Exercise 1

This exercise may be done during the Tutorium

Write a class `MonteCarloPiFromHypersphere` extending `MonteCarloEvaluationsWithExactResultAbstract` in

`com.andreamazzon.exercise3.montecarloevaluations`

and providing approximations of π from the Monte-Carlo approximation of the volume of the unit hypersphere

$$\{(x_1, \dots, x_d) \in \mathbb{R}^d | x_1^2 + \cdots + x_d^2 \leq 1\}$$

for a given dimension d. The dimension must be a field of the class initialized in the constructor.

Hints: this is basically a generalization for higher dimensions of the Monte-Carlo implementation in `com.andreamazzon.exercise3.montecarlopi.MonteCarloPi`. The volume of the hypersphere is

$$V_d = \int_{-1}^{1} \cdots \int_{-1}^{1} \mathbf{1}_{\{x_1^2+\cdots+x_d^2 \leq 1\}} dx_1 \dots dx_d = 2^d \int_{0}^{1} \cdots \int_{0}^{1} \mathbf{1}_{\{(2(x_1-0.5))^2+\cdots+(2(x_d-0.5))^2 \leq 1\}} dx_1 \dots dx_d. \quad (1)$$

It holds

$$V_{2k} = \frac{\pi^k}{k!}, \quad (2)$$

$$V_{2k+1} = \frac{2(4\pi)^k k!}{(2k+1)!}. \quad (3)$$

for $k \geq 1$ natural number.

Exercise 2

This exercise may be done during the Tutorium

Write a class `HaltonSequencePiFromHypersphere`, providing an approximation of the value of π via the approximation of the integral (1) and the equations (2) - (3), where the evaluations points $(x_1^i, \dots, x_d^i)$, $i = 1, \dots, n$, with n number of sample points, are now provided by an Halton sequence with a given d-dimensional base.

You can write a class `HaltonSequence`, with a method providing the sample points, or directly use the one in

`info.quantlab.numericalmethods.lecture.randomnumbers`

in the `numerical-methods-lecture` project.

The class `HaltonSequencePiFromHypersphere` must also provide a `public` method which returns the error in the approximation (note that here, for a given base, only one value of the approximation is produced, so it does not make sense to consider a vector of approximations as for the Monte-Carlo method).

Experiment on the quality of the approximation of the two methods by printing the average error produced by `MonteCarloPiFromHypersphere` for 100 computations and the error given by `HaltonSequencePiFromHypersphere`, for 100000 sample points, for different dimensions.

Regarding the choice of the base of `HaltonSequencePiFromHypersphere`, consider the following cases:

- all the elements of the base are equal to each other (for example, `base =` $\{2,2,2,2\}$ for dimension 4);
- the elements of the base are different to each other, but share common divisors (for example, `base =` $\{2,4,6,8\}$ for dimension 4);

- the elements of the base are different to each other, and do not share common divisors (for example, `base = ` $\{$`2,3,5,7`$\}$ for dimension 4).

What do you observe regarding the approximation error? How can you explain this behaviour?

Exercise 3

Find $D(A_i)$ and $D^*(A_i)$, $i = 1, 2$, for the sets:

$$A_1 = \{1/8, 1/4, 1/2, 3/4\}$$

and

$$A_2 = \{1/4, 1/2, 5/8, 3/4\}.$$