Exercise 1

Let $P(t,T)$ be the continuously compounded time-t price of a bond maturing at time T, and assume that it is a *deterministic* function of t and T: in other words

$$P(t,T) = e^{-\int_t^T r(u)du}$$

for some deterministic positive *short rate* function $r(t)$.

(a) Prove, via discussion of arbitrage possibilities, that for $t \le T \le S$ it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Define the *continuously compounded forward rate prevailing at t of reset date T and maturing S* as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)},$$

and the instantaneous forward rate as

$$f(t,T) = \lim_{S \to T} f(t,T,S).$$

Prove that it holds

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

and

$$r(t) = \lim_{T \to t} f(t,T).$$

(c) Conclude from (a) and (b) that

$$f(t,S) = r(S)$$

for all $t \le S$.

(d) Establish in which of the points (a), (b), (c), the assumption of deterministic rates is necessary or can be relaxed to some class of stochastic short rates $r(t)$.

Solution to exercise 1

(a) Suppose that $P(t,S) > P(t,T)P(T,S)$, for some times $t \le T \le S$. Then we can apply the following strategy:

- at time t, we sell an S-bond and buy $P(T,S)$ units of a T-bond: the total cost is

$$-P(t,S) + P(T,S)P(t,T) < 0,$$

by assumption.

- At time T, we receive $P(T,S)$ euros from the T-bond we have bought in t, and buy an S-bond: the total cost is
$$-P(T,S) + P(T,S) = 0.$$

- At time S, we receive one euro (from the S-bond we have bought in T) and pay one euro (for the S-bond we have sold in t).

The strategy above gives us a net gain of

$$P(t, S) - P(T, S)P(t, T) > 0,$$

so it is an arbitrage.

If $P(t, S) < P(t, T)P(T, S)$, the same profit can be made just changing the signs in the strategy. We have then seen that in order to avoid arbitrage opportunities, it has to hold

$$P(t, S) = P(t, T)P(T, S).$$

(b) Suppose again $t \leq T \leq S$.

The *continuously compounded forward rate prevailing at t of reset date T and maturing S*, called $f(t, T, S)$, is defined as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t, T)}{P(t, S)}, \tag{1}$$

and the *instantaneous forward rate $f(t, T)$* by

$$f(t, T) := \lim_{S \to T} f(t, T, S).$$

We want first to see that

$$P(t, T) = e^{-\int_t^T f(t,u)du}.$$

From (1) we get

$$f(t, T, S)(S - T) = \ln\left(\frac{P(t, T)}{P(t, S)}\right),$$

and so

$$f(t, T, S) = -\frac{\ln P(t, S) - \ln P(t, T)}{S - T}.$$

Hence

$$f(t, T) := \lim_{S \to T} f(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T}.$$

Since at the same way we can show that

$$f(t, u) = \frac{\partial \ln P(t, u)}{\partial u}$$

for every $u \geq t$, integrating we get

$$-\int_t^T f(t, u)du = \int_t^T \frac{\partial \ln P(t, u)}{\partial u} du = \ln P(t, T) - \ln P(t, t) = \ln P(t, T).$$

Then we have

$$P(t, T) = e^{-\int_t^T f(t,u)du}. \tag{2}$$

We now want to see that

$$r(t) = \lim_{T \to t} f(t, T).$$

From (2) and since

$$e^{-\int_t^T r(u)du} = P(t, T),$$

taking the derivative of the logarithms and then the limit we get

$$-r(t) = \lim_{T \to t} \frac{\partial \ln P(t, T)}{\partial T} = -\lim_{T \to t} f(t, T),$$

and we have what we wanted.

(c) We have that

$$e^{-\int_t^S f(t,u)du} = P(t,S) = e^{-\int_t^S r(u)du},$$

that is,

$$-\int_t^S f(t,u)du = \ln P(t,S) = -\int_t^S r(u)du.$$

Thus, taking the derivative it follows

$$-f(t,S) = \frac{\partial \ln P(t,S)}{\partial S} = -r(S).$$

(d) What do you think? Try to give an answer after having seen the solution and then I'll upload the answer.

Exercise 2

A *swap* is an exchange payment of fixed rate for a floating rate. In particular, let

$$0 = T_0 < T_1 < T_2 < \cdots < T_n$$

denote a given tenor structure. A swap pays

$$N \left(L(T_i, T_{i+1}; T_i) - S_i \right) L(T_i, T_{i+1}; T_i)$$

in T_{i+1} for $i = 1, \ldots, n-1$, where $S_i \in \mathbb{R}$, $i = 1, \ldots, n-1$, denote the so called fixed swap rates, $L(T_i, T_{i+1}; T_i)$ denotes the forward rate from Definition 111 of the script, and N denotes the notional. The value of the swap is given by Theorem 139 in the script.

The par swap rate S at time t is the unique rate for which a swap with $S_i := S$ for every $i = 1, \ldots, n-1$ has value 0 at time t. Analytically compute the value of S at time t as a function of the times of the tenor structure and of the curve $P(T_i; t)$, $i = 1, \ldots, n$.

Solution to exercise 2

The solution to Exercise 2 follows from Remark 140 at page 257 of the script. You get the last equality by writing

$$L(T_i, T_{i+1}; t) = \frac{1}{T_{i+1} - T_i} \frac{P(T_i; t) - P(T_{i+1}; t)}{P(T_{i+1}; t)},$$

and noting that you have a telescoping sum.