Exercise 3

Consider the tenor discretization $T_0 < T_1 < \cdots < T_n$ and the displaced LIBOR market model where the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \dots, n-1$ follow the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i,$$

where $d\langle W_i, W_j\rangle(t) = \rho_{i,j}(t)dt$, under the real-world measure $\mathbb{P}$. Also assume here that $d > 0$ and that $\sigma_i^D(\cdot)$ are deterministic functions.
Derive an analytical approximation for the price of a swaption in this setting, in a similar way to what you have seen in the lecture for the log-nornal case, see pages 669-681 of the script.
Hint 1: in order to solve the exercise, you first have to *guess* the dynamics of the par swap rate S. In particular, you can guess S to have displaced dynamics as well, with a displacement $d_S = d$. Try to see why having a look at Lemma 175 at page 121 of the script.
Hint 2: the computation of coefficients w_k might be quite lengthy. In case, you don't have to compute them in detail (this is not the point of the exercise).

Solution

We want to find an approximated analytic formula for the swaption with swap tenor $T_a < T_{a+1} < \cdots < T_b$ which is a subset of the tenor discretization $T_0 < T_1 < \cdots < T_n$. We suppose by simplicity that the swap tenor is as coarse as the original tenor discretization.
Proceeding as in the script (pages 197-201) we find that

$$V_{\text{swaption}}(T_a) = A_{a,b}(T_a)\max\left(S_{a,b}(T_a) - K, 0\right),$$

where $A_{a,b}$ is the annuity for the swap tenor above and $(S_{a,b}(t))_{0\le t\le T_a}$ is the process representing the par swap rate associated to the swap tenor above, i.e.,

$$S_{a,b}(t) = S(T_a, \dots, T_b; t) := \frac{P(T_a; t) - P(T_b; t)}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1}; t)}, \quad 0 \le t \le T_a.$$

Then

$$V_{\text{swaption}}(0) = A_{a,b}(0)\mathbb{E}^{Q^A}\left[\max\left(S_{a,b}(T_a) - K, 0\right)\right], \tag{1}$$

where Q^A is the probability measure associated to the annuity, under which $S_{a,b}$ is a martingale.
We have seen that, *if* $S_{a,b}$ follows log-normal dynamics with a deterministic log-volatility function $\sigma_{a,b}(\cdot)$, we can use the Black formula to compute (1). So the question now is: is it possible to assume such dynamics for $S_{a,b}$, at least approximately?
Remember that we assume displaced log-normal dynamics for the underlying LIBOR market model, i.e., we introduce the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \dots, n-1$ with

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i. \tag{2}$$

From Lemma 175 at page 121 of the script, we know that

$$S_{a,b}(t) = \sum_{k=a}^{b-1}\alpha_k(t)L_k(t), \quad 0 \le t \le T_a \tag{3}$$

where the weights are defined by

$$\alpha_k(t) := \frac{(T_{k+1} - T_k)P(T_{k+1}; t)}{\sum_{j=a}^{b-1}(T_{j+1} - T_j)P(T_{j+1}; t)}, \quad 0 \le t \le T_a \tag{4}$$

and satisfy $\alpha_k(t) \geq 0$, $k = a, \dots, b-1$, and $\sum_{k=a}^{b-1} \alpha_k(t) = 1$. For this reason and from (2) we get that

$$S_{a,b}(t) = \sum_{k=a}^{b-1} \alpha_k(t) L_k(t) \geq \sum_{k=a}^{b-1} \alpha_k(t)(-d) = -d, \quad 0 \leq t \leq T_a.$$

We can then make our guess and assume displaced log-normal dynamics for $S_{a,b}$ as well, with same displacement d as in (2). In particular, we can model the evolution of $S_{a,b}$ as given by

$$dS_{a,b}(t) = (S_{a,b}(t) + d)\sigma_S(t) dW^A(t), \quad 0 \leq t \leq T_a, \tag{5}$$

where W^A is a Q^A-Brownian motion and $\sigma_S(\cdot)$ is a deterministic function. If this is the case, using the same arguments we have seen in Exercise 1 of Handout 9, from (1) we get

$$V_{\text{swaption}}(0) = A(0) BS\left(S_{a,b}(0) + d, K + d, \bar{\sigma}_S, T_a\right), \tag{6}$$

i.e., we determine the price of the option as given by the value of the annuity at zero times the Black-Scholes formula with initial value $S_{a,b}(0) + d$, strike $K + d$, maturity T_a and integrated volatility $\bar{\sigma}_S$ defined as

$$\bar{\sigma}_S = \frac{1}{T_a} \int_0^{T_a} \sigma_S^2(t) dt, \quad 0 \leq t \leq T_a. \tag{7}$$

