Cambridge Part III Maths

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Non-Newtonian Fluid Mechanics

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Contents

1 Introduction to Non-Newtonian Fluids

Lecture 1
08/10/20

Newtonian fluids are typically characterised by 2 material properties: viscosity and density. We may also refer to Newtonian fluids as 'simple fluids'.

Non-Newtonian fluids may have many other material properties, for example intrinisic time, length and stress scales, and an intrinsic orientation. They often exhibit a mix of fluid and solid behaviour. We may also refer to non-Newtonian fluids as 'complex fluids'.
There are many examples of complex fluids readily apparent in our everyday lives, for example sand (wet or dry) and mud; lava and glass, both of which experience phase changes; ketchup; foam; paint; emulsions such as milk; liquid crystals used in screens; blood on very small scales.
The goal of this course is to cover three main areas.

- Phenomenology of non-Newtonian fluids – how do they behave?
- Mathematical modelling – how do we quantify the behaviour?
- Predictions (and limits) of models

2 Summary of Newtonian Fluid Mechanics

2.1 Continuum approximation

We describe fluids in terms of two main fields: *density* $\rho(\boldsymbol{x}, t)$ and *velocity* $\boldsymbol{u}(\boldsymbol{x}, t)$. We use the continuum approximation whereby the fluid is assumed to be a continuum rather than made up of discrete fluid particles. Under this assumption, the macroscopic properties of density and velocity are well-defined as 'averages' of infinitesimal volume elements.
The velocity field is *Eulerian*, meaning it is measured at a specific point in space and time, as opposed to following a material element (Lagrangian).

2.2 Conservation of mass

Conservation of mass can be expressed in the classical form of a conservation equation as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \boldsymbol{u}] = 0$$

Expanding the flux term, this may be expressed in a form which relates the rate of change of density of a fluid element with the divergence of the flow:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}$$

In this course we will assume that all fluids are incompressible, which is expressed mathematically as:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \iff \nabla \cdot \boldsymbol{u} = 0$$

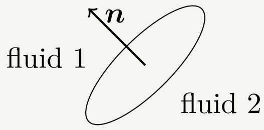
2.3 Mechanical equilibrium

Newton's second law, i.e. conservation of momentum is expressed for a fluid using the *Cauchy momentum equation*:

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right] = \nabla \cdot \sigma + \boldsymbol{F}$$

where $\nabla \cdot \sigma$ are the surface forces acting on the fluid and $\boldsymbol{F}$ are body forces which we will assume to be negligible unless otherwise stated.
The Cauchy momentum equation is valid for all continuum fluids. To close the equation, we require an expression for σ.

Definition. The *stress tensor* σ is a symmetric second-rank tensor.

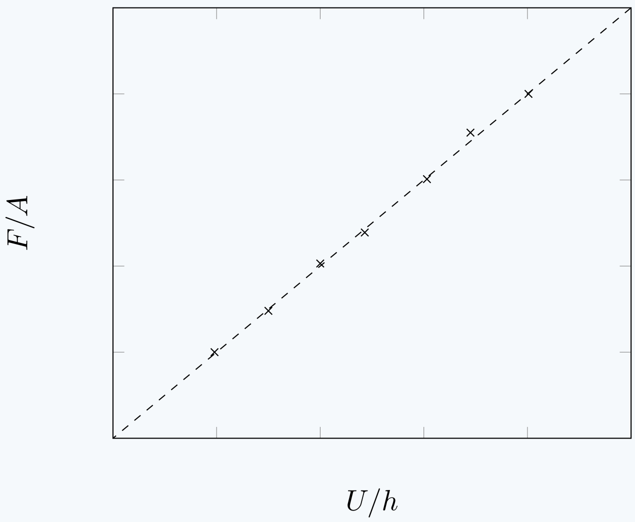


Physically, $\sigma_{ij} n_j$ is the i^{th} component of the force per unit area from the motion of fluid 1 on fluid 2, where $\boldsymbol{n}$ is the surface normal pointing into fluid 1.

2.4 Constitutive modelling

The stress tensor is specified in terms of the deformation via *constitutive modelling*.

Example. Newton's experiment
Consider two parallel plates of area A separated a distance h by a fluid. We consider the force F required on the top plate to induce motion at speed U. Note that there is no force perpendicular to the plates if the fluid is Newtonian. This is not necessarily true for complex fluids.



From experiment, we find $\sigma = F/A \propto U/h$. Note that U/h has dimensions of time^{-1}. We define the *shear rate* $\dot{\gamma} = U/h$ and the *viscosity* η via $\sigma = \eta\dot{\gamma}$. Viscosity is a constant material property, for example in water $\eta = 10^{-3}\,\text{Pa}\cdot\text{s}$.

We can now generalise for all Newtonian flows. We start by separating out an isotropic component of the shear tensor:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

where p is the *dynamic pressure* and τ is the *deviatoric stress*. The deviatoric stress may be a function of $u_i, \frac{\partial u_i}{\partial x_j}, \dots$; local or non-local in time or space; or a function of other material properties and parameters. For Newtonian fluids, we make five assumptions.

1. Galilean invariance: the deviatoric stress cannot depend on u_i

2. Instaneous response: there is no dependence on the history of deformation
3. Locality: no dependence on second or higher spatial derivatives
4. Linearity: τ_{ij} is linearly related to $\frac{\partial u_m}{\partial x_n}$
5. Isotropy: the relationship is independent of reference frame, i.e. isotropic

We can satisfy 1, 2, 3, and 4 by writing

$$\tau_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l}$$

where A_{ijkl} is a fourth rank tensor. Using the form of the most general isotropic fourth rank tensor we may enforce isotropy.

$$A_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}$$

Since σ is symmetric, τ is symmetric, therefore $A_{ijkl} = A_{jikl}$. This requires $B = C \equiv \eta$. Thus

$$\begin{aligned} \tau_{ij} &= A\delta_{ij}\frac{\partial u_k}{\partial x_k} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= 2\eta e_{ij} \end{aligned}$$

since $\frac{\partial u_k}{\partial x_k} = \nabla\cdot\boldsymbol{u} = 0$. Note $e = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the *rate of strain* tensor. This is the *Newtonian constitutive relationship.*

> **Definition.** To exclude the factors of 2 we define the *shear rate* $\dot\gamma_{ij} = 2e_{ij}$ so that $\tau_{ij} = \eta\dot\gamma_{ij}$.

