

Exercise 3

Consider the tenor discretization $T_0 < T_1 < \dots < T_n$ and the displaced LIBOR market model where the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \dots, n-1$ follow the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \leq t \leq T_i,$$

where $d\langle W_i, W_j \rangle(t) = \rho_{i,j}(t)dt$, under the real-world measure \mathbb{P} . Also assume here that $d > 0$ and that $\sigma_i^D(\cdot)$ are deterministic functions.

- Derive an analytical approximation for the price of a swaption in this setting, in a similar way to what you have seen in the lecture for the log-normal case, see pages 553-565 of the script.

Hint: in order to solve the exercise, you first have to *guess* the dynamics of the par swap rate S . In particular, you can guess S to have displaced dynamics as well, with a displacement $d_S = d$. Try to see why having a look at Lemma 142 at page 258 of the script.

- Assume now that

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad 0 \leq t \leq T_i, \quad 1 \leq i \leq n-1,$$

with $d_i \neq d_j$ for at least one $i \neq j$. How would your analytical approximation change? Would you need one more approximation? Why?

Solution

We want to find an approximated analytic formula for the swaption with swap tenor $T_a < T_{a+1} < \dots < T_b$ which is a subset of the tenor discretization $T_0 < T_1 < \dots < T_n$. We suppose by simplicity that the swap tenor is as coarse as the original tenor discretization. Consider the par swap rate $(S_{a,b}(t))_{0 \leq t \leq T_a}$ associated to the swap tenor above, i.e.,

$$S_{a,b}(t) = S(T_a, \dots, T_b; t) := \frac{P(T_a; t) - P(T_b; t)}{\sum_{j=a}^{b-1} (T_{j+1} - T_j)P(T_{j+1}; t)}, \quad 0 \leq t \leq T_a.$$

As done in the script for the log-normal and normal case, our aim is to find an approximated Black formula for the swaption. Proceeding as in the script, we find that

$$V_{\text{swaption}}(0) = A_{a,b}(0)\mathbb{E}^{Q^A}[\max(S_{a,b}(T_a) - K, 0)], \quad (1)$$

where $A_{a,b}$ is the annuity for the swap tenor above and Q^A is the probability measure associated to the annuity, under which $S_{a,b}$ is a martingale.

In order to evaluate the expectation above, we have of course to define the dynamics of the process driving $S_{a,b}$. Remember that we assume displaced log-normal dynamics for the underlying LIBOR market model, i.e., we introduce the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \dots, n-1$ with

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \leq t \leq T_i. \quad (2)$$

From Lemma 142 at page 258 of the script, we know that

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t)L_k(t), \quad 0 \leq t \leq T_a \quad (3)$$

where the weights are defined by

$$\alpha_k(t) := \frac{(T_{k+1} - T_k)P(T_{k+1}; t)}{\sum_{j=a}^{b-1} (T_{j+1} - T_j)P(T_{j+1}; t)}, \quad 0 \leq t \leq T_a \quad (4)$$

and satisfy $\alpha_k(t) \geq 0$, $k = a, \dots, b-1$, and $\sum_{k=a}^{b-1} \alpha_k(t) = 1$. For this reason and from (2) we get that

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t) L_k(t) \geq \sum_{k=a}^{b-1} \alpha_k(t) (-d) = -d, \quad 0 \leq t \leq T_a.$$

We can then make our guess and assume displaced log-normal dynamics for $S_{a,b}$ as well, with same displacement d as in (2). In particular, we can model the evolution of $S_{a,b}$ as given by

$$dS_{a,b}(t) = (S_{a,b}(t) + d) \sigma_S(t) dW^A(t), \quad 0 \leq t \leq T_a, \quad (5)$$

where W^A is a Q^A -Brownian motion and $\sigma_S(\cdot)$ is a deterministic function. If this is the case, using the same arguments we have seen in Exercise 1 of Handout 11, from (1) we get

$$V_{\text{swaption}}(0) = A(0)BS(S_{a,b}(0) + d, K + d, \bar{\sigma}_S, T_a), \quad (6)$$

i.e., we determine the price of the option as given by the value of the annuity at zero times the Black-Scholes formula with initial value $S_{a,b}(0) + d$, strike $K + d$, maturity T_a and integrated volatility $\bar{\sigma}_S$ defined as

$$\bar{\sigma}_S = \frac{1}{T_a} \int_0^{T_a} \sigma_S^2(t) dt, \quad 0 \leq t \leq T_a. \quad (7)$$

So, we just have to derive an (approximated, as we will see) expression for $\bar{\sigma}_S$, which will be our goal from now on. Since the value of $S_{a,b}$ clearly depends on the values of the LIBOR rates L_k , $k = a, \dots, b-1$ (see for example (3)) we can apply Itô's formula and get

$$\begin{aligned} dS_{a,b}(t) &= (\dots)dt + \sum_{k=a}^{b-1} \frac{\partial S_{a,b}}{\partial L_k}(t) dL_k(t) \\ &= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial(S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} dL_k(t) \\ &= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial(S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} (L_k(t) + d) \sigma_k^D(t) dW_k(t) \\ &= (\dots)dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} w_k(t) \sigma_k^D(t) dW_k(t), \quad 0 \leq t \leq T_a \end{aligned} \quad (8)$$

with

$$w_k(t) = \frac{\partial(S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} (L_k(t) + d) = \frac{\partial \log(S_{a,b} + d)}{\partial L_k}(t) (L_k(t) + d), \quad (9)$$

for any $t \in [0, T_a]$. Once we compute the expression of the weights in (9), we can use it as done in the script in order to determine an approximation for $\sigma_S(\cdot)$ in (5) and then for $\bar{\sigma}_S$ in (7): indeed, equations (5) and (8) imply that $\sigma_S(\cdot)$ satisfies

$$\sigma_S^2(t) dt = \frac{d\langle S_{a,b}, S_{a,b} \rangle(t)}{S_{a,b}^2(t)} = \sum_{k,\ell=a}^{b-1} w_k(t) w_\ell(t) \sigma_k(t) \sigma_\ell(t) \rho_{k,\ell}(t) dt, \quad 0 \leq t \leq T_a.$$

