

# Cambridge Part III Maths

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## Fluid Dynamics of Climate

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Lecture 1  
12/10/20

### 1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \nabla \phi + \rho \mathbf{F}$$

The body forces are assumed to be conservative with potential  $\phi$ , e.g.  $\phi = gz$  for gravitational force.  $\mathbf{F}$  is the frictional force.

Consider a reference frame rotating about the  $z$ -axis with constant angular velocity  $\boldsymbol{\Omega}$ . Axes in the inertial frame are denoted with a subscript  $I$  and axes in the rotating frame are denoted with a subscript  $R$ .

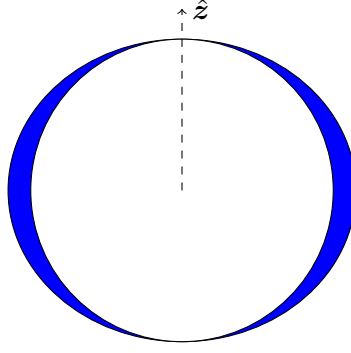
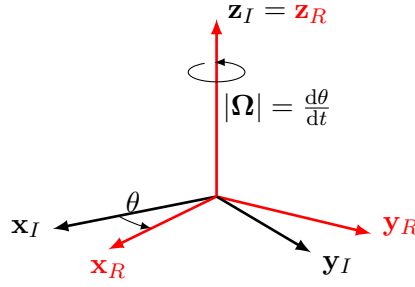


Figure 1: Geopotential ocean surface relative to a spherical Earth.



For a point with position vector  $\mathbf{x}$  and velocity  $\mathbf{u}_R = \left(\frac{d\mathbf{x}}{dt}\right)_R$  in the rotating reference frame

$$\left(\frac{d\mathbf{x}}{dt}\right)_I = \left(\frac{d\mathbf{x}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{x}$$

or equivalently  $\mathbf{u}_I = \mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{x}$ . Hence the acceleration is

$$\begin{aligned} \left(\frac{d\mathbf{u}}{dt}\right)_I &= \left(\frac{d}{dt} [\mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{x}]\right)_R + \boldsymbol{\Omega} \times (\mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{x})_R \\ &= \left(\frac{d\mathbf{u}_R}{dt}\right)_R + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) \end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal acceleration*. Note that we can write the centrifugal acceleration in the form of a conservative force

$$\begin{aligned} \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) &= \nabla \phi_c \\ \phi_c &= -\frac{1}{2} |\boldsymbol{\Omega} \times \mathbf{x}|^2 \end{aligned}$$

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho \left( \frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} \right) = -\nabla p - \rho \nabla (\phi + \phi_c) + \rho \mathbf{F} \quad (1)$$

We group the potential terms into a *geopotential*  $\Phi \equiv \phi + \phi_c$ . The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.

Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

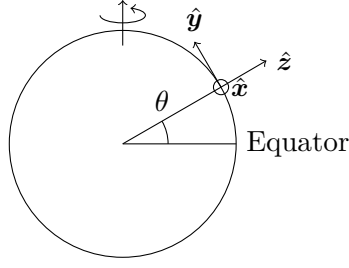


Figure 2: Local Cartesian coordinates

### 1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2). In this coordinate system  $\mathbf{\Omega} = (0, \Omega \cos \theta, \Omega \sin \theta)$ . Hence if  $\mathbf{u} = (u, v, w)$  then

$$\begin{aligned} 2\mathbf{\Omega} \times \mathbf{u} &= (2\Omega w \cos \theta - 2\Omega v \sin \theta, 2\Omega u \sin \theta, -2\Omega u \cos \theta) \\ &= (-fv + f^*w, fu - f^*u) \end{aligned}$$

where  $f \equiv 2\Omega \sin \theta$  is the *Coriolis parameter* and  $f^* \equiv 2\Omega \cos \theta$ .

**Example.** In Cambridge,  $\theta = 52.1^\circ N$  so

$$\begin{aligned} f &= 2\Omega \sin \theta \\ &= 2 \cdot \frac{2\pi}{3600 \cdot 24} \cdot 0.79 s^{-1} \\ &\approx 1.14 \times 10^{-4} s^{-1} \end{aligned}$$

At mid-latitudes,  $f \sim 10^{-4}$  is a good approximation.

