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多元正态分布

THE MULTIVARIATE NORMAL DISTRIBUTION

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Random Vector 随机向量

- Suppose $X_1, X_2, \ldots, X_p$ are random variables.
- An p-dimensional random vector is a function

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = [X_1, \quad X_2, \quad \cdots, \quad X_p]'$$

Random Vector 随机向量

- Multivariate distribution function

$$F(x_1, x_2, \cdots x_p) = P(X_1 \le x_1, X_2 \le x_2, \cdots, X_p \le x_p)$$

- Multivariate density function

$$f(x_1, \cdots, x_p) = \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \cdots, x_p) \qquad f(x_1, \cdots, x_p) \ge 0$$

We have $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_p) dx_1 \cdots dx_p = 1$

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Random Vector 随机向量

If $\boldsymbol{X}$, $\boldsymbol{Y}$ are random vector, $\boldsymbol{A}$, $\boldsymbol{B}$ are constant matrices, then we have

$$E(\boldsymbol{X}) = [E(X_1), E(X_2), \cdots, E(X_p)]'$$

$$E(\boldsymbol{AX}) = \boldsymbol{A}E(\boldsymbol{X})$$

$$E(\boldsymbol{AXB}) = \boldsymbol{A}E(\boldsymbol{X})\boldsymbol{B}$$

$$E(\boldsymbol{AX} + \boldsymbol{BY}) = \boldsymbol{A}E(\boldsymbol{X}) + \boldsymbol{B}E(\boldsymbol{Y})$$

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Random Vector 随机向量

随机向量X的协方差矩阵

$$D(X)=E\left[\left(X-E(X)\right)\left(X-E(X)\right)'\right]$$

$$=\begin{pmatrix} Cov(X_1,X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_p) \\ Cov(X_2,X_1) & Cov(X_2,X_2) & \cdots & Cov(X_2,X_p) \\ \vdots & \vdots & & \vdots \\ Cov(X_p,X_1) & Cov(X_p,X_2) & \cdots & Cov(X_p,X_p) \end{pmatrix}$$

$$=\left(\sigma_{ij}\right)_{p\times p}\overset{def}{=}\Sigma$$

Random Vector 随机向量

If $\boldsymbol{a}$ is a constant vector, $\boldsymbol{A}$, $\boldsymbol{B}$ are constant matrices, then we have

$$D(\boldsymbol{X}+\boldsymbol{a})=D(\boldsymbol{X})$$

$$D(\boldsymbol{A}\boldsymbol{X})=\boldsymbol{A}D(\boldsymbol{X})\boldsymbol{A}'$$

$$\mathrm{Cov}(\boldsymbol{A}\boldsymbol{X},\boldsymbol{B}\boldsymbol{Y})=\boldsymbol{A}\mathrm{Cov}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{B}'$$

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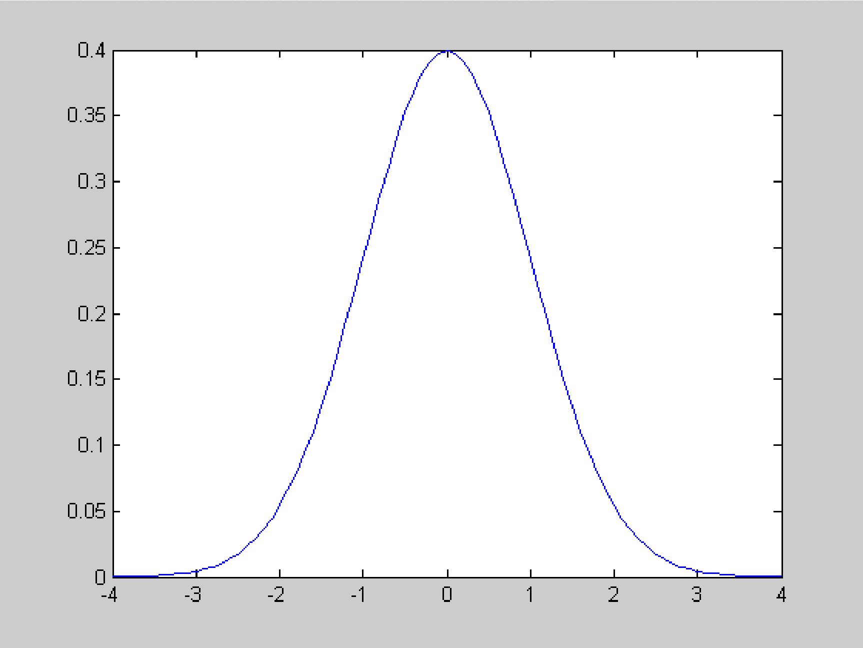
Univariate Normal Distribution 一元正态分布

