Cambridge Part III Maths

Lent 2020

Hydrodynamic Stability

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Contents

1 Introduction

We are typically interested in whether a given flow solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{u}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t)$ and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2}\hat{\boldsymbol{u}}^2(\boldsymbol{x}, t)\,\mathrm{d}V$$

A solution is said to be stable if

$$\lim_{t\to\infty} \frac{E(t)}{E(0)} = 0$$

for all perturbations $\hat{\boldsymbol{u}}$. Conversely, if there exists $\hat{\boldsymbol{u}}$ such that $E(t) \nrightarrow 0$ then $\boldsymbol{u}$ is unstable. The nature of $E(0)$ determines the type of perturbation:

- If $E(0) \to 0$ we have an infinitesimal disturbance
- If $E(0) < \delta$ then we probe finite amplitude disturbances
- If $E(0) \to \infty$ this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

If $\boldsymbol{U}(\boldsymbol{x})$ is a steady (basic) solution then

$$\boldsymbol{U} \cdot \nabla \boldsymbol{U} + \nabla P = \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{U}$$

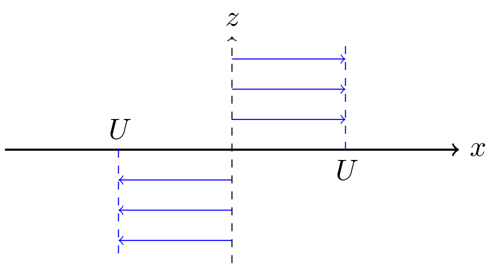
Let $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t), p = P + \hat{p}$. Then

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \hat{\boldsymbol{u}} + \cancel{\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}}} + \nabla \hat{p} = \frac{1}{\mathrm{Re}} \nabla^2 \hat{\boldsymbol{u}}$$

The term $\hat{u} \cdot \nabla \boldsymbol{U}$ is stabilising whilst the term $\nabla^2 \hat{u}/\mathrm{Re}$ is stabilising. Therefore, we expect stability as $\mathrm{Re} \to 0$ as this term dominates, and instability as $\mathrm{Re} \to \infty$. Thus there exists some value $\mathrm{Re}_{\text{crit}}$ at which instability arises. We will ask what this value is, and what is the form of initial instability/mode/pattern?

2 Kelvin-Helmholtz instability

See Drazen (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazen).



Consider a flow $\boldsymbol{u} = U(z)\hat{\boldsymbol{x}}$ where

$$U(z) = \begin{cases} U & z > 0 \\ -U & z < 0 \end{cases}$$

The linearised, *inviscid* equation for perturbation $\hat{\boldsymbol{u}}$ is

$$\begin{aligned} \frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{w} U' \hat{\boldsymbol{x}} + U \frac{\partial \hat{\boldsymbol{u}}}{\partial x} + \nabla \hat{p} &= 0 \\ \nabla \cdot \hat{\boldsymbol{u}} &= 0 \end{aligned}$$

The boundary conditions are $\hat{\boldsymbol{u}} \to 0$ as $z \to \pm\infty$, i.e. no energy radiated in from infinity. We will work in 2D $(\hat{u}, \hat{w}) = (\psi_z, -\psi_x)$ and let $\psi(x,z,t) = \phi(z)e^{i\alpha(x-ct)}$ where c is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U-c)\begin{pmatrix}\phi' \\ -i\alpha\phi\end{pmatrix} + \begin{pmatrix}-i\alpha U'\phi \\ 0\end{pmatrix} + \begin{pmatrix} i\alpha p \\ \frac{\partial p}{\partial z}\end{pmatrix} = 0$$

We can eliminate p via $\partial_z(\text{top}) - i\alpha(\text{bottom})$ to get

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

with boundary conditions $\phi \to 0$ as $z \to \pm\infty$. This is *Rayleigh's equation.* Note that c is the crucal eigenvalue. We wish to know when $c_i = \Im(c) > 0$ as a function of $U(z)$, as c_i is the growth rate:

$$\hat{u} \propto e^{i\alpha(x-ct)} = e^{i\alpha(c - c_r t - ic_i t)} = e^{i\alpha(x-c_r t) + \alpha c_i t}$$

Note the following:

- There is a symmetry $\alpha \mapsto -\alpha$, so without loss of generality we consider $\alpha > 0$.
- The complex conjugate is also a solution with $c \mapsto c^*$. Hence an unstable mode has a damped partner, so we have stability only if all modes are 'neutral' i.e. $c_i = 0$.
- There is a possible singularity at y where $U(y) = c$, called the *critical layer.* If c is real, see later.

We now solve Rayleigh's equation with $U(z)$ defined as before. We solve above and below $z = 0$ and piece the solutions together. Since $U'' = 0$, we have

$$\phi'' = \alpha^2\phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ Be^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at $z = 0$ are

1. Pressure $\hat{p}$ continuous at $z = 0$, with $\hat{p}$ given by:

$$\hat{p} = U'\phi - (U-c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z - \zeta(x,t)\right) = 0$$

   where $z = \zeta(x,t)$ is the position of the surface. After linearising, we have

$$w - \frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} = 0$$

   Inserting the form of w and U we require that

$$\zeta = -\frac{\phi}{U-c}$$

   is continuous across $z = 0$.

