Initial sigma-level equations

For a nice formulation of the hydrostatic primitive equations in sigma coordinates: see page 41 of https://ethz.ch/content/dam/ethz/special-interest/usys/iac/iac-dam/documents/edu/courses/weather_and_climate_models/FS2023/Slides/03b_WCM_VertCoord.pdf

Momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \dot{\sigma} \frac{\partial \mathbf{u}}{\partial \sigma} + f\mathbf{k} \times \mathbf{u} = -(\nabla \Phi + RT \nabla \ln p_s) \tag{1}$$

Thermodynamic equation:

$$\frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} + \dot{\sigma} \frac{\partial \mathbf{T}}{\partial \sigma} = \frac{\kappa T}{\sigma p_s} \omega \tag{2}$$

Continuity equation:

$$\frac{\partial p_s}{\partial t} + \mathbf{u} \cdot \nabla p_s + p_s \nabla \cdot \mathbf{u} + p_s \frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \tag{3}$$

Hydrostatic equation:

$$\frac{\partial \Phi}{\partial \sigma} = -\frac{RT}{\sigma} \tag{4}$$

Boundary conditions: $\dot{\sigma}(\sigma = 0) = 0$, $\dot{\sigma}(\sigma = 1) = 0$, $\Phi(\sigma = 1) = \Phi_N = \Phi_{sfc}$

Transforming the equation set

Dividing (3) by p_s and integrating it vertically, defining $\tilde{(\cdot)} = \int_0^1 (\cdot) d\sigma$ and using the BCs on $\dot{\sigma}$, one obtains:

$$\frac{\partial \ln p_s}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \ln p_s + \nabla \cdot \tilde{\mathbf{u}} = 0 \tag{5}$$

Substituting $\omega = \frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\mathrm{D}(\sigma p_s)}{\mathrm{D}t} = \dot{\sigma} p_s + \sigma \frac{\mathrm{D}p_s}{\mathrm{D}t}$ in (2):

$$\frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} + \dot{\sigma} \frac{\partial \mathbf{T}}{\partial \sigma} = \kappa T \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \ln p_s + \kappa T \frac{\dot{\sigma}}{\sigma} \tag{6}$$

and using (5),

$$\frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} + \dot{\sigma} \frac{\partial \mathbf{T}}{\partial \sigma} = \kappa T (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla \ln p_s - \kappa T \nabla \cdot \tilde{\mathbf{u}} + \kappa T \frac{\dot{\sigma}}{\sigma} \tag{7}$$

Finally, $\dot{\sigma}$ is diagnosed using (3) and (5):

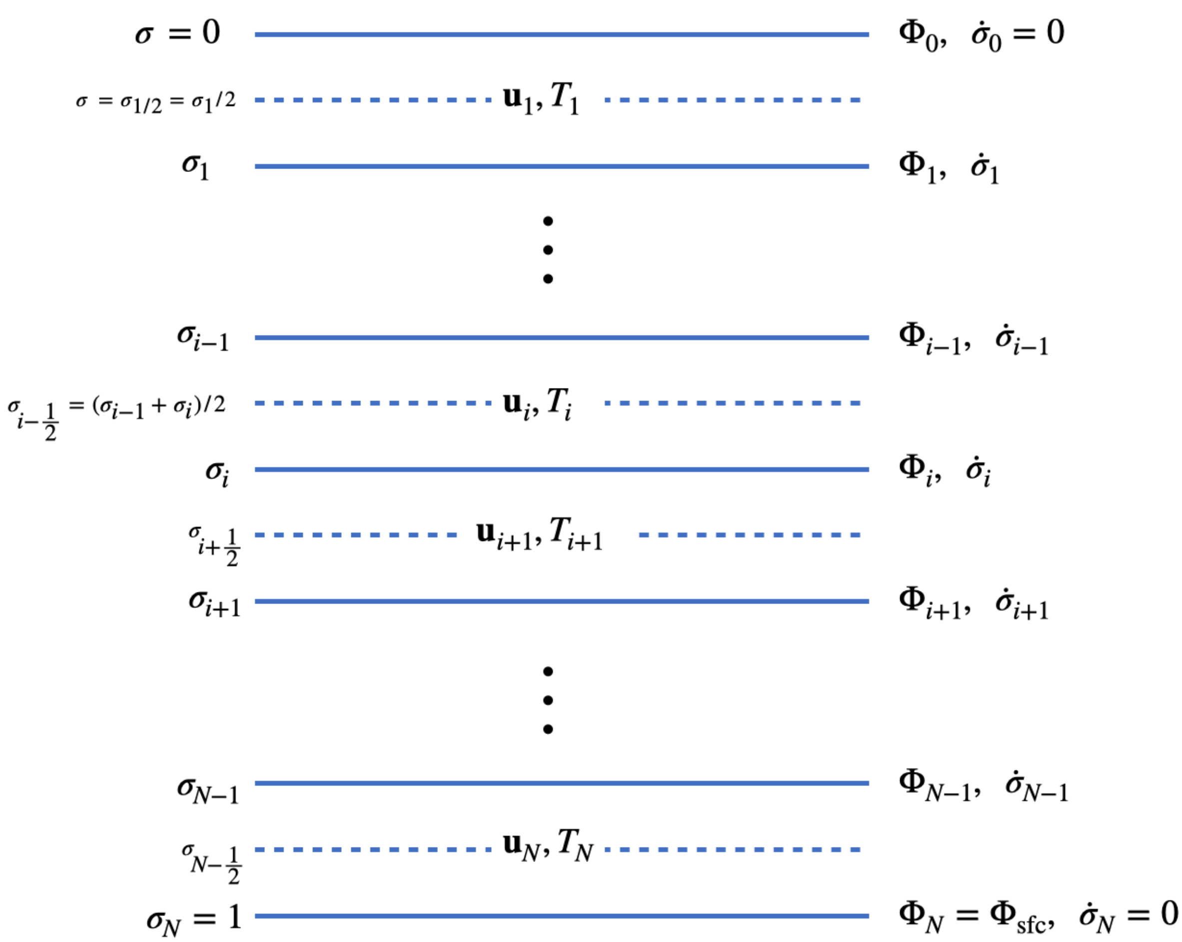
$$\dot{\sigma}(\sigma) = -\int_0^{\sigma} \left[(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla \ln p_s + \nabla \cdot \tilde{\mathbf{u}} \right] \mathrm{d}\sigma' \tag{8}$$

