Cambridge Part III Maths

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Astrophysical Fluid Dynamics

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Lecture 1
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1 Introduction

1.1 Areas of application

Astrophysical fluid dynamics (AFD) is relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, the interstellar medium, the intergalactic medium, and cosmology itself. A fluid description is not applicable in regions that are solidified, such as the rocky or icy cores of giant planets and the crusts of neutron stars, and also in very genuous regions where the medium is not sufficiently collisional.

1.2 Theoretical varieties

Various flavours of AFD are in use. The basic models we will consider are:

Hydrodynamics (HD) / Newtonian gas dynamics: This model is non-relativistic, compressible, ideal (inviscid and adiabatic), self-gravitating, and usually assumes a perfect gas.

Magnetohydrodynamics (MHD): This model is the same as above, with the addition of a magnetic field. We will often use ideal MHD, which assumes a perfectly conducting fluid.

1.3 Characteristic features

The elements of theory often important in AFD are compressibility, gravitation, and thermal physics. Sometimes, magnetic fields, radiation forces, and relativity are important. Rarely important aspects are viscosity, surface tension, and solid boundaries.

1.4 Useful data

Some useful data for the course, in CGS (centimetres, grams, seconds) units:

Newton's constant	$G = 6.674 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$
Boltzmann's constant	$k = 1.381 \times 10^{-16}\,\mathrm{erg}\,\mathrm{K}^{-1}$
Stefan's constant	$\sigma = 5.670 \times 10^{-5}\,\mathrm{erg}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{K}^{-4}$
Speed of light	$c = 2.998 \times 10^{1-}\,\mathrm{cm}\,\mathrm{s}^{-1}$
Hydrogen mass	$m_H = 1.674 \times 10^{-24}\,\mathrm{g}$
Solar mass	$M_s = 1.988 \times 10^{33}\,\mathrm{g}$
Solar radius	$R_s = 6.957 \times 10^{10}\,\mathrm{cm}$
Solar luminosity	$L_s = 3.828 \times 10^{33}\,\mathrm{ergs}^{-1}$
Parsec	$pc = 3.086 \times 10^{18}\,\mathrm{cm}$
Astronomial unit (AU)	$au = 1.496 \times 10^{13}\,\mathrm{cm}$
Joule erg conversion	$1\mathrm{J} = 10^7\mathrm{erg}$

2 Ideal gas dynamics

2.1 Fluid variables

A fluid is characterised by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The *specific volume* (volume per unit mass) is $v = 1/\rho$.

We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionisation and recombination.

2.2 Eulerian and Lagrangian viewpoints

In the *Eulerian* viewpoint we consider how fluid properties vary in time at a point which is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian time derivative is simply ∂_t.

In the *Lagrangian* viewpoint we consider how fluid properties vary in time at a point which moves with the fluid at velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The Lagrangian time derivative (or material derivative) is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

2.3 Material points and structures

A material point is an idealised fluid element, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of the fluid. Note that the true particles of which the fluid is composed also have random thermal motion.

Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are deformed by it. An infinitesimal material line element $\delta\boldsymbol{x}$ evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x} \cdot \nabla\boldsymbol{u}$$

It changes its length and/or orientation in the presence of a velocity gradient.

Infinitesimal material surface or volume elements can be defined from two or three material line elements according to the vector product and the triple scalar product.

$$\delta \boldsymbol{S} = \delta \boldsymbol{x}^{(1)} \times \delta \boldsymbol{x}^{(2)}$$
$$\delta V = \delta \boldsymbol{x}^{(1)} \cdot \delta \boldsymbol{x}^{(2)} \times \delta \boldsymbol{x}^{(3)}$$

They evolve according to

$$\frac{\mathrm{D}\delta \boldsymbol{S}}{\mathrm{D}t} = (\nabla \cdot \boldsymbol{u})\delta \boldsymbol{S} - \nabla \boldsymbol{u} \cdot \delta \boldsymbol{S}$$
$$\frac{\mathrm{D}\delta V}{\mathrm{D}t} = (\nabla \cdot \boldsymbol{u})\delta V$$

The second result is easier to understand: the volume element increases when the flow is divergent.

2.4 Equation of mass conservation

The equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

has typical form of conservation law: rate of change of a density and divergence of a flux. Here, ρ is mass density and $\rho \boldsymbol{u}$ is mass flux density. An alternative form of the same equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}$$

If $\delta m = \rho \delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0$$

2.5 Equation of motion

The equation of motion

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p$$

derives from Newton's second law per unit volume with gravitational and pressure forces. The gravitational potential is $\Phi(\boldsymbol{x}, t)$ and $\boldsymbol{g} = -\nabla \Phi$ is the gravitational field.
The force due to pressure acting on a volume V with bounding surface S is

$$-\int_S p \, \mathrm{d}\boldsymbol{S} = \int_V (-\nabla p) \, \mathrm{d}V$$

Viscous forces are neglected in ideal gas dynamics.

2.6 Poisson's equation

The gravitational potential is related to the mass density by Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho$$

where G is Newton's constant. The solution

$$\Phi = \Phi_{\text{int}} + \Phi_{\text{ext}} = -G \int_V \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, \mathrm{d}^3 \boldsymbol{x}' - G \int_{\hat{V}} \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, \mathrm{d}^3 \boldsymbol{x}'$$

generally involves contributions form both the fluid region V under consideration and the exterior region $\hat{V}$. A *non-self-gravitating* fluid is one of negligible mass for which Φ_{int} can be neglected. More generally, the *Cowling approximation* consists of treating Φ as being specified in advance, so that Poisson's equation is not coupled to the other equations.

2.7 Thermal energy equation and equation of state

In the absence of non-adiabatic heating (e.g. by viscous dissipation or nuclear reactions) and cooling (e.g. by radiation or conduction),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0$$

where s is the *specific entropy* (entropy per unit mass). Fluid element undergo reversible thermodynamic changes and preserve their entropy (adiabatic flow). This condition is violated in shocks (see section 6).

The thermal variables (T, s) can be related to the dynamical variables (P, ρ) via an *equation of state* and standard thermodynamic identities. The most important case is that of an *ideal gas* with *blackbody radiation*

$$p = p_g + p_r = \frac{k\rho T}{\mu m_H} + \frac{4\sigma T^4}{3c}$$

where k is Boltzmann's constant, m_H is mass of a hydrogen atom, σ is Stefan's constant, c is the speed of light, and μ is the mean molecular weight, defined as the average mass of the particles in units of m_H, equal to

- 2.0 for molecular hydrogen
- 1.0 for atomic hydrogen
- 0.5 for fully ionised hydrogen
- about 0.6 for ionised matter of typical cosmic abundances.

The component p_g is the *gas pressure* and p_r is the *radiation pressure*. Radiation pressure is usually negligible except in the centres of high mass stars and in the immediate environments of neutron stars and black holes. The pressure of an ideal gas is often written in the form $\mathcal{R}\rho T/\mu$ where $\mathcal{R} = k/m_H$ is a version of the universal gas constant.

