Exercise 2

Consider two dates $0 < T_1 < T_2$. Do the following experiments about convexity adjustments:

- Consider first the natural floater paying

$$N(T_2 - T_1)L(T_1, T_2; T_1) \tag{1}$$

in T_2, and then the LIBOR in arrears paying the value (1) in T_1. Write a `Junit` test class where you set $T_1 = 1$, $T_2 = 2$, let the LIBOR follow log-normal dynamics with volatility $\sigma = 0.25$, let the notional be $N = 10000$ and the prices of the zero coupon bonds $P(T_1; 0) = 0.95$, $P(T_2; 0) = 0.9$. For such parameters, do the following:

  (a) compute the analytic value of a natural floater;

  (b) simulate the process $L = (L_t)_{t\in[0,T_1]}$ with $L_t := L(T_1, T_2; t)$ under $Q^{P(T_2)}$, and test if the resulting Monte-Carlo value of a natural floater is equal (up to a given tolerance) to the value computed in (a);

  (c) find the analytic formula for the value of the floater in arrears and compute the value in the present case;

  (d) compute the value of the floater in arrears by simulating the process L under $Q^{P(T_2)}$ and compare this value with the one found in (c).

- A caplet is said to be paid *in arrears* if the payment of the option on the observed LIBOR rate $L(T_1, T_2; T_1)$ is made at T_1 instead of T_2. Find the formula for the price of a caplet in arrears by a suitable convexity adjustment, and use it to write a method that prices this product.

Compute then the difference between the valuation of the caplet in Exercise 1 of Handout 4 and the current one, setting $T_1 = 1$, $T_2 = 2$, $L(T_1, T_2; 0) = 0.05$, LIBOR volatility $\sigma = 0.3$, strike $K = 0.044$, discount factor $P(T_2; 0) = 0.91$, notional $N = 10000$: this is the market price of the convexity adjustment.
Expected result for the difference: 5.7819.

Solution of the theoretical parts

We always assume here that the notional is equal to 1. For a general notional, we just need to multiply everything by a constant.
For a given numéraire $N = (N_t)_{t\geq 0}$, we introduce the measure Q^N such that for any traded asset $X^i = (X^i_t)_{t\geq 0}$ we have that the discounted value X^i/N is a martingale under Q^N.
Consider now the underlying $L = (L_t)_{t\geq 0}$ defined by

$$L_t := L(T_1, T_2; t) = \frac{1}{T_2 - T_1}\frac{P(T_1; t) - P(T_2; t)}{P(T_2; t)}, \quad t \geq 0.$$

Let $N_t = P(T_2; t)$, $t \geq 0$. Then L is a martingale under Q^N since

$$L_t = \frac{1}{T_2 - T_1}\frac{P(T_1; t) - P(T_2; t)}{P(T_2; t)} = \frac{P(T_1; t) - P(T_2; t)}{T_2 - T_1}\frac{1}{N_t}, \quad t \geq 0,$$

and $P(T_1; \cdot)$, $P(T_2; \cdot)$ are traded assets.
Consider a product with payoff $V(T_1) = f(L_{T_1})$, and assume first it is payed in T_2. Then we have

$$\frac{V(0)}{N_0} = \mathbb{E}^{Q^N}\left[\frac{V(T_1)}{N_{T_2}}\middle|\mathcal{F}_0\right] = \mathbb{E}^{Q^N}\left[\frac{V(T_1)}{P(T_2; T_2)}\middle|\mathcal{F}_0\right] = \mathbb{E}^{Q^N}\left[V(T_1)|\mathcal{F}_0\right],$$

so that

$$V(0) = P(T_2;0)\mathbb{E}^{Q^N}\left[V(T_1)|\mathcal{F}_0\right].$$

The value of a natural floater comes then as a very simple application of the evaluation described above, when $V(T_1) = f(L_{T_1}) := (T_2 - T_1)L_{T_1}$: in this case, we get indeed

$$\begin{aligned}
V(0) &= P(T_2;0)\mathbb{E}^{Q^N}\left[(T_2 - T_1)L_{T_1}|\mathcal{F}_0\right] = P(T_2;0)(T_2 - T_1)L_0 \\
&= P(T_2;0)(T_2 - T_1)\frac{1}{T_2 - T_1}\frac{P(T_1;0) - P(T_2;0)}{P(T_2;0)} \\
&= P(T_1;0) - P(T_2;0).
\end{aligned} \tag{2}$$

Assume now that $V(T_1) = f(L_{T_1})$ is payed in T_1. In this case, we have

$$\frac{V(0)}{N_0} = \mathbb{E}^{Q^N}\left[\frac{V(T_1)}{N_{T_1}}\bigg|\mathcal{F}_0\right] = \mathbb{E}^{Q^N}\left[\frac{V(T_1)}{P(T_2;T_1)}\bigg|\mathcal{F}_0\right].$$

