Cambridge Part III Maths

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Astrophysical Fluid Dynamics

based on a course given by
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Contents

Lecture 1
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1 Introduction

1.1 Areas of application

Astrophysical fluid dynamics (AFD) is relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, the interstellar medium, the intergalactic medium, and cosmology itself. A fluid description is not applicable in regions that are solidified, such as the rocky or icy cores of giant planets and the crusts of neutron stars, and also in very genuous regions where the medium is not sufficiently collisional.

1.2 Theoretical varieties

Various flavours of AFD are in use. The basic models we will consider are:

Hydrodynamics (HD) / Newtonian gas dynamics: This model is non-relativistic, compressible, ideal (inviscid and adiabatic), self-gravitating, and usually assumes a perfect gas.

Magnetohydrodynamics (MHD): This model is the same as above, with the addition of a magnetic field. We will often use ideal MHD, which assumes a perfectly conducting fluid.

1.3 Characteristic features

The elements of theory often important in AFD are compressibility, gravitation, and thermal physics. Sometimes, magnetic fields, radiation forces, and relativity are important. Rarely important aspects are viscosity, surface tension, and solid boundaries.

1.4 Useful data

Some useful data for the course, in CGS (centimetres, grams, seconds) units:

Newton's constant	$G = 6.674 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
Boltzmann's constant	$k = 1.381 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$
Stefan's constant	$\sigma = 5.670 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$
Speed of light	$c = 2.998 \times 10^{1-}\,\mathrm{cm\,s^{-1}}$
Hydrogen mass	$m_H = 1.674 \times 10^{-24}\,\mathrm{g}$
Solar mass	$M_s = 1.988 \times 10^{33}\,\mathrm{g}$
Solar radius	$R_s = 6.957 \times 10^{10}\,\mathrm{cm}$
Solar luminosity	$L_s = 3.828 \times 10^{33}\,\mathrm{ergs^{-1}}$
Parsec	$pc = 3.086 \times 10^{18}\,\mathrm{cm}$
Astronomial unit (AU)	$au = 1.496 \times 10^{13}\,\mathrm{cm}$
Joule erg conversion	$1\mathrm{J} = 10^7\mathrm{erg}$

2 Ideal gas dynamics

2.1 Fluid variables

A fluid is characterised by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The *specific volume* (volume per unit mass) is $v = 1/\rho$.
We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionisation and recombination.

2.2 Eulerian and Lagrangian viewpoints

In the *Eulerian* viewpoint we consider how fluid properties vary in time at a point which is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian time derivative is simply ∂_t.
In the *Lagrangian* viewpoint we consider how fluid properties vary in time at a point which moves with the fluid at velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The Lagrangian time derivative (or material derivative) is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

2.3 Material points and structures

A material point is an idealised fluid element, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of the fluid. Note that the true particles of which the fluid is composed also have random thermal motion.
Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are deformed by it. An infinitesimal material line element $\delta\boldsymbol{x}$ evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u}$$

It changes its length and/or orientation in the presence of a velocity gradient.
Infinitesimal material surface or volume elements can be defined from two or three material line elements according to the vector product and the triple scalar product.

$$\begin{aligned}\delta\boldsymbol{S} &= \delta\boldsymbol{x}^{(1)} \times \delta\boldsymbol{x}^{(2)} \\ \delta V &= \delta\boldsymbol{x}^{(1)} \cdot \delta\boldsymbol{x}^{(2)} \times \delta\boldsymbol{x}^{(3)}\end{aligned}$$

They evolve according to

$$\begin{aligned}\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta\boldsymbol{S} - \nabla\boldsymbol{u}\cdot\delta\boldsymbol{S} \\ \frac{\mathrm{D}\delta V}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta V\end{aligned}$$

The second result is easier to understand: the volume element increases when the flow is divergent.