

## Computation for Exercise 2

Consider a market with a risk free rate  $r > 0$  and a risky asset following a Black-Scholes model, i.e., represented by a process  $S = (S_t)_{t \geq 0}$  which has dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad t \geq 0,$$

where  $W^Q$  is a one-dimensional Brownian motion with respect to the pricing measure  $Q$ , under which the discounted process  $(e^{-rt}S_t)_{t \geq 0}$  is a martingale.

Consider a call option on  $S$  with maturity  $T > 0$ , i.e., an option with payoff

$$\psi(S_T) = (S_T - K)^+,$$

and an *asset or nothing* option, i.e., an option with payoff

$$\phi(S_T) = S_T \mathbf{1}_{\{S_T > K\}}.$$

We want to prove that the delta of the call option, i.e., the derivative with respect to  $S_0$  of  $e^{-rT} \mathbb{E}[\psi(S_T)]$ , is equal to the valuation of a portfolio of  $1/S_0$  *asset or nothing* options. That is, we want to prove

$$\partial_{S_0} (e^{-rT} \mathbb{E}^Q[(S_T - K)^+]) = \frac{1}{S_0} e^{-rT} \mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}],$$

that is,

$$\partial_{S_0} (\mathbb{E}^Q[(S_T - K)^+]) = \frac{1}{S_0} \mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}].$$

We have that

$$\begin{aligned} \mathbb{E}^Q[(S_T - K)^+] &= \mathbb{E}^Q \left[ \left( S_0 e^{(r-\sigma^2/2)T + \sigma W_T^Q} - K \right)^+ \right] \\ &= \int_{-\infty}^{+\infty} \left( S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} - K \right)^+ g(z) dz, \end{aligned} \quad (1)$$

where  $g(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ . Differentiating (1), we get (by using Leibniz integral rule)

$$\begin{aligned} \partial_{S_0} \mathbb{E}^Q[(S_T - K)^+] &= \partial_{S_0} \int_{-\infty}^{+\infty} \left( S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} - K \right)^+ g(z) dz \\ &= \int_{-\infty}^{+\infty} \partial_{S_0} \left( S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} - K \right)^+ g(z) dz \\ &= \int_{-\infty}^{+\infty} e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} \mathbf{1}_{\{S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} > K\}} g(z) dz \\ &= \frac{1}{S_0} \int_{-\infty}^{+\infty} S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} \mathbf{1}_{\{S_0 e^{(r-\sigma^2/2)T + \sigma \sqrt{T}z} > K\}} g(z) dz \\ &= \frac{1}{S_0} \mathbb{E}^Q[S_T \mathbf{1}_{\{S_T > K\}}]. \end{aligned} \quad (2)$$

Thus we have proved our result.