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WE START WITH A CORRELATION MATRIX

$$R = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \vdots & \vdots & & \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix},$$

SUCH THAT

$$d\langle W_i, W_j \rangle = \rho_{ij}\, dt.$$

IT CAN BE FOUND n EIGENVALUES $(\lambda_1, \ldots, \lambda_n)$

AND n EIGENVECTORS

$$v_1 = (v_{11}, \ldots, v_{1n})^T,$$
$$v_2 = (v_{21}, \ldots, v_{2n})^T,$$
$$\vdots$$
$$v_n = (v_{n1}, \ldots, v_{nn})^T$$

SUCH THAT

$$R = VDV^T,$$

WITH $V \doteq (v_1, \ldots, v_n)$ AND $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \ddots \; \lambda_n \end{pmatrix}.$

IN THIS WAY, WE CAN EXPRESS A DEPENDENCY STRUCTURE OF $(W_1, \ldots, W_n)$, I.E, THE CORRELATED BROWNIAN MOTIONS, WITH RESPECT TO n INDEPENDENT BROWNIAN MOTIONS

$(U_1, \ldots, U_n)$.

IN PARTICULAR, WE HAVE

$$dW = \begin{pmatrix} dW_1 \\ \vdots \\ dW_n \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & & & \\ v_{n1} & v_{n2} & & v_{nn} \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}}_{\sqrt{D}}}_{:= F} \begin{pmatrix} dU_1 \\ \vdots \\ dU_n \end{pmatrix}$$

$$\sqrt{D}\begin{pmatrix} dU_1 \\ \vdots \\ dU_n \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1}\, dU_1 \\ \vdots \\ \sqrt{\lambda_n}\, dU_n \end{pmatrix}$$

INDEED, IN THIS WAY WE HAVE THAT

$$dW_i = v_{i1}\sqrt{\lambda_1}\, dU_1 + v_{i2}\sqrt{\lambda_2}\, dU_2 + \dots + v_{in}\, dU_n,$$

$$dW_j = v_{j1}\sqrt{\lambda_1}\, dU_1 + v_{j2}\sqrt{\lambda_2}\, dU_2 + \dots + v_{jn}\, dU_n,$$

SO THAT

$$d\langle W_i, W_j \rangle = (v_{i1} v_{j1} \lambda_1 + v_{i2} v_{j2} \lambda_2 + \dots + v_{jn} v_{in} \lambda_n)\, dt$$

WHICH IS THE (i,j) ENTRY OF THE MATRIX

$$VDV^T = R,$$

AS WE WANTED.

IF WE NOW WANT TO TAKE INTO ACCOUNT ONLY THE FIRST (= "MOST IMPORTANT" m FACTORS), WE CAN TAKE THE FIRST m COLUMNS OF THE MATRIX

$$F := V\sqrt{D},$$

RE-NORMALIZE THEM AS IN THE SCRIPT AND OBTAIN THE $n \times m$ MATRIX F^m.

SO AT THIS POINT WE HAVE A STRUCTURE (3)

$$d\overline{W} = d\begin{pmatrix}\overline{W}_1\\ \vdots \\ \overline{W}_n\end{pmatrix} = \begin{pmatrix} F_{11}^r & F_{12}^r & \cdots & F_{1m}^r \\ F_{21}^r & F_{22}^r & \cdots & F_{2m}^r \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1}^r & F_{n2}^r & \cdots & F_{nm}^r \end{pmatrix}\begin{pmatrix} dU_1 \\ \vdots \\ dU_m\end{pmatrix},$$

SO THAT

$$d\overline{W}_i = F_{i1}^r\, dU_1 + F_{i2}^r\, dU_2 + \dots + F_{im}^r\, dU_m,$$

$$d\overline{W}_j = F_{j1}^r\, dU_1 + F_{j2}^r\, dU_2 + \dots + F_{jm}^r\, dU_m,$$

AND

$$d\langle \overline{W}_i, \overline{W}_j\rangle = (\underbrace{F_{i1}^r F_{j1}^r + F_{i2}^r F_{j2}^r + \dots + F_{im}^r F_{jm}^r}_{=:\ \rho_{ij}^r})\,dt,$$

WHICH DEFINES THEN A NEW CORRELATION MATRIX (REDUCED) GIVEN BY $R^r = (\rho_{ij}^r)_{\substack{1\le i\le n\\ 1\le j\le n}}$

WITH

$$\rho_{ij}^r = \sum_{k=1}^{m} F_{ik}^r F_{jk}^r. \quad (1)$$

THEN THE STEPS ARE:

- GET $F = V\sqrt{D}$ FROM THE ORIGINAL CORRELATION MATRIX $R = (\rho_{ij})_{1\le i,j\le n}$
- GET F^r FROM F WITH FACTOR REDUCTION
- GET $R^r = (\rho_{ij}^r)_{1\le i,j\le n}$ FROM F^r, WITH ρ_{ij}^r AS IN (1)