0 Perturbation Methods (16 Lectures)

0.1 Introduction

- 24 lectures prised into 16.
- Any corrections and suggestions should be emailed to me at `S.J.Cowley@maths.cam.ac.uk`.
- Closed book examination. *Likely* rubric:

  Attempt no more than TWO questions.
  There are THREE questions in total.
  The questions carry equal weight.

- **Books**
  - Hinch, *Perturbation methods.*
  - Van Dyke, *Perturbation methods in fluid mechanics.*
  - Kevorkian & Cole, *Perturbation methods in applied mathematics.*
  - Bender & Orszag, *Advanced mathematical methods for scientists and engineers.*
- **Philosophy**
  - Many physical processes are described by equations that cannot be solved analytically.
  - One approach is to solve the equations numerically; however, often there exists a 'small' parameter, ε, e.g.
    - in low Mach number flows $\varepsilon = M = \frac{u}{c}$, where u is the fluid velocity and c is the speed of sound;
    - in fast flows $\varepsilon = \frac{1}{Re}$, where Re is the Reynolds number.
  - We can use the smallness of ε to simplify the equations, and then find analytic (or simpler numerical) solutions.
- Primarily interested in differential equations, but a number of the ideas can be illustrated for algebraic equations and/or integrals. We will use algebraic equations to motivate some of the ideas.
- The only pre-requisites are (a) a course in 'Sums' (i.e. a competency to perform moderately messy calculations), and (b) an ability to solve simple differential equations and evaluate simple integrals (e.g. using integration by parts).

1 Algebraic Equations

1.1 Regular Expansions and Iteration

Consider

$$x^2 + \varepsilon x - 1 = 0\,. \tag{1.1}$$

Exact solution:

$$x = -\frac{1}{2}\varepsilon \pm \left(1 + \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\,.$$

If $|\varepsilon| < 2$, then can expand in a convergent series:

$$x = \begin{cases} 1 - \dfrac{1}{2}\varepsilon + \dfrac{1}{8}\varepsilon^2 - \dfrac{1}{128}\varepsilon^4 + \cdots \\[2ex] -1 - \dfrac{1}{2}\varepsilon - \dfrac{1}{8}\varepsilon^2 + \dfrac{1}{128}\varepsilon^4 + \cdots \end{cases}$$

Since the series is convergent for $|\varepsilon| < 2$, for small ε we can increase the accuracy by taking more terms. We have

solved the equation and then **approximated the solution**

However, we cannot always solve the equation exactly, so can we

approximate and then **solve the equation?**

1.1.1 Iterative method (liked by Pure Mathematicians)

Based on

$$x_{n+1} = g(x_n)\,.$$

Suppose $x_n = x^* + \delta_n$ where $x^* = g(x^*)$. Then by Taylor Series

$$\delta_{n+1} = g'(x^*)\,\delta_n + \mathcal{O}\left(\delta_n^2\right)\,.$$

If we have a good guess, so that $|\delta_n|$ is small, this is convergent if

$$|g'(x^*)| < 1\,.$$

Rearrange (1.1):

$$x^2 = 1 - \varepsilon x\,.$$

For the root near $x = 1$ try

$$\begin{aligned} x_{n+1} &= (1 - \varepsilon x_n)^{\frac{1}{2}} \\ x_0 &= 1 \\ x_1 &= (1-\varepsilon)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots \\ x_2 &= \left(1 - \varepsilon\,(1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \frac{1}{8}\varepsilon^3 + \cdots \\ x_3 &= \left(1 - \varepsilon\left(1 - \varepsilon\,(1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + 0 + \mathcal{O}\left(\varepsilon^4\right) + \cdots \end{aligned}$$

Hard work for the higher terms — also, how many terms are correct?

