Cambridge Part III Maths

Lent 2020

Hydrodynamic Stability

based on a course given by
Prof. Richard Kerswell

written up by
Charles Powell

Notes created based on Josh Kirklin's LaTeX packages & classes. Please do not distribute these notes other than to fellow Part III students. Please send errors and suggestions to cwp29@cam.ac.uk.

Contents

1 Introduction

We are typically interested in whether a given flow solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{u}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t)$ and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2}\hat{\boldsymbol{u}}^2(\boldsymbol{x}, t)\,\mathrm{d}V$$

A solution is said to be stable if

$$\lim_{t\to\infty}\frac{E(t)}{E(0)}=0$$

for all perturbations $\hat{\boldsymbol{u}}$. Conversely, if there exists $\hat{\boldsymbol{u}}$ such that $E(t)\not\to 0$ then $\boldsymbol{u}$ is unstable. The nature of $E(0)$ determines the type of perturbation:

- If $E(0)\to 0$ we have an infinitesimal disturbance
- If $E(0)<\delta$ then we probe finite amplitude disturbances
- If $E(0)\to\infty$ this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}$$

If $\boldsymbol{U}(\boldsymbol{x})$ is a steady (basic) solution then

$$\boldsymbol{U}\cdot\nabla\boldsymbol{U}+\nabla P=\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{U}$$

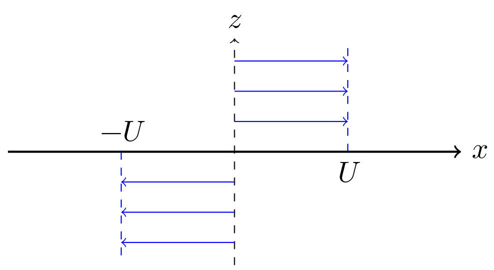
Let $\boldsymbol{u}=\boldsymbol{U}(\boldsymbol{x})+\hat{\boldsymbol{u}}(\boldsymbol{x},t), p=P+\hat{p}$. Then

$$\frac{\partial\hat{\boldsymbol{u}}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{U}+\boldsymbol{U}\cdot\nabla\hat{\boldsymbol{u}}+\cancel{\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}}+\nabla\hat{p}=\frac{1}{\mathrm{Re}}\nabla^2\hat{\boldsymbol{u}}$$

The term $\hat{u}\cdot\nabla\boldsymbol{U}$ is stabilising whilst the term $\nabla^2\hat{u}/\mathrm{Re}$ is destabilising. Therefore, we expect stability as $\mathrm{Re}\to 0$ the stabilising term dominates, and instability as $\mathrm{Re}\to\infty$ when the destablising term dominates. Thus there exists some value $\mathrm{Re}_{\mathrm{crit}}$ at which instability arises. We will ask what this value is, and what is the form of the initial instability/mode/pattern?

2 Kelvin-Helmholtz instability

See Drazin (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazin).



Consider a flow $\boldsymbol{u}=U(z)\hat{\boldsymbol{x}}$ where

$$U(z)=\begin{cases}U & z>0\\ -U & z<0\end{cases}$$

The linearised, *inviscid* equation for perturbation $\hat{\boldsymbol{u}}$ is

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{w} U' \hat{\boldsymbol{x}} + U \frac{\partial \hat{\boldsymbol{u}}}{\partial x} + \nabla \hat{p} = 0$$
$$\nabla \cdot \hat{\boldsymbol{u}} = 0$$

The boundary conditions are $\hat{\boldsymbol{u}} \to 0$ as $z \to \pm\infty$, i.e. no energy is radiated in from infinity. We will work in 2D with velocity components $(\hat{u}, \hat{w}) = (\psi_z, -\psi_x)$ and let $\psi(x,z,t) = \phi(z) e^{i\alpha(x-ct)}$ where c is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U-c)\begin{pmatrix}\phi' \\ -i\alpha\phi\end{pmatrix} + \begin{pmatrix}-i\alpha U'\phi \\ 0\end{pmatrix} + \begin{pmatrix} i\alpha p \\ \frac{\partial p}{\partial z}\end{pmatrix} = 0$$

We can eliminate p via $\partial_z(\text{top}) - i\alpha(\text{bottom})$ to get

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

with boundary conditions $\phi \to 0$ as $z \to \pm\infty$. This is *Rayleigh's equation.* Note that c is the crucial eigenvalue. We wish to know when $c_i = \Im(c) > 0$ as a function of $U(z)$, as c_i is the growth rate:

$$\hat{u} \propto e^{i\alpha(x-ct)} = e^{i\alpha(c - c_r t - i c_i t)} = e^{i\alpha(x - c_r t) + \alpha c_i t}$$

Note the following:

- There is a symmetry $\alpha \mapsto -\alpha$, so without loss of generality we consider $\alpha > 0$.
- The complex conjugate is also a solution with $c \mapsto c^*$. Hence an unstable mode has a damped partner, so we have stability only if all modes are 'neutral' i.e. $c_i = 0$.
- There is a possible singularity at y where $U(y) = c$, called the *critical layer.* If c is real, see later.

We now solve Rayleigh's equation with $U(z)$ defined as before. We solve above and below $z = 0$ and piece the solutions together. Since $U'' = 0$, we have

$$\phi'' = \alpha^2\phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ Be^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at $z = 0$ are

1. Pressure $\hat{p}$ continuous at $z = 0$, with $\hat{p}$ given by:

$$\hat{p} = U'\phi - (U-c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - \zeta(x,t)) = 0$$

where $z = \zeta(x,t)$ is the position of the surface. After linearising, we have

$$w - \frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} = 0$$

Inserting the form of w and U we require that

$$\zeta = -\frac{\phi}{U-c}$$

is continuous across $z = 0$.

Requiring p continuous gives

$$-(U-c)A(-\alpha) = -(-U-c)B(\alpha)$$

Requiring ζ continuous gives

$$\frac{A}{U-c} = \frac{B}{-U-c}$$

Hence we have

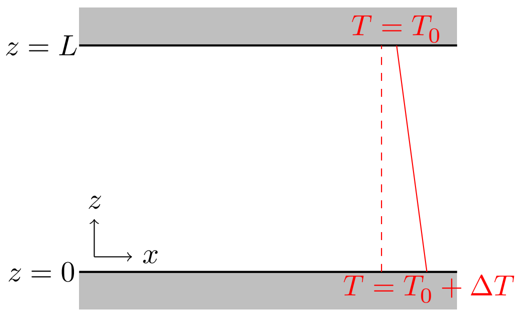
$$(U-c)^2 = -(U+c)^2$$

i.e. $c = \pm iU$ so the growth rate is αU. Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small α)
- by adding surface tension, which stabilises short wavelengths (large α), e.g. Drazin page 50 equation 3.21.

3 Thermal instabilities: Rayleigh-Bernard convection

Consider two parallel plates separated by distance L with fluid subject to gravity and temperatue difference ΔT between the plates. The lower plate is heated to $T_0 + \Delta T$ whilst the upper plate is fixed at temperature T_0.



The basic state consists of no motion, with heat transfer by conduction only.

Governing equations. The governing equations are those of momentum, mass, and (thermal) energy conservation.