We define the *first adiabatic exponent*

$$\Gamma_1 = \left(\frac{\partial \log p}{\partial \log \rho}\right)_S$$

which is related to the ratio of specific heat capacities

$$\gamma = \frac{c_p}{c_v} = \frac{T\left(\frac{\partial s}{\partial T}\right)_p}{T\left(\frac{\partial s}{\partial T}\right)_V}$$

by $\Gamma_1 = \chi_\rho \gamma$ where

$$\chi_\rho = \left(\frac{\partial \log p}{\partial \log \rho}\right)_T$$

can be found from the equation of state. We can then rewrite the thermal energy equation as

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\Gamma_1 p \nabla \cdot \boldsymbol{u}$$

For an ideal gas with negligible radiation pressure, $\chi_\rho = 1$ and so $\Gamma_1 = \gamma$. Adopting this very common assumption, we write

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \boldsymbol{u}$$

2.8 Simplified models

A *perfect gas* may be defined as an ideal gas with constant c_v, c_p, γ and μ. Equipartition of energy for a classical gas with n degrees of freeodm per particle gives

$$\gamma = 1 + \frac{2}{n}$$

For a classical monatomic gas with $n = 3$ translational degrees of freedom, $\gamma = 5/3$. This is relevant for fully ionised matter. For a classical diatomic gas with two addditional rotational degrees of freeodm, $n = 5$ and $\gamma = 7/5$. This is relevant for molecular hydrogen. In reality, Γ_1 is variable when the gas undergoes ionisation or when the gas and radiation pressures are comparable.
The specific *internal energy* (or *thermal energy*) of a perfect gas is

$$e = \frac{p}{(\gamma - 1)\rho} = \frac{n}{\mu m_H} \frac{1}{2} kT$$

Note that each particle has an internal energy of $\frac{1}{2}kT$ per degree of freedom, and the number of particles per unit mass is $1/\mu m_H$.
A *barotropic* fluid is an idealised situation in which the relation $p(\rho)$ is known in advance. We can then dispense with the thermal energy equation. For example, if the gas is strictly isothermal and perfect, $p = c_s^2 \rho$ with the constant c_s being the isothermal sound speed. Alternatively, if the gas is strictly homentropic (constant s) and perfect, then $p = K\rho^\gamma$ with K constant.
An *incompressible* fluid is an idealised situation in which $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, implying $\nabla \cdot \boldsymbol{u} = 0$. This can be achieved formally by taking the limit $\gamma \to \infty$. The approximation of incompressibility eliminates acoustic phenomena from the dynamics. The ideal gas law itself is not valid at very high densities or where quantum degeneracy is important.

2.9 Microphysical basis

It is useful to understand the way in which the fluid-dynamical equations are derived microphysical considerations. The simplest model involves identical neutral particles of mass m of negligible size with no internal degrees of freedom.

2.9.1 Boltzmann equation

Between collisions, particles follow Hamiltonian trajectories in their six dimensional $(\boldsymbol{x}, \boldsymbol{v})$ phase space:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = a_i - \frac{\partial \Phi}{\partial x_i}$$

The distribution funnction $f(\boldsymbol{x}, \boldsymbol{v}, t)$ specifies the number density of particles in phase space. The velocity moments of f define the number $n(\boldsymbol{x}, t)$ in real space, the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the velocity dispersion $c(\boldsymbol{x}, t)$ according to

$$\int f \mathrm{d}^3\boldsymbol{v} = n$$

$$\int \boldsymbol{v} f \mathrm{d}^3\boldsymbol{v} = n\boldsymbol{u}$$

$$\int |\boldsymbol{v} - \boldsymbol{u}|^2 f \mathrm{d}^3\boldsymbol{v} = 3nc^2$$

Equivalently,

$$\int v^2 f \, \mathrm{d}^3\boldsymbol{v} = n(\boldsymbol{u}^2 + 3c^2)$$

The relation between velocity dispersion and temperature is $kT = mc^2$. In the absence of collisions, f is conserved following the Hamiltonian flow in phase space. This is because particles are conserved and the flow in phase space is incompressible. More generally, f evolves according to *Boltzmann's equation*

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_c$$

The collision term on the RHS is a complicated integral operator but has 3 simple properties corresponding to the conservation of mass, momentum and energy in collisions.

$$\int m \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

$$\int m\boldsymbol{v} \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

$$\int \frac{1}{2} m\boldsymbol{v}^2 \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

The collision term is local in $\boldsymbol{x}$ (not even involving derivatives) although it does involve integrals over $\boldsymbol{v}$. The equation $\left(\frac{\partial f}{\partial t}\right)_c = 0$ has the general solution

$$f = f_M = (2\pi c^2)^{-3/2} n \exp\left(-\frac{|\boldsymbol{v} - \boldsymbol{u}|^2}{2c^2}\right)$$

with parameters $n, \boldsymbol{u}$ and c that may depend on $\boldsymbol{x}$. This is the *Maxwellian distribution.*

2.9.2 Derivation of fluid equations

A crude but illuminating model of the collision operator is the *BGK approximation*

$$\left(\frac{\partial f}{\partial t}\right)_c \approx -\frac{1}{\tau}(f - f_M)$$

where f_M is a Maxwellian distribution with the same $n, \boldsymbol{u}$ and c as f, and τ is the *relaxation time.* This timescale τ can be identified approximately with the mean flight time of particles between collisions. The collisions attempt to restore a Maxwellian distribution on a characteristic timescale τ. They do this by randomising the particle velocities in a way consistent with the conservation of momentum and energy. If the characteristic timescale of the fluid flow is much greater than τ, then the collision term dominates the Boltzmann equation and f is very close to f_m. This is the *hydrodynamic limit.*

The velocity moments of f_M can be determined from standard Gaussian integrals, in particular

$$\int f_M \, \mathrm{d}^3\boldsymbol{v} = n$$

$$\int v_i f_M \, \mathrm{d}^3\boldsymbol{v} = n u_i$$

$$\int v_i v_j f_M \, \mathrm{d}^3\boldsymbol{v} = n(u_i u_j + c^2 \delta_{ij})$$

$$\int \boldsymbol{v}^2 v_i f_M \, \mathrm{d}^3\boldsymbol{v} = n(\boldsymbol{u}^2 + 5c^2) u_i$$

We obtain equations for mass, momentum and energy by taking moments of the Boltzmann equation weighted by $(m, mv_i, \frac{1}{2}m\boldsymbol{v}^2)$. In each case the collision term integrates to 0 because of its conservative properties, and the $\partial/\partial v_j$ term can be integrated by parts. We replace f with f_M when evaluating the LHS and note that $mn = \rho$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left[\rho(u_i u_j + c^2\delta_{ij})\right] - \rho a_i = 0$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\boldsymbol{u}^2 + \frac{3}{2}\rho c^2\right] + \frac{\partial}{\partial x_i}\left[(\frac{1}{2}\rho\boldsymbol{u}^2 + \frac{5}{2}\rho c^2)u_i\right] - \rho u_i a_i = 0$$

These are equivalent to the equations of ideal gas dynamics in conservative form (see section 4) for a monatomic ideal gas ($\gamma = 5/3$). The specific internal energy is $\rho = \frac{3}{2}c^2 = \frac{3}{2}\frac{kT}{m}$.

2.10 Validity of fluid approach

Deviations from Maxwellian distribution are small when collisions are frequent compared to the characteristic timescale of the flow. In higher-order approximations these deviations can be estimated, leading to the equations of dissipative gas dynamics including transport effects (viscosity and heat conduction).

The fluid approach breaks down if the mean flight time τ is not much less than the characteristic timescale of the flow, or if the mean free path $\lambda \approx c\tau$ between collisions is not much less than the characteristic lengthscale of the flow.

The Coulomb cross-section for ‘collisions’ (i.e. large-angle scatterings) between charged particles (electrons or ions) is

$$\sigma \approx 1 \times 10^{-4}(T/K)^{-2}\mathrm{cm}^2$$

The mean free path is $\lambda = 1/n\sigma$.

3 Ideal magnetohydrodynamics

3.1 Elementary derivation of MHD equations

Magnetohydrodynamics (MHD) is the dynamics of an electrically conducting fluid (a full or partially ionised gas or a liquid metal) containing a magnetic field. It is a fushion of fluid dynamics and electromagnetism.

