Exercise 3

Let $P(t,T)$ be the continuously compounded time-t price of a bond maturing at time T, and assume that it is a *deterministic* function of t and T: in other words

$$P(t,T) = e^{-\int_t^T r(u)du}$$

for some deterministic positive *short rate* function $r(t)$.

(a) Prove, via discussion of arbitrage possibilities, that for $t \le T \le S$ it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Define the *continuously compounded forward rate prevailing at t of reset date T and maturing S* as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)},$$

and the instantaneous forward rate as

$$f(t,T) = \lim_{S \to T} f(t,T,S).$$

Prove that

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

and

$$r(t) = \lim_{T \to t} f(t,T).$$

(c) Establish in which of the points (a) and (b) the assumption of deterministic rates is necessary or can be relaxed to some class of stochastic short rates $r(t)$.

Solution to exercise 3

(a) Suppose that $P(t,S) > P(t,T)P(T,S)$, for some times $t \le T \le S$. Then we can apply the following strategy:

- at time t, we sell an S-bond and buy $P(T,S)$ units of a T-bond: the total cost is

$$-P(t,S) + P(T,S)P(t,T) < 0,$$

  by assumption.
- At time T, we receive $P(T,S)$ euros from the T-bond we have bought in t, and buy an S-bond: the total cost is

$$-P(T,S) + P(T,S) = 0.$$

- At time S, we receive one euro (from the S-bond we have bought in T) and pay one euro (for the S-bond we have sold in t).

The strategy above gives us a net gain of

$$P(t,S) - P(T,S)P(t,T) > 0,$$

so it is an arbitrage.

If $P(t,S) < P(t,T)P(T,S)$, the same profit can be made just changing the signs in the strategy. We have then seen that in order to avoid arbitrage opportunities, it has to hold

$$P(t,S) = P(t,T)P(T,S).$$

(b) Suppose again $t \le T \le S$.

The *continuously compounded forward rate prevailing at t of reset date T and maturing S*, called $f(t,T,S)$, is defined as the unique solution to the equation

$$e^{f(t,T,S)(S-T)} := \frac{P(t,T)}{P(t,S)}, \tag{1}$$

and the *instantaneous forward rate* $f(t,T)$ by

$$f(t,T) := \lim_{S\to T} f(t,T,S).$$

We want first to see that

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

From (1) we get

$$f(t,T,S)(S-T) = \ln\left(\frac{P(t,T)}{P(t,S)}\right),$$

and so

$$f(t,T,S) = -\frac{\ln P(t,S) - \ln P(t,T)}{S-T}.$$

Hence

$$f(t,T) := \lim_{S\to T} f(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{2}$$

Since at the same way we can show that

$$f(t,u) = -\frac{\partial \ln P(t,u)}{\partial u}$$

for every $u \ge t$, integrating we get

$$-\int_t^T f(t,u)du = \int_t^T \frac{\partial \ln P(t,u)}{\partial u} du = \ln P(t,T) - \ln P(t,t) = \ln P(t,T).$$

Then we have

$$P(t,T) = e^{-\int_t^T f(t,u)du}. \tag{3}$$

We now want to see that

$$r(t) = \lim_{T\to t} f(t,T). \tag{4}$$

From

$$e^{-\int_t^T r(u)du} = P(t,T)$$

we get

$$-\int_t^T r(u)du = \ln P(t,T) = \ln P(t,T) - \ln P(t,t),$$

so that dividing both sides by $T-t$ and taking the limit for $T \to t$ we get

$$-r(t) = \lim_{T\to t} \frac{\partial \ln P(t,T)}{\partial T} = -\lim_{T\to t} f(t,T),$$

where the last equality follows by equation (2).

(c) In point (a), we have to require that the short rate $(r(t))_{t\geq 0}$ is deterministic when we set up the strategy to buy $P(T,S)$ units of the S bond: in order to be able to do this, we have to know the value of $P(T,S)$ at t. However, if r is stochastic, we have

$$P(T,S) = \mathbb{E}^Q\left[e^{-\int_T^S r(u)du}\middle|\mathcal{F}_T\right], \quad 0 \leq T \leq S$$

for a given pricing measure Q and a given filtration $(\mathcal{F}_t)_{t\geq 0}$. This is of course not $\mathcal{F}_t$-measurable if $T > t$, so we don't know this value at t.

In point (b), we have that (3) also holds if $(r(t))_{t\geq 0}$ is stochastic: note that we did not use equation (1), nor any hypothesis on r, to prove it. However, we used (1) in order to prove (4): this does not hold if $(r(t))_{t\geq 0}$ is stochastic.