Cambridge Part III Maths

Michaelmas 2020

Fluid Dynamics of Climate

based on a course given by
Dr. John Taylor & Prof. Peter Haynes

written up by
Charles Powell

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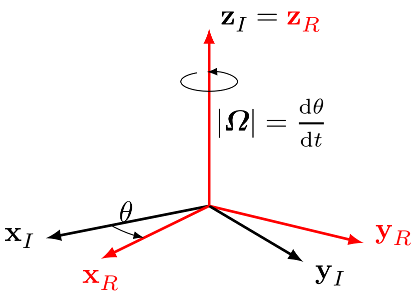
1 Introduction

1.1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p - \rho\nabla\phi + \rho\boldsymbol{F}$$

The body forces are assumed to be conservative with potential ϕ, e.g. $\phi = gz$ for gravitational force. $\boldsymbol{F}$ is the frictional force.
Consider a reference frame rotating about the z-axis with constant angular velocity $\boldsymbol{\Omega}$. Axes in the inertial frame are denoted with a subscript $_I$ and axes in the rotating frame are denoted with a subscript $_R$.



For a point with position vector $\boldsymbol{x}$ and velocity $\boldsymbol{u}_R = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R$ in the rotating reference frame

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{x}$$

or equivalently $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}$. Hence the acceleration is

$$\begin{aligned}\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}\right]\right)_R + \boldsymbol{\Omega}\times\left(\boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}\right)_R \\ &= \left(\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{u}_R + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{x}\right)\end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal accelertion.* Note that we can write the centrifugal acceleration in the form of a conservative force

$$\begin{aligned}\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{x}\right) &= \nabla\phi_c \\ \phi_c &= -\frac{1}{2}\left|\boldsymbol{\Omega}\times\boldsymbol{x}\right|^2\end{aligned}$$

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{u}\right) = -\nabla p - \rho\nabla\left(\phi + \phi_c\right) + \rho\boldsymbol{F} \tag{1}$$

We group the potential terms into a *geopotential* $\Phi \equiv \phi + \phi_c$. The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.
Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

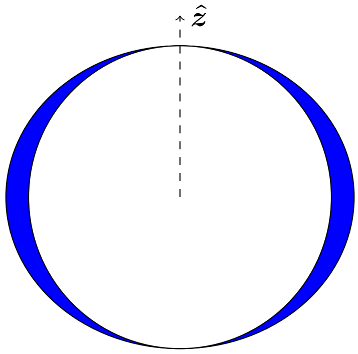


Figure 1: Geopotential ocean surface relative to a spherical Earth.

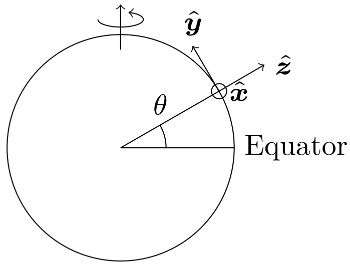


Figure 2: Local Cartesian coordinates

1.1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2). In this coordinate system $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$. Hence if $\boldsymbol{u} = (u, v, w)$ then

$$\begin{aligned} 2\boldsymbol{\Omega} \times \boldsymbol{u} &= (2\Omega w\cos\theta - 2\Omega v\sin\theta, 2\Omega u\sin\theta, -2\Omega u\cos\theta) \\ &= (-fv + f^*w, fu - f^*u) \end{aligned}$$

where $f \equiv 2\Omega\sin\theta$ is the *Coriolis parameter* and $f^* \equiv 2\Omega\cos\theta$.

Example. In Cambridge, $\theta = 52.1^\circ N$ so

$$\begin{aligned} f &= 2\Omega\sin\theta \\ &= 2 \cdot \frac{2\pi}{3600 \cdot 24} \cdot 0.79 s^{-1} \\ &\approx 1.14 \times 10^{-4} s^{-1} \end{aligned}$$

At mid-latitudes, $f \sim 10^{-4}$ is a good approximation.

We can simplify the Coriolis acceleration expression; often $f^*w \ll fv$ and $f^*u \ll g$. Hence

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \approx (-fv, fu, 0) = f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

This is the *traditional approximation.* This is *not* always a good approximation, particularly at intermediate scales.

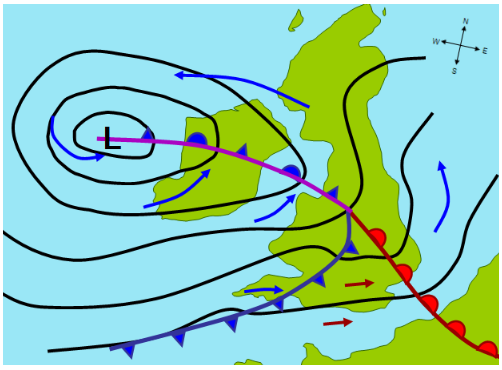


Figure 3: Lines of constant pressure p act as streamlines for the horizontal flow.

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1.1.2 Scale analysis.

Define characteristic scales for length L, time T, and velocity U. Non-dimensional variables are denoted with a superscript star: $\boldsymbol{u}^* = \boldsymbol{u}/U$, etc.
Using these scalings with $\boldsymbol{F} = \nu\nabla^2\boldsymbol{u}$ we have

$$\frac{U}{T}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \frac{U^2}{L}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + fU\hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho}\nabla\left(p + \rho\Phi\right) + \frac{\nu U}{L^2}\nabla_*^2 \boldsymbol{u}^*$$

Dividing through by fU leaves the Coriolis acceleration term ord(1) with other terms scaled relatively.

$$\frac{1}{fT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \mathrm{Ro}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + \hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho f U}\nabla\left(p + \rho\Phi\right) + \mathrm{E}\nabla_*^2 \boldsymbol{u}^*$$

where $\mathrm{Ro} \equiv \frac{U}{fL}$ is the *Rossby number* and $\mathrm{E} \equiv \frac{\nu}{fL^2}$ is the *Ekman number*.