3.1.1 Galilean electromagnetism

The equations of Newtonian gas dynamics are invariant under the Galilean transformation to a frame of reference moving with uniform velocity $\boldsymbol{v}$,

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t, \quad t' = t$$

Under this change of frame, the fluid velocity transforms according to

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}$$

while scalar variables such as p, ρ, and Φ are invariant. The Lagrangian time derivative $\mathrm{D}/\mathrm{D}t$ is also invariant, because the partial derivatives transform according to

$$\nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$$

To derive a consistent Newtonian theory of MHD, valid for situations in which the fluid motions are slow compared to the speed of light, we use Maxwell's equations without the displacement current:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

We will not require the fourth Maxwell equation, involving $\nabla \cdot \boldsymbol{E}$, because the charge density will be found to be unimportant. The equation of energy conservation is

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) = -\boldsymbol{E} \cdot \boldsymbol{J}$$

in which the energy density $B^2/2\mu_0$ is purely magnetic, while the energy flux density has the usual form of the *Poynting vector* $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$. These 'pre-Maxwell' equations are invariant under the Galilean transformation, provided that the fields transform according to

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \quad \boldsymbol{B}' = \boldsymbol{B}, \quad \boldsymbol{J}' = \boldsymbol{J}$$

3.1.2 Induction equation

In the *ideal MHD approximation* we regard the fluid as a perfect electrical conductor. The electric field in the rest frame of the fluid therefore vanishes, implying that

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$$

in a frame in which the fluid velocity is $\boldsymbol{u}(\boldsymbol{x}, t)$. This condition can be regarded as the limit of a constitutive relationship such as Ohm's law, in which the effects of resistivity (i.e. finite conductivity) are neglected. From Maxwell's equations, we then obtain the *ideal induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

This is an evolutionary equation for $\boldsymbol{B}$ alone, having elimnated $\boldsymbol{E}$ and $\boldsymbol{J}$. The divergence of the induction equation,

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

ensures that $\boldsymbol{B}$ remains solenoidal.

3.1.3 Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a Lorentz force

$$\boldsymbol{F}_m = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. This can be understood as the sum of the Lorentz forces on individual particles of charge q and velocity $\boldsymbol{v}$,

$$\sum q\boldsymbol{v} \times \boldsymbol{B} = \left(\sum q\boldsymbol{v}\right) \times \boldsymbol{B}$$

The electrostatic force can be shown to be negligible in the Newtonian limit. In Cartesian coordinates we have

$$
\begin{aligned}
(\mu_0 \boldsymbol{F}_m)_i &= \varepsilon_{ijk}\varepsilon_{jlm}\frac{\partial B_m}{\partial x_l}B_k \\
&= \left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right)B_k \\
&= B_k\frac{\partial B_i}{\partial x_k} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right)
\end{aligned}
$$

Thus the Lorentz force can be written as

$$\boldsymbol{F}_m = \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

The first term can be interpreted as a *curvature force* due to a *magnetic tension* $T_m = B^2/\mu_0$ per unit area in the field lines. The second term is the gradient of an isotropic *magnetic pressure* $p_m = B^2/2\mu_0$ which is also the magnetic energy density. The magnetic tension gives rise to *Alfvén waves* (see later), which travel parallel to the magnetic field with characteristic speed

$$v_a = \left(\frac{T_m}{\rho}\right)^{1/2} = \frac{B}{\sqrt{\mu_0\rho}}$$

the *Alfvén speed*, or (vector) *Alfvén velocity*

$$\boldsymbol{v}_a = \frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}}$$

The magnetic pressure also affects the propagation of sound waves, which become *magnetoacoustic waves* (see later). The combination

$$\Pi = p + \frac{B^2}{2\mu_0}$$

is often referred to as the *total pressure*, while the ratio

$$\beta = \frac{p}{B^2/2\mu_0}$$

is known as the *plasma beta*.

3.1.4 Summary of the MHD equations

The full set of ideal MHD equations is

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= 0 \\
\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \\
\frac{\mathrm{D}s}{\mathrm{D}t} &= 0 \\
\frac{\partial\boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) \\
\nabla\cdot\boldsymbol{B} &= 0
\end{aligned}
$$

together with the equation of state, Poisson's equation, etc., as required. Most of these equations can be written in at least one other way that may be useful in different circumstances.

3.2 Physical interpretation of MHD

There are two aspects to MHD:

- Advection of $\boldsymbol{B}$ by $\boldsymbol{u}$ (induction equation)
- Dynamical back-reaction of $\boldsymbol{B}$ on $\boldsymbol{u}$.

3.2.1 Kinematics of the magnetic field

The ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

has a beautiful geometric interpretation: magnetic field lines are 'frozen in' to the fluid and can be identified with material curves. One way to show this result is to use the identity

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \boldsymbol{u}(\nabla \cdot \boldsymbol{B})$$

to write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u})$$

and use the equation of mass conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right) \cdot \nabla \boldsymbol{u}$$

This is exactly the same equation satisfied by a material line element (section 2.3). Therefore a magnetic field line (an integral curve of $\boldsymbol{B}/\rho$) is advected and distorted by the fluid in the same way as a material curve. A complementary property is that the magnetic flux $\delta\boldsymbol{\Phi} = \boldsymbol{B} \cdot \delta\boldsymbol{S}$ through a material surface is conserved:

$$\begin{aligned}\frac{\mathrm{D}\boldsymbol{\Phi}}{\mathrm{D}t} &= \frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} \cdot \delta\boldsymbol{S} + \boldsymbol{B} \cdot \frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} \\ &= \left(B_j \frac{\partial u_i}{\partial x_j} - B_i \frac{\partial u_j}{\partial x_j}\right)\delta S_i + B_i \left(\frac{\partial u_j}{\partial x_j}\delta S_i - \frac{\partial u_j}{\partial x_i}\delta S_j\right) \\ &= 0\end{aligned}$$

By extension, we have conservation of the magnetic flux $\boldsymbol{\Phi} = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ passing through any material surface.

3.2.2 (Partial) analogy with vorticity

In homentropic (uniform entropy) or barotropic ideal fluid dynamics in the absence of a magnetic field, the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ satisfies

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})$$

Vortex lines are then 'frozen in' to the fluid. Conserved quantity

$$\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}$$

is analogous to

$$\int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x}$$

However, $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ whereas in MHD $\boldsymbol{B}$ and $\boldsymbol{u}$ are indirectly related through the equation of motion and the Lorentz force, so the analogy between vorticity dynamics and MHD is limited in scope.

3.2.3 Lorentz force

The Lorentz force per unit volume,

$$\boldsymbol{F}_m = \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla(B^2/2\mu_0)$$

can also be written as the divergence of the *Maxwell stress tensor*

$$\boldsymbol{F}_m = \nabla\cdot\mathsf{M}, \qquad \mathsf{M} = \frac{1}{\mu_0}\left(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\mathsf{I}\right)$$

where I is the identity tensor. In Cartesian components,

$$(\boldsymbol{F}_m)_i = \frac{\partial \mathsf{M}_{ji}}{\partial x_j}, \qquad \mathsf{M}_{ij} = \frac{1}{\mu_0}\left(B_iB_j - \frac{B^2}{2}\delta_{ij}\right)$$

If $\boldsymbol{B}$ is locally aligned with the x-axis, then

$$\mathsf{M} = \begin{pmatrix} T_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} p_m & 0 & 0 \\ 0 & p_m & 0 \\ 0 & 0 & p_m \end{pmatrix}$$

showing the magnetic tension and pressure. Combining the ideas of magnetic tension and a frozen-in field leads to the picture of field lines as elastic strings embedded in the fluid. There is a close analogy between MHD and the dynamics of polymer solutions. The magnetic field imparts elasticity to the fluid.

3.2.4 Differential rotation and torsional Alfvén waves

First consider the kinematic behaviour of a magnetic field in the presence of a prescribed velocity field involving differential rotation. In cylindrical polar coordinates (r,ϕ,z) let

$$\boldsymbol{u} = r\Omega(r,z)\hat{\boldsymbol{\phi}}$$

Consider an axisymmetric magnetic field, which we separate into *poloidal* (meridional: r and z) and *toroidal* (azimuthal: ϕ) parts:

$$\boldsymbol{B} = \boldsymbol{B}_p(r,z,t) + B_\phi(r,z,t)\hat{\boldsymbol{\phi}}$$

The ideal induction equation reduces to

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0, \qquad \frac{\partial B_\phi}{\partial t} = r\boldsymbol{B}_p\cdot\nabla\Omega$$

Differential rotation winds the poloidal field to generate a torsional field. To obtain a steady state without winding, we require the angular velocity to be constant along each magnetic field line:

$$\boldsymbol{B}_p\cdot\nabla\Omega = 0$$

a result known as *Ferraro's law of isorotation*.
There is an energetic cost to winding the field, as work is done against magnetic tension. In a dynamical situation a strong magnetic field tends to enforce isorotation along its length. Generalise the analysis to allow for axisymmetric torsional oscillations:

$$\boldsymbol{u} = r\Omega(r, z, t)\hat{\boldsymbol{\phi}}$$

The azimuthal component of the equation of motion is then

$$\rho r \frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0 9r} \boldsymbol{B}_p \cdot \nabla(r\boldsymbol{B}_\phi)$$

Combine with the induction equation above to give

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\mu_0 \rho r^2} \boldsymbol{B}_p \cdot \nabla(r^2 \boldsymbol{B}_p \cdot \nabla\Omega)$$

This equation describes *torsional Alfvén waves*. For example, if $\boldsymbol{B}_p = B_z\hat{\boldsymbol{z}}$ is vertical and uniform, then

$$\frac{\partial^2 \Omega}{\partial t^2} = v_a^2 \frac{\partial^2 \Omega}{\partial z^2}$$

This is not strictly an exact non-linear analysis because we have neglected the force balance (and indeed motion) in the meridional plane.