1.1.2 Expansion method

For $\varepsilon = 0$, the roots are $x = \pm 1$. For the root near $x = 1$ try

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Substitute into equation (1.1):

$$\begin{array}{llllllll} & 1 + 2\varepsilon x_1 & + 2\varepsilon^2 x_2 + \varepsilon^2 x_1^2 & + 2\varepsilon^3 x_3 + 2\varepsilon^3 x_1 x_2 & + \cdots & \\ & + \varepsilon & + \varepsilon^2 x_1 & + \varepsilon^3 x_2 & + \cdots & \\ -1 & & & & & = 0 \end{array}$$

Equate powers of ε:

$$\begin{array}{lrlcr} \varepsilon^0 : & 1 - 1 & = 0 & & \\ \varepsilon^1 : & 2x_1 + 1 & = 0 & , & x_1 = -\tfrac{1}{2} \\ \varepsilon^2 : & 2x_2 + x_1^2 + x_1 & = 0 & , & x_2 = \tfrac{1}{8} \\ \varepsilon^3 : & 2x_3 + 2x_1 x_2 + x_2 & = 0 & , & x_3 = 0 \end{array}$$

Easier than the iterative method for higher terms, but you need to guess the expansion correctly.

1.2 Singular Perturbations and Rescaling

Consider

$$\varepsilon x^2 + x - 1 = 0 \, . \tag{1.2}$$

$$\begin{array}{lcl} \varepsilon = 0 & : & \text{one solution} \\ \varepsilon \neq 0 & : & \text{two solutions} \end{array}$$

The limit process $\varepsilon \to 0$ is said to be *singular*.

Exact solution: $\quad \dfrac{-1 \pm (1 + 4\varepsilon)^{\frac{1}{2}}}{2\varepsilon}$.

Expansion for $|\varepsilon| < \frac{1}{4}$:

$$x = \left\{ \begin{array}{l} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + \cdots \end{array} \right. \tag{1.3}$$

The singular (i.e. extra) root $\to \mp\infty$ as $\varepsilon \to 0\pm$.

1.2.1 Iterative method

(a) For the non-singular root try

$$x_{n+1} = 1 - \varepsilon x_n^2 \, .$$

(b) For the singular root, we need to keep the 'εx^2' term as a major player. The leading order approximation is

$$\varepsilon x^2 + x \approx 0 \, ;$$

so try rearranging (1.2) to

$$x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n} \, .$$

Exercise. Confirm (1.3) by iteration.

Note that in (b)

$$x_{n+1} = g(x_n) \, , \qquad \text{where} \qquad g(x) = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x} \, .$$

Hence

$$g'(x) = -\frac{1}{\varepsilon x^2} \, , \quad \left| g'\left(-\frac{1}{\varepsilon}\right) \right| = \varepsilon < 1 \quad \text{if} \quad 0 < \varepsilon < 1 \, .$$

1.2.2 Expansion method

For one root try

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \ , \tag{1.4a}$$

and for the other try

$$x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \cdots \ . \tag{1.4b}$$

Substitute (1.4b) into (1.2):

$$\begin{array}{rcccl} & \dfrac{x_{-1}^2}{\varepsilon} + & 2x_{-1}x_0 + & \varepsilon\left(x_0^2 + 2x_{-1}x_1\right) + \cdots & \\ + & \dfrac{x_{-1}}{\varepsilon} + & x_0 + & \varepsilon x_1 + \cdots & \\ & - & 1 & & = 0 \end{array}$$

Equate powers

$$\begin{array}{llcllll} \varepsilon^{-1}: & x_{-1}^2 + x_{-1} & = 0 & ; \quad x_{-1} & = & 0 & , \quad -1 \\ \varepsilon^{0}\ : & (2x_{-1}+1)\,x_0 - 1 & = 0 & ; \quad x_0 & = & 1 & , \quad -1 \\ \varepsilon\ \ : & x_0^2 + 2x_{-1}x_1 + x_1 & = 0 & ; \quad x_1 & = & -1 & , \quad 1 \\ & & & & & \uparrow & \quad \uparrow \\ & & & & & (1.3a) & \quad (1.3b) \end{array}$$

1.2.3 Rescaling before expansion

How do you decide on the expansion if you do not know the solution?