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \mu\nabla^2\boldsymbol{u} + \rho g\hat{\boldsymbol{z}}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \kappa\nabla^2 T$$
$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{u} = 0$$

To close the set of equations we need a relationship between ρ and T. Most cases of interest have ΔT and $\Delta\rho$ small, i.e. $\Delta\rho \ll \rho_0, \Delta T \ll T_0$. Two consequences of this assumption are:

1. We can Taylor expand $\rho = \rho(T)$:

$$\rho \approx \rho(T_0)\left[1 - \alpha(T - T_0)\right]$$

   where $\alpha > 0$ is the coefficient of thermal expansion, such that T increases when ρ decreases. We write $\rho_0 = \rho(T_0)$.

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term $\rho g\hat{\boldsymbol{z}}$. Importantly, we can assume the fluid is incompressible.

Define $\theta = T - T_0$. The governing equations are now

$$\begin{aligned}
\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p &= \mu\nabla^2\boldsymbol{u} + \rho_0(1 - \alpha\theta)g\hat{\boldsymbol{z}} \\
\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta &= \kappa\nabla^2\theta \\
\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}$$

The basic state is $u = 0, \theta = \Delta T(1 - z/L)$ and

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_0(1 - \alpha\Delta T(1 - z/L))g$$

We now non-dimensionalise using scalings $t \sim L^2/\kappa, u \sim \kappa/L, \theta \sim \Delta T$, e.g. $\theta = \Delta T\theta^*$ where θ^* is the non-dimensionalised variable. We normalise the $\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}$ term, to get:

$$\begin{aligned}
\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*} + \nabla^* p^* &= \frac{\mu}{\rho_0\kappa}\nabla^{*2}\boldsymbol{u}^* + \frac{\alpha g\Delta T L^3}{\kappa^2}\theta^*\hat{\boldsymbol{z}} \\
\frac{\partial\theta^*}{\partial t^*} + \boldsymbol{u}^*\cdot\nabla^*\theta^* &= \nabla^{*2}\theta^*
\end{aligned}$$

Define the *Prandtl number*

$$\sigma \equiv \frac{\nu}{\kappa} = \frac{\mu}{\rho_0\kappa}$$

which is the ratio of viscous/momentum diffusion to thermal diffusion. Typical values are 0.72 in air, 7 in water, 10^5 in magma. We also define the *Rayleigh number*

$$\mathrm{Ra} \equiv \frac{\alpha\Delta T g L^3}{\kappa\nu}$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the * notation, we have

$$\begin{aligned}
\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \sigma\nabla^2\boldsymbol{u} + \sigma\mathrm{Ra}\theta\hat{\boldsymbol{z}} \\
\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta &= \nabla^2\theta \\
\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}$$

Boundary conditions. There are three combinations of boundary condition available in this problem, with the choice fixed wall (no slip) or stress free (free slip).

$\theta = 0$, $z = 1$	Fixed wall $\boldsymbol{u} = 0$	Free slip $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	Free slip $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$, $z = 0$	$\boldsymbol{u} = 0$ Fixed wall	$\boldsymbol{u} = 0$ Fixed wall	$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ Free slip

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

Basic state. In the basic state we have conductive profile $\boldsymbol{u}_0 = 0, \theta_0 = 1 - z$ and from integration $p_0 = \sigma\mathrm{Ra}(z - \frac{1}{2}z^2)$. We generate linearised equations for perturbations $\theta = \theta_0 + \theta', \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', p = p_0 + p'$. As usual with linear stability analysis, we assume $(\theta', \boldsymbol{u}', p')$ are small.

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}'}{\partial t} + \cancel{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} + \nabla p' &= \sigma \nabla^2 \boldsymbol{u}' + \sigma \mathrm{Ra} \theta' \hat{\boldsymbol{z}} \\
\frac{\partial \theta'}{\partial t} - w' + \cancel{\boldsymbol{u}' \cdot \nabla \theta'} &= \nabla^2 \theta' \\
\nabla \cdot \boldsymbol{u}' &= 0
\end{aligned}
$$

Dropping the $'$ notation for clarity we have perturbation equations

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{u} + \nabla p = \sigma\mathrm{Ra}\theta\hat{\boldsymbol{z}} \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = w \tag{3}$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g. $\theta = \theta_0 + \theta' = 1$ at $z = 0$ combined with $\theta_0 = 1$ at $z = 0$ gives $\theta' = 0$. Similarly, $\theta' = 0$ at $z = 1$ and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.
From $\nabla \times$ (1) we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{\omega} = \sigma\mathrm{Ra}\nabla \times \theta\hat{\boldsymbol{z}}$$

Taking the curl again and using $\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2\boldsymbol{u}$ we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2\boldsymbol{u}) = \sigma\mathrm{Ra}\nabla \times (\nabla \times \theta\hat{\boldsymbol{z}}) = \sigma\mathrm{Ra}\left(\nabla\frac{\partial\theta}{\partial z} - \hat{\boldsymbol{z}}\nabla^2\theta\right)$$

The z component is

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2 w) = \sigma\mathrm{Ra}\nabla_H^2\theta \tag{4}$$

where $\nabla_H^2 = \partial_x^2 + \partial_y^2$. Now (3) can be used to eliminate θ by applying the operator $(\partial_t - \nabla^2)$:

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \sigma\mathrm{Ra}\nabla_H^2 w \tag{5}$$

This is a 6^{th} order PDE for w, hence we need three boundary conditions at each wall $z = 0, 1$. We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad \text{at } \; z = 0, 1$$

The second set of conditions comes from incompressibility. Taking $\partial_z(\nabla \cdot \boldsymbol{u})$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right) + \frac{\partial^2 w}{\partial z^2} = 0 \implies w_{zz} = 0$$

The third and final set of conditions comes from requiring $\theta = 0$ at $z = 0, 1$. From (4), $\nabla_H^2 \theta = 0$ implies

$$\left(\frac{\partial}{\partial t} - \sigma \nabla^2\right) \nabla^2 w = 0$$

We now have 6 boundary conditions to supplement the PDE.

Normal mode solution. Seek a solution $w(x, y, z, t) = W(z)e^{ik_1 x + ik_2 y + \lambda t}$ where k_1, k_2 are wavenumbers and $\lambda \in \mathbb{C}$ is the growth rate. Write $D = \mathrm{d}/\mathrm{d}z$ and $k = \sqrt{k_1^2 + k_2^2}$ since the problem is rotationally symmetric in the (x, y) plane. Substituting into (5) we have

$$(\lambda - [D^2 - k^2])(\lambda - \sigma\,[D^2 - k^2])(D^2 - k^2)W = -\sigma \mathrm{Ra} k^2 W$$

with boundary conditions at $z = 0, 1$:

$$\begin{aligned} W(0) &= W(1) = 0 \\ D^2 W(0) &= D^2 W(1) = 0 \\ \left[\lambda - \sigma(D^2 - k^2)\right]\left[D^2 - k^2\right] W = 0 \implies D^4 W(0) &= D^4 W(1) = 0 \end{aligned}$$

