Cambridge Part III Maths

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Hydrodynamic Stability

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Notes created based on Josh Kirklin's LaTeX packages & classes. Please do not distribute these notes other than to fellow Part III students. Please send errors and suggestions to cwp29@cam.ac.uk.

Contents

1 Introduction

We are typically interested in whether a given flow solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{u}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t)$ and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2}\hat{\boldsymbol{u}}^2(\boldsymbol{x}, t)\,\mathrm{d}V$$

A solution is said to be stable if

$$\lim_{t\to\infty} \frac{E(t)}{E(0)} = 0$$

for all perturbations $\hat{\boldsymbol{u}}$. Conversely, if there exists $\hat{\boldsymbol{u}}$ such that $E(t) \nrightarrow 0$ then $\boldsymbol{u}$ is unstable. The nature of $E(0)$ determines the type of perturbation:

- If $E(0) \to 0$ we have an infinitesimal disturbance
- If $E(0) < \delta$ then we probe finite amplitude disturbances
- If $E(0) \to \infty$ this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}$$

If $\boldsymbol{U}(\boldsymbol{x})$ is a steady (basic) solution then

$$\boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla P = \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{U}$$

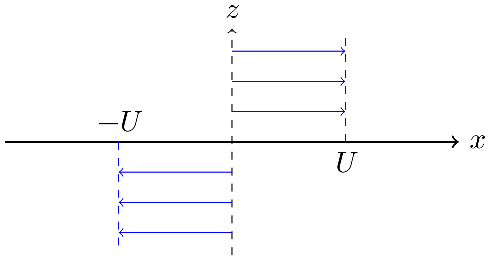
Let $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t), p = P + \hat{p}$. Then

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\hat{\boldsymbol{u}} + \cancel{\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}} + \nabla\hat{p} = \frac{1}{\mathrm{Re}}\nabla^2\hat{\boldsymbol{u}}$$

The term $\hat{u}\cdot\nabla\boldsymbol{U}$ is stabilising whilst the term $\nabla^2\hat{u}/\mathrm{Re}$ is destabilising. Therefore, we expect stability as $\mathrm{Re} \to 0$ the stabilising term dominates, and instability as $\mathrm{Re} \to \infty$ when the destablising term dominates. Thus there exists some value $\mathrm{Re}_{\mathrm{crit}}$ at which instability arises. We will ask what this value is, and what is the form of the initial instability/mode/pattern?

2 Kelvin-Helmholtz instability

See Drazin (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazin).



Consider a flow $\boldsymbol{u} = U(z)\hat{\boldsymbol{x}}$ where

$$U(z) = \begin{cases} U & z > 0 \\ -U & z < 0 \end{cases}$$

The linearised, *inviscid* equation for perturbation $\hat{\boldsymbol{u}}$ is

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{w}U'\hat{\boldsymbol{x}} + U\frac{\partial \hat{\boldsymbol{u}}}{\partial x} + \nabla \hat{p} = 0$$
$$\nabla \cdot \hat{\boldsymbol{u}} = 0$$

The boundary conditions are $\hat{\boldsymbol{u}} \to 0$ as $z \to \pm\infty$, i.e. no energy is radiated in from infinity. We will work in 2D with velocity components $(\hat{u}, \hat{w}) = (\psi_z, -\psi_x)$ and let $\psi(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$ where c is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U-c)\begin{pmatrix} \phi' \\ -i\alpha\phi \end{pmatrix} + \begin{pmatrix} -i\alpha U'\phi \\ 0 \end{pmatrix} + \begin{pmatrix} i\alpha p \\ \frac{\partial p}{\partial z} \end{pmatrix} = 0$$

We can eliminate p via $\partial_z(\text{top}) - i\alpha(\text{bottom})$ to get

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

with boundary conditions $\phi \to 0$ as $z \to \pm\infty$. This is *Rayleigh's equation.* Note that c is the crucial eigenvalue. We wish to know when $c_i = \Im(c) > 0$ as a function of $U(z)$, as c_i is the growth rate:

$$\hat{u} \propto e^{i\alpha(x-ct)} = e^{i\alpha(c-c_r t - ic_i t)} = e^{i\alpha(x-c_r t)+\alpha c_i t}$$

Note the following:

- There is a symmetry $\alpha \mapsto -\alpha$, so without loss of generality we consider $\alpha > 0$.
- The complex conjugate is also a solution with $c \mapsto c^*$. Hence an unstable mode has a damped partner, so we have stability only if all modes are 'neutral' i.e. $c_i = 0$.
- There is a possible singularity at y where $U(y) = c$, called the *critical layer.* If c is real, see later.

We now solve Rayleigh's equation with $U(z)$ defined as before. We solve above and below $z = 0$ and piece the solutions together. Since $U'' = 0$, we have

$$\phi'' = \alpha^2\phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ Be^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at $z = 0$ are

1. Pressure $\hat{p}$ continuous at $z = 0$, with $\hat{p}$ given by:

$$\hat{p} = U'\phi - (U-c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z-\zeta(x,t)\right)=0$$

where $z=\zeta(x,t)$ is the position of the surface. After linearising, we have

$$w-\frac{\partial\zeta}{\partial t}-U\frac{\partial\zeta}{\partial x}=0$$

Inserting the form of w and U we require that

$$\zeta=-\frac{\phi}{U-c}$$

is continuous across $z=0$.

Requiring p continuous gives

$$-(U-c)A(-\alpha)=-(-U-c)B(\alpha)$$

Requiring ζ continuous gives

$$\frac{A}{U-c}=\frac{B}{-U-c}$$

Hence we have

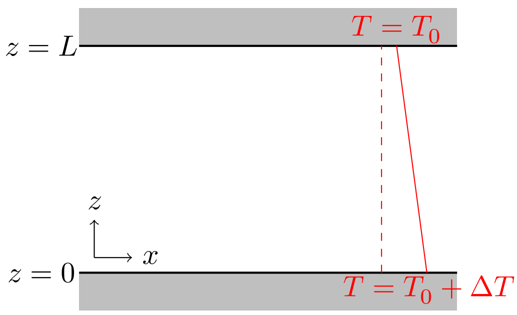
$$(U-c)^2=-(U+c)^2$$

i.e. $c=\pm iU$ so the growth rate is αU. Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small α)
- by adding surface tension, which stabilises short wavelengths (large α), e.g. Drazin page 50 equation 3.21.

3 Thermal instabilities: Rayleigh-Bernard convection

Consider two parallel plates separated by distance L with fluid subject to gravity and temperatue difference ΔT between the plates. The lower plate is heated to $T_0+\Delta T$ whilst the upper plate is fixed at temperature T_0.



The basic state consists of no motion, with heat transfer by conduction only.

Governing equations. The governing equations are those of momentum, mass, and (thermal) energy conservation.

$$\begin{aligned}\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\nabla p&=\mu\nabla^2\boldsymbol{u}+\rho g\hat{\boldsymbol{z}}\\ \frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T&=\kappa\nabla^2 T\\ \frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{u}&=0\end{aligned}$$

To close the set of equations we need a relationship between ρ and T. Most cases of interest have ΔT and $\Delta\rho$ small, i.e. $\Delta\rho\ll\rho_0, \Delta T\ll T_0$. Two consequences of this assumption are:

1. We can Taylor expand $\rho=\rho(T)$:

$$\rho\approx\rho(T_0)\left[1-\alpha(T-T_0)\right]$$

   where $\alpha>0$ is the coefficient of thermal expansion, such that T increases when ρ decreases. We write $\rho_0=\rho(T_0)$.

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term $\rho g\hat{\boldsymbol{z}}$. Importantly, we can assume the fluid is incompressible.

Define $\theta=T-T_0$. The governing equations are now

$$\begin{aligned}\rho_0\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\nabla p&=\mu\nabla^2\boldsymbol{u}+\rho_0(1-\alpha\theta)g\hat{\boldsymbol{z}}\\ \frac{\partial\theta}{\partial t}+\boldsymbol{u}\cdot\nabla\theta&=\kappa\nabla^2\theta\\ \nabla\cdot\boldsymbol{u}&=0\end{aligned}$$

The basic state is $u=0,\theta=\Delta T(1-z/L)$ and

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\rho_0(1-\alpha\Delta T(1-z/L))g$$

We now non-dimensionalise using scalings $t\sim L^2/\kappa, u\sim\kappa/L,\theta\sim\Delta T$, e.g. $\theta=\Delta T\theta^*$ where θ^* is the non-dimensionalised variable. We normalise the $\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}$ term, to get:

$$\begin{aligned}\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}+\nabla^*p^*&=\frac{\mu}{\rho_0\kappa}\nabla^{*2}\boldsymbol{u}^*+\frac{\alpha g\Delta TL^3}{\kappa^2}\theta^*\hat{\boldsymbol{z}}\\ \frac{\partial\theta^*}{\partial t^*}+\boldsymbol{u}^*\cdot\nabla^*\theta^*&=\nabla^{*2}\theta^*\end{aligned}$$

Define the *Prandtl number*

$$\sigma\equiv\frac{\nu}{\kappa}=\frac{\mu}{\rho_0\kappa}$$

which is the ratio of viscous/momentum diffusion to thermal diffusion. Typical values are 0.72 in air, 7 in water, 10^5 in magma. We also define the *Rayleigh number*

$$\mathrm{Ra}\equiv\frac{\alpha\Delta TgL^3}{\kappa\nu}$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the * notation, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \sigma\nabla^2\boldsymbol{u} + \sigma \mathrm{Ra}\theta\hat{\boldsymbol{z}}$$
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \nabla^2\theta$$
$$\nabla\cdot\boldsymbol{u} = 0$$

Boundary conditions. There are three combinations of boundary condition available in this problem, with the choice fixed wall (no slip) or stress free (free slip).