Combining the Cauchy momentum equation and Newtonian constitutive relationship yields the *Navier-Stokes equations*

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \eta\nabla^2\boldsymbol{u}$$

Consider a general body with intrinsic length scale L, velocity length scale U and unsteady motion frequency scale ω. Two dimensionless numbers are used to quantify the importance of inertia:

$$\mathrm{Re} = \frac{\rho U L}{\eta}, \qquad \mathrm{Re}_\omega = \frac{\rho\omega L^2}{\eta}$$

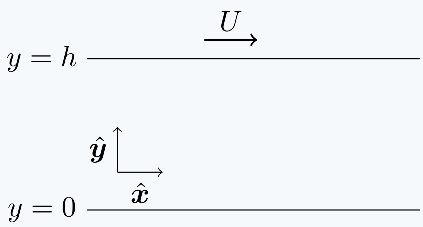
We will assume throughout this course that $\mathrm{Re} \ll 1$ and $\mathrm{Re}_\omega \ll 1$ unless otherwise stated. In this no-inertia limit, the Navier-Stokes equations simplify to the *Stokes equations.* For a general fluid these are:

$$\nabla\cdot\sigma = 0$$

Using the Newtonian constitutive relationship, the Stokes equations are:

$$\mathbf{0} = -\nabla p + \eta\nabla^2\boldsymbol{u}$$

Example. Newton's experiment revisited
We will apply the above theory to Newton's experiment and show the same results are obtained.



Assume the flow is unidirectional: $\boldsymbol{u} = u(y)\hat{\boldsymbol{x}}$. The Stokes equations become

$$\begin{cases} \frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2} & = \text{const.} \\ \frac{\partial p}{\partial y} = 0 & \implies p = p(x) \end{cases}$$

Assuming there is no net pressure drop, $p = p(x) \equiv 0$. Applying the boundary conditions we find

$$u(y) = Uy/h \implies \dot{\gamma} = U/h$$

as before. This is a *shear* or *Couette* flow.

Lecture 2
13/10/20

3 Phenomenology

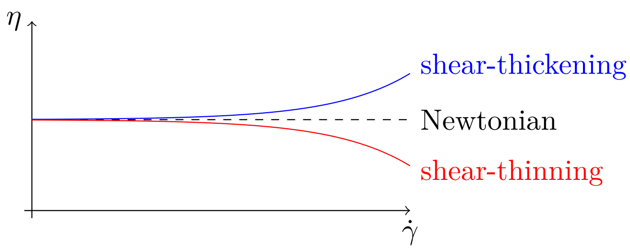
There are four distinguishing properties of non-Newtonian fluids which differ to Newtonian fluids.

3.1 Shear-dependent viscosity

Newtonian fluids have a constant viscosity η which is a constant of proportionality between shear stress σ and shear rate $\dot{\gamma}$. For a complex fluid, η is *not* constant. We re-define viscosity as an implicit function of shear rate.

Definition. Viscosity is defined via $\eta \equiv \frac{\sigma}{\dot{\gamma}} = \eta(\dot{\gamma})$.

We can broadly categorise complex fluids into two types: *shear thinning* and *shear thickening* fluids.



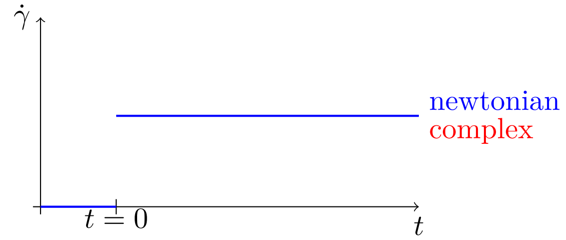
Examples of shear-thinning fluids are polymer suspensions; paint; blood. These complex fluids have $\frac{\partial \eta}{\partial \dot{\gamma}} < 0$.
Examples of shear-thickening fluids are: cornstarch in water; suspesions of colloidal particles. These complex fluids have $\frac{\partial \eta}{\partial \dot{\gamma}} > 0$.

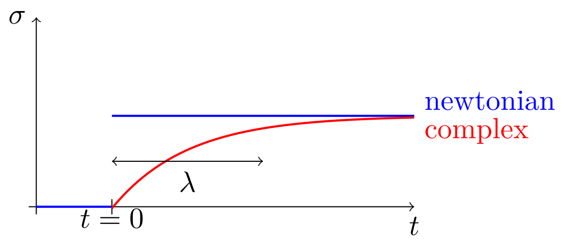
Definition. The zero-shear rate viscosity is $\eta_0 = \lim_{\dot\gamma\to 0}\eta$.

Note that some complex fluids have approximately constant viscosity (Boger fluids).