Example. For an atmospheric storm, $U \sim 10ms^{-1}, L \sim 1000km, f \sim 10^{-4}s^{-1}$. Thus $\mathrm{Ro} \sim 0.1, \mathrm{E} \sim 10^{-13}$.

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Further, if $T = L/U$, then $\mathrm{Ro} = U/fL = 1/fT$. For small Ro, E, on surfaces of constant Φ, $f\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla p$. This is *geostrophic balance*. In components, we have

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The equations of geostrophic balance can be arranged to give the horizontal velocity: $\boldsymbol{u}_H$

$$\boldsymbol{u}_H \equiv (u, v) = \frac{1}{\rho f}\hat{\boldsymbol{z}} \times \nabla p$$

Horizontal velocity is perpendicular to ∇p and hence parallel to isobars (lines of constant p), i.e. pressure acts like a streamfunction (see figure 3).
In the Northern Hemisphere, air moves clockwise around high p and anticlockwise around low p. A *cyclonic* rotation is in the same sense as $\boldsymbol{\Omega}$, *anticyclonic* in the opposite sense as $\boldsymbol{\Omega}$.

1.1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small Ro, E)

$$\nabla \cdot \boldsymbol{u} = 0$$
$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \tag{2}$$

Taking the curl of (2) we have

$$\begin{aligned} \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\Omega_l u_m \\ &= \varepsilon_{kij}\varepsilon_{klm}\Omega_l \partial_j u_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\Omega_l \partial_j u_m \\ &= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i \end{aligned}$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0$$

For $\boldsymbol{\Omega} = (0, 0, \Omega)$, this implies $\frac{\partial w}{\partial z} = 0$. If $w = 0$ on some horizontal surface (e.g. ground) then $w = 0$ everywhere.
Also, $u_x + v_y = 0$, i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in 'columns' parallel to $\boldsymbol{\Omega}$, called *Taylor columns*.

1.2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density ρ_0 with angular velocity $\boldsymbol{\Omega} = (0, 0, f/2)$. Assume small ampltiude motions (i.e. $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$), i.e. neglect $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$. From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \tag{3}$$
$$v_t + fu = -\frac{p_y}{\rho_0} \tag{4}$$
$$w_t = -\frac{p_z}{\rho_0} \tag{5}$$
$$u_x + v_y + w_z = 0 \tag{6}$$

We will eliminate variables in favour of p.

$$\begin{aligned} \nabla \cdot ((3) - (5)) &\implies \nabla^2 p = \rho_0 f \left(v_x - u_y\right) \\ \partial_x(4) - \partial_y(3) \& (6) &\implies (v_x - u_y)_t = f w_z \end{aligned}$$

Combining these and using (5) we have

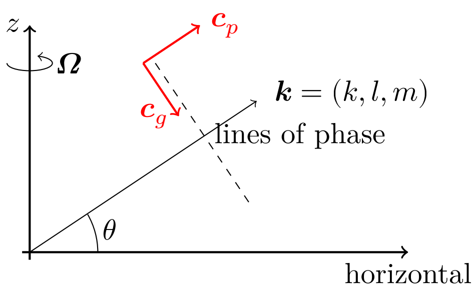
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for p. Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed $\boldsymbol{c}_p = w/\boldsymbol{k}$ and group velocity

$$\boldsymbol{c}_g = \frac{\partial w}{\partial \boldsymbol{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\boldsymbol{k}|^{3/2}}$$

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Note that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$. Also note $|\omega| \leq |f|$.

1.2.1 Inertial (free) oscillations

Assume $\nabla p = \mathbf{0}$. The x and y components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus $u = U \sin ft$ where f is the *inertial frequency*. Similarly, we have $v = U \cos ft$. For a particle with position (x_p, y_p) floating on an ocean surface $z = 0$ moving with the fluid velocity, we have

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u \implies x_p = -\frac{U}{f} \cos ft + x_0$$
$$\frac{\mathrm{d}y_p}{\mathrm{d}t} = v \implies y_p = -\frac{U}{f} \sin ft + y_0$$

Thus the motion of fluid particles describes describes *inertial circles* with radius $\frac{2U}{f}$.

1.2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress $\boldsymbol{\tau}_w$. Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e. $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e. $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e. $w(z = 0) = 0$
4. No flow in deep ocean, i.e. $\lim_{z \to -\infty} \boldsymbol{u} = \mathbf{0}$
5. Constant density ρ
6. Traditional approximation

Continuity (incompressibility) says $u_x + v_y + w_z = 0$. Assumptions 2 and 3 then imply $w = 0$ everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$
$$fu = \nu v_{zz} \tag{8}$$

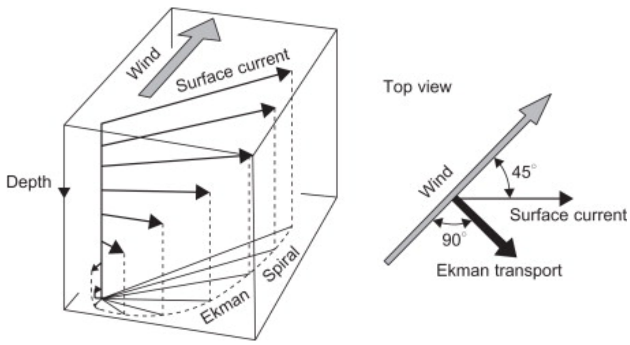

Figure 4: Ekman spiral.