$\theta = 0$, $z = 1$	Fixed wall: $\boldsymbol{u} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$, $z = 0$	$\boldsymbol{u} = 0$: Fixed wall	$\boldsymbol{u} = 0$: Fixed wall	$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$: Free slip

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

Basic state. In the basic state we have conductive profile $\boldsymbol{u}_0 = 0, \theta_0 = 1 - z$ and from integration $p_0 = \sigma\mathrm{Ra}(z - \frac{1}{2}z^2)$. We generate linearised equations for perturbations $\theta = \theta_0 + \theta', \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', p = p_0 + p'$. As usual with linear stability analysis, we assume $(\theta', \boldsymbol{u}', p')$ are small.

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \cancel{\boldsymbol{u}'\cdot\nabla\boldsymbol{u}'} + \nabla p' = \sigma\nabla^2\boldsymbol{u}' + \sigma\mathrm{Ra}\theta'\hat{\boldsymbol{z}}$$
$$\frac{\partial \theta'}{\partial t} - w' + \cancel{\boldsymbol{u}'\cdot\nabla\theta'} = \nabla^2\theta'$$
$$\nabla\cdot\boldsymbol{u}' = 0$$

Dropping the $'$ notation for clarity we have perturbation equations

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{u} + \nabla p = \sigma\mathrm{Ra}\theta\hat{\boldsymbol{z}} \tag{1}$$
$$\nabla\cdot\boldsymbol{u} = 0 \tag{2}$$
$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = w \tag{3}$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g. $\theta = \theta_0 + \theta' = 1$ at $z = 0$ combined with $\theta_0 = 1$ at $z = 0$ gives $\theta' = 0$. Similarly, $\theta' = 0$ at $z = 1$ and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.
From $\nabla\times$(1) we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{\omega} = \sigma\mathrm{Ra}\nabla\times\theta\hat{\boldsymbol{z}}$$

Taking the curl again and using $\nabla\times\boldsymbol{\omega} = \nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$ we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2\boldsymbol{u}) = \sigma\mathrm{Ra}\nabla\times(\nabla\times\theta\hat{\boldsymbol{z}}) = \sigma\mathrm{Ra}\left(\nabla\frac{\partial\theta}{\partial z} - \hat{\boldsymbol{z}}\nabla^2\theta\right)$$

The z component is

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2 w) = \sigma \mathrm{Ra} \nabla_H^2 \theta \tag{4}$$

where $\nabla_H^2 = \partial_x^2 + \partial_y^2$. Now (3) can be used to eliminate θ by applying the operator $(\partial_t - \nabla^2)$:

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \sigma \mathrm{Ra} \nabla_H^2 w \tag{5}$$

This is a 6^{th} order PDE for w, hence we need three boundary conditions at each wall $z = 0, 1$. We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad \text{at } z = 0, 1$$

The second set of conditions comes from incompressibility. Taking $\partial_z(\nabla \cdot \boldsymbol{u})$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right) + \frac{\partial^2 w}{\partial z^2} = 0 \implies w_{zz} = 0$$

The third and final set of conditions comes from requiring $\theta = 0$ at $z = 0, 1$. From (4), $\nabla_H^2 \theta = 0$ implies

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\nabla^2 w = 0$$

We now have 6 boundary conditions to supplement the PDE.

Normal mode solution. Seek a solution $w(x, y, z, t) = W(z)e^{ik_1 x + ik_2 y + \lambda t}$ where k_1, k_2 are wavenumbers and $\lambda \in \mathbb{C}$ is the growth rate. Write $D = \mathrm{d}/\mathrm{d}z$ and $k = \sqrt{k_1^2 + k_2^2}$ since the problem is rotationally symmetric in the (x, y) plane. Substituting into (5) we have

$$(\lambda - [D^2 - k^2])(\lambda - \sigma[D^2 - k^2])(D^2 - k^2)W = -\sigma \mathrm{Ra} k^2 W$$

with boundary conditions at $z = 0, 1$:

$$\begin{aligned} W(0) &= W(1) = 0 \\ D^2 W(0) &= D^2 W(1) = 0 \\ [\lambda - \sigma(D^2 - k^2)]\,[D^2 - k^2]\,W = 0 \implies D^4 W(0) &= D^4 W(1) = 0 \end{aligned}$$

The objective is to find

$$\max_k \Re\{\lambda(k; \mathrm{Ra}, \sigma)\}$$

The onset of linear instability (for a given σ) at $\mathrm{Ra} = \mathrm{Ra}_{\text{crit}}$ is defined by

$$\max_k \Re\{\lambda(k; \mathrm{Ra}_{\text{crit}}, \sigma)\} = 0$$

In general, $\lambda \in \mathbb{C}$, but for this problem it can be proven that at marginality $\Im(\lambda) = 0$ as well as $\Re(\lambda) = 0$; a condition called the *principle of exchange of stabilities*. Hence setting $\lambda = 0$ in the above, we get

$$(D^2 - k^2)^3 W = -\mathrm{Ra}\, k^2 W \tag{6}$$

Note that σ drops out of the problem! It's easy to see $W(z) = \sin(n\pi z)$ solves (6) and satisfies the free-slip BCs. Hence

$$(n^2\pi^2 + k^2)^3 = \mathrm{Ra}\, k^2$$

Criticality is then given by

$$\mathrm{Ra}_{\mathrm{crit}} = \min_{n,k} \frac{(n^2\pi^2 + k^2)^3}{k^2}$$

We find the minimum in the usual way:

$$\begin{aligned}
\frac{\partial \mathrm{Ra}}{\partial k} &= \frac{3(2k)(n^2\pi^2 + k^2)^2 k^2 - 2k(n^2\pi^2 + k^2)^3}{k^4} \\
&= \frac{2k(n^2\pi^2 + k^2)^2(3k^2 - (n^2\pi^2 + k^2))}{k^4} = 0 \\
\implies 2k^2 &= n^2\pi^2 \\
\implies k &= \frac{n\pi}{\sqrt{2}}
\end{aligned}$$

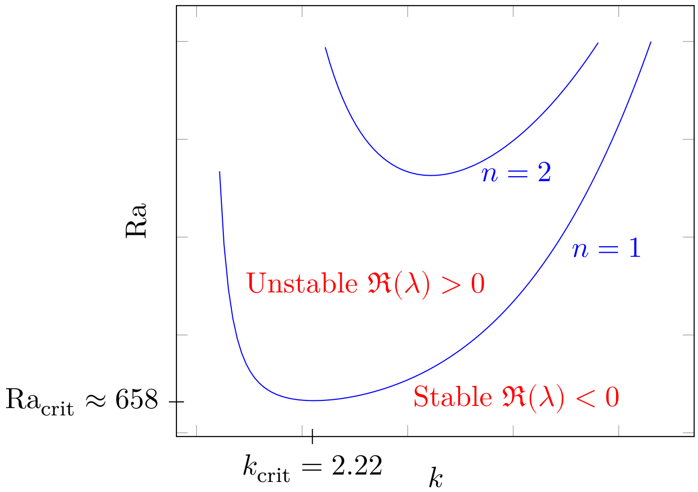
Given $k = n\pi/\sqrt{2}$ the Rayleigh number is

$$\mathrm{Ra}(k = \frac{n\pi}{\sqrt{2}}) = \frac{(n^2\pi^2 + \frac{1}{2}n^2\pi^2)^3}{n^2\pi^2/2} = \frac{27}{4}n^4\pi^4$$

Clearly the critical Rayleigh number is given by $n = 1$, hence

$$\begin{aligned}
\mathrm{Ra}_{\mathrm{crit}} &= \frac{27}{4}\pi^4 \sim 658 \\
k_{\mathrm{crit}} &= \frac{\pi}{\sqrt{2}} \sim 2.22
\end{aligned}$$

Thermal convection Rayleigh number



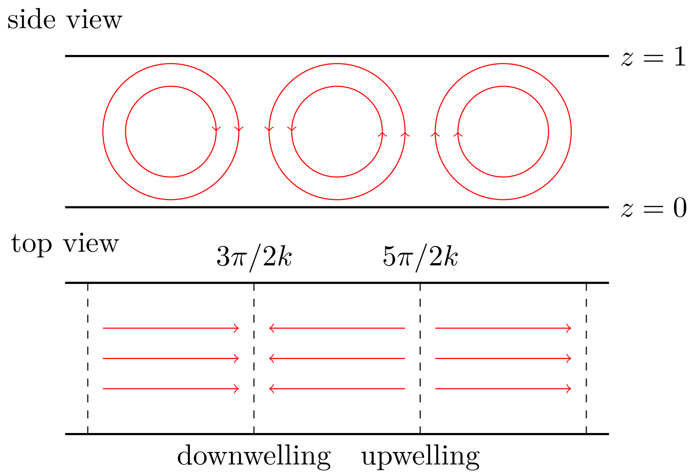
Results for other boundary conditions are:

- Free–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1101, k_c = 2.68$
- Rigid–rigid boundary: $\mathrm{Ra}_{\mathrm{crit}} \sim 1708, k_c = 3.117$

Notice that at criticality only the size of k is specified, *not* its direction. Hence there are an infinite number of possibilities $\boldsymbol{k} = (k\cos\phi, k\sin\phi)$. Various different patterns which tesselate are as follows.

