Solution to exercise 1

Let $X = (X_t)_{t\in[0,T]}$ be the geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad 0 \le t \le T,$$

with $X_0 = x \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$.
We start by showing that

$$\mathbb{E}[X_t] = xe^{\mu t}, \qquad Var(X_t) = xe^{2\mu t}(e^{\sigma^2 t} - 1), \quad 0 \le t \le T.$$

We apply Itô's formula for $Y = \log(X)$. We have

$$\begin{aligned} dY_t &= \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2} d[X,X]_t \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t, \quad 0 \le t \le T. \end{aligned}$$

Thus it holds

$$\log(X_t) = \log(x) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \quad 0 \le t \le T,$$

from which it follows

$$X_t = xe^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad 0 \le t \le T.$$

We then easily obtain that

$$\mathbb{E}[X_t] = xe^{(\mu - \frac{1}{2}\sigma^2)t}\mathbb{E}[e^{\sigma W_t}], \quad 0 \le t \le T.$$

In order to compute $\mathbb{E}[e^{\sigma W_t}]$, we exploit the fact that $\sigma W_t \sim \mathcal{N}(0, \sigma^2 t)$, and we know the density function. We have

$$\begin{aligned} \mathbb{E}[X_t] &= \int_{-\infty}^{+\infty} e^y \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{y^2}{2\sigma^2 t}} dy \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-y^2 + 2y\sigma^2 t}{2\sigma^2 t}} dy \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{\frac{-(y-\sigma^2 t)^2}{2\sigma^2 t}} e^{\frac{\sigma^2 t}{2}} dy \\ &= \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{\sigma^2 t}{2}} \int_{-\infty}^{+\infty} e^{\frac{-(y-\sigma^2 t)^2}{2\sigma^2 t}} dy \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma^2 t}{2}} \int_{-\infty}^{+\infty} e^{\frac{-z^2}{2}} dz \\ &= e^{\frac{\sigma^2 t}{2}}, \quad 0 \le t \le T. \end{aligned} \tag{1}$$

where in the last equality we have taken the change of variables $z = \frac{y - \sigma^2 t}{\sigma\sqrt{t}}$, with $dy = \sigma\sqrt{t}dz$. Then we have

$$\mathbb{E}[X_t] = xe^{(\mu - \frac{1}{2}\sigma^2)t}\mathbb{E}[e^{\sigma W_t}] = xe^{\mu t}, \quad 0 \le t \le T.$$

Moreover,

$$Var(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2, \quad 0 \le t \le T,$$

where

$$\mathbb{E}[X_t]^2 = x^2 e^{2\mu t}, \quad 0 \leq t \leq T$$

and

$$\begin{aligned}
\mathbb{E}[X_t^2] &= \mathbb{E}[x^2 e^{2(\mu - \frac{1}{2}\sigma^2)t + 2\sigma W_t}] \\
&= x^2 e^{2(\mu - \frac{1}{2}\sigma^2)t} \mathbb{E}[e^{2\sigma W_t}] \\
&= x^2 e^{2(\mu - \frac{1}{2}\sigma^2)t} e^{2\sigma^2 t} \qquad [\text{since } \mathbb{E}[e^{2\sigma W_t}] = e^{\frac{(2\sigma)^2 t}{2}} = e^{2\sigma^2 t}] \\
&= x^2 e^{2\mu t} e^{\sigma^2 t}, \quad 0 \leq t \leq T.
\end{aligned}$$

We obtain therefore

$$Var(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = x^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1 \right), \quad 0 \leq t \leq T.$$

Let now $\mathcal{T}^\Delta$ be a time-discretization with step size $\Delta > 0$, that is,

$$\mathcal{T}^\Delta = \{t_0 = 0, t_1, \dots, t_n = T\},$$

with $\Delta = t_{i+1} - t_i, i = 1, \dots, n$. Then the Euler-Maruyama scheme for X with time-discretization $\mathcal{T}^\Delta$ is

$$X_k^\Delta = (1 + \mu\Delta) X_{k-1}^\Delta + \sigma X_{k-1}^\Delta \Delta W_k,$$

where we denote $X_k^\Delta = X_{t_k}^\Delta$ and $W_k = W_{t_k} - W_{t_{k-1}}$, $k = 1, \dots, n$.
We want to prove that it holds

$$\mathbb{E}[X_k^\Delta] = x(1 + \mu\Delta)^k, \quad k = 1, \dots, n, \tag{2}$$

and

$$Var(X_k^\Delta) = x^2 \left[\left((1 + \mu\Delta)^2 + \sigma^2 \Delta \right)^k - (1 + \mu\Delta)^{2k} \right], \quad k = 1, \dots, n. \tag{3}$$

We start from the expectation and prove the statement by induction. We have that

$$\mathbb{E}[X_1^\Delta] = \mathbb{E}[(1 + \mu\Delta)x + \sigma x \Delta W_1] = x(1 + \mu\Delta),$$

as we wanted. We now fix $k > 1$ and suppose that $\mathbb{E}[X_{k-1}^\Delta] = x(1 + \mu\Delta)^{k-1}$. Then it holds

$$\begin{aligned}
\mathbb{E}[X_k^\Delta] &= \mathbb{E}[(1 + \mu\Delta) X_{k-1}^\Delta + \sigma X_{k-1}^\Delta \Delta W_k] \\
&= (1 + \mu\Delta) \mathbb{E}[X_{k-1}^\Delta] \\
&= x(1 + \mu\Delta)(1 + \mu\Delta)^{k-1} \\
&= x(1 + \mu\Delta)^k.
\end{aligned}$$

Note that this implies that

$$\mathbb{E}[X_n^\Delta] = x(1 + \mu\Delta)^n = \left(1 + \mu \frac{T}{n} \right)^n \longrightarrow e^{\mu T} = \mathbb{E}[X_T]$$

as $n \to \infty$. We now compute the variance

$$Var(X_k^\Delta) = \mathbb{E}[(X_k^\Delta)^2] - \mathbb{E}[X_k^\Delta]^2, \quad k = 1, \dots, n.$$

In order to get (3), we use (2) and prove that

$$\mathbb{E}[(X_k^\Delta)^2] = x^2 \left((1 + \mu\Delta)^2 + \sigma^2 \Delta \right)^k, \quad k = 1, \dots, n.$$

We have

$$
\begin{aligned}
\mathbb{E}[(X_k^\Delta)^2] &= \mathbb{E}\left[\left((1+\mu\Delta)X_{k-1}^\Delta + \sigma X_{k-1}^\Delta \Delta W_k\right)^2\right] \\
&= \mathbb{E}\left[(1+\mu\Delta)^2 (X_{k-1}^\Delta)^2\right] + \mathbb{E}\left[2\sigma(1+\mu\Delta)(X_{k-1}^\Delta)^2 \Delta W_k\right] + \mathbb{E}\left[\sigma^2 (X_{k-1}^\Delta)^2 (\Delta W_k)^2\right] \\
&= (1+\mu\Delta)^2 \mathbb{E}\left[(X_{k-1}^\Delta)^2\right] + 2\sigma(1+\mu\Delta)\mathbb{E}\left[(X_{k-1}^\Delta)^2\right]\mathbb{E}\left[\Delta W_k\right] + \sigma^2 \mathbb{E}\left[(X_{k-1}^\Delta)^2\right]\mathbb{E}\left[(\Delta W_k)^2\right] \\
&= (1+\mu\Delta)^2 \mathbb{E}\left[(X_{k-1}^\Delta)^2\right] + \sigma^2 \mathbb{E}\left[(X_{k-1}^\Delta)^2\right]\Delta \\
&= \left((1+\mu\Delta)^2 + \sigma^2\Delta\right)\mathbb{E}\left[(X_{k-1}^\Delta)^2\right] \\
&= \left((1+\mu\Delta)^2 + \sigma^2\Delta\right)\left((1+\mu\Delta)^2 + \sigma^2\Delta\right)\mathbb{E}\left[(X_{k-2}^\Delta)^2\right] \\
&= \dots \\
&= \left((1+\mu\Delta)^2 + \sigma^2\Delta\right)\left((1+\mu\Delta)^2 + \sigma^2\Delta\right)^{k-1} x^2 \\
&= x^2 \left((1+\mu\Delta)^2 + \sigma^2\Delta\right)^k .
\end{aligned}
$$

In the third equality we have used the fact that X_{k-1}^Δ is independent of ΔW_k because it depends on the increments $\Delta W_1, \dots \Delta W_{k-1}$, which are indeed independent of ΔW_k.