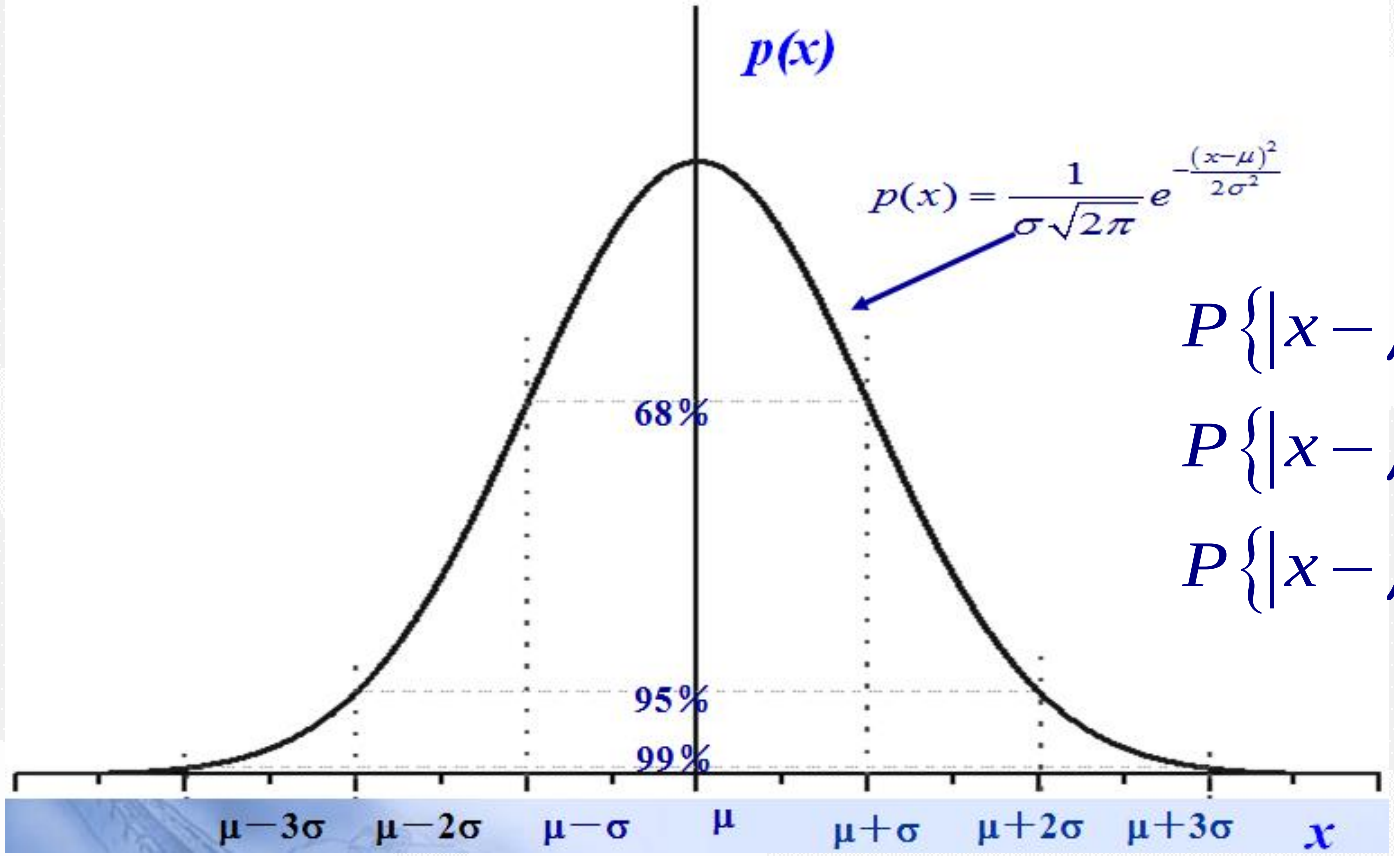
If $\quad X \sim N(0,1), \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (-\infty < x < +\infty)$



$$P\{|x| \le 1\} = 0.6826, P\{|x| \le 2\} = 0.9545, P\{|x| \le 3\} = 0.9973$$

Univariate Normal Distribution 一元正态分布

If $X \sim N(\mu, \sigma^2)$, $\quad f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (-\infty < x < +\infty)$



$$P\{|x-\mu| \le \sigma\} = 0.6826$$
$$P\{|x-\mu| \le 2\sigma\} = 0.9545$$
$$P\{|x-\mu| \le 3\sigma\} = 0.9973$$

Univariate Normal Distribution 一元正态分布

The density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (-\infty < x < +\infty)$$

can be rewritten as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(x-\mu)'(\sigma^2)^{-1}(x-\mu)} \qquad (-\infty < x < +\infty)$$