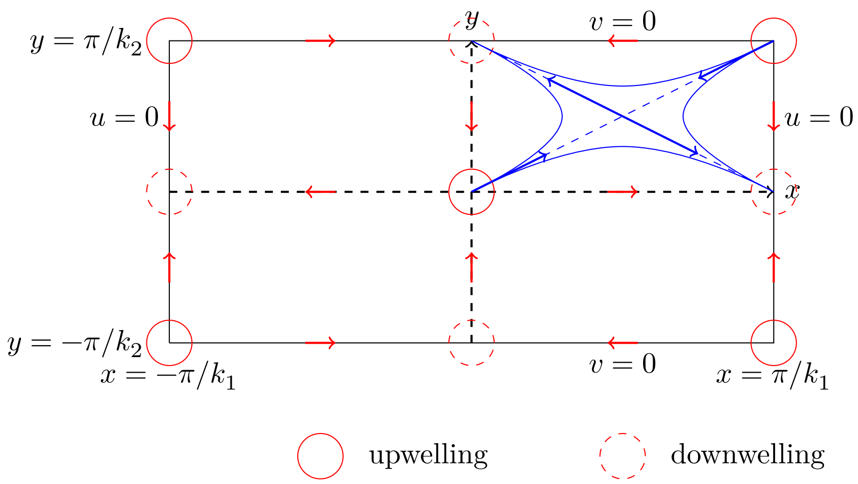
1. **2D rolls.** Orientate x-axis along k such that $k_2 = 0$. We have velocity components (w specified in problem, u follows from incompressibility)

$$w = W(z)\sin kx$$
$$v = 0$$
$$u = \frac{\pi \cos \pi z \cos kx}{k}$$



2. **Rectangles.** Velocity components are

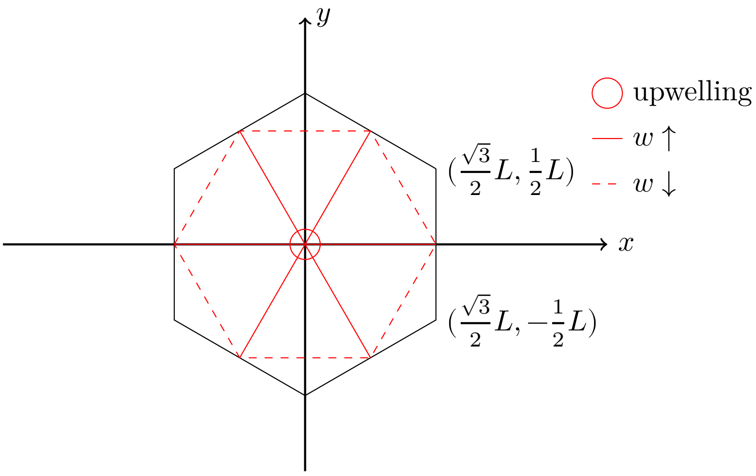
$$w = W(z)\cos k_1 x \cos k_2 y$$
$$v = -\frac{k_2}{k^2} W' \cos k_1 x \sin k_2 y$$
$$u = -\frac{k_1}{k^2} W' \sin k_1 x \cos k_2 y$$



3. **Hexagons.** Vertical velocity component

$$w = W(z)\left[\cos\frac{k}{2}(\sqrt{3}x + y) + \cos\frac{k}{2}(\sqrt{3}x - y) + \cos ky\right]$$

This is flow in a hexagon of side length $L = 4\pi/3k$.



4 Centrifugal instabilities

Flows with curved streamlines can be unstable due to centrifugal effects.

4.1 Rayleigh's criterion

We will concentrate on axisymmetric flows. Consider an azimuthal flow

$$\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}} = r\Omega(r)\hat{\boldsymbol{\theta}}$$

The inviscid, axisymmetric equations for a general flow $\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}}$ are

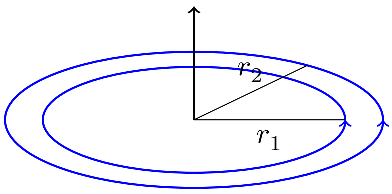
$$\begin{aligned}
\frac{\partial u_r}{\partial t} + \boldsymbol{u}\cdot\nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + \boldsymbol{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} &= \cancel{-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}} \\
\frac{\partial u_z}{\partial t} + \boldsymbol{u}\cdot\nabla u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned}$$

where $\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial}{\partial \theta}} + u_z\frac{\partial}{\partial z}$. Cancelled terms are absent in the axisymmetric setting. The *centrifugal* term is $-u_\theta^2/r$ in the r-momentum equation. The θ-momentum equation can be rearranged, and multiplied by r to give a material conservation equation:

$$\begin{aligned}
\frac{\partial}{\partial r}(ru_\theta) + ru_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial}{\partial z}(ru_\theta) + r\left(\frac{u_r u_\theta}{r}\right) &= 0 \\
\implies \frac{\partial}{\partial t}(ru_\theta) + u_r\frac{\partial}{\partial r}(ru_\theta) + u_z\frac{\partial}{\partial z}(ru_\theta) &= 0 \\
\implies \frac{\mathrm{D}}{\mathrm{D}t}(ru_\theta) &= 0
\end{aligned}$$

This expresses conservation of angular momentum: the angular momentum per unit mass is $I = ru_\theta$, hence $\frac{\mathrm{D}I}{\mathrm{D}t} = 0$. This result also follows from Kelvin's circulation theorem, using the circulation $\Gamma = 2\pi ru_\theta$ for an inviscid fluid. The statement says that if $\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}}$ (i.e. axisymmetric azimuthal flow) then $I = I(r)$ is a basic state.

What distributions of $I(r)$ could be stable? Rayleigh's argument considers 2 rings of fluid at radius r_1 and $r_2(> r_1)$ respectively.



The kinetic energy is

$$E = \frac{1}{2}\rho\left(\frac{I_1^2}{r_1^2} + \frac{I_2^2}{r_2^2}\right)$$

Now suppose the rings swap places due to a perturbation, but they keep their angular momentum (since it is materially conserved). The new KE is

$$E_{\text{new}} = \frac{1}{2}\left(\frac{I_2^2}{r_1^2} + \frac{I_1^2}{r_2^2}\right)$$

Hence the swap has resulted in an energy change

$$\Delta E = (I_2^2 - I_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

We can expect instability if $\Delta E < 0$. Since $r_2 > r_1$, the second factor is positive hence

$$\Delta E < 0 \iff I_2^2 < I_1^2$$

Hence Rayleigh's criterion for stability is $I_2^2 \geq I_1^2$ or equivalently

$$\frac{\mathrm{d}I^2}{\mathrm{d}r} \geq 0$$

i.e. angular momentum does not increase outwards. Note that with $I = ru_\theta = r^2\Omega$ we have the condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) \geq 0$$

for stability. This is often written using the *Rayleigh determinant*

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right)$$

Hence stability is predicted if $\Phi \geq 0$.

4.2 Derivation via linear stability analysis

Consider Taylor-Couette geometry: cylindrical walls at r_1 and r_2 with an inviscid base state $\boldsymbol{u} = r\Omega(r)\hat{\boldsymbol{\theta}}$, with axisymmetric perturbations $\boldsymbol{u}'$. We have incompressibility

$$\nabla \cdot \boldsymbol{u}' = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(ru_r') + \frac{\partial u_z'}{\partial z} = 0$$

The Euler equations for this perturbation are

$$\begin{aligned}
\frac{\partial u_r'}{\partial t} - \frac{2r\Omega u_\theta'}{r} &= -\frac{1}{\rho}\frac{\partial p'}{\partial r} \\
\frac{\partial u_\theta'}{\partial t} + u_r'\frac{\mathrm{d}}{\mathrm{d}r}(r\Omega) + \frac{u_r' r\Omega}{r} &= 0 \\
\frac{\partial u_z'}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial z}
\end{aligned}$$

Now specify normal mode decomposition

$$\begin{pmatrix} u_r' \\ u_\theta' \\ u_z' \\ p' \end{pmatrix} = \begin{pmatrix} \hat{u}_r(r) \\ \hat{u}_\theta(r) \\ \hat{u}_z(r) \\ \hat{p}(r) \end{pmatrix} e^{ikz+\sigma t}$$

Only axisymmetric perturbations are considered. The Euler equations become

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r) + ik\hat{u}_z &= 0 \\
\sigma\hat{u}_r - 2\Omega\hat{u}_\theta &= -\frac{1}{\rho}\frac{\mathrm{d}\hat{p}}{\mathrm{d}r} \\
\sigma\hat{u}_\theta + \hat{u}_r(\Omega + (r\Omega)_r) &= 0 \\
\sigma\hat{u}_z &= -\frac{1}{\rho}ik\hat{p}
\end{aligned}$$

We can reduce this system down to a single equation for $\hat{u}_r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\right)\hat{u}_r - k^2\hat{u}_r - 2\frac{k^2}{\sigma^2}\Omega(2\Omega + r\Omega')\hat{u}_r = 0$$

This is a second order ODE for $\hat{u}_r$ with BCs $\hat{u}_r = 0$ at $r = r_1, r_2$. For this flow, Rayleigh's determinant is

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = 4\Omega^2 + 2r\Omega'\Omega$$

Hence the ODE for $\hat{u}_r$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right) - k^2\hat{u}_r = \frac{k^2}{\sigma^2}\Phi(r)\hat{u}_r \tag{7}$$

Multiply (7) by $r\hat{u}_r^*$ (complex conjugate) and integrate from r_1 to r_2:

$$\int_{r_1}^{r_2} r\hat{u}_r^*\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term may be integrated by parts to give:

$$\cancel{\left[r\hat{u}_r^*\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right]_{r_1}^{r_2}} - \int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r^*)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term vanishes since $\hat{u}_r = 0$ at $r = r_1, r_2$. Labelling the first integral as $H_1 > 0$ and the second as $H_2 > 0$, we have

$$\frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r = -H_1 - k^2H_2 < 0$$

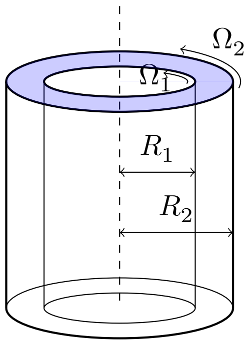
If $\Phi \geq 0$ then $\sigma^2 < 0$, i.e. σ is imaginary and we have stability. If instead $\Phi < 0$ somewhere in the domain, then potentially

$$\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2 \,\mathrm{d}r < 0$$

in which case $\sigma^2 > 0$ and we have instability. Hence $\Phi < 0$ somewhere in the domain is *necessary* (but not sufficient) condition for instability. So this formal analysis confirms Rayleigh's heuristic criterion. Note, really we need to consider non-axisymmetric perturbations too.

