

Cambridge Part III Maths

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Slow Viscous Flow

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1 Basic Fluid Mechanics

Lecture 1
09/10/20

‘Infinitesimal’ fluid particles have well-defined density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ where $\mathbf{x}(t)$ is the position of the fluid particles.

Definition. The *Eulerian* or *material* derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

is the rate of change following the fluid particle.

1.1 Mass Conservation

In general, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \iff \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

For an incompressible fluid, $\frac{D\rho}{Dt} = 0 \iff \nabla \cdot \mathbf{u} = 0$

1.2 The Stress Tensor

The *stress* $\boldsymbol{\tau}$ is the force per unit area acting across a surface. Force balance on an ‘infinitesimal’ fluid tetrahedron shows that the stress $\boldsymbol{\tau}$ is linearly related to the surface normal \mathbf{n} :

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

where $\boldsymbol{\sigma}$ is the *stress tensor* and $\boldsymbol{\tau}$ is stress exerted by the outside fluid on the inside of a surface with outward normal \mathbf{n} . Angular momentum balance shows that $\boldsymbol{\sigma}$ is symmetric in most fluids.

1.3 Momentum equation

The *Cauchy momentum equation* states in general

$$\frac{D\mathbf{u}}{Dt} = \mathbf{F} + \nabla \cdot \boldsymbol{\sigma}$$

1.4 Energy equation

In the case of an incompressible fluid, the rate of local inertial *viscous dissipation* is derived by contracting the Cauchy momentum equation with the fluid velocity and integrating over a volume. We have

$$\mathcal{D} = \int_V e_{ij} \sigma_{ij} dV = \int_V \mathbf{e} : \boldsymbol{\sigma} dV$$

where $e_{ij} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the *rate of strain* tensor. Note $e_{ii} = 0$ by incompressibility and $e_{ij} = e_{ji}$.

The rate of working by external surface forces on the fluid is

$$\int_{\partial V} u_i \sigma_{ij} n_j dS$$

1.5 Newtonian Fluids

Definition. Fluid deformation produces internal viscous stresses. If the relationship between fluid deformation $\frac{\partial u_i}{\partial x_j}$ and stress σ_{ij} is local, linear, instantaneous and isotropic, then the fluid is *Newtonian*.

If the fluid is also incompressible, then the stress tensor takes the form

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

where μ is the *dynamic viscosity* and $2\mu e_{ij}$ is the *deviatoric stress*. Note that there is no dependence on the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

For an incompressible Newtonian fluid with uniform viscosity we have the *Navier-Stokes equations*

$$\begin{aligned}\rho \frac{D\mathbf{u}}{Dt} &= -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

The rate of viscous dissipation is

$$\mathcal{D} = 2\mu \int e_{ij} e_{ij} dV$$

Often body forces are conservative $\mathbf{F} = -\nabla\phi$ and we incorporate \mathbf{F} into a *modified pressure* $p + \phi$.

1.6 Boundary conditions

Kinematic boundary conditions on a fluid-fluid interface are

- $[\mathbf{u} \cdot \mathbf{n}]_-^+ = 0$ by mass conservation
- $[\mathbf{u} \times \mathbf{n}]_-^+ = \mathbf{0}$ to avoid infinite stresses

Kinematic boundary conditions on a rigid boundary are

- No flux: $\mathbf{u} \cdot \mathbf{n} = 0$
- No slip: $\mathbf{u} \times \mathbf{n} = \mathbf{0}$

Dynamic boundary conditions in the absence of surface tension are

$$[\sigma \cdot \mathbf{n}]_-^+ = \mathbf{0}$$

Note that modified pressure should not be used here.

With surface tension included, the condition becomes

$$[\sigma \cdot \mathbf{n}]_-^+ = \gamma \kappa \mathbf{n} - \nabla_s \gamma$$

where $\kappa = \nabla_s \cdot \mathbf{n}$ is the *curvature* and γ is the *surface tension*.

1.7 Reynolds number

Suppose $U, L, L/U$ are representative velocity, length, and time scales of the flow. Then

$$\rho \frac{D\mathbf{u}}{Dt} \sim \rho \frac{U^2}{L}$$

$$\mu \nabla^2 \mathbf{u} \sim \mu \frac{U}{L^2}$$

Definition. The *Reynolds number* is the ratio of these quantities and determines the important of inertial vs. viscous stresses.

$$Re = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$

If $Re \ll 1$ then inertia is negligible and we have the *Stokes equations*

$$\mu \nabla^2 \mathbf{u} = \nabla p - \mathbf{F}$$

$$\nabla \cdot \mathbf{u} = 0$$

Stokes equations are useful in many regimes.

- Large μ , e.g. magma, glass, ice sheets
- Small L , e.g. microorganisms, microfluid devices
- Thin film flows, e.g. lubrication theory

Lecture 2
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Example. Sperm cell – intrinsic length scales $L \sim 5\mu m, U \sim 100\mu m \cdot s^{-1}, \nu \sim 10^{-2} cm^2 \cdot s^{-1} = 10^6 \mu m^2 \cdot s^{-1}$. Therefore $Re \sim 5 \times 10^{-4}$ so can be described by the Stokes equations.