We can simplify the Coriolis acceleration expression; often  $f^*w \ll fv$  and  $f^*u \ll g$ . Hence

$$2\mathbf{\Omega} \times \mathbf{u} \approx (-fv, fu, 0) = f\hat{\mathbf{z}} \times \mathbf{u}$$

This is the *traditional approximation*. This is *not* always a good approximation, particularly at intermediate scales.

### 1.2 Scale analysis.

Define characteristic scales for length  $L$ , time  $T$ , and velocity  $U$ . Non-dimensional variables are denoted with a superscript star:  $\mathbf{u}^* = \mathbf{u}/U$ , etc.

Using these scalings with  $\mathbf{F} = \nu \nabla^2 \mathbf{u}$  we have

$$\frac{U}{T} \frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{U^2}{L} \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + fU \hat{\mathbf{z}} \times \mathbf{u}^* = -\frac{1}{\rho} \nabla (p + \rho\Phi) + \frac{\nu U}{L^2} \nabla_*^2 \mathbf{u}^*$$

Dividing through by  $fU$  leaves the Coriolis acceleration term  $\text{ord}(1)$  with other terms scaled relatively.

$$\frac{1}{fT} \frac{\partial \mathbf{u}^*}{\partial t^*} + \text{Ro} \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + \hat{\mathbf{z}} \times \mathbf{u}^* = -\frac{1}{\rho fU} \nabla (p + \rho\Phi) + \text{E} \nabla_*^2 \mathbf{u}^*$$

where  $\text{Ro} \equiv \frac{U}{fL}$  is the *Rossby number* and  $\text{E} \equiv \frac{\nu}{fL^2}$  is the *Ekman number*.

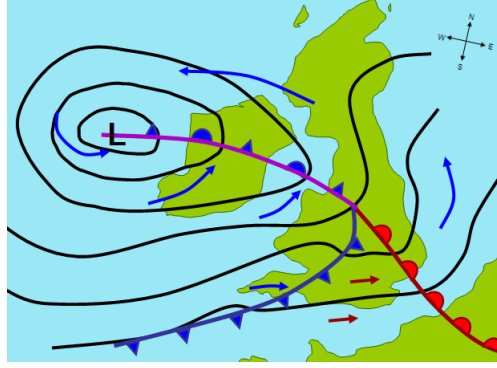


Figure 3: Lines of constant pressure  $p$  act as streamlines for the horizontal flow.

**Example.** For an atmospheric storm,  $U \sim 10 \text{ m s}^{-1}$ ,  $L \sim 1000 \text{ km}$ ,  $f \sim 10^{-4} \text{ s}^{-1}$ . Thus  $\text{Ro} \sim 0.1$ ,  $\text{E} \sim 10^{-13}$ .

Further, if  $T = L/U$ , then  $\text{Ro} = U/fL = 1/fT$ . For small  $\text{Ro}$ ,  $\text{E}$ , on surfaces of constant  $\Phi$ ,  $f\hat{\mathbf{z}} \times \mathbf{u} \approx -\frac{1}{\rho}\nabla p$ . This is *geostrophic balance*. In components, we have

$$\begin{aligned} -fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}$$

The equations of geostrophic balance can be arranged to give the horizontal velocity:  $\mathbf{u}_H$

$$\mathbf{u}_H \equiv (u, v) = \frac{1}{\rho f} \hat{\mathbf{z}} \times \nabla p$$

Horizontal velocity is perpendicular to  $\nabla p$  and hence parallel to isobars (lines of constant  $p$ ), i.e. pressure acts like a streamfunction (see figure 3).

In the Northern Hemisphere, air moves clockwise around high  $p$  and anticlockwise around low  $p$ . A *cyclonic* rotation is in the same sense as  $\mathbf{\Omega}$ , *anticyclonic* in the opposite sense as  $\mathbf{\Omega}$ .

