locations must be chosen amongst sites 2, 4, 6, 8, 12, 14, and 16.

$$y_1 + y_{10} + y_{20} \leq 1$$
$$y_2 + y_4 + y_6 + y_8 + y_{12} + y_{14} + y_{16} \leq 4$$

Tree Planting

Formulating the constraints

If planting location 30 is chosen then the sites 31 and 32 cannot be chosen.

Tree Planting

Formulating the constraints

If planting location 30 is chosen then the sites 31 and 32 cannot be chosen.

$$y_{30} \leq 1 - y_{31}$$

$$y_{30} \leq 1 - y_{32}$$

Tree Planting

Formulating the constraints

If planting location 21 is chosen then the sites 22 and 23 must be chosen.

Tree Planting

Formulating the constraints

If planting location 21 is chosen then the sites 22 and 23 must be chosen.

$$2y_{21} \leq y_{22} + y_{23}$$

Tree Planting

Formulating the constraints

The sum of all trees planted at sites 1-18 equals the trees planted at sites 19-36.

Tree Planting

Formulating the constraints

The sum of all trees planted at sites 1-18 equals the trees planted at sites 19-36.

$$\sum_{i=1}^{18} x_i = \sum_{i=19}^{36} x_i$$

Tree Planting

Formulating the constraints

Plant exactly 10 million trees in 2024.

Tree Planting

Formulating the constraints

Plant exactly 10 million trees in 2024.

$$\sum_{i=1}^{36} x_i = 10{,}000{,}000$$

Tree Planting

Minimize $\mathbf{Z} = \sum_{i=1}^{36} c_i x_i$

Subject to:

$\sum_{i=1}^{36} y_i \geq 13$	(Min number of locations)
$x_i \geq 33111 y_i \quad \forall i$	(Min trees per location)
$x_i \leq 668457 y_i \quad \forall i$	(Max trees per location)
$y_1 + y_{10} + y_{20} \leq 1$	(Planning requirement #1)
$y_2 + y_4 + y_6 + y_8 + y_{12} + y_{14} + y_{16} \leq 4$	(Planning requirement #2)
$y_{30} \leq 1 - y_{31}$	(Planning requirement #3)
$y_{30} \leq 1 - y_{32}$	(Planning requirement #4)
$2y_{21} \leq y_{22} + y_{23}$	(Planning requirement #5)
$\sum_{i=1}^{18} x_i = \sum_{i=19}^{36} x_i$	(Planning requirement #6)
$\sum_{i=1}^{36} x_i = 10000000$	(Tree planting requirement)
$x_{i,} \geq 0 \; and \; Integer \qquad for \; i = 1,..,36$	(Integrality constraint)
$y_i \in \{0,1\} \qquad for \; i = 1,..,36$	(Binary constraint)

Tree Planting: Python Solution

- **Recall:** the modulo operator in Python is `a % b`
- Because of the maximum number of trees/location, the constraint that at least 13 planting locations must be chosen in Ontario is redundant.
- While the objective only contains variable costs, we still need the binary decision variables to ensure that the relevant laws are upheld in the constraints.

What managerial intuition do you get from the Python solution?

Next Class: Large Models

Programming is a tool that helps manage scalability!

- Mathematical programs are typically large with thousands of decision variables and constraints.
- Gurobi is a powerful tool that can be used to solve large-scale problems. The complexity associated with creating these models are no greater than simpler ones.
  - Leverage the power of dictionaries and loop structures.
- After solving the problem, it's important to understand how to query the model to extract actionable insights.
  - Printing out the entire solution is not tractable nor is it helpful.
  - Teasing out the relevant bits of the solution is important for further scrutiny or to be used in downstream tasks.
  - Fosters a better understanding of the Gurobi library.