Graceful Trees and Parking Functions

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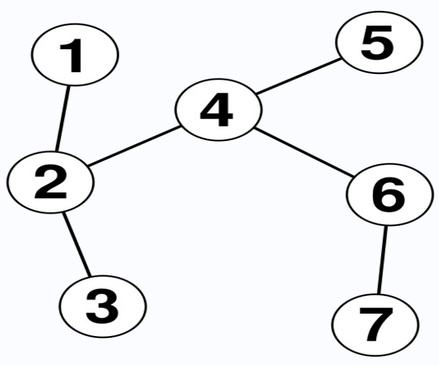
Abstract

A math problem (Graceful Tree Conjecture) asks **'whether all trees have a graceful labeling'**. Posed in the 1960s by Rosa [1], it remains unsolved. Besides being an elegant problem, graceful labellings have applications in **coding theory** and **communication network addressing** (MPLS multi-casting using gracefully labeled caterpillar trees [2]). We explore the connection between labeled trees and parking functions, hoping to get insight into the elusive GTC.

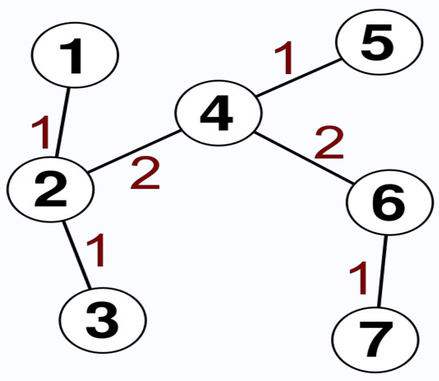
Graceful Trees

A **tree** is a an undirected graph that is acyclic and connected:



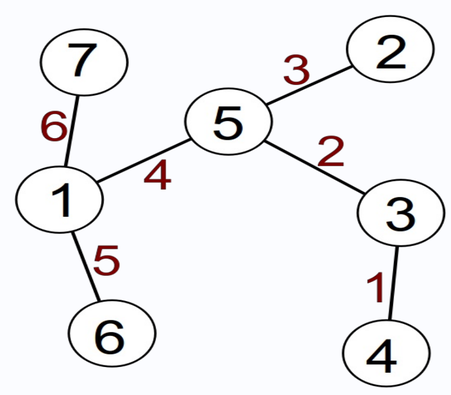
A Labeled Tree A:

Its vertices are arbitrarily labeled 1 to 7.



We can get an edge-labeling (difference between endpoints) from the given vertex labels..

Is it possible to get *better* edge labels for this tree: one where each edge label is unique?



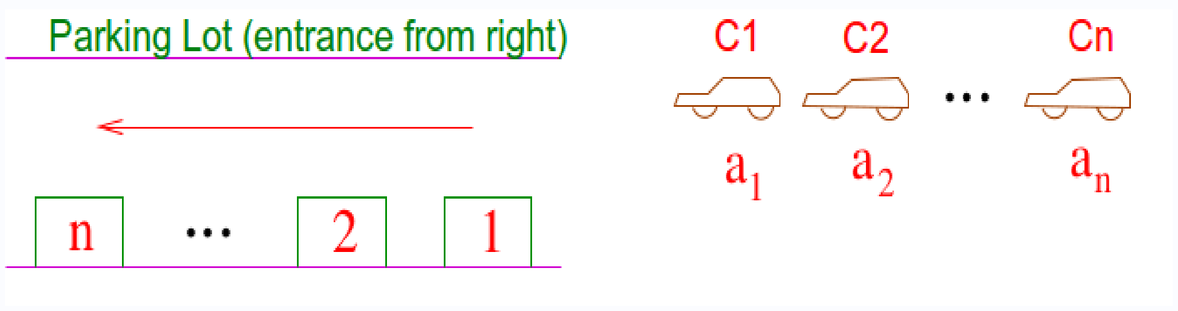
Yes; the new vertex labels on the left allow us to get unique edge labels!

This vertex labeling is a **graceful labeling** of the (unlabeled) tree.

An unlabeled tree is a **graceful tree** if it has a graceful labeling**.** Thus, Tree A (with labels removed) is a graceful tree.

Rosa conjectured that **all trees are graceful** (1967)

Parking Functions



Taken from [3]

We have **n** cars that want to park in a parking lot of size **n.**

For each car, C_i , it's preferred parking space is a_i. The cars enter the parking lot one after another.

If a_i is occupied, then C_i parks in the first empty space after a_i. If there is none, then it is unable to park.

We call $(a_1, a_2, \ldots, a_n)$ a **parking function** (of length **n**) if all the **n** cars are able to park in the **n-sized** parking lot.

Some examples are (1,1), (1,2), (2,1), (1,1,1), (1,1,2), (1,2,1).