The objective is to find

$$\max_k \Re\{\lambda(k; \mathrm{Ra}, \sigma)\}$$

The onset of linear instability (for a given σ) at $\mathrm{Ra} = \mathrm{Ra}_{\text{crit}}$ is defined by

$$\max_k \Re\{\lambda(k; \mathrm{Ra}_{\text{crit}}, \sigma)\} = 0$$

In general, $\lambda \in \mathbb{C}$, but for this problem it can be proven that at marginality $\Im(\lambda) = 0$ as well as $\Re(\lambda) = 0$; a condition called the *principle of exchange of stabilities*. Hence setting $\lambda = 0$ in the above, we get

$$(D^2 - k^2)^3 W = -\mathrm{Ra}\, k^2 W \tag{6}$$

Note that σ drops out of the problem! It's easy to see $W(z) = \sin(n\pi z)$ solves (6) and satisfies the free-slip BCs. Hence

$$(n^2\pi^2 + k^2)^3 = \mathrm{Ra}\, k^2$$

Criticality is then given by

$$\mathrm{Ra}_{\text{crit}} = \min_{n,k} \frac{(n^2\pi^2 + k^2)^3}{k^2}$$

We find the minimum in the usual way:

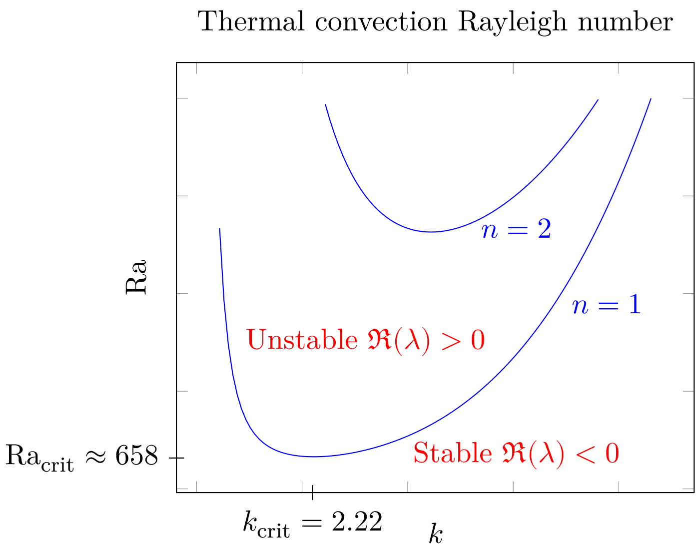
$$\begin{aligned}
\frac{\partial \mathrm{Ra}}{\partial k} &= \frac{3(2k)(n^2\pi^2+k^2)^2k^2 - 2k(n^2\pi^2+k^2)^3}{k^4} \\
&= \frac{2k(n^2\pi^2+k^2)^2(3k^2-(n^2\pi^2+k^2))}{k^4} = 0 \\
\implies 2k^2 &= n^2\pi^2 \\
\implies k &= \frac{n\pi}{\sqrt{2}}
\end{aligned}$$

Given $k = n\pi/\sqrt{2}$ the Rayleigh number is

$$\mathrm{Ra}(k = \frac{n\pi}{\sqrt{2}}) = \frac{(n^2\pi^2 + \frac{1}{2}n^2\pi^2)^3}{n^2\pi^2/2} = \frac{27}{4}n^4\pi^4$$

Clearly the critical Rayleigh number is given by $n = 1$, hence

$$\begin{aligned}
\mathrm{Ra}_{\mathrm{crit}} &= \frac{27}{4}\pi^4 \sim 658 \\
k_{\mathrm{crit}} &= \frac{\pi}{\sqrt{2}} \sim 2.22
\end{aligned}$$



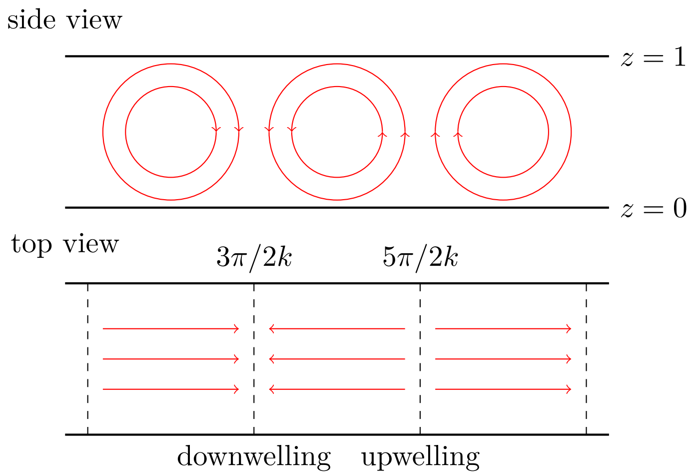
Results for other boundary conditions are:

- Free–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1101, k_c = 2.68$
- Rigid–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1708, k_c = 3.117$

Notice that at criticality only the size of k is specified, *not* its direction. Hence there are an infinite number of possibilities $\boldsymbol{k} = (k\cos\phi, k\sin\phi)$. Various different patterns which tesselate are as follows.

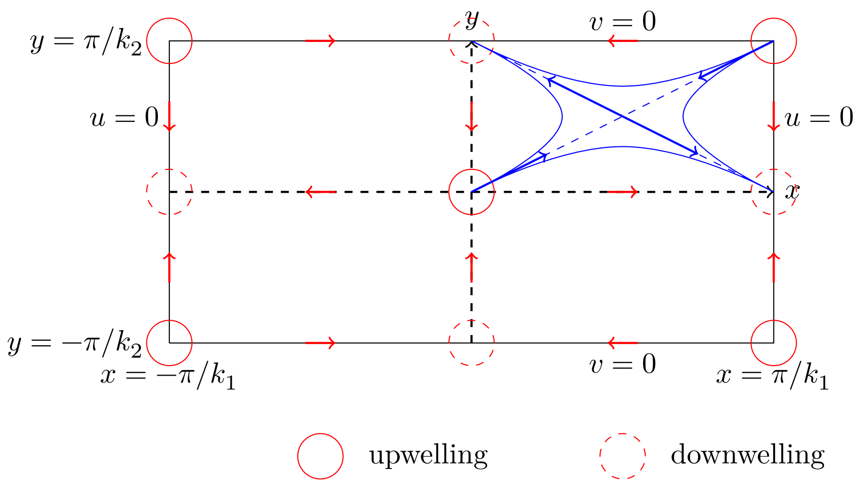
1. **2D rolls.** Orientate x-axis along k such that $k_2 = 0$. We have velocity components (w specified in problem, u follows from incompressibility)

$$\begin{aligned}
w &= W(z)\sin kx \\
v &= 0 \\
u &= \frac{\pi\cos\pi z\cos kx}{k}
\end{aligned}$$



2. **Rectangles.** Velocity components are

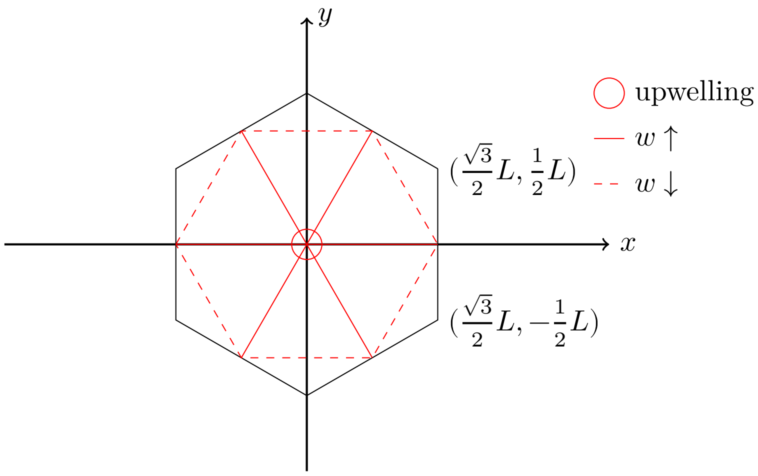
$$w = W(z)\cos k_1 x \cos k_2 y$$
$$v = -\frac{k_2}{k^2} W' \cos k_1 x \sin k_2 y$$
$$u = -\frac{k_1}{k^2} W' \sin k_1 x \cos k_2 y$$



3. **Hexagons.** Vertical velocity component

$$w = W(z)\left[\cos\frac{k}{2}(\sqrt{3}x + y) + \cos\frac{k}{2}(\sqrt{3}x - y) + \cos ky\right]$$

This is flow in a hexagon of side length $L = 4\pi/3k$.