Requiring p continuous gives

$$-(U-c)A(-\alpha) = -(-U-c)B(\alpha)$$

Requiring ζ continuous gives

$$\frac{A}{U-c} = \frac{B}{-U-c}$$

Hence we have

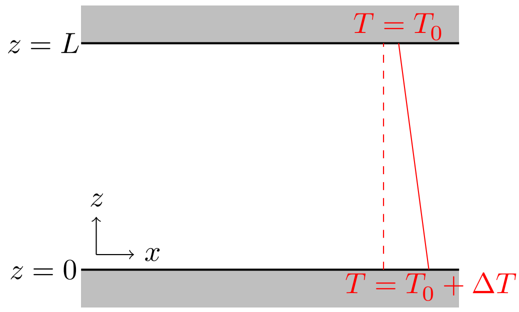
$$(U-c)^2 = -(U+c)^2$$

i.e. $c = \pm iU$ so the growth rate is αU. Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small α)
- by adding surface tension, which stabilises short wavelengths (large α), e.g. Drazen page 50 equation 3.21.

3 Thermal instabilities: convection

Consider two parallel plates separated by distance L with fluid subject to gravity and temperatue difference ΔT between the plates. The lower plate is heated.



The basic state consists of no motion, with heat transfer by conduction only.

Governing equations. The governing equations are

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \mu \nabla^2 \boldsymbol{u} + \rho g \hat{\boldsymbol{z}}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$
$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = 0$$

We need a relationship between ρ and T. Most cases of interest have ΔT and $\Delta\rho$ small, i.e. $\Delta\rho \ll \rho_0, \Delta T \ll T_0$. Two consequences of this assumption are:

1. We can Taylor expand $\rho = \rho(T)$:

$$\rho \approx \rho(T_0)\left[1 - \alpha(T - T_0)\right]$$

   where $\alpha > 0$ is the coefficient of thermal expansion, such that T increases when ρ decreases. We write $\rho_0 = \rho(T_0)$.

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term $\rho g\hat{\boldsymbol{z}}$. Importantly, we can assume the fluid is incompressible.

Define $\theta = T - T_0$. The governing equations are now

$$
\begin{aligned}
\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p &= \mu \nabla^2 \boldsymbol{u} + \rho_0 (1-\alpha\theta) g\hat{\boldsymbol{z}} \\
\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta &= \kappa\nabla^2\theta \\
\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}
$$

The basic state is $u = 0, \theta = \Delta T(1 - z/L)$ and

$$
\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_0(1 - \alpha\Delta T(1-z/L))g
$$

We now non-dimensionalise using scalings $t \sim L^2/\kappa, u \sim \kappa/L, \theta \sim \Delta T$, e.g. $\theta = \Delta T\theta^*$ where θ^* is the non-dimensionalised variable. We normalise the $\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}$ term, to get:

$$
\begin{aligned}
\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*} + \nabla^* p^* &= \frac{\mu}{\rho_0\kappa}\nabla^{*2}\boldsymbol{u}^* + \frac{\alpha g \Delta T L^3}{\kappa^2}\theta^*\hat{\boldsymbol{z}} \\
\frac{\partial \theta^*}{\partial t^*} + \boldsymbol{u}^*\cdot\nabla^*\theta^* &= \nabla^{*2}\theta^*
\end{aligned}
$$

Define the *Prandtl number*

$$
\sigma \equiv \frac{\nu}{\kappa} = \frac{\mu}{\rho_0\kappa}
$$

which is the ratio of viscous/momentum diffusion to thermal diffusion. Typical values are 0.72 in air, 7 in water, 10^5 in magma. We also define the *Rayleigh number*

$$
\mathrm{Ra} \equiv \frac{\alpha\Delta T g L^3}{\kappa\nu}
$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the * notation, we have

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \sigma\nabla^2\boldsymbol{u} + \sigma\mathrm{Ra}\theta\hat{\boldsymbol{z}} \\
\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta &= \nabla^2\theta \\
\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}
$$

Boundary conditions. There are three combinations of boundary condition available in this problem.

$\theta = 0$, $z = 1$	Fixed wall: $\boldsymbol{u} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$, $z = 0$	$\boldsymbol{u} = 0$: Fixed wall	$\boldsymbol{u} = 0$: Fixed wall	$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$: Free slip

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

Basic state. In the basic state we have conductive profile $\boldsymbol{u}_0 = 0, \theta_0 = 1 - z$ and from integration $p_0 = \sigma \mathrm{Ra}(z - \frac{1}{2}z^2)$. We generate linearised equations for perturbations $\theta = \theta_0 + \theta', \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', p = p_0 + p'$. As usual with linear stability analysis, we assume $(\theta, \boldsymbol{u}', p')$ are small.