Bivariate Normal Distribution 二元正态分布

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]}$$

$$(X_1,X_2)\sim N_2(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$$

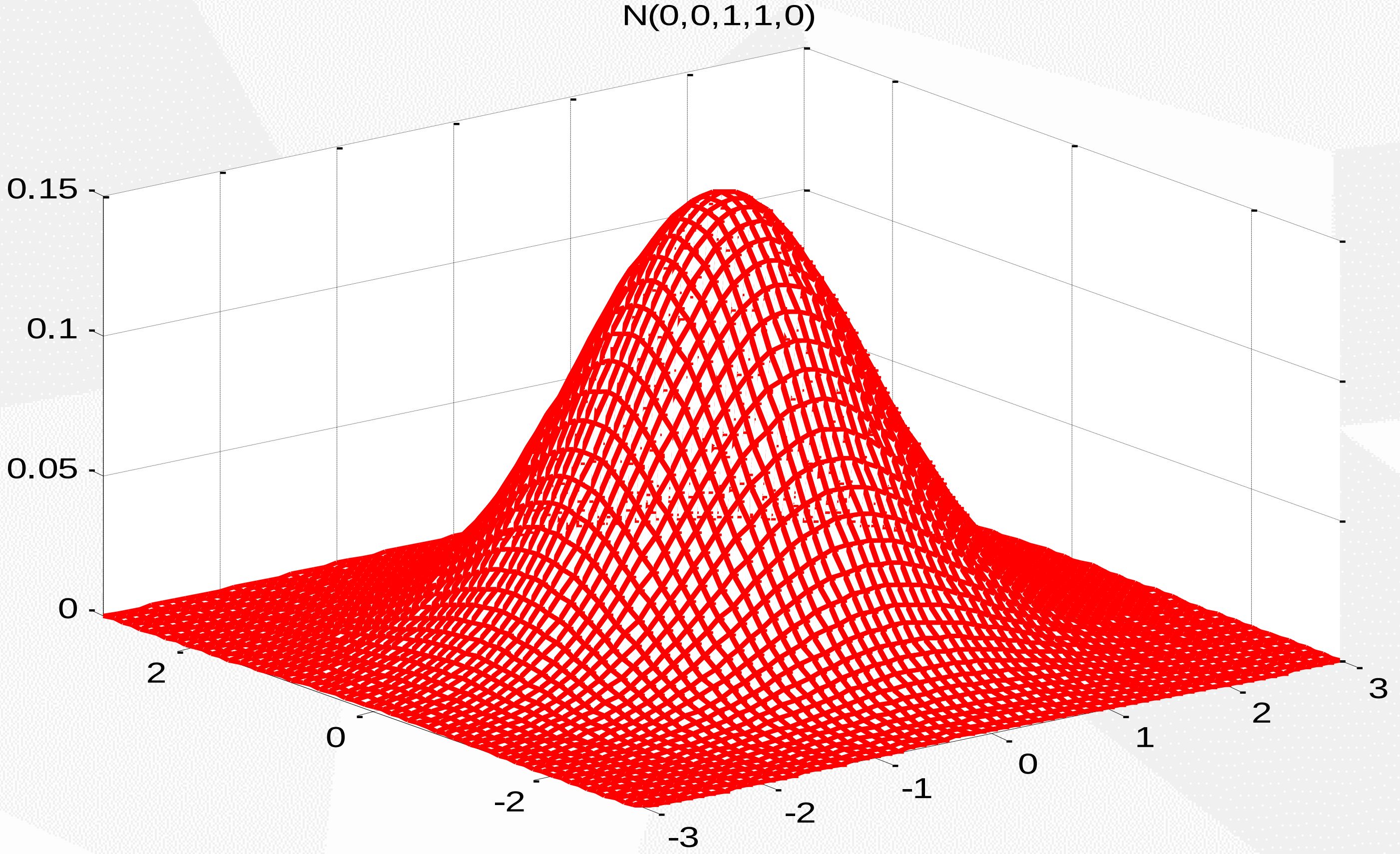
$$EX_1=\mu_1,\quad DX_1=\sigma_1{}^2,$$

$$EX_2=\mu_2,\quad DX_2=\sigma_2{}^2$$

$$\rho(X_1\ X_2)=\frac{Cov(X_1\ X_2)}{\sqrt{D(X_1)D(X_2)}}=\frac{Cov(X_1\ X_2)}{\sigma_1\sigma_2},\quad Cov(X_1\ X_2)=\rho\,\sigma_1\sigma_2$$

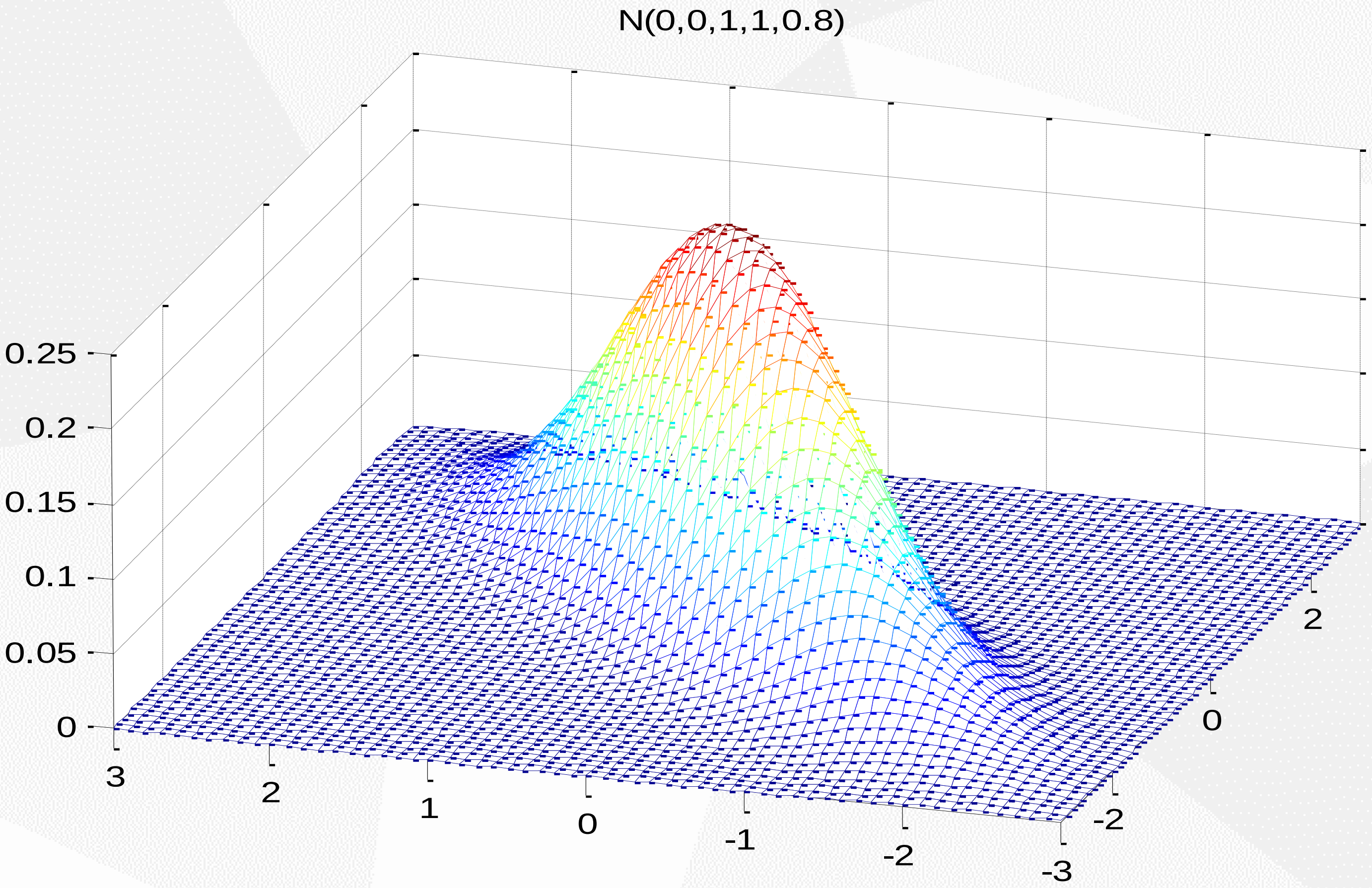
Bivariate Normal Distribution 二元正态分布

$$(X_1, X_2) \sim N_2(0,0,1,1,0), \quad f(x_1,x_2) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x_1^2 + x_2^2)}$$



Bivariate Normal Distribution 二元正态分布

$$(X_1, X_2) \sim N_2(0,0,1,1,0.8),\ p(x_1, x_2) = \frac{1}{2\pi \cdot 0.6} \cdot e^{-\frac{1}{0.72}(x_1^2 - 1.6x_1x_2 + x_2^2)}$$



