QSIURP Report

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July 26th 2015

Abstract :

A tree is a type of graph that is connected (there is a path between every pair of vertices) and is acyclic (there is a unique path between any pair of vertices). A tree on n vertices always has $n-1$ edges. Let a graph be denoted by $G = (V, E)$ where V is the set of vertices and E is the set of edges. An edge between two vertices u and v is denoted by (u, v), A vertex labelling of G with the numbers from 1 to n is an injective function $f : V \to \{1, 2, ..., n\}$. A graph is called 'graceful' if there exists some labelling of it's vertices (with numbers from 1 to n) such that the induced edge labelling is graceful meaning that $g : E \to \{1, 2, ..., n-1\}$ given by $g((u, v)) = |f(u) - f(v)|$ is a bijection. You can also equivalently define the vertex labelling to begin from 0 instead of 1 (and end at $n-1$), and the edge labels will remain unchanged.

The Graceful-Tree Conjecture (GTC) is a famous, unsolved problem in Graph Theory; it states that all trees are graceful (they admit a graceful labelling). Posed first by Rosa in 1967 [1], the problem remains unsolved despite continuous efforts over the years. In summer 2015, we, firstly, explored the work already done in this area and, secondly, experimented with an indirect approach to the problem. In this paper, we provide several proofs, in our own words, of the gracefulness of several classes of trees as shown by mathematicians over time. We also present the method that we tried to tackle this infamous problem.

Classes of trees proven graceful :

There are two general approaches that have been taken to tackle this problem. The first, which is the more mathematical one and the one that I focused on, is to show that all trees having a specific structure (belonging to a "class") are graceful, the idea being that you gradually prove this for all possible structures. The second approach is to use computers to brute force all possible labellings of trees to show that all trees upto certain number of vertices are graceful. For example, it has been shown that all trees with upto 35 vertices are graceful [2]. Of course, the larger a tree is, the harder it is to find a graceful labelling, so this approach has computational limitations.

We will now look at proofs for some classes of trees that have been shown to be graceful. We start with the simplest.

Paths :

A path is a tree that has only two leaves (vertices with degree 1); all other vertices have degree 2.

Proof of gracefulness:

Let T be a path with n vertices. Then, starting from one end, label the vertices along the path alternating between the highest and lowest unused label from the set $\{1, 2, 3, ..., n\}$; one sequence of vertex labels can be $(n, 1, n-1, 2, n-2, 3, ...)$. This will result in the corresponding edge labels along the path being $(n-1, n-2, n-3, ..., 1)$ which is a graceful labelling. Thus, every path has a graceful labelling.

Caterpillars :

A caterpillar is a tree such that if you remove all of it's leaves (vertices with degree 1), then the remaining graph is a path. Every caterpillar has a central path that can be obtained by removing it's leaves (there may be multiple such paths). They were first proved graceful by Rosa, 1967, himself [1].

Proof of gracefulness:

Let T be a caterpillar with n vertices. If $n \leq 2$, then T is just a path which we know to be graceful, so assume $n \geq 3$. Let P be the central path obtained when you remove the leaves from T, and let it's vertices be denoted by $v_1, v_2, v_3, .., v_k$. Considering the graph T, we can have any number of leaves (vertices with degree 1) attached to any of the vertices in the path P. If a vertex, v_i, in path P has h leaves attached to it (ignoring the neighbors from P itself) then we denote those vertices (leaves) by $u_{i,1}, u_{i,2}, ..., u_{i,h}$.

We need to label the vertices of T from the set $\{1, 2, 3, ..., n\}$. Because $n \geq 3$, we know that the central path, P, has at least one vertex. We partition the vertex set, V, of T into two based on the parity of their distance from v_1(the first vertex in the path). So, we let

$T_{odd} = \{x |\ x \in V(T),\ d(x, v_1) \equiv 0\ (mod\ 2)\}$ and $T_{even} = \{x |\ x \in V(T),\ d(x, v_1) \equiv 1\ (mod\ 2)\}$.