3.2.5 Force-free fields

In regions of low density, e.g. the solar corona, $\boldsymbol{B}$ may be dynamically dominant over inertia, gravity and gas pressure. We then have (approximately) a force-free magnetic field such that

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = 0$$

Since $\nabla \times \boldsymbol{B}$ must be parallel to $\boldsymbol{B}$ in this case, we may write

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B} \tag{1}$$

for some scalar field $\lambda(\boldsymbol{x})$. Taking the divergence of (1) gives

$$0 = \boldsymbol{B} \cdot \nabla\lambda$$

so λ is constant along each magnetic field line. In the special case where λ is constant, known as a *linear force-free magnetic field*, the curl of (1) results in the *Helmholtz equation*

$$-\nabla^2 \boldsymbol{B} = \lambda^2 \boldsymbol{B}$$

which admits a wide variety of solutions. A subset of force-free magnetic fields consists of *potential* or *current-free* magnetic fields for which

$$\nabla \times \boldsymbol{B} = 0$$

In a true vacuum, $\boldsymbol{B}$ must be potential. However, only an extremely low density of electrons is needed to make the force-free description more relevant.
An example of a force-free field in cylindrical polar coordinates (r, ϕ, z) is

$$\boldsymbol{B} = B_\phi(r)\hat{\boldsymbol{\phi}} + B_z(r)\hat{\boldsymbol{z}}$$
$$\nabla \times \boldsymbol{B} = -\frac{\mathrm{d}B_z}{\mathrm{d}r}\hat{\boldsymbol{\phi}} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\phi\right)\hat{\boldsymbol{z}}$$

Now $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ implies

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B_z}{\mathrm{d}r}\right) = \lambda^2 B_z$$

which is the z component of the Helmholtz equation. The solution regular at $r = 0$ is

$$B_z = B_0 J_0(\lambda r), \qquad B_\phi = B_0 J_1(\lambda r)$$

where J_n is the Bessel function of order n. Note that $J_0(x)$ satisfies

$$(xJ_0')' + xJ_0 = 0, \qquad J_1(x) = -J_0'(x)$$

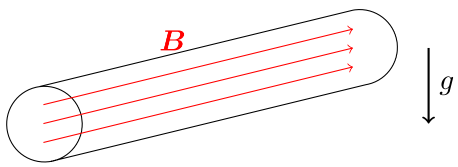
The helical nature of this field is typical of force-free fields with $\lambda \neq 0$.

3.2.6 Magnetostatic equilibrium and magnetic buoyancy

A *magnetostatic equilibrium* is a static solution ($\boldsymbol{u} = 0$) of the equation of motion, i.e. one satisfying

$$0 = -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

together with $\nabla \cdot \boldsymbol{B} = 0$. While solutions do exist, inhomogeneities in $\boldsymbol{B}$ often result in a lack of equilibrium. A magnetic flux tube is an idealised situation in which the magnetic field is localised to the interior of a tube and vanishes outside.



To balance the total pressure at the interface, the gas pressure must be lower inside. Unless the temperatures are different, the density is lower inside. In a gravitational field the tube experiences an upward buoyancy force and tends to rise.

4 Conservation laws & hyperbolic structure

4.1 Introduction

An equation in conservative form is written

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

where $q(\boldsymbol{x}, t)$ is the density of some property and $\boldsymbol{F}(\boldsymbol{x}, t)$ is the flux density of the same quantity. The total amount in a time-independent volume V is

$$Q = \int_V q\,\mathrm{d}V$$

and evolves due to the flux through the bounding surface

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_V \nabla \cdot \boldsymbol{F}\mathrm{d}V = -\int_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

The prototypical choice for this equation is mass with $q = \rho$, $\boldsymbol{F} = \rho\boldsymbol{u}$.

A *material invariant* is a scalar field $f(\boldsymbol{x},t)$ for which $\frac{\mathrm{D}f}{\mathrm{D}t}=0$. Then f is constant for each fluid element, and is therefore conserved following the fluid motion. An example is specific entropy s in ideal fluid dynamics. Combined with mass conservation we obtain an equation in conservative form:

$$\frac{\partial}{\partial t}(\rho f)+\nabla\cdot(\rho f\boldsymbol{u})=0$$

4.2 Total energy equation

Starting from the ideal MHD equations, we construct the total energy equation piece by piece. First, consider *kinetic energy*:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}u^2\right)=\rho\boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\rho\boldsymbol{u}\cdot\nabla\Phi-\boldsymbol{u}\cdot\nabla p+\frac{1}{\mu_0}\boldsymbol{u}\cdot(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

Gravitational energy (assuming a non-self-gravitating fluid and Φ independent of t):

$$\rho\frac{\mathrm{D}\Phi}{\mathrm{D}t}=\rho\boldsymbol{u}\cdot\nabla\Phi$$

Internal (thermal) energy (using the fundamental thermodynamic identity $\mathrm{d}e=T\mathrm{d}s-p\mathrm{d}v$):

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t}=\cancel{\rho T\frac{\mathrm{D}s}{\mathrm{D}t}}+p\frac{\mathrm{D}\log\rho}{\mathrm{D}t}=-\rho\nabla\cdot\boldsymbol{u}$$

Summing these three equations gives

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{u}^2+\Phi+e\right)=-\nabla\cdot(p\boldsymbol{u})+\frac{1}{\mu_0}\boldsymbol{u}\cdot(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

Rewrite the Lorentz force term:

$$\frac{1}{\mu_0}\boldsymbol{u}\cdot(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot(-\boldsymbol{u}\times\boldsymbol{B})=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}$$

Using mass conservation:

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2+\Phi+\rho)\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2+\Phi+\rho)+p\boldsymbol{u}\right]=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}$$

Finally, consider the magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right)=\frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}=-\frac{1}{\mu_0}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E})$$

The total energy equation is then

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2+\Phi+\rho)+\frac{\boldsymbol{B}^2}{2\mu_0}\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2+\Phi+h)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]=0$$

where $h=e+p/\rho$ is the *specific enthalpy* and we have used the identity

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})=\boldsymbol{B}\cdot\nabla\times\boldsymbol{E}-\boldsymbol{E}\cdot\nabla\times\boldsymbol{B}$$

to include the *Poynting vector* (EM energy flux density) $\boldsymbol{E}\times\boldsymbol{B}/\mu_0$. The total energy is therefore conserved.

For a self-gravitating system satisfying Poisson's equation, the gravitational energy can instead be regarded as $-g^2/8\pi G$:

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) = -\frac{1}{4\pi G}\nabla\Phi \cdot \frac{\partial \nabla\Phi}{\partial t}$$

using the fact $g = -\nabla\Phi$. We can 'integrate by parts' by writing

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla \cdot \left(\frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t}\right) = \frac{\Phi}{4\pi G}\frac{\partial \nabla^2\Phi}{\partial t} = \Phi\frac{\partial \rho}{\partial t} = -\Phi\nabla \cdot (\rho \boldsymbol{u})$$

Hence in conservative form we have

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0$$

The total energy equation is then

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2 + \rho) - \frac{g^2}{8\pi G} + \frac{\boldsymbol{B}^2}{2\mu_0}\right] + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0$$

It is important to note that some of the gravitational and magnetic energy of an astrophysical body is stored in the exterior region, even if mass density vanishes there.