Bivariate Normal Distribution 二元正态分布

$$X=\begin{pmatrix}X_1\\X_2\end{pmatrix},$$

$$\mu=EX=\begin{pmatrix}EX_1\\EX_2\end{pmatrix}=\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix},$$ X_1和X_2的均值向量

$$\Sigma=\operatorname{cov}(X)=\begin{pmatrix}DX_1 & \operatorname{cov}(X_1,X_2)\\\operatorname{cov}(X_2,X_1) & DX_2\end{pmatrix}=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\\rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$$ X_1和X_2的协方差矩阵

$$|\Sigma|=\sigma_1^2\sigma_2^2\left(1-\rho^2\right)$$

$$\Sigma^{-1}=\begin{pmatrix}\dfrac{1}{\sigma_1^2(1-\rho^2)} & \dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)}\\\dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} & \dfrac{1}{\sigma_2^2(1-\rho^2)}\end{pmatrix}$$

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Bivariate Normal Distribution 二元正态分布

$$\frac{1}{(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]$$

$$=\begin{pmatrix} x_1-\mu_1 & x_2-\mu_2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_1^2(1-\rho^2)} & \frac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} \\ \frac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} & \frac{1}{\sigma_2^2(1-\rho^2)} \end{pmatrix}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$=(x-\mu)'\Sigma^{-1}(x-\mu)$$

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]}$$

$$=\frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

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Bivariate Normal Distribution 二元正态分布

$$X=\begin{pmatrix}X_1\\X_2\end{pmatrix},\quad \mu=\begin{pmatrix}0\\0\end{pmatrix},\quad \Sigma=\begin{pmatrix}1&0.8\\0.8&1\end{pmatrix}$$

$$|\Sigma|^{\frac{1}{2}}=0.6,\quad \Sigma^{-1}=\begin{pmatrix}\dfrac{1}{\sigma_1^2(1-\rho^2)}&\dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)}\\[2ex]\dfrac{-\rho}{\sigma_1\sigma_2(1-\rho^2)}&\dfrac{1}{\sigma_2^2(1-\rho^2)}\end{pmatrix}=\begin{pmatrix}\dfrac{1}{0.36}&\dfrac{-0.8}{0.36}\\[2ex]\dfrac{-0.8}{0.36}&\dfrac{1}{0.36}\end{pmatrix}$$

$$\frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}=\frac{1}{2\pi\cdot 0.6}\cdot e^{-\frac{1}{0.72}(x_1^2-1.6x_1x_2+x_2^2)}$$

$$X\sim N\left(\begin{pmatrix}0\\0\end{pmatrix},\ \begin{pmatrix}1&0.8\\0.8&1\end{pmatrix}\right).$$

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Bivariate Normal Distribution 二元正态分布

$$(X_1, X_2) \sim N_2(0,0,1,1,0.8),$$

$$\begin{cases} Y_1 = 2X_1 + X_2 \\ Y_2 = 2X_1 - X_2 \end{cases}$$

$$EY_1 = 2EX_1 + EX_2 = 0,$$

$$DY_1 = 4DX_1 + DX_2 + 2\operatorname{cov}(2X_1, X_2) = 5 + 4 \times 0.8 = 8.2$$

$$EY_2 = 2EX_1 - EX_2 = 0,$$

$$DY_2 = 4DX_1 + DX_2 - 2\operatorname{cov}(2X_1, X_2) = 5 - 4 \times 0.8 = 1.8$$

$$\operatorname{cov}(Y_1, Y_2) = \operatorname{cov}(2X_1 + X_2, 2X_1 - X_2)$$

$$= \operatorname{cov}(2X_1, 2X_1) + \operatorname{cov}(X_2, 2X_1) + \operatorname{cov}(2X_1, -X_2) + \operatorname{cov}(X_2, -X_2) = 3$$

Bivariate Normal Distribution 二元正态分布

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mu_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$EY = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} EX = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_Y = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}' = \begin{pmatrix} 8.2 & 3 \\ 3 & 1.8 \end{pmatrix}$$

$$Y \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8.2 & 3 \\ 3 & 1.8 \end{pmatrix} \right)$$

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Bivariate Normal Distribution 二元正态分布

$$X=\begin{pmatrix}X_1\\X_2\end{pmatrix},\quad EX=\begin{pmatrix}EX_1\\EX_2\end{pmatrix},\quad DX=\begin{pmatrix}DX_1 & \operatorname{cov}(X_1,X_2)\\ \operatorname{cov}(X_1,X_2) & DX_2\end{pmatrix}$$

$$Y=\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}X_1\\X_2\end{pmatrix}+\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$EY=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}EX+\begin{pmatrix}b_1\\b_2\end{pmatrix},\quad \Sigma_Y=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}DX_1 & \operatorname{cov}(X_1,X_2)\\ \operatorname{cov}(X_1,X_2) & DX_2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}'$$

$$Y\sim N_2\left(\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}EX+\begin{pmatrix}b_1\\b_2\end{pmatrix},\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}DX_1 & \operatorname{cov}(X_1,X_2)\\ \operatorname{cov}(X_1,X_2) & DX_2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}'\right)$$

Multivariate Normal Distribution 多元正态分布

p元正态分布的密度函数

$$p(x)=\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right),$$

$$X=(X_1,\cdots,X_p)',X\sim N_p(\mu,\ \Sigma).$$

$$X=\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix},\quad EX=\begin{pmatrix}1\\2\\1\end{pmatrix},\quad DX=\begin{pmatrix}2&0&0\\0&3&0\\0&0&3\end{pmatrix}$$

$$X\sim N_3\left(\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2&0&0\\0&3&0\\0&0&3\end{pmatrix}\right)$$