However, the expression above is stochastic, so $\sigma_S^2(\cdot)$ is not deterministic as we would have needed. We then freeze the weights to their initial values and obtain the approximation

$$\sigma_S^2(t) dt \approx \sum_{k,\ell=a}^{b-1} w_k(0) w_\ell(0) \sigma_k(t) \sigma_\ell(t) \rho_{k,\ell}(t) dt, \quad 0 \leq t \leq T_a,$$

which gives us

$$\bar{\sigma}_S^2 \approx \tilde{\sigma}_S^2 := \frac{1}{T_a} \sum_{k,\ell=a}^{b-1} w_k(0) w_\ell(0) \int_0^{T_a} \sigma_k(t) \sigma_\ell(t) \rho_{k,\ell}(t) dt. \quad (10)$$

From (6) and (10) we then get the approximation

$$V_{\text{swaption}}(0) \approx A(0)BS(S_{a,b}(0) + d, K + d, \tilde{\sigma}_S, T_a), \quad (11)$$

with $\tilde{\sigma}_S$ defined in (10).

It remains now to compute the values of the coefficients in (9). For any $t \in [0, T_a]$ we have

$$\begin{aligned}
w_k(t) &= \frac{\partial \log(S_{a,b} + d)}{\partial L_k}(t)(L_k(t) + d) \\
&= \frac{\partial}{\partial L_k} \log \left(\frac{P(T_a) - P(T_b)}{\sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1})} + d \right) (t)(L_k(t) + d) \\
&= \frac{\partial}{\partial L_k} \left(\log \left(P(T_a) - P(T_b) + d \sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}) \right) - \log \left(\sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}) \right) \right) (t) \\
&\quad \cdot (L_k(t) + d) \\
&= \frac{L_k(t) + d}{P(T_a; t) - P(T_b; t) + d \sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)} \frac{\partial}{\partial L_k} \left(P(T_a) - P(T_b) + d \sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}) \right) (t) \\
&\quad + \frac{L_k(t) + d}{\sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)} \sum_{j=a}^{b-1} (T_{j+1} - T_j) \frac{\partial}{\partial L_k} P(T_{j+1}; t) \\
&= \left(\frac{P(T_b; t) + d \sum_{j=k}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)}{P(T_a; t) - P(T_b; t) + d \sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)} + \frac{\sum_{j=k}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)}{\sum_{j=a}^{b-1} (T_{j+1} - T_j) P(T_{j+1}; t)} \right) \\
&\quad \cdot (L_k(t) + d) \frac{(T_{k+1} - T_k)}{1 + (T_{k+1} - T_k) L_k(t)} \\
&= \left(\frac{P(T_b; t) + d A_{k,b}(t)}{P(T_a; t) - P(T_b; t) + d A_{a,b}(t)} + \frac{A_{k,b}(t)}{A_{a,b}(t)} \right) \cdot \left(\frac{P(T_k; t) - P(T_{k+1}; t)}{P(T_k; t)} + d \frac{(T_{k+1} - T_k) P(T_{k+1}; t)}{P(T_k; t)} \right) \\
&= \left(\frac{P(T_b; t) + d A_{k,b}(t)}{P(T_a; t) - P(T_b; t) + d A_{a,b}(t)} + \frac{A_{k,b}(t)}{A_{a,b}(t)} \right) \cdot \frac{P(T_k; t) - P(T_{k+1}; t) + d (T_{k+1} - T_k) P(T_{k+1}; t)}{P(T_k; t)},
\end{aligned}$$

where $A_{a,b}$ and $A_{k,b}$ are the annuities of the tenors $T_a < \dots < T_b$ and $T_k < \dots < T_b$, respectively.

We can plug the above expression evaluated at time $t = 0$ in (10) to obtain our approximated integrated volatility, from which we then get the approximated price via (11).

Let's now consider the case when

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad 0 \leq t \leq T_i, \quad 1 \leq i \leq n-1,$$

with $d_i \neq d_j$ for at least one $i \neq j$. In this case,

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t) L_k(t) \geq - \sum_{k=a}^{b-1} \alpha_k(t) d_k, \quad 0 \leq t \leq T_a,$$

so we can guess displaced log-normal dynamics for $S_{a,b}$ with displacement now given by

$$d_S(t) := \sum_{k=a}^{b-1} \alpha_k(t) d_k, \quad 0 \leq t \leq T_a.$$

The problem is now that the coefficients $\alpha_k(t)$ in the expression above are not deterministic, see (9). So one has to freeze them at time $t = 0$ and then proceed as before. This gives a second approximation that can impact the goodness of the price estimate.