Define the *complex velocity* $\mathcal{V} \equiv u + iv$. Then

$$\mathcal{V}_{zz} = \frac{if}{\nu}\mathcal{V} \tag{9}$$

Without loss of generality, assume $\boldsymbol{\tau}_w$ is aligned with the x-axis: $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$. Boundary conditions for (9) are

$$\mathcal{V}_z = \left(\frac{\tau_w}{\rho\nu}, 0\right) \quad \text{at } z = 0$$
$$\mathcal{V} = (0,0) \quad \text{as } z \to -\infty$$

Thus $\mathcal{V} = Ae^{(1+i)z/\delta}$ where $\delta = \sqrt{\frac{2\nu}{f}}$, $A = \frac{\tau_w\delta(1-i)}{2\rho\nu}$. In terms of the velocity components, we have

$$u = \frac{\tau_w}{\rho\sqrt{\nu f}}e^{z/\delta}\cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$
$$v = -\frac{\tau_w}{\rho\sqrt{\nu f}}e^{z/\delta}\sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

1.2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where $\nu\boldsymbol{u}_z \approx 0$ at $z = -h$. Since $\nu\boldsymbol{u}_z(z=0) = (\tau_w/\rho, 0)$, the *Ekman transport* $\boldsymbol{U}_T$ is

$$U_T \equiv \int_{-h}^{0} u \, \mathrm{d}z = 0$$
$$V_T \equiv \int_{-h}^{0} v \, \mathrm{d}z = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented 90° to the right of the applied wind shear stress (in the Northern Hemisphere).

1.2.4 Ekman pumping

Consider a wind stress $\tau_w(y)$ that varies over large scales. Then from incompressibility

$$\int_{-h}^{0} w_z \, \mathrm{d}z = -\int_{-h}^{0} u_x \, \mathrm{d}z - \int_{-h}^{0} v_y \, \mathrm{d}z$$

Thus for h constant,

$$-w(z=-h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\tau_w}{\rho f}\right)$$

In general we have

$$w(z=-h) = \hat{\boldsymbol{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$

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2 Shallow Water Systems

2.1 Rotating shallow water equations

Consider a thin layer of fluid with constant density ρ. Define characteristic scales

- length $L=$ horiz., $H=$ vert.
- velocity U
- time T
- pressure P

such that $\partial_x, \partial_y \sim \frac{1}{L}, \partial_z \sim \frac{1}{H}$. Define the *aspect ratio* $\delta \equiv H/L$. We will assume $\delta \ll 1$. From continuity (incompressibility) we have

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \implies \frac{w}{H} &= \mathcal{O}(U/L) \\ \implies w &= \mathcal{O}(\delta U) \end{aligned}$$

Using the traditional approximation and assuming the fluid is inviscid, the x-momentum equation

$$\begin{array}{rcccccccc} & \dfrac{\partial u}{\partial t} & +u\dfrac{\partial u}{\partial x} & +v\dfrac{\partial u}{\partial y} & +w\dfrac{\partial u}{\partial z} & -fv & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \text{scaling:} & \dfrac{U}{T} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{wU}{H} & fU & = & \dfrac{P}{\rho L} \end{array} \tag{10}$$

Thus if p_x appears at leading order then

$$P \sim \rho U \max(L/T, U, fL)$$

Similarly the z-momentum equation and its scalings are

$$\begin{array}{rccccccc} & \dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} - g \\ \text{scaling:} & \dfrac{w}{T} & \dfrac{Uw}{L} & \dfrac{Uw}{L} & \dfrac{w^2}{H} & = & \dfrac{P}{\rho H} \end{array} \tag{11}$$

Hence $\frac{\mathrm{D}w}{\mathrm{D}t} \sim \max(\frac{w}{T}, \frac{Uw}{L})$. Comparing with the pressure term, we have

$$\begin{aligned} \frac{\frac{\mathrm{D}w}{\mathrm{D}t}}{\frac{1}{\rho}\frac{\partial p}{\partial z}} &\sim \frac{\max(\frac{w}{T}, \frac{Uw}{L})}{\frac{U}{H}\max(\frac{L}{T}, \frac{U}{L}, f)} \\ &\sim \delta^2 \frac{\max(\frac{1}{T}, \frac{U}{L})}{\max(\frac{1}{T}, \frac{U}{L}, f)} \end{aligned}$$

Therefore to $\mathcal{O}(\delta^2)$ we have *hydrostatic balance*. To this order, (11) becomes

$$\frac{\partial p}{\partial z} - \rho g \implies p = \rho g(\eta - z)$$

assuming $p = 0$ at $z = \eta(x, y, t)$. Similarly, we have $\frac{1}{\rho}p_x = g\eta_x$ and $\frac{1}{\rho}p_y = g\eta_y$. Hence horizontal acceleration (i.e. the LHS of (10)) is independent of z. Motivated by this, we *assume* that horizontal velocity is also independent of z. For $Ro \ll 1$, this follows from the Tayor-Proudman theorem. Re-writing (10) with these results we have

$$u_t + uu_x + vu_y - fv = -g\eta_x \tag{12}$$
$$v_t + uv_x + vv_y + fu = -g\eta_y \tag{13}$$

since $u_z = v_z = 0$ by assumption. Integrating the continuity equation gives

$$w = -z(u_x + v_y) + A(x, y, t)$$

where A is to be determined by the boundary conditions. Requiring no normal flow at $z = -H_0 + h_b$ is imposed by $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ where $\boldsymbol{n} = \nabla(z - h_b)$. Thus

$$-u\frac{\partial h_b}{\partial x} - v\frac{\partial h_b}{\partial y} + w = 0$$

Hence

$$A(x, y, t) = u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$$

The kinematic boundary condition at $z = \eta$ is $\frac{\mathrm{D}\eta}{\mathrm{D}t} = w$ which may be written as

$$\eta_t + u\eta_x + v\eta_y - w = 0$$

where $w = -\eta(u_x+v_y)+u\frac{\partial h_b}{\partial x}+v\frac{\partial h_b}{\partial y}+(-H_0+h_b)(u_x+v_y)$. Combining these boundary conditions gives

$$\eta_t + \left[(H_0 - h_b + \eta)u\right]_x + \left[(H_0 - h_b + \eta)v\right]_y = 0 \tag{14}$$

If $H \equiv H_0 - h_b + \eta$ is the total depth of the fluid, then since $H_t = \eta_t$,

$$H_t + (uH)_x + (vH)_y = 0 \tag{15}$$

which is a statement of the conservation of volume (equivalently mass, since ρ is constant). Equations (12), (13), and (14) are the *rotating shallow water* (SW) equations.