Example. Mantle convection – intrinsic length scales $L \sim 1000 km = 10^8 cm, U \sim 2 cm year^{-1} \sim 10^7 cm \cdot s^{-1}, \nu \sim 10^{21} cm^2 \cdot s^{-1}$. Thus $Re \sim 10^{-20}$.

There are some caveats which come with the use of intrinsic length scales.

- $\mathbf{u} \cdot \nabla$ and ∇^2 may not involve the same length scale L , e.g. in lubrication theory there is a short length scale for the depth of the flow, which is small compared to other length scales of the flow.
- L may vary in the flow e.g. in the far field of a moving body $Re \sim \frac{Ur}{\nu}$.
- T may not equal L/U if there is an external time scale, e.g. oscillating body with $T \sim \omega^{-1}$.

2 The Stokes Equations

$$\nabla \cdot \sigma = \mu \nabla^2 \mathbf{u} - \nabla p = -\mathbf{F}$$

$$\nabla \cdot \mathbf{u} = 0$$

2.1 Simple Properties

2.1.1 Instantaneous

The Stokes equations involve no ∂_t term, so there is no inertia, no memory, and the flow only ‘knows’ about the current boundary conditions and applied forces, and responds immediately to changes. With moving boundaries (i.e. changing boundary conditions) the flow is *quasi-steady*.

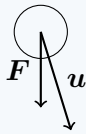
2.1.2 Linear

The Stokes equations are linear in \mathbf{F} , p , and \mathbf{u} . Therefore the fluid response is proportional to forcing and solutions for a given geometry can be superposed.

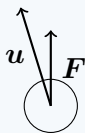
2.1.3 Reversible

If all the forces change sign, then \mathbf{u} changes sign. Thus if we reverse all the forces and the history of their application, the flow returns to its original state. Reversibility can sometimes be used with a symmetry to rule out certain behaviours of the flow.

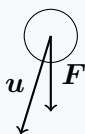
Example. Sedimenting sphere – consider a sphere sedimenting in a Stokes flow next to a rigid wall. Will the sphere migrate laterally?



Applying reversibility, change $\mathbf{F} \rightarrow -\mathbf{F}$ so $\mathbf{u} \rightarrow -\mathbf{u}$:



Now apply symmetry: reflect the geometry top to bottom.



Comparing with the original situation, we see there can be no lateral component of \mathbf{u} .

2.1.4 Forces balance

Since there is no inertia, the forces must balance. From the equations,

$$\nabla \cdot \sigma = -\mathbf{F} \implies \int_{\partial V} \sigma \cdot \mathbf{n} \, dS + \int_V \mathbf{F} \, dV = \mathbf{0}$$

This is a consistency check on stress boundary conditions.

Similarly, in the absence of fluid sources,

$$\nabla \cdot \mathbf{u} = 0 \implies \int_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS = 0$$

This is a consistency check on velocity boundary conditions.

Likewise, torques balance, giving another consistency check on stress boundary conditions.

2.1.5 Work balances dissipation

Intuitively, the flow has no kinetic energy (no inertia) so any work done on the fluid must be viscously dissipated instantaneously. We have

$$\begin{aligned} \mathcal{D} &= 2\mu \int_V e_{ij} e_{ij} \, dV \\ &= \int (\sigma_{ij} + p\delta_{ij}) e_{ij} \, dV \\ &= \int \sigma_{ij} \frac{\partial u_i}{\partial x_j} + p e_{ii} \, dV \\ &= \int \frac{\partial}{\partial x_j} \sigma_{ij} u_i - u_i \frac{\partial \sigma_{ij}}{\partial x_j} \, dV \\ &= \int_{\partial V} \mathbf{u} \cdot \sigma \cdot \mathbf{n} \, dS + \int_V \mathbf{u} \cdot \mathbf{F} \, dV \end{aligned}$$

The first term is the work done by surface forces at the boundary, and the second term is the work done by body forces.

2.2 Three Theorems Based on Dissipation Integrals

Lemma 1. *If \mathbf{u}^I is an incompressible flow and \mathbf{u}^S is a Stokes flow with body force \mathbf{F}^S then*

$$2\mu \int e^I : e^S \, dV = \int_{\partial V} \mathbf{u}^I \cdot \sigma^S \cdot \mathbf{n} \, dS + \int_V \mathbf{u}^I \cdot \mathbf{F}^S \, dV$$

Proof. Same as ‘work balances dissipation’.

Theorem 1. *Uniqueness theorem. Suppose $\mathbf{u}_1, \mathbf{u}_2$ are Stokes flows with the same boundary conditions and body forces, i.e. $\mathbf{F}_1 = \mathbf{F}_2$ in V and either $\mathbf{u}_1 = \mathbf{u}_2$ or $\sigma_1 \cdot \mathbf{n} = \sigma_2 \cdot \mathbf{n}$ on ∂V . Then $\mathbf{u}_1 = \mathbf{u}_2$.*

Proof. Let $\mathbf{u}^* = \mathbf{u}_1 - \mathbf{u}_2$. From lemma 1,

$$2\mu \int_V \mathbf{e}^* : \mathbf{e}^* dV = 0$$

Thus $\mathbf{e}^* = 0$ in V . Hence we can deduce \mathbf{u}^* consists entirely of rigid body motion: $\mathbf{u}^* = \mathbf{U} + \mathbf{\Omega} \times \mathbf{u}$. Using the boundary conditions, we have $\mathbf{U} = \mathbf{\Omega} = 0$ thus $\mathbf{u}_1 = \mathbf{u}_2$, i.e. Stokes flows are unique.