For each of the v_i (the vertices on the path P), v_i is in T_{even} if i is even and in T_{odd} otherwise. We will label T by walking along the vertices in path P, as below:

We label $v_1$with n. Then, supposing v_1 has h_1 leaves attached to it, we label these leaves with $1, 2, 3, ..., h_1$ and then v_2 with $h_1 + 1$. So we label the neighbors of v_1 with the smallest unused labels ending with the largest label on v_2. Then, we label the neighbors of v_2 with the largest unused labels (which begins with $n - 1$) and end with the smallest label on v_3.

In general, after we have labelled vertex v_{2i}, we label it's neighbors (in decreasing order) starting with the largest unused label and ending with the smallest label on v_{2i+1}. Then, assign labels to the neighbors of v_{2i+1}(except v_{2i} which is of course already labelled) in increasing order starting with the smallest unused label and ending with the largest label on v_{2i+2}.

The overall vertex labelling gives the labels $n, n-1, n-2, n-|T_{odd}|+1$ to the vertices in T_{odd}, and the labels $1, 2, 3, ..., |T_{even}|$ to the vertices in T_{even}. Also, as $n - |T_{odd}| = |T_{even}|$, every vertex got a distinct label.

The edge labelling that is induced starts from $n-1$ (the edge between $v_1$and it's first neighbor) and then decreased by 1 for the next edge label induced by the subsequent vertex label we give out. For example, if v_1 has h_1 leaves (they get labels $1, 2, .., h_1$) attached to it, then the h_1 edges to these leaves get the labels $n-1, n-2, ..., n-h_1$ and the edge (v_1, v_2) gets the label $n - h_1 - 1$. Then, if v_2 (labelled $h_1 + 1$) has h_2 leaves (they get labels $n-1, n-2, .., n-h_2$) attached to it, then the h_2 edges to these leaves get the labels $n - h_1 - 2, n - h_1 - 3, ..., n - h_1 - h_2 - 1$ and the edge (v_2, v_3) gets the label $n - h_1 - h_2 - 2$. After this we will continue assigning edge labels in decreasing order.

Thus, the order that we label the vertices in, is also a decreasing order on the edge labels induced beginning at $n - 1$ and ending at 1. Thus, the edge labelling is graceful, and every caterpillar is graceful.

A class of Spiders :

A spider is defined to be a tree where a set of paths (called legs) are joined together by a central vertex, which has degree greater than 2 (otherwise it's just a path). Certain kinds of spiders have been shown to be graceful. One important such class is the set of all spiders with each of their legs having length m or $m+1$ (for some m). The algorithm that gracefully labels such a spider is given and proved below, attributed to [3].

Proof of gracefulness (of spiders having l legs all of which are length m or $m+1$):

Let T be a spider with l legs (assume $l \geq 3$) where length of each leg is m or $m+1$. There are two cases:

1) Case 1 (l is odd):

Let l_s be the legs with length m and let l_l be the remaining legs of length $m+1$; note that $l = l_s + l_l$. This implies that$T$ has $n = lm + l_l + 1$ total vertices which need to be labeled from $\{1, 2, ...n\}$. We can consider the tree T to be rooted at it's central vertex, denoted by $v*$, and it's l legs running downwards. We will position the l_l longer legs ($m+1$ length) all to the left and then come the l_s shorter legs (m length) to the right. We will name each leg left to right by $L_1, L_2, ..., L_l$. This means that legs $L_1, L_2, ..., L_{l_l}$ are the legs of length $m+1$ and then $L_{l_l+1}, L_{l_l+2}, ..., L_l$ are of length m. We denote by $v_{i,j}$ the vertex that is in leg L_i at a distance j from $v*$(the central vertex). Now, we label each vertex in the tree using the function f below:

