Exercise 1

Consider the Monte-Carlo simulation of a log-normal forward rate term structure model (LIBOR Market Model) defined on a tenor $\{T_0, T_1, \dots, T_n\}$ discretized as

$$L_i(t_{j+1}) = L_i(t_j) \exp\left(\left(\mu_i(t_j, L) - \frac{1}{2}\sigma_i^2(t_j)\right)\Delta t_j + \sigma_i(t_j)\Delta W(t_j)\right), \qquad i = 1, \dots, n.$$

Assume that $\sigma_i(\cdot)$ is constant for any $i = 1, \dots, n$ and let $0 < k < n$ be fixed.
Specify a choice of numéraire N such that the valuation of a caplet on the rate L_k does not suffer from a time-discretization error when using the Euler scheme above. Explain why this is a good choice.

Solution

Let's start discussing the discretization error for a general stochastic process $X = (X(t))_{t\geq 0}$ with log-normal dynamics

$$dX(t) = \mu(t, X(t))X(t)dt + \sigma X(t)dW(t), \quad t \geq 0, \tag{1}$$

where $\sigma > 0$ is a constant, $W = (W(t))_{t\geq 0}$ is a standard Brownian motion and $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying conditions under which (1) has a unique strong solution.
Defining the process $Y = (Y(t))_{t\geq 0}$ by

$$Y(t) := \log X(t), \quad t \geq 0$$

and applying Itô's formula we get

$$dY(t) := \left(\mu(t, X(t)) - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t), \quad t \geq 0,$$

so that we can write the dynamics of X as

$$X(t) = X(0) \exp\left\{\int_0^t \left(\mu(s, X(s)) - \frac{1}{2}\sigma^2\right)ds + \sigma W(t)\right\}, \quad t \geq 0.$$

In particular, for any points in time $r < z$ we can write

$$\begin{aligned} X(z) =& X(0) \exp\left\{\int_0^r \left(\mu(s, X(s)) - \frac{1}{2}\sigma^2\right)ds + \int_r^z \left(\mu(s, X(s)) - \frac{1}{2}\sigma^2\right)ds + \sigma W(r) + \sigma(W(z) - W(r))\right\} \\ =& X(r) \exp\left\{\int_r^z \left(\mu(s, X(s)) - \frac{1}{2}\sigma^2\right)ds + \sigma(W(z) - W(r))\right\}. \end{aligned} \tag{2}$$

Based on (2), we can discretize the process X on a (fine enough) time discretization $0 < t_1 < \cdots < t_m$ by

$$\tilde{X}(t_{j+1}) = \tilde{X}(t_j) \exp\left\{\left(\mu(t_j, \tilde{X}(t_j)) - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))\right\}, \quad j = 0, \dots, m-1.$$

This introduces a time discretization error $X(t_j) - \tilde{X}(t_j)$, $j = 1, \dots, m$, which could be not negligible and also increase in time (due to the red terms).
However: what if $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is constant (so, if it does not depend on time and on space)? In this case, we have no need to approximate the blue integral in (2), so that

$$X(t_{j+1}) = X(t_j) \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j))\right\}, \quad j = 0, \dots, m-1$$

and

$$\tilde{X}(t_{j+1}) = \tilde{X}(t_j) \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right)(t_{j+1} - t_j) + \sigma(W(t_{j+1}) - W(t_j)) \right\}, \quad j = 0, \dots, m-1.$$

Thus (having of course the same starting value) we have $X = \tilde{X}$ and no discretization error.
So: our goal in this exercise is to make constant (or better, zero) the drift μ_k of L_k in

$$L_k(t_{j+1}) = L_k(t_j) \exp\left(\left(\mu_k(t_j, L) - \frac{1}{2}\sigma_i^2 \right) \Delta t_j + \sigma_i \Delta W(t_j) \right). \tag{3}$$

In this case, indeed, this discretization is not affected by the time-discretization error, and since the value of caplet on L_k only depends from the evolution of L_k until T_k, we have that the valuation itself of the option via the above Euler scheme is also not affected by the time-discretization error.
Remember that

$$L_k(t) = L(T_k, T_{k+1}; t) = \frac{1}{T_{k+1} - T_k} \frac{P(T_k; t) - P(T_{k+1}; t)}{P(T_{k+1}; t)}, \quad 0 \leq t \leq T_k.$$

Therefore, we have to choose a numéraire $N = (N(t))_{t \in [0, T_n]}$ such that $N(t) = P(T_{k+1}; t)$ for any $t \in [0, T_k]$: indeed, for in this case, we would have that L_k is the N-relative value of a portfolio of zero coupon bonds (which are traded assets) and thus a martingale under the measure $\mathbb{Q}^N$ associated to this numéraire. This implies of course that the drift of L_k under $\mathbb{Q}^N$ is zero, so we would be done.
An easy choice is the T_{k+1}-forward drift, defined as

$$N(t) = \begin{cases} P(T_{k+1}; t), & t \leq T_{k+1}, \\ P(T_{m(t)+1;t}) \Pi_{j=k+1}^{m(t)+1} \frac{1}{P(T_j; T_{j-1})}, & t > T_{k+1}, \end{cases} \tag{4}$$

where $m(t) := \max\{i : \ T_i \leq t\}$.