Exercise 1

This exercise might be done during the Tutorium

Add the following methods to the class `NormalRandomVariable`:

- `public double[] generateBoxMuller()` which generates and returns a pair of independent normal random variables by the Box-Müller algorithm. This algorithm exploits the fact that if U, V are independent uniform random variables in $[0,1]$ then

$$Z_1 = \sqrt{-2\log(U)}\cos(2\pi V), \qquad Z_2 = \sqrt{-2\log(U)}\sin(2\pi V)$$

  are independent normal $\mathcal{N}(0,1)$.

- `public double[] generateARBoxMuller()`, which generates and returns a pair of independent normal random variables by the following acceptance-rejection refinement of the Box-Müller algorithm:
  - generate $W = U^2 + V^2$ for two uniformly distributed random variables U and V in $[-1,1]$;
  - if $W > 1$ discard W, otherwise set $S = \sqrt{-2\log(W)/W}$;
  - in this way,

$$Z_1 = SU, \qquad Z_2 = SV$$

    are independent normal $\mathcal{N}(0,1)$.

Add these two generation methods to the tests of Exercise 2.(a) of Handout 7.

Exercise 2

Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(2.5, \alpha)$. Also let h be the function $h(x) = I_{\{x>2.5\}}$, f the density of X and g the density of Y. Investigate further the behaviour you have observed in Exercise 2.(b) of Handout 7 by:

- Finding an expression for the difference

$$\mathrm{Var}(h(X)) - \mathrm{Var}(h(Y)f(Y)/g(Y)), \tag{1}$$

  that can help you understanding why this weighted Monte-Carlo approximation reduces the variance of your estimator.

- Empirically testing the variance reduction in (1) using the appropriate method(s) of the interface `RandomVariableInterface`.

Exercise 3

Consider an Itô process satisfying the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad 0 \le t \le T,$$

with $X_0 = x \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and let $\mathcal{T}^\Delta$ be a time-discretization with step size $\Delta > 0$, that is,

$$\mathcal{T}^\Delta = \{t_0 = 0, t_1, \dots, t_n = T\},$$

with $\Delta = t_{i+1} - t_i, i = 1, \dots, n$.
Write down the Euler-Maruyama scheme for $(X_t)_{t\in[0,T]}$ with discretization step-size Δ, i.e., the way you derive $X^\Delta_{t_{k+1}}$ from $X^\Delta_{t_k}$, where $(X^\Delta_{t_i})_{i=0,\dots,n}$ is the approximated process. Also derive analytic expressions for $\mathbb{E}[X_T]$, $Var[X_T]$, $\mathbb{E}[X^\Delta_{t_n}]$ and $Var[X^\Delta_{t_n}]$.