i) $f(v*) = 1$

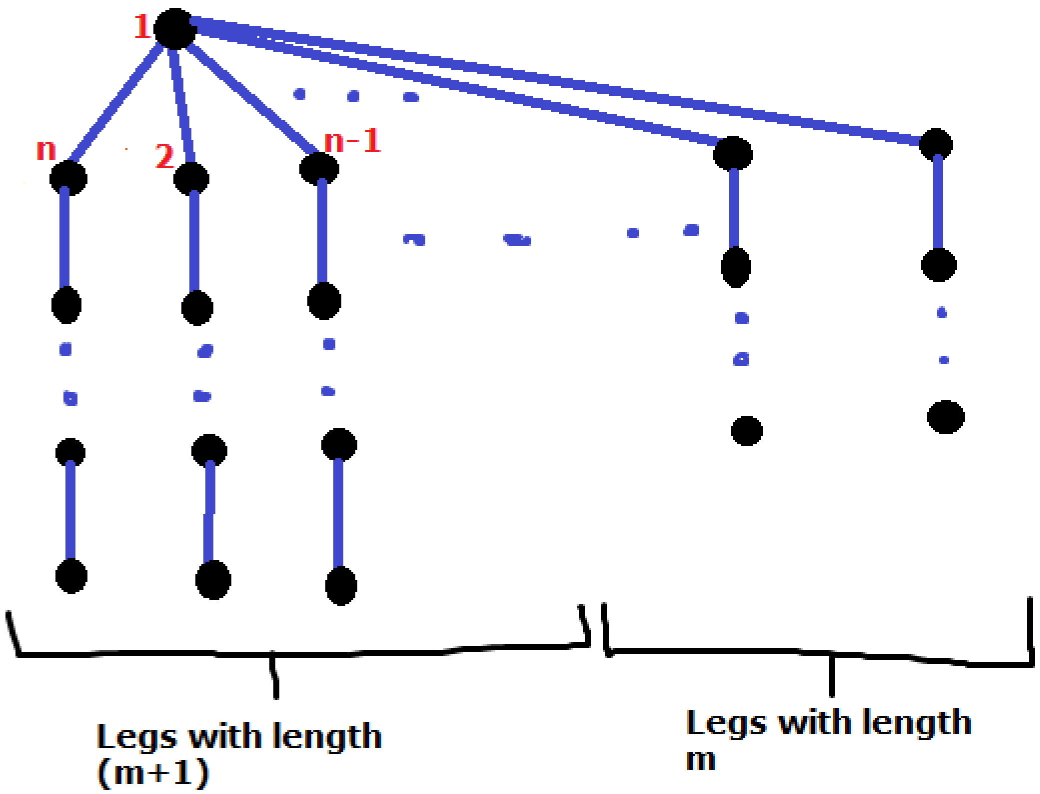
ii) If i and j are both odd, then $f(v_{i,j}) = n - \frac{i-1}{2} - \frac{(j-1)l}{2}$

iii) If i and j are both even, then $f(v_{i,j}) = n - \frac{l-1}{2} - \frac{(j-2)l}{2}$

iv) If i is even and j is odd, then $f(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2} + 1$

v) If i is odd and j is even, then $f(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} + \frac{(j-2)l}{2} + 1$

The piecewise function above looks complicated but it's actually simple in practice. It assigns labels from $\{1, 2, 3, ..., n\}$ to vertices in T by allocating 1 to the root ($v*$) and then it traverses left to right (the longer length legs first) and downwards on the legs of the spider spiralling away from the root of the tree alternating between the highest and the lowest unused label. A diagram to illustrate this process is given (first 4 vertices are labelled only):



Now, we need to show that the edge labelling is graceful.

First off, it is clear that all edge labels, will come from the set $\{1, 2, ..., n-1\}$ as the vertices all come from the set $\{1, 2, ..., n\}$. And we know that the total number of edges in T is $n-1$. So, if we can show that all edge labels induced by the vertex labelling f are distinct, then by the pigeonhole principle, we would have shown that the edge labelling is graceful (i.e. it is in a bijection with $\{1, 2, ..., n-1\}$).

For that purpose, notice that for each leg of the spider, the vertex labelling f places the maxima of the leg at those positions where $i \equiv j \ (mod\ 2)$. So, for example for L_1, the maxima will lie at odd distances (1,3,...) from the root. For such a vertex (a maxima), we can compute the label for two edges coincident (assuming vertex is not the last one in it's leg so it has degree 2) on it as follows:

$f(v_{i,j}) - f(v_{i,j+1}) = n - \frac{l-1}{2} - i + (1-j)l$

and $f(v_{i,j}) - f(v_{i,j-1}) = n - \frac{l-1}{2} - i + (2-j)l$

Both of the above are positive values because $v_{i,j}$ is a local maxima (while $v_{i,j+1}$ and $v_{i,j-1}$ are minima).