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \cancel{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} + \nabla p' = \sigma \nabla^2 \boldsymbol{u}' + \sigma \mathrm{Ra} \theta' \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta'}{\partial t} - w' + \cancel{\boldsymbol{u}' \cdot \nabla \theta'} = \nabla^2 \theta'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

Dropping the $'$ notation for clarity we have perturbation equations

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) \boldsymbol{u} + \nabla p = \sigma \mathrm{Ra} \theta \hat{\boldsymbol{z}} \tag{1}$$
$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$
$$\left(\frac{\partial}{\partial t} - \nabla^2\right) \theta = w \tag{3}$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g. $\theta = \theta_0 + \theta' = 1$ at $z = 0$ combined with $\theta_0 = 1$ at $z = 0$ gives $\theta' = 0$. Similarly, $\theta' = 0$ at $z = 1$ and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.
From $\nabla \times$ (1) we have

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) \boldsymbol{\omega} = \sigma \mathrm{Ra} \nabla \times \theta \hat{\boldsymbol{z}}$$

Taking the curl again and using $\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$ we have

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) (-\nabla^2 \boldsymbol{u}) = \sigma \mathrm{Ra} \nabla \times (\nabla \times \theta \hat{\boldsymbol{z}}) = \sigma \mathrm{Ra} \left(\nabla \frac{\partial \theta}{\partial z} - \hat{\boldsymbol{z}} \nabla^2 \theta\right)$$

The z component is

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) (-\nabla^2 w) = \sigma \mathrm{Ra} \nabla_H^2 \theta \tag{4}$$

where $\nabla_H^2 = \partial_x^2 + \partial_y^2$. Now (3) can be used to eliminate θ by applying the operator $(\partial_t - \nabla^2)$:

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) \left(\frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 w = \sigma \mathrm{Ra} \nabla_H^2 w \tag{5}$$

This is a 6^{th} order PDE for w, hence we need three boundary conditions at each wall $z = 0, 1$. We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad \text{at } z = 0, 1$$

The second set of conditions comes from incompressibility. Taking $\partial_z(\nabla \cdot \boldsymbol{u})$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right) + \frac{\partial^2 w}{\partial z^2} = 0 \implies w_{zz} = 0$$

The third and final set of conditions comes from requiring $\theta = 0$ at $z = 0, 1$. From (4), $\nabla_H^2 \theta = 0$ implies

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) \nabla^2 w = 0$$

We now have 6 boundary conditions to supplement the PDE.

Normal mode solution. Seek a solution $w(x,y,z,t) = W(z)e^{ik_1x+ik_2y+\lambda t}$ where k_1, k_2 are wavenumbers and $\lambda \in \mathbb{C}$ is the growth rate. Write $D = \mathrm{d}/\mathrm{d}z$ and $k = \sqrt{k_1^2+k_2^2}$ since the problem is rotationally symmetric in the (x,y) plane. Substituting into (5) we have

$$(\lambda - [D^2-k^2])(\lambda - \sigma[D^2-k^2])(D^2-k^2)W = -\sigma \mathrm{Ra} k^2 W$$

with boundary conditions at $z = 0, 1$:

$$\begin{aligned} W(0) &= W(1) = 0 \\ D^2W(0) &= D^2W(1) = 0 \\ [\lambda - \sigma(D^2-k^2)]\,[D^2-k^2]\,W = 0 \implies D^4W(0) &= D^4W(1) = 0 \end{aligned}$$

The objective is to find

$$\max_k \Re\{\lambda(k; \mathrm{Ra}, \sigma)\}$$

The onset of linear instability (for a given σ) at $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{crit}}$ is defined by

$$\max_k \Re\{\lambda(k; \mathrm{Ra}_{\mathrm{crit}}, \sigma)\} = 0$$

In general, $\lambda \in \mathbb{C}$, but for this problem it can be proven that at marginality $\Im(\lambda) = 0$ as well as $\Re(\lambda) = 0$; a condition called the *principle of exchange of stabilities*. Hence setting $\lambda = 0$ in the above, we get

$$(D^2-k^2)^3W = -\mathrm{Ra}\,k^2W \tag{6}$$

Note that σ drops out of the problem! It's easy to see $W(z) = \sin(n\pi z)$ solves (6) and satisfies the free-slip BCs. Hence

$$(n^2\pi^2+k^2)^3 = \mathrm{Ra}\,k^2$$

Criticality is then given by

$$\mathrm{Ra}_{\mathrm{crit}} = \min_{n,k} \frac{(n^2\pi^2+k^2)^3}{k^2}$$

We find the minimum in the usual way:

