2 Asymptotic Approximations

2.1 Convergence and Asymptoticness

An expansion $\sum_{n=0}^{\infty} f_n(z)$ converges for a fixed z if, given $\varepsilon > 0$, $\exists\ N(z,\varepsilon)$ s.t.

$$\left|\sum_{\ell}^{m} f_n(z)\right| < \varepsilon \qquad \forall\ \ell, m > N\ .$$

Convergent series can be useful analytically, but hopeless in practice. For instance, consider

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt\ .$$

We know that

$$e^{-t^2} = \sum_{0}^{\infty} \frac{\left(-t^2\right)^n}{n!}$$

is analytic in the entire complex plane. Hence we have uniform convergence on any bounded part of the plane $\Rightarrow$ we can integrate term by term:

$$\operatorname{erf}(z) \quad = \tfrac{2}{\sqrt{\pi}} \textstyle\sum_0^\infty \tfrac{(-)^n z^{2n+1}}{(2n+1)n!}\ .$$

$\downarrow$ also has ∞ radius of convergence

To obtain an accuracy of 10^{-5} we need

8 terms up to z =1
16 terms up to z =2
31 terms up to z =3
75 terms up to z =5

However, intermediate terms can be large $\Rightarrow$ problems due to round-off error on computers.

An alternative for large z is to proceed as follows. First rewrite the integral:

$$\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt\ .$$

Then repeatedly integrate by parts:

$$\begin{aligned}
\int_z^\infty e^{-t^2}\, dt &= \int_z^\infty \left(-\frac{1}{2t}\right) d\left(e^{-t^2}\right)\\
&= \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{1}{2t^2} e^{-t^2}\, dt\\
&\ \ \vdots\\
&= \left(1 - \frac{1}{2z^2} + \frac{1.3}{\left(2z^2\right)^2} - \frac{1.3.5}{\left(2z^2\right)^3}\right) \frac{e^{-z^2}}{2z} + R_5
\end{aligned}$$

where

$$\begin{aligned}
R_5 &= \int_z^\infty \frac{105}{16} \frac{e^{-t^2}}{t^8}\, dt = \int_z^\infty \frac{105}{32\, t^9}\, d\left(-e^{-t^2}\right)\\
&\leqslant \frac{105}{32 z^9} \int_z^\infty d\left(-e^{-t^2}\right) = \frac{105}{32} \frac{e^{-z^2}}{z^9}\ .
\end{aligned}$$

The series in z^{-1} is divergent (due to the odd factorial in the numerator), but the truncated series is useful, e.g. 10^{-5} accuracy with 3 terms for $z = 2.5$
2 terms for $z = 3$.

"First term is essentially the answer, while subsequent terms are minor corrections."

Problem: What if the leading term is not sufficiently accurate (e.g. in reality ε is not sufficiently small)? Adding a few extra terms *may* help, but there is a limit to the number of useful extra terms if the series diverges as $N \to \infty$ at fixed ε. It is not sensible to include extra terms once they stop decreasing in magnitude. By suitable truncation, one can obtain exponential accuracy (see §3.1 and the first example
01/04 sheet).

2.2 Definitions

The expansion $\sum_0^N f_n(\varepsilon)$ is an *asymptotic approximation* of $f(\varepsilon)$ as $\varepsilon \to 0$, if $\forall\, m \leqslant N$,

$$\frac{\sum_0^m f_n(\varepsilon) - f(\varepsilon)}{f_m(\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0$$

i.e. the remainder is less than the last included term.

If we can let $N \to \infty$ (in principle) then we have an *asymptotic expansion.*

If $f_n = a_n \varepsilon^n$, then we have an *asymptotic power series*; however we frequently need more general expansions involving terms like ε^α, $\left(\ln \frac{1}{\varepsilon}\right)^{-1}$, etc. We write these as

$$\sum_{n=0}^{N} a_n \delta_n(\varepsilon) \tag{2.1}$$

where the δ_n form an asymptotic sequence:

$$\frac{\delta_{n+1}}{\delta_n} \to 0 \qquad \text{as } \varepsilon \to 0\ .$$

Note that sometimes we need to restrict to one sector of the complex ε plane to keep the δ_n single valued.

Often ε is real and positive. A useful set of asymptotic functions are then Hardy's logarithm–exponential functions obtained by a finite number of $+, -, *, /, \exp$ & $\log$ operations, with all intermediate quantities real.

This class has the property that it can be ordered, i.e. either $f(\varepsilon) = \mathrm{o}(g(\varepsilon))$, or $g(\varepsilon) = \mathrm{o}(f(\varepsilon))$ or
02/01 $f(\varepsilon) = \mathrm{ord}\,(g(\varepsilon))$.