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Multivariate Normal Distribution 多元正态分布

$$Y=\begin{pmatrix}Y_1\\Y_2\\Y_3\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix}+\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=AX+B$$

$$EY=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}EX+\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},\quad \Sigma_Y=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}DX_1&\operatorname{cov}(X_1,X_2)&\operatorname{cov}(X_1,X_3)\\\operatorname{cov}(X_2,X_1)&DX_2&\operatorname{cov}(X_2,X_3)\\\operatorname{cov}(X_3,X_1)&\operatorname{cov}(X_3,X_2)&DX_3\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}'$$

$$Y\sim N_2\left(\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}EX+\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},\ \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}DX_1&\operatorname{cov}(X_1,X_2)&\operatorname{cov}(X_1,X_3)\\\operatorname{cov}(X_2,X_1)&DX_2&\operatorname{cov}(X_2,X_3)\\\operatorname{cov}(X_3,X_1)&\operatorname{cov}(X_3,X_2)&DX_3\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}'\right)$$

$$Y\sim N_2\left(AEX+B,\ ADXA'\right)$$

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Multivariate Normal Distribution 多元正态分布

$$X \sim N_3(\mu, 2I_3)$$

$$\mu = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Y = AX + B$$

$$\sim N_2\left(\begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix}' \right)$$

$$Y \sim N_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right)$$

Multivariate Normal Distribution 多元正态分布

设$X=\begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}\begin{matrix} r \\ p-r \end{matrix} \sim N_p(\mu,\Sigma)$，将$\mu$，$\Sigma$分为

$$\mu=\begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}\begin{matrix} r \\ p-r \end{matrix}, \quad \Sigma=\begin{array}{cc} \begin{matrix} \quad r & \quad p-r \end{matrix} & \\ \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} & \begin{matrix} r \\ p-r \end{matrix} \end{array},$$

则$X^{(1)} \sim N_r\left(\mu^{(1)},\Sigma_{11}\right), X^{(2)} \sim N_{p-r}\left(\mu^{(2)},\Sigma_{22}\right)$.

Multivariate Normal Distribution 多元正态分布

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 & 2 & 0 & 0 \\ 2 & 10 & 0 & 0 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 3 & 7 \end{pmatrix}\right)$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 2 & 10 \end{pmatrix}\right)$$

$$\begin{pmatrix} X_3 \\ X_4 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 9 & 3 \\ 3 & 7 \end{pmatrix}\right)$$

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Multivariate Normal Distribution 多元正态分布

随机向量X和Y的协方差矩阵

$$Cov(X,Y)=E\left[(X-EX)(Y-EY)'\right]$$

$$=\begin{bmatrix} Cov(X_1,Y_1) & Cov(X_1,Y_2) & \cdots & Cov(X_1,Y_q) \\ Cov(X_2,Y_1) & Cov(X_2,Y_2) & \cdots & Cov(X_2,Y_q) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_p,Y_1) & Cov(X_p,Y_2) & \cdots & Cov(X_p,Y_q) \end{bmatrix}$$

为随机向量X和Y的协方差矩阵。若$Cov(X,Y)=O$，(其中O表示零矩阵)，则称随机向量X和Y不相关。

设正态随机向量X和Y相互独立$\Leftrightarrow Cov(X,Y)=O_{p\times q}$;

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Multivariate Normal Distribution 多元正态分布

随机向量X的相关矩阵

$R=\left(r_{ij}\right)_{p\times p}$为$X$的相关矩阵，其中

$$r_{ij}=\frac{Cov(X_i,X_j)}{\sqrt{Var\left(X_i\right)}\sqrt{Var\left(X_j\right)}}=\frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}},(i,j=1,2,\cdots,p)$$

若记 $V^{\frac{1}{2}}=diag\left(\sqrt{\sigma_{11}},\sqrt{\sigma_{22}},\cdots,\sqrt{\sigma_{pp}}\right)$为标准差矩阵，

则记 $\Sigma=V^{\frac{1}{2}}RV^{\frac{1}{2}}$，或$R=\left(V^{\frac{1}{2}}\right)^{-1}\Sigma\left(V^{\frac{1}{2}}\right)^{-1}$.

Multivariate Normal Distribution 多元正态分布

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 8 & 2 & 0 & 0 \\ 2 & 10 & 0 & 0 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 3 & 7 \end{pmatrix}\right)$$

$$R = \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{9}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{7}} \end{pmatrix} \begin{pmatrix} 8 & 2 & 0 & 0 \\ 2 & 10 & 0 & 0 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 3 & 7 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{9}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{7}} \end{pmatrix}$$

Multivariate Normal Distribution 多元正态分布

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \mu_X = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$EZ =$$

$$\Sigma_Z =$$

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Multivariate Normal Distribution 多元正态分布

$$EX=\begin{pmatrix}2\\4\\-1\\3\\0\end{pmatrix},\quad DX=\begin{pmatrix}4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0\\ -1 & 3 & 1 & -1 & 0\\ \frac{1}{2} & 1 & 6 & 1 & -1\\ -\frac{1}{2} & -1 & 1 & 4 & 0\\ 0 & 0 & -1 & 0 & 2\end{pmatrix}$$

$$A=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},B=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & -2\end{pmatrix}$$

$$E\left(AX^{(1)}\right),E\left(BX^{(2)}\right)$$

$$Cov\left(X^{(1)}\right),Cov\left(AX^{(1)}\right),Cov\left(BX^{(2)}\right)$$

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Multivariate Normal Distribution 多元正态分布

$$X_1 \sim N_p(\mu_1, \Sigma_{11}), \qquad X_2 \sim N_q(\mu_2, \Sigma_{22})$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_{p+q}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}\right)$$

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样本统计量的特征

总体$X \sim N\left(\mu,\sigma^2\right)$容量为n的随机样本$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

样本平均数 $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ 样本方差 $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

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估计量的评价标准(样本均值和方差的性质)

一、无偏性

若$\hat{\theta}=\hat{\theta}(X_1,\cdots,X_n)$的数学期望存在， 且$E\hat{\theta}=\theta$,

则称$\hat{\theta}$是θ的无偏估计量.