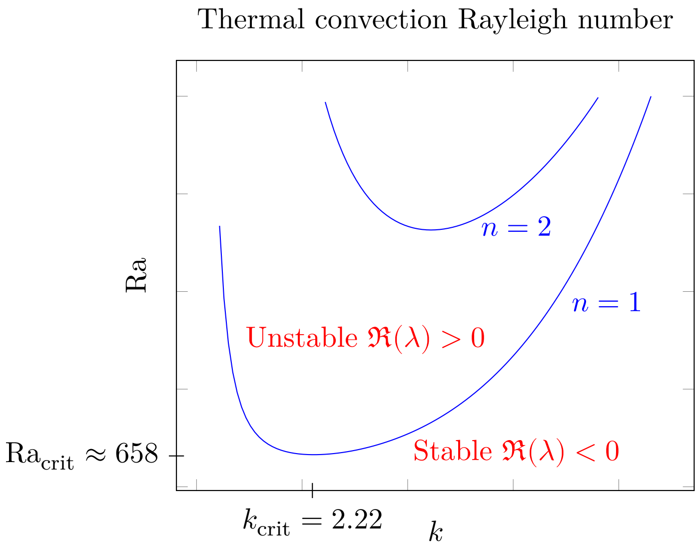
$$\begin{aligned} \frac{\partial \mathrm{Ra}}{\partial k} &= \frac{3(2k)(n^2\pi^2+k^2)^2k^2 - 2k(n^2\pi^2+k^2)^3}{k^4} \\ &= \frac{2k(n^2\pi^2+k^2)^2(3k^2-(n^2\pi^2+k^2))}{k^4} = 0 \\ \implies 2k^2 &= n^2\pi^2 \\ \implies k &= \frac{n\pi}{\sqrt{2}} \end{aligned}$$

Given $k = n\pi/\sqrt{2}$ the Rayleigh number is

$$Ra(k = \frac{n\pi}{\sqrt{2}}) = \frac{(n^2\pi^2 + \frac{1}{2}n^2\pi^2)^3}{n^2\pi^2/2} = \frac{27}{4}n^4\pi^4$$

Clearly the critical Rayleigh number is given by $n = 1$, hence

$$\begin{aligned} \mathrm{Ra}_{\mathrm{crit}} &= \frac{27}{4}\pi^4 \sim 658 \\ k_{\mathrm{crit}} &= \frac{\pi}{\sqrt{2}} \sim 2.22 \end{aligned}$$



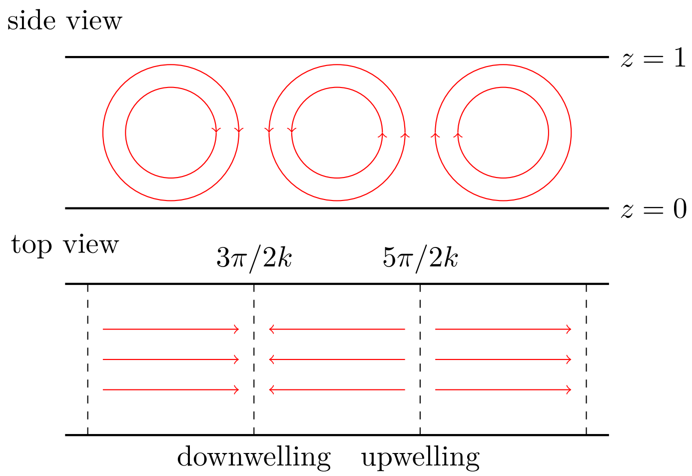
Results for other boundary conditions are:

- Free–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1101, k_c = 2.68$
- Rigid–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1708, k_c = 3.117$

Notice that at criticality only the size of k is specified, *not* its direction. Hence there are an infinite number of possibilities $\boldsymbol{k} = (k\cos\phi, k\sin\phi)$. Various different patterns seen which tesselate are as follows.

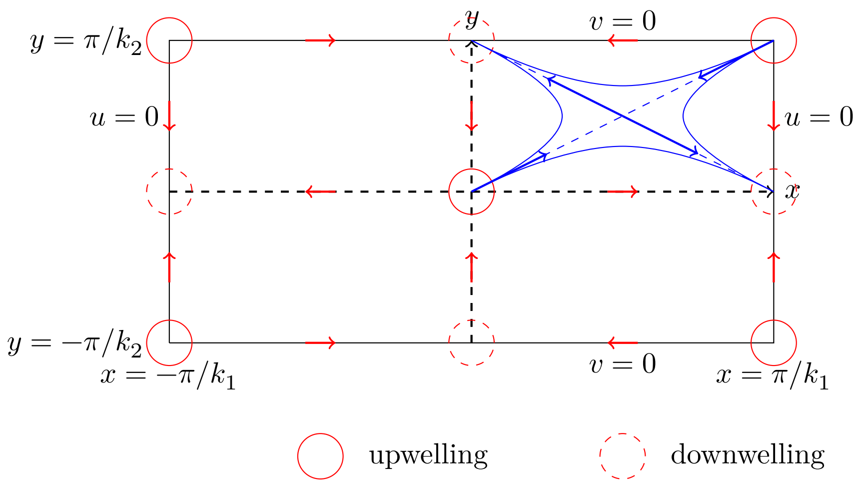
1. **2D rolls.** Orientate x-axis along k such that $k_2 = 0$. We have velocity components (w specified in problem, u follows from incompressibility)

$$
\begin{aligned}
w &= W(z)\sin kx \\
v &= 0 \\
u &= \frac{\pi\cos\pi z\cos kx}{k}
\end{aligned}
$$