4.3 Taylor vortices

Apply Rayleigh's criterion to Taylor-Couette flow.



When viscosity is present, the general solution with $\partial_\theta = \partial_z = 0$ is

$$u_\theta(r) = Ar + \frac{B}{r}$$

No-slip boundary conditions at $r = R_1, R_2$ give

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}}$$

Note this solves $(\nabla^2 - 1/r^2)u_\theta = 0$ where $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r)$. In this case $\Omega = u_\theta/r = A + B/r^2$ hence Rayleigh's determinant is

$$\Phi = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4\left(A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}\right)\right] = 4A^2\left(1 + \frac{B}{Ar^2}\right)$$

For convenience we define $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2 < 1$. Then

$$\Phi = 4A^2\left[1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2}\right]$$

For stability, i.e. $\Phi \geq 0$ everywhere, we require for all $r \in [R_1, R_2]$

$$1 \geq \frac{(1-\mu)R_1^2}{(\eta^2-\mu)r^2} \geq \frac{1-\mu}{\eta^2-\mu}$$

where the last inequality follows since $R_1^2/r^2 \geq 1$ for all $r \in [R_1, R_2]$. There are now two cases:

- If $\eta^2 > \mu$ then

  $$\eta^2 - \mu \geq 1 - \mu \implies \eta^2 \geq 1$$

  This is a contradiction since $\eta < 1$.

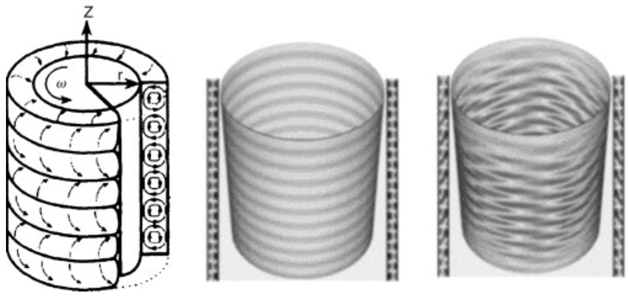
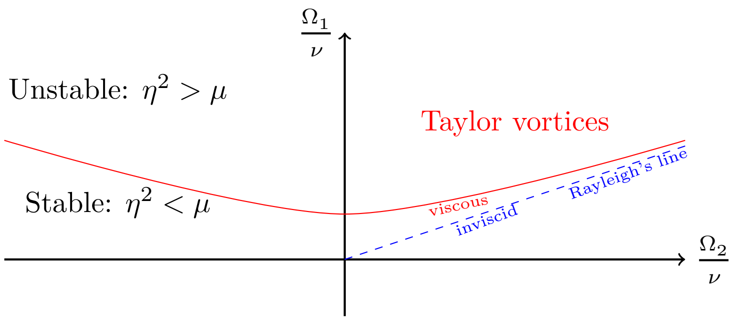


Figure 1: Taylor vortices, from Dutta and Ray, 2004.

- Otherwise $\eta^2 < \mu$, so

$$\eta^2 - \mu \leq 1 - \mu \implies \eta^2 \leq 1$$

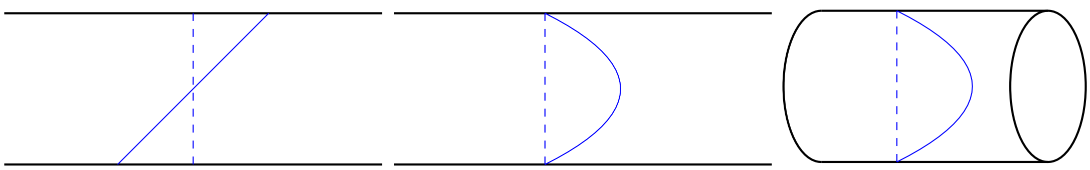
Thus Rayleigh's criterion is $\eta^2 < \mu$ for stability.



For a fixed geometry (i.e. fixed η) we can plot a stability diagram, with Rayleigh's line $\eta^2 = \mu = \Omega_2/\Omega_1$ marking the stability heuristic. In Taylor-Couette geometry, the instability often manifests itself as *Taylor vortices*, though there are many different modes of instability depending on Ω_1, Ω_2, ν.

5 Parallel shear flows

For some flows, inviscid analysis gives a good approximation to the stability properties of a viscous fluid (e.g. Kelvin-Helmholtz, Taylor-Couette flow) but for others, it does not (e.g. plane Couette flow, channel flow, pipe flow). In these flows, viscosity can be *de*stabilising.



Plane Couette flow   Channel flow   Pipe flow

5.1 Inviscid analysis

Consider a parallel shear flow $U(z)\hat{\boldsymbol{x}}$. The non-dimensionalised Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

with boundary conditions $\boldsymbol{u} \cdot \hat{\boldsymbol{z}} = 0$ at $z = z_1, z_2$. The basic flow is $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with P constant – any constant form of the pressure is valid. Add small perturbations

$$\boldsymbol{u} = U(z)\hat{\boldsymbol{x}} + \boldsymbol{u}', \quad p = P + p'$$

The Euler equations become

$$\frac{\partial \boldsymbol{u}'}{\partial t} + U\frac{\partial \boldsymbol{u}'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z}\hat{\boldsymbol{x}} = -\nabla p'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

with boundary conditions $w' = 0$ at $z = z_1, z_2$. All equations have coefficients independent of x, y, t so we can separate the variables by taking normal modes of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \hat{\boldsymbol{u}}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$
$$p'(\boldsymbol{x}, t) = \hat{p}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$

Note we have replaced the usual σ with $-i\alpha c$. It is understood that the physical fluid perturbation velocity $\boldsymbol{u}'$ is represented by the real part, e.g.

$$w' = \left[\Re(\hat{w})\cos(\alpha x + \beta y - \alpha c_r t) - \Im(\hat{w})\sin(\alpha x + \beta y - \alpha c_r t)\right] e^{\alpha c_i t}$$

This mode is a wave travelling with phase speed $\alpha c_r/\sqrt{\alpha^2 + \beta^2}$ in the $(\alpha, \beta, 0)$ direction and it decays like $e^{\alpha c_i t}$ for $c_i < 0$, or grows if $c_i > 0$. The equations are now

$$i\alpha(U - c)\hat{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\alpha\hat{p} = 0 \tag{8}$$
$$i\alpha(U - c)\hat{v} + i\beta\hat{p} = 0 \tag{9}$$
$$i\alpha(U - c)\hat{w} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}z} = 0 \tag{10}$$
$$i\alpha\hat{u} + i\beta\hat{v} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0 \tag{11}$$

with boundary conditions $\hat{w} = 0$ at $z = z_1, z_2$. This is an eigenvalue problem in $c \in \mathbb{C}$. Instability corresponds to $c_i > 0$ and $c_i \leq 0$ for stability.

5.1.1 Squire's transformation (Squire, 1933)

Before attempting to solve (8)–(11), we consider Squire's transformation. Define the transformed variables

$$\tilde{\alpha} = \sqrt{\alpha^2 + \beta^2}, \quad \tilde{u} = \frac{\alpha\hat{u} + \beta\hat{v}}{\tilde{\alpha}}, \quad \tilde{p} = \frac{\tilde{\alpha}\hat{p}}{\alpha}$$

Construct $(\alpha(8) + \beta(9))/\alpha$:

$$i\tilde{\alpha}(U - c)\tilde{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\tilde{\alpha}\tilde{p} = 0 \tag{12}$$

Similarly $\tilde{\alpha}(10)/\alpha$:

$$i\tilde{\alpha}(U - c)\hat{w} + \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = 0 \tag{13}$$

Incompressibility is now expressed as

$$i\tilde{\alpha}\tilde{u} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0$$

The transformed system has the same form as (8)–(11) with $\beta = \hat{v} = 0$ and $\alpha \to \tilde{\alpha}, \hat{u} \to \tilde{u}, \hat{p} \to \tilde{p}$ but c unchanged. Thus the eigenvalue c depends on $\sqrt{\alpha^2 + \beta^2}$ but the growth rate is αc_i. So the largest growth rate αc_i is given by $\beta = 0$ for all wavenumber pairs (α, β) with $\sqrt{\alpha^2 + \beta^2}$ constant. Hence it is sufficient to consider $\beta = 0$ disturbances only. To any unstable 3D mode $\alpha \neq 0, \beta \neq 0$ there corresponds a more unstable 2D mode with $\beta = 0$.