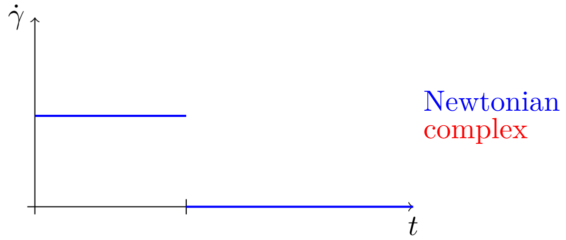
3.2 Fluid memory

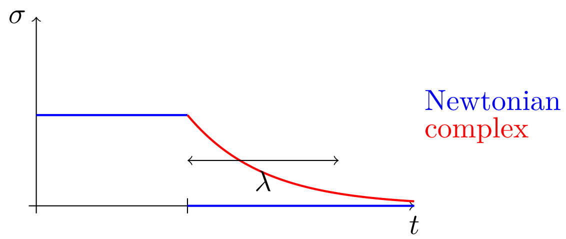
Consider a step-shear flow, that is, Newton's experiment where the top place impulsively starts motion at $t = 0^+$. The response of a Newtonian fluid is on an inertial timescale $\tau \sim h^2/\nu$ which is almost instantaneous, whilst a non-Newtonian fluid takes time to respond to adjust to the change in deformation: the jump occurs over some *relaxation timescale* λ.





The relationship between stress and deformation is *history dependent*. Impulsively removing the applied shear (i.e. the plate impulsively comes to rest) results in *stress relaxation* with $\sigma \sim e^{-t/\lambda}$.





The reverse is also true. Imposing a stress σ and measuring the deformation $\dot\gamma$, complex fluids in general wil have a history-dependent response. This is called *strain retardation* and λ' is the *retardation timescale*.

3.3 Normal stress differences

In Newton's experiment, a Newtonian fluid exerts no normal force F_N on the moving plate, since linearity and reversibility ($U \to -U$) impies $F_N = -F_N \equiv 0$. We previously calculated

$$\boldsymbol{u} = \frac{Uy}{h}\hat{\boldsymbol{x}}, \qquad \dot\gamma = \frac{U}{h}$$

Thus in the Newtonian case the stress tensor has the form

$$\sigma = \begin{pmatrix} -p_0 & \eta\dot\gamma & 0 \\ \eta\dot\gamma & -p_0 & 0 \\ 0 & 0 & -p_0 \end{pmatrix}$$

where p_0 is the external pressure. Note that the normal stresses are equal and constant.

In the non-Newtonian case we have

$$\sigma = \begin{pmatrix} \sigma_{xx} & \eta)(\dot{\gamma})\dot{\gamma} & 0 \\ \eta(\dot{\gamma})\dot{\gamma} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

where in general normal stresses are not equal and not constant. Normal stresses include the external pressure, so the relevant quantity is the difference between normal stresses.

Definition. The first and second *normal stress differences* are

$$N_1 = \sigma_{xx} - \sigma_{yy}, \qquad N_2 = \sigma_{yy} - \sigma_{zz}$$

Note the following.

- For a Newtonian fluid, $N_1 = N_2 = 0$
- Polymeric fluids (for example) have $N_1 > 0, N_2 < 0, |N_2/N_1| \sim 0.1$
- N_1 and N_2 are defined only for steady shear flow
- N_1 and N_2 will, in general, depend on $\dot{\gamma}$
- Boger fluids have constant viscosity but they have non-zero N_1 and N_2

Reversibility implies N_1 and N_2 have to be *even* functions of $\dot{\gamma}$. In the limit $\dot{\gamma} \to 0$, i.e. the Newtonian limit, we should have $N_1 \to 0, N_2 \to 0$. Thus the Taylor expansion of N_1, N_2 near $\dot{\gamma} = 0$ is

$$N_{1,2} = A_{1,2}\dot{\gamma}^2 + B_{1,2}\dot{\gamma}^4 + \dots$$

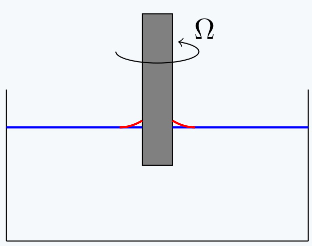
Definition. The *normal stress coefficients* are

$$\Psi_1 = \frac{N_1}{\dot{\gamma}^2}, \qquad \Psi_2 = \frac{N_2}{\dot{\gamma}^2}$$

The physical consequence of having normal stress differences is the introduction of elastic tension along flow streamlines.
Suppose $\sigma_{xx} = -p$. Thus compression $p > 0 \implies \sigma_{xx} < 0$ and $N_1 > 0 \implies \sigma_{xx} > 0$ which can be thought of as 'negative pressure' which acts as tension. An intuitive example is stretching of polymer molecules. This has many consequences on experiments and flow behaviour.

Example. Two examples of the consequences of normal stress differences are as follows.

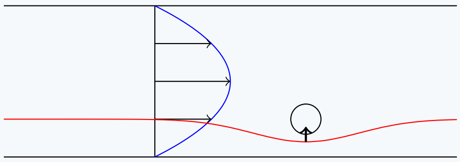
1. Rod-climbing (*Weissenberg effect*).



Consider a vertical rod rotating at a constant rate Ω placed into a fluid. In a Newtonian fluid, viscous enough for Stokes equations to apply, there is no change in the position of the interface (blue).

In a non-Newtonian fluid, the interface climbs up the rod (red). This is due to elastic tension: rotation creates circular streamlines. Tension along circles creates *hoop stress* which 'squeezes' the fluid and thus climbs up the rod.

2. Particle migration in a pipe flow.



In the case of Newtonian Stokes flow, the flow has no component perpendicular to the walls so a particle remains the same distance from the wall as it moves along the pipe. In the non-Newtonian case, hoop stress caused by curved streamlines lifts the particle away from the wall.