Theorem 2. *Reciprocal theorem. If \mathbf{u}_1 and \mathbf{u}_2 are Stokes flows in V then*

$$\int_{\partial V} \mathbf{u}_1 \cdot \boldsymbol{\sigma}_2 \cdot \mathbf{n} dS + \int_V \mathbf{u}_1 \cdot \mathbf{F}_2 dV = \int_{\partial V} \mathbf{u}_2 \cdot \boldsymbol{\sigma}_1 \cdot \mathbf{n} dS + \int_V \mathbf{u}_2 \cdot \mathbf{F}_1 dV$$

That is, work done by forces of flow 1 against flow 2 = work done by forces of flow 2 against flow 1.

Proof. Apply the lemma twice.

Theorem 3. *Minimum Dissipation theorem. Among all the incompressible flows in V that satisfy given velocity boundary conditions, the dissipation is minimised by the Stokes flow \mathbf{u}^S with $\mathbf{F}^S = \mathbf{0}$ satisfying the same velocity boundary conditions.*

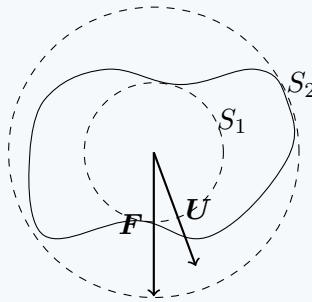
Proof. We have

$$\begin{aligned} 0 &\leq 2\mu \int (e - e^S) : (e - e^S) dV \\ &\leq 2\mu \int e : e - e^S : e^S dV + 4\mu \int e^S : (e^S - e) dV \end{aligned}$$

Applying the lemma with $\mathbf{u}^I = \mathbf{u}^S - \mathbf{u}$, the last term is 0 since $\mathbf{u}^I = 0$ on ∂V and $\mathbf{F}^S = 0$ on V . Thus

$$0 \leq \mathcal{D} - \mathcal{D}^S$$

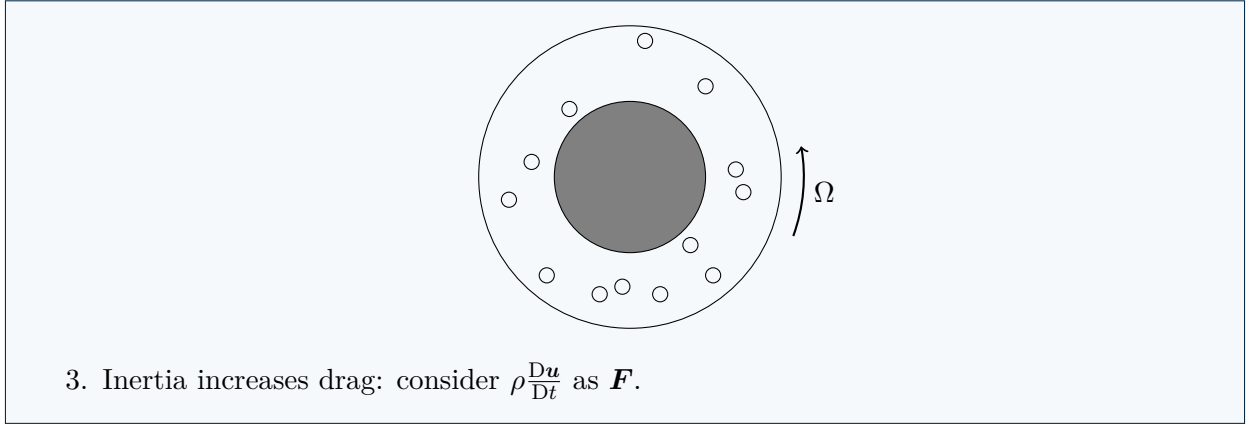
Example. 1. Consider an irregularly shaped body in a Stokes flow with inscribing circle S_1 with radius a_1 and circumscribing circle S_2 with radius a_2 . Suppose the body experiences a force \mathbf{F} and has uniform velocity \mathbf{U} .



Applying the theorem by taking \mathbf{U}^S to be the Stokes flow past S_1 and \mathbf{U}^I to be the Stokes flow past S_2 superposed with solid body motion in the gap between S_1 and S_2 , we have

$$(6\pi\mu a_1 U)U \leq \mathbf{F} \cdot \mathbf{U} \leq (6\pi\mu a_2 U)U$$

2. Adding *rigid* particles to a Stokes flow with given *external* velocity boundary conditions increases dissipation and, if the particles are *force-free* and *torque-free*, the apparent viscosity also increases.