### 1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small  $\text{Ro}$ ,  $\text{E}$ )

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ 2\mathbf{\Omega} \times \mathbf{u} &= -\frac{1}{\rho}\nabla p \end{aligned} \tag{2}$$

Taking the curl of (2) we have

$$\begin{aligned} \nabla \times (\mathbf{\Omega} \times \mathbf{u}) &= \varepsilon_{ijk} \partial_j \varepsilon_{klm} \Omega_l u_m \\ &= \varepsilon_{kij} \varepsilon_{klm} \Omega_l \partial_j u_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \Omega_l \partial_j u_m \\ &= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i \end{aligned}$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \mathbf{u}) = \boldsymbol{\Omega} \cdot \nabla \mathbf{u} = 0$$

For  $\boldsymbol{\Omega} = (0, 0, \Omega)$ , this implies  $\frac{\partial w}{\partial z} = 0$ . If  $w = 0$  on some horizontal surface (e.g. ground) then  $w = 0$  everywhere.

Also,  $u_x + v_y = 0$ , i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in ‘columns’ parallel to  $\boldsymbol{\Omega}$ , called *Taylor columns*.

## 2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density  $\rho_0$  with angular velocity  $\boldsymbol{\Omega} = (0, 0, f/2)$ . Assume small amplitude motions (i.e.  $|\mathbf{u}|^2 \ll |\mathbf{u}|$ ), i.e. neglect  $\mathbf{u} \cdot \nabla \mathbf{u}$  and  $\nu \nabla^2 \mathbf{u}$ . From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \quad (3)$$

$$v_t + fu = -\frac{p_y}{\rho_0} \quad (4)$$

$$w_t = -\frac{p_z}{\rho_0} \quad (5)$$

$$u_x + v_y + w_z = 0 \quad (6)$$

We will eliminate variables in favour of  $p$ .

$$\begin{aligned} \nabla \cdot ((3) - (5)) &\implies \nabla^2 p = \rho_0 f (v_x - u_y) \\ \partial_x (4) - \partial_y (3) &\implies (v_x - u_y)_t = fw_z \end{aligned}$$

Combining these and using (5) we have

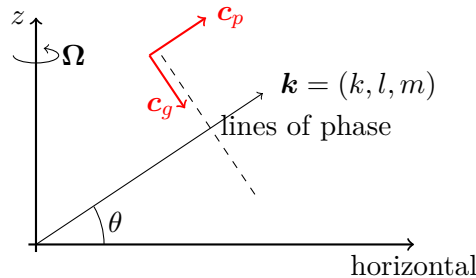
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for  $p$ . Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed  $c_p = \omega/k$  and group velocity

$$\mathbf{c}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\mathbf{k}|^{3/2}}$$

Note that  $\mathbf{c}_p \cdot \mathbf{c}_g = 0$ . Also note  $|\omega| \leq |f|$ .

## 2.1 Inertial (free) oscillations

Assume  $\nabla p = \mathbf{0}$ . The  $x$  and  $y$  components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus  $u = U \sin ft$  where  $f$  is the *inertial frequency*. Similarly, we have  $v = U \cos ft$ . For a particle with position  $(x_p, y_p)$  floating on an ocean surface  $z = 0$  moving with the fluid velocity, we have

$$\begin{aligned} \frac{dx_p}{dt} = u &\implies x_p = -\frac{U}{f} \cos ft + x_0 \\ \frac{dy_p}{dt} = v &\implies y_p = -\frac{U}{f} \sin ft + y_0 \end{aligned}$$

Thus the motion of fluid particles describes *inertial circles* with radius  $\frac{2U}{f}$ .