4.3 Helicity conservation

In ideal fluid dynamics there are geometrical or topological invariants:

- (potential) vorticity/circulation
- kinetic helicity $\boldsymbol{u}\cdot\boldsymbol{\omega}$

The Lorentz force breaks these conservation laws, but new topological invariants associated with $\boldsymbol{B}$ appear. The *magnetic helicity* in a volume V with bounding surface S is defined as

$$H_m = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. Now

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\Phi_e = \boldsymbol{u}\times\boldsymbol{B} - \nabla\Phi_e$$

where Φ_e is the *electrostatic potential* ('uncurl' of the induction equation). Thus

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) = -\boldsymbol{B}\cdot\nabla\Phi_e + \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$$

So H_m is conserved in ideal MHD:

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) + \nabla\cdot[\Phi_e\boldsymbol{B} + \boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})] = 0$$

Under a *gauge transformation*

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\chi$$
$$\Phi_e \to \Phi_e - \frac{\partial\chi}{\partial t}$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, but H_m changes by

$$\int_V \boldsymbol{B}\cdot\nabla\chi\,\mathrm{d}V = \int_V \nabla\cdot(\chi\boldsymbol{B})\,\mathrm{d}V = \int_S \chi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S$$

So H_m is not uniquely defined unless $\boldsymbol{B}\cdot\boldsymbol{n}=0$ on the surface S. H_m is a *pseudoscalar*, i.e. it changes sign under a reflection. Non-zero helicity $H_m \neq 0$ occurs only when $\boldsymbol{B}$ lacks reflection symmetry. H_m can be interpreted topologically in terms of the twistedness and knottedness of $\boldsymbol{B}$ (see ES2, Q4). Since $\boldsymbol{B}$ is 'frozen in' to the fluid and is deformed continuously by it, the topological properties of $\boldsymbol{B}$ are conserved.
The equivalent conserved quantity in homentropic or barotropic ideal gas dynamics (without $\boldsymbol{B}$) is the *kinetic helicity*

$$H_k = \int_V \boldsymbol{u}\cdot(\nabla\times\boldsymbol{u})\,\mathrm{d}V$$

The *cross-helicity* in a volume V is

$$H_c = \int_V \boldsymbol{u}\cdot\boldsymbol{B}\,\mathrm{d}V$$

It is helpful to write the equation of motion in ideal MHD in the form

$$\frac{\partial u}{\partial t} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \nabla(\frac{1}{2}\boldsymbol{u}^2+\Phi+h) = T\nabla s + \frac{1}{\mu_0\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \tag{2}$$

using $\mathrm{d}h = T\mathrm{d}s + v\mathrm{d}p$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{B}) + \nabla\cdot\left[\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{B}) + (\frac{1}{2}\boldsymbol{u}^2+\Phi+h)\boldsymbol{B}\right] = T\boldsymbol{B}\cdot\nabla s$$

So H_c is conserved in homentropic/barotropic flow, i.e. when $T\boldsymbol{B}\cdot\nabla s = 0$.
Bernoulli's theorem follows from the inner product of the equation of motion (2) with $\boldsymbol{u}$. In steady flow,

$$\boldsymbol{u}\cdot\nabla(\frac{1}{2}\boldsymbol{u}^2+\Phi+h) = 0$$

i.e. the *Bernoulli function* $\frac{1}{2}\boldsymbol{u}^2+\Phi+h$ is constant along streamlines, but only if $\boldsymbol{u}\cdot\boldsymbol{F}_m = 0$, i.e. if $\boldsymbol{B}$ does no work on the flow, for example if $\boldsymbol{u}$ is parallel to $\boldsymbol{B}$.

4.4 Symmetries

The equations of ideal gas dynamics and MHD have numerous symmetries. For an isolated, self-gravitating system:

- Translations of time and space, and rotations of space: related (via Noether's theorem) to the conservation of energy, momentum and angular momentum
- Reversal of time: related to the absence of dissipation
- Reflections of space (but note that $\boldsymbol{B}$ is a pseudovector and behaves oppositely to $\boldsymbol{u}$ under a reflection)
- Galilean transformations
- Reversal of the sign of $\boldsymbol{B}$

- Similarity transformations: if space and time are rescaled by independent factors λ and μ, i.e.

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, \qquad t \to \mu t$$

  then (for a perfect gas) we have $\boldsymbol{u} \to \lambda\mu^{-1}\boldsymbol{u}, \rho \to \mu^{-2}\rho, p \to \lambda^2\mu^{-4}p, \Phi \to \lambda^2\mu^{-2}\Phi, \boldsymbol{B} \to \lambda\mu^{-2}\boldsymbol{B}$.

For a non-isolated system with an external potential Φ_{ext}, these symmetries (other than $\boldsymbol{B} \to -\boldsymbol{B}$) apply only if Φ_{ext} has them. But for a non-self-gravitating perfect gas, the mass can be rescaled by any factor λ:

$$\rho \to \lambda\rho, \quad p \to \lambda p, \quad \boldsymbol{B} \to \lambda^{1/2}\boldsymbol{B}$$

4.5 Hyperbolic structure

The hyperbolic structure is one way of understanding wave modes and information propagation in a fluid. It is fundamental to the construction of some numerical methods. We neglect gravity here, because it involves instantaneous action at a distance, i.e. not a finite wave speed. We write the equations of ideal gas dynamics as

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} &= 0 \\
\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{u} &= 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\rho}\nabla p &= 0
\end{aligned}$$

and combine in the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0$$

where $\boldsymbol{U}$ is a 5D 'state vector'

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ p \\ u_x \\ u_y \\ u_z \end{pmatrix}$$

and $\mathsf{A}_x, \mathsf{A}_y, \mathsf{A}_z$ are 5×5 matrices. This works because every term in the equation involves a first derivative with respect to either time or space.

$$\mathsf{A}_x = \begin{pmatrix} u_x & & \rho & & \\ & u_x & \gamma p & & \\ & 1/\rho & u_x & & \\ & & & u_x & \\ & & & & u_x \end{pmatrix} \qquad \mathsf{A}_y = \begin{pmatrix} u_y & & & \rho & \\ & u_y & & \gamma p & \\ & & u_y & & \\ & 1/\rho & & u_y & \\ & & & & u_y \end{pmatrix}$$

$$\mathsf{A}_z = \begin{pmatrix} u_z & & & & \rho \\ & u_z & & & \gamma p \\ & & u_z & & \\ & & & u_z & \\ & 1/\rho & & & u_z \end{pmatrix}$$

The system of equations is hyperbolic if the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector $\boldsymbol{n}$ and if the eigenvectors span the 5D space. The eigenvalues can be identified as wave speeds, and the eigenvectors as wave modes, with $\boldsymbol{n}$ being the unit wavevector, locally normal to the wavefronts.

Taking $\boldsymbol{n} = \hat{\boldsymbol{x}}$ WLOG, we find

$$\det\left(\mathsf{A}_x - v\mathsf{I}\right) = -(v-u_x)^3\left[(v-u_x)^2 - v_s^2\right]$$

where $v_s = \sqrt{\gamma p/\rho}$ is the *adiabatic sound speed*. The wave speeds v are real and the system is indeed hyperbolic. Two modes are *sound waves* (acoustic waves) which have speed $u_x \pm v_s$ and propagate at the sound speed relative to the fluid. Their eigenvectors

$$\begin{pmatrix} \rho \\ \gamma p \\ \pm v_s \\ 0 \\ 0 \end{pmatrix}$$

involve perturbations of density, pressure and longitudinal velocity. The other 3 modes have $v = u_x$ and do not propagate relative to the fluid. Their eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The *entropy wave* perturbs the density but not the pressure. Since $s = s(\rho, p)$, the entropy is perturbed. The *vortical waves* perturb the tranverse velocity and therefore the vorticity. These waves propagate at the fluid velocity because entropy and vorticity are conserved.
To extend the analysis to ideal MHD, consider the induction equation in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} + \boldsymbol{B}(\nabla\cdot\boldsymbol{u}) = 0$$

and include the Lorentz force in the equation of motion:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho}\left(p + \frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B} = 0$$