2.1.1 Potential vorticity (PV)

Denote the vertical vorticity by $\zeta = v_x - u_y$. Consider $\partial_x(13) - \partial_y(12)$, which gives

$$\zeta_t + u\zeta_x + v\zeta_y + vf_y = -(\zeta + f)(u_x + v_y)$$

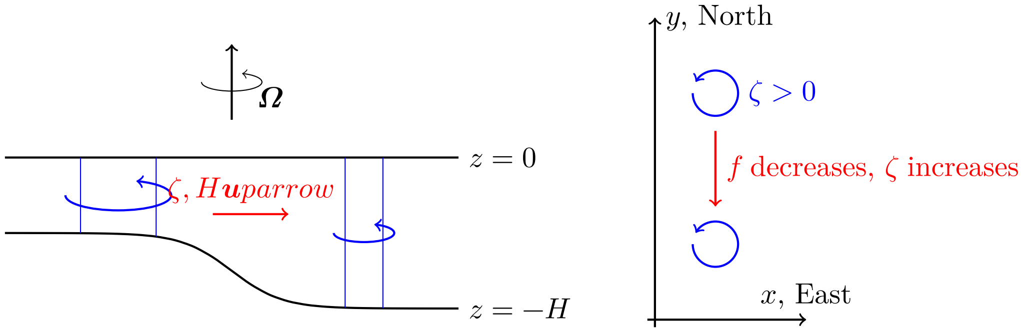
Now from conservation of volume (15),

$$u_x + v_y = -\frac{1}{H}\frac{\mathrm{D}H}{\mathrm{D}t}$$

Combining these relates the material derivative of ζ and H by

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\zeta + f}{H}\frac{\mathrm{D}H}{\mathrm{D}t} \implies \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{16}$$

Let $q \equiv \frac{\zeta+f}{H}$, the *shallow water potential vorticity* (SWPV). SWPV is conserved following fluid motion. We call ζ the *relative vorticity* and f the *planetary vorticity.* ζ and f will change as a fluid moves to conserve SWPV (changing f) and angular momentum (changing depth).



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2.2 Small amplitude motions in rotating SW

Consider a stationary fluid with depth $H_s(x,y) = H_0 - h_b$. The fluid surface is then perturbed by $\eta(x,y,t)$ where $\eta \ll H_s$. The total depth is $H(x,y,t) = H_s + \eta$. For $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$, linearise the shallow water equations:

$$u_t - fv = -g\eta_x \tag{17}$$

$$v_t + fu = -g\eta_y \tag{18}$$

$$\eta_t + (uH_s)_x + (vH_s)_y = 0$$

Assuming f is constant, we have from $\partial_x(17) + \partial_y(18)$ and $\partial_y(17) - \partial_x(18)$:

$$\partial_t\left[\left(\partial_t^2 + f^2\right)\eta - \nabla\cdot(gH_s\nabla\eta)\right] - fgJ(H_s,\eta) = 0 \tag{19}$$

where the Jacobian $J(a,b) = a_x b_y - a_y b_x$. For the velocity components we have

$$\left(\partial_t^2 + f^2\right)u = -g\left(\eta_{xt} + f\eta_y\right) \tag{20}$$

$$\left(\partial_t^2 + f^2\right)v = -g\left(\eta_{yt} + f\eta_x\right) \tag{21}$$

2.2.1 Steady flows

We now assume $\partial_t = 0$. From (20), (21),

$$u = -\frac{g}{f}\eta_y, \qquad v = \frac{g}{f}\eta_x$$

This is *shallow water geostrophic balance*: the surface displacement η acts as a streamfunction. Applying the steady assumption to (19) gives $J(H_s,\eta) = 0$ which implies $\eta = \eta(H_s(x,y))$. Hence linearised steady geostrophic flow in shallow water follows contours of constant depth. Steady PV conservation follows from (16) with $\partial_t = 0$ and assuming $\zeta \ll f$

$$\boldsymbol{u}\cdot\nabla\frac{f}{H_s} = 0$$

Thus when f varies, the flow follows contours of constant f/H_s.

2.2.2 Waves in an unbounded domain

Assume H_s is constant. From (19), we have

$$\left(\partial_t^2 + f^2\right)\eta - gH_s\nabla^2\eta = 0$$

Seek plane wave solutions to this wave equation with ansatz $\eta = \eta_0 \exp(i(kx + ly - \omega t))$. The dispersion relation is then

$$\omega^2 = f^2 + gH_s(k^2 + l^2) \tag{22}$$

If $f = 0$, i.e. no rotation, then the frequency is $\omega = \pm\sqrt{gH_s}|\boldsymbol{k}| = \omega_0$ and the phase speed is $|c_p| = \frac{|\omega|}{|\boldsymbol{k}|} = \sqrt{gH_s} = c_0$. For $f \neq 0$, we get *Poincaré* waves with

$$\omega^2 > \omega_0^2, \qquad |c_p| > c_0$$

i.e. rotation increases the frequency and phase speed. Define the *Rossby deformation scale* $R_D \equiv \frac{c_0}{f}$. From (22),

$$\frac{\omega^2}{f^2} = 1 + R_D^2|\boldsymbol{k}|^2$$

Without loss of generality, let $l = 0$, by reorienting x and y. If $\eta = \eta_0\cos(kx - \omega t)$ then (20), (21) imply the fluid velocity is

$$u = \frac{\omega_0\eta_0}{kH_s}\cos(kx - \omega t)$$
$$v = \frac{f\eta_0}{kH_s}$$

Thus the motion is an ellipse, also known as a *tidal ellipse*, which reduces to intertial circles if $\omega_0 = f$:

$$u^2 + \frac{\omega_0^2}{f^2}v^2 = \frac{\omega_0^2\eta_0^2}{k^2H_s^2}$$

Since $\omega > f$, the fluid moves anticylonically. The Rossby deformation scale R_D is the length scale for which rotation becomes important. Consider short and long waves:

- Short waves: $|\boldsymbol{k}|R_D \gg 1$. We have $\omega^2 \to gH_s|\boldsymbol{k}|^2$ i.e. non-rotating shallow water gravity waves.
- Long waves: $|\boldsymbol{k}|R_D \ll 1$. We have $\omega^2 \to f^2$ i.e. inertial waves where fluid moves in inertial circles. Gravity is not involved.