4 Centrifugal instabilities

Flows with curved streamlines can be unstable due to centrifugal effects.

4.1 Rayleigh's criterion

We will concentrate on axisymmetric flows. Consider an azimuthal flow

$$\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}} = r\Omega(r)\hat{\boldsymbol{\theta}}$$

The inviscid, axisymmetric equations for a general flow $\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}}$ are

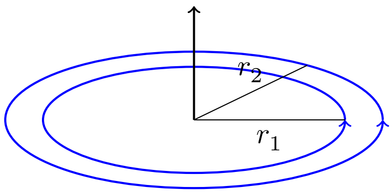
$$\begin{aligned}
\frac{\partial u_r}{\partial t} + \boldsymbol{u}\cdot\nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}\\
\frac{\partial u_\theta}{\partial t} + \boldsymbol{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} &= \cancel{-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}}\\
\frac{\partial u_z}{\partial t} + \boldsymbol{u}\cdot\nabla u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\\
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned}$$

where $\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial}{\partial\theta}} + u_z\frac{\partial}{\partial z}$. Cancelled terms are absent in the axisymmetric setting. The *centrifugal* term is $-u_\theta^2/r$ in the r-momentum equation. The θ-momentum equation can be rearranged, and multiplied by r to give a material conservation equation:

$$\begin{aligned}
\frac{\partial}{\partial r}(ru_\theta) + ru_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial}{\partial z}(ru_\theta) + r\left(\frac{u_r u_\theta}{r}\right) &= 0\\
\implies \frac{\partial}{\partial t}(ru_\theta) + u_r\frac{\partial}{\partial r}(ru_\theta) + u_z\frac{\partial}{\partial z}(ru_\theta) &= 0\\
\implies \frac{\mathrm{D}}{\mathrm{D}t}(ru_\theta) &= 0
\end{aligned}$$

This expresses conservation of angular momentum: the angular momentum per unit mass is $I = ru_\theta$, hence $\frac{\mathrm{D}I}{\mathrm{D}t} = 0$. This result also follows from Kelvin's circulation theorem, using the circulation $\Gamma = 2\pi r u_\theta$ for an inviscid fluid. The statement says that if $\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}}$ (i.e. axisymmetric azimuthal flow) then $I = I(r)$ is a basic state.

What distributions of $I(r)$ could be stable? Rayleigh's argument considers 2 rings of fluid at radius r_1 and $r_2 (> r_1)$ respectively.



The kinetic energy is

$$E = \frac{1}{2}\rho\left(\frac{I_1^2}{r_1^2} + \frac{I_2^2}{r_2^2}\right)$$

Now suppose the rings swap places due to a perturbation, but they keep their angular momentum (since it is materially conserved). The new KE is

$$E_{\text{new}} = \frac{1}{2}\left(\frac{I_2^2}{r_1^2} + \frac{I_1^2}{r_2^2}\right)$$

Hence the swap has resulted in an energy change

$$\Delta E = (I_2^2 - I_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

We can expect instability if $\Delta E < 0$. Since $r_2 > r_1$, the second factor is positive hence

$$\Delta E < 0 \iff I_2^2 < I_1^2$$

Hence Rayleigh's criterion for stability is $I_2^2 \geq I_1^2$ or equivalently

$$\frac{\mathrm{d}I^2}{\mathrm{d}r} \geq 0$$

i.e. angular momentum does not increase outwards. Note that with $I = ru_\theta = r^2\Omega$ we have the condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) \geq 0$$

for stability. This is often written using the *Rayleigh determinant*

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right)$$

Hence stability is predicted if $\Phi \geq 0$.

4.2 Derivation via linear stability analysis

Consider Taylor-Couette geometry: cylindrical walls at r_1 and r_2 with an inviscid base state $\boldsymbol{u} = r\Omega(r)\hat{\boldsymbol{\theta}}$, with axisymmetric perturbations $\boldsymbol{u}'$. We have incompressibility

$$\nabla \cdot \boldsymbol{u}' = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(ru_r') + \frac{\partial u_z'}{\partial z} = 0$$

The Euler equations for this perturbation are

$$\begin{aligned}
\frac{\partial u_r'}{\partial t} - \frac{2r\Omega u_\theta'}{r} &= -\frac{1}{\rho}\frac{\partial p'}{\partial r} \\
\frac{\partial u_\theta'}{\partial t} + u_r'\frac{\mathrm{d}}{\mathrm{d}r}(r\Omega) + \frac{u_r' r\Omega}{r} &= 0 \\
\frac{\partial u_z'}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial z}
\end{aligned}$$

Now specify normal mode decomposition

$$\begin{pmatrix} u_r' \\ u_\theta' \\ u_z' \\ p' \end{pmatrix} = \begin{pmatrix} \hat{u}_r(r) \\ \hat{u}_\theta(r) \\ \hat{u}_z(r) \\ \hat{p}(r) \end{pmatrix} e^{ikz+\sigma t}$$

Only axisymmetric perturbations are considered. The Euler equations become

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r) + ik\hat{u}_z &= 0 \\
\sigma\hat{u}_r - 2\Omega\hat{u}_\theta &= -\frac{1}{\rho}\frac{\mathrm{d}\hat{p}}{\mathrm{d}r} \\
\sigma\hat{u}_\theta + \hat{u}_r(\Omega + (r\Omega)_r) &= 0 \\
\sigma\hat{u}_z &= -\frac{1}{\rho}ik\hat{p}
\end{aligned}$$

We can reduce this system down to a single equation for $\hat{u}_r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\right)\hat{u}_r - k^2\hat{u}_r - 2\frac{k^2}{\sigma^2}\Omega(2\Omega + r\Omega')\hat{u}_r = 0$$

This is a second order ODE for $\hat{u}_r$ with BCs $\hat{u}_r = 0$ at $r = r_1, r_2$. For this flow, Rayleigh's determinant is

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = 4\Omega^2 + 2r\Omega'\Omega$$

Hence the ODE for $\hat{u}_r$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right) - k^2\hat{u}_r = \frac{k^2}{\sigma^2}\Phi(r)\hat{u}_r \tag{7}$$