Every term still involves a first derivative, so the MHD equations can be written as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i\frac{\partial \boldsymbol{U}}{\partial x_i} = 0$$

where $\boldsymbol{U}$ is now an 8D state vector and the A_i are three 8×8 matrices. The characteristic polynomial of A_x is

$$\det\left(\mathsf{A}_x - v\mathsf{I}\right) = (v-u_x)^2\left((v-u_x)^2 - v_{ax}^2\right)\left((v-u_x)^4 - (v_s^2+v_a^2)(v-u_x)^2 + v_s^2 v_{ax}^2\right)$$

The wave speeds v are real and the system is hyperbolic. The MHD wavemodes will be examined in section 5. In this representation, there are two modes with $v = u_x$ that do not propagate relative to the fluid. One is the entropy wave, which is physical and involves only a density perturbation. The other is the '$\nabla\cdot\boldsymbol{B}$' mode which is unphysical and involves a perturbation of $\nabla\cdot\boldsymbol{B}$ (i.e. of B_x, in the case $\boldsymbol{n} = \hat{\boldsymbol{x}}$). This must be eliminated by imposing the constraint $\nabla\cdot\boldsymbol{B} = 0$. The vortical waves are replaced by Alfvén waves with speeds $u_x \pm v_{ax}$.

4.6 Stress tensor & virial theorem

In the absence of external forces, the equation of motion of a fluid can usually be written as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \mathsf{T}$$

where T is the *stress tensor*, a symmetric second rank tensor field. Using mass conservation, we can relate this to the equation of momentum conservation:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} - \mathsf{T}) = 0$$

So $-\mathsf{T}$ is the momentum flux density, excluding the advective flux. For a self-gravitating system in ideal MHD, the stress tensor is

$$\mathsf{T} = -p\mathsf{I} - \frac{1}{4\pi G}(\boldsymbol{g}\boldsymbol{g} - \frac{1}{2}g^2\mathsf{I}) + \frac{1}{\mu_0}(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathsf{I})$$

The gravitational stress tensor works for a self-gravitating system in which $\boldsymbol{g} = -\nabla\Phi$ and ρ are related through Poisson's equation

$$-\nabla \cdot \boldsymbol{g} = \nabla^2\Phi = 4\pi G\rho$$

For a general vector field $\boldsymbol{v}$, it can be shown that

$$\begin{aligned}\nabla \cdot (\boldsymbol{v}\boldsymbol{v} - \frac{1}{2}v^2\mathsf{I}) &= (\nabla \cdot \boldsymbol{v})\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla(\frac{1}{2}v^2) \\ &= (\nabla \cdot \boldsymbol{v})\boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v}\end{aligned}$$

In the magnetic case ($\boldsymbol{v} = \boldsymbol{B}$) this simplifies to $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$). In the gravitational case ($\boldsymbol{v} = \boldsymbol{g}$) it simplifies to $(\nabla \cdot \boldsymbol{g})\boldsymbol{g} = -4\pi G\rho\boldsymbol{g}$, which becomes the force per unit volume $\rho\boldsymbol{g}$ when divided by $-4\pi G$.
The *virial equations* are the spatial moments of the equation of motion. They provide integral measures of the balance of forces acting on the fluid. THe first moments are generally most useful. Recall

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial \mathsf{T}_{ji}}{\partial x_j}$$

Consider

$$\begin{aligned}\rho \frac{\mathrm{D}^2}{\mathrm{D}t^2}(x_i x_j) &= \rho \frac{\mathrm{D}}{\mathrm{D}t}(u_i x_j + x_i u_j) \\ &= 2\rho u_i u_j + x_j \frac{\partial \mathsf{T}_{ki}}{\partial x_k} + x_i \frac{\partial \mathsf{T}_{kj}}{\partial x_k}\end{aligned}$$

Consider a material volume V bounded by a material surface S. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f \,\mathrm{d}m = \int_V \frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}m$$

where f is any function and $\mathrm{d}m = \rho \mathrm{d}V$ is the material-invariant mass element. Integrate over V to get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V x_i x_j \,\mathrm{d}m &= \int_V \left(2\rho u_i u_j + x_j \frac{\partial \mathsf{T}_{ki}}{\partial x_k} + x_i \frac{\partial \mathsf{T}_{kj}}{\partial x_k}\right) \mathrm{d}V \\ &= \int_V \left(2\rho u_i u_j - \mathsf{T}_{ji} - \mathsf{T}_{ij}\right) \mathrm{d}V + \int_S \left(x_j \mathsf{T}_{ki} + x_i \mathsf{T}_{kj}\right) n_k \,\mathrm{d}S\end{aligned}$$

where we have integrated by parts using the divergence theorem:

$$\begin{aligned}\int_V f\frac{\partial g_k}{\partial x_k}\,\mathrm{d}V &= \int_V \left[\frac{\partial}{\partial x_k}(fg_k) - g_k\frac{\partial f}{\partial x_k}\right]\mathrm{d}V \\ &= \int_S fg_kn_k\,\mathrm{d}S - \int_V g_k\frac{\partial f}{\partial x_k}\,\mathrm{d}V\end{aligned}$$

For an isolated system with no external sources of gravity or magnetic field, $\boldsymbol{g}$ decays as $|\boldsymbol{x}|^{-2}$ at large distance, and $\boldsymbol{B}$ decays faster. Therefore T_{ij} decays as $|\boldsymbol{x}|^{-4}$ and the surface integral can be eliminated if we let V occupy the whole space. Divide by 2 to obtain the *tensor virial theorem*:

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = 2K_{ij} - \mathcal{T}_{ij}$$

where

$$I_{ij} = \int x_i x_j\,\mathrm{d}m$$

is related to the inertia tensor of the system,

$$K_{ij} = \int \frac{1}{2}u_i u_j\,\mathrm{d}m$$

is a kinetic energy tensor and

$$\mathcal{T}_{ij} = \int T_{ij}\,\mathrm{d}V$$

is the integrated stress tensor. Note if the above conditions are not satisfied, there will be an additional contribution from the surface integral. The *scalar virial theorem* is the trace of this equation:

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K - \mathcal{T}$$

Note that K is the total KE, and

$$\begin{aligned}-\mathcal{T} &= \int\left(3p - \frac{g^2}{8\pi G} + \frac{B^2}{2\mu_0}\right)\mathrm{d}V \\ &= 3(\gamma-1)U + W + M\end{aligned}$$

for a perfect gas with no external gravity, where U, W and M are the total internal, gravitational and magnetic energies. Thus

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + 3(\gamma-1)U + W + M$$

On the right-hand side, only W is negative. For the system to be bound (i.e. to noy fly apart) the kinetic, internal and magnetic energies are limited by

$$2K + 3(\gamma-1)U + M \leq |W|$$

In fact, equality must hold, at least on average, unless the system is collapsing or contracting.
The tensor virial theorem provides more specific information relating to the energies associated with individual directions. This is particularly relevant in cases where anisotropy is introduced by rotation or a magnetic field.