3.4 Extensional Viscosity

A shear flow is a *weak flow*: there is algebraic growth of distances between particles. In a *strong flow*, distances grow exponentially.
Consider a fluid with extension in the x and y directions and compression in the z direction: $\boldsymbol{u} + \dot{\varepsilon}\left(\frac{1}{2}x, \frac{1}{2}y, -z\right)$. We call $\dot{\varepsilon}$ the *extension rate*. Note that this flow is incompressible. The shear rate tensor is

$$\underline{\underline{\dot{\gamma}}} = \dot{\varepsilon}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

In the Newtonian case, the shear rate tensor is

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p_0 + \eta\dot{\varepsilon} & 0 & 0 \\ 0 & -p_0 + \eta\dot{\varepsilon} & 0 \\ 0 & 0 & -p_0 - 2\eta\dot{\varepsilon} \end{pmatrix}$$

Definition. The *extensional viscosity* is

$$\eta_{\text{ext}} = \frac{\sigma_{xx} - \sigma_{zz}}{\dot{\varepsilon}}$$

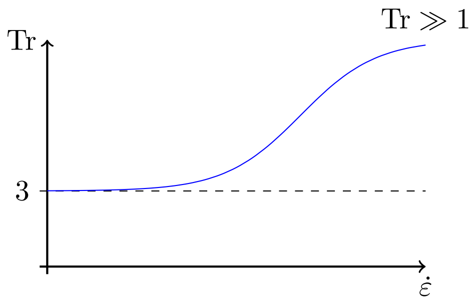
which has units of viscosity.

Thus for the Newtonian fluid, $\eta_{\text{ext}} = 3\eta$ which is constant.

Definition. The *Troutou ratio* is

$$\text{Tr} = \frac{\eta_{\text{ext}}}{\eta}$$

By the above calculations, Newtonian fluids have $\mathrm{Tr} = 3$. Non-Newtonian fluids tend to have $\mathrm{Tr} \gg 1$ in some range of shear rates. Thus complex fluids have a very different response to strong flows compared with weak flows.



4 Generalised Newtonian Fluids

We will focus on steady flows and fluids which are *inelastic*. We will find how to incorporate shear-dependent viscosity into the constitutive relationship. One way is to generalise the Newtonian constitutive relationship to

$$\underline{\underline{\sigma}} = -p\underline{\underline{1}} + \eta(\underline{\underline{\dot{\gamma}}})\underline{\underline{\dot{\gamma}}}$$

where $\eta(\underline{\underline{\dot{\gamma}}})$ is found empirically. These are called *generalised Newtonian fluids* (GNF).
We have a scalar function η of a tensor $\underline{\underline{\dot{\gamma}}}$, which is not in general coordinate invariant. Thus η must be a function of the *invariants* of $\underline{\underline{\dot{\gamma}}}$. A rank 2 tensor in 3 dimensions has 3 invariants. These are combinations of trace, determinants, and eigenvalues. We choose as the three invariants:

$$\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}}\right), \qquad \mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right), \qquad \mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right)$$

We always have $\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}}\right) = 0$ because the flow is incompressible:

$$\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}}\right) = \dot{\gamma}_{ii} = \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 2\nabla \cdot \boldsymbol{u} = 0$$

Consider a simple shear flow $\boldsymbol{u} = \dot{\gamma} y \hat{\boldsymbol{x}}$. Then

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}} = \begin{pmatrix} \dot{\gamma}^2 & 0 & 0 \\ 0 & \dot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\gamma}^3 & 0 \\ \dot{\gamma}^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right) = 0$ and $\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right) = 2\dot{\gamma}^2$. We assume all flows are approximately steady shear flow. Then $\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right)$ is the only non-zero invariant and $\eta(\underline{\underline{\dot{\gamma}}}) = \eta(\mathrm{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right))$.

Definition. The *magnitude of shear rate* is

$$\dot{\gamma} \equiv \left(\frac{\operatorname{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right)}{2}\right)^{1/2} = \left(\frac{\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}}{2}\right)^{1/2}$$

Note that $\dot{\gamma} \geq 0$ and for a simple shear flow the magnitude of shear rate $\dot{\gamma} = |\dot{\gamma}|$ which is the shear rate from steady shear flow. Thus the definitions coincide.

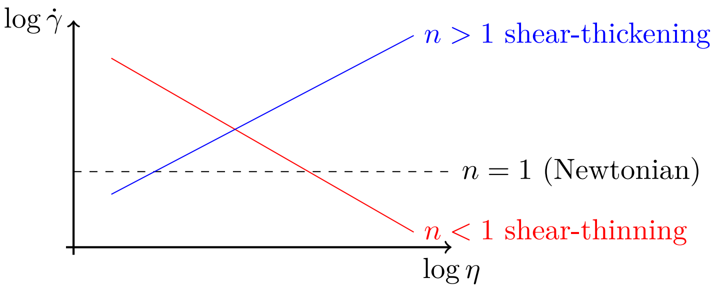
Note the second invariant $\operatorname{tr}\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\dot{\gamma}}}\right) = \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \geq 0$ and is zero only when there is no deformation (since $\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}$ is proportional to viscous dissipation), i.e. rigid body motion.
Recall for a simple shear flow, the shear stress is $\sigma = \eta(\dot{\gamma})\dot{\gamma}$, thus the above definitions agree with measurements of shear-dependent viscosity.

4.1 Power-law fluids

There are many choices for the function $\eta(\dot{\gamma})$ for a generalised Newtonian fluid. Here, we choose a *power-law*

$$\eta(\dot{\gamma}) \equiv \kappa \dot{\gamma}^{n-1}$$

where $\kappa > 0$ and $n \in \mathbb{Z}$ is the *power-index* of the fluid. Note $n = 1$ for a Newtonian fluid. For dimensional consistency, we require $[\kappa] = \mathrm{Pa} \cdot \mathrm{s}^n$.



Note that in the limit $\dot{\gamma} \to 0$, we cannot define a zero-shear rate viscosity η_0 unless $n = 1$. Thus the model is problematic at small shear rates. The model is appropriate only for a finite range of shear rates.