Multiply (7) by $r\hat{u}_r^*$ (complex conjugate) and integrate from r_1 to r_2:

$$\int_{r_1}^{r_2} r\hat{u}_r^*\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term may be integrated by parts to give:

$$\cancel{\left[r\hat{u}_r^*\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right]_{r_1}^{r_2}} - \int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r^*)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term vanishes since $\hat{u}_r = 0$ at $r = r_1, r_2$. Labelling the first integral as $H_1 > 0$ and the second as $H_2 > 0$, we have

$$\frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r = -H_1 - k^2H_2 < 0$$

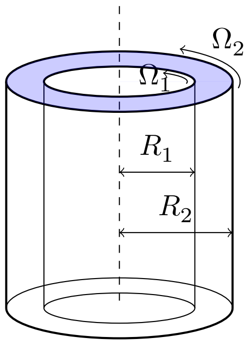
If $\Phi \geq 0$ then $\sigma^2 < 0$, i.e. σ is imaginary and we have stability. If instead $\Phi < 0$ somewhere in the domain, then potentially

$$\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2 \, \mathrm{d}r < 0$$

in which case $\sigma^2 > 0$ and we have instability. Hence $\Phi < 0$ somewhere in the domain is *necessary* (but not sufficient) condition for instability. So this formal analysis confirms Rayleigh's heuristic criterion. Note, really we need to consider non-axisymmetric perturbations too.

4.3 Taylor vortices

Apply Rayleigh's criterion to Taylor-Couette flow.



When viscosity is present, the general solution with $\partial_\theta = \partial_z = 0$ is

$$u_\theta(r) = Ar + \frac{B}{r}$$

No-slip boundary conditions at $r = R_1, R_2$ give

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}}$$

Note this solves $(\nabla^2 - 1/r^2)u_\theta = 0$ where $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r)$. In this case $\Omega = u_\theta/r = A + B/r^2$ hence Rayleigh's determinant is

$$\Phi = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4\left(A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}\right)\right] = 4A^2\left(1 + \frac{B}{Ar^2}\right)$$

For convenience we define $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2 < 1$. Then

$$\Phi = 4A^2\left[1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2}\right]$$

For stability, i.e. $\Phi \geq 0$ everywhere, we require for all $r \in [R_1, R_2]$

$$1 \geq \frac{(1-\mu)R_1^2}{(\eta^2-\mu)r^2} \geq \frac{1-\mu}{\eta^2-\mu}$$

where the last inequality follows since $R_1^2/r^2 \geq 1$ for all $r \in [R_1, R_2]$. There are now two cases:

- If $\eta^2 > \mu$ then

  $$\eta^2 - \mu \geq 1 - \mu \implies \eta^2 \geq 1$$

  This is a contradiction since $\eta < 1$.

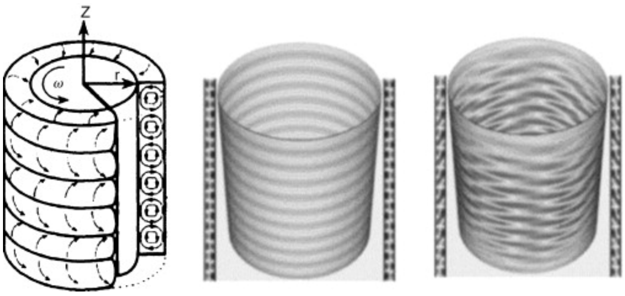
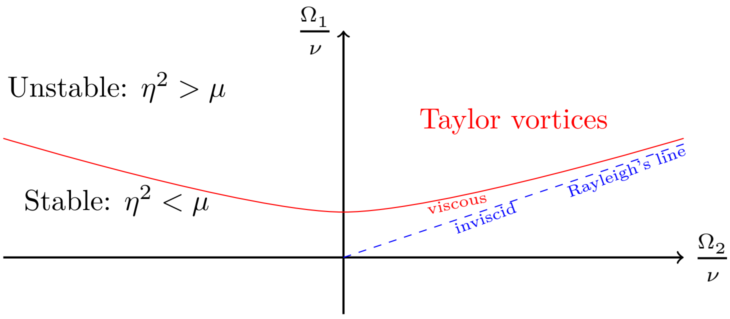


Figure 1: Taylor vortices, from Dutta and Ray, 2004.

- Otherwise $\eta^2 < \mu$, so
$$\eta^2 - \mu \leq 1 - \mu \implies \eta^2 \leq 1$$

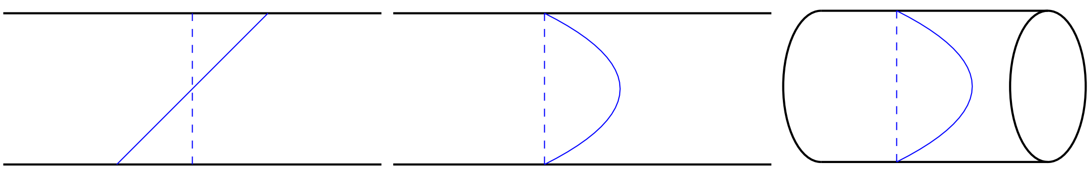
Thus Rayleigh's criterion is $\eta^2 < \mu$ for stability.



For a fixed geometry (i.e. fixed η) we can plot a stability diagram, with Rayleigh's line $\eta^2 = \mu = \Omega_2/\Omega_1$ marking the stability heuristic. In Taylor-Couette geometry, the instability often manifests itself as *Taylor vortices*, though there are many different modes of instability depending on Ω_1, Ω_2, ν.

5 Parallel shear flows

For some flows, inviscid analysis gives a good approximation to the stability properties of a viscous fluid (e.g. Kelvin-Helmholtz, Taylor-Couette flow) but for others, it does not (e.g. plane Couette flow, channel flow, pipe flow). In these flows, viscosity can be *de*stabilising.



Plane Couette flow Channel flow Pipe flow

5.1 Inviscid analysis

Consider a parallel shear flow $U(z)\hat{\boldsymbol{x}}$. The non-dimensionalised Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with boundary conditions $\boldsymbol{u} \cdot \hat{\boldsymbol{z}} = 0$ at $z = z_1, z_2$. The basic flow is $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with P constant – any constant form of the pressure is valid. Add small perturbations

$$\boldsymbol{u} = U(z)\hat{\boldsymbol{x}} + \boldsymbol{u}', \quad p = P + p'$$

The Euler equations become

$$\frac{\partial \boldsymbol{u}'}{\partial t} + U\frac{\partial \boldsymbol{u}'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z}\hat{\boldsymbol{x}} = -\nabla p'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

with boundary conditions $w' = 0$ at $z = z_1, z_2$. All equations have coefficients independent of x, y, t so we can separate the variables by taking normal modes of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \hat{\boldsymbol{u}}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$
$$p'(\boldsymbol{x}, t) = \hat{p}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$

Note we have replaced the usual σ with $-i\alpha c$. It is understood that the physical fluid perturbation velocity $\boldsymbol{u}'$ is represented by the real part, e.g.

$$w' = [\Re(\hat{w})\cos(\alpha x + \beta y - \alpha c_r t) - \Im(\hat{w})\sin(\alpha x + \beta y - \alpha c_r t)]\, e^{\alpha c_i t}$$

This mode is a wave travelling with phase speed $\alpha c_r/\sqrt{\alpha^2 + \beta^2}$ in the $(\alpha, \beta, 0)$ direction and it decays like $e^{\alpha c_i t}$ for $c_i < 0$, or grows if $c_i > 0$. The equations are now

$$i\alpha(U - c)\hat{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\alpha\hat{p} = 0 \tag{8}$$
$$i\alpha(U - c)\hat{v} + i\beta\hat{p} = 0 \tag{9}$$
$$i\alpha(U - c)\hat{w} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}z} = 0 \tag{10}$$
$$i\alpha\hat{u} + i\beta\hat{v} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0 \tag{11}$$

with boundary conditions $\hat{w} = 0$ at $z = z_1, z_2$. This is an eigenvalue problem in $c \in \mathbb{C}$. Instability corresponds to $c_i > 0$ and $c_i \leq 0$ for stability.

