Based on Strang's *Introduction to Applied Mathematics*

Operation Counts for Gaussian Elimination

Gaussian Elimination. $A = LU$ if no row exchanges are needed:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} p_1 & u_{12} & u_{13} & u_{14} \\ 0 & p_2 & u_{23} & u_{24} \\ 0 & 0 & p_3 & u_{34} \\ 0 & 0 & 0 & p_4 \end{bmatrix}$$

where $p_1, \ldots, p_4$ are the pivots of A. With row exchanges, $PA = LU$. Gaussian elimination involves multiplying the pivot row j by l_{ij} and subtracting from row $i > j$ to produce a 0 in the ij position.

(Elimination matrix E_{ij} is lower triangular, 1's on diagonal, single off-diagonal element $-l_{ij}$. E_{ij}^{-1} is lower triangular, 1's on diagonal, single off-diagonal element l_{ij}. For a 3×3 matrix A, $L = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$.)

To solve $Ax = b$, $Ax = b \to Ux = c$ where $Lc = b$.

Operation Counts for $n \times n$ matrix A. (i) $A \to U$: The first stage of elimination (column 1) produces 0's beneath the first pivot. Each new 0 below the pivot requires 1 multiplication and 1 subtraction $= n^2 - n \approx n^2$ flops[1]. The second stage of elimination requires approximately $(n-1)^2$ flops. Thus the complexity for $A \to U$ is

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 = \frac{1}{3}n\left(n + \frac{1}{2}\right)(n+1) \approx \frac{1}{3}n^3$$

(ii) Forward elimination on the right-hand side $b \to c$ requires

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{1}{2}(n-1)n$$

flops, and then (iii) to back solve for x from $Ux = c$ requires

$$1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$$

[1] *flop* = floating point operation = add-multiply.

operation	*matrix type*	*flops*
$A = LU$	full	$n^3/3$
$Ux = c$	full	n^2
$A = LU$	banded	$w^2 n$
$Ux = c$	banded	$2wn$

Table 1: Asymptotic operation counts for $n \gg 1$, $w \gg 1$. The bandwidth w = number of bands above (and below) the main diagonal. To find A^{-1} (which we almost never need) requires n^3 flops.

flops, for a total of n^2 flops.

For a banded matrix, there are only w nonzero diagonals below (and above) the main diagonal. Note: We must allow for fill-in out to the outermost sub- and superdiagonals. The 0's outside the band remain 0 during elimination in U and L.

For a tridiagonal matrix, $w = 1$, and the operation count for Gaussian elimination is $5n$ (see tridisolve( )).