5.1.2 Rayleigh's equation

Work in 2D (Squires). Use streamfunction ψ' such that

$$u' = \psi'_z,\ v' = 0,\ w' = -\psi'_x$$

Further, let $\psi'(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$ so that it is now clear that c_r is the phase speed in the x direction. Now $\hat{u} = \frac{\mathrm{d}\phi}{\mathrm{d}z}$ and $\hat{w} = -i\alpha\phi$ (notice the phase difference). Then (12) becomes

$$\begin{aligned} i\alpha(U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}(-i\alpha\phi) + i\alpha\hat{p} &= 0 \\ \implies \hat{p} &= \frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} \end{aligned}$$

Substituting into (13) gives

$$\begin{aligned} i\alpha(U-c)(-i\alpha\phi) + \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}\right] &= 0 \\ \implies (U-c)(\phi'' - \alpha^2\phi) - U''\phi &= 0 \end{aligned} \tag{14}$$

with boundary conditions $\phi = 0$ at $z = z_1, z_2$. This is *Rayleigh's equation (1880).*

Comments.

- Rayleigh's equation involves α^2 only so need only consider $\alpha > 0$.
- If (ϕ, c) solves the problem then so does (ϕ^*, c^*). So if there exists a growing mode, there also exists a corresponding decaying mode. Hence stability means $c \in \mathbb{R}$ for all α.
- A singularity exists at $U(z_c) = c$ – this is called a critical layer and only occurs when $c \in \mathbb{R}$. Critical layers are important in solving IVPs and relating Rayleigh's equation to its viscous analogue, the Orr-Sommerfield equation (see later).
- There are two types of eigensolution:
  - Continuous spectrum $c \in [\min U, \max U]$ and ϕ has a discontinuous derivative at z_c. This type of solution is never unstable.
  - Discrete spectrum of complex conjugate pairs. This solution can be unstable.

5.1.3 Properties of Rayleigh's equation.

Inflection point criterion. Suppose $c_i > 0$, i.e. consider an unstable mode. Multiply Rayleigh's equation by ϕ^* and integrate from z_1 to z_2:

$$\int_{z_1}^{z_2} \left[\phi^*\phi'' - \alpha^2|\phi|^2 - \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0$$

Integrate the first term by parts and note $\phi = \phi^* = 0$ at z_1 and z_2. Hence

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0 \tag{15}$$

Take imaginary part:

$$\Im\left[\int_{z_1}^{z_2}\frac{U''(U-c^*)}{|U-c|^2}|\phi|^2\mathrm{d}z\right]=0$$
$$\implies -c_i\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

But $c_i>0$ so we must have

$$\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

Now $|U-c|^2>0$ and $|\phi|^2>0$ so U'' must change sign somewhere in $[z_1,z_2]$. Thus $U''=0$ at least once is a necessary condition for inviscid instability, called the *inflection point criterion*.

Fjørtoft's condition. A stronger form of the inflection point criterion was obtained by Fjørtoft (1950): given a monotonic mean velocity profile $U(z)$, a necessary condition for instability is that $U''(U-U_s)<0$ for some $z\in[z_1,z_2]$ with $U_s=U(z_s)$ where $U''(z_2)=0$.
To see this, take the real part of (15) to get

$$\int_{z_1}^{z_2}\frac{U''(U-c_r)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

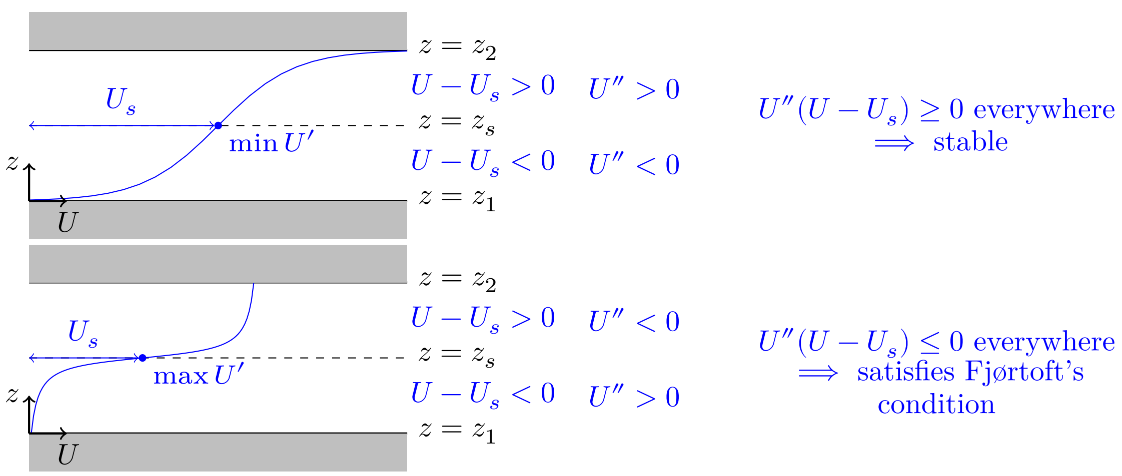
Add the term

$$(c_r-U_s)\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

which vanishes if $c_i>0$ by above. Then

$$\int_{z_1}^{z_2}\frac{U''(U-U_s)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

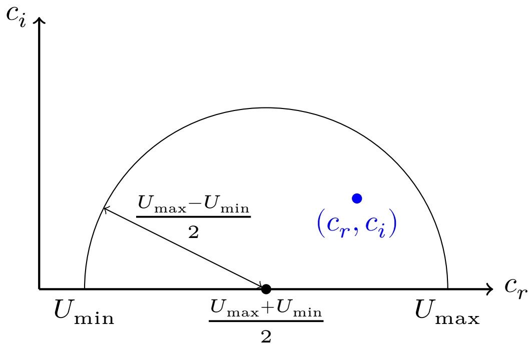
The RHS terms are negative definite, and $\left|\phi^2\right|>0$ as well as $|U-c|^2>0$. Hence $U''(U-U_s)<0$ somewhere in $[z_1,z_2]$. This means that the inflection point has to be a maximum (rather than a minimum) of the spanwise vorticity $U'(z)\hat{\boldsymbol{y}}$.



Howard's semicircle theorem Due to Howard (1961). The unstable eigenvalues of the Rayleigh equation satisfy

$$\left[c_r - \frac{1}{2}(U_{\max} + U_{\min})\right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{\max} - U_{\min})\right]^2$$

This is best viewed as a geometric condition: the unstable eigenvalues lie in a semicircle centred at $\frac{1}{2}(U_{\max} + U_{\min})$ of radius $\frac{1}{2}(U_{\max} - U_{\min})$.



Let $\Psi = \frac{\phi}{U-c}$. Rayleigh's equation (14) in terms of Ψ is

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}[(U-c)\Psi] - \alpha^2(U-c)\Psi\right) = U''(U-c)\Psi$$

Evaluating the derivative and simplifying gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(U-c)^2\frac{\mathrm{d}\Psi}{\mathrm{d}z}\right] = \alpha^2(U-c)^2\Psi$$

Multiply the equation by Ψ^* and integrate over $[z_1, z_2]$:

$$\int_{z_1}^{z_2} \Psi^* \left[(U-c)^2\Psi'\right]' \mathrm{d}z = \alpha^2 \int_{z_1}^{z_2} (U-c)^2 |\Psi|^2 \mathrm{d}z$$

We then integrate by parts and note that $\Psi = \phi/(U-c) = 0$ on $z = z_1, z_2$. Hence

$$\int_{z_1}^{z_2} (U-c)^2 \left[|\Psi'|^2 + \alpha^2|\Psi|^2\right] \mathrm{d}z = 0$$

Denote the $[\ldots]$ factor by Q. We have $Q > 0$ and $c \in \mathbb{C}$. Taking real and imaginary parts gives

$$\int_{z_1}^{z_2} \left[(U-c_r)^2 - c_i^2\right] Q \,\mathrm{d}z = 0$$

$$-2c_i \int_{z_1}^{z_2} (U-c_r) Q \,\mathrm{d}z = 0$$

Since Q is strictly positive, $U - c_r$ has to change sign in $[z_1, z_2]$. Hence

$$U_{\min} < c_r < U_{\max}$$

Rewrite the imaginary part as

$$\int_{z_1}^{z_2} UQ \,\mathrm{d}z = c_r \int_{z_1}^{z_2} Q \,\mathrm{d}z \tag{16}$$

and the real part as

$$\begin{aligned}
\int_{z_1}^{z_2} U^2 Q \,\mathrm{d}z &= 2c_r \int_{z_1}^{z_2} UQ\,\mathrm{d}z + (-c_r^2 + c_i^2)\int_{z_1}^{z_2} Q\,\mathrm{d}z \\
&\overset{(16)}{=} 2c_r^2 \int_{z_1}^{z_2} Q\,\mathrm{d}z + (c_i^2 - c_r^2)\int_{z_1}^{z_2} Q\,\mathrm{d}z \\
&= (c_r^2 + c_i^2)\int_{z_1}^{z_2} Q\,\mathrm{d}z
\end{aligned} \tag{17}$$

Now 'notice' that

$$\int_{z_1}^{z_2} (U - U_{\min})(U - U_{\max}) Q\,\mathrm{d}z \le 0$$

since the first factor is ≥ 0, the second is ≤ 0 and $Q > 0$. Expanding the terms we have

$$\int_{z_1}^{z_2} [U^2 Q - (U_{\min} + U_{\max})UQ + U_{\min}U_{\max}Q]\,\mathrm{d}z \le 0$$

Now using (16) and (17) we can rewrite as

$$\begin{aligned}
&\int_{z_1}^{z_2} [(c_r^2 + c_i^2) - (U_{\min} + U_{\max})c_r + U_{\min}U_{\max}]\, Q\,\mathrm{d}z \le 0 \\
\implies &\int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + U_{\min}U_{\max} - \left(\frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2\right] Q\,\mathrm{d}z \le 0 \\
\implies &\int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2\right] Q\,\mathrm{d}z \le 0
\end{aligned}$$

Equivalently we can write

$$\left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2\right] \int_{z_1}^{z_2} Q\,\mathrm{d}z \le 0$$

But $\int_{z_1}^{z_2} Q\,\mathrm{d}z > 0$ so

$$\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2 \le 0$$

which establishes the semicircle theorem.