So, we just have to derive an (approximated, as we will see) expression for $\bar{\sigma}_S$, which will be our goal from now on. Since the value of $S_{a,b}$ clearly depends on the values of the LIBOR rates L_k, $k = a, \dots, b-1$ (see (3)) we can apply Itô's formula and get

$$\begin{aligned}
dS_{a,b}(t) &= (\dots) dt + \sum_{k=a}^{b-1} \frac{\partial S_{a,b}}{\partial L_k}(t) dL_k(t) \\
&= (\dots) dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} dL_k(t) \\
&= (\dots) dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} (L_k(t) + d) \sigma_k^D(t) dW_k(t) \\
&= (\dots) dt + (S_{a,b}(t) + d) \sum_{k=a}^{b-1} w_k(t) \sigma_k^D(t) dW_k(t), \quad 0 \leq t \leq T_a
\end{aligned} \tag{8}$$

with

$$w_k(t) = \frac{\partial (S_{a,b} + d)}{\partial L_k}(t) \frac{1}{S_{a,b}(t) + d} (L_k(t) + d) = \frac{\partial \log(S_{a,b} + d)}{\partial L_k}(t)(L_k(t) + d), \tag{9}$$

for any $t \in [0, T_a]$. Once we compute the expression of the weights in (9), we can use it as done in the script in order to determine an approximation for $\sigma_S(\cdot)$ in (5) and then for $\bar{\sigma}_S$ in (7): indeed, equations (5) and (8) imply that $\sigma_S(\cdot)$ satisfies

$$\sigma_S^2(t) dt = \frac{d\langle S_{a,b}\rangle(t)}{(S_{a,b}(t) + d)^2} = \sum_{k,\ell=a}^{b-1} w_k(t) w_\ell(t) \sigma_k^D(t) \sigma_\ell^D(t) \rho_{k,\ell}(t) dt, \quad 0 \leq t \leq T_a.$$

However, the expression above is stochastic, so $\sigma_S^2(\cdot)$ is not deterministic as we would have needed. We then freeze the weights to their initial values and obtain the approximation

$$\sigma_S^2(t) dt \approx \sum_{k,\ell=a}^{b-1} w_k(0) w_\ell(0) \sigma_k^D(t) \sigma_\ell^D(t) \rho_{k,\ell}(t) dt, \quad 0 \leq t \leq T_a,$$

which gives us

$$\bar{\sigma}_S^2 \approx \tilde{\sigma}_S^2 := \frac{1}{T_a} \sum_{k,\ell=a}^{b-1} w_k(0) w_\ell(0) \int_0^{T_a} \sigma_k^D(t) \sigma_\ell^D(t) \rho_{k,\ell}(t) dt. \tag{10}$$

From (6) and (10) we then get the approximation

$$V_{\text{swaption}}(0) \approx A(0) BS\left(S_{a,b}(0) + d, K + d, \tilde{\sigma}_S, T_a\right), \tag{11}$$

with $\tilde{\sigma}_S$ defined in (10).
It remains now to compute the values of the coefficients in (9). For any $t \in [0, T_a]$ we have

$$
\begin{aligned}
w_k(t) &= \frac{\partial \log(S_{a,b}+d)}{\partial L_k}(t)(L_k(t)+d) \\
&= \frac{\partial}{\partial L_k} \log\left(\frac{P(T_a)-P(T_b)}{\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}+d\right)(t)(L_k(t)+d) \\
&= \frac{\partial}{\partial L_k}\left(\log\left(P(T_a)-P(T_b)+d\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})\right)-\log\left(\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})\right)\right)(t) \\
&\quad \cdot (L_k(t)+d) \\
&= \frac{L_k(t)+d}{P(T_a)-P(T_b)+d\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}\frac{\partial}{\partial L_k}\left(P(T_a)-P(T_b)+d\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})\right)(t) \\
&\quad - \frac{L_k(t)+d}{\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}\sum_{j=a}^{b-1}(T_{j+1}-T_j)\frac{\partial}{\partial L_k}P(T_{j+1}) \\
&= \left(\frac{P(T_b)+d\sum_{j=k}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}{P(T_a)-P(T_b)+d\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}-\frac{\sum_{j=k}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}{\sum_{j=a}^{b-1}(T_{j+1}-T_j)P(T_{j+1})}\right) \\
&\quad \cdot (L_k(t)+d)\frac{(T_{k+1}-T_k)}{1+(T_{k+1}-T_k)L_k(t)} \\
&= \left(\frac{P(T_b)+dA_{k,b}(t)}{P(T_a)-P(T_b)+dA_{a,b}(t)}-\frac{A_{k,b}(t)}{A_{a,b}(t)}\right)\cdot\left(\frac{P(T_k)-P(T_{k+1})}{P(T_k)}+d\frac{(T_{k+1}-T_k)P(T_{k+1})}{P(T_k)}\right) \\
&= \left(\frac{P(T_b)+dA_{k,b}(t)}{P(T_a)-P(T_b)+dA_{a,b}(t)}-\frac{A_{k,b}(t)}{A_{a,b}(t)}\right)\cdot\frac{P(T_k)-P(T_{k+1})+d(T_{k+1}-T_k)P(T_{k+1})}{P(T_k)},
\end{aligned}
$$

where $A_{a,b}$ and $A_{k,b}$ are the annuities of the tenors $T_a < \cdots < T_b$ and $T_k < \cdots < T_b$, respectively.
We can plug the above expression evaluated at time $t = 0$ in (10) to obtain our approximated integrated volatility, from which we then get the approximated price via (11).