Example. Newton's experiment with a power-law fluid. We assume there are no external pressures, and the flow is unidirectional: $\boldsymbol{u} = u(y)\hat{\boldsymbol{x}}$. We have

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the magnitude of shear rate $\dot{\gamma} = \left|\frac{\partial u}{\partial y}\right|$. To find the flow, we use the Cauchy equation in 2D:

$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial x} = \frac{\partial \sigma_{xy}}{\partial y}$$

since σ_{xy} is the only non-zero component of the shear stress. The first equation implies $p = p(x)$ only, which combined with the second implies $\sigma_{xy} = \text{const.}$. Note we have not used the constitutive relationship yet: this is true for all GNFs. We have

$$\sigma_{xy} = \kappa\dot{\gamma}^{n-1}\dot{\gamma} = \kappa\dot{\gamma}^n = \kappa\left|\frac{\partial u}{\partial y}\right|^n = \text{const.}$$

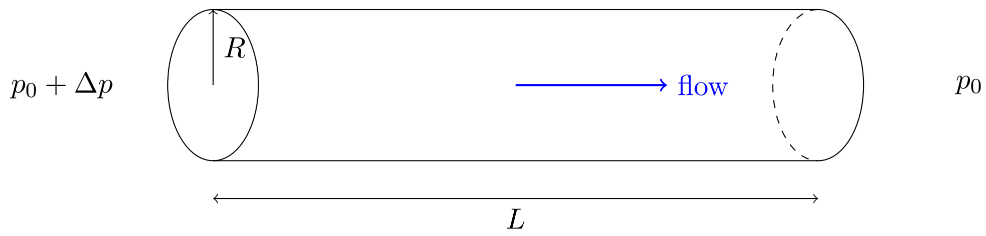
Thus u_y is constant, i.e. u is linear in y, as with a Newtonian fluid.

If $\eta(\dot{\gamma})\dot{\gamma} = \sigma$ is a one-to-one function of $\dot{\gamma}$ then the result is the same: u varies linearly. A Couette flow $\boldsymbol{u} = Uy/h\hat{\boldsymbol{x}}$ is a *viscometric flow*: this flow is realised for all constitutive relationships provided σ is indeed a one-to-one function of $\dot{\gamma}$.
How do experimentally measure $\eta(\dot{\gamma})$? One method is a *shear flow rheometer*:

1. Impose $\dot{\gamma}$, measure σ (or vice versa)
2. Measure $\eta = \sigma/\dot{\gamma}$
3. Repeat varying $\dot{\gamma}$ or σ

4.1.1 Pipe flow of a power-law fluid

Consider axisymmetric pressure-driven flow in a pipe of a power-law GNF. If the fluid was Newtonian, we would get Poiseuille flow with a parabolic flow profile.



We will use cylindrical coordinates (r, θ, z) and assume the flow is unidirectional: $\boldsymbol{u} = u(r)\hat{\boldsymbol{z}}$. Then

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & 0 & \frac{\partial u}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial r} & 0 & 0 \end{pmatrix}$$

The magnitude of shear rate $\dot{\gamma} = \left|\frac{\partial u}{\partial r}\right|$. The Cauchy equations in cylindrical coordinates are

$$\frac{\partial p}{\partial r} = 0$$
$$\frac{\partial p}{\partial \theta} = 0$$
$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})$$

since σ_{rz} is the only non-zero component of shear stress. The first two equations imply $p = p(z)$. Then $\frac{\partial p}{\partial z}$ is a function of z only, but the RHS is a function of r only. Thus each must be constant. Then

$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{L} \implies \sigma_{rz} = -\frac{\Delta p}{2L}r + \frac{A}{r}$$

The A/r term is singular at $r = 0$ thus $A \equiv 0$. Note these results are true for all guids - we have not yet used the fact this is a power-law fluid.
Denote the *magnitude of wall shear stress* as σ_w. In this case, $\sigma_w = \frac{\Delta p}{2L}R$. To find the flow field, we need the constitutive relationship. For a GNF we have

$$\sigma_{rz} = \eta\left(\left|\frac{\partial u}{\partial r}\right|\right)\frac{\partial u}{\partial r} = -\frac{\Delta p}{2L}r$$

We expect u to be at maximum when $r = 0$, so expect $\frac{\partial u}{\partial r} < 0$. Thus $|u_r| = -u_r$. For a power-law fluid, $\sigma = \kappa\dot{\gamma}^n$, thus

$$\begin{aligned}
\kappa\left|\frac{\partial u}{\partial r}\right|^n &= \frac{\Delta p}{2L}r \\
\implies \left|\frac{\partial u}{\partial r}\right| &= \left(\frac{\Delta p}{2L\kappa}\right)^{1/n} r^{1/n} = -\frac{\partial u}{\partial r} \\
\implies u(r) &= C - \left(\frac{\Delta p}{2L\kappa}\right)^{1/n}\frac{n}{n+1}r^{\frac{n+1}{n}}
\end{aligned}$$

Enforcing no-slip boundary conditions on the pipe wall $u(R) = 0$ and re-writing in terms of the wall shear stress we have

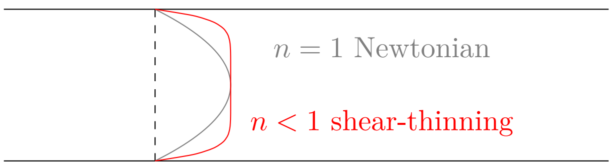
$$u(r) = \left(\frac{\sigma_w}{\kappa R}\right)^{1/n}\frac{n}{n+1}\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)$$

We can calculate the mean flow speed $\bar{U}$:

$$\bar{U} \equiv \frac{1}{\pi R^2}\iint u\,\mathrm{d}S = \frac{2}{R^2}\int_0^R ru(r)\,\mathrm{d}r = \left(\frac{\sigma_w}{\kappa}\right)^{1/n}\frac{nR}{3n+1}$$