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2.3 Geostrophic adjustment

Consider the response of rotating shallow water to an initial state *not* in geostrophic balance. Here, we consider $\eta(x, y,) = \eta_0\mathrm{sgn}(x)$, $\boldsymbol{u}(x, y, 0) = \boldsymbol{0}$, so the initial PV is 0.
Assume f is constant, the perturbation is small $\eta_0 \ll H$, the PV is small $\zeta \ll f$, and the bottom is flat $H_s = H_0$. Linearise the shallow water PV:

$$q = \frac{f + \zeta}{H_0 + \eta} = \frac{f}{H_0}\left(1 + \frac{\zeta}{f} + ...\right)\left(1 - \frac{\eta}{H_0} + ...\right) \approx \frac{f}{H_0}\left(1 + \frac{\zeta}{f} - \frac{\eta}{H_0}\right)$$

Since PV is conserved, we have

$$\frac{\zeta}{f} - \frac{\eta}{H_0} = -\frac{\eta_0}{H_0}\mathrm{sgn}(x) \qquad \forall t \tag{23}$$

By symmetry, $\partial_y \equiv 0$ so the PV is $\zeta = v_x$. The linearised shallow water equations in this case

$$\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= 0 \\ \eta_t + H_0 u_x &= 0 \end{aligned}$$

Using these equations we have

$$\zeta = v_x = \frac{u_{xt} + g\eta_{xx}}{f} = -\frac{1}{fH_0}\eta_{tt} + \frac{g}{f}\eta_{xx}$$

Now conservation of potential vorticity (23) gives

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = f^2\eta_0\mathrm{sgn}(x)$$

where $c^2 \equiv gH_0$. This is a *Klein-Gordon equation* where the $f^2\eta$ term adds elasticity to the waves.

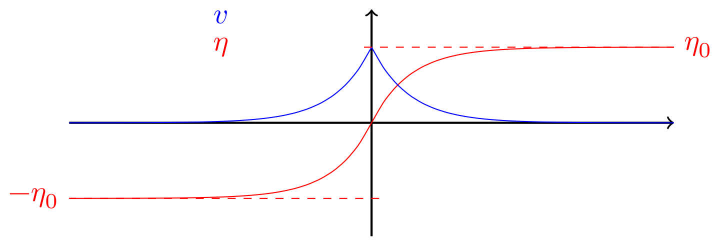
2.3.1 Steady solutions

Consider steady solutions. Owing to the step forcing, our BCs are to match η_x and η at $x = 0$. We find

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R_d} & x > 0 \\ -1 + e^{x/R_d} & x < 0 \end{cases} \tag{24}$$

where $R_d \equiv \sqrt{gH_0}/f$ is the *deformation radius*. From the equations of geostrophic balance we have the velocity components

$$u = 0, \qquad v = \frac{g\eta_0}{fR_d}e^{-|x|/R_d}$$



2.3.2 Transients

The steady solution (24) solves the geostrophic adjustment equation, but it does not match the initial conditions. We add this particular solution to a solution to the homogeneous equation

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = 0$$

with initial condition

$$\eta = \eta_0\mathrm{sgn}(x) - \eta_{\text{steady}} = \eta_0 e^{-|x|/R_d}\mathrm{sgn}(x)$$

We seek solutions of plane wave form

$$\eta = \hat{\eta}e^{i(kx - \omega t)}$$

with $\omega^2 = f^2 + c^2k^2$. These are Poincaré waves.

2.3.3 Energetics

The change in potential energy per unit length in the y direction is

$$\begin{aligned}
PE_{\text{initial}} - PE_{\text{final}} &= \int_{-\infty}^{\infty}\int_{0}^{\eta_i} \rho_0 g z \,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty} \rho_0 g z \,\mathrm{d}z\,\mathrm{d}x \\
&= 2\rho_0 g \left[\int_0^\infty \frac{\eta_i^2}{2}\,\mathrm{d}x - \int_0^\infty \frac{\eta_f^2}{2}\,\mathrm{d}x\right] \\
&= \rho_0 g \eta_0^2 \int_0^\infty \left[1 - (1 - e^{-x/R_d})^2\right]\,\mathrm{d}x \\
&= \frac{3}{2}\rho_0 g \eta_0^2 R_d
\end{aligned}$$

The change in kinetic energy per unit length in the y direction is

$$\begin{aligned}
KE_{\text{initial}} - KE_{\text{final}} &= \int_{-\infty}^{\infty}\int_{-H}^{\eta_i} \frac{1}{2}\rho_0 v_i^2\,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty}\int_{-H}^{\eta_f} \frac{1}{2}\rho_0 v_f^2\,\mathrm{d}z\,\mathrm{d}x \\
&\approx 0 - \frac{1}{2}\rho_0 \int_{-\infty}^{\infty} H_s v_f^2\,\mathrm{d}x \\
&= -\rho_0 H_s \int_0^\infty \frac{g^2\eta_0^2}{f^2 R_d^2} e^{-2x/R_d}\,\mathrm{d}x \\
&= -\rho_0 \frac{R_d^2 g \eta_0^2}{R_d^2} \cdot -\frac{R_d}{2} \cdot \left[e^{-2x/R_d}\right]_0^\infty \\
&= -\rho_0 g \eta_0^2 \frac{R_d}{2}
\end{aligned}$$

Only $\frac{1}{3}$ of the potential energy released is converted into kinetic energy of the geostrophic flow.
Lecture 7 26/10/20 The remainder is radiated away by Poincaré waves.

