Exercise 3

Consider the tenor discretization $T_0 < T_1 < \cdots < T_n$ and the displaced LIBOR market model where the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \ldots, n-1$ follow the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i,$$

where $d\langle W_i, W_j\rangle(t) = \rho_{i,j}(t)dt$, under the real-world measure $\mathbb{P}$. Also assume here that $d > 0$ and that $\sigma_i^D(\cdot)$ are deterministic functions.

- Derive an analytical approximation for the price of a swaption in this setting, in a similar way to what you have seen in the lecture for the log-nornal case, see pages 488-500 of the script.

  Hint: in order to solve the exercise, you first have to *guess* the dynamics of the par swap rate S. In particular, you can guess S to have displaced dynamics as well, with a displacement $d_S = d$. Try to see why having a look at Lemma 142 at page 107 of the script.

- Assume now that

  $$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i, \quad 1 \le i \le n-1,$$

  with $d_i \neq d_j$ for at least one $i \neq j$. How would your analytical approximation change? Would you need one more approximation? Why?

Solution

We want to find an approximated analytic formula for the swaption with swap tenor $T_a < T_{a+1} < \cdots < T_b$ which is a subset of the tenor discretization $T_0 < T_1 < \cdots < T_n$. We suppose by simplicity that the swap tenor is as coarse as the original tenor discretization.
Proceeding as in the script (pages 176-180) we find that

$$V_{\text{swaption}}(T_a) = A_{a,b}(T_a) \max\left(S_{a,b}(T_a) - K, 0\right),$$

where $A_{a,b}$ is the annuity for the swap tenor above and $(S_{a,b}(t))_{0 \le t \le T_a}$ is the process representing the par swap rate associated to the swap tenor above, i.e.,

$$S_{a,b}(t) = S(T_a, \ldots, T_b; t) := \frac{P(T_a;t) - P(T_b;t)}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1};t)}, \quad 0 \le t \le T_a.$$

Then

$$V_{\text{swaption}}(0) = A_{a,b}(0)\mathbb{E}^{Q^A}\left[\max\left(S_{a,b}(T_a) - K, 0\right)\right], \tag{1}$$

where Q^A is the probability measure associated to the annuity, under which $S_{a,b}$ is a martingale.
We have seen that, *if* $S_{a,b}$ follows log-normal dynamics with a deterministic log-volatility function $\sigma_{a,b}(\cdot)$, we can use the Black formula to compute (1). So the question now is: is it possible to assume such dynamics for $S_{a,b}$, at least approximately?
Remember that we assume displaced log-normal dynamics for the underlying LIBOR market model, i.e., we introduce the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \ldots, n-1$ with

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i. \tag{2}$$

From Lemma 142 at page 107 of the script, we know that

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t)L_k(t), \quad 0 \le t \le T_a \tag{3}$$

where the weights are defined by

$$\alpha_k(t) := \frac{(T_{k+1} - T_k)P(T_{k+1};t)}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1};t)}, \quad 0 \le t \le T_a \tag{4}$$

and satisfy $\alpha_k(t) \ge 0$, $k = a, \dots, b-1$, and $\sum_{k=a}^{b-1} \alpha_k(t) = 1$. For this reason and from (2) we get that

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t) L_k(t) \ge \sum_{k=a}^{b-1} \alpha_k(t)(-d) = -d, \quad 0 \le t \le T_a.$$

We can then make our guess and assume displaced log-normal dynamics for $S_{a,b}$ as well, with same displacement d as in (2). In particular, we can model the evolution of $S_{a,b}$ as given by

$$dS_{a,b}(t) = (S_{a,b}(t) + d)\sigma_S(t)dW^A(t), \quad 0 \le t \le T_a, \tag{5}$$

where W^A is a Q^A-Brownian motion and $\sigma_S(\cdot)$ is a deterministic function. If this is the case, using the same arguments we have seen in Exercise 2 of Handout 8, from (1) we get

$$V_{\text{swaption}}(0) = A(0)BS\left(S_{a,b}(0) + d, K + d, \bar{\sigma}_S, T_a\right), \tag{6}$$

i.e., we determine the price of the option as given by the value of the annuity at zero times the Black-Scholes formula with initial value $S_{a,b}(0) + d$, strike $K + d$, maturity T_a and integrated volatility $\bar{\sigma}_S$ defined as

$$\bar{\sigma}_S = \frac{1}{T_a} \int_0^{T_a} \sigma_S^2(t)dt, \quad 0 \le t \le T_a. \tag{7}$$

So, we just have to derive an (approximated, as we will see) expression for $\bar{\sigma}_S$, which will be our goal from now on. Since the value of $S_{a,b}$ clearly depends on the values of the LIBOR rates L_k, $k = a, \dots, b-1$ (see (3) and (9)) we can apply Itô's formula and get

$$\begin{aligned}
dS_{a,b}(t) &= (\dots)dt + \sum_{k=a}^{b-1} \frac{\partial S_{a,b}}{\partial L_k}(t)dL_k(t) \\
&= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} dL_k(t) \\
&= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} (L_k(t) + d)\sigma_k^D(t)dW_k(t) \\
&= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} w_k(t)\sigma_k^D(t)dW_k(t), \quad 0 \le t \le T_a
\end{aligned} \tag{8}$$