Seek rescaling[s] to convert the singular equation into a regular equation. Try

$$x = \underbrace{\delta(\varepsilon)}_{\text{need to choose suitable } \delta} \underbrace{X}_{\text{strictly order `unity'; say } X = \mathrm{ord}(1).}$$

(1.2) becomes

$$\varepsilon \delta^2 X^2 + \delta X - 1 = 0 \ .$$

Consider the possibilities for different choices of δ ($|\varepsilon| \ll 1$):

$\delta \ll 1$:			small	+	small	−	1	=	0	※
$\delta = 1$:			small	+	X	−	1	=	0	regular root
$1 \ll \delta \ll \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	=	small	+	X	+	small	=	0	※ (since $X = \mathrm{ord}(1)$)
$\delta = \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	=	X^2	+	X	+	small	=	0	singular root
$\delta \gg \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\varepsilon\delta^2}$	=	X^2	+	small	+	small	=	0	※

The distinguished choices are therefore:

$$\begin{array}{lccl} \delta = 1: & \varepsilon X^2 + X - 1 = 0 & ; & X = 1 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots \\ \delta = \frac{1}{\varepsilon}: & X^2 + X - \varepsilon = 0 & ; & X = -1 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots \end{array}$$

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1.3 Non Integral Powers

Inter alia, double roots can cause problems. Consider

$$(1-\varepsilon)\,x^2 - 2x + 1 = 0\ . \tag{1.5}$$

When $\varepsilon = 0$, there is a double root at $x = 1$. Try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

then

$$\begin{array}{cccccccc} & 1 & + & 2\varepsilon x_1 & + & \varepsilon^2\left(2x_2 + x_1^2\right) & + & \cdots \\ & & - & \varepsilon & - & \varepsilon^2\,(2x_1) & & \\ - & 2 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + & \cdots \\ + & 1 & & & & & & = \ 0 \end{array}$$

and equating powers of ε:

$$\begin{array}{lrll} \varepsilon^0: & 1-2+1 & = 0 & \\ \varepsilon^1: & 2x_1 - 1 - 2x_1 & = 0 & ⁂ \end{array}$$

We need 'εx_1' to be larger.

From the exact solution:

$$x = \frac{1 \pm \varepsilon^{\frac{1}{2}}}{1-\varepsilon}\,,$$

we see that we should have expanded in powers of $\varepsilon^{\frac{1}{2}}$:

$$x = 1 + \varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \varepsilon x_1 + \varepsilon^{\frac{3}{2}} x_{\frac{3}{2}} + \cdots$$

$$\begin{array}{cccccccc} & 1 & + & 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & + & 2\varepsilon x_1 & + & \varepsilon x_{\frac{1}{2}}^2 \\ & & & & - & \varepsilon & & \\ - & 2 & - & 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & - & 2\varepsilon x_1 & & \\ + & 1 & & & & & & = \ 0 \end{array}$$

This time on equating powers of ε we see that

$$\begin{array}{lrll} \varepsilon^0: & 1-2+1 & = 0 & \\ \varepsilon^{\frac{1}{2}}: & 2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} & = 0 & \textbf{no information} \\ \varepsilon^1: & 2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 & = 0 & x_{\frac{1}{2}} = \pm 1 \end{array}$$

We must work to $\mathcal{O}(\varepsilon)$ to obtain the solution to $\mathcal{O}(\varepsilon^{\frac{1}{2}})$.

From the original equation

$$(x-1)^2 = \varepsilon x^2\ ,$$

we see that a change in the ordinate by $\mathrm{ord}(\varepsilon)$ changes the position of the root by $\mathrm{ord}(\varepsilon^{\frac{1}{2}})$.