2.3 Representation by Potentials

Assume $\mathbf{F} = 0$, or that \mathbf{F} is conservative and absorbed by the modified pressure. Consider the Stokes equations

$$\mu \nabla^2 \mathbf{u} = \nabla p \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

From these equations we have

$$\nabla \cdot (1) \& (2) \implies \nabla^2 p = 0 \implies p \text{ is harmonic}$$

$$\nabla \times (1) \implies \nabla^2 \boldsymbol{\omega} = 0 \implies \text{vorticity } \boldsymbol{\omega} = \nabla \times \mathbf{u} \text{ is harmonic}$$

$$\nabla^2 (1) \implies \nabla^4 \mathbf{u} = 0 \implies \mathbf{u} \text{ is bi-harmonic}$$

In two dimensions, we can use a stream-function so that $\mathbf{u} = \nabla \times (0, 0, \psi)$. Then

$$\omega_z = -\nabla^2 \psi \implies \nabla^4 \psi = 0$$

Similarly, in axisymmetric spherical polars, $\mathbf{u} = \nabla \times (0, 0, \frac{\Psi}{r \sin \theta})$. Then $\omega_\phi = -\frac{E^2 \Psi}{r \sin \theta}$ and $E^4 \Psi = 0$ where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right)$$

Many exact solutions can be found in coordinate systems where the operators ∇^2, ∇^4, E^2 , etc are separable.

2.4 Complex Variable Theory in 2D Flow

Writing $z = x + iy, \bar{z} = x - iy$ gives

$$\nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Thus $f(x, y)$ analytic implies $f = f(z)$ or equivalently $\frac{\partial f}{\partial \bar{z}} = 0$. Thus $\text{Re } f$ and $\text{Im } f$ are harmonic. Similarly $\nabla^4 \psi = 0$ implies ψ can be written as $\psi = \text{Im}(\bar{z}\phi + \chi)$ where $\phi(z), \chi(z)$ are analytic.

We can find clever exact solutions to difficult problems using this theory, but it is limited to 2D.

2.5 Papkovitch-Neuber Solution

Let $p = \nabla^2 \pi$ where

$$\pi(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{p(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV$$

Then the Stokes equations can be written $\nabla^2(\mu \mathbf{u} - \nabla \pi) = 0$. Thus

$$\mu \mathbf{u} = \nabla \pi - \Phi$$

where $\nabla^2 \Phi = 0$. Now $\nabla \cdot \mathbf{u} = 0$ implies $\nabla^2 \pi = \nabla \cdot \Phi$. Then

$$\pi = \frac{1}{2}(\mathbf{x} \cdot \Phi + \chi)$$

where $\nabla^2 \chi = 0$. Thus *any* Stokes flow with $\mathbf{F} = 0$ can be written in terms of a harmonic vector Φ and a harmonic scalar χ . The Stokes equations are then

$$\begin{aligned} 2\mu \mathbf{u} &= \nabla(\mathbf{x} \cdot \Phi + \chi) - 2\Phi \\ p &= \nabla \cdot \Phi \end{aligned}$$

which may also be re-written with the 2μ factor absorbed by p . Note the following.

1. Any irrotational flow $2\mu \mathbf{u} = \nabla \chi$ is also a Stokes flow, though $p = 0$ and $\underline{\underline{\sigma}} = \nabla \nabla \chi$ which is different from an inviscid irrotational flow.
2. It is sometimes possible to find a harmonic scalar ϕ with $\chi = \mathbf{x} \cdot \nabla \phi - 2\phi$. If so, χ can be eliminated by writing $\Phi' = \Phi + \nabla \phi$. For example, if χ has a spherical harmonic expansion we can eliminate all of the terms except the uniform strain $\chi/2\mu = \frac{1}{2} \mathbf{x} \cdot \underline{\underline{E}} \cdot \mathbf{x} \iff \mathbf{u} = \underline{\underline{E}} \cdot \mathbf{x}$, since $\chi = r^n Y_n^m(\theta, \phi) \iff \phi = \frac{r^n}{n-2} Y_n^m(\theta, \phi)$ which fails for $n = 2$.
3. Conversely, if $\Phi = \nabla \phi$ then we can get the same \mathbf{u} from $\chi = \mathbf{x} \cdot \nabla \phi - 2\phi$, which is easier to calculate.

2.6 Solutions for points, spheres, and cylinders

A point or sphere has no intrinsic direction or orientation, thus solutions on these geometries should also have no intrinsic direction or orientation.