2. **Rectangles.** Velocity components are

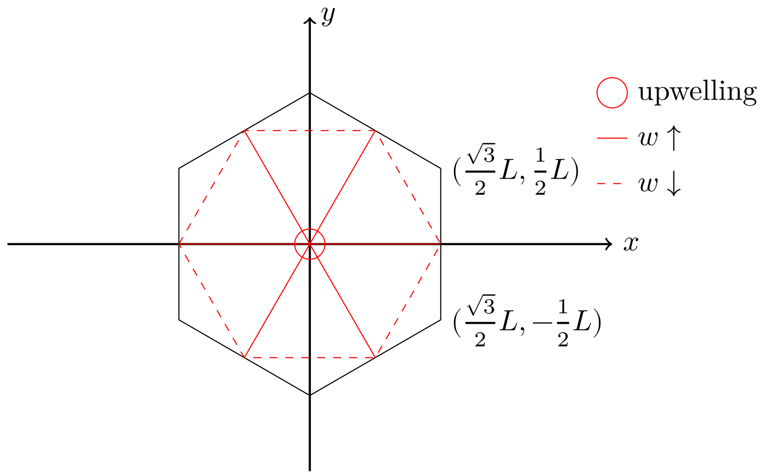
$$w = W(z)\cos k_1 x \cos k_2 y$$
$$v = -\frac{k_2}{k^2}W' \cos k_1 x \sin k_2 y$$
$$u = -\frac{k_1}{k^2}W' \sin k_1 x \cos k_2 y$$



3. **Hexagons.** Vertical velocity component

$$w = W(z)\left[\cos\frac{k}{2}(\sqrt{3}x + y) + \cos\frac{k}{2}(\sqrt{3}x - y) + \cos ky\right]$$

This is flow in a hexagon of side length $L = 4\pi/3k$.



4 Centrifugal instabilities

Flows with curved streamlines can be unstable due to centrifugal effects.

4.1 Rayleigh's criterion

We will concentrate on axisymmetric flows. Consider an azimuthal flow

$$\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}} = r\Omega(r)\hat{\boldsymbol{\theta}}$$

The inviscid, axisymmetric equations for a general flow $\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}}$ are

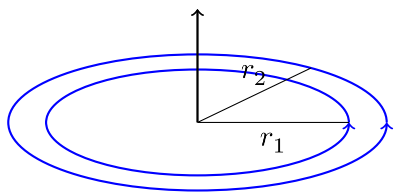
$$
\begin{aligned}
\frac{\partial u_r}{\partial t} + \boldsymbol{u}\cdot\nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + \boldsymbol{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} &= \cancel{-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}} \\
\frac{\partial u_z}{\partial t} + \boldsymbol{u}\cdot\nabla u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned}
$$

where $\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial}{\partial\theta}} + u_z\frac{\partial}{\partial z}$. Cancelled terms are absent in the axisymmetric setting. The *centrifugal* term is $-u_\theta^2/r$ in the r-momentum equation. The θ-momentum equation can be rearranged, and multiplied by r to give a material conservation equation:

$$
\begin{aligned}
\frac{\partial}{\partial r}(ru_\theta) + ru_r\frac{\partial u_\theta}{\partial r} + u - z\frac{\partial}{\partial z}(ru_\theta) + r\left(\frac{u_r u_\theta}{r}\right) &= 0 \\
\implies \frac{\partial}{\partial t}(ru_\theta) + u_r\frac{\partial}{\partial r}(ru_\theta) + u_z\frac{\partial}{\partial z}(ru_\theta) &= 0 \\
\implies \frac{\mathrm{D}}{\mathrm{D}t}(ru_\theta) &= 0
\end{aligned}
$$

This expresses conservation of angular momentum: the angular momentum per unit mass is $I = ru_\theta$, hence $\frac{\mathrm{D}I}{\mathrm{D}t} = 0$. This result also follows from Kelvin's circulation theorem, using the circulation $\Gamma = 2\pi r u_\theta$ for an inviscid fluid. The statement says that if $\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}}$ (i.e. axisymmetric azimuthal flow) then $I = I(r)$ is a basic state.

What distributions of $I(r)$ could be stable? Rayleigh's argument considers 2 rings of fluid at radius r_1 and $r_2 (> r_1)$ respectively.



The kinetic energy is

$$E = \frac{1}{2}\rho\left(\frac{I_1^2}{r_1^2} + \frac{I_2^2}{r_2^2}\right)$$

Now suppose the rings swap places due to a perturbation, but they keep their angular momentum (since it is materially conserved). The new KE is

$$E_{\text{new}} = \frac{1}{2}\left(\frac{I_2^2}{r_1^2} + \frac{I_1^2}{r_2^2}\right)$$

Hence the swap has resulted in an energy change

$$\Delta E = (I_2^2 - I_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

We can expect instability if $\Delta E < 0$. Since $r_2 > r_1$, the second factor is positive hence

$$\Delta E < 0 \iff I_2^2 < I_1^2$$

Hence Rayleigh's criterion for stability is $I_2^2 \geq I_1^2$ or equivalently

$$\frac{\mathrm{d}I^2}{\mathrm{d}r} \geq 0$$

i.e. angular momentum does not increase outwards. Note that with $I = ru_\theta = r^2\Omega$ we have the condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) \geq 0$$

for stability. This is often written using the *Rayleigh determinant*

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right)$$

Hence stability is predicted if $\Phi \geq 0$.