## 2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress  $\boldsymbol{\tau}_w$ . Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e.  $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e.  $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e.  $w(z=0) = 0$
4. No flow in deep ocean, i.e.  $\lim_{z \rightarrow -\infty} \mathbf{u} = \mathbf{0}$
5. Constant density  $\rho$
6. Traditional approximation

Continuity (incompressibility) says  $u_x + v_y + w_z = 0$ . Assumptions 2 and 3 then imply  $w = 0$  everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$

$$fu = \nu v_{zz} \tag{8}$$

Define the *complex velocity*  $\mathcal{V} \equiv u + iv$ . Then

$$\mathcal{V}_{zz} = \frac{if}{\nu} \mathcal{V} \tag{9}$$

Without loss of generality, assume  $\boldsymbol{\tau}_w$  is aligned with the  $x$ -axis:  $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$ . Boundary conditions for (9) are

$$\mathcal{V}_z = \left( \frac{\tau_w}{\rho\nu}, 0 \right) \quad \text{at } z = 0$$

$$\mathcal{V} = (0, 0) \quad \text{as } z \rightarrow -\infty$$

Thus  $\mathcal{V} = Ae^{(1+i)z/\delta}$  where  $\delta = \sqrt{\frac{2\nu}{f}}$ ,  $A = \frac{\tau_w \delta(1-i)}{2\rho\nu}$ . In terms of the velocity components, we have

$$\begin{aligned} u &= \frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right) \\ v &= -\frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right) \end{aligned}$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

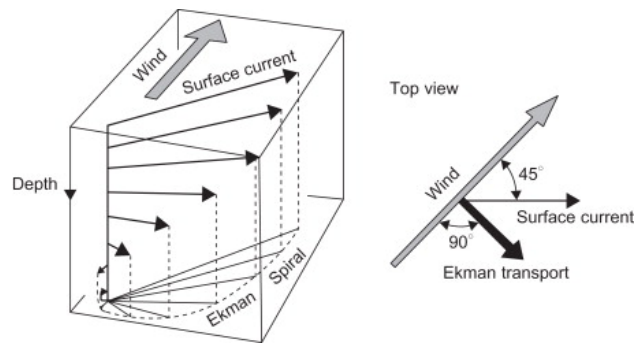


Figure 4: Ekman spiral.

### 2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where  $\nu \mathbf{u}_z \approx 0$  at  $z = -h$ . Since  $\nu \mathbf{u}_z(z = 0) = (\tau_w/\rho, 0)$ , the *Ekman transport*  $\mathbf{U}_T$  is

$$U_T \equiv \int_{-h}^0 u \, dz = 0$$

$$V_T \equiv \int_{-h}^0 v \, dz = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented  $90^\circ$  to the right of the applied wind shear stress (in the Northern Hemisphere).

### 2.4 Ekman pumping

Consider a wind stress  $\tau_w(y)$  that varies over large scales. Then from incompressibility

$$\int_{-h}^0 w_z \, dz = -\int_{-h}^0 u_x \, dz - \int_{-h}^0 v_y \, dz$$

Thus for  $h$  constant,

$$-w(z = -h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\tau_w}{\rho f} \right)$$

In general we have

$$w(z = -h) = \hat{\mathbf{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$

## 3 Rotating shallow water equations

Consider a thin layer of fluid with constant density  $\rho$ . Define characteristic scales

- length  $L = \text{horiz.}, H = \text{vert.}$
- velocity  $U$
- time  $T$
- pressure  $P$

such that  $\partial_x, \partial_y \sim \frac{1}{L}, \partial_z \sim \frac{1}{H}$ . Define the *aspect ratio*  $\delta \equiv H/L$ . We will assume  $\delta \ll 1$ . From continuity (incompressibility) we have

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \implies \frac{w}{H} &= \mathcal{O}(U/L) \\ \implies w &= \mathcal{O}(\delta U) \end{aligned}$$

Using the traditional approximation and assuming the fluid is inviscid, the  $x$ -momentum equation

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (10) \\ \text{scaling: } \frac{U}{T} \quad \frac{U^2}{L} \quad \frac{U^2}{L} \quad \frac{wU}{H} \quad fU &= \frac{P}{\rho L} \end{aligned}$$

Thus if  $p_x$  appears at leading order then

$$P \sim \rho U \max(L/T, U, fL)$$

Similarly the  $z$ -momentum equation and its scalings are

$$\begin{aligned} \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (11) \\ \text{scaling: } \frac{w}{T} \quad \frac{Uw}{L} \quad \frac{Uw}{L} \quad \frac{w^2}{H} &= \frac{P}{\rho H} \end{aligned}$$