2.4 Quasi-geostrophic equations

Large scale motions in the ocean and atmosphere are associated with small Rossby number $Ro \equiv \frac{U}{fL} \ll 1$. In this limit, the rotating shallow water equations are approximated by the SW quasi-geostrophic (SW QG) equation. Start from the SW PV equation:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{25}$$

Assumption 1: $Ro \ll 1$ Assuming a small Rossby number implies the flow is close to geostrophic balance with

$$f\hat{\boldsymbol{k}} \times \boldsymbol{u} \approx -g\nabla\eta$$

where $\hat{\boldsymbol{k}}$ is the vertical unit vector. Define the *geostrophic streamfunction* $\psi \equiv \frac{g\eta}{f}$. In terms of this streamfunction we have

$$\begin{aligned}
\boldsymbol{u} &\approx -\nabla \times (\psi\hat{\boldsymbol{k}}) \\
\zeta &= (\nabla \times \boldsymbol{u})\hat{\boldsymbol{k}} \approx \nabla^2\psi
\end{aligned}$$

Assumption 2: small changes in f Recall the Coriolis parameter $f = 2\Omega \sin\theta$ where θ is latitude. Expand in a Taylor series about $\theta = \theta_0$ to get

$$f = f_0 + y\frac{\mathrm{d}f}{\mathrm{d}y}|_{\theta_0} + \cdots \approx f_0 + \beta y$$

where y is in the direction of local North, $f_0 = 2\Omega \sin\theta_0$ and β is defined as

$$\beta = \frac{1}{R}\frac{\mathrm{d}f}{\mathrm{d}\theta}|_{\theta_0} = \frac{2\Omega}{R}\cos\theta_0$$

with R the radius of Earth. For characteristic length scale L, assume $\frac{\beta L}{f_0} \ll 1$. This is the *$\beta$-plane approximation*.

Assumption 3: small changes in fluid height. This is consistent with small Rossby number: from geostrophic balance, we know $\eta \sim \frac{fUL}{g}$ and $\frac{\eta}{H_0} \sim \frac{fUL}{gH_0} = \frac{U}{fL}\frac{L^2}{R_D^2}$. Therefore $\eta/H_0 \ll 1$ if $Ro \ll \frac{R_D^2}{L^2}$. For $L \sim R_D$, $Ro \ll 1$ implies $\eta/H_0 \ll 1$. Further, we assume $h_b/H_0 \ll 1$.

Quasi-geostrophic equations. With these assumptions, SWPV becomes

$$\begin{aligned}\frac{\zeta + f}{H_0 - h_b + \eta} &\approx \frac{f_0}{H_0}\frac{1 + \frac{\beta y}{f_0} + \frac{\zeta}{f_0}}{1 - \frac{h_b}{H_0} + \frac{\eta}{H_0}} \\ &\approx \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\nabla^2\psi}{f_0} + \frac{h_b}{H_0} - \frac{f_0\psi}{gH_0}\right) \\ &= \frac{f_0}{H_0}P_g\end{aligned}$$

where P_g is the *quasi-geostrophic potential vorticity* and $\zeta = \nabla^2\psi, \eta = \frac{f_0\psi}{g}$. Hence from SWPV conservation (25),

$$\frac{\partial P_g}{\partial t} + \boldsymbol{u}\cdot\nabla P_g \approx 0$$

Using $\boldsymbol{u} \approx -\nabla \times (\psi\hat{\boldsymbol{k}})$, $\boldsymbol{u} = -\psi_y, v = \psi_x$ so

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) \approx 0 \tag{26}$$

This is the *shallow water Quasi-geostrophic* (SWQG) equation, which is one equaiton for one unknown ψ, as opposed to SWPV with 2 unknowns ζ, η.

2.4.1 Waves in QG

Assume a flat bottom $h_b = 0$. Linearise (26) about a state of rest (i.e. neglect terms $\mathcal{O}(\psi^2)$). Then

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{f_0^2}{gH_0}\psi\right) + \frac{\partial\psi}{\partial x}\beta = 0$$

Seek plane wave solutions of the form

$$\psi = \psi_0 e^{i(kx+ly-\omega t)}$$

with dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2 + R_D^{-2}}, \qquad R_D \equiv \frac{\sqrt{gH_0}}{f_0}$$

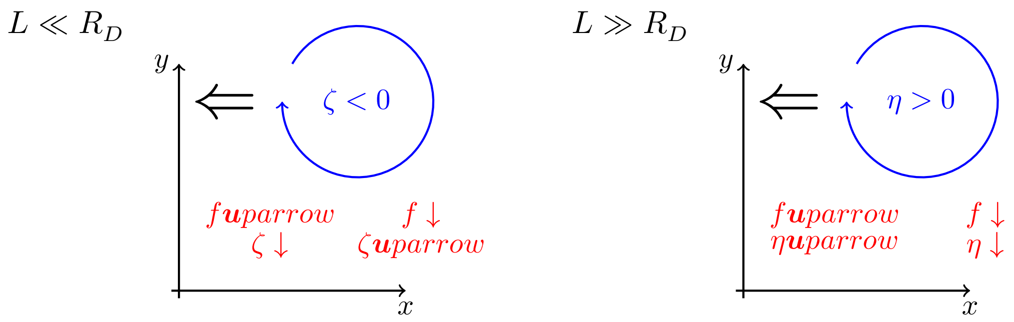
This is the *Rossby wave dispersion relation.* Note $\omega = 0$ (i.e. no waves) if $\beta = 0$. Also, if $h_b = 0$ and $\beta = 0$ there are no wave solutions unlike rotating SW. Thus the QG system 'filters' out Poincaré waves. Note that $\beta = \frac{2\Omega}{R}\cos\theta \geq 0$, hence $c_p = \frac{\omega}{k} \leq 0$. Rossby wave speed is always directed to the *west.*
Consider the size of the dynamic terms in P_g, specifically the ratio of relative vorticity to surface height

$$\frac{\nabla^2\psi}{-\frac{f_0^2\psi}{gH_0}} \sim \frac{R_D^2}{L^2}$$

Hence relativity vorticity dominates at scales small compared to R_D whilst surface height dominates at scales large compared to R_D.