2.3 Uniqueness and Manipulation

If f can be expanded asymptotically for a given asymptotic sequence, then the expansion is unique. For if the expansion exists it has the form

$$f(\varepsilon) \sim \sum_n a_n \delta_n(\varepsilon)\ ,$$

then by construction

$$\begin{aligned} a_0 &= \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)} \\ a_n &= \lim_{\varepsilon \to 0} \left\{ \frac{f(\varepsilon) - \sum_0^{n-1} a_m \delta_m}{\delta_n} \right\}\ . \end{aligned}$$

However, a single function can have different asymptotic expansions for different sequences:

$$\begin{aligned}\tan(\varepsilon) &\sim \varepsilon + \frac{1}{3}\varepsilon^3 + \frac{2}{15}\varepsilon^5 + \cdots \\ &\sim \sin\varepsilon + \frac{1}{2}(\sin\varepsilon)^3 + \frac{3}{8}(\sin\varepsilon)^5 + \cdots \\ &\sim \varepsilon\cosh\sqrt{\frac{2}{3}}\varepsilon + \frac{31}{270}\left(\varepsilon\cosh\sqrt{\frac{2}{3}}\varepsilon\right)^5 + \cdots .\end{aligned}$$

Part of the 'art' of obtaining an effective asymptotic solution is choosing the most appropriate asymptotic sequence.

Worse: two functions can have the same asymptotic expansion:

$$\begin{aligned}\exp\varepsilon &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad \text{as } \varepsilon \to 0 \\ \exp\varepsilon + \exp\left(-\frac{1}{\varepsilon}\right) &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad \text{as } \varepsilon \searrow 0 .\end{aligned}$$

Exercise. Does $f = x^2 + e^{-x^2(1-\sin x)}$ have an asymptotic expansion as $x \to \infty$?

- Asymptotic expansions can be added, multiplied and divided to produce asymptotic expansions for the sum, product and quotient (if necessary one may need to enlarge the asymptotic sequence).
- If appropriate, one can try to substitute an asymptotic expansion into another – but care is needed, e.g. if

  $$f(z) = e^{z^2}, \qquad z(\varepsilon) = \frac{1}{\varepsilon} + \varepsilon$$

  then

  $$\begin{aligned}f(z(\varepsilon)) &= \exp\left[\frac{1}{\varepsilon^2} + 2 + \varepsilon^2\right] \\ &\sim e^{1/\varepsilon^2}e^2\left\{1 + \varepsilon^2 + \frac{\varepsilon^4}{2} + \cdots\right\},\end{aligned}$$

  but if we just work to leading order

  $$\begin{aligned}z &\sim \frac{1}{\varepsilon} \\ f(z) &\not\sim e^{1/\varepsilon^2}\end{aligned}$$

  $\uparrow$missing e^2

  The leading-order approximation in z is inadequate for the leading-order approximation in $f(z)$.
- Integration w.r.t. ε of asymptotic expansions is allowed term-by-term producing the correct result.
- Differentiation is not allowed in principle because $\mathcal{O}$ and o estimates do not survive differentiation. For instance:

  (a)

  $$\begin{aligned}f &= e^{ix^2} = \mathcal{O}(1) \qquad \text{as } x \to \infty \\ \frac{df}{dx} &= 2ixe^{ix^2} = \mathcal{O}(x) \qquad \text{as } x \to \infty\end{aligned}$$

  (b)

  $$\begin{aligned}f &= 1 + e^{-1/x^2}\sin\left(e^{1/x^2}\right) \sim 1 + \cdots \qquad \text{as } x \to 0 \\ \frac{df}{dx} &= \underbrace{-\frac{2}{x^3}\cos\left(e^{1/x^2}\right)} + \frac{2}{x^3}e^{-1/x^2}\sin\left(e^{1/x^2}\right)\end{aligned}$$

  No asymptotic expansion as $x \to 0$.

01/03

(c)

$$f = t^2 + t\sin t \sim t^2\,, \quad f' = (2+\cos t)t + \sin t \not\sim 2t \quad \text{as} \quad t \to \infty.$$

However:

(i) If $f'(x)$ exists and is integrable, and $f(x) \sim \sum_{n=0}^{N} a_n x^n$ as $x \to 0$, then

$$f' \sim \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{as } x \to 0.$$

(ii) If $f(z)$ is analytic in $\theta_1 \leqslant \arg z \leqslant \theta_2$, $0 < |z| < R$ and

$$f \quad \sim \quad \sum_{n=0}^{\infty} a_n z^n \qquad \text{as } z \to 0 \ (\theta_1 \leqslant \arg z \leqslant \theta_2)$$

then

$$f' \quad \sim \quad \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{as } z \to 0 \ (\theta_1 \leqslant \arg z \leqslant \theta_2).$$

(iii) There are lots more special cases. For instance, consider asymptotic expansions of solutions to differential equations.

