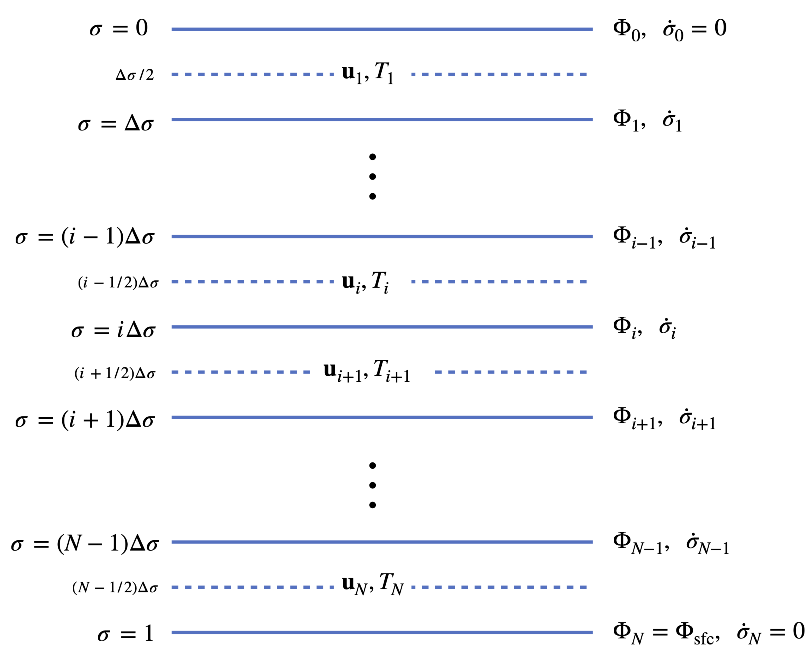
Stationary wave model

For a nice formulation of the hydrostatic primitive equations in sigma coordinates: see page 41 of https://ethz.ch/content/dam/ethz/special-interest/usys/iac/iac-dam/documents/edu/courses/weather_and_climate_models/FS2023/Slides/03b_WCM_VertCoord.pdf

All fields are separated into a basic state (overbars) and a perturbation (would usually have primes, but we omit them here). For example, the full velocity field is $\overline{\mathbf{u}_i} + \mathbf{u}_i$.

Vertical staggering: $\dot\sigma$ and Φ on full levels, $\mathbf{u}, T$ on half levels. Index $i = 0$ is at the top of the domain; $i = N$ is at the surface. See sketch below:



Boundary conditions: $\dot\sigma(\sigma = 0) = \dot\sigma_0 = 0$, $\dot\sigma(\sigma = 1) = \dot\sigma_N = 0$, $\Phi(\sigma = 1) = \Phi_N = \Phi_{sfc}$

Momentum equations:

$$
\begin{aligned}
\frac{\partial \mathbf{u}_i}{\partial t} \quad & + \quad \overline{\mathbf{u}_i} \cdot \nabla \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \overline{\mathbf{u}_i} \\
& + \quad \frac{1}{2\Delta\sigma}\left[\overline{\dot\sigma_i}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \overline{\dot\sigma_{i-1}}(\mathbf{u}_i - \mathbf{u}_{i-1})\right] + \frac{1}{2\Delta\sigma}\left[\dot\sigma_i\overline{(\mathbf{u}_{i+1} - \mathbf{u}_i)} + \dot\sigma_{i-1}\overline{(\mathbf{u}_i - \mathbf{u}_{i-1})}\right] \\
& + \quad f\mathbf{k} \times \mathbf{u}_i \\
& = \quad -\left\{\nabla \frac{\Phi_i + \Phi_{i-1}}{2} + \epsilon_i \mathbf{u}_i + \nu \nabla^4 \mathbf{u}_i + R\overline{T}\nabla \ln p_s + RT\nabla \overline{\ln p_s}\right\}
\end{aligned}
$$

Thermodynamic equations:

$$
\begin{aligned}
\frac{\partial T_i}{\partial t} &+ \overline{\mathbf{u}_i} \cdot T_i + \mathbf{u}_i \cdot \nabla \overline{T_i} \\
&+ \frac{1}{2\Delta\sigma}\left[\overline{\dot{\sigma}_i}(T_{i+1} - T_i) + \overline{\dot{\sigma}_{i-1}}(T_i - T_{i-1})\right] + \frac{1}{2\Delta\sigma}\left[\dot{\sigma}_i\overline{(T_{i+1} - T_i)} + \dot{\sigma}_{i-1}\overline{(T_i - T_{i-1})}\right] \\
&- \frac{\kappa}{(i-1/2)\Delta\sigma}\left[\overline{T}_i \frac{\dot{\sigma}_i + \dot{\sigma}_{i-1}}{2} + T_i \frac{\overline{\dot{\sigma}_i + \dot{\sigma}_{i-1}}}{2}\right] \\
&- \kappa \overline{T_i} \frac{\partial \ln p_s}{\partial t} - \kappa T_i \overline{\mathbf{u}_i} \cdot \nabla \overline{\ln p_s} - \kappa \overline{T_i} \mathbf{u}_i \cdot \nabla \overline{\ln p_s} - \kappa \overline{T_i} \overline{\mathbf{u}_i} \cdot \ln p_s \\
&= -\left\{\epsilon_i T_i + \nu \nabla^4 T_i\right\} \\
&\quad +Q_{\mathrm{diab},i}
\end{aligned}
$$

Prognostic equation for $\ln(p_s)$: $\dfrac{\partial \ln(p_s)}{\partial t} = -\Delta\sigma\left[\nabla \cdot \left(\sum_{i=1}^{N} \mathbf{u}_i\right) + \left(\sum_{i=1}^{N} \mathbf{u}_i\right) \cdot \nabla \overline{\ln(p_s)} + \left(\sum_{i=1}^{N} \overline{\mathbf{u}_i}\right) \cdot \nabla \ln(p_s)\right]$

Diagnosis of $\dot{\sigma}$: $\dot{\sigma}_i = -i\Delta\sigma \dfrac{\partial \ln(p_s)}{\partial t} - \Delta\sigma\left[\nabla \cdot \left(\sum_{j=1}^{i} \mathbf{u}_j\right) + \left(\sum_{j=1}^{i} \mathbf{u}_j\right) \cdot \nabla \overline{\ln(p_s)} + \left(\sum_{j=1}^{i} \overline{\mathbf{u}_j}\right) \cdot \nabla \ln(p_s)\right]$

Diagnosis of Φ: $\dfrac{\partial \Phi}{\partial \sigma} = -\dfrac{RT}{\sigma}$ so $\dfrac{\Phi_i - \Phi_{i-1}}{\Delta\sigma} = -\dfrac{RT_i}{(i-1/2)\Delta\sigma}$ with $\Phi_N = \Phi_{sfc}$

Hence $\Phi_i = \Phi_{sfc} + R\sum_{j=i+1}^{N} \dfrac{T_j}{j-1/2}$