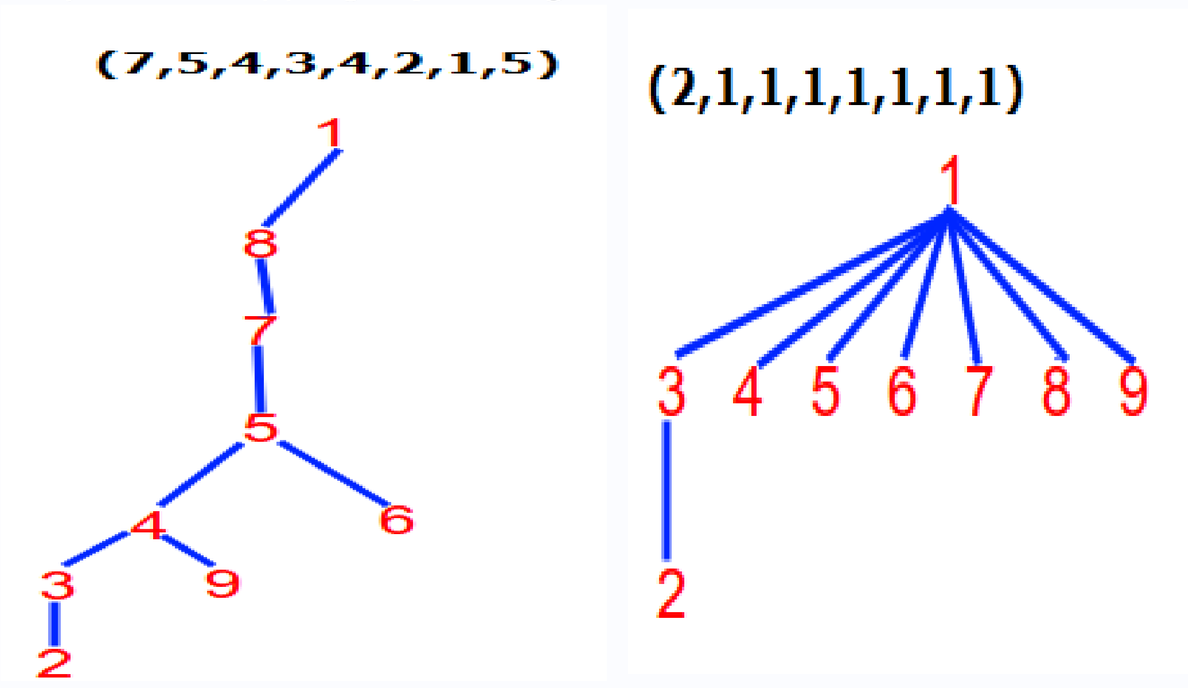
Their relationship

Cayley's formula is a proved result that tells us that there are $n^{(n-2)}$ different trees on **n** labeled vertices.

We know that the set of all labeled trees on **n** vertices has the same size as the set of all **Prufer sequences** of size **(n-2)**. Heinz Prufer used these sequences to prove Cayley's formula.

Quite similarly, **parking functions** of length **(n-1)** are in bijection with all labeled trees of length **n**. The number of parking functions of length (n-1) is also $n^{(n-2)}$.

If two sets have the same size, bijections exist between them. We considered a particular known **bijection** (call it **f()**) between parking functions and labeled trees. Details of the bijection are omitted here, but below are two examples of that bijection mapping a **parking function** to a **labeled tree**:



Our Contribution

1) We gave our **own proof** for why **f()** is a bijection.

2) Our **aim** was to look at all labeled trees on **n** vertices, get corresponding parking functions (PF) of length **(n-1)** for each of them, and study these. We wrote a python program that **partitioned** the set of PFs into those that mapped to graceful trees and otherwise. We were able to do this for values of **n** up to **10**. We wanted to (a) form the concept of what makes a PF **graceful**, and (b) find concretely the number of graceful parking functions on **n**.

Some patterns we found that may help in the future:

(i) PFs of the form (2,1,1,1,...) are graceful

(ii) The non-decreasing PFs (1,1,1,1, ...) with even length are always graceful, and always non-graceful with odd length

(iii) Replacing vertex label ‘i’ (for all) with the label ‘n-i' preserves gracefulness, as edge labels remain the same

(iv) A PF and its permutation have the same tree structure

(v) It is not the case that you can always take a bad (no permutation graceful), non-decreasing PF and get a graceful, non-decreasing PF by switching children's ordering at some node in the tree representation. A counter-example is (1,1,2,2,3)

We could not make significant progress towards our main aim. Further exploration of this and other bijections can be done to improve our understanding of PF and gracefulness.

References

[1] A. Rosa, On Certain Valuations of the Vertices of a Graph, Theory of Graphs, Gordong and Breach, N. Y. and Dunod Paris 1967, 349-355
[2] Basak A. 2004. MPLS multicasting using caterpillars and a graceful labelling scheme. Proceedings: Eighth International Conference on Information Visualisation.
[3] R. P. Stanley, MIT, available at http://www-math.mit.edu/~rstan