Suppose that y is the solution to

$$y'' + qy = 0 \tag{2.2}$$

where q has an asymptotic expansion as $x \to 0$.

Assume y has an asymptotic expansion as $x \to 0$;

then from (2.2) y'' has an asymptotic expansion (multiplication OK)
thus y' has an asymptotic expansion (integration OK)
thus y has an asymptotic expansion (integration OK) .

Hence if y has an asymptotic expansion, the equation ensures that its differentials have asymptotic expansions (the *proof* that y has an asymptotic expansion in the first place is often tricky).

2.4 Parametric Expansions

For functions of two (or more) variables, e.g. $f(x,\varepsilon)$ (as might arise in solutions to pdes, etc.), we make the obvious generalisation of (2.1) to allow the a_n to be functions of x:

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{2.3}$$

If the approximation is asymptotic as $\varepsilon \to 0$ for each x, then it is called a Poincaré, or classical, asymptotic approximation.

The above pointwise asymptoticness may not be uniform in x, e.g. it may require $\varepsilon < x$ (restrictive as $x \to 0$). Such problems sometimes need a further extension:

$$f(x,\varepsilon) \sim \sum_n a_n(x,\varepsilon)\delta_n(\varepsilon) \tag{2.4}$$
$$\text{e.g. } a_n(x,\varepsilon) = b_n\left(\tfrac{x}{\varepsilon}\right).$$

Uniqueness extends to (2.3), but not to (2.4), etc.

3 Integral Methods

3.1 Elementary Examples

Example 1. Rewrite an integral so that we can use a Taylor series. For instance:

$$I = \int_x^\infty e^{-t^4}\,dt \qquad \text{as} \qquad x \to 0\ .$$

Then

$$\begin{aligned} I &= \int_0^\infty e^{-t^4}\,dt - \int_0^x e^{-t^4}\,dt \\ &= \Gamma(5/4) - \int_0^x \sum_{n=0}^\infty \frac{(-t^4)^n}{n!}\,dt \\ &= \Gamma(5/4) - \sum_{n=0}^\infty \frac{(-)^n x^{4n+1}}{(4n+1)n!}\ . \end{aligned}$$

Example 2. Use a Taylor series even when we cannot! For instance:

$$I = \int_0^\infty \frac{e^{-t}}{x+t}\,dt \qquad \text{as} \qquad x \to \infty\ .$$

Then

$$\begin{aligned} I &= \frac{1}{x}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{-1}\,dt \\ &= \frac{1}{x}\int_0^\infty e^{-t}\left(1-\frac{t}{x}+\frac{t^2}{x^2}-\frac{t^3}{x^3}+\dots\right)\,dt \\ &= \frac{1}{x}\left(1-\frac{1!}{x}+\frac{2!}{x^2}-\frac{3!}{x^3}+\dots\right)\ . \end{aligned}$$

↑dubious, since invalid for $t > x$.

↑Divergent

Estimate the remainder using

$$1-\frac{t}{x}+\frac{t^2}{x^2}+\dots+\left(-\frac{t}{x}\right)^{m-1} = \frac{1-\left(-\frac{t}{x}\right)^m}{1+\frac{t}{x}}\ .$$

Then

$$I = \frac{1}{x}\sum_{n=0}^{m-1}\int_0^\infty \left(-\frac{t}{x}\right)^n e^{-t}\,dt + R_m(x)\ ,$$

where

$$R_m(x) = \frac{1}{x^{m+1}}\int_0^\infty \frac{(-t)^m e^{-t}}{(1+\frac{t}{x})}\,dt\ ,$$

and

$$|R_m(x)| \leqslant \frac{1}{|x^{m+1}|}\int_0^\infty t^m e^{-t}\,dt = \frac{m!}{x^{m+1}}\ .$$

Hence

$$I = \frac{1}{x}\left(1-\frac{1}{x}+\frac{2!}{x^2}+\dots+\frac{m!}{(-x)^m}+\mathcal{O}\left(\frac{(m+1)!}{x^{m+1}}\right)\right)$$

Truncate the series when the remainder has the smallest bound, i.e. stop one before smallest term when $x \sim m$. The error when we truncate is then (after using Stirling's formula)

$$|R_m| \sim \frac{x!}{x^{x+1}} \sim \frac{(2\pi)^{1/2}e^{-x}}{x^{1/2}}\ ,$$

03/01 i.e. the error is exponentially small for large x (so the 'dubious' step wasn't too bad).

02/04

02/20