Exercise 1

Consider the Monte-Carlo simulation of a log-normal forward rate term structure model (LIBOR Market Model) defined on a tenor $\{T_0, T_1, \ldots, T_n\}$ discretized as

$$L_i(t_{j+1}) = L_i(t_j)\exp\left(\left(\mu_i(t_j, L) - \frac{1}{2}\sigma_i^2(t_j)\right)\Delta t_j + \sigma_i(t_j)\Delta W(t_j)\right), \qquad i = 1, \ldots, n.$$

Assume that $\sigma_i(\cdot)$ is constant for any $i = 1, \ldots, n$ and let $0 < k < n$ be fixed.
Specify a choice of numéraire N such that the valuation of a caplet on the rate L_k does not suffer from a time-discretization error when using the Euler scheme above. Explain why this is a good choice.

Exercise 2

In the first part of the lecture we have seen analytic formulas for the valuation of some European Options on interest rates. Some of such formulas are given under the Black Model. We now consider instead the Libor Market Model, i.e., we assume log-normal dynamics for n Libors $L_i(\cdot) := L(T_i, T_{i+1}; \cdot)$. Find at least one formula derived under the Black model that still holds in the Libor Market Model setting and at least one that does not hold (because its assumption cannot be valid in general) in the Libor Market Model setting. Motivate your answer.

Exercise 3

Consider the tenor discretization $T_0 < T_1 < \cdots < T_n$ and the displaced LIBOR market model where the processes $L_i := L(T_i, T_{i+1})$, $i = 1, \ldots, n-1$ follow the dynamics

$$dL_i(t) = \mu_i(t)dt + (L_i(t) + d)\sigma_i^D(t)dW_i(t), \quad 0 \le t \le T_i,$$

where $d\langle W_i, W_j\rangle(t) = \rho_{i,j}(t)dt$, under the real-world measure $\mathbb{P}$. Also assume here that $d > 0$ and that $\sigma_i^D(\cdot)$ are deterministic functions.
Derive an analytical approximation for the price of a swaption in this setting, in a similar way to what you have seen in the lecture for the log-nornal case, see pages 669-681 of the script.
Hint 1: in order to solve the exercise, you first have to *guess* the dynamics of the par swap rate S. In particular, you can guess S to have displaced dynamics as well, with a displacement $d_S = d$. Try to see why having a look at Lemma 175 at page 121 of the script.
Hint 2: the computation of coefficients w_k might be quite lengthy. In case, you don't have to compute them in detail (this is not the point of the exercise).