5.1.1 Squire's transformation (Squire, 1933)

Before attempting to solve (8)–(11), we consider Squire's transformation. Define the transformed variables

$$\tilde{\alpha} = \sqrt{\alpha^2 + \beta^2}, \quad \tilde{u} = \frac{\alpha\hat{u} + \beta\hat{v}}{\tilde{\alpha}}, \quad \tilde{p} = \frac{\tilde{\alpha}\hat{p}}{\alpha}$$

Construct $(\alpha(8) + \beta(9))/\alpha$:

$$i\tilde{\alpha}(U - c)\tilde{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\tilde{\alpha}\tilde{p} = 0 \tag{12}$$

Similarly $\tilde{\alpha}(10)/\alpha$:

$$i\tilde{\alpha}(U - c)\hat{w} + \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = 0 \tag{13}$$

Incompressibility is now expressed as

$$i\tilde{\alpha}\tilde{u} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0$$

The transformed system has the same form as (8)–(11) with $\beta = \hat{v} = 0$ and $\alpha \to \tilde{\alpha}, \hat{u} \to \tilde{u}, \hat{p} \to \tilde{p}$ but c unchanged. Thus the eigenvalue c depends on $\sqrt{\alpha^2 + \beta^2}$ but the growth rate is αc_i. So the largest growth rate αc_i is given by $\beta = 0$ for all wavenumber pairs (α, β) with $\sqrt{\alpha^2 + \beta^2}$ constant. Hence it is sufficient to consider $\beta = 0$ disturbances only. To any unstable 3D mode $\alpha \neq 0, \beta \neq 0$ there corresponds a more unstable 2D mode with $\beta = 0$.

5.1.2 Rayleigh's equation

Work in 2D (Squires). Use streamfunction ψ' such that

$$u' = \psi'_z, \ v' = 0, \ w' = -\psi'_x$$

Further, let $\psi'(x,z,t) = \phi(z)e^{i\alpha(x-ct)}$ so that it is now clear that c_r is the phase speed in the x direction. Now $\hat{u} = \frac{\mathrm{d}\phi}{\mathrm{d}z}$ and $\hat{w} = -i\alpha\phi$ (notice the phase difference). Then (12) becomes

$$\begin{aligned} i\alpha(U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}(-i\alpha\phi) + i\alpha\hat{p} = 0 \\ \implies \hat{p} = \frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} \end{aligned}$$

Substituting into (13) gives

$$\begin{aligned} i\alpha(U-c)(-i\alpha\phi) + \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}\right] = 0 \\ \implies (U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \end{aligned} \tag{14}$$

with boundary conditions $\phi = 0$ at $z = z_1, z_2$. This is *Rayleigh's equation (1880)*.

Comments.

- Rayleigh's equation involves α^2 only so need only consider $\alpha > 0$.
- If (ϕ, c) solves the problem then so does (ϕ^*, c^*). So if there exists a growing mode, there also exists a corresponding decaying mode. Hence stability means $c \in \mathbb{R}$ for all α.
- A singularity exists at $U(z_c) = c$ – this is called a critical layer and only occurs when $c \in \mathbb{R}$. Critical layers are important in solving IVPs and relating Rayleigh's equation to its viscous analogue, the Orr-Sommerfield equation (see later).
- There are two types of eigensolution:
  - Continuous spectrum $c \in [\min U, \max U]$ and ϕ has a discontinuous derivative at z_c. This type of solution is never unstable.
  - Discrete spectrum of complex conjugate pairs. This solution can be unstable.

5.1.3 Properties of Rayleigh's equation.

Inflection point criterion. Suppose $c_i > 0$, i.e. consider an unstable mode. Multiply Rayleigh's equation by ϕ^* and integrate from z_1 to z_2:

$$\int_{z_1}^{z_2} \left[\phi^*\phi'' - \alpha^2|\phi|^2 - \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0$$

Integrate the first term by parts and note $\phi = \phi^* = 0$ at z_1 and z_2. Hence

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0 \tag{15}$$

Take imaginary part:

$$\Im\left[\int_{z_1}^{z_2}\frac{U''(U-c^*)}{|U-c|^2}|\phi|^2\mathrm{d}z\right]=0$$
$$\implies -c_i\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

But $c_i>0$ so we must have

$$\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

Now $|U-c|^2>0$ and $|\phi|^2>0$ so U'' must change sign somewhere in $[z_1,z_2]$. Thus $U''=0$ at least once is a necessary condition for inviscid instability, called the *inflection point criterion*.

Fjørtoft's condition. A stronger form of the inflection point criterion was obtained by Fjørtoft (1950): given a monotonic mean velocity profile $U(z)$, a necessary condition for instability is that $U''(U-U_s)<0$ for some $z\in[z_1,z_2]$ with $U_s=U(z_s)$ where $U''(z_2)=0$.
To see this, take the real part of (15) to get

$$\int_{z_1}^{z_2}\frac{U''(U-c_r)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

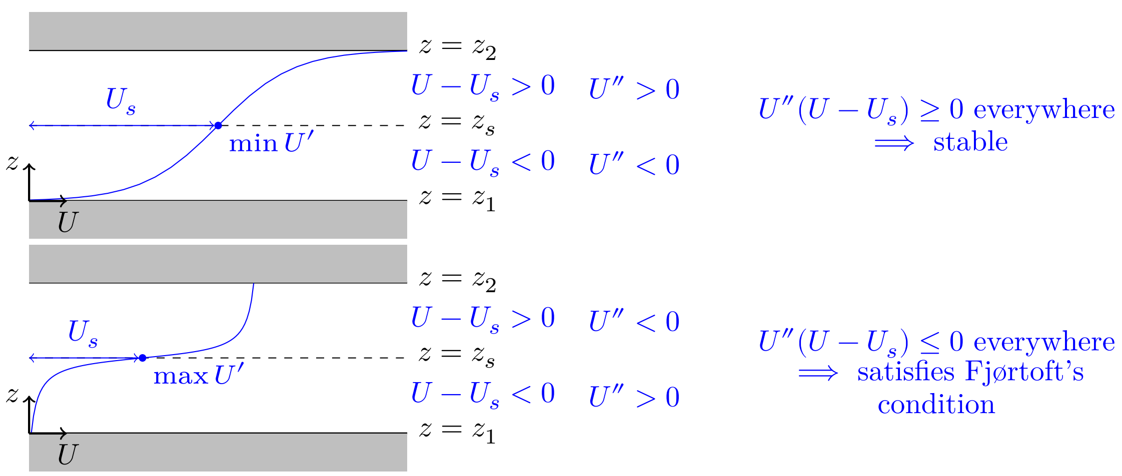
Add the term

$$(c_r-U_s)\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

which vanishes if $c_i>0$ by above. Then

$$\int_{z_1}^{z_2}\frac{U''(U-U_s)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