4.2 Derivation via linear stability analysis

Consider Taylor-Couette geometry: cylindrical walls at r_1 and r_2. Consider an inviscid base state $\boldsymbol{u} = r\Omega(r)\widehat{\boldsymbol{\theta}}$, with axisymmetric perturbations $\boldsymbol{u}'$. We have incompressibility

$$\nabla \cdot \boldsymbol{u}' = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(ru_r') + \frac{\partial u_z'}{\partial z} = 0$$

The Euler equations for this perturbation are

$$\begin{aligned}
\frac{\partial u_r'}{\partial t} - \frac{2r\Omega u_\theta'}{r} &= -\frac{1}{\rho}\frac{\partial p'}{\partial r} \\
\frac{\partial u_\theta'}{\partial t} + u_r'\frac{\mathrm{d}}{\mathrm{d}r}(r\Omega) + \frac{u_r' r\Omega}{r} &= 0 \\
\frac{\partial u_z'}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial z}
\end{aligned}$$

Now specify normal mode decomposition

$$\begin{pmatrix} u_r' \\ u_\theta' \\ u_z' \\ p' \end{pmatrix} = \begin{pmatrix} \hat{u}_r(r) \\ \hat{u}_\theta(r) \\ \hat{u}_z(r) \\ \hat{p}(r) \end{pmatrix} e^{ikz+\sigma t}$$

Only axisymmetric perturbations are considered. The Euler equations become

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r) + ik\hat{u}_z &= 0 \\
\sigma\hat{u}_r - 2\Omega\hat{u}_\theta &= -\frac{1}{\rho}\frac{\mathrm{d}\hat{p}}{\mathrm{d}r} \\
\sigma\hat{u}_\theta + \hat{u}_r(\Omega + (r\Omega)_r) &= 0 \\
\sigma\hat{u}_z &= -\frac{1}{\rho}ik\hat{p}
\end{aligned}$$

We can reduce this system down to a single equation for $\hat{u}_r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\right)\hat{u}_r - k^2\hat{u}_r - 2\frac{k^2}{\sigma^2}\Omega(2\Omega + r\Omega')\hat{u}_r = 0$$

This is a second order ODE for $\hat{u}_r$ with BCs $\hat{u}_r = 0$ at $r = r_1, r_2$. For this flow, Rayleigh's determinant is

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = 4\Omega^2 + 2r\Omega'\Omega$$

Hence the ODE for $\hat{u}_r$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right) - k^2\hat{u}_r = \frac{k^2}{\sigma^2}\Phi(r)\hat{u}_r \tag{7}$$

Multiply (7) by $r\hat{u}_r^*$ (complex conjugate) and integrate from r_1 to r_2:

$$\int_{r_1}^{r_2} r\hat{u}_r^*\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term may be integrated by parts to give:

$$\cancel{\left[r\hat{u}_r^*\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right]_{r_1}^{r_2}} - \int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r^*)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term vanishes since $\hat{u}_r = 0$ at $r = r_1, r_2$. Hence

$$-\int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r^*)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

Labelling the first integral as $H_1 > 0$ and the second as $H_2 > 0$, we have

$$\frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r = -H_1 - k^2H_2 < 0$$

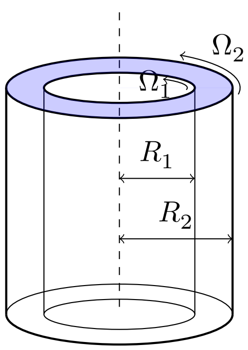
If $\Phi \geq 0$ then $\sigma^2 < 0$, i.e. σ is imaginary and we have stability. If instead $\Phi < 0$ somewhere in the domain, then potentially

$$\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r < 0$$

in which case $\sigma^2 > 0$ and we have instability. Hence $\Phi < 0$ somewhere in the domain is *necessary* (but not sufficient) condition for instability. So this formal analysis confirms Rayleigh's heuristic criterion. Note, really we need to consider non-axisymmetric perturbations too.

4.3 Taylor vortices

Apply Rayleigh's criterion to Taylor-Couette flow.



When viscosity is present, the only solution with $\partial_\theta = \partial_z = 0$ is

$$u_\theta(r) = Ar + \frac{B}{r}$$

where

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}}$$

Note this solves $(\nabla^2 - 1/r^2)u_\theta = 0$ where $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r)$. In this case $\Omega = u_\theta/r = A + B/r^2$ hence Rayleigh's determinant is

$$\Phi = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4\left(A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}\right)\right] = 4A^2\left(1 + \frac{B}{Ar^2}\right)$$

For convenience we define $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2 < 1$. Then

$$\Phi = 4A^2\left[1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2}\right]$$