2.4.2 Physical interpretation of Rossby waves

Consider $L \ll R_d$ ($L \gg R_D$) and a small perturbation in the dominant term for the scale, ζ (η). For $L \ll R_D$, the planetary vorticity increases (thus ζ decreases) on the westward side, whilst the planetary vorticity decreases (thus ζ increases) on the eastward side. Hence the perturbation propagates westwards. For $L \gg R_D$, the planetary vorticity increases (η increases) on the westward size and decreases (η decreases) on the eastward side as before. Thus the perturbation propagates to the west also. These are Rossby waves.



2.5 Large scale ocean circulation

2.5.1 Sverdrup flow

Seek steady solutions for rotating shallow water driven by a wind stress $\boldsymbol{\tau}_w$. We have

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} \tag{27}$$

$$H_t + \nabla \cdot (\boldsymbol{u}H) = 0 \tag{28}$$

Consider $\nabla \times (27) \cdot \hat{\boldsymbol{k}}$ and (28) which implies modified PV conservation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = \frac{1}{H}\nabla \times \left(\frac{\boldsymbol{\tau}_w}{\rho H}\right) \cdot \hat{\boldsymbol{k}} \tag{29}$$

Thus we see frictional forcing modifies PV conservation. Assuming H is constant, $\zeta \ll f$ ($Ro \ll 1$), and using the β-plane approximation $f = f_0 + \beta y$, (29) becomes

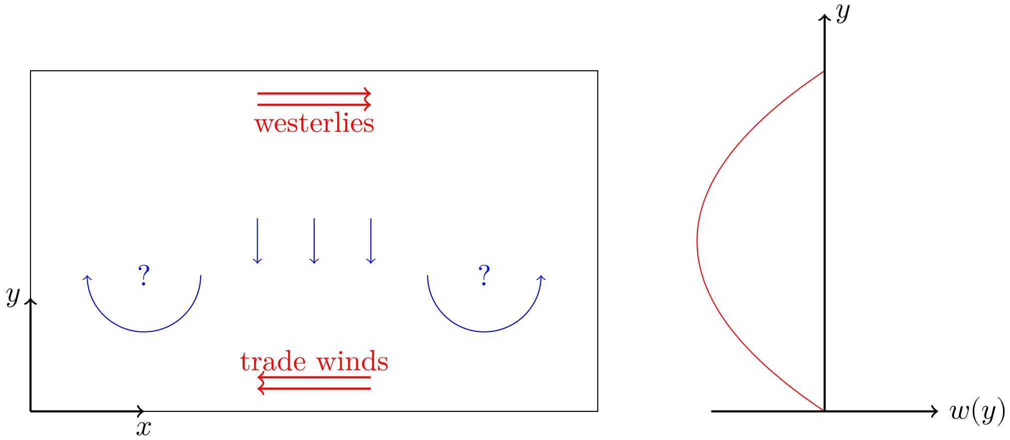
$$\beta v = \frac{1}{\rho H} (\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}} \tag{30}$$

This is called *Sverdrup balance*. Physically, the North/South advection of planetary vorticity $\boldsymbol{u} \cdot \nabla f$ balances the vorticity input by wind.

2.5.2 Western boundary currents

Consider steady circulation in a rectangular basin, driven by a wind stress curl

$$w(y) = \frac{(\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}}}{\rho H}$$



From (30), $w < 0 \implies v < 0$. Recall $\boldsymbol{u} = -\nabla \times \psi \hat{\boldsymbol{k}}$. Boundary conditions are no normal flow at the boundaries, i.e. ψ is constant. Sverdrup balance (30) $\beta \psi_x = w(y)$ gives

$$\psi = \frac{x w(y)}{\beta} + G(y)$$

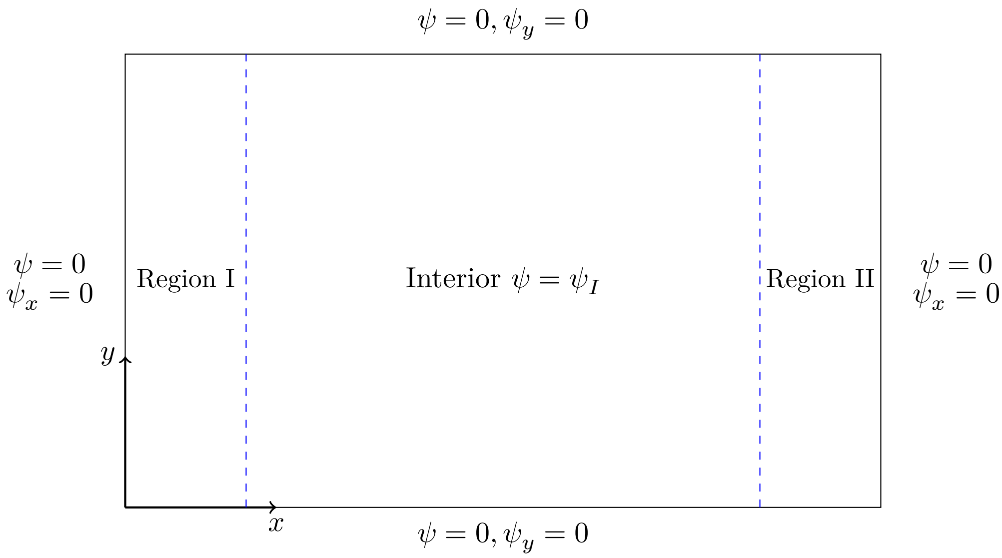
for some arbitary function $G(y)$. This presents a problem: we cannot meet the boundary conditions at both $x = 0$ and $x = L$. Hence we need extra terms and boundary layers. Following Musk, we include horizontal friction in (27):

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} + \nu \nabla^2 \boldsymbol{u} \tag{31}$$

Note here we are using the horizontal gradient $\nabla \equiv (\partial_x, \partial_y)$. Consider $\nabla \times (31) \cdot \hat{\boldsymbol{k}}$ with $\zeta \ll f$. Then

$$\beta \psi_x = w(y) + \nu \nabla^4 \psi \tag{32}$$

The PDE is now fourth order, so we need four boundary conditions.