2.6.1 Spherical harmonic functions

Let $r = |\mathbf{x}|$. Recall $\nabla^2(\frac{1}{r}) = 0$ for $r \neq 0$. All other spherical harmonic functions ϕ with $\phi \rightarrow 0$ as $r \rightarrow \infty$ are obtained from

$$\frac{1}{r}, \quad \nabla \frac{1}{r}, \quad \nabla \nabla \frac{1}{r}, \quad \text{etc.}$$

The harmonic functions which are bounded as $r \rightarrow 0$ are obtained from

$$r \cdot \frac{1}{r} = 1, \quad r^3 \nabla \frac{1}{r} = -\mathbf{x}, \quad r^5 \nabla \nabla \frac{1}{r}, \quad \dots, \quad r^{2n+1} \nabla^n \frac{1}{r}$$

Compare with separable solutions, for example the $2n + 1$ solutions in spherical polars given by

$$\begin{pmatrix} r^n \\ r^{-n-1} \end{pmatrix} P_n^m(\theta) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

where P_n^m are *associated Legendre functions* and $0 \leq m \leq n$. Recall the following results.

$$\begin{aligned}\nabla \mathbf{x} &= \underline{\underline{I}} \\ \nabla r &= \frac{\mathbf{x}}{r} \\ \nabla f(r) &= f'(r) \nabla r = f'(r) \frac{\mathbf{x}}{r}\end{aligned}$$

Hence we have

$$\begin{aligned}\nabla \frac{1}{r} &= -\frac{\mathbf{x}}{r^3} \\ \nabla \nabla \frac{1}{r} &= -\frac{\underline{\underline{I}}}{r^3} + \frac{3\mathbf{x} \cdot \mathbf{x}}{r^5} \\ \nabla_i \nabla_j \nabla_k \frac{1}{r} &= \nabla_i \left(-\frac{\delta_{jk}}{r^3} + \frac{3x_j \cdot x_k}{r^5} \right) \\ &= \frac{3(x_i \delta_{jk} + x_j \delta_{ik} + x_k \delta_{ij})}{r^5} - \frac{15x_i x_j x_k}{r^7}\end{aligned}$$

Note: these depend only on \mathbf{x} and r and thus have no preferred direction, as hoped. We can use these functions to form Papkovitch-Neuber potentials Φ and χ by multiplying the harmonic functions above by constant scalars, vectors or tensors and taking an appropriate number of dot products, e.g. the following are all harmonic vectors

$$\mathbf{A} \frac{1}{r}, \quad \underline{\underline{B}} \cdot \nabla \frac{1}{r}, \quad C \nabla \frac{1}{r}, \quad (\mathbf{D} \cdot \nabla) \nabla \frac{1}{r}, \quad (\underline{\underline{E}} : \nabla \nabla) \nabla \frac{1}{r}, \quad \boldsymbol{\Omega} \times \nabla \frac{1}{r}$$

It is useful to distinguish between *true* and *pseudo* tensors. True / pseudo tensors keep / change sign upon reflection, e.g.

$$T'_{ijk} = \pm R_{il} R_{jm} R_{kn} T_{lmn}$$

Examples of true vectors are velocity \mathbf{u} ; force \mathbf{F} ; position \mathbf{x} ; del ∇ ; identity $\underline{\underline{I}}$. Examples of pseudo vectors are angular velocity $\boldsymbol{\Omega}$; torque \mathbf{G} ; $\mathbf{u} \times \mathbf{x}$; vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Products obey the obvious parity rules, e.g. helicity $\mathbf{u} \cdot \boldsymbol{\Omega}$ is a pseudo scalar.

2.6.2 Solution due to a point force

The Papkovitch-Neuber solution due to a point force is a Green's function for the Stokes equations. This problem is also known as a 'Stokeslet'. Consider the problem

$$\begin{aligned}\nabla \cdot \underline{\underline{\sigma}} &= \mu \nabla^2 \mathbf{u} - \nabla p = -\mathbf{F} \delta(\mathbf{x}) \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

with $\mathbf{u} \rightarrow 0$ at infinity. The answer must be linear in \mathbf{F} , but otherwise has no orientation. The only choice is $\Phi = \alpha \frac{\mathbf{F}}{r}$. We could have tried $\mathbf{F} \times \nabla \frac{1}{r}$, but this is a pseudo vector whilst Φ and χ need to be true since \mathbf{u} is true. Similar arguments rule out other harmonic functions.

We have

$$\begin{aligned}2\mu \mathbf{u} &= \alpha \left(\nabla \left(\frac{\mathbf{F} \cdot \mathbf{x}}{r} \right) - 2 \frac{\mathbf{F}}{r} \right) \\ &= \alpha \left(\frac{\mathbf{F} \cdot \underline{\underline{I}}}{r} - \frac{(\mathbf{F} \cdot \mathbf{x}) \mathbf{x}}{r^3} - 2 \frac{\mathbf{F}}{r} \right) \\ &= -\alpha \left(\frac{\mathbf{F}}{r} + \frac{(\mathbf{F} \cdot \mathbf{x}) \mathbf{x}}{r^3} \right)\end{aligned}$$

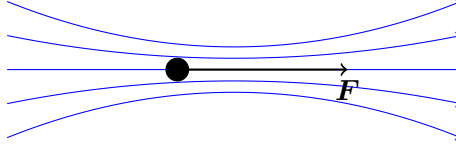


Figure 1: Stokeslet solution for a point force.

