Cambridge Part III Maths

Lent 2020

Fluid Dynamics of Environment

based on a course given by
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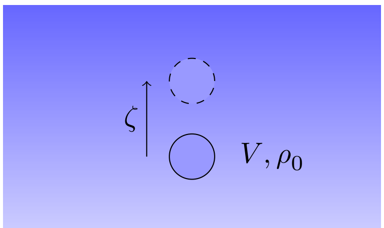
Contents

Lecture 1
22/01/21

1 Internal waves

1.1 Intuitive version



Consider a fluid parcel of volume V and density ρ_0 in a fluid with density profile $\hat{\rho}(z)$. Suppose the parcel is moved upwards by ζ. The parcel experiences a *buoyancy force* $B = gV\zeta\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$. Newton's second law gives

$$\ddot{\zeta} + \left(-\frac{g}{\rho_0}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}\right)\zeta = 0$$

The *buoyancy frequency* (or Brunt-Väisälä frequency) is defined as

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$$

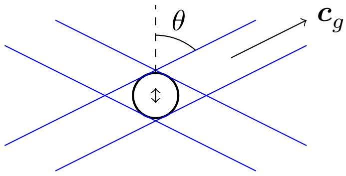
which has general solution

$$\zeta = A\cos Nt + B\sin Nt$$

If we instead consider a fluid slab inclined at angle θ with the vertical rather than a fluid parcel, the slab can fall in its plane much more easily than in the vertical. Hence in this situation we have

$$\ddot{\zeta} + N^2\cos^2\theta\zeta = 0$$

The dispersion relation is thus $\omega/N = \cos\theta$.
Now consider a sphere oscillating at frequency ω in the vertical in a stratified fluid with density $\rho(z)$. The fluid resonates in bands at angle θ satisfying the dispersion relation, provided $\omega < N$. Intuitively, the group velocity must be out of the beams as energy is radiated away.



At the leading edge of the rays, baroclinic vorticity is generated by the movement of fluid of different density to its surroundings. This provides the mechanism for the instability.

1.2 Rigorous derivation

Consider a fluid in which the mean pressure $p_0(z)$ and the mean density $\rho_0(z)$ are in hydrostatic balance when the fluid is at rest:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g$$

Assume that the vertical lengthscale for ρ_0 variation is L. Motion is governed by the Navier-Stokes equations (1) and (2) with $\nu = 0$, and mass conservation (3).

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p - \rho g\hat{\boldsymbol{z}} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\rho = 0 \tag{3}$$

Following the Boussinesq approximation, assume small perturbations to the mean state: $\rho = \rho_0(z) + \tilde{\rho}$ and $p = p_0(z) + \tilde{p}$ where $\tilde{p} \ll p_0, \tilde{\rho} \ll \rho_0$. Under this approximation, the momentum equation (2) becomes

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}g\hat{\boldsymbol{z}} \\ &= -\frac{1}{\rho_0}\nabla(p + \rho_0 g z) - g'\hat{\boldsymbol{z}}\end{aligned}$$

where $g' = g(\rho - \rho_0)/\rho_0$ is the *reduced gravity*. We now linearise $\boldsymbol{u}$ about a state of rest, ignoring second order quantities in the velocity disturbance $\boldsymbol{u}'$. It is now further desirable to split the disturbance components into $\tilde{\rho} = \hat{\rho} + \rho', \tilde{p} = \hat{p} + p'$ where $\hat{\rho}, \hat{p}$ are in hydrostatic balance. We have

$$\begin{aligned}\nabla \cdot \boldsymbol{u}' &= 0 \\ \frac{\partial \rho'}{\partial t} + w'\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z} &= \frac{\partial \rho'}{\partial t} - w'\frac{\rho_0}{g}N^2 = 0 \\ \frac{\partial \boldsymbol{u}'}{\partial t} &= -\frac{1}{\rho_0}\nabla(p_0 + \hat{p}) - \frac{g\hat{\rho}}{\rho_0}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}\end{aligned}$$

Hydrostatic balance eliminates the first two RHS terms of the momentum equation; the hydrostatic pressure field is

$$p_0 + \hat{p} = -\int g\hat{\rho}\,\mathrm{d}z$$

Finally we have

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}$$

Define buoyancy $b = -g\rho'/\rho_0$. The governing equations are now

$$\frac{\partial b}{\partial t} = -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}}$$
$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}}$$

To eliminate pressure, we take the curl of the momentum equation to get

$$\frac{\partial \boldsymbol{\zeta}'}{\partial t} = -\hat{\boldsymbol{z}} \times \nabla b$$

where $\boldsymbol{\zeta}' = \nabla \times \boldsymbol{u}'$ is the disturbance vorticity. Using the buoyancy equation we have

$$\left[\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2\right] w' = 0$$

where $\nabla_H = (\partial_x, \partial_y)$ is the horizontal gradient operator. This equation admits plane wave solutions

$$w'(\boldsymbol{x}, t) = \Re\left[\hat{w}(t)e^{i(k_x x + k_y y - \omega t)}\right]$$

where $\hat{w}$ satisfies

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)\hat{w} = 0$$

which has general solution

$$\hat{w} = \Re\left[Ae^{-inz} + Be^{inz}\right]$$
$$n^2 = (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)$$

If $\omega > N$, n is imaginary and, defining $\gamma = \sqrt{1 - N^2/\omega^2}$, we have

$$w' = (Ae^{-\gamma kz} + Be^{\gamma kz})e^{i(k_x x + k_y y - \omega t)}$$

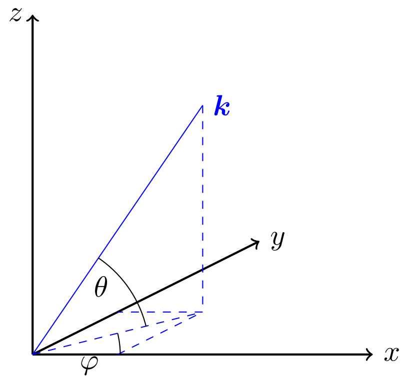
When $N = 0$, we get potential flow. If $0 < N < \omega$ then we have rescaled potential flow, with scaling γ. If $\omega < N$ then n is real and solutions are oscillatory with

$$n^2 = kz^2 = (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)$$

The wavenumber vector is $\boldsymbol{k} = (k_x, k_y, k_z) = (k, l, m)$. Hence

$$\frac{\omega^2}{N^2} = \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} = 1 - \frac{k_z^2}{|\boldsymbol{k}|^2} = \cos^2\theta$$

where θ is the angle between $\boldsymbol{k}$ and the horizontal plane. Note that N is assumed constant.



We will use θ as the angle between the horizontal plane and $\boldsymbol{k}$, hence $\omega/N = \cos\theta$. Some authors use the polar inclination angle between $\hat{\boldsymbol{z}}$ and $\boldsymbol{k}$, in which case $\omega/N = \sin\theta$. The azimuthal angle between the x axis and the projection of $\boldsymbol{k}$ onto the horizontal plane is denoted by φ. We will also assume $\omega \geq 0$ going forward. The wavevector takes the form

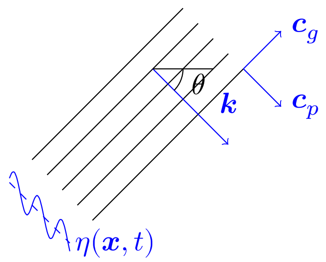
$$\boldsymbol{k} = |\boldsymbol{k}| \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$

The *phase* of the wave is defined to be $\phi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$. Note the following:

$$\begin{aligned} e^{i\phi} &= e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \\ \Re\left[e^{i\phi}\right] &= \Re\left[e^{-i\phi}\right] \\ \Re\left[\tilde{\eta}e^{i\phi}\right] &= \Re\left[\tilde{\eta}^* e^{-i\phi}\right] \end{aligned}$$

We will focus on 2D waves. By suitable choice of coordinate system, 3D waves can be reduced to 2D. In the (x,z) plane, from $\nabla\cdot\boldsymbol{u} = 0$ and assuming $w(\boldsymbol{x},t) = \tilde{w}e^{i\phi}$ with $\tilde{w}\in\mathbb{C}$ we have

$$u = \int -\frac{\partial w}{\partial z}\,\mathrm{d}x = -\frac{m}{k}\tilde{w}e^{i\phi} = -\tan\theta\tilde{w}e^{i\phi}$$



Denoting the wave displacement by $\eta(\boldsymbol{x},t) = \tilde{\eta}e^{i\phi}$ we have the following:

$$\begin{aligned} \frac{\partial\eta}{\partial t} &= -i\omega\tilde{\eta}e^{i\phi} \\ u(\boldsymbol{x},t) = \tilde{u}e^{i\phi} &= -i\omega\sin\theta\tilde{\eta}e^{i\phi} \\ w(\boldsymbol{x},t) = \tilde{w}e^{i\phi} &= i\omega\cos\theta\tilde{\eta}e^{i\phi} \end{aligned}$$

Using the buoyancy equation $\partial b/\partial t = -wN^2$ we also find

$$i\omega\tilde{b} = -\tilde{w}N^2 \implies b = -\tilde{\eta}\frac{\omega^2}{\cos\theta}e^{i\phi} = -\tilde{\eta}\omega N e^{i\phi}$$

Similarly the pressure field follows from the momentum equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \implies \tilde{p} = i\frac{\omega^2}{k}\tilde{\eta}\sin\theta$$

1.3 Wave velocities

1.3.1 Phase velocity

For the phase $\phi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ we can write

$$\begin{aligned}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x_i} &= 0\\ \implies k_i\frac{\partial\phi}{\partial t} + \omega\frac{\partial\phi}{\partial x_i} &= 0\\ \implies \frac{\partial\phi}{\partial t} + \frac{\omega}{|\boldsymbol{k}|^2}k_i\frac{\partial\phi}{\partial x_i}&\end{aligned}$$

Defining the phase velocity $\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k}$ as the velocity at which the phase is advected, we have

$$\frac{\partial\phi}{\partial t} + (\boldsymbol{c}_p\cdot\nabla)\phi = 0$$

For all waves, this is the speed at which wave crests move. For deep water waves with dispersion relation $\omega = \sqrt{gk}$ we find $c_p = \sqrt{g/k}$.

1.3.2 Group velocity

By symmetry of partial differentiation, we may write

$$\begin{aligned}\frac{\partial^2\phi}{\partial t\partial x_i} - \frac{\partial^2\phi}{\partial t\partial x_i} &= 0\\ \implies \frac{\partial k_i}{\partial t} + \frac{\partial\omega}{\partial x_i} &= 0 \qquad (4)\end{aligned}$$

If $\omega = \omega(\boldsymbol{k})$ then from chain rule

$$\frac{\partial\omega}{\partial x_i} = \frac{\partial\omega}{\partial k_j}\frac{\partial k_j}{\partial x_i}$$

We also know from definition of phase

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_j}\right) = \frac{\partial k_i}{\partial x_j}$$

Hence (4) may be written as

$$\frac{\partial k_i}{\partial t} + \frac{\partial\omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = 0$$

Thus we define the group velocity as the velocity at which the wavevector is advected, i.e. $\boldsymbol{c}_g = \frac{\partial\omega}{\partial k_i}$ and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{c}_g\cdot\nabla\right)\boldsymbol{k} = 0$$

For deep water waves, we thus have $c_g = \sqrt{g/k}/2 = c_p/2$.

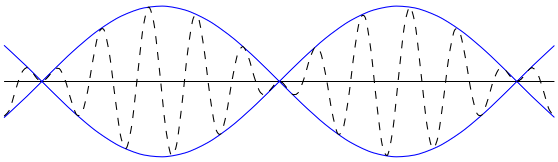
1.3.3 Superposition

Consider two waves with slightly different wavenumber and frequency superposed:

$$\begin{aligned}\eta &= \cos\left((k+\delta k)x - (\omega+\delta\omega)t\right) + \cos\left((k-\delta k)x - (\omega-\delta\omega)t\right)\\ &= 2\cos\left(\delta k x - \delta\omega t\right)\cos\left(kx - \omega t\right)\end{aligned}$$

This is referred to as *modulation* with $|\delta k| \ll |k|, |\delta\omega| \ll |\omega|$. Equivalently, in the limit as $|\delta k| \to 0, |\delta\omega| \to 0$, we have

$$\eta = 2\cos\left((x - \frac{\partial\omega}{\partial k}t)\delta k\right)\cos\left(kx - \omega t\right)$$



1.3.4 Internal gravity wave velocities

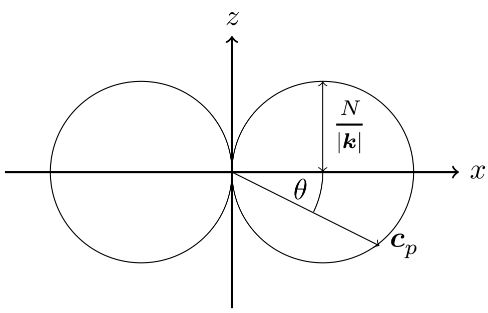
For internal gravity waves we have dispersion relation

$$\frac{\omega^2}{N^2} = \frac{k^2+l^2}{k^2+l^2+m^2} = \cos^2\theta$$

Hence the phase velocity is

$$\begin{aligned}\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k} &= N\frac{(k^2+l^2)^{1/2}}{(k^2+l^2+m^2)^{3/2}}\boldsymbol{k}\\ &= \frac{N\cos\theta}{|\boldsymbol{k}|^2}\begin{pmatrix}|\boldsymbol{k}|\cos\varphi\cos\theta\\ |\boldsymbol{k}|\sin\varphi\cos\theta\\ |\boldsymbol{k}|\sin\theta\end{pmatrix}\\ &= \frac{N|\cos\theta|}{|k|}\begin{pmatrix}\cos\varphi\cos\theta\\ \sin\varphi\cos\theta\\ \sin\theta\end{pmatrix}\end{aligned}$$

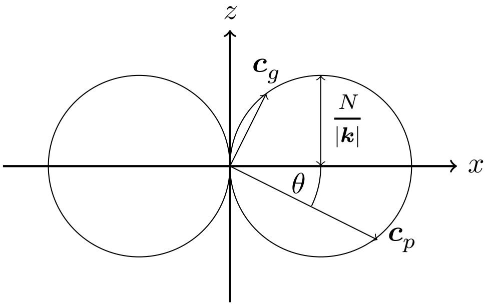
The prefactor is the magnitude of the phase velocity. The locus of possible phase velocities for given N and $\boldsymbol{k}$ is two circles:



The group velocity is

$$\boldsymbol{c}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{1}{2\omega}\frac{\partial \omega^2}{\partial \boldsymbol{k}} = \frac{\omega}{|\boldsymbol{k}|^2}\left(\frac{N^2}{\omega^2}K\boldsymbol{k} - m\hat{\boldsymbol{z}}) - \boldsymbol{k}\right) = \frac{N}{|\boldsymbol{k}|}|\sin\theta| \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ -\cos\theta \end{pmatrix}$$

Hence the magnitude of the group velocity is $N|\sin\theta|/|\boldsymbol{k}|$. The group velocity is perpendicular to the phase velocity:



Note that

$$\boldsymbol{c}_g + \boldsymbol{c}_p = \frac{N}{|\boldsymbol{k}|}\left[|\cos\theta|\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix} + |\sin\theta|\begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ -\cos\theta \end{pmatrix}\right] = \frac{N}{|\boldsymbol{k}|}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Hence $\left|\boldsymbol{c}_p + \boldsymbol{c}_g\right| = N/|\boldsymbol{k}|$ and $c_{pz} = -c_{gz}$. Note also that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$.

1.4 Equipartition of energy

We can form an energy equation from the momentum equation dotted with $\boldsymbol{u}$:

$$\boldsymbol{u} \cdot \left(\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p + \rho g\hat{\boldsymbol{z}}\right) = 0$$

Using the Boussinesq approximation and linearising, we get

$$\boldsymbol{u} \cdot \left(\rho_0\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p' + \rho' g\hat{\boldsymbol{z}}\right) = 0$$
$$\frac{1}{2}\rho_0\frac{\partial}{\partial t}|\boldsymbol{u}|^2 + \boldsymbol{u}\cdot p' + w\rho' g = 0$$

The first term is the rate of change of kinetic energy, the second is the work against pressure gradients, and the last is the rate of change of potential energy.
Using linearised conservation of mass we have

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z} = \frac{\partial \rho'}{\partial t} - w\frac{\rho_0}{g}N^2 = 0 \implies w = \frac{g}{\rho_0 N^2}\frac{\partial \rho'}{\partial t}$$

Hence also using $\nabla\cdot\boldsymbol{u} = 0$ we have the energy equation

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \frac{1}{2}\frac{g^2}{\rho_0 N^2}\rho'^2\right] + \nabla\cdot(p'\boldsymbol{u}) = 0$$

The term proportional to ρ'^2 is *potential energy.* To see this, consider a parcel of fluid raised vertically by some amount ζ. The buoyant force on the parcel is

$$F = Vg\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}(z - z_0)$$

Hence the potential energy gained is

$$E = \int_{z_0}^{z_0+\zeta} g\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}(z - z_0)\,\mathrm{d}z = \frac{1}{2}\rho_0 N^2 \zeta^2$$

Using the hydrostatic relation shows that this value matches the potential energy term in the energy equiation.