The system can be solved by using $\mathbf{u}, T$, and $\ln p_s$ as prognostic variables, with equations (1), (7), and (5). Φ and $\dot{\sigma}$ are respectively diagnosed using (4) and (8).

Basic-state / perturbation separation and vertical discretization

All fields are separated into a basic state (overbars) and a perturbation (would usually have primes, but we omit them here). For example, the full velocity field is $\overline{\mathbf{u}} + \mathbf{u}$. We also discretize the fields in the vertical using the following staggered grid.

Vertical staggering: $\dot{\sigma}$ and Φ on full levels, $\mathbf{u}, T$ on half levels. Index $i = 0$ is at the top of the domain; $i = N$ is at the surface. The σ_i (called the full levels) are arbitrary; the $\sigma_{i+1/2}$ (called the half levels) are at the midpoints of full levels.



The full equations are:

Momentum equations:

$$
\begin{aligned}
\frac{\partial \mathbf{u}_i}{\partial t} \quad &+ \quad \overline{\mathbf{u}_i} \cdot \nabla \mathbf{u}_i + \mathbf{u}_i \cdot \nabla \overline{\mathbf{u}_i} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \\
&+ \quad \frac{1}{2}\left[\overline{\dot{\sigma}_i} \frac{\mathbf{u}_{i+1}-\mathbf{u}_i}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \overline{\dot{\sigma}_{i-1}} \frac{\mathbf{u}_i-\mathbf{u}_{i-1}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&+ \quad \frac{1}{2}\left[\dot{\sigma}_i \frac{\overline{\mathbf{u}_{i+1}}-\overline{\mathbf{u}_i}}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \dot{\sigma}_{i-1} \frac{\overline{\mathbf{u}_i}-\overline{\mathbf{u}_{i-1}}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&+ \quad \frac{1}{2}\left[\dot{\sigma}_i \frac{\mathbf{u}_{i+1}-\mathbf{u}_i}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \dot{\sigma}_{i-1} \frac{\mathbf{u}_i-\mathbf{u}_{i-1}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&+ \quad f\mathbf{k} \times \mathbf{u}_i \\
&= \quad -\left\{\nabla \frac{\Phi_i + \Phi_{i-1}}{2} + \epsilon_i \mathbf{u}_i + \nu \nabla^4 \mathbf{u}_i + R\overline{T} \nabla \ln p_s + RT \nabla \overline{\ln p_s} + RT \nabla \ln p_s\right\}
\end{aligned}
$$

