Exercise 1

Consider the Acceptance-Rejection method for the generation of realizations of a random variable X which has Inhomogeneous Exponential distribution with intensity function $\lambda : [0, \infty) \to [0, \infty)$ defined by

$$\lambda(x) = \frac{1}{2} + \frac{2x}{1+x^2}, \quad x \geq 0.$$

Suppose we use an exponential distributed random variable Y as at page 251 of the script. How many uniform random numbers are required, on average, to generate one realization of X? Motivate your answer.

Exercise 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual assumptions. Introduce an $(\mathcal{F}_t)_{0 \leq t \leq T}$-Brownian motion $(W_t)_{0 \leq t \leq T}$. Consider an Itô process satisfying the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad 0 \leq t \leq T,$$

with $X_0 = x \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and let $\mathcal{T}^\Delta$ be a time-discretization with step size $\Delta > 0$, that is,

$$\mathcal{T}^\Delta = \{t_0 = 0, t_1, \dots, t_n = T\},$$

with $\Delta = t_{i+1} - t_i, i = 1, \dots, n$.
Write down the Euler-Maruyama scheme for $(X_t)_{t \in [0,T]}$ with discretization step-size Δ, i.e., the way you derive $X^\Delta_{t_{k+1}}$ from $X^\Delta_{t_k}$, where $(X^\Delta_{t_i})_{i=0,\dots,n}$ is the approximated process. Also derive analytic expressions for $\mathbb{E}[X_T]$, $Var[X_T]$, $\mathbb{E}[X^\Delta_{t_n}]$ and $Var[X^\Delta_{t_n}]$.

Exercise 3

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual assumptions. Introduce an $(\mathcal{F}_t)_{0 \leq t \leq T}$-Brownian motion $(W_t)_{0 \leq t \leq T}$ and a continuous adapted stochastic process $(X_t)_{0 \leq t \leq T}$ which solves the SDE

$$\begin{aligned} dX_t &= (a - bX_t)dt + cX_t^\alpha dW_t, \quad 0 \leq t \leq T, \\ X_0 &= x_0, \end{aligned}$$

for $a, b, c > 0$, $\alpha \geq 0$.

- Find a set for α for which SDE above admits a unique strong solution.
- Find a set for α for which the Euler-Maruyama discretisation converges strongly.
- Provide the Milstein discretisation of the SDE.