X为任意总体　$X_1,\cdots,X_n$ 为简单随机样本

$\hat{\mu}=\bar{X}$　μ的矩估计量。

$$E(\bar{X})=E\left(\frac{\sum_{i=1}^{n}X_i}{n}\right)=\frac{\sum_{i=1}^{n}E(X_i)}{n}=\mu$$

样本的中心矩是否是总体中心矩的无偏估计?以*k*=2为例证明

$$E(X)=\mu,\quad D(X)=\sigma^2$$

$$\Rightarrow E(X^2)=\sigma^2+\mu^2$$

$$E(\bar{X})=\mu,\quad D(\bar{X})=D\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n^2}D\left(\sum_{i=1}^{n}X_i\right)=\frac{D(X)}{n}=\frac{\sigma^2}{n}$$

$$\Rightarrow E(\bar{X}^2)=\frac{\sigma^2}{n}+\mu^2$$

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$$\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

$$E\left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right\} = \frac{1}{n}\sum_{i=1}^{n} E\left(X_i^2\right) - E\left(\bar{X}^2\right)$$

$$= \mu^2 + \sigma^2 - \left(\mu^2 + \frac{\sigma^2}{n}\right) = \frac{(n-1)}{n}\sigma^2$$

$$E(S^2) = E\left(\frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}\right) = \sigma^2$$

二、有效性

若$\hat{\theta}_1=\hat{\theta}_1(X_1,\cdots,X_n)$，$\hat{\theta}_2=\hat{\theta}_2(X_1,\cdots,X_n)$都是$\theta$的无偏估计量,且$D(\hat{\theta}_1)\le D(\hat{\theta}_2)$，则称$\hat{\theta}_1$较$\hat{\theta}_2$有效.

X_1,X_2,X_3 是从该总体中抽取的一个样本.

总体的数学期望是 μ

$$X_1 \qquad \frac{X_1+X_2}{2} \qquad \frac{X_1+X_2+X_3}{3} \qquad \frac{5}{10}X_1+\frac{4}{10}X_2+\frac{1}{10}X_3$$

都是未知参数 μ 的无偏估计，并指出哪一个最有效？

解：$E(X_1)=\mu$ $E\left(\dfrac{X_1+X_2}{2}\right)=\mu$

$$E\left(\frac{1}{3}X_1+\frac{1}{3}X_2+\frac{1}{3}X_3\right)=\mu \quad E\left(\frac{5}{10}X_1+\frac{4}{10}X_2+\frac{1}{10}X_3\right)=\mu$$

这表明都是未知参数 μ 的无偏估计.

$D(X_1)=\sigma^2$ $D\left(\dfrac{X_1+X_2}{2}\right)=\dfrac{\sigma^2}{2}$

$$D\left(\frac{1}{3}X_1+\frac{1}{3}X_2+\frac{1}{3}X_3\right)=\frac{\sigma^2}{3}$$

$$D\left(\frac{5}{10}X_1+\frac{4}{10}X_2+\frac{1}{10}X_3\right)=\left(\frac{25}{100}+\frac{16}{100}+\frac{1}{100}\right)\sigma^2$$

三、一致性

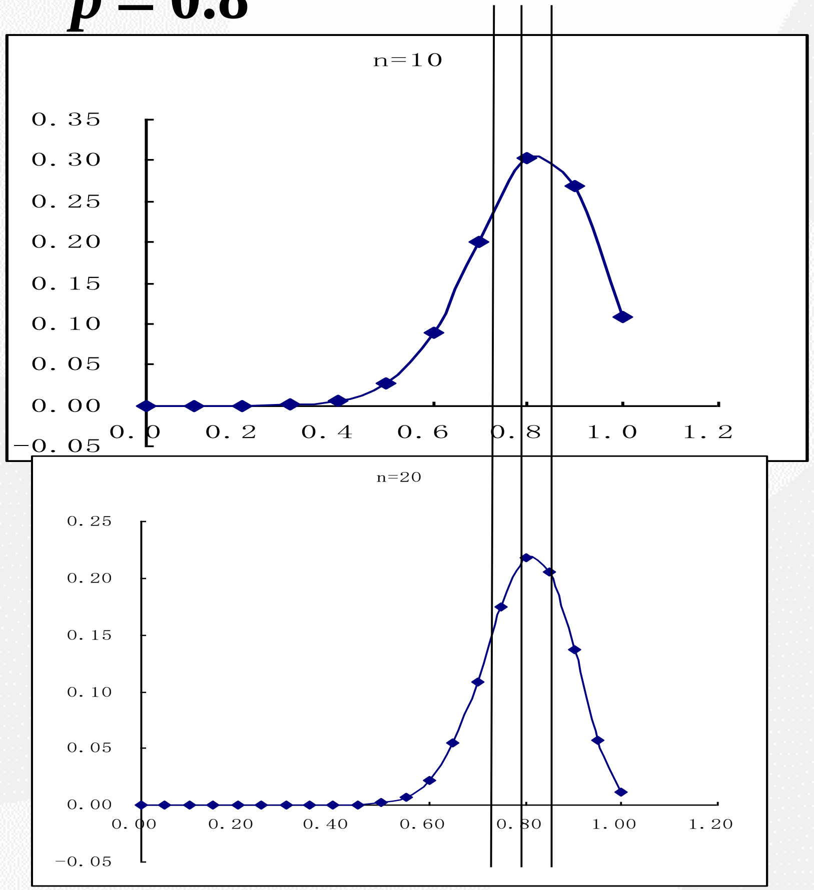
若$\hat{\theta} = \hat{\theta}_n(X_1,\cdots,X_n)$为参数$\theta$ 的估计量,如果

