Exercise 1

Given the tenor discretization $T_0 < T_1 < \cdots < T_n$, consider the caplets paying

$$\max\left(L(T_i, T_{i+1}; T_i) - K, 0\right)(T_{i+1} - T_i) \quad \text{in } T_{i+1},$$

$i = 1, \dots, n-1$, for a given LIBOR market model represented by the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \dots, n-1$.
Derive the analytic formula of such caplets under the displaced lognormal model, that is, assuming that for any $i = 1, \dots, n-1$, the process L_i follows the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad t \geq 0,$$

under the real-world measure $\mathbb{P}$. Also assume here that $d_i > 0$ and that $\sigma_i^D(\cdot)$ are deterministic functions.

Solution

Fix an index $i = 1, \dots, n-1$. Consider then the process L_i following the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d_i)\sigma_i^D(t)dW_i(t), \quad t \geq 0,$$

under the real-world measure $\mathbb{P}$. Note that L_i is a martingale under the measure $Q^{P(T_{i+1})}$, i.e., has the dynamics

$$dL_i(t) = (L_i(t) + d_i)\sigma_i^D(t)dW_i^{Q^{P(T_{i+1})}}(t), \quad t \geq 0$$

where $W^{Q^{P(T_{i+1})}}$ is a $Q^{P(T_{i+1})}$-Brownian motion.
Also note that

$$\begin{aligned}
\max\left(L(T_i, T_{i+1}; T_i) - K, 0\right)(T_{i+1} - T_i) &= \max\left(L_i(T_i) - K, 0\right)(T_{i+1} - T_i) \\
&= \max\left((L_i(T_i) + d_i) - (K + d_i), 0\right)(T_{i+1} - T_i),
\end{aligned}$$

so that evaluating the caplet with strike K written on the underlying L_i is equivalent to evaluate a caplet with strike $K + d_i$ written on the underlying $\bar{L}_i := L_i + d_i$. We have that $\bar{L}_i$ satisfies

$$\begin{aligned}
&d\bar{L}_i(t) = dL_i(t) = (L_i(t) + d_i)\sigma_i^D(t)dW_i^{Q^{P(T_{i+1})}}(t) = \bar{L}_i(t)\sigma_i^D(t)dW_i^{Q^{P(T_{i+1})}}(t), \quad t \geq 0, \\
&\bar{L}_i(0) = L_i(0) + d.
\end{aligned}$$

This implies that we can get the price of the caplet written on $\bar{L}_i$ with strike $K + d_i$ from formula (83) of the script, where the initial value is the initial value of $\bar{L}_i$, i.e., $\bar{L}_i(0) = L_i(0) + d_i$, and the strike is $K + d_i$. This gives us the formula of the caplet written on L_i and solves the exercise.