Now, suppose for the sake of contradiction that there exists an $(i', j') \neq (i, j)$, where $i \equiv j \pmod 2$ and $i' \equiv j' \pmod 2$. Now, there are three sub-cases to consider:

i) $f(v_{i,j}) - f(v_{i,j+1}) = f(v_{i',j'}) - f(v_{i',j'+1})$

Then, we get from the equations above that $i - i' + (j - j')l = 0$

$l = \frac{i-i'}{j'-j}$

We know that $j' \neq j$, since otherwise i must also equal i' which is impossible because $(i, j) \neq (i', j')$. Thus, $|j' - j| \geq 1$ and $|i - i'| < l$ (this is obvious since i and i' are leg numbers coming from 1 to l).

This implies, that $l = \frac{i-i'}{j'-j} < \frac{l}{1} = l$ which is a contradiction.

ii) $f(v_{i,j}) - f(v_{i,j+1}) = f(v_{i',j'}) - f(v_{i',j'-1})$

Then, we get from equations above that $i - i' + l(j' - j - 1) = 0$.

$l = \frac{i-i'}{j'-j-1}$

Now, $j' \neq j+1$, because otherwise $i' = i$ which is a bad case because that means that the two edges we are considering are actually the same: it is the edge between $v_{i,j}$and $v_{i',j'}$.

Thus, $j' \neq j+1$, which then means that $|i - i'| < l$ and $|j' - j - 1| \geq 1$, and now the same contradiction arises as case (ii):

$l = \frac{i-i'}{j'-j-1} < \frac{l}{1} = l$

iii) $f(v_{i,j}) - f(v_{i,j-1}) = f(v_{i',j'}) - f(v_{i',j'-1})$

Then, we get from equations above that $i' - i + l(j' - j) = 0$

$l = \frac{i-i'}{j-j'}$

We know that $j' \neq j$, since otherwise i must also equal i' which is impossible because $(i, j) \neq (i', j')$. Thus, $|j' - j| \geq 1$ and $|i - i'| < l$ (this is obvious since i and i' are leg numbers coming from 1 to l).

This once again implies, that $l = \frac{i-i'}{j'-j} < \frac{l}{1} = l$ which is a contradiction.

And this means that no two edges hold the same labels and thus the labelling induced by the function f is graceful.

2) Case 2 (l is even):

We have a spider, T, with $l \geq 4$ legs, where l is even, and all legs are length m or $m + 1$. Now, after we have named the legs from L_1 to L_l, simply remove L_l from the graph to get T',which is a spider with an odd $l - 1$ number of legs and satisfying the other conditions for Case (1). Thus, we can get a vertex labelling f' for T' that is graceful. We will use this to construct a labelling for T.

First, we will prove an intermediate statement:

"If f (a vertex labelling) is a graceful labelling on a graph T (with n vertices), then $f'(v) = n - f(v) + 1$ is also a graceful labelling on the graph T."

Proof:

f' is graceful because this mapping does not change any of the edge labels (the original labelling was graceful by assumption) and because every vertex label is still distinct.

Firstly, Suppose for the sake of contradiction that two different vertices (u and v) now have the same labels. Then, $f'(u) = f'(v)$

$n - f(u) + 1 = n - f(v) + 1 \rightarrow f(u) = f(v)$ which is a contradiction because we know that f was a valid graceful labelling.

Secondly, let edge (u, v) have label x under the labelling f. Then, $|f(u) - f(v)| = x$. After the new mapping, f', this edge will have the label,

$|f'(u) - f'(v)| = |n - f(u) + 1 - n + f(v) - 1| = |f(v) - f(u)| = x$. So, the edge labels do not change. Thus, f' is also a graceful labelling.