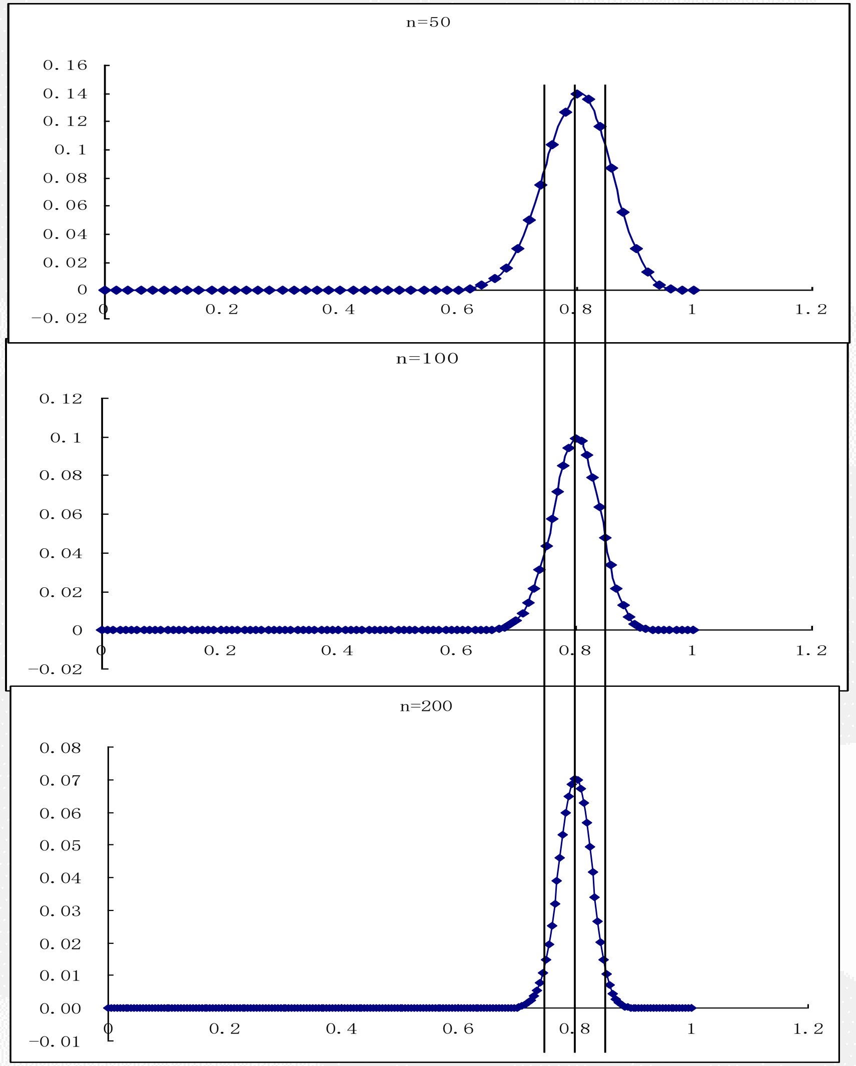
对于任意$\varepsilon > 0, \lim\limits_{n\to\infty} P\left\{\left|\hat{\theta}_n - \theta\right| > \varepsilon\right\} = 0$

或$\lim\limits_{n\to\infty} P\left\{\left|\hat{\theta}_n - \theta\right| \le \varepsilon\right\} = 1$

则称$\hat{\theta}$是θ的一致估计.

$p = 0.8$





$$P\left\{\left|\hat{p} - p\right| \le \varepsilon\right\} \ge 1 - \frac{\frac{p(1-p)}{n}}{\varepsilon^2} \to 1$$

频率是概率的一致估计

$$\bar{X}=\frac{\sum_{i=1}^{n}X_i}{n}\rightarrow E(X)$$

$$E\bar{X}=\mu,\quad D\bar{X}=\frac{\sigma^2}{n}$$

由切比雪夫不等式 $P\left(\left|\bar{X}-\mu\right|<\varepsilon\right)\geq 1-\frac{\sigma^2}{n\varepsilon^2}\xrightarrow{n\to\infty}1$

样本一阶原点矩是总体一阶原点矩的一致估计

同理,样本k阶原点矩是总体k阶原点矩的一致估计

$$\frac{1}{n}\sum_{i=1}^{n}X_i^k\rightarrow EX^k$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2\to EX^2-(EX)^2=\sigma^2$$

样本的二阶中心矩是总体二阶中心矩的一致估计

$$\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{n}{n-1}\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\to\sigma^2$$

样本方差是总体二阶中心矩的一致估计

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正态总体的样本均值与样本方差的分布：

定理1 设$X_1,\cdots,X_n$是总体$N(\mu,\sigma^2)$的样本，$\bar{X},S^2$分别是样本均值与样本方差，则有：

(1) $\bar{X} \sim N(\mu,\frac{\sigma^2}{n})$; $\qquad \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$,

(2) $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$;

可以表示成n-1个相互独立的标准正态的平方和

(3) $\bar{X}$与S^2相互独立

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把来自p元总体$X \sim N_p(\mu, \Sigma)$容量为n的随机样本排成一个$n \times p$矩阵的X：

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{pmatrix} \overset{def}{=} \begin{pmatrix} X'_{(1)} \\ X'_{(2)} \\ \vdots \\ X'_{(n)} \end{pmatrix}$$

其中$X_{(i)}(i = 1, 2, \cdots, n)$是来自$p$元总体的一个样本,则样本数据阵就是一个随机阵.

Multivariate Normal Distribution 多元正态分布

一．多元正态总体样本的数字特征

对于多元统计分析，我们常引入以下多元正态总体样本的相关量.

(1)　样本的均值向量$\bar{X}$：$\bar{X}=\begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n}X_{i1} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n}X_{ip} \end{pmatrix}$

(2)　样本离差阵（交叉乘积阵）L：

$$L=\begin{pmatrix} X_{11}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{12}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{1p}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \\ X_{21}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{22}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{2p}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{n2}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{np}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \end{pmatrix}^{T}\begin{pmatrix} X_{11}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{12}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{1p}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \\ X_{21}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{22}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{2p}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1}-\frac{1}{n}\sum_{i=1}^{n}X_{i1} & X_{n2}-\frac{1}{n}\sum_{i=1}^{n}X_{i2} & \cdots & X_{np}-\frac{1}{n}\sum_{i=1}^{n}X_{ip} \end{pmatrix}$$

Multivariate Normal Distribution 多元正态分布

(3)样本协方差阵

$$S=\frac{1}{n-1}L=\left(s_{ij}\right)_{p\times p}\quad\left(或S^{*}=\frac{1}{n}L\right),$$

称为随机变量X_i的样本方差；样本方差的平方根$\sqrt{s_{ii}}$称为X_i的样本标准差.

(4) 样本相关矩阵R

$$R=(r_{ij})_{p\times p}$$

其中 $r_{ij}=\dfrac{s_{ij}}{\sqrt{s_{ii}}\sqrt{s_{jj}}}\overset{或}{=\!=\!=}\dfrac{l_{ij}}{\sqrt{l_{ii}}\sqrt{l_{jj}}}\quad\left(i,j=1,2,\cdots,p\right).$

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Multivariate Normal Distribution 多元正态分布

以下是μ和Σ的估计的一些性质.