with

$$w_k(t) = \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d}(L_k(t) + d) = \frac{\partial \log(S_{a,b} + d)}{\partial L_k}(t)(L_k(t) + d), \tag{9}$$

for any $t \in [0, T_a]$. Once we compute the expression of the weights in (9), we can use it as done in the script in order to determine an approximation for $\sigma_S(\cdot)$ in (5) and then for $\bar{\sigma}_S$ in (7): indeed, equations (5) and (8) imply that $\sigma_S(\cdot)$ satisfies

$$\sigma_S^2(t)dt = \frac{d\langle S_{a,b}\rangle(t)}{(S_{a,b}(t) + d)^2} = \sum_{k,\ell=a}^{b-1} w_k(t)w_\ell(t)\sigma_k^D(t)\sigma_\ell^D(t)\rho_{k,\ell}(t)dt, \quad 0 \le t \le T_a.$$

However, the expression above is stochastic, so $\sigma_S^2(\cdot)$ is not deterministic as we would have needed. We then freeze the weights to their initial values and obtain the approximation

$$\sigma_S^2(t)dt \approx \sum_{k,\ell=a}^{b-1} w_k(0)w_\ell(0)\sigma_k^D(t)\sigma_\ell^D(t)\rho_{k,\ell}(t)dt, \quad 0 \le t \le T_a,$$

which gives us

$$\bar{\sigma}_S^2 \approx \tilde{\sigma}_S^2 := \frac{1}{T_a} \sum_{k,\ell=a}^{b-1} w_k(0) w_\ell(0) \int_0^{T_a} \sigma_k^D(t) \sigma_\ell^D(t) \rho_{k,\ell}(t) dt. \tag{10}$$

From (6) and (10) we then get the approximation

$$V_{\text{swaption}}(0) \approx A(0) BS\left(S_{a,b}(0) + d, K + d, \tilde{\sigma}_S, T_a\right), \tag{11}$$

with $\tilde{\sigma}_S$ defined in (10).
It remains now to compute the values of the coefficients in (9). For any $t \in [0, T_a]$ we have

$$\begin{aligned}
w_k(t) &= \frac{\partial \log(S_{a,b} + d)}{\partial L_k}(t)(L_k(t) + d) \\
&= \frac{\partial}{\partial L_k} \log\left(\frac{P(T_a) - P(T_b)}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})} + d\right)(t)(L_k(t) + d) \\
&= \frac{\partial}{\partial L_k}\left(\log\left(P(T_a) - P(T_b) + d\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})\right) - \log\left(\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})\right)\right)(t) \\
&\quad \cdot (L_k(t) + d) \\
&= \frac{L_k(t) + d}{P(T_a) - P(T_b) + d\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})} \frac{\partial}{\partial L_k}\left(P(T_a) - P(T_b) + d\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})\right)(t) \\
&\quad - \frac{L_k(t) + d}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})} \sum_{j=a}^{b-1}(T_{j+1} - T_j)\frac{\partial}{\partial L_k}P(T_{j+1}) \\
&= \left(\frac{P(T_b) + d\sum_{j=k}^{b-1}(T_{j+1} - T_j)P(T_{j+1})}{P(T_a) - P(T_b) + d\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})} - \frac{\sum_{j=k}^{b-1}(T_{j+1} - T_j)P(T_{j+1})}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1})}\right) \\
&\quad \cdot (L_k(t) + d)\frac{(T_{k+1} - T_k)}{1 + (T_{k+1} - T_k)L_k(t)} \\
&= \left(\frac{P(T_b) + dA_{k,b}(t)}{P(T_a) - P(T_b) + dA_{a,b}(t)} - \frac{A_{k,b}(t)}{A_{a,b}(t)}\right) \cdot \left(\frac{P(T_k) - P(T_{k+1})}{P(T_k)} + d\frac{(T_{k+1} - T_k)P(T_{k+1})}{P(T_k)}\right) \\
&= \left(\frac{P(T_b) + dA_{k,b}(t)}{P(T_a) - P(T_b) + dA_{a,b}(t)} - \frac{A_{k,b}(t)}{A_{a,b}(t)}\right) \cdot \frac{P(T_k) - P(T_{k+1}) + d(T_{k+1} - T_k)P(T_{k+1})}{P(T_k)},
\end{aligned}$$

where $A_{a,b}$ and $A_{k,b}$ are the annuities of the tenors $T_a < \cdots < T_b$ and $T_k < \cdots < T_b$, respectively.
We can plug the above expression evaluated at time $t = 0$ in (10) to obtain our approximated integrated volatility, from which we then get the approximated price via (11).
Let's now consider the case when

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i, \quad 1 \le i \le n-1,$$

with $d_i \neq d_j$ for at least one $i \neq j$. In this case,

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t) L_k(t) \ge -\sum_{k=a}^{b-1} \alpha_k(t) d_k, \quad 0 \le t \le T_a,$$

so we can guess displaced log-normal dynamics for $S_{a,b}$ with displacement now given by

$$S_{a,b}(t) := \sum_{k=a}^{b-1} \alpha_k(t) d_k, \quad 0 \le t \le T_a.$$

The problem is now that the coefficients $\alpha_k(t)$ in the expression above are not deterministic, see (9). So one has to freeze them at time $t = 0$ and then proceed as before. This gives a second approximation that can impact the goodness of the price estimate.