In region I we have $\psi \approx \psi_I + \psi^{(1)}$ and in region II we have $\psi \approx \psi_I + \psi^{(2)}$. The full solution is $\psi = \psi_I + \psi^{(1)} + \psi^{(2)}$ with interior flow $\psi_I = x\frac{w(y)}{\beta} + G(y)$.

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Region I. Let $\varepsilon = \nu$ with $\varepsilon \ll 1$. Define a rescaled coordinate $\tilde{x} \equiv \frac{x}{\varepsilon^a}$ with $\partial_x = \varepsilon^{-a}\partial_{\tilde{x}}$. Note: if $a > 0$ then $\partial_x \gg \partial_y$. This is the *method of undetermined coefficients.* From the PDE (32) for ψ we have

$$\cancel{\beta\psi^I_x} + \beta\varepsilon^{-a}\tilde{\psi}^{(1)}_{\tilde{x}} = \varepsilon^{1-4a}\tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \cancel{w}$$

Matching exponents, we have $a = \frac{1}{3}$. Hence

$$\beta\tilde{\psi}^{(1)}_{\tilde{x}} = \tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$$

Seek solutions of the form $\tilde{\psi} = \tilde{\psi}_0 e^{r\tilde{x}}$. Then $r^4 - \beta r = 0$ so $r = 0, \beta^{1/3}, -\frac{1}{2}\beta^{1/3} \pm i\frac{\sqrt{3}}{2}\beta^{1/3}$. The general solution is therefore

$$\tilde{\psi}^{(1)} = A(y) + B(y)e^{\beta^{1/3}\tilde{x}} + C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}} + D(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{-i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}}$$

In order for the interior and boundary layer flows to match asymptotically, we apply the *matching condition* $\lim_{\tilde{x}\to\infty}\tilde{\psi}^{(1)} = 0$. Thus $A(y) = B(y) = 0$. For convenience we re-define C and D to get

$$\tilde{\psi}^{(1)} = C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} + D(y)\right)$$

We now apply the boundary conditions. $\psi = 0$ at $x = 0$ gives $\tilde{\psi}^{(1)} = -\psi^I\mid_{x=0}$. Hence

$$C(y)\cos D(y) = -G(y)$$

$\psi_x = 0$ at $x = 0$ gives $\psi^{(1)}_x = -\psi^I_x\mid_{x=0}$. Hence

$$\varepsilon^{-1/3}\tilde{\psi}^{(1)}_{\tilde{x}} = -\frac{w(y)}{\beta}$$

$$\varepsilon^{-1/3}(-\frac{1}{2}\beta^{1/3})C(y)\cos D(y) - \varepsilon^{-1/3}\frac{\sqrt{3}}{2}\beta^{1/3}C(y)\sin D(y) = -\frac{w(y)}{\beta}$$

Since $\varepsilon \ll 1$ and can be taken arbitrarily small, we require

$$\begin{aligned} -\frac{1}{2}\cos D(y) &= \frac{\sqrt{3}}{2}\sin D(y) \\ \implies \tan D(y) &= -\frac{1}{\sqrt{3}} \\ \implies D(y) &= -\frac{\pi}{6} \end{aligned}$$

Combining the boundary conditions we also have $C(y) = -\frac{2}{\sqrt{3}}G(y)$. Finally we have

$$\tilde{\psi}^{(1)} = -\frac{2}{\sqrt{3}}G(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} - \frac{\pi}{6}\right)$$

Region II. Here, we define a rescaled coordinate $\tilde{x} = \frac{x-L}{\varepsilon^{1/3}}$. The same PDE is satisfied in region II, so the general solution is the same. Here, the matching condition is $\lim_{\tilde{x}\to-\infty}\tilde{\psi}^{(2)} = 0$ which gives $A(y) = C(y) = D(y) = 0$, so

$$\tilde{\psi}^{(2)} = B(y)e^{\beta^{1/3}\tilde{x}}$$

We now apply the boundary conditions. $\psi_x = 0$ at $x = L$ gives

$$\begin{aligned} \varepsilon^{-1/3}\tilde{\psi}^{(2)} &= -\psi_x^I \quad \text{at} \quad x = L \\ \varepsilon^{-1/3}\beta^{1/3}B(y) &= -\frac{w(y)}{\beta} \\ \implies B(y) &= -\frac{\varepsilon^{1/3}w(y)}{\beta^{4/3}} \end{aligned}$$

To enforce $\psi = 0$ at $x = L$, note $\lim_{\varepsilon\to 0} B(y) = 0$, so $\tilde{\psi}^{(2)} \mid_{x=L} \to 0$ as $\varepsilon \to 0$ so we instead require $\psi^I \mid_{x=L} = 0$.

$$\implies G(y) = -\frac{w(y)L}{\beta}$$

Hence we have

$$\tilde{\psi}^{(2)} = -\varepsilon^{1/3}w(y)\beta^{-4/3}e^{\beta^{1/3}\tilde{x}}$$

Full solution. The full solution $\psi = \psi^I + \psi^{(1)} + \psi^{(2)}$ is

$$\begin{aligned} \psi = {} & \frac{x-L}{\beta}w(y) && \text{interior} \\ & + \frac{2w(y)L}{\sqrt{3}\beta}e^{-\beta^{1/3}\frac{x}{2\nu^{1/3}}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\nu^{-1/3}x - \frac{\pi}{6}\right) && \text{western boundary correction} \\ & - \nu^{1/3}\beta^{-4/3}w(y)e^{