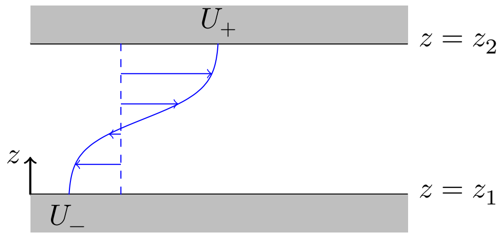
5.1.4 Predictions

For channel flow $\boldsymbol{U}(z) = (1 - z^2)\hat{\boldsymbol{x}}$ we have $U'' \neq 0$, i.e. no inflection points, so no inviscid instability predicted. However, channel flow is linearly unstable at sufficiently high Reynolds number. We must add viscosity to gain a more accurate stability heuristic.

5.2 Viscous analysis

Consider a basic state $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with $P = p_0 - Gx$ and $U(z_1) = U_-, U(z_2) = U_+$. At leading order, Navier-Stokes gives

$$-G = \frac{1}{\mathrm{Re}} U''$$



Special cases are

- Plane Poiseuille flow (PPF) $U(z) = 1 - z^2$ in $[-1, 1]$ and $G = 2/\mathrm{Re}, U_+ = U_- = 0$.
- Plane Couette flow (PCF) with $U(z) = z$ in $[-1, 1]$ and $G = 0$, $U_+ = 1$, $U_- = -1$.

The linearised Navier-Stokes equations for a perturbation $\boldsymbol{u}', p'$ are

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z} = -\frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u' \tag{18}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v' \tag{19}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w' \tag{20}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

The divergence of the first three equations is

$$\nabla \cdot \begin{pmatrix} (18) \\ (19) \\ (20) \end{pmatrix} \implies \frac{\partial w'}{\partial x}\frac{\mathrm{d}U}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x} = -\nabla^2 p'$$

Hence $\nabla^2 p' = -2U'w'_x$. Now consider ∇^2(20):

$$\nabla^2 \left[\frac{\partial w'}{\partial z} + U\frac{\partial w'}{\partial x}\right] = -\frac{\partial}{\partial z}\nabla^2 p' + \frac{1}{\mathrm{Re}}\nabla^4 w'$$

Combining these results we have

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\nabla^2 w' + U''\frac{\partial w'}{\partial x} + 2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial^2 w'}{\partial x \partial z} = -\frac{\partial}{\partial z}\left(-2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x}\right)$$

$$\implies \left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right)\nabla^2 - U''\frac{\partial}{\partial x}\right] w' = 0 \tag{21}$$

with boundary conditions $w' = w'_z = 0$ on the boundaries. This is a fourth order PDE with 4 boundary conditions, so w' is fully determined. To close the problem we need another equation: first define the *normal vorticity*

$$\eta' \equiv \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$$

Now ∂_y(18) $- \partial_x$(19) gives

$$\frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} = \frac{1}{\mathrm{Re}}\nabla^2 \eta'$$

$$\implies \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta' = -\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} \tag{22}$$

with boundary conditions $\eta' = 0$ on the boundaries since tangential velocities vanish at the boundaries. We have reduced $(u', v', w', p') \to (w', \eta')$. Given w' and η' determined from (21) and (22), we can generate v', w', p' from

$$\begin{aligned} u'_x + v'_y &= -w'_z \\ u'_y - v'_x &= \eta' \\ \nabla^2 p' &= -2U' w'_x \end{aligned}$$

5.2.1 Orr-Sommerfeld & Squire Equations

Introduce normal modes / wavelike disturbances / apply a Fourier transform:

$$(w', \eta')(x, y, z, t) = (\hat{w}(z), \hat{\eta}(z))e^{i(\alpha x + \beta y - \alpha c t)}$$

Let $k^2 = \alpha^2 + \beta^2$ be the total horizontal wavenumber. Then (21) and (22) become

$$\left[i\alpha(U - c)(D^2 - k^2) - i\alpha U'' - \frac{1}{\mathrm{Re}}(D^2 - k^2)^2 \right] \hat{w} = 0 \tag{23}$$

$$\left[i\alpha(U - c) - \frac{1}{\mathrm{Re}}(D^2 - k^2) \right] \hat{\eta} = -i\beta U' \hat{w} \tag{24}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}z}$ as usual. Equation (23) is the *Orr-Sommerfeld equation* (Orr 1907, Sommerfeld 1908) and equation (24) is the Squire equation (Squire 1933).

- The Orr-Sommerfeld (OS) equation is the viscous extension of the Rayleigh equation.
- System (23) and (24) has two types of solution:
  1. OS modes $(\hat{w}, \hat{\eta})$ where $\hat{w}$ solves (23) and $\hat{\eta}$ is the forced response in (24).
  2. Squire modes $(0, \hat{\eta})$ which are always damped. Consider (24)/$(-i\alpha)$:

     $$c\hat{\eta} = U\hat{\eta} + \frac{i}{\alpha \mathrm{Re}}(D^2 - k^2)\hat{\eta}$$

     Multiply by $\hat{\eta}^*$:

     $$c|\hat{\eta}|^2 = U|\hat{\eta}|^2 + \frac{i}{\alpha \mathrm{Re}}\hat{\eta}^*(D^2 - k^2)\hat{\eta}$$

     Take the imaginary part and integrate over $[z_1, z_2]$:

     $$\begin{aligned} c_i \int_{z_1}^{z_2} |\hat{\eta}|^2 \,\mathrm{d}z &= \frac{1}{\alpha \mathrm{Re}} \int_{z_1}^{z_2} \frac{i\hat{\eta}^*(D^2 - k^2)\hat{\eta} - (-i)\hat{\eta}(D^2 - k^2)\hat{\eta}^*}{2i} \,\mathrm{d}z \\ &= \frac{1}{\alpha \mathrm{Re}} \int_{z_1}^{z_2} \frac{1}{2}(\hat{\eta}^* D^2 \hat{\eta} + \hat{\eta} D^2 \hat{\eta}^*) - k^2 |\hat{\eta}|^2 \,\mathrm{d}z \\ &= -\frac{1}{\alpha \mathrm{Re}} \int_{z_1}^{z_2} |D\hat{\eta}|^2 + k^2 |\hat{\eta}|^2 \,\mathrm{d}z < 0 \end{aligned}$$

     Thus $c_i < 0$ so solutions are damped.

  Hence we just need to consider the OS equation to establish instability.

- Squire's theorem holds for the OS equation. The 3D version is

$$(U-c)(D^2-k^2)\widehat{w}-U''\widehat{w}-\frac{1}{i\alpha\mathrm{Re}}(D^2-k^2)^2\widehat{w}=0$$

  Compare with the 2D version

$$(U-c)(D^2-\hat{\alpha}^2)\widehat{w}-U''\widehat{w}-\frac{1}{i\hat{\alpha}\hat{\mathrm{Re}}}(D^2-\hat{\alpha}^2)^2\widehat{w}=0$$

  where $\hat{\alpha}=k^2=\alpha^2+\beta^2$ and

$$\hat{\mathrm{Re}}=\frac{\alpha\mathrm{Re}_{3D}}{\hat{\alpha}}=\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\mathrm{Re}_{3D}\leq\mathrm{Re}_{3D}$$

  Thus each 3D OS mode corresponds to a 2D OS mode at a *lower* Re. Note this is a slightly different result from the inviscid case where 2D always had a larger growth rate. We can instead note that if the critical Reynolds number for linear stability is Re_c then

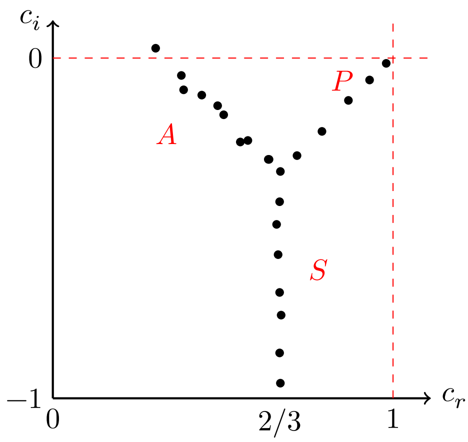
$$\mathrm{Re}_c=\min_{\alpha,\beta}\mathrm{Re}_c(\alpha,\beta)=\min_{\alpha}\mathrm{Re}_c(\alpha,0)$$

  where the first equality defines Re_c and the second is Squire's theorem. This led to a focus on the 2D OS equation.

- What is the connection between Rayleigh and OS equations?
  - OS is non-singular and has a countably infinite number of eigenvalues and its eigenfunctions are complete (Scheisted 1960). Note if the interval of flow is unbounded, there is a continuous spectrum of neutrally stable eigenfunctions in addition to the discrete spectrum (Herron 1987).
  - OS equation is fourth order whilst Rayleigh's equation is second order. 2 OS modes approximate Rayleigh modes, the other 2 modes fix the boundary conditions at the walls (lots of work on this – see Drazin & Reid (1981)).
  - Today it is absolutely routine to numerically solve the OS eigenvalue problem for $\mathrm{Re}\leq 10^7$. Very famous paper by Orszag (1971) used spectral methods as opposed to shooting techniques or finite difference to predict Re_c in channel flow.