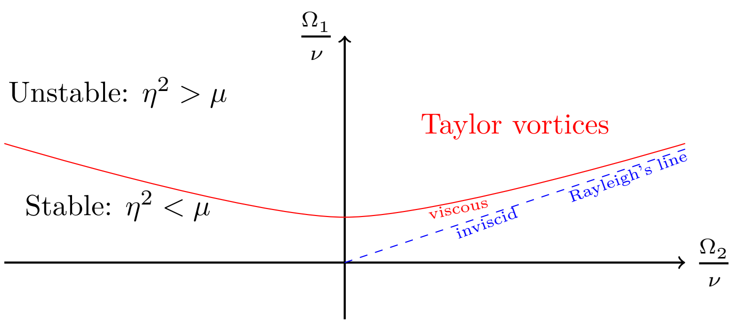
For stability, i.e. $\Phi \geq 0$ everywhere, we require for all $r \in [R_1, R_2]$

$$1 \geq \frac{(1-\mu)R_1^2}{(\eta^2-\mu)r^2} \geq \frac{1-\mu}{\eta^2-\mu}$$

where the last inequality follows since $R_1^2/r^2 \geq 1$ for all $r \in [R_1, R_2]$. There are now two cases:

- If $\eta^2 > \mu$ then
$$\eta^2 - \mu \geq 1 - \mu \implies \eta^2 \geq 1$$
This is a contradiction since $\eta < 1$.
- Otherwise $\eta^2 < \mu$, so
$$\eta^2 - \mu \leq 1 - \mu \implies \eta^2 \leq 1$$

Thus Rayleigh's criterion is $\eta^2 < \mu$ for stability.



For a fixed geometry (i.e. fixed η) we can plot a stability diagram, with Rayleigh's line $\eta^2 = \mu = \Omega_2/\Omega_1$ marking the stability heuristic. In Taylor-Couette geometry, the instability manifests itself as *Taylor vortices*.

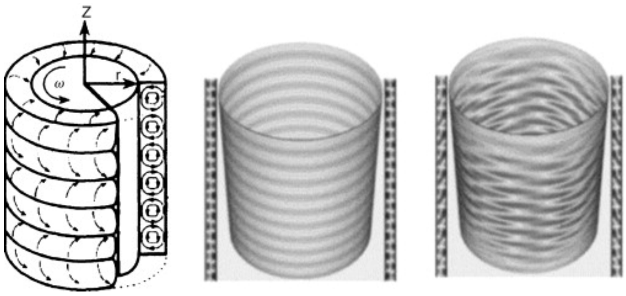
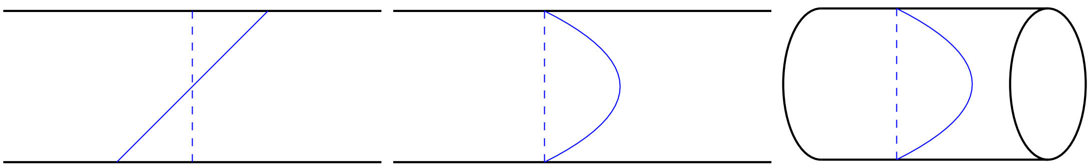


Figure 1: Taylor vortices, from Dutta and Ray, 2004.

5 Parallel shear flows

For some flows, inviscid analysis gives a good approximation to the stability properties of a viscous fluid (e.g. Kelvin-Helmholtz, Taylor-Couette flow) but for others, it does not (e.g. plane Couette flow, channel flow, pipe flow). Here, viscosity can be *de*stabilising.



5.1 Inviscid analysis

Consider a parallel shear flow $U(z)\hat{\boldsymbol{x}}$. The non-dimensionalised Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with boundary conditions $\boldsymbol{u} \cdot \hat{\boldsymbol{z}} = 0$ at $z = z_1, z_2$. The basic flow is $U(z)\hat{\boldsymbol{x}}$ with p constant – any form is valid. Add small perturbations

$$\boldsymbol{u} = U(z)\hat{\boldsymbol{x}} + \boldsymbol{u}', \quad p = P + p'$$

The Euler equations become

$$\frac{\partial \boldsymbol{u}'}{\partial t} + U\frac{\partial \boldsymbol{u}'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z}\hat{\boldsymbol{x}} = -\nabla p'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

with boundary conditions $w' = 0$ at $z = z_1, z_2$. All equations have coefficients independent of x, y, t so we can separate the variables by taking normal modes of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \hat{\boldsymbol{u}}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$
$$p'(\boldsymbol{x}, t) = \hat{p}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$

Note we have replaced the usual σ with $-i\alpha c$. It is understood that the physical fluid perturbation velocity $\boldsymbol{u}'$ is represented by the real part, e.g.

$$w' = [\Re(\hat{w})\cos(\alpha x + \beta y - \alpha c_r t) - \Im(\hat{w})\sin(\alpha x + \beta y - \alpha c_r t)]\, e^{\alpha c_i t}$$

This mode is a wave travelling with phase speed $\alpha c_r/\sqrt{\alpha^2+\beta^2}$ in the $(\alpha,\beta,0)$ direction. While it decays like $e^{\alpha c_i t}$, or growth if $c_i > 0$. The equations are now

$$i\alpha(U-c)\hat{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\alpha\hat{p} = 0 \tag{8}$$

$$i\alpha(U-c)\hat{v} + i\beta\hat{p} = 0 \tag{9}$$

$$i\alpha(U-c)\hat{w} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}z} = 0 \tag{10}$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0 \tag{11}$$

with boundary conditions $\hat{w} = 0$ at $z = z_1, z_2$. This is an eigenvalue problem in $c \in \mathbb{C}$. Instability corresponds to $c_i > 0$ and $c_i \leq 0$ for stability.