Now, we have a graceful labelling f' for T'. And we know from case (1) that this labelling places the vertex label 1 at the central vertex (the root). Then, get the complementary labelling of f' ($f''(u) = n - f'(u) + 1$), and call it f''. Under this graceful labelling (proved above why f'' is graceful), we will have label n at the root ($v*$) of T'.

Now, a leg is missing from T' that we removed earlier. We will attach one vertex to $v*$, call it v_1. Add 1 to all the other vertex labels (given by f'') and give v_1 the label 1. Call this labelling f_{v1}. This, is a graceful labelling because all the older edges have the same labels as f'' and the new edge between v_1 and $v*$ has the label n (which is what was missing since before the highest edge label was $n-1$). Now, we can take the complementary labelling of f_{v1}, call it $f_{v1'}$, and under this we get $n+1$ as the label of v_1. Now, we can repeat the above process by appending another vertex, v_2, to v_1, labelling it 1 and adding 1 to all older labels to get a graceful labelling for the extended tree, and then get the complementary labelling again. Continuing in this way, we can add however many vertices we can, all the while maintaing a graceful labelling. Thus, we can add the number of vertices that were in the leg of T that was missing from T' (Note that we can add as many vertices as we want, not just the number required for the length of the leg (m/ m+1)), and arrive at a graceful labelling of T. Thus, T is graceful and case (2) holds as well.

Symmetrical trees :

A symmetrical tree is a rooted tree, such that all vertices within a level are of the same degree, for every level of the tree. It was shown first in [4] that all symmetrical trees are graceful: I give here the proof instead by Robeva, 2011, given in [5].

Before giving the proof of gracefulness, note a lemma: For any symmetrical tree, T, with root v, if we consider the vertices $v_1, v_2, ..., v_k$ at it's first level, and then take the k sub-trees $T_1, T_2, ..., T_k$, then these k sub-trees will be symmetric and isomorphic (equivalent graphs) to each other. The proof of this lemma is omitted here.

Proof of gracefulness:

We will show by induction on the number of levels in a symmetrical tree that every symmetrical tree has a graceful labelling which places the label 1 at the root.

Base case:

Let levels $l = 0$, then the tree is just 1 vertex with no edges and is ofcourse graceful.

Let levels $l = 1$, then the tree is a root connected to an arbitrary number of vertices (a star graph). This is graceful because you place 1 at the root and then assign labels 2 through n to the rest of the vertices arbitrarily giving a graceful labelling.

Inductive case:

Suppose as the Induction Hypothesis (IH) that for some $l \geq 1$, all symmetrical trees with l levels are graceful and have a graceful labelling that places 1 at their roots.

Now, consider any symmetrical tree, T, with $l+1$ levels. Let $v*$ be the root of T, and let $v_1, v_2, .., v_k$ be the vertices in the first level of T, and let $T_{1,...}, T_k$ be the symmetrical, isomorphic rooted trees at the vertices, each of which has l levels. Then, by the IH we have a graceful labelling f for T_1(which places 1 at it's root) which also ofcourse works as a graceful labelling for $T_2, ..., T_k$. We label T in the following way:

1) We initially label each of the sub-trees $T_1, ..., T_k$ with f (gracefully label each sub-tree with an identical labelling) and then we add certain numbers to each of the vertex labels in the sub-trees.

2) Let the sub-trees $(T_1, ..., T_k)$ be ordered left to right beneath the root. Then, letting n be the number of vertices in each sub-tree (total number of vertices in T would then be $nk+1$), we start from the 1^{st} layer (the layer with $v_1, ..., v_k$, so 0^{th} layer for $T_1, ..., T_k$) and moving left to right we add $n(k-1)$ to v_1, $n(k-2)$ to v_2,..., $n(k-k) = 0$ to v_k. Then, for the 2^{nd} layer of T, we move from

right to left now and add $n(k-1)$ to each vertex in the 1^{st} layer of T_k, $n(k-2)$ to each vertex in the 1^{st} layer of T_{k-1},..., and 0 to the vertices in 1^{st}layer of T_1. Then, we move to the 3^{rd} layer of T, move from left to right now and do the process. We do this for each layer downwards alternating the direction in which we move.