5 Linear waves in homogeneous media

In ideal MHD the density, pressure and magnetic field evolve according to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\boldsymbol{u}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{u}\\ \frac{\partial p}{\partial t} &= -\boldsymbol{u}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{u}\\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{u}\times\boldsymbol{B})\end{aligned}$$

Consider a magnetostatic equilibrium in which

$$\rho = \rho_0(\boldsymbol{x}), \quad p = p_0(\boldsymbol{x}), \quad \boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{x}), \quad \boldsymbol{u} = 0$$

Now consider small perturbations from equilibrium, such that

$$\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}) + \delta\rho(\boldsymbol{x}, t)$$

with $|\delta\rho| \ll \rho_0$, etc. The linearised equations are

$$\begin{aligned}\frac{\partial \delta\rho}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla\rho_0 - \rho_0\nabla\cdot\delta\boldsymbol{u}\\ \frac{\partial \delta p}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla p_0 - \gamma p_0\nabla\cdot\delta\boldsymbol{u}\\ \frac{\partial \delta\boldsymbol{B}}{\partial t} &= \nabla\times(\delta\boldsymbol{u}\times\boldsymbol{B}_0)\end{aligned}$$

Introduce the *displacement* $\boldsymbol{\xi}(\boldsymbol{x}, t)$ such that $\delta\boldsymbol{u} = \frac{\partial\boldsymbol{\xi}}{\partial t}$. Integrate to obtain

$$\begin{aligned}\delta\rho &= -\boldsymbol{\xi}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{\xi}\\ \delta p &= -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi}\\ \delta\boldsymbol{B} &= \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})\\ &= \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B}\end{aligned}$$

We can now drop the subscript 0 without danger of confusion. Note arbitrary additive functions of $\boldsymbol{x}$ can be discarded if all variables have the same harmonic time dependence. The linearised equation of motion is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\nabla\delta\boldsymbol{\Phi} - \delta\rho\nabla\boldsymbol{\Phi} - \nabla\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\delta\boldsymbol{B})$$

where the total pressure perturbation is

$$\begin{aligned}\delta\Pi &= \delta p + \frac{1}{\mu_0}\boldsymbol{B}\cdot\delta\boldsymbol{B}\\ &= -\boldsymbol{\xi}\cdot\nabla\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi})\end{aligned}$$

The gravitational potential perturbation satisfies

$$\nabla^2\delta\boldsymbol{\Phi} = 4\pi G\delta\rho$$

Consider a basic state of uniform ρ, p and $\boldsymbol{B}$, in the absence of gravity. This system is homogeneous but anisotropic, because $\boldsymbol{B}$ distinguishes a particular direction. The problem simplifies to

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla\delta\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}))$$

with

$$\delta\Pi = -\left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi})$$

Since the basic state is independent of $\boldsymbol{x}$ and t, there are plane wave solutions of the form

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \Re\left[\tilde{\boldsymbol{\xi}}\exp(i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t))\right]$$

where ω and $\boldsymbol{k}$ are the frequency and wavevector, and $\tilde{\boldsymbol{\xi}}$ is a constant vector representing the amplitude of the wave. For such solutions (omitting the tilde)

$$\rho\omega^2\boldsymbol{\xi} = \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]\boldsymbol{k} + \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\left[(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{\xi} - \boldsymbol{B}(\boldsymbol{k}\cdot\boldsymbol{\xi})\right] \tag{3}$$

For transverse displacements $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ hence this simplifies to

$$\rho\omega^2\boldsymbol{\xi} = \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2\boldsymbol{\xi}$$

These solutiosn are Alfvén waves with dispersion relation

$$\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{v}_a)^2$$

Given the dispersion relation $\omega(\boldsymbol{k})$ of any wave, the *phase and group velocities* are

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}}, \qquad \boldsymbol{v}_g = \frac{\partial\omega}{\partial\boldsymbol{k}} = \nabla_{\boldsymbol{k}}\omega$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. The phase velocity $\boldsymbol{v}_p$ is the velocity at which the phase of the wave travels, whilst the group velocity $\boldsymbol{v}_g$ is the velocity at which the energy of the wave (or the centre of a wavepacket) is transported. For Alfvén waves, $\omega = \pm\boldsymbol{k}\cdot\boldsymbol{v}_a$:

$$\boldsymbol{v}_p = \pm v_a\cos\theta\hat{\boldsymbol{k}}, \qquad \boldsymbol{v}_g = \pm\boldsymbol{v}_a$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$. To find the other solutions, consider $\boldsymbol{k}\cdot$(3) and $\boldsymbol{B}\cdot$(3):

$$\begin{aligned}\rho\omega^2\boldsymbol{k}\cdot\boldsymbol{\xi} &= \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]k^2 \\ \rho\omega^2\boldsymbol{B}\cdot\boldsymbol{\xi} &= \gamma p(\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{k}\cdot\boldsymbol{B}\end{aligned}$$

Write these together as

$$\begin{pmatrix}\rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2 & \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})k^2 \\ -\gamma p(\boldsymbol{k}\cdot\boldsymbol{B}) & \rho\omega^2\end{pmatrix}\begin{pmatrix}\boldsymbol{k}\cdot\boldsymbol{\xi} \\ \boldsymbol{B}\cdot\boldsymbol{\xi}\end{pmatrix} = 0$$

The 'trivial solution' $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ corresponds to the Alfvén wave. The other solutions satisfy

$$\rho\omega^2\left[\rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2\right] + \gamma pk^2\frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2 = 0$$

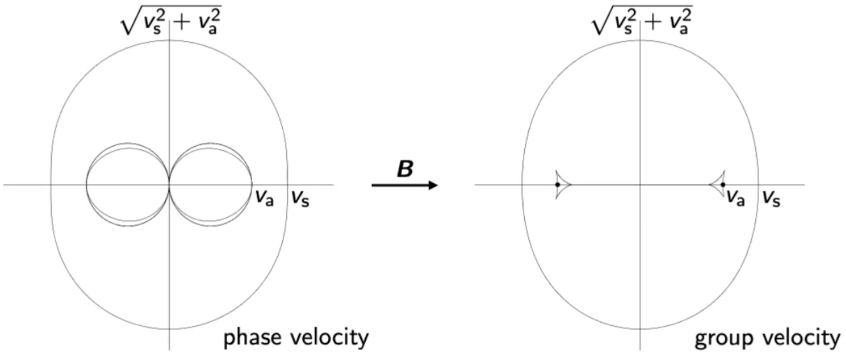


Figure 1: Friedrichs diagram for the case $v_a < v_s$ ($v_a = 0.7v_s$)

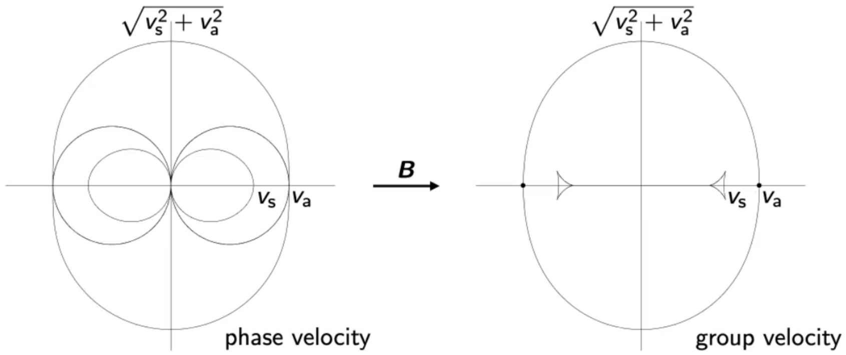


Figure 2: Friedrichs diagram for the case $v_a > v_s$ ($v_s = 0.7v_a$)

which simplifies to (divide by $\rho^2 k^4$):

$$v_p^4 - (v_s^2 + v_a^2)v_p^2 + v_s^2 v_a^2 \cos^2\theta = 0$$

The two solutions

$$v_p^2 = \frac{1}{2}(v_s^2 + v_a^2) \pm \left[\frac{1}{4}(v_s^2 + v_a^2)^2 - v_s^2 v_a^2 \cos^2\theta\right]^{1/2}$$

are called *fast and slow magnetoacoustic waves* respectively. In the special case $\theta = 0$ (i.e. $\boldsymbol{k} \parallel \boldsymbol{B}$) we have

$$v_p^2 = v_s^2 \quad \text{or} \quad v_p^2 = v_a^2$$

together with $v_p^2 = v_a^2$ for the Alfvén wave. Note the fast wave could be either $v_p^2 = v_s^2$ or $v_p^2 = v_a^2$, whichever is greater. In the special case $\theta = \pi/2$ (i.e. $\boldsymbol{k} \perp \boldsymbol{B}$), we have

$$v_p^2 = v_s^2 + v_a^2 \quad \text{or} \quad v_p^2 = 0$$

together with $v_p^2 = 0$ for the Alfvén wave.
Magnetic tension gives rise to Alfvén waves, similar to waves on an elastic string. Magnetic pressure responds to compression, so modifies the propagation of acoustic waves. *Friedrichs diagrams* are parametric plots of $\boldsymbol{v}_p(\theta)$ and $\boldsymbol{v}_g(\theta)$ for all θ: see figure 1 and 2.

Interpretation.

- The fast wave is a quasi-isotropic acoustic-type wave in which both gas and magnetic pressure contribute.