Thus the stress tensor is

$$\underline{\underline{\sigma}} = \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - (\nabla \cdot \Phi) \underline{\underline{I}} = 3\alpha (\mathbf{F} \cdot \mathbf{x}) \frac{\mathbf{x}\mathbf{x}}{r^5}$$

On any sphere $r = R$, $\mathbf{n} = \frac{\mathbf{x}}{R}$, and

$$\begin{aligned} 2\mu \mathbf{u} \cdot \mathbf{n} &= -\frac{2\alpha}{R} \mathbf{F} \cdot \mathbf{n} \\ \underline{\underline{\sigma}} \cdot \mathbf{n} &= 3\alpha \frac{(\mathbf{F} \cdot \mathbf{n})\mathbf{n}}{R^2} \end{aligned}$$

To determine the constant α we can consider the surface volume flux and the surface stress. The surface volume flux is

$$\int_{r=R} \mathbf{u} \cdot \mathbf{n} dS = -\frac{\alpha \mathbf{F}}{\mu R} \cdot \int_{r=R} \mathbf{n} dS = 0$$

which does not provide any information on α . The surface forces should equal $-\mathbf{F}$. We have

$$-\mathbf{F} = \int_{r=R} \underline{\underline{\sigma}} \cdot \mathbf{n} dS = 3\alpha \mathbf{F} \cdot \int_{r=R} \mathbf{n}\mathbf{n} \frac{dS}{R^2} = 3\alpha \mathbf{F} \cdot \frac{4\pi}{3} \underline{\underline{I}} = 4\pi\alpha \mathbf{F}$$

Hence we choose $\alpha = -1/4\pi$. Thus the final solution is

$$\mathbf{u} = \mathbf{F} \cdot \underline{\underline{J}}(\mathbf{x}), \quad \underline{\underline{\sigma}} = \mathbf{F} \cdot \underline{\underline{K}}(\mathbf{x}), \quad p = \frac{\mathbf{F} \cdot \mathbf{x}}{4\pi r^3}$$

where $\underline{\underline{J}}$ is the *Oseen tensor*:

$$\underline{\underline{J}} = \frac{1}{8\pi\mu} \left(\frac{\underline{\underline{I}}}{r} + \frac{\mathbf{x}\mathbf{x}}{r^3} \right), \quad \underline{\underline{K}} = -\frac{3}{4\pi} \frac{\mathbf{x}\mathbf{x}\mathbf{x}}{r^5}$$

Finally, from incompressibility $\nabla \cdot \mathbf{u} = 0$ and the Stokes equations $\nabla \cdot \underline{\underline{\sigma}} = -\mathbf{F}\delta(\mathbf{x})$, we deduce

$$\nabla \cdot \underline{\underline{J}} = 0, \quad \nabla \cdot \underline{\underline{K}} = -\underline{\underline{I}}\delta(\mathbf{x})$$

Note that the velocity scales as $u \propto \frac{1}{r}$: see figure 1. This is slowly decaying compared to many forces e.g. gravity which scales as $\frac{1}{r^2}$. Thus particle interactions in Stokes flow can occur on much larger scales than (for example) charge interactions.

2.6.3 Source flow

Consider a source of strength Q with $\nabla \cdot \mathbf{u} = Q\delta(\mathbf{x})$. This is referred to as a *point volume source*. One can show the solution is

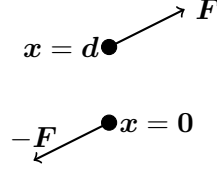
$$\mathbf{u} = \frac{Q\mathbf{x}}{4\pi r^3}$$

which is obtained using Papkovitch-Neuber potentials

$$\chi = \alpha \frac{Q}{r}, \quad \Phi = \beta Q \nabla \frac{1}{r}$$

2.6.4 Force dipole, stresslet, rotlet

Further solutions for dipoles, quadrupoles, etc. can be found by taking gradients of the Stokeslet and source solutions. For example, consider a dipole.



The Stokeslet solution is

$$\mathbf{u} = \mathbf{F} \cdot \underline{\underline{J}}(\mathbf{x} - \mathbf{d}) - \mathbf{F} \cdot \underline{\underline{J}}(\mathbf{x}) = \mathbf{F} \cdot (-\mathbf{d} \cdot \nabla) \underline{\underline{J}}(\mathbf{x}) + \text{h.o.t.}$$

Take the limit $\mathbf{d} \rightarrow \mathbf{0}$ with $\mathbf{F}\mathbf{d}$ fixed and split $-F_i d_j$ into

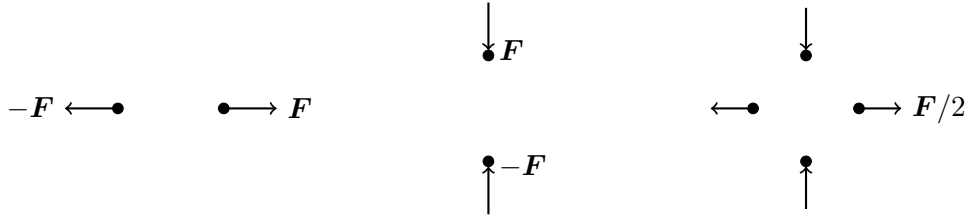
1. An isotropic part $-\frac{1}{3}F_k d_k \delta_{ij}$
2. A symmetric traceless part

$$s_{ij} = -\frac{1}{2}(F_i d_j + F_j d_i) + \frac{1}{3}F_k d_k \delta_{ij}$$

3. An antisymmetric part $-\frac{1}{2}\varepsilon_{ijk}G_k$ where $\mathbf{G} = \mathbf{d} \times \mathbf{F}$

The flow contribution from each of these components may then be calculated.

