Computation for Exercise 1

Consider a market with a risk free rate $r > 0$ and a risky asset following a Black-Scholes model, i.e., represented by a process $S = (S_t)_{t\geq 0}$ which has dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad t \geq 0,$$

where W^Q is a one-dimensional Brownian motion with respect to the pricing measure Q, under which the discounted process $(e^{-rt}S_t)_{t\geq 0}$ is a martingale.
Consider a call option on S with maturity $T > 0$, i.e., an option with payoff

$$\psi(S_T) = (S_T - K)^+,$$

and an *asset or nothing* option, i.e., an option with payoff

$$\phi(S_T) = S_T \mathbf{1}_{\{S_T > K\}}.$$

We want to prove that the delta of the call option, i.e., the derivative with respect to S_0 of $e^{-rT}\mathbb{E}[\psi(S_T)]$, is equal to the valuation of a portfolio of $1/S_0$ *asset or nothing* options. That is, we want to prove

$$\partial_{S_0}\left(e^{-rT}\mathbb{E}^Q[(S_T - K)^+]\right) = \frac{1}{S_0}e^{-rT}\mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}],$$

that is,

$$\partial_{S_0}\left(\mathbb{E}^Q[(S_T - K)^+]\right) = \frac{1}{S_0}\mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}].$$

We have that

$$\begin{aligned}
\mathbb{E}^Q[(S_T - K)^+] &= \mathbb{E}^Q\left[\left(S_0 e^{(r-\sigma^2/2)T + \sigma W_T^Q} - K\right)^+\right] \\
&= \int_{-\infty}^{+\infty}\left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z} - K\right)^+ g(z)dz, \qquad (1)
\end{aligned}$$

where $g(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2}$. Differentiating (1), we get (by using Leibniz integral rule)

$$\begin{aligned}
\partial_{S_0}\mathbb{E}^Q[(S_T - K)^+] &= \partial_{S_0}\int_{-\infty}^{+\infty}\left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z} - K\right)^+ g(z)dz \\
&= \int_{-\infty}^{+\infty}\partial_{S_0}\left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z} - K\right)^+ g(z)dz \\
&= \int_{-\infty}^{+\infty} e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z}\mathbf{1}_{\{S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z} > K\}} g(z)dz \\
&= \frac{1}{S_0}\int_{-\infty}^{+\infty} S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z}\mathbf{1}_{\{S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z} > K\}} g(z)dz \\
&= \frac{1}{S_0}\mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}]. \qquad (2)
\end{aligned}$$

Thus we have proved our result.