**Exercise 1**

Let $P(t,T)$ be the continuously compounded time-$t$ price of a bond maturing at time $T$, and assume that it is a *deterministic* function of $t$ and $T$: in other words

$$P(t,T) = e^{-\int_t^T r(u)du}$$

for some deterministic positive *short rate* function $r(t)$.

(a) Prove, via discussion of arbitrage possibilities, that for $t \le T \le S$ it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Define the *continuously compounded forward rate prevailing at $t$ of reset date $T$ and maturing $S$* as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)},$$

and the instantaneous forward rate as

$$f(t,T) = \lim_{S \to T} f(t,T,S).$$

Prove that

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

and

$$r(t) = \lim_{T \to t} f(t,T).$$

(c) Establish in which of the points (a) and (b) the assumption of deterministic rates is necessary or can be relaxed to some class of stochastic short rates $r(t)$ .

## Solution to exercise 1

(a) Suppose that $P(t,S) > P(t,T)P(T,S)$, for some times $t \le T \le S$. Then we can apply the following strategy:

- at time $t$, we sell an $S$-bond and buy $P(T,S)$ units of a $T$-bond: the total cost is

$$-P(t,S) + P(T,S)P(t,T) < 0,$$

by assumption.

- At time $T$, we receive $P(T,S)$ euros from the $T$-bond we have bought in $t$, and buy an $S$-bond: the total cost is

$$-P(T,S) + P(T,S) = 0.$$

- At time $S$, we receive one euro (from the $S$-bond we have bought in $T$) and pay one euro (for the $S$-bond we have sold in $t$).

The strategy above gives us a net gain of

$$P(t,S) - P(T,S)P(t,T) > 0,$$

so it is an arbitrage.

If $P(t,S) < P(t,T)P(T,S)$, the same profit can be made just changing the signs in the strategy. We have then seen that in order to avoid arbitrage opportunities, it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Suppose again $t \leq T \leq S$.

The *continuously compounded forward rate prevailing at $t$ of reset date $T$ and maturing $S$*, called $f(t,T,S)$, is defined as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)}, \tag{1}$$

and the *instantaneous forward rate* $f(t,T)$ by

$$f(t,T) := \lim_{S \to T} f(t,T,S).$$

We want first to see that

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

From (1) we get

$$f(t,T,S)(S-T) = \ln\left(\frac{P(t,T)}{P(t,S)}\right),$$

and so

$$f(t,T,S) = -\frac{\ln P(t,S) - \ln P(t,T)}{S-T}.$$

Hence

$$f(t,T) := \lim_{S \to T} f(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{2}$$

Since at the same way we can show that

$$f(t,u) = -\frac{\partial \ln P(t,u)}{\partial u}$$

for every $u \geq t$, integrating we get

$$-\int_t^T f(t,u)du = \int_t^T \frac{\partial \ln P(t,u)}{\partial u} du = \ln P(t,T) - \ln P(t,t) = \ln P(t,T).$$

Then we have

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

We now want to see that

$$r(t) = \lim_{T \to t} f(t,T).$$

From

$$e^{-\int_t^T r(u)du} = P(t,T)$$

we get

$$-\int_t^T r(u)du = \ln P(t,T) = \ln P(t,T) - \ln P(t,t),$$

so that dividing both sides by $T-t$ and taking the limit for $T \to t$ we get

$$-r(t) = \lim_{T \to t} \frac{\partial \ln P(t,T)}{\partial T} = -\lim_{T \to t} f(t,T),$$

where the last equality follows by equation (2).