Exercise 1

Let $P(t,T)$ be the continuously compounded time-t price of a bond maturing at time T, and assume that it is a *deterministic* function of t and T: in other words

$$P(t,T) = e^{-\int_t^T r(u)du}$$

for some deterministic positive *short rate* function $r(t)$.

(a) Prove, via discussion of arbitrage possibilities, that for $t \le T \le S$ it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Define the *continuously compounded forward rate prevailing at t of reset date T and maturing S* as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)},$$

and the instantaneous forward rate as

$$f(t,T) = \lim_{S \to T} f(t,T,S).$$

Prove that it holds

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

and

$$r(t) = \lim_{T \to t} f(t,T).$$

(c) Conclude from (a) and (b) that

$$f(t,S) = r(S)$$

for all $t \le S$.

(d) Establish in which of the points (a), (b), (c), the assumption of deterministic rates is necessary or can be relaxed to some class of stochastic short rates $r(t)$.

Solution to exercise 1

(a) Suppose that $P(t,S) > P(t,T)P(T,S)$, for some times $t \le T \le S$. Then we can apply the following strategy:

- at time t, we sell an S-bond and buy $P(T,S)$ units of a T-bond: the total cost is

$$-P(t,S) + P(T,S)P(t,T) < 0,$$

  by assumption.
- At time T, we receive $P(T,S)$ euros from the T-bond we have bought in t, and buy an S-bond: the total cost is

$$-P(T,S) + P(T,S) = 0.$$

- At time S, we receive one euro (from the S-bond we have bought in T) and pay one euro (for the S-bond we have sold in t).

The strategy above gives us a net gain of

$$P(t,S) - P(T,S)P(t,T) > 0,$$

so it is an arbitrage.

If $P(t,S) < P(t,T)P(T,S)$, the same profit can be made just changing the signs in the strategy. We have then seen that in order to avoid arbitrage opportunities, it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Suppose again $t \leq T \leq S$.

The *continuously compounded forward rate prevailing at t of reset date T and maturing S*, called $f(t,T,S)$, is defined as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)}, \tag{1}$$

and the *instantaneous forward rate* $f(t,T)$ by

$$f(t,T) := \lim_{S \to T} f(t,T,S).$$

We want first to see that

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

From (1) we get

$$f(t,T,S)(S-T) = \ln\left(\frac{P(t,T)}{P(t,S)}\right),$$

and so

$$f(t,T,S) = -\frac{\ln P(t,S) - \ln P(t,T)}{S-T}.$$

Hence

$$f(t,T) := \lim_{S \to T} f(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

Since at the same way we can show that

$$f(t,u) = -\frac{\partial \ln P(t,u)}{\partial u}$$

for every $u \geq t$, integrating we get

$$-\int_t^T f(t,u)du = \int_t^T \frac{\partial \ln P(t,u)}{\partial u} du = \ln P(t,T) - \ln P(t,t) = \ln P(t,T).$$

Then we have

$$P(t,T) = e^{-\int_t^T f(t,u)du}. \tag{2}$$

We now want to see that

$$r(t) = \lim_{T \to t} f(t,T).$$

From (2) and since

$$e^{-\int_t^T r(u)du} = P(t,T),$$

taking the derivative of the logarithms and then the limit we get

$$-r(t) = \lim_{T \to t} \frac{\partial \ln P(t,T)}{\partial T} = -\lim_{T \to t} f(t,T),$$

and we have what we wanted.

(c) We have that

$$e^{-\int_t^S f(t,u)du} = P(t,S) = e^{-\int_t^S r(u)du},$$

that is,

$$-\int_t^S f(t,u)du = \ln P(t,S) = -\int_t^S r(u)du.$$

Thus, taking the derivative it follows

$$-f(t,S) = \frac{\partial \ln P(t,S)}{\partial S} = -r(S).$$

This follows actually from the very end of the solution of point (b), before taking the limit.

(d) The only point where we have to require that the bond is deterministic is point (a), specifically when we set up the strategy to buy $P(T,S)$ units of the S bond: in order to be able to do this, we have to know the value of $P(T,S)$ at t. We don't use this assumption at points (b) and (c).

Exercise 2

A *swap* is an exchange payment of fixed rate for a floating rate. In particular, let

$$0 = T_0 < T_1 < T_2 < \cdots < T_n$$

denote a given tenor structure. A swap pays

$$N\left(L(T_i, T_{i+1}; T_i) - S_i\right) L(T_i, T_{i+1}; T_i)$$

in T_{i+1} for $i = 1, \dots, n-1$, where $S_i \in \mathbb{R}$, $i = 1, \dots, n-1$, denote the so called fixed swap rates, $L(T_i, T_{i+1}; T_i)$ denotes the forward rate from Definition 111 of the script, and N denotes the notional. The value of the swap is given by Theorem 139 in the script.

The par swap rate S at time t is the unique rate for which a swap with $S_i := S$ for every $i = 1, \dots, n-1$ has value 0 at time t. Analytically compute the value of S at time t as a function of the times of the tenor structure and of the curve $P(T_i; t)$, $i = 1, \dots, n$.

Solution to exercise 2

The solution to Exercise 2 follows from Remark 140 at page 257 of the script. You get the last equality by writing

$$L(T_i, T_{i+1}; t) = \frac{1}{T_{i+1} - T_i} \frac{P(T_i; t) - P(T_{i+1}; t)}{P(T_{i+1}; t)},$$

and noting that you have a telescoping sum.