5.1.1 Squire's transformation (Squire, 1933)

Before attempting to solve (8)–(11), we consider the Squire transformation. Define the transformed variables

$$\tilde{\alpha} = \sqrt{\alpha^2+\beta^2}, \quad \tilde{u} = \frac{\alpha\hat{u} + \beta\hat{v}}{\tilde{\alpha}}, \quad \tilde{p} = \frac{\tilde{\alpha}\hat{p}}{\alpha}$$

Construct $(\alpha(8) + \beta(9))/\alpha$:

$$i\tilde{\alpha}(U-c)\tilde{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\tilde{\alpha}\tilde{p} = 0 \tag{12}$$

Similarly $\tilde{\alpha}(10)/\alpha$:

$$i\tilde{\alpha}(U-c)\hat{w} + \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = 0 \tag{13}$$

Incompressibility is now expressed as

$$i\tilde{\alpha}\tilde{u} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0$$

The transformedd system has the same form as (8)–(11) with $\beta = \hat{v} = 0$ and $\alpha \to \tilde{\alpha}, \hat{u} \to \tilde{u}, \hat{p} \to \tilde{p}$ but c unchanged. Thus the eigenvalue c depends on $\sqrt{\alpha^2+\beta^2}$ but the growth rate is αc_i. So the largest growth rate αc_i is given by $\beta = 0$ for all wavenumber pairs (α,β) with $\sqrt{\alpha^2+\beta^2}$ constant. Hence it is sufficient to consider $\beta = 0$ disturbances only. To any unstable 3D mode $\alpha \neq 0, \beta \neq 0$ there corresponds a more unstable 2D mode with $\beta = 0$.

5.1.2 Rayleigh's equation

Work in 2D (Squires). Use streamfunction ψ' such that

$$u' = \psi'_z, \;\; v' = 0, \;\; w' = -\psi'_x$$

Further, let $\psi'(x,z,t) = \phi(z)e^{i\alpha(x-ct)}$ so that it is now clear that c_r is the phase speed in the x direction. Now $\hat{u} = \frac{\mathrm{d}\phi}{\mathrm{d}z}$ and $\hat{w} = -i\alpha\phi$ (notice the phase difference). Then (12) becomes

$$i\alpha(U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}(-i\alpha\phi) + i\alpha\hat{p} = 0$$

$$\implies \hat{p} = \frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}$$

Substituting into (13) gives

$$
\begin{aligned}
i\alpha(U-c)(-i\alpha\phi) + \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}\right] &= 0 \\
\implies (U-c)(\phi'' - \alpha^2\phi) - U''\phi &= 0
\end{aligned}
$$

with boundary conditions $\phi = 0$ at $z = z_1, z_2$. This is Rayleigh's equation (1880).

Comments.

- Rayleigh's equation involves α^2 only so need only consider $\alpha > 0$.
- If (ϕ, c) solves the problem then so does (ϕ^*, c^*). So if there exists a growing mode, there also exists a corresponding decaying mode. Hence stability means $c \in \mathbb{R}$ for all α.
- A singularity exists at $U(z_c) = c$ – this is called a critical layer and only occurs when $c \in \mathbb{R}$. Critical layers are important in solving IVPs and relating Rayleigh's equation to its viscous analogue, the Orr-Sommerfield equation (see later).
- There are two types of eigensolution:
  - Continuous spectrum $c \in [\min U, \max U]$ and ϕ has a discontinuous derivative at z_c. This type of solution is never unstable.
  - Discrete spectrum of complex conjugate pairs. This solution can be unstable.

5.1.3 Properties of Rayleigh's equation.

Inflection point criterion. Suppose $c_i > 0$, i.e. consider an unstable mode. Integrate $\phi^* \times$ Rayleigh's equation from z_1 to z_2:

$$
\int_{z_1}^{z_2} \left[\phi^*\phi'' - \alpha^2|\phi|^2 - \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0
$$

Integrate the first term by parts and note $\phi = \phi^* = 0$ at z_1 and z_2. Hence

$$
\int_{z_1}^{z_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0
$$

Take imaginary part:

$$
\begin{aligned}
\Im\left[\int_{z_1}^{z_2} \frac{U''(U-c^*)}{|U-c|^2}|\phi|^2 \mathrm{d}z\right] &= 0 \\
\implies -c_i \int_{z_1}^{z_2} \frac{U''}{|U-c|^2}|\phi|^2 \mathrm{d}z &= 0
\end{aligned}
$$

But $c_i > 0$ so we must have

$$
\int_{z_1}^{z_2} \frac{U''}{|U-c|^2}|\phi|^2 \mathrm{d}z = 0
$$

Now $|U-c|^2 > 0$ and $|\phi|^2 > 0$ so U'' must change sign somewhere in $[z_1, z_2]$. Thus $U'' = 0$at least once is a necessary condition for inviscid instability, called the *inflection point criterion.*