Thermodynamic equations:

$$
\begin{aligned}
\frac{\partial T_i}{\partial t} \quad &+ \quad \overline{\mathbf{u}_i} \cdot T_i + \mathbf{u}_i \cdot \nabla \overline{T_i} + \mathbf{u}_i \cdot \nabla T_i \\
&+ \quad \frac{1}{2}\left[\overline{\dot{\sigma}_i} \frac{T_{i+1}-T_i}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \overline{\dot{\sigma}_{i-1}} \frac{T_i-T_{i-1}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&+ \quad \frac{1}{2}\left[\dot{\sigma}_i \frac{\overline{T_{i+1}}-\overline{T_i}}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \dot{\sigma}_{i-1} \frac{\overline{T_i}-\overline{T_{i-1}}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&+ \quad \frac{1}{2}\left[\dot{\sigma}_i \frac{T_{i+1}-T_i}{\sigma_{i+\frac{1}{2}}-\sigma_{i-\frac{1}{2}}} + \dot{\sigma}_{i-1} \frac{T_i-T_{i-1}}{\sigma_{i-\frac{1}{2}}-\sigma_{i-\frac{3}{2}}}\right] \\
&- \quad \frac{\kappa}{\sigma_{i-\frac{1}{2}}}\left[\overline{T}_i \frac{\dot{\sigma}_i + \dot{\sigma}_{i-1}}{2} + T_i \frac{\overline{\dot{\sigma}_i} + \overline{\dot{\sigma}_{i-1}}}{2} + T_i \frac{\dot{\sigma}_i + \dot{\sigma}_{i-1}}{2}\right] \\
&- \quad \kappa \overline{T_i}(\overline{\mathbf{u}_i} - \overline{\tilde{\mathbf{u}}_i}) \cdot \nabla \ln p_s - \kappa \overline{T_i}(\mathbf{u}_i - \tilde{\mathbf{u}}_i) \cdot \nabla \overline{\ln p_s} - \kappa T_i(\overline{\mathbf{u}_i} - \overline{\tilde{\mathbf{u}}_i}) \cdot \nabla \overline{\ln p_s} \\
&- \quad \kappa \overline{T_i}(\mathbf{u}_i - \tilde{\mathbf{u}}_i) \cdot \nabla \ln p_s - \kappa T_i(\overline{\mathbf{u}_i} - \overline{\tilde{\mathbf{u}}_i}) \cdot \nabla \ln p_s - \kappa T_i(\mathbf{u}_i - \tilde{\mathbf{u}}_i) \cdot \nabla \overline{\ln p_s} \\
&- \quad \kappa T_i(\mathbf{u}_i - \tilde{\mathbf{u}}_i) \cdot \nabla \ln p_s \\
&+ \quad \kappa \overline{T_i} \nabla \cdot \mathbf{u}_i + \kappa T_i \nabla \cdot \overline{\mathbf{u}_i} + \kappa T_i \nabla \cdot \mathbf{u}_i \\
&= \quad -\left\{\epsilon_i + \nu \nabla^4\right\}\left[T_i - \frac{\partial \overline{T_i}}{\partial \ln \sigma} \ln p_s\right] \\
&\qquad +Q_{\text{diab},i}
\end{aligned}
$$

where

$$
\tilde{(\cdot)} = \sum_{i=1}^{n} (\sigma_i - \sigma_{i-1})(\cdot)_i
$$

Recall that $\dot{\sigma}_0 = \dot{\sigma}_N = 0$, hence some vertical advection & expansion terms drop out for layers $i = 1$ and $i = N$.

Prognostic equation for $\ln(p_s)$:

$$\frac{\partial \ln p_s}{\partial t} + \bar{\tilde{\mathbf{u}}} \cdot \nabla \ln p_s + \tilde{\mathbf{u}} \cdot \nabla \overline{\ln p_s} + \tilde{\mathbf{u}} \cdot \nabla \ln p_s + \nabla \cdot \tilde{\mathbf{u}} = 0$$

Diagnosis of $\dot{\sigma}$:

$$\dot{\sigma}(\sigma) = -\int_0^{\sigma} \left[(\overline{\mathbf{u}_i} - \overline{\tilde{\mathbf{u}}_i}) \cdot \nabla \ln p_s + (\mathbf{u}_i - \tilde{\mathbf{u}}_i) \cdot \nabla \overline{\ln p_s} + (\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla \ln p_s + \nabla \cdot \tilde{\mathbf{u}} \right] \mathrm{d}\sigma' \tag{8}$$

Diagnosis of Φ:

$$\Phi_i = \Phi_{sfc} + R \sum_{j=i+1}^{N} (\sigma_j - \sigma_{j-1}) \frac{T_j}{\sigma_{j-\frac{1}{2}}}$$