

**Exercise 1**

Let  $P(t, T)$  be the continuously compounded time- $t$  price of a bond maturing at time  $T$ , and assume that it is a *deterministic* function of  $t$  and  $T$ : in other words

$$P(t, T) = e^{-\int_t^T r(u) du}$$

for some deterministic positive *short rate* function  $r(t)$ .

- (a) Prove, via discussion of arbitrage possibilities, that for  $t \leq T \leq S$  it has to hold

$$P(t, S) = P(t, T)P(T, S).$$

- (b) Define the *continuously compounded forward rate prevailing at  $t$  of reset date  $T$  and maturing  $S$*  as the unique solution to the equation

$$e^{f(t, T, S)(S-T)} := \frac{P(t, T)}{P(t, S)},$$

and the instantaneous forward rate as

$$f(t, T) = \lim_{S \rightarrow T} f(t, T, S).$$

Prove that it holds

$$P(t, T) = \exp \left( - \int_t^T f(t, u) du \right)$$

and

$$r(t) = \lim_{T \rightarrow t} f(t, T).$$

- (c) Conclude from (a) and (b) that

$$f(t, S) = r(S)$$

for all  $t \leq S$ .

- (d) Establish in which of the points (a), (b), (c), the assumption of deterministic rates is necessary or can be relaxed to some class of stochastic short rates  $r(t)$ .

**Exercise 2**

A *swap* is an exchange payment of fixed rate for a floating rate. In particular, let

$$0 = T_0 < T_1 < T_2 < \dots < T_n$$

denote a given tenor structure. A swap pays

$$N (L(T_i, T_{i+1}; T_i) - S_i) P(T_{i+1}; t)$$

in  $T_{i+1}$  for  $i = 1, \dots, n-1$ , where  $S_i \in \mathbb{R}$ ,  $i = 1, \dots, n-1$ , denote the so called fixed swap rates,  $L(T_i, T_{i+1}; T_i)$  denotes the forward rate from Definition 111 of the script, and  $N$  denotes the notional. The value of the swap is given by Theorem 139 in the script.

The par swap rate  $S$  at time  $t$  is the unique rate for which a swap with  $S_i := S$  for every  $i = 1, \dots, n-1$  has value 0 at time  $t$ . Analytically compute the value of  $S$  at time  $t$  as a function of the times of the tenor structure and of the curve  $P(T_i; t)$ ,  $i = 1, \dots, n$ .

**Exercise 3**

**This exercise (or some points of this exercise) may be done during the Tutorium**

Write a `SwapWithoutFinmath`, implementing the interface

you find in the exercises repository.

In particular, you have to write methods that calculate the value of an interest rate Swap (for given fixed swap rates  $S_i$ ,  $i = 0, \dots, n-1$ ) as well as the par swap rate  $S$ , in correspondence to a tenure structure  $0 = T_0 < T_1 < \dots < T_n = T$ , at evaluation time  $t = 0$ .

Take into account the fact that the user can provide the forward rates  $L(T_i, T_{i+1}; 0)$ ,  $i = 0, \dots, n-1$  or the initial zero-bond curve  $P(T_i; 0)$ ,  $i = 0, \dots, n-1$ , both as array of doubles.

Also try to give a different implementation to your methods in the case when the settlement dates  $0 = T_0 < T_1 < \dots < T_n = T$  are evenly distributed, i.e.,  $\Delta_i := T_{i+1} - T_i = \delta > 0 \forall i = 1, \dots, n$ : in this case, you can save time avoiding to compute  $\Delta_i$  more than once.

Write then a JUnit test class which checks if the value for the Swap you get by choosing  $S_i = S \forall i$  is actually 0 (again, up to a suitable tolerance). For this purpose, take  $T = 3$ ,  $\Delta_i = \delta = 0.5$  (half year), and the following zero coupon bond curve:

Maturity	Price
6m	0.9986509108
1yr	0.9949129829
1.5yr	0.9897033769
2 yr	0.9835370208
2.5 yr	0.9765298116
3 yr	0.9689909565