Hence  $\frac{Dw}{Dt} \sim \max(\frac{w}{T}, \frac{Uw}{L})$ . Comparing with the pressure term, we have

$$\begin{aligned} \frac{\frac{Dw}{Dt}}{\frac{1}{\rho} \frac{\partial p}{\partial z}} &\sim \frac{\max(\frac{w}{T}, \frac{Uw}{L})}{\frac{U}{H} \max(\frac{L}{T}, \frac{U}{L}, f)} \\ &\sim \delta^2 \frac{\max(\frac{1}{T}, \frac{U}{L})}{\max(\frac{1}{T}, \frac{U}{L}, f)} \end{aligned}$$

Therefore to  $\mathcal{O}(\delta^2)$  we have *hydrostatic balance*. To this order, (11) becomes

$$\frac{\partial p}{\partial z} - \rho g \implies p = \rho g(\eta - z)$$

assuming  $p = 0$  at  $z = \eta(x, y, t)$ . Similarly, we have  $\frac{1}{\rho} p_x = g\eta_x$  and  $\frac{1}{\rho} p_y = g\eta_y$ . Hence horizontal acceleration (i.e. the LHS of (10)) is independent of  $z$ . Motivated by this, we *assume* that horizontal velocity is also independent of  $z$ . For  $Ro \ll 1$ , this follows from the Taylor-Proudman theorem.

Re-writing (10) with these results we have

$$u_t + uu_x + vu_y - fv = -g\eta_x \quad (12)$$

$$v_t + uv_x + vv_y + fu = -g\eta_y \quad (13)$$

since  $u_z = v_z = 0$  by assumption. Integrating the continuity equation gives

$$w = -z(u_x + v_y) + A(x, y, t)$$



where  $A$  is to be determined by the boundary conditions. Requiring no normal flow at  $z = -H_0 + h_b$  is imposed by  $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$  where  $\mathbf{n} = \nabla(z - h_b)$ . Thus

$$-u \frac{\partial h_b}{\partial x} - v \frac{\partial h_b}{\partial y} + w = 0$$

Hence

$$A(x, y, t) = u \frac{\partial h_b}{\partial x} + v \frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$$

The kinematic boundary condition at  $z = \eta$  is  $\frac{D\eta}{Dt} = w$  which may be written as

$$\eta_t + u\eta_x + v\eta_y - w = 0$$

where  $w = -\eta(u_x + v_y) + u \frac{\partial h_b}{\partial x} + v \frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$ . Combining these boundary conditions gives

$$\eta_t + [(H_0 - h_b + \eta)u]_x + [(H_0 - h_b + \eta)v]_y = 0 \quad (14)$$

If  $H \equiv H_0 - h_b + \eta$  is the total depth of the fluid, then since  $H_t = \eta_t$ ,

$$H_t + (uH)_x + (vH)_y = 0 \quad (15)$$

which is a statement of the conservation of volume (equivalently mass, since  $\rho$  is constant). Equations (12), (13), and (14) are the *rotating shallow water* (SW) equations.

### 3.1 Potential vorticity (PV)

Denote the vertical vorticity by  $\zeta = v_x - u_y$ . Consider  $\partial_x(13) - \partial_y(12)$ , which gives

$$\zeta_t + u\zeta_x + v\zeta_y + vf_y = -(\zeta + f)(u_x + v_y)$$

Now from conservation of volume (15),

$$u_x + v_y = -\frac{1}{H} \frac{DH}{Dt}$$

Combining these relates the material derivative of  $\zeta$  and  $H$  by

$$\frac{D\zeta}{Dt} = \frac{\zeta + f}{H} \frac{DH}{Dt} \implies \frac{D}{Dt} \left( \frac{\zeta + f}{H} \right) = 0$$

Let  $q \equiv \frac{\zeta + f}{H}$ , the *shallow water potential vorticity* (SWPV). SWPV is conserved following fluid motion. We call  $\zeta$  the *relative vorticity* and  $f$  the *planetary vorticity*.  $\zeta$  and  $f$  will change as a fluid moves to conserve SWPV (changing  $f$ ) and angular momentum (changing depth).