3) We then label $v*$, the root of T, with $nk+1$ (the highest label which is also unused so far: the highest label in a sub-tree was n and the highest number we added was $n(k-1)$ so the highest label we've assigned is nk).

4) The resulting labelling (f_t) is graceful (proof comes later below). And finally to get 1 as the label of the root ($v*$), we get the complementary labelling by applying the mapping $f_t'(x) = (nk+1)+1-x$ which ofcourse is also graceful and places 1 at the root as desired.

Now, it should be made clear that every vertex in T gets a distinct label. This is true because we know there are n vertices in T_i for each i, and that the labelling f used a distinct label from $1, 2, ..., n$ for each vertex. Now, we can look at it this way: we have k copies of the sub-tree T_1 all having the same labelling (f) on them initially, and we have nk vertices in total. Now take any arbitrary vertex in T_1, call it v_s, and let it's original label be x ($1 \leq x \leq n$). Then, there is an equivalent of this vertex in each of the other sub-trees $T_2, ..., T_k$: a vertex in the same level in the same position with the label x. The transformation above takes k vertices labelled x and re-labels them as $x, n+x, 2n+x, ..., (k-1)n+x$. All these are different labels (but they're all x modulo n:). Also another label's k copies are mapped to some other class modulo n so there's no overlap there either. So, with the k copies of labels $1, ..., n$, we get all labels $1, ..., nk$, and then we place $nk+1$ at the root of T.

Now, we show how the edge labelling is graceful. We will show that the edges cover all values in the set $\{1, 2, ..., nk\}$. f is the original labelling of the sub-trees. For the vertices in level l of T_i (l starts at 0), the new labelling (f_t) is defined as $f_t(v) = f(v)+(k-i)n$ if l is even and $f_t(v) = f(v)+(i-1)n$ if l is odd. Consider first the edges between the root ($v*$) of T and the first layer $v_1, ..., v_k$. These have the labels: $|f_t(v*) - f_t(v_i)| = |nk+1-(1+n(k-i))| = |ni|$ for $i \in \{1, 2, ..., k\}$. These then form the edge labels $\{n, 2n, ..., kn\}$.

Now, consider two adjacent vertices, $u_{1,x}, u_{1,y}$ in T_1, where $u_{1,x}$ is in some level l and $u_{1,y}$ is in level $l+1$. Let the corresponding pair of these vertices in sub-tree $T_i$be $u_{i,x}$ and $u_{i,y}$.

Now, if l is odd, we get the edges,

$\{|f_t(u_{i,x}) - f_t(u_{i,y})| : \ i \in \{1, 2, ..., k\}\} = \{|f(u_{i,x}) + (i-1)n - f(u_{i,y}) - (k-i)n| : \ i \in \{1, 2, ..., k\}\}$
$= \{|f(u_{i,x}) - f(u_{i,y}) - n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\} = \{|f(u_{i,y}) - f(u_{i,x}) + n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\}$

If l is even, we get the edges,

$\{|f_t(u_{i,x}) - f_t(u_{i,y})| : \ i \in \{1, 2, ..., k\}\} = \{|f(u_{i,x}) + (k-i)n - f(u_{i,y}) - (i-1)n| : \ i \in \{1, 2, ..., k\}\}$
$= \{|f(u_{i,x}) - f(u_{i,y}) + n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\}$

Notice that $1 \leq |f(u_{i,x}) - f(u_{i,y})| \leq n-1$, because these are edge labels from the original labelling f.

Now, consider that for $1 \leq |m| \leq n-1$, we have,

$\{|m+n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\}$
$= \{m+n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{-m-n(k-2i+1) : \ \frac{k+1}{2} < i \leq k\} \cup \{|m| : \ i = \frac{k+1}{2}, i \in \mathbb{N}\}$

Now, the second componet (set) in the above expression can be rewritten as,

$\{-m-n(k-2i+1) : \ \frac{k+1}{2} < i \leq k\} = \{-m-n(k-2(k+1-i)+1) : \ 1 \leq i < \frac{k+1}{2}\}$

$= \{-m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\}$

Now, plugging this back into the original 3-set union, we get

$\{m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{-m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{|m| : \ i = \frac{k+1}{2}, i \in \mathbb{N}\}$

$= \{|m| + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - |m| + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{|m| : \ i = \frac{k+1}{2}, i \in \mathbb{N}\}$.