In general we must derive (guess) the expansion required, e.g. try

$$\begin{aligned} x(\varepsilon) &= 1 + \delta_1(\varepsilon)\, x_1 + \delta_2(\varepsilon)\, x_2 + \cdots \\ & \qquad 1 \gg \delta_1 \gg \delta_2 \gg \cdots \\ & \qquad x_j = \mathrm{ord}(1). \end{aligned}$$

Substitute into (1.5):

$$\begin{array}{ccccccccccccccccc} & 1 & + & 2\delta_1 x_1 & + & 2\delta_2 x_2 & + & \cdots & + & \delta_1^2 x_1^2 & + & \cdots & + & 2\delta_1\delta_2 x_1 x_2 & + & \cdots \\ & & & & - & \varepsilon & & & - & 2\varepsilon\delta_1 x_1 & + & \cdots & & & & \\ - & 2 & - & 2\delta_1 x_1 & - & 2\delta_2 x_2 & + & \cdots & & & & & & & & \\ + & 1 & & & & & & & & & & & = & 0 & & \end{array}$$

The leading order terms are $\quad \delta_1^2 x_1^2 \quad$ and $\quad -\varepsilon$.
Hence take $\quad \delta_1 = \varepsilon^{\frac{1}{2}}$

allow x_1 to absorb any multiple roots.

Exercise. Show that the choices $\delta_1^2 \gg \varepsilon$, or $\delta_1^2 \ll \varepsilon$, lead to a ⋇.

Cancelling off these two terms, the leading-order terms become

$$2\delta_1\delta_2 x_1 x_2 \qquad \text{and} \qquad -\,2\varepsilon\delta_1 x_1\ .$$

Repeating the argument $\Rightarrow \delta_2 = \varepsilon$ (and $x_2 = 1$).

1.4 Logarithms

Solve

$$xe^{-x} = \varepsilon\ . \tag{1.6}$$

One root is close to $x = \varepsilon$, the other root is between

$$x = \ln\frac{1}{\varepsilon} \qquad\qquad (xe^{-x} = \varepsilon \ln \tfrac{1}{\varepsilon} > \varepsilon)$$

and

$$x = 2\ln\frac{1}{\varepsilon} \qquad (xe^{-x} = 2\varepsilon^2 \ln \tfrac{1}{\varepsilon} < \varepsilon, \text{ for } \varepsilon \text{ small}).$$

Note: doubling x reduces the e^{-x} factor by an order of magnitude.

The expansion method is unclear, so try the iteration scheme. Consider a rearrangement that emphasises the e^{-x} factor:

$$e^x = \frac{x}{\varepsilon}$$

so try

$$x_{n+1} = \log\frac{1}{\varepsilon} + \log x_n \ .$$

Then

$$\begin{aligned}
x_0 &= \log\frac{1}{\varepsilon} \\
x_1 &= \underbrace{\log\frac{1}{\varepsilon}}_{L_1} + \underbrace{\log\log\frac{1}{\varepsilon}}_{L_2} \\
x_2 &= L_1 + \log\left(L_1 + L_2\right) \\
&= L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots \\
x_3 &= L_1 + \log\left(L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots\right) \\
&= L_1 + L_2 + \frac{L_2}{L_1} + \frac{-\frac{1}{2}L_2^2 + L_2}{L_1^2} + \frac{\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2}{L_1^3} + \cdots
\end{aligned}$$

The iterative method can give more than one term per iteration.

Numerical disaster. Percentage errors for the truncated series:

ε	L_1	L_2	L_2/L_1	$-L_2^2/2L_1^2$	L_2/L_1^2
10^{-1}	36%	12%	2%	4%	0.03%
10^{-3}	24%	3%	0.02%	0.04%	0.04%
10^{-5}	19%	1%	0.04%	0.1%	0.001%

Do not separate terms like $-L_2^2/2L_1^2$ & L_2/L_1^2.

A very small ε is needed before this is tolerably accurate.

Check convergence.

$$\begin{aligned}
x_{n+1} &= g(x_n) \\
g(x) &= \log\frac{1}{\varepsilon} + \log x \\
g'(x) &= \frac{1}{x} \\
g'(x^*) &\approx \frac{1}{\log\frac{1}{\varepsilon}}
\end{aligned}$$

$\uparrow$need ε very small for $|g'| \ll 1$.

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