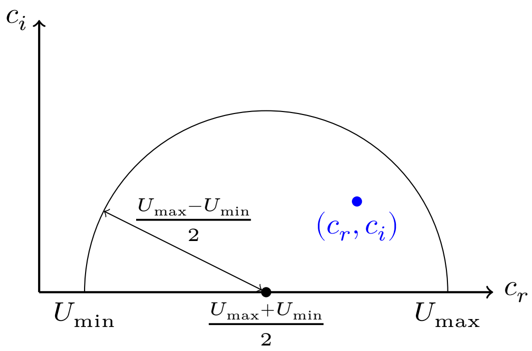
The RHS terms are negative definite, and $\left|\phi^2\right|>0$ as well as $|U-c|^2>0$. Hence $U''(U-U_s)<0$ somewhere in $[z_1,z_2]$. This means that the inflection point has to be a maximum (rather than a minimum) of the spanwise vorticity $U'(z)\hat{\boldsymbol{y}}$.



Howard's semicircle theorem Due to Howard (1961). The unstable eigenvalues of the Rayleigh equation satisfy

$$\left[c_r - \frac{1}{2}(U_{\max} + U_{\min})\right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{\max} - U_{\min})\right]^2$$

This is best viewed as a geometric condition: the unstable eigenvalues lie in a semicircle centred at $\frac{1}{2}(U_{\max} + U_{\min})$ of radius $\frac{1}{2}(U_{\max} - U_{\min})$.



Let $\Psi = \frac{\phi}{U-c}$. Rayleigh's equation (14) in terms of Ψ is

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}[(U-c)\Psi] - \alpha^2(U-c)\Psi\right) = U''(U-c)\Psi$$

Evaluating the derivative and simplifying gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(U-c)^2\frac{\mathrm{d}\Psi}{\mathrm{d}z}\right] = \alpha^2(U-c)^2\Psi$$

Multiply the equation by Ψ^* and integrate over $[z_1, z_2]$:

$$\int_{z_1}^{z_2} \Psi^*\left[(U-c)^2\Psi'\right]' \,\mathrm{d}z = \alpha^2 \int_{z_1}^{z_2} (U-c)^2|\Psi|^2 \,\mathrm{d}z$$

We then integrate by parts and note that $\Psi = \phi/(U-c) = 0$ on $z = z_1, z_2$. Hence

$$\int_{z_1}^{z_2} (U-c)^2\left[|\Psi'|^2 + \alpha^2|\Psi|^2\right] \mathrm{d}z = 0$$

Denote the [...] factor by Q. We have $Q > 0$ and $c \in \mathbb{C}$. Taking real and imaginary parts gives

$$\int_{z_1}^{z_2} \left[(U-c_r)^2 - c_i^2\right] Q \,\mathrm{d}z = 0$$

$$-2c_i \int_{z_1}^{z_2} (U-c_r) Q \,\mathrm{d}z = 0$$

Since Q is strictly positive, $U - c_r$ has to change sign in $[z_1, z_2]$. Hence

$$U_{\min} < c_r < U_{\max}$$

Rewrite the imaginary part as

$$\int_{z_1}^{z_2} UQ \,\mathrm{d}z = c_r \int_{z_1}^{z_2} Q \,\mathrm{d}z \tag{16}$$

and the real part as

$$\begin{aligned}\int_{z_1}^{z_2} U^2 Q \,\mathrm{d}z &= 2c_r \int_{z_1}^{z_2} UQ \,\mathrm{d}z + (-c_r^2 + c_i^2) \int_{z_1}^{z_2} Q \,\mathrm{d}z \\ &\overset{(16)}{=} 2c_r^2 \int_{z_1}^{z_2} Q \,\mathrm{d}z + (c_i^2 - c_r^2) \int_{z_1}^{z_2} Q \,\mathrm{d}z \\ &= (c_r^2 + c_i^2) \int_{z_1}^{z_2} Q \,\mathrm{d}z \end{aligned} \tag{17}$$

Now 'notice' that

$$\int_{z_1}^{z_2} (U - U_{\min})(U - U_{\max}) Q \,\mathrm{d}z \leq 0$$

since the first factor is ≥ 0, the second is ≤ 0 and $Q > 0$. Expanding the terms we have

$$\int_{z_1}^{z_2} [U^2 Q - (U_{\min} + U_{\max}) UQ + U_{\min} U_{\max} Q] \,\mathrm{d}z \leq 0$$

Now using (16) and (17) we can rewrite as

$$\begin{aligned} &\int_{z_1}^{z_2} [(c_r^2 + c_i^2) - (U_{\min} + U_{\max}) c_r + U_{\min} U_{\max}] \, Q \,\mathrm{d}z \leq 0 \\ \implies &\int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2} \right)^2 + U_{\min} U_{\max} - \left(\frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 \right] Q \,\mathrm{d}z \leq 0 \\ \implies &\int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2} \right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2} \right)^2 \right] Q \,\mathrm{d}z \leq 0 \end{aligned}$$

Equivalently we can write

$$\left[\left(c_r - \frac{U_{\max} + U_{\min}}{2} \right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2} \right)^2 \right] \int_{z_1}^{z_2} Q \,\mathrm{d}z \leq 0$$

But $\int_{z_1}^{z_2} Q \,\mathrm{d}z > 0$ so

$$\left(c_r - \frac{U_{\max} + U_{\min}}{2} \right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2} \right)^2 \leq 0$$

which establishes the semicircle theorem.

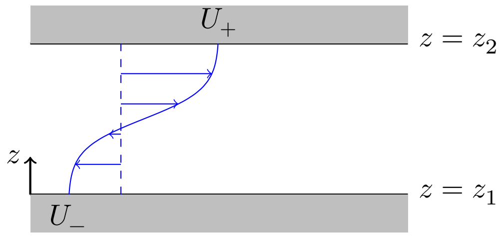
5.1.4 Predictions

For channel flow $\boldsymbol{U}(z) = (1 - z^2)\hat{\boldsymbol{x}}$ we have $U'' \neq 0$, i.e. no inflection points, so no inviscid instability predicted. However, channel flow is linearly unstable at sufficiently high Reynolds number. We must add viscosity to gain a more accurate stability heuristic.

5.2 Viscous analysis

Consider a basic state $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with $P = p_0 - Gx$ and $U(z_1) = U_-, U(z_2) = U_+$. At leading order, Navier-Stokes gives

$$-G = \frac{1}{\mathrm{Re}} U''$$



Special cases are

- Plane Poiseuille flow (PPF) $U(z) = 1 - z^2$ in $[-1, 1]$ and $G = 2/\mathrm{Re}, U_+ = U_- = 0$.
- Plane Couette flow (PCF) with $U(z) = z$ in $[-1, 1]$ and $G = 0$, $U_+ = 1$, $U_- = -1$.