- The slow wave is an acoustic-type wave that is strongly guided by $\boldsymbol{B}$.
- The Alfvén wave is similar to a wave on an elastic string, propagating via magnetic tension and perfectly guided by $\boldsymbol{B}$.

6 Non-linear waves, shocks and discontinuities

6.1 1D gas dynamics

The equations of mass conservation and motion in 1D are

$$\begin{aligned}\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}&=-\rho\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\end{aligned}$$

We assume the gas is homentropic (s constant) and perfect. This eliminates the entropy wave and leaves only the two sound waves. Then $p\propto\rho^\gamma$ and $v_s^2=\gamma p/\rho\propto\rho^{\gamma-1}$. We use v_s as a variable in place of ρ or p:

$$\mathrm{d}p=v_s^2\,\mathrm{d}\rho,\qquad \mathrm{d}\rho=\frac{\rho}{v_s}\frac{2\,\mathrm{d}v_s}{\gamma-1}$$

Then

$$\begin{aligned}\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v_s\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right)&=0\\ \frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right)+v_s\frac{\partial u}{\partial x}&=0\end{aligned}$$

Add and subtract the two equations:

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(u+v_s)\frac{\partial}{\partial x}\right]\left(u+\frac{2v_s}{\gamma-1}\right)&=0\\ \left[\frac{\partial}{\partial t}+(u-v_s)\frac{\partial}{\partial x}\right]\left(u-\frac{2v_s}{\gamma-1}\right)&=0\end{aligned}$$

Define the two *Riemann invariants*

$$R_\pm=u\pm\frac{2v_s}{\gamma-1}$$

Then we deduce that $R_\pm=$ const. along a *characteristic (curve)* $C_\pm$ of gradient

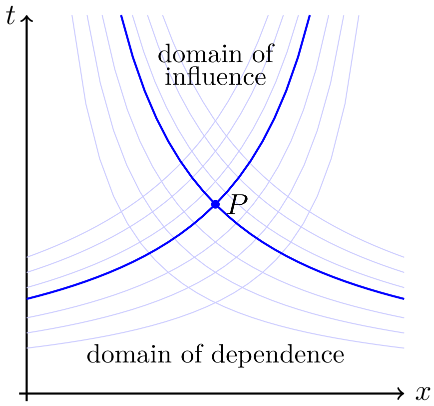
$$\frac{\mathrm{d}x}{\mathrm{d}t}=u\pm v_s$$

in the (x,t) plane. The $\pm$ characteristics form an interlocking web covering the space-time diagram. Both R_+ and R_- are needed to reconstruct the solution (u and v_s). Half the information is propagated along C_+ and half along C_-. In general $C_\pm$ are not known in advance but must be determined along with the solution. $C_\pm$ propagate at speed v_s to the right and left, *with respsect to the moving fluid*. This may be viewed as a non-linear geneeralisation of the solution of the classical wave equation $f(x-v_st)+g(x+v_st)$.

6.1.1 Method of characteristics

Sketch of a numerical method of solution:

1. Start with initial data (u and v_s) for all relevant x at $t = t_i$.
2. Determine characteristic slopes at t_i.
3. Propagate $R_\pm$ to $t = t_{i+1} = t_i + \delta t$, neglecting variation of characteristic slopes.
4. Combine $R_\pm$ to find u and v_s at each x at $t = t_{i+1}$.
5. Re-evaluate slopes and repeat.



The *domain of dependence* of a point P in the spacetime diagram is the region bounded by the $C_\pm$ through P and located in the past of P. The *domain of influence* of P is the region bounded by the $C_\pm$ through P and located in the future. The solution at P cannot depend on anything that occurs outside the domain of dependence. Similarly, the solution at P cannot influence anything outside the domain of influence.

6.1.2 A simple wave

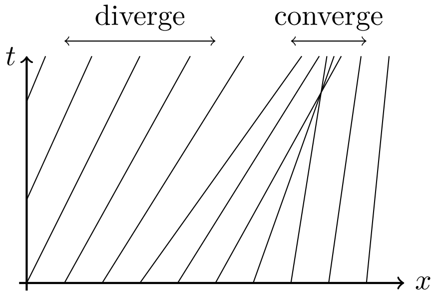
Suppose R_- is uniform: the sme constant value on every C_- characteristic emanating from an undisturbed region to the right. Its value everywhere is that of the undisturbed region:

$$u - \frac{2v_s}{\gamma - 1} = u_0 - \frac{2v_{s0}}{\gamma - 1}$$

Then, along the C_+, both $R_\pm$ and so u and v_s, are constant. The C_+ therefore have constant slope $v = u + v_s$, so they are straight lines. The statement that the wavespeed v is constant along the family of straight lines $\frac{\mathrm{d}x}{\mathrm{d}t} = v$ is expressed by the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

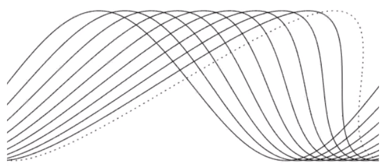
known as the *inviscid Burgers equation* or *non-linear advection equation*.
This equation has only one set of characteristics, with slope $\mathrm{d}x/\mathrm{d}t = v$, which is easily solved by the method of characteristics. The initial data define $v_0(x) = v(x, 0)$ and the characteristics are straight lines.



In regions where $\mathrm{d}v_0/\mathrm{d}x > 0$ the characteristics diverge in the future. In regions where $\mathrm{d}v_0/\mathrm{d}x < 0$ the characteristics converge and will form a *shock* at some point. Contradictory information arrives at the same event, leading to a breakdown of the solution.

6.1.3 Wave steepening

Another viewpoint is *wave steepening*. Each point of the graph $v(x)$ moves at its wave speed v. The crest moves fastest and eventually overtakes the trough to the right of it. The profile would



become multiple-valued, but the wave breaks, forming a discontinuity. The formal solution of the inviscid Burgers equation is

$$v(x,t) = v_0(x_0)$$

with $x = x_0 + v_0(x_0)t$. By the chain rule,

$$\frac{\partial v}{\partial x} + \frac{v_0'}{1 + v_0' t}$$

which diverges first at the breaking time

$$t^* = \frac{1}{\max(-v_0')}$$

6.2 Simple non-linear waves

Recall the hyperbolic structure of the equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0, \qquad \boldsymbol{U} = [\rho, p, \boldsymbol{u}, \boldsymbol{B}]^T$$

These are hyperbolic because the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector n_i. The eigenvalues are wave speeds, eigenvectors are wave modes.
In a simple wave propagating in the x-direction, all physical quantities are functions of a single variable, the phase $\psi(x,t)$. Then $\boldsymbol{U} = \boldsymbol{U}(\psi)$ and so

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\psi}\frac{\partial \psi}{\partial t} + \mathsf{A}_x \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\psi}\frac{\partial \psi}{\partial x} = 0$$

This equation is satisfied if $\mathrm{d}\boldsymbol{U}/\mathrm{d}\psi$ is an eigenvector of the hyperbolic system and if

$$\frac{\partial \psi}{\partial t} + v\frac{\partial \psi}{\partial x} = 0$$

where v is the corresponding wave speed (eigenvalue). But since $v = v(\psi)$ we again find

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

which is the inviscid Burgers equation. Steepening is therefore generic for simple waves, but waves do not always steepen in practice. For example, linear dispersion arising from Coriolis or buoyancy forces can counteract nonlinear wave steepening. Waves propagating on a non-uniform background are not simple waves. Waves may be damped by diffusive processes (viscosity, thermal conduction or resistivity) before they can steepen.
Even some simple waves do not steepen. This happens if the wave speed v does not depend on the variables that actually vary in the wave mode, for which there are two simple examples: the entropy wave ($v = u_x$) in which ρ varies but not p or u_x. Also, the Alfvén wave ($v = u_x \pm v_{ax}$) in which u_y, u_z, B_y, B_z vary but not ρ, p, u_x, B_x. In these cases the relevant solution of the inviscid Burgers equation is just $v = \text{const}$. The slow and fast magnetoacoustic waves, though, are 'generally nonlinear' and undergo steepening.