5.2.2 Channel flow (PPF)

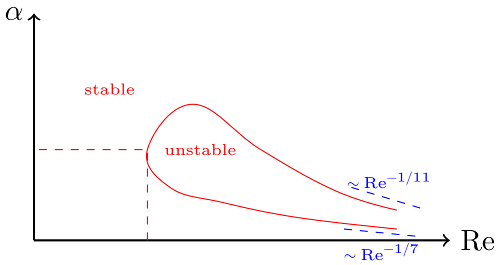
Thomas (1953) found $\mathrm{Re}_c=5780$ at $\alpha_c=1.026$ using finite differences (FD). Further FD estimates came from Nachtshen (1964) with $(\mathrm{Re}_c,\alpha_c)=(5767,1.02)$ and Grosch & Salwen (1968) with $(\mathrm{Re}_c,\alpha_c)=(5750,1.025)$. The accepted result now is from Orszag (1971) with $\mathrm{Re}_c=5772.22$ at $\alpha_c=1.02056$ using spectral methods.
Solving the Orrfield-Sommerfield equation with $\mathrm{Re}=7000,\alpha=1,\beta=0$ gives the following plot.



Note the single unstable eigenvalue with $c_i > 0$. The eigenvalues arise in 3 families, denoted A for Airy, P for Pekeris, and S for Scheisted.

- A: $c_r \to 0$, wall modes, advected towards wall
- P: $c_r \to 1$, centre modes, advected towards centreline
- S: $c_r \approx 2/3$, identified by Mach (1976).

Note that a parabolic base state does not have an inflection point, but *does* have an unstable mode for large Re. Hence viscosity must be destabilising. The unstable mode is called a Tollmien-Schlichting mode/wave (Tollmien 1935, Schlichting 1933). Tollmien was the first to show the OS equation has instability for non-inflection point profiles.
However, as $\mathrm{Re} \to \infty$ we are left with the Rayleigh equation and stability, so somewhere in between they must match up. The stability diagram for the OS equation with finite Reynolds number appears as follows:



The neutral curve closes as there is no instability for $\mathrm{Re} \to \infty$.

5.2.3 Other flows

Notes. Blasius boundary layer (BL) profile f solves $f''' + ff'' = 0$ subject to $f(0) = f'(0) = 0, f'(\infty) = 1$. This is boundary layer flow over an infinite plate.
HPF stands for Hagen-Poiseuille flow. Pipe flow observed to be unstable at $\mathrm{Re} \approx \mathcal{O}(2000)$ (Reynolds, 1883). The transition is at $\mathcal{O}(2000)$ with reasonable care, $\mathcal{O}(12,000)$ with a very careful experiment to minimise disturbances. The world record is $\mathrm{Re} \sim 10^5$ accredited to Pfenniger 1961. Conclusion: pipe flow is unstable to finite amplitude disturbances and threshold for instability *decreases* as $\mathrm{Re} \to \infty$.

Type of flow	Profile	Stable?	$\mathrm{Re}_{\mathrm{crit}}$	α_{crit}	Proof?
Uniform	$U = \text{const.}$	Yes	∞	–	trivial
PCF	$U = z,\ z \in [-1, 1]$	Yes	∞	–	Romanov 1973
PPF	$U = 1 - z^2,\ z \in [-1, 1]$	No	5772	1.02	–
Blasius BL	$U = f'(z), z \geq 0*$	No	520	0.3	–
Shear layer	$U = \tanh z, -\infty < z < \infty$	No	0	0	–
Jet/wake	$U = \operatorname{sech}^2 z, -\infty < z < \infty$	No	4.02	0.17	–
HPF (pipe flow)	$\boldsymbol{U} = (1 - r^2)\hat{\boldsymbol{z}},\ 0 < r < 1$	Yes	∞	–	Chen et al., 2019?

6 Transient Growth & IVPs

So far in this course, the analysis has been 'modal' – identifying eigenfunctions and eigenvalues of linear operators around basic states. This can miss interesting features of the linearised dynamics over 'short' times. Need to consider initial value problems (IVPs).

6.1 Example of IVP analysis

Consider the initial value problem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ \eta \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix} + \begin{pmatrix} \eta^2 \\ -v\eta \end{pmatrix} \\ &= \mathsf{L}(\mathrm{Re})\begin{pmatrix} v \\ \eta \end{pmatrix} + \boldsymbol{N}(v, \eta)\end{aligned} \tag{25}$$

The first term is the linear part, and the second is the nonlinear part emulating the nonlinearities of the dynamical equations. The eigenfunctions of L are $-1/\mathrm{Re}$ and $-2/\mathrm{Re}$ so a basic state

$$\begin{pmatrix} v \\ \eta \end{pmatrix} = \mathbf{0}$$

is linearly stable. Then we must ask if all disturbances decay exponentially? Certainly asymptotically ($t \to \infty$) *but* not over short times.
We can solve (25) linearised:

$$\begin{aligned} \dot{v} &= -\frac{1}{\mathrm{Re}}v & &\implies v(t) = v_0 e^{-t/\mathrm{Re}} \\ \dot{\eta} &= v - \frac{2}{\mathrm{Re}}\eta & &\implies (\eta e^{\frac{2t}{\mathrm{Re}}})_t = v_0 e^{t/\mathrm{Re}} \\ & & &\implies \eta = \mathrm{Re}v_0 e^{-t/\mathrm{Re}} + (\eta_0 - \mathrm{Re}v_0)e^{-2t/\mathrm{Re}} \end{aligned}$$

Hence the solution is

$$\begin{aligned}\begin{pmatrix} v \\ \eta \end{pmatrix} &= v_0\begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix} e^{-t/\mathrm{Re}} + (\eta_0 - \mathrm{Re}v_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/\mathrm{Re}} \\ &= \begin{pmatrix} v_0\left(1 - \frac{t}{\mathrm{Re}}\right) + \mathcal{O}(t^2) \\ \eta_0 + t\left(v_0 - \frac{2\eta_0}{\mathrm{Re}}\right) + \mathcal{O}(t^2) \end{pmatrix}\end{aligned}$$

The linearised solution demonstrates the possibility for short term *algebraic* growth of η provided $v_0 - 2\eta_0/\mathrm{Re} > 0$. To make this more specific, define a norm

$$E \equiv \frac{1}{2}\left(v^2 + \eta^2\right)$$

and assume $\eta_0 = v_0 = 1$. Then

$$\begin{aligned} E(t) &= \frac{1}{2}\left((1 - t/\mathrm{Re} + \dots)^2 + (1 + t(1 - 2/\mathrm{Re}) + \dots)^2\right) \\ &= 1 + \left(1 - \frac{3}{\mathrm{Re}}\right) t + \mathcal{O}(t^2) \end{aligned}$$

So there is energy grwoth at least initially for $\mathrm{Re} > 3$. What is going on? The eigenvectors of L are

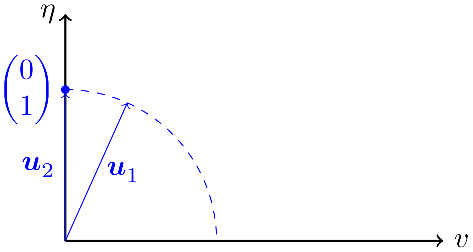
$$\begin{aligned} (\lambda_1, \boldsymbol{u}_1) &= (-\frac{1}{\mathrm{Re}}, \frac{1}{\sqrt{1+\mathrm{Re}^2}} \begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix}) \\ (\lambda_2, \boldsymbol{u}_2) &= (-\frac{2}{\mathrm{Re}}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \end{aligned}$$

Note that these eigenvectors *overlap*. They satisfy $\boldsymbol{u}_1^T \cdot \boldsymbol{u}_1 = \boldsymbol{u}_2^T \cdot \boldsymbol{u}_2 = 1$ and also

$$\boldsymbol{u}_1^T \cdot \boldsymbol{u}_2 = \frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}} \to 1 \text{ as } \mathrm{Re} \to \infty$$

Hence the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is very inefficient in representing disturbances directed along v. For example,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{1+\mathrm{Re}^2}\boldsymbol{u}_1 - \mathrm{Re}\boldsymbol{u}_2$$



Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ decay at different rates, 'growth' appears as large coefficients $\sqrt{1+\mathrm{Re}^2}, \mathrm{Re}$ no longer largely cancel.

6.2 Key points

Matrix L is non-normal.

> **Definition.** A matrix L is non-normal if $L^T L \neq L L^T$, otherwise L is normal.

Note this definition is extensible to operators, i.e. an operator L is normal if

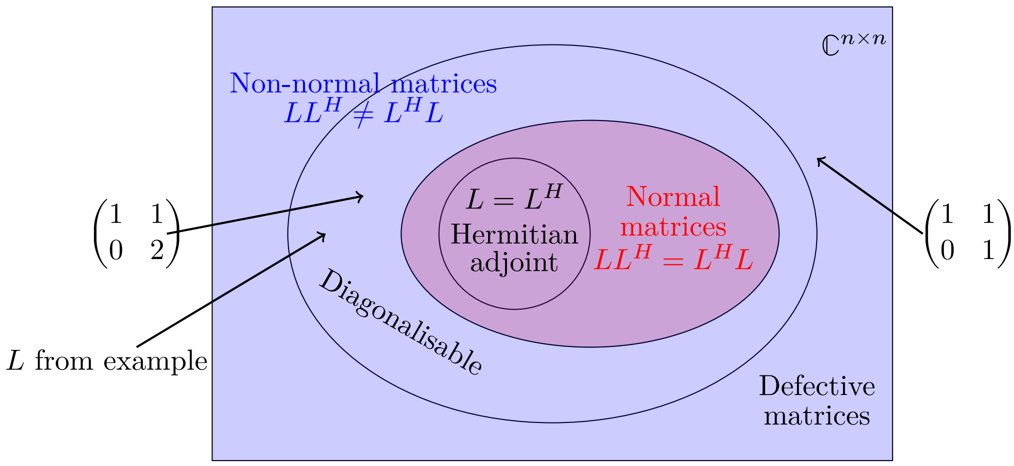
$$\langle u, Lv \rangle = \langle L^T u, v \rangle \implies \langle u, L L^T v \rangle = \langle u, L^T L v \rangle$$

For the matrix L defined in (25), we have

$$\begin{aligned} L^T L &= \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{2}{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mathrm{Re}^2} + 1 & -\frac{2}{\mathrm{Re}} \\ -\frac{2}{\mathrm{Re}} & \frac{4}{\mathrm{Re}^2} \end{pmatrix} \\ L L^T &= \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{2}{\mathrm{Re}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathrm{Re}^2} & -\frac{1}{\mathrm{Re}} \\ -\frac{1}{\mathrm{Re}} & \frac{4}{\mathrm{Re}^2} + 1 \end{pmatrix} \end{aligned}$$

So L is non-normal.

