Exercise 1

Write a class `HaltonSequencePiFromHypersphere`, providing an approximation of the value of π via the approximation of the integral

$$V_d = \int_{-1}^{1} \cdots \int_{-1}^{1} \mathbf{1}_{\{x_1^2+\cdots+x_d^2 \leq 1\}} dx_1 \ldots dx_d = 2^d \int_{0}^{1} \cdots \int_{0}^{1} \mathbf{1}_{\{(2(x_1-0.5))^2+\cdots+(2(x_d-0.5))^2 \leq 1\}} dx_1 \ldots dx_d$$

and the equations

$$V_{2k} = \frac{\pi^k}{k!},$$
$$V_{2k+1} = \frac{2(4\pi)^k k!}{(2k+1)!}.$$

for $k \geq 1$ natural number, where the evaluations points $(x_1^i, \ldots, x_d^i)$, $i = 1, \ldots, n$, with n number of sample points, are now provided by an Halton sequence with a given d-dimensional base.

You can write a class `HaltonSequence`, with a method providing the sample points, or directly use the one in

`info.quantlab.numericalmethods.lecture.randomnumbers`

in the `numerical-methods-lecture` project.

The class `HaltonSequencePiFromHypersphere` must also provide a `public` method which returns the error in the approximation (note that here, for a given base, only one value of the approximation is produced, so it does not make sense to consider a vector of approximations as for the Monte-Carlo method).

Experiment on the quality of the approximation of the two methods by printing the average error produced by `MonteCarloPiFromHypersphere` of Exercise 2 of Handout 4 for 100 computations and the error given by `HaltonSequencePiFromHypersphere`, for 100000 sample points, for different dimensions.

Regarding the choice of the base of `HaltonSequencePiFromHypersphere`, consider the following cases:

- all the elements of the base are equal to each other (for example, `base =` $\{$`2,2,2,2`$\}$ for dimension 4);
- the elements of the base are different to each other, but share common divisors (for example, `base =` $\{$`2,4,6,8`$\}$ for dimension 4);
- the elements of the base are different to each other, and do not share common divisors (for example, `base =` $\{$`2,3,5,7`$\}$ for dimension 4).

What do you observe regarding the approximation error? How can you explain this behaviour?

Exercise 2

Compute (on paper) $D(A_i)$ and $D^*(A_i)$, $i = 1, 2$, for the sets:

$$A_1 = \{1/8, 1/4, 1/2, 3/4\}$$

and

$$A_2 = \{1/4, 1/2, 5/8, 3/4\}.$$

Exercise 3

Write a class with two `public` methods, returning the discrepancy and the star discrepancy, respectively, of a set $\{x_1, \ldots, x_n\}$ of one-dimensional points. These points are not supposed to be sorted when given. Test your implementation computing the discrepancy and the star discrepancy of the sets of Exercise 2, i.e.,

$$A_1 = \{1/8, 1/4, 1/2, 3/4\}$$

and

$$A_2 = \{1/4, 1/2, 5/8, 3/4\}.$$

As seen in the exercise solution, you have to get

$$D(A_1) = \frac{3}{8}, \quad D^*(A_1) = \frac{1}{4}, \quad D(A_2) = \frac{1}{2}, \quad D^*(A_2) = \frac{1}{4}.$$

Hint:

The discrepancy may be computed as

$$D(\{x_1, \ldots, x_n\}) = \max_{a \in \{0, x_1, \ldots, x_n\}} \max_{b \in \{x_1, \ldots, x_n, 1\}, b > a} \max\left(b - a - \frac{|x_i \in (a,b)|}{n}, \frac{|x_i \in [a,b]|}{n} - (b-a)\right). \quad (1)$$

One can then use representation (1) by first computing

$$\max_{b \in \{x_1, \ldots, x_n, 1\}, b > a} \max\left(b - a - \frac{|x_i \in (a,b)|}{n}, \frac{|x_i \in [a,b]|}{n} - (b-a)\right) \quad (2)$$

for $a \in \{x_1, \ldots, x_n\}$ fixed, and then computing the discrepancy as the maximum between the star discrepancy, which is

$$\max_{b \in \{x_1, \ldots, x_n, 1\}, b > a} \max\left(b - \frac{|x_i \in (0,b)|}{n}, \frac{|x_i \in [0,b]|}{n} - b\right),$$

and the maximum of the values of (2).