1. The isotropic component gives no flow since $\nabla \cdot \underline{\underline{J}} = 0$
2. This component is a *stresslet* representing the following components of motion



3. This component is a *rotlet* due to a point torque \mathbf{G} .



Both the stresslet and rotlet decay as $\frac{1}{r^2}$.

2.6.5 Rigid sphere with velocity \mathbf{U}

Consider a rigid sphere of radius a moving uniformly with velocity \mathbf{U} in a Stokes flow. We have $\nabla \cdot \mathbf{u} = 0$ and $\mu \nabla^2 \mathbf{u} = \nabla p$ in $r > a$. We require $\mathbf{u} \rightarrow 0$ as $r \rightarrow \infty$ and $\mathbf{u} = \mathbf{U}$ on the sphere's

surface $r = a$. The sphere is isotropic, so we need harmonic functions of \mathbf{x}, \mathbf{U} which are linear in \mathbf{U} ; decay at ∞ ; and are true tensors. We choose

$$\frac{1}{2\mu}\Phi = \alpha \mathbf{U} \frac{1}{r}, \quad \frac{1}{2\mu}\chi = \beta \mathbf{u} \cdot \nabla \frac{1}{r}$$

This gives a solution which is a superposition of a Stokeslet and a source dipole

$$\mathbf{u} = -\alpha \left(\frac{\mathbf{U}}{r} + \frac{(\mathbf{U} \cdot \mathbf{x})\mathbf{x}}{r^3} \right) + \beta \left(-\frac{\mathbf{U}}{r^3} + 3\frac{(\mathbf{U} \cdot \mathbf{x})\mathbf{x}}{r^5} \right)$$

Enforcing the boundary condition $\mathbf{u} = \mathbf{U}$ on $r = a$ requires

$$\begin{aligned} -\frac{\alpha}{a} - \frac{\beta}{a^3} &= 1, & -\frac{\alpha}{a} + 3\frac{\beta}{a^3} &= 0 \\ \implies \alpha &= -\frac{3a}{4}, & \beta &= -\frac{a^3}{4} \end{aligned}$$

Thus the final solution for a sphere in a Stokes flow is

$$\mathbf{u} = \frac{3}{4}\mathbf{U} \left(\frac{a}{r} + \frac{a^3}{3r^3} \right) + \frac{3}{4} \frac{(\mathbf{U} \cdot \mathbf{x})\mathbf{x}}{r^2} \left(\frac{a}{r} - \frac{a^3}{r^3} \right)$$

The corresponding pressure and vorticity, both of which are harmonic and due to the Stokeslet, are

$$p = \frac{3}{2}\mu a \frac{\mathbf{U} \cdot \mathbf{x}}{r^3}, \quad \boldsymbol{\omega} = -\frac{1}{\mu} \nabla \times \Phi = \frac{3a}{2} \frac{\mathbf{U} \times \mathbf{x}}{r^3}$$

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2.6.6 Force and stress on a translating sphere

The flow at large distances is dominated by the $\frac{1}{r}$ Stokeslet term with strength $\mathbf{F} = 6\pi\mu a\mathbf{U}$. Since the force across $r = \infty$ must balance the force across $r = a$, we can see without further work that the force by the fluid on the sphere i.e. drag $= -6\pi\mu a\mathbf{U}$. In more detail:

$$\begin{aligned} \underline{\underline{\sigma}} &= -p\underline{\underline{I}} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \\ &= -\frac{3}{2}\mu a \frac{\mathbf{U} \cdot \mathbf{x}}{r} + \frac{3}{4}\mu \left[\left(-\frac{a}{r^3} - \frac{a^3}{r^5} \right) (\mathbf{x}\mathbf{U} + \mathbf{U}\mathbf{x}) \right. \\ &\quad \left. + (\mathbf{U}\mathbf{x} + \mathbf{x}\mathbf{U} + 2\mathbf{U} \cdot \mathbf{x}\underline{\underline{I}}) \left(\frac{a}{r^3} - \frac{a^3}{r^5} \right) + (\mathbf{U} \cdot \mathbf{x}) \left(-\frac{3a}{r^5} + \frac{5a^3}{r^7} \right) 2\mathbf{x}\mathbf{x} \right] \\ \implies \underline{\underline{\sigma}} \cdot \mathbf{n} |_{r=a} &= -\frac{3}{2}\frac{\mu}{a}(\mathbf{U} \cdot \mathbf{n})\mathbf{n} + \frac{3\mu}{4a}[-2(\mathbf{n}(\mathbf{U} \cdot \mathbf{n}) + \mathbf{U}) + (\mathbf{U} \cdot \mathbf{n})(-2)2\mathbf{n}] \\ &= -\frac{3\mu}{2a}\mathbf{U} \end{aligned}$$

Thus the drag on the sphere is

$$\mathbf{F} = \int_{r=a} \underline{\underline{\sigma}} \cdot \mathbf{n} dS = -\frac{3\mu}{2a}\mathbf{U} \cdot 4\pi a^2 = -6\pi\mu a\mathbf{U}$$

as expected.