The linearised Navier-Stokes equations for a perturbation $\boldsymbol{u}', p'$ are

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z} = -\frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u' \tag{18}$$
$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v' \tag{19}$$
$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w' \tag{20}$$
$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

The divergence of the first three equations is

$$\nabla \cdot \begin{pmatrix} (18) \\ (19) \\ (20) \end{pmatrix} \implies \frac{\partial w'}{\partial x}\frac{\mathrm{d}U}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x} = -\nabla^2 p'$$

Hence $\nabla^2 p' = -2U' w'_x$. Now consider ∇^2(20):

$$\nabla^2 \left[\frac{\partial w'}{\partial z} + U\frac{\partial w'}{\partial x}\right] = -\frac{\partial}{\partial z}\nabla^2 p' + \frac{1}{\mathrm{Re}}\nabla^4 w'$$

Combining these results we have

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\nabla^2 w' + U''\frac{\partial w'}{\partial x} + 2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial^2 w'}{\partial x \partial z} = -\frac{\partial}{\partial z}\left(-2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x}\right)$$
$$\implies \left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right)\nabla^2 - U''\frac{\partial}{\partial x}\right] w' = 0 \tag{21}$$

with boundary conditions $w' = w'_z = 0$ on the boundaries. This is a fourth order PDE with 4 boundary conditions, so w' is fully determined. To close the problem we need another equation: first define the *normal vorticity*

$$\eta' \equiv \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$$

Now ∂_y(18) $- \partial_x$(19) gives

$$\frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} = \frac{1}{\mathrm{Re}}\nabla^2 \eta'$$
$$\implies \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta' = -\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} \tag{22}$$

with boundary conditions $\eta' = 0$ on the boundaries since tangential velocities vanish at the boundaries. We have reduced $(u', v', w', p') \to (w', \eta')$. Given w' and η' determined from (21) and (22), we can generate v', w', p' from

$$\begin{aligned} u'_x + v'_y &= -w'_z \\ u'_y - v'_x &= \eta' \\ \nabla^2 p' &= -2U' w'_x \end{aligned}$$

5.2.1 Orr-Sommerfeld & Squire Equations

Introduce normal modes / wavelike disturbances / apply a Fourier transform:

$$(w', \eta')(x, y, z, t) = (\hat{w}(z), \hat{\eta}(z)) e^{i(\alpha x + \beta y - \alpha c t)}$$

Let $k^2 = \alpha^2 + \beta^2$ be the total horizontal wavenumber. Then (21) and (22) become

$$\left[i\alpha(U - c)(D^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - k^2)^2 \right] \hat{w} = 0 \tag{23}$$

$$\left[i\alpha(U - c) - \frac{1}{\mathrm{Re}}(D^2 - k^2) \right] \hat{\eta} = -i\beta U' \hat{w} \tag{24}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}z}$ as usual. Equation (23) is the *Orr-Sommerfeld equation* (Orr 1907, Sommerfeld 1908) and equation (24) is the Squire equation (Squire 1933).

- The Orr-Sommerfeld (OS) equation is the viscous extension of the Rayleigh equation.
- System (23) and (24) has two types of solution:
  1. OS modes $(\hat{w}, \hat{\eta})$ where $\hat{w}$ solves (23) and $\hat{\eta}$ is the forced response in (24).
  2. Squire modes $(0, \hat{\eta})$ which are always damped. Consider (24)/$(-i\alpha)$:

     $$c\hat{\eta} = U\hat{\eta} + \frac{i}{\alpha\mathrm{Re}}(D^2 - k^2)\hat{\eta}$$

     Multiply by $\hat{\eta}^*$:

     $$c|\hat{\eta}|^2 = U|\hat{\eta}|^2 + \frac{i}{\alpha\mathrm{Re}}\hat{\eta}^*(D^2 - k^2)\hat{\eta}$$

     Take the imaginary part and integrate over $[z_1, z_2]$:

     $$\begin{aligned} c_i \int_{z_1}^{z_2} |\hat{\eta}|^2 \,\mathrm{d}z &= \frac{1}{\alpha\mathrm{Re}} \int_{z_1}^{z_2} \frac{i\hat{\eta}^*(D^2 - k^2)\hat{\eta} - (-i)\hat{\eta}(D^2 - k^2)\hat{\eta}^*}{2i} \,\mathrm{d}z \\ &= \frac{1}{\alpha\mathrm{Re}} \int_{z_1}^{z_2} \frac{1}{2}(\hat{\eta}^* D^2\hat{\eta} + \hat{\eta} D^2 \hat{\eta}^*) - k^2|\hat{\eta}|^2 \,\mathrm{d}z \\ &= -\frac{1}{\alpha\mathrm{Re}} \int_{z_1}^{z_2} |D\hat{\eta}|^2 + k^2|\hat{\eta}|^2 \,\mathrm{d}z < 0 \end{aligned}$$

     Thus $c_i < 0$ so solutions are damped.

  Hence we just need to consider the OS equation to establish instability.

- Squire's theorem holds for the OS equation. The 3D version is

$$(U-c)(D^2-k^2)\hat{w} - U''\hat{w} - \frac{1}{i\alpha \mathrm{Re}}(D^2-k^2)^2\hat{w} = 0$$

  Compare with the 2D version

$$(U-c)(D^2-\hat{\alpha}^2)\hat{w} - U''\hat{w} - \frac{1}{i\hat{\alpha}\hat{\mathrm{Re}}}(D^2-\hat{\alpha}^2)^2\hat{w} = 0$$

  where $\hat{\alpha} = k^2 = \alpha^2 + \beta^2$ and

$$\hat{\mathrm{Re}} = \frac{\alpha \mathrm{Re}_{3D}}{\hat{\alpha}} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\mathrm{Re}_{3D} \leq \mathrm{Re}_{3D}$$

  Thus each 3D OS mode corresponds to a 2D OS mode at a *lower* Re. Note this is a slightly different result from the inviscid case where 2D always had a larger growth rate. We can instead note that if the critical Reynolds number for linear stability is Re_c then

$$\mathrm{Re}_c = \min_{\alpha,\beta} \mathrm{Re}_c(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_c(\alpha,0)$$

  where the first equality defines Re_c and the second is Squire's theorem. This led to a focus on the 2D OS equation.

- What is the connection between Rayleigh and OS equations?
  - OS is non-singular and has a countably infinite number of eigenvalues and its eigenfunctions are complete (Scheisted 1960). Note if the interval of flow is unbounded, there is a continuous spectrum of neutrally stable eigenfunctions in addition to the discrete spectrum (Herron 1987).
  - OS equation is fourth order whilst Rayleigh's equation is second order. 2 OS modes approximate Rayleigh modes, the other 2 modes fix the boundary conditions at the walls (lots of work on this – see Drazin & Reid (1981)).
  - Today it is absolutely routine to numerically solve the OS eigenvalue problem for $\mathrm{Re} \leq 10^7$. Very famous paper by Orszag (1971) used spectral methods as opposed to shooting techniques or finite difference to predict Re_c in channel flow.

5.2.2 Channel flow (PPF)

Thomas (1953) found $\mathrm{Re}_c = 5780$ at $\alpha_c = 1.026$ using finite differences (FD). Further FD estimates came from Nachtshen (1964) with $(\mathrm{Re}_c, \alpha_c) = (5767, 1.02)$ and Grosch & Salwen (1968) with $(\mathrm{Re}_c, \alpha_c) = (5750, 1.025)$. The accepted result now is from Orszag (1971) with $\mathrm{Re}_c = 5772.22$ at $\alpha_c = 1.02056$ using spectral methods.