These are the new edges that we get.

We take two sub-cases:

1) We consider all $1 \leq |m| \leq n-1$ such that $m \neq n-m$. Then, there are two different edges $uu' \in E(T_1)$ and $vv' \in E(T_1)$ such that $|f(u) - f(u')| = m$ and $|f(v) - f(v')| = n - m$. Let u and v be in odd layers of T_1, so then $u'$and v' are in the even layers of T_1.

We then get the following two sets of labels in T induced by f_t (the labelling for the whole tree):

$\{|f(u) - f(u') + n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\} \cup \{|f(v) - f(v') + n(k-2i+1)| : \ i \in \{1, 2, ..., k\}\}$

By the expansion we found above, we get

$= \{m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\}$

$\cup \{m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{m : \ i = \frac{k+1}{2}\} \cup \{n - m : \ i = \frac{k+1}{2}\}$.

Now, if k is odd then

$\{m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{m : \ i = \frac{k+1}{2}\} = \{m + in : \ 0 \leq i < k\}$

On the other hand, even if k is even, we still get (we lose the 3rd set in the union because i is never $\frac{k+1}{2}$):

$\{m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} = \{m + in : \ 0 \leq i < k\}$.

In the same way, in both cases we get (from the other 3 sets):

$\{n - m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - m : \ i = \frac{k+1}{2}\} = \{n - m + in : \ 1 \leq i < k\}$

So, for each $1 \leq m \leq n-1$ (given that $m \neq n - m\}$, the set $\{m + in : \ 1 \leq i < k\}$ is in the induced edge labels of T.

2) If on the contrary, $m = n - m$.

Now, there is $uu' \in E(T_1)$ such that $|f(u) - f(u'| = m$, and u is in an odd layer of T_1. Then, under f_t, we get in the edge labels of T,

$\{|f(u) - f(u') + n(k-2i+1) : \ 1 \leq i \leq k|\}$

$= \{m + n(k-2i+1) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{n - m + n(k-2i) = m + n(k-2i) : \ 1 \leq i < \frac{k+1}{2}\} \cup \{m : \ i = \frac{k+1}{2}\}$

Once again, whether k is odd or *even*, this becomes the set, $\{m + in : \ 0 \leq i \leq k\}$.

Thus, after the transformation, from the edges in $T_1, ..., T_k$, we get all edge labels $\{m + in : \ 0 \leq i < k\}$ for $1 \leq m \leq n-1$ and from the edges between the root of T (v*) and $v_1, v_2, ..., v_k$ we get the edge labels $\{in : \ 1 \leq i \leq k\}$. Thus, all edge labels in the set $\{1, ..., nk\}$ are covered by one of the edges in T, and as the total number of edges in T are nk, f_t is in face a bijection between the edge set $E(T)$ and $\{1, ..., nk\}$. Thus, this is a graceful labelling. And by a lemma shown before the complementary labelling that we produce, f'_t, is also graceful and places 1 as $v*$'s vertex label.

Thus, the inductive case holds, and all symmetrical trees are graceful.

The connection between parking functions and labelled trees :

We explored parking functions and their one to one correspondence with labelled trees. We had hoped to draw some insights into the Graceful-Tree problem by working in the parking functions' universe instead of directly on trees.

Definition:

A parking function of size $n (\in \mathbb{N})$ is a sequence of positive integers of size n, $(a_1, a_2, ..., a_n)$, such that if $(b_1, b_2, ..., b_n)$ is the non-decreasing re-arrangement of that sequence then $b_i \leq i$ for each i.

By the well known Cayley's formula, we know that the number of labeled trees on n vertices for any positive integer n is equal to n^{n-2}.

For parking functions, it is known that the number of parking functions of size n is $(n+1)^{n-1}$ [6] . The proof is omitted here.