2.6.7 Gravitational settling

The force balance of weight, buoyancy, and drag gives the settling velocity of a rigid sphere in a Stokes flow.

$$\begin{aligned} \frac{4}{3}\pi a^3 \rho_s \mathbf{g} - \frac{4}{3}\pi a^3 \rho \mathbf{g} - 6\pi\mu a \mathbf{U} &= 0 \\ \implies \mathbf{U} &= \frac{2a^2}{9\mu}(\rho_s - \rho)\mathbf{g} \propto a^2 \end{aligned}$$

2.6.8 2D potentials

The harmonic functions in two-dimensions which are bounded as $r \rightarrow \infty$ are

$$\ln r, \quad \nabla \ln r = \frac{\mathbf{x}}{r^2}, \quad \nabla \nabla \ln r = \frac{\underline{\underline{I}}}{r^2} - \frac{2\mathbf{x}\mathbf{x}}{r^4}, \quad \dots$$

Similar to the 3D spherical case, the harmonic functions in two-dimensions which are bounded as $r \rightarrow 0$ are

$$1, \quad r^2 \nabla \ln r = \mathbf{x}, \quad \dots, \quad r^{2n} \nabla^n \ln r$$

The 2D Stokeslet solution follows from the Papkovitch-Neuber potentials $\Phi = \frac{\mathbf{F}}{2\pi} \ln r$ which gives

$$\begin{aligned} \mathbf{u} &= \mathbf{F} \cdot \underline{\underline{J}}^{2D} \\ p &= \frac{\mathbf{F} \cdot \mathbf{x}}{2\pi r^2} \\ \underline{\underline{\sigma}} &= \mathbf{F} \cdot \underline{\underline{K}}^{2D} \\ \underline{\underline{J}}^{2D} &= \frac{1}{4\pi\mu} \left(-\ln r \underline{\underline{I}} + \frac{\mathbf{x}\mathbf{x}}{r^2} \right) \\ \underline{\underline{K}}^{2D} &= -\frac{1}{\pi} \frac{\mathbf{x}\mathbf{x}\mathbf{x}}{r^4} \end{aligned}$$

The 2D Stokeslet solution corresponds to a line force \mathbf{F} per unit length. Note that $\mathbf{u} \not\rightarrow 0$ as $r \rightarrow \infty$ for a line force, though $\mathbf{u} \rightarrow 0$ for line dipoles, line quadrupoles, etc.

2.7 Motion of rigid particles

2.7.1 The resistance matrix

A rigid particle moving with velocity $\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{x}$ through fluid otherwise at rest exerts a force \mathbf{F} and a couple \mathbf{G} on the fluid. By linearity,

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & \underline{\underline{D}} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\Omega} \end{pmatrix}$$

where the tensors $\underline{\underline{A}} - \underline{\underline{D}}$ depend on the size, shape, and orientation of the body. We refer to this matrix as the *resistance matrix*. We have

$$\int_{\text{body}} \mathbf{u} \cdot \underline{\underline{\sigma}} \cdot \mathbf{n} \, dS = \int (\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{x}) \cdot \underline{\underline{\sigma}} \cdot \mathbf{n} \, dS = \mathbf{U} \cdot \mathbf{F} + \boldsymbol{\Omega} \cdot \mathbf{G}$$

Similarly, from the reciprocal theorem (with $\mathbf{F} = 0$), for all $\mathbf{U}_1, \boldsymbol{\Omega}_1, \mathbf{U}_2, \boldsymbol{\Omega}_2$,

$$(\mathbf{U}_1, \boldsymbol{\Omega}_1) \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & \underline{\underline{D}} \end{pmatrix} \begin{pmatrix} \mathbf{U}_2 \\ \boldsymbol{\Omega}_2 \end{pmatrix} = (\mathbf{U}_2, \boldsymbol{\Omega}_2) \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{C}} & \underline{\underline{D}} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \boldsymbol{\Omega}_1 \end{pmatrix} = (\mathbf{U}_1, \boldsymbol{\Omega}_1) \begin{pmatrix} \underline{\underline{A}}^T & \underline{\underline{C}}^T \\ \underline{\underline{B}}^T & \underline{\underline{D}}^T \end{pmatrix} \begin{pmatrix} \mathbf{U}_2 \\ \boldsymbol{\Omega}_2 \end{pmatrix}$$

Hence $\underline{\underline{A}} = \underline{\underline{A}}^T$, $\underline{\underline{D}} = \underline{\underline{D}}^T$ so $\underline{\underline{A}}$ and $\underline{\underline{D}}$ are diagonalisable, and $\underline{\underline{B}} = \underline{\underline{C}}^T$, i.e. the force from pure rotation is equal to the couple from pure translation. Since the viscous dissipation is positive, the resistance matrix is positive definite, so invertible.