Exercise 3

Consider the quanto on the Libor rate of foreign currency $L^f(T_1, T_2)$, for $T_1 \leq T_2$, assuming now that the process L^f defined in Exercise 2 has normal dynamics, i.e.,

$$dL_t^f = \mu_L(t)dt + \sigma_L(t)dW_t^L, \quad 0 \leq t \leq T_1,$$

where $\mu_L(\cdot)$ and $\sigma_L(\cdot)$ are deterministic functions of time.
Also suppose that $f = (f_t)_{0 \leq t \leq T_1}$ has dynamics given by

$$df_t = \sigma_f(t) f_t dW_t^f, \quad 0 \leq t \leq T_1,$$

where $\sigma_f(\cdot)$ is a deterministic function of time and W^f is a $\mathbb{Q}^{P(T_2)}-$Brownian motion such that $d\langle W_t^{P(T_2)}, W_t^f \rangle = \rho dt$. Derive a formula for the pricing of the Quanto Caplet under this setting.
Suppose now that f has normal dynamics as well, i.e.,

$$df_t = \sigma_f(t)dW_t^f, \quad 0 \leq t \leq T_1.$$

Can you derive a similar pricing formula as in the first part of the exercise? If not, what is the problem?

Solution

In the script, you can see the valuation of the Quanto Caplet assuming log-normal dynamics of the foreign Libor rate. We want to derive a similar valuation supposing now that the foreign Libor rate evolves along normal dynamics.
Remember that the payoff of the quanto caplet is given by

$$V(T_2) = \max\left(L^f(T_1, T_2; T_1) - K, 0\right)(T_2 - T_1)C \text{ in } T_2,$$

where C is the quanto rate.
As we have seen in the script, even if we have

$$dL_t^f = \sigma_L(t)dW_t^{P^f(T_2)}, \quad 0 \leq t \leq T_1, \quad \text{under } \mathbb{Q}^{P^f(T_2)},$$

where $W^{P^f(T_2)}$ is a $\mathbb{Q}^{P^f(T_2)}$-Brownian motion and $\mathbb{Q}^{P^f(T_2)}$ is the probability measure such that all the traded assets *in the foreign economy* are martingales when divided by the numéraire $P^f(T_2)$, we cannot choose such a numéraire once we convert back to the domestic economy, because the foreign T_2-bond $P^f(T_2)$ is not a traded asset in the domestic economy.
We then choose as a numéraire the domestic T_2-bond $P(T_2)$, since its value is $P(T_2; T_2) = 1$ at the payment date T_2. On the other hand, the drawback is that L^f is in general not a martingale under $\mathbb{Q}^{P(T_2)}$, but has dynamics

$$dL_t^f = \mu^{P(T_2)}(t)dt + \sigma_L(t)dW_t^{P(T_2)}, \quad 0 \leq t \leq T_1. \tag{1}$$

Our first and main goal is then to derive the drift $\mu^{P(T_2)}(\cdot)$ of L^f under $\mathbb{Q}^{P(T_2)}$. To this purpose, we now consider the Forward FX rate f, i.e., the process defined by

$$f_t = \frac{P^f(T_2; t)}{P(T_2; t)} FX(t), \quad 0 \leq t \leq T_2,$$

where $FX(t)$ is the (foreign) exchange rate at time t, i.e., such that $P^f(T_2; t)FX(t)$ is the foreign T_2-bond *converted to domestic currency.*

As also highlighted in the script, the process

$$L^f \cdot f = \frac{1}{T_2 - T_1} \frac{P^f(T_1) - P^f(T_2)}{P^f(T_2)} \cdot \frac{P^f(T_2)}{P(T_2)} FX = \frac{1}{T_2 - T_1} \frac{P^f(T_1) - P^f(T_2)}{P(T_2)} FX$$

is a martingale under $\mathbb{Q}^{P(T_2)}$ because it can be seen as a constant times the difference of two foreign bonds *converted to domestic currency* (which are therefore traded assets in the domestic economy) all divided by the numéraire $P(T_2)$. In other terms, the drift of such a process under $\mathbb{Q}^{P(T_2)}$ is zero, so we want to recover the drift of L^f under $\mathbb{Q}^{P(T_2)}$ by just expressing the drift of $L^f \cdot f$ using Itô's formula. In order to do that, we have to specify of course the dynamics of f, which in our case are