Therefore, here we cannot exploit the fact that the numéraire cancels out at the payment time. However, we can write

$$\frac{1}{N_{T_1}} = \frac{1}{P(T_2;T_1)} = 1 + \frac{1 - P(T_2;T_1)}{P(T_2;T_1)} = 1 + \frac{P(T_1;T_1) - P(T_2;T_1)}{P(T_2;T_1)} = 1 + \frac{M_{T_1}}{N_{T_1}},$$

with $M = (M_t)_{t\geq 0}$ defined by

$$M_t := P(T_1;t) - P(T_2;t), \quad t \geq 0.$$

Thus we get

$$\frac{V(0)}{N_0} = \mathbb{E}^{Q^N}\left[V(T_1)\left(1 + \frac{M_{T_1}}{N_{T_1}}\right)\bigg|\mathcal{F}_0\right] = \mathbb{E}^{Q^N}\left[V(T_1)|\mathcal{F}_0\right] + \underbrace{\mathbb{E}^{Q^N}\left[V(T_1)\frac{M_{T_1}}{N_{T_1}}\bigg|\mathcal{F}_0\right]}_{\text{convexity adjustment}}.$$

Therefore, we have to compute

$$\mathbb{E}^{Q^N}\left[V(T_1)\frac{M_{T_1}}{N_{T_1}}\bigg|\mathcal{F}_0\right] = \mathbb{E}^{Q^N}\left[f(L_{T_1})\frac{M_{T_1}}{N_{T_1}}\bigg|\mathcal{F}_0\right].$$

We are then in the context of Theorem 213 of the script, see page 268. We now check if the hypothesis of the theorem hold in our case. First, we want to see that $L\frac{N}{M}$ is a martingale under Q^M. This is trivially true since

$$\frac{M_t}{N_t} = \frac{P(T_1;t) - P(T_2;t)}{P(T_2;t)} = L(T_1,T_2;t)(T_2 - T_1) = L_t(T_2 - T_1), \quad t \geq 0,$$

so that

$$L_t\frac{N_t}{M_t} = \frac{1}{T_2 - T_1},$$

for any $t \geq 0$.

We assume now log-normal dynamics for the process L, i.e., we assume that L is driven by the SDE

$$dL_t = \sigma_L L_t dW_t^N, \quad t \geq 0,$$

where W^N is Q^N-Brownian motion. Under this assumption, we want to check that there exists a process $\gamma = (\gamma_t)_{t\geq 0}$ such that

$$d\left\langle L, \frac{M}{N}\right\rangle_t = L_t\frac{M_t}{N_t}\gamma_t, \quad t \geq 0, \tag{3}$$

and compute the value of γ such that (3) holds. Note first that

$$d\left(\frac{M_t}{N_t}\right) = (T_2 - T_1)dL_t = (T_2 - T_1)\sigma_L L_t dW_t^N = \sigma_L\frac{M_t}{N_t}dW_t^N, \quad t \geq 0.$$

Then we have

$$d\left\langle L, \frac{M}{N}\right\rangle_t = \sigma_L^2 L_t\frac{M_t}{N_t}dt,$$

and (3) holds with $\gamma_t = \sigma_L^2$ for all $t \geq 0$.
Then by Theorem 213 of the script we get

$$\begin{aligned}
V(L_0, f, M) &:= N_0 \mathbb{E}^{Q^N}\left[f(L_{T_1}) \frac{M_{T_1}}{N_{T_1}} \middle| \mathcal{F}_0 \right] \\
&= \frac{M_0}{N_0} V(\tilde{L}_0, f, N) \\
&= \frac{P(T_1;0) - P(T_2;0)}{P(T_2;0)} V(\tilde{L}_0, f, N) \\
&= (T_2 - T_1) L_0 V(\tilde{L}_0, f, N),
\end{aligned}$$

where $V(\tilde{L}_0, f, N)$ is the value of the option if is is payed in T_2 and

$$\tilde{L}_0 = L_0 e^{\sigma_L^2 T_1}.$$

This is the general formula. Let's now have a look at our two cases, i.e., the floater in arrears and the caplet in arrears.

(a) For the floater we have

$$V(\tilde{L}_0, f, N) = (T_2 - T_1)\tilde{L}_0 P(T_2;0) = (T_2 - T_1) L_0 e^{\sigma_L^2 T_1} P(T_2;0).$$

Then the convexity adjustment is

$$\begin{aligned}
V(L_0, f, M) &= (T_2 - T_1) L_0 V(\tilde{L}_0, f, N) = (T_2 - T_1) L_0 (T_2 - T_1) L_0 e^{\sigma_L^2 T_1} P(T_2;0) \\
&= \left((T_2 - T_1) L_0\right)^2 e^{\sigma_L^2 T_1} P(T_2;0).
\end{aligned}$$

Then the value of the floater in arrears is

$$P(T_2;0) - P(T_1;0) + P(T_2;0)(T_2 - T_1)^2 L_0^2 e^{\sigma_L^2 T_1}.$$

(b) For the caplet we have instead that the convexity adjustment is

$$(T_2 - T_1) L_0 V(L_0 e^{\sigma_L^2 T_1}, f, N) = P(T_2;0)(T_2 - T_1)^2 L_0 BS(L_0 e^{\sigma_L^2 T_1}, 0, \sigma_L, T_1, K),$$

and the value of the caplet in arrears is thus

$$P(T_2;0)(T_2 - T_1) BS(L_0, 0, \sigma_L, T_1, K) + P(T_2;0)(T_2 - T_1)^2 L_0 BS(L_0 e^{\sigma_L^2 T_1}, 0, \sigma_L, T_1, K),$$