This means that the number of parking functions of size n is equal to the number of labeled trees of size $n+1$. Now, each labeled tree, can have it's edges labeled as the absolute difference of their end-points. So, you can ascertain whether a labelled tree, is 'gracefully' labeled or not by checking if no two edges have the same label (Note however a negative result here does not mean the tree itself, 'the unlabeled graph', is not graceful because there may be another labelling for the same tree that is graceful).

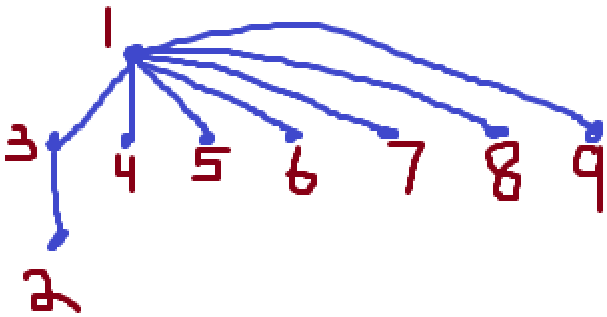
We studied an explicit bijection, f, between labelled trees and parking functions. Several bijections between the two sets have been studied over the years.

The one we used worked like this:

Given a parking function,$P = (x_1, x_2, ..., x_n)$, we can construct a labelled tree of size $n+1$. We create a vertex labelled 1 which will act as the root of the tree, T. Now, the number at index i in P signifies the position of the vertex labelled $i+1$ in the tree T, relative to the root (which is labelled 1). We start from the lowest numbers in P to the largest numbers. First we take indices that have label 1 (Note that there has to be at least one otherwise it's not a parking function!), suppose they are x_i, x_j, x_k where $i < j < k$, then we attach vertices labelled $i+1, j+1, k+1$ to the root (labelled 1) and keep them in that order from left to right. Now, we see if there are any indices with label 2. If there are any, say x and y $(x < y)$, then we attach vertices with label $x+1$ and $y+1$ to the first vertex (from left) at level 1 (root is at level 0), which in this case is the vertex labelled $i+1$. Then, we for all indices with label 3, we attach the relevant vertices (add 1 to the index) to the vertex labelled $j+1$. And so on. We keep all vertices within a level in increasing order from left to right. In short, once we are done, children of the root correspond to label 1 in the parking function. Children of the left most child of the root correspond to label 2. Children of the next vertex correspond to label 3. When we reach the last vertex on level 1, we then move to the left-most vertex on level 2. The transformation from labelled tree back to the parking function is basically the same process interpreted in the opposite direction.

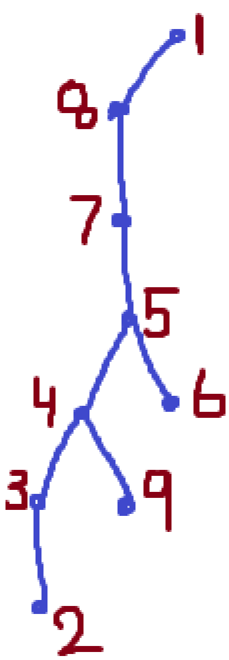
As an example, $(2, 1, 1, 1, 1, 1, 1, 1)$ will be mapped to:

(2,1,1,1,1,1,1,1)



and $(7, 5, 4, 3, 4, 2, 1, 5)$ will be mapped to:

(7,5,4,3,4,2,1,5)



We were able to form our own proof for why this function is a bijection. The proof, however, is omitted here due to a time constraint.

Our idea was to look at all labeled trees, T, for small values of n, get all parking functions, P_f, corresponding to that set (size $n-1$), have a routine that paritioned the set P_f into two parts according to whether the parking function corresponds under f to a gracefully labeled tree or otherwise. I wrote a Python script that did this for values of n upto 10.

We were able to learn about 'graceful' and 'non-graceful' parking functions. We tried to form idea as to what causes a parking function to be graceful (that information could've been useful for the GTC), but were not able to make significant progress (besides small observations like all parking functions of the form [2,1,1,1,...] are graceful). Further thought is needed on this matter: both in the differences between graceful and non-graceful parking functions and in how these differences can help us in the realm of labeled trees.

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