$$df_t = \sigma_f(t) f_t dW_t^f, \quad 0 \le t \le T_1,$$

where W^f is a $P(T_2)$-Brownian motion. Then we have that

$$\begin{aligned} d(L_t^f \cdot f_t) &= dL_t^f \cdot f_t + L_t^f \cdot df_t + d\langle L^f, f\rangle_t \\ &= f_t \mu^{P(T_2)}(t)dt + f_t \sigma_L(t) dW_t^{P(T_2)} + L_t^f \sigma_f(t) f_t dW_t^f + \rho(t)\sigma_L(t)\sigma_f(t) f_t dt \\ &= f_t \left(\mu^{P(T_2)}(t) + \rho(t)\sigma_L(t)\sigma_f(t)\right) dt + f_t \sigma_L(t) dW_t^{P(T_2)} + L_t^f \sigma_f(t) f_t dW_t^f, \quad 0 \le t \le T_1, \end{aligned}$$

so since the drift has to be zero, we find

$$\mu^{P(T_2)}(t) = -\rho \sigma_L(t) \sigma_f(t), \quad 0 \le t \le T_1.$$

Substituting in (1), we get

$$dL_t^f = -\rho \sigma_L(t)\sigma_f(t) dt + \sigma_L(t) dW_t^{P(T_2)}, \quad 0 \le t \le T_1,$$

that is,

$$L_{T_1}^f \sim \mathcal{N}\left(L_0^f - \rho \int_0^{T_1} \sigma_L(s)\sigma_f(s) ds,\ \bar{\sigma}_L \sqrt{T_1}\right), \tag{2}$$

where

$$\bar{\sigma}_L := \left(\frac{1}{T_1} \int_0^{T_1} \sigma_L(s) ds\right)^{1/2}.$$

At this point, we can proceed as in the script simply replacing the initial value with $L_0^f - \rho \int_0^{T_1} \sigma_L(s)\sigma_f(s)ds$ in equation (90) at page 318: we get

$$V(0) = P(T_2;0) \left[\left(L_0^f - \rho \int_0^{T_1} \sigma_L(s)\sigma_f(s) ds - K\right) \Phi(d_+) + \bar{\sigma}_L \sqrt{T_1} \Phi(d_+)\right] (T_2 - T_1),$$

where

$$\Phi(x) := \int_0^x \varphi(y) dy, \qquad \varphi(y) := \frac{1}{2\pi} \exp\left(-\frac{x^2}{2}\right),$$

and

$$d_+ = \frac{L_0^f - \rho \int_0^{T_1} \sigma_L(s)\sigma_f(s) ds - K}{\bar{\sigma}_L \sqrt{T_1}}.$$

Let us now suppose that the dynamics of f are given by

$$df_t = \sigma_f(t) dW_t^f, \quad 0 \le t \le T_1.$$

In this case, we have

$$\begin{aligned} d(L_t^f \cdot f_t) &= dL_t^f \cdot f_t + L_t^f \cdot df_t + d\langle L^f, f\rangle_t \\ &= f_t \mu^{P(T_2)}(t)dt + f_t \sigma_L(t) dW_t^{P(T_2)} + L_t^f \sigma_f(t) dW_t^f + \rho(t)\sigma_L(t)\sigma_f(t) dt \\ &= \left(f_t \mu^{P(T_2)}(t) + \rho(t)\sigma_L(t)\sigma_f(t)\right) dt + f_t \sigma_L(t) dW_t^{P(T_2)} + L_t^f \sigma_f(t) dW_t^f, \quad 0 \le t \le T_1, \end{aligned}$$

and we find now

$$\mu^{P(T_2)}(t) = -\frac{\rho \sigma_L(t)\sigma_f(t)}{f_t}, \quad 0 \le t \le T_1.$$

Substituting in (1), we get

$$dL_t^f = -\frac{\rho\sigma_L(t)\sigma_f(t)}{f_t}dt + \sigma_L(t)dW_t^{P(T_2)}, \quad 0 \le t \le T_1.$$

With such dynamics for f, we have now a stochastic term in the drift, which gives integrability problems as f may become 0. However, note that normal dynamics for the Forward FX rate are not realistic since in this case it could be negative.