Normality implies complete set of orthonormal eigenvectors. Hence a non-normal matrix has non-orthogonal eigenvectors. Consider all complex square matrices $\mathbb{C}^{n\times n}$ and denote the Hermitian conjugate by superscript ^{H}. The space may be split into normal and non-normal categories as follows.



Choice of norm is important. Consider a general matrix L which is diagonalisable, i.e. $\exists Q$ such that $Q^{-1}LQ = \Lambda$, a diagonal matrix. Then there is no growth in the norm

$$E' \equiv \boldsymbol{x}^T(Q^{-1})^T Q^{-1}\boldsymbol{x} = \boldsymbol{x}^T W \boldsymbol{x}$$

where $W = (Q^{-1})^T Q^{-1}$ is the *weight*.
Proof: let $\boldsymbol{y} = Q^{-1}\boldsymbol{x}$ and consider

$$\begin{aligned}\frac{\mathrm{d}E'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{y}^T\cdot\boldsymbol{y}\right) = \boldsymbol{y}^T\cdot\dot{\boldsymbol{y}} + \dot{\boldsymbol{y}}^T\cdot\dot{\boldsymbol{y}} &= 2\boldsymbol{y}^T\cdot\dot{\boldsymbol{y}}\\ &= 2(Q^{-1}\boldsymbol{x})^T Q^{-1}\dot{\boldsymbol{x}}\\ &= 2\boldsymbol{x}^T(Q^{-1})^T Q^{-1}\dot{\boldsymbol{x}}\\ &= 2\boldsymbol{x}^T(Q^{-1})^T Q^{-1}\dot{\boldsymbol{x}}\\ &= 2\boldsymbol{x}^T(Q^{-1})^T Q^{-1}L\boldsymbol{x} \qquad \text{since } \dot{\boldsymbol{x}} = L\boldsymbol{x}\\ &= 2\boldsymbol{x}^T(Q^{-1})^T Q^{-1}LQQ^{-1}\boldsymbol{x}\\ &= 2\boldsymbol{x}^T(Q^{-1})^T\Lambda Q^{-1}\boldsymbol{x}\\ &= 2\boldsymbol{y}^T\Lambda\boldsymbol{y}\end{aligned}$$

Hence $\dot{E}'$ is negative if all eigenvalues of $\Re(\Lambda)$ are negative. The key here is that $\boldsymbol{y} = Q^{-1}\boldsymbol{x}$ transforms $\boldsymbol{x}$ into a basis of eigenvectors. For example, for L in the previous example we have

$$Q = \begin{pmatrix}\frac{1}{\sqrt{1+\mathrm{Re}^2}} & 0\\ \frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}} & 1\end{pmatrix} \implies Q^{-1} = \begin{pmatrix}\sqrt{1+\mathrm{Re}^2} & 0\\ -\mathrm{Re} & 1\end{pmatrix}$$

Hence in the basis of eigenvectors $(v\ \ \eta)^T$ is

$$Q^{-1}\boldsymbol{x} = Q^{-1}\begin{pmatrix}v\\ \eta\end{pmatrix} = \begin{pmatrix}v\sqrt{1+\mathrm{Re}^2}\\ \eta - \mathrm{Re}v\end{pmatrix}$$

So above result indicates

$$(1+\mathrm{Re}^2)^2_v + (\eta - \mathrm{Re}v)^2 = E'$$

decays. We can calculate explicitly:

$$\begin{aligned}\frac{\mathrm{d}E'}{\mathrm{d}t} &= 2(1+\mathrm{Re}^2)v\dot{v} + 2(\eta - \mathrm{Re}v)(\dot{\eta} - \mathrm{Re}\dot{v}) \\ &= 2(1+\mathrm{Re}^2)\left(\frac{-1}{\mathrm{Re}^2}v^2\right) + 2(\eta - \mathrm{Re}v)\left(2v - \frac{2\eta}{\mathrm{Re}}\right) \\ &= -\frac{2(1+\mathrm{Re}^2)}{\mathrm{Re}}v^2 - \frac{4(\eta - \mathrm{Re}v)^2}{\mathrm{Re}} \\ &< 0 \ \text{ for } v \neq 0, \eta \neq 0\end{aligned}$$

Non-normality is necessary but not sufficient for transient growth. Non-normality does not imply transient growth, but transient growth does imply non-normality. For example, consider the norm E defined in section 6.1, $E = (v^2 + \eta^2)/2$. We have

$$\begin{aligned}\dot{E} &= (v \quad \eta)\begin{pmatrix}\dot{v}\\ \dot{\eta}\end{pmatrix} \\ &= -\frac{1}{\mathrm{Re}}v^2 - \frac{2}{\mathrm{Re}}\eta^2 + v\eta \\ &= -\left(\frac{1}{\sqrt{\mathrm{Re}}}v - \frac{\sqrt{\mathrm{Re}}}{2}\eta\right)^2 + \left(\frac{\mathrm{Re}^2 - 8}{4\mathrm{Re}}\right)\eta^2 \end{aligned} \tag{26}$$

So provided $\mathrm{Re}^2 < 8$, no growth is possible even though L is non-normal. Note from (26) we can see that if $\mathrm{Re}^2 > 8$ the maximum *initial* growth is obtained for $v_0 = \frac{\mathrm{Re}}{2}\eta_0$ so if $E_0 = 1$, then

$$(v_0, \eta_0) = \left(\frac{\mathrm{Re}\sqrt{2}}{\sqrt{4+\mathrm{Re}^2}}, \frac{\sqrt{8}}{\sqrt{4+\mathrm{Re}^2}}\right)$$

maximises the initial growth.
We can do the same analysis for the whole system

$$\begin{aligned}\dot{v} &= -\frac{1}{\mathrm{Re}}v + \eta^2 \\ \dot{\eta} &= v - \frac{2}{\mathrm{Re}}\eta - v\eta\end{aligned}$$

as the non-linearity is energy preserving (exactly as in the Navier-Stokes equations). We have

$$(v,\eta)\begin{pmatrix}\dot{v}\\ \dot{\eta}\end{pmatrix} = (v,\eta)L\begin{pmatrix}v\\ \eta\end{pmatrix}$$

as before. Thus the non-linear terms satisfy

$$(v,\eta)N(v,\eta) = 0 = v(\eta^2) + \eta(-v\eta)$$

Hence *any* initial condition will decay monotonically for $\mathrm{Re} < \sqrt{8}$, i.e. $(v,\eta) = \mathbf{0}$ is then a global attractor.
Exactly the same type of analysis (energy stability analysis) can be done for the Navier-Stokes equations (hope to revisit later).

How to find max growth and optimal ICs. We can find the optimal initial conditions as a function of T, a chosen time. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ \eta \end{pmatrix} = L \begin{pmatrix} v \\ \eta \end{pmatrix} \implies \begin{pmatrix} v \\ \eta \end{pmatrix} = e^{Lt}\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}$$

Let $A \equiv e^{Lt}$. By direct calculation, we have

$$A = \begin{pmatrix} e^{-t/\mathrm{Re}} & 0 \\ \mathrm{Re}(e^{-t/\mathrm{Re}} - e^{-2t/\mathrm{Re}}) & e^{-2t/\mathrm{Re}} \end{pmatrix}$$

or more generally

$$e^{\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}^n t^n = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & e^{\lambda_2 t} \end{pmatrix}$$

Define the *energy gain* at time T

$$\begin{aligned}
G(T; \mathrm{Re}) &\equiv \max_{v_0, \eta_0} \frac{E(T)}{E(0)} \\
&= \max_{v_0, \eta_0} \frac{(v(T) \ \ \eta(T))\begin{pmatrix} v(T) \\ \eta(T) \end{pmatrix}}{(v_0 \ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\
&= \max_{v_0, \eta_0} \frac{\left[A(T)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}\right]^T \left[A(T)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}\right]}{(v_0 \ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\
&= \max_{v_0, \eta_0} \frac{(v_0 \ \ \eta_0)A^T A\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}}{(v_0 \ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\
&= \|A\|_2^2
\end{aligned}$$

The maximum over v_0, η_0 is equivalent to the maximum eigenvalue of $A^T A$. Since $A^T A$ is real and symmetric, it is normal, so all eigenvalues are real. $A^T A$ is also positive definite, so all eigenvalues are positive. The norm

$$\|A\|_2^2$$

is usually computed using singular value decomposition (SVD). From the above, we have

$$G = \sigma_1^2 = \text{largest eigenvalue of } A^T A$$

and σ_1 is the largest singular value of A. The corresponding eigenvector is the optimal initial condition.