(1) $\bar{X}$是μ的无偏估计，$\dfrac{L}{n-1}$是Σ的无偏估计

(2) $\bar{X}$是μ的有效估计，$\dfrac{L}{n-1}$是Σ的有效估计

(3) $\bar{X}$是μ的一致估计，$\dfrac{L}{n-1}$是Σ的一致估计

Multivariate Normal Distribution 多元正态分布

设$\bar{X}$和L分别为p元正态总体$N_p(\mu,\Sigma)$的样本均值向量和样本离差阵，则

(1) $\bar{X} \sim N_p\left(\mu, \frac{1}{n}\Sigma\right)$;

(2) $L = \sum_{t=1}^{n} Z_t Z_t'$，其中$Z_1, Z_2, \cdots, Z_n$ 独立且服从$N_p(O,\Sigma)$分布;

(3) $\bar{X}$和L相互独立;

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χ^2-分布和Wishart分布

定义1 设 $X_1, X_2, \cdots, X_n$ 为 相互独立且同服从于 $N(0,1)$ 分布的随机变量。则

$$\chi^2 = \sum_{i=1}^{n} X_i^2$$

所服从的分布叫做χ^2 –分布,n 称为自

由度且记为$\chi^2 \sim \chi^2(n)$ 。

定理．设 $X_1 \sim \chi^2(n_1)$ $X_2 \sim \chi^2(n_2)$，

且 X_1 **与** X_2 **相互独立，则**

$$X_1 + X_2 \sim \chi^2(n_1 + n_2)$$

Wishart分布

定义1 设 $X_1, X_2, \cdots, X_n$ 为 相互独立且同服从于分布 $N_p(0, \Sigma)$ ，令 $\boldsymbol{X} = [X_1, X_2, \cdots, X_n]'$ 则

$$W = XX' = \sum_{i=1}^{n} X_i X_i'$$

所服从的分布叫做 自由度为 n 的p维-维希特分布，记作 $W \sim W_p(n, \Sigma)$

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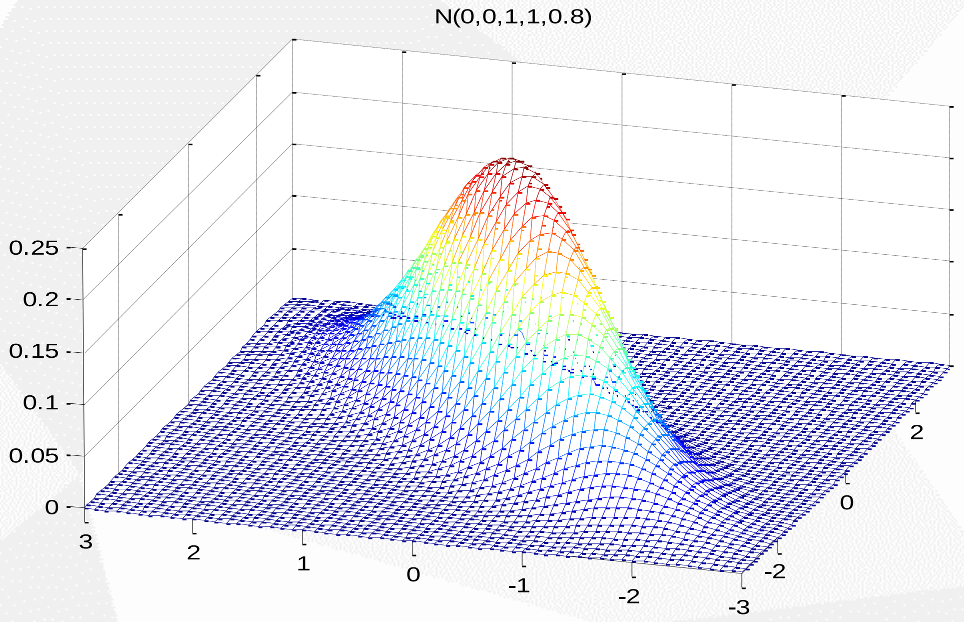

$$X_i \sim N_2\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1 & 0.8\\0.8 & 1\end{pmatrix}\right), i=1,2,3,4,5$$

$$X_1X_1^{'}+X_2X_2^{'}+X_3X_3^{'}+X_4X_4^{'}+X_5X_5^{'} \sim W_2\left(5,\begin{pmatrix}1 & 0.8\\0.8 & 1\end{pmatrix}\right)$$

显然，当 *p*=1， $\Sigma=\sigma^2$ **时，有**

$$W_1(n,\sigma^2)=\sigma^2\chi^2(n)$$

Wishart分布像卡方分布一样具有加法性质，若

$$W_1\sim W_p(n_1,\Sigma),W_2\sim W_p(n_2,\Sigma)$$

相互独立，则

$$W_1+W_2\sim W_p(n_1+n_2,\Sigma)$$

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Multivariate Normal Distribution 多元正态分布

测定稻谷每亩穗数X1,每穗粒数X2,每亩稻谷产量X3,

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3(\mu, \Sigma)$$ 求参数的无偏估计。

X1	26.7	31.3	30.4	33.9	34.6	33.8	30.4	27	33.3	30.4	31.5	33	34
X2	73.4	59	65.9	58.2	64.6	64.6	62.1	71.4	64.5	64.1	61.1	56	59.8
X3	1008	959	1051	1022	1097	1103	992	945	1074	1029	1004	995	1045

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Multivariate Normal Distribution 多元正态分布

$$\bar{X} = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{pmatrix} = \begin{pmatrix} 31.56 \\ 63.44 \\ 1024.92 \end{pmatrix}$$

$$S = \frac{1}{n-1} L$$

$$= \frac{1}{13-1} \begin{pmatrix} 26.7-31.56 & \cdots & 34-31.56 \\ 73.4-63.44 & \cdots & 59.8-63.44 \\ 1008-1024.92 & \cdots & 1045-1024.92 \end{pmatrix} \begin{pmatrix} 26.7-31.56 & 73.4-63.44 & 1008-1024.92 \\ \vdots & \vdots & \vdots \\ 34-31.56 & 59.8-63.44 & 1045-1024.92 \end{pmatrix}$$

$$= \begin{pmatrix} 6.1 & -8.41 & 72.83 \\ -8.41 & 22.76 & 3.00 \\ 72.83 & 3.00 & 2172.69 \end{pmatrix}$$

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