Solution to exercise 2

Let $f(\cdot)$ be a smooth payoff function, let Y represent an underlying process and consider a parameter θ on which $f(Y)$ may depend.
The pathwise differentiation method approximates the derivative of the payoff $f(Y(\theta))$ at given time with respect to θ as

$$\frac{\partial}{\partial \theta}\mathbb{E}^Q\left[f(Y(\theta))|\mathcal{F}_{t_0}\right] \approx \frac{1}{n}\sum_{i=1}^{n} f'\left(Y(\omega_i,\theta)\right)\cdot\frac{\partial Y(\omega_i,\theta)}{\partial\theta}, \tag{1}$$

with Q pricing measure, where we have stressed the dependence of the value of the underlying at given time on the θ and on the *state of the world* ω_i, $1 \le i \le n$.
In our exercise, we are considering an European call option with strike $K > 0$ written on an underlying following a Black-Scholes model. That is, $f(x) = (x-K)^+$, and Y has the following dynamics under the pricing measure Q:

$$dY_t = rY_t dt + \sigma Y_t dB_t, \quad t \ge 0, \tag{2}$$

where $B = (B_t)_{t\ge 0}$ is a Q-Brownian motion, $r \ge 0$ is the risk-free rate and $\sigma > 0$.
We consider the derivative with respect to the parameter σ, i.e., the *Vega* of the option. We know that the solution to (2) is given by

$$Y_t = y_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma B_t}, \quad t \ge 0.$$

We then have $f'(x) = \mathbf{1}_{\{x>K\}}$ and

$$\frac{\partial Y_T}{\partial \sigma} = \frac{\partial}{\partial\sigma} y_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+sB_T} = (-\sigma T + B_T)Y_T.$$

Thus, applying (3) we get

$$\frac{\partial}{\partial \sigma}\mathbb{E}^Q\left[(Y_T-K)^+|\mathcal{F}_{t_0}\right] \approx \frac{1}{n}\sum_{i=1}^{n}(-\sigma T + B_T(\omega_i))Y_T(\omega_i)\mathbf{1}_{\{Y_T(\omega_i)>K\}}.$$