Finally, we can rewrite the solution as flow relative to the mean flow speed:

$$\frac{u(r)}{\bar{U}} = \frac{3n+1}{n+1}\left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$



Physically, we have high shear near the pipe walls, so low viscosity in a shear-thinning complex fluid. The flow rate is

$$Q = \iint u\,\mathrm{d}S = \frac{\pi n}{3n+1}\left(\frac{\Delta p}{2L\kappa}\right)^{1/n} R^{3+\frac{1}{n}}$$

For a Newtonian fluid, $Q \sim \Delta p R^4$ whereas for a power-law fluid $Q \sim \Delta p^{1/n}R^{3+\frac{1}{n}}$. For a shear-thinning fluid with $n < 1$, we thus have a very strong dependence on Δp and R. In a device with Q fixed, $\Delta p^{1/n}R^{3+\frac{1}{n}} = \text{const.}$ so $\Delta p \sim R^{-(3n+1)}$. For $n < 1$, it is therefore easier to push fluid through a pipe than with a Newtonian fluid.

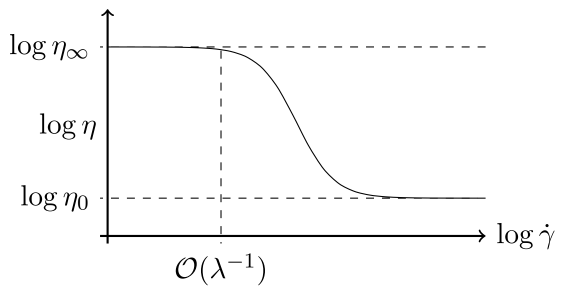
4.2 Other models & problems

4.2.1 Carreau-Yasuda model

The Carreau-Yasuda model can be written

$$\frac{\eta(\dot{\gamma}) - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\lambda\dot{\gamma})^a]^{n-1}\, a$$

for $n \leq 1$. In this model, η transitions smoothly from η_0 to η_∞. In some finite range of shear rates, the model resembles a power-law fluid. In fact, a power-law fluid can be thought of as a 'subset' of Carreau-Yasuda. The constant parameter a is related to the curvature of the η vs. $\dot{\gamma}$ curve, whilst n is related to the steepness of the curve.



4.2.2 Powell-Eyring & Ellis models

The Powell-Eyring model is given by

$$\frac{\eta(\dot{\gamma}) - \eta_\infty}{\eta_0 - \eta_\infty} = \frac{\sinh^{-1}(\lambda\dot{\gamma})}{\lambda\dot{\gamma}}$$

Some models specify $\eta(\sigma)$ instead of $\eta(\dot{\gamma})$, for example the Ellis model.

$$\eta = \eta_0 \left[1 + \left|\frac{\sigma}{\sigma_0}\right|^{1-n}\right]^{-1}$$

There are several properties and issues associated with generalised Newtonian fluids.

1. Empirical: GNFs are based on experiments rather than derived.

2. Instantaneous: GNFs have no memory (recall the step shear experiment), so are not suitable for modelling unsteady flows. This also means GNFs are instantaneously reversible which is undesired.

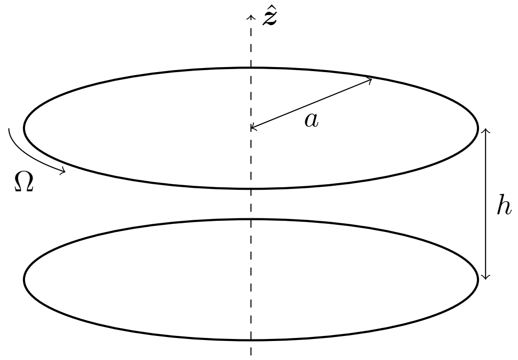
3. No normal stress differences.

4. Under extension, the Troutou ratio is $\mathrm{Tr} = 3$ for GNFs, i.e. there is no increase in extensional viscosity.

5. The behaviour of stress when $\dot{\gamma} \to 0$ is important. If η_0 can be defined, then $\sigma \to 0$. If not, such as with a power-law fluid, there is problematic behavoiur near 0. The limit $\dot{\gamma} \to 0$ is often used to recover Newtonian behaviour.

4.3 Rheometry

Rheometry is the science and engineering of measuring material properties of fluids. For generalised Newtonian fluids, we need to measure $\eta(\dot{\gamma})$. The 'easy' method is to use a steady shear flow, which is simple in principle. However, we can only use a finite volume of fluid. A common solution is the *parallel plate rheometer*, also known as a parallel disc rheometer.
Consider two coaxial rigid circular discs of radius a held at a constant separation h. The bottom plate is held stationary whilst the upper plate rotates at a prescribed angular velocity Ω. The variable $\dot{\gamma}$ is controlled in this experiment.



The input-output relationship can be inverted to infer $\eta(\dot{\gamma})$. Use cylindrical coordinates (r, θ, z) and look for an axisymmetric flow field $\boldsymbol{u} = u(r, z)\hat{\boldsymbol{\theta}}$. The shear rate tensor is

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) & 0 \\ r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) & 0 & \frac{\partial u}{\partial z} \\ 0 & \frac{\partial u}{\partial z} & 0 \end{pmatrix}$$

Consider the steady Cauchy equation in cylindrical coordinates:

$$\begin{aligned} \frac{\partial p}{\partial \theta} &= 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right) + \frac{\partial}{\partial z}\left(\tau_{\theta z}\right) \\ \frac{\partial p}{\partial r} &= 0 \\ \frac{\partial p}{\partial z} &= 0 \end{aligned}$$

We wish to find a viscometric flow solution. In the Newtonian case, we have

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right) + \frac{\partial^2 u}{\partial z^2}$$

To find a solution we require both terms vanish. Thus $u = A(r)z + B(r)$. The no-slip boundary condition $u = 0$ at $z = 0$ and $u = \Omega r$ at $z = h$ gives

$$u(r, z) = \Omega r \frac{z}{h}$$

Given this flow, the shear rate tensor becomes

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{r\Omega}{h} \\ 0 & \frac{r\Omega}{h} & 0 \end{pmatrix}$$

Assume $\Omega > 0$, so the magnitude of the shear rate is $\dot{\gamma} = |r\Omega/h|$. This solution is also a solution for a GNF: we have $\tau_{r\theta} = 0$ and

$$\tau_{\theta z} = \eta(\dot{\gamma})\dot{\gamma} = \eta(\frac{r\Omega}{h})\frac{r\Omega}{h} = \tau_{\theta z}(r)$$

The torque exerted to rotate the top plate is $\boldsymbol{T} = T\hat{\boldsymbol{z}}$ where

$$T = \int r\tau_{\theta z}\,\mathrm{d}S = 2\pi\int_0^a r^2\eta(\dot{\gamma})\dot{\gamma}\,\mathrm{d}r$$

This relationship can be used to infer $\eta(\dot{\gamma})$ by a change of variable. Let $\dot{\gamma}_a = a\Omega/h$ be the shear rate at the edge of the disc. Substitute $r = a\dot{\gamma}/\dot{\gamma}_a$:

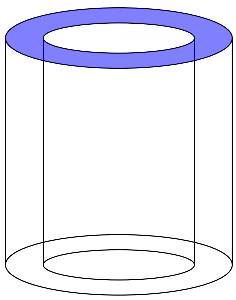
$$T = 2\pi\int_0^{\dot{\gamma}_a}\frac{a^2}{\dot{\gamma}_a^2}\dot{\gamma}^2\eta(\dot{\gamma})\dot{\gamma}\frac{a}{\dot{\gamma}_a}\,\mathrm{d}\dot{\gamma} = 2\pi\frac{a^3}{\dot{\gamma}_a^3}\int_0^{\dot{\gamma}_a}\eta(\dot{\gamma})\dot{\gamma}^3\,\mathrm{d}\dot{\gamma}$$

This is valid for all values of $\dot{\gamma}_a$. Rearranging and differentiating with respect to $\dot{\gamma}_a$ we have

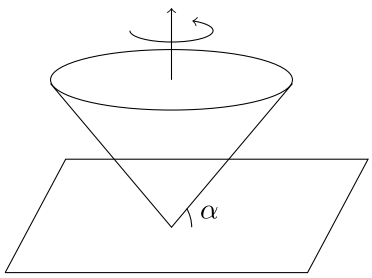
$$\eta(\dot{\gamma}_a) = \frac{1}{\dot{\gamma}_a^3}\frac{\mathrm{d}}{\mathrm{d}\dot{\gamma}^a}\left[\frac{T\dot{\gamma}_a^3}{2\pi a^3}\right]$$

In an experiment, we control $\dot{\gamma}_a$, measure T, slightly change $\dot{\gamma}_a$, and repeat. Application of the above result then yields $\eta(\dot{\gamma}_a)$. Other geometries can also be used:

- Taylor-Couette.



- Cone and plate. Useful since α small implies the shear rate $\dot{\gamma}$ is uniform.



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4.4 Variational approach

So far, we have found exact solutions only. For many GNF, no exact solutions are available so we must find approximate solutions. Here we consider a method relying on a form of energy minimisation. Conider a fluid volume V, bounded by a surface S, in the Stokes flow limit (no inertia). Assume $\boldsymbol{u} = \boldsymbol{u}_0$ is prescribed on S. The total rate of energy dissipation is

$$P = \int_V \frac{1}{2}\underline{\underline{\sigma}} : \underline{\underline{\dot{\gamma}}}\,\mathrm{d}V = \int_V \frac{\eta}{2}\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}\,\mathrm{d}V \geq 0$$

Recall the minimum dissipation theorem: solutions to Stokes equations minimise P over all incompressible flows satisfying the same boundary conditions. The inequality $P_{\text{Stokes}} \leq P$ can be shown directly, or we can use the calculus of variations. Define a Lagrangian

$$\mathcal{L} \equiv P + \int_S \lambda \nabla \cdot \boldsymbol{u} \, \mathrm{d}V$$

Variations with respect to λ give $\nabla \cdot \boldsymbol{u} = 0$, and variations with respect to $\boldsymbol{u}$ give Stokes equations with pressure $p \propto \lambda$.
To generalise this to GNF, consider the power density $f = \eta \dot{\gamma}^2$ for a Newtonian fluid. Analogously, for a GNF we define

$$f = \int_0^{\dot{\gamma}} \eta(x) x \, \mathrm{d}x$$

and a new Lagrangian

$$\mathcal{L} = \mathcal{L}[\boldsymbol{u}, \lambda] = \int_V f \, \mathrm{d}V + \int_V \lambda \nabla \cdot \boldsymbol{u} \, \mathrm{d}V$$

4.4.1 Calculus of variations

Consider a functional J of a vector field $\boldsymbol{y}(\boldsymbol{x})$ and of its spatial derivative.

$$J[\boldsymbol{y}] = \int_V F(x_i, y_j, y_{m,n}) \, \mathrm{d}V$$

where $y_{m,n} \equiv \frac{\partial y_m}{\partial x_n}$ and V is the total domain. The first variation of J is δJ where

$$J + \delta J = J[\boldsymbol{y} + \delta \boldsymbol{y}] = \int_V F(\boldsymbol{x}, \boldsymbol{y} + \delta \boldsymbol{y}) \, \mathrm{d}V$$

Taylor expand the integrand:

$$F(\boldsymbol{x}, \boldsymbol{y} + \delta \boldsymbol{y}) = F + \delta y_j \frac{\partial F}{\partial y_j} + \delta y_{m,n} \frac{\partial F}{\partial y_{m,n}} + \dots$$

Using integration by parts and writing $\delta y_{m,n} = \frac{\partial}{\partial x_n} \delta y_m$ we have

$$\delta y_{m,n} \frac{\partial F}{\partial y_{m,n}} = \frac{\partial}{\partial x_n} \left(\delta y_m \frac{\partial F}{\partial y_{m,n}} \right) - \delta y_m \frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial y_{m,n}} \right)$$

The first term becomes a surface integral via Stokes' theorem, which may or may not vanish depending on boundary conditions. The first variation of F is thus

$$\delta F = \delta y_j \left[\frac{\partial F}{\partial y_j} - \frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial y_{j,n}} \right) \right] + \text{boundary terms}$$

Now $\delta J = 0$ for all δy implies $\delta F = 0$ for all δy, giving the *Euler-Lagrange equation.*

$$\frac{\partial F}{\partial y_j} - \frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial y_{j,n}} \right) = 0$$

4.4.2 Variational solution

Returning to our original set-up, denote $F_2 = \lambda \nabla \cdot \boldsymbol{u}$, $F_1 = \int_0^{\dot{\gamma}} \eta(x) x \, \mathrm{d}x$, so

$$\mathcal{L}[\boldsymbol{u}, \lambda] = \int_V F_1 + F_2 \, \mathrm{d}V$$

The Euler-Lagrange equation for λ gives

$$\frac{\partial}{\partial \lambda} (F_1 + F_2) = 0 \implies \nabla \cdot \boldsymbol{u} = 0$$

For variations with respect to $\boldsymbol{u}$, we have boundary terms involving $\delta \boldsymbol{u}$. Since $\delta \boldsymbol{u} = \boldsymbol{0}$ on S, the boundary terms vanish in this case. There is no explicit dependence on $\boldsymbol{u}$ in F_1, F_2 so the Euler-Lagrange equation becomes

$$\frac{\partial}{\partial x_n} \left(\frac{\partial F}{\partial u_{j,n}} \right) = 0$$

where $F \equiv F_1 + F_2$. By chain rule,

$$\frac{\partial F_1}{\partial u_{j,n}} = \frac{\partial \dot{\gamma}}{\partial u_{j,n}} \frac{\partial F_1}{\partial \dot{\gamma}}$$

We have $\frac{\partial F_1}{\partial \dot{\gamma}} = \eta(\dot{\gamma})\dot{\gamma}$ and

$$\begin{aligned}
\frac{\partial \dot{\gamma}}{\partial u_{j,n}} &= \frac{1}{2\dot{\gamma}} \frac{\partial}{\partial u_{j,n}} \left(\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \right) \\
&= \frac{1}{2\dot{\gamma}} \frac{\partial}{\partial u_{j,n}} \left(\frac{1}{2} \dot{\gamma}_{pq} \dot{\gamma}_{pq} \right) \\
&= \frac{1}{2\dot{\gamma}} \frac{\partial}{\partial u_{j,n}} \left(u_{p,q} u_{p,q} + u_{p,q} u_{q,p} \right) \\
&= \frac{1}{2\dot{\gamma}} \left(2u_{j,n} + 2u_{n,j} \right) \\
&= \frac{\dot{\gamma}_{jn}}{\dot{\gamma}}
\end{aligned}$$

Now for $F_2 = \lambda \nabla \cdot \boldsymbol{u}$ we have

$$\begin{aligned}
\frac{\partial}{\partial x_n} \left(\frac{\partial F-2}{\partial u_{j,n}} \right) &= \frac{\partial}{\partial x_n} \left(\lambda \frac{\partial}{\partial u_{j,n}} \frac{\partial u_i}{\partial x_i} \right) \\
&= \frac{\partial}{\partial x_n} \left(\lambda \delta_{ij} \delta_{in} \right) \\
&= \frac{\partial \lambda}{\partial x_j}
\end{aligned}$$

Thus the full Euler-Lagrange equation for $\boldsymbol{u}$ is

$$\frac{\partial}{\partial x_n} \left[\eta(\dot{\gamma}) \dot{\gamma}_{jn} \right] + \frac{\partial \lambda}{\partial x_j} = 0$$

for $j = 1, 2, 3$. Given $\lambda = -p$, this becomes the Cauchy equation for a GNF.

$$-\nabla p + \nabla \cdot \left[\eta(\dot{\gamma}) \underline{\underline{\dot{\gamma}}} \right] = \boldsymbol{0}$$

Pressure is used as a Lagrange multiplier to enforce incompressibility. Note: if the boundary conditions are different, e.g. $\underline{\underline{\sigma}} \cdot \boldsymbol{n}$ is prescribed, we must distinguish surfaces where $\boldsymbol{u}$ is described and surfaces where $\underline{\underline{\sigma}} \cdot \boldsymbol{n}$ is prescribed, and add a surface integral to the Lagrangian.
To generate approximate solutions to the Cauchy GNF equations, we note that the closer the value of $\mathcal{L}$ to its minimum, the better the approximation. Thus

1. Use incompressible test functions $\boldsymbol{u}_\alpha$ where α is a parameter
2. Substitute into Lagrangian, minimise with respect to α
3. If the minimum of $\mathcal{L}$ is at $\alpha = \alpha^*$ then $\boldsymbol{u}_{\alpha^*}$ is an approximate solution

Integrals involved in $\mathcal{L}$ in general have to be evaluated numerically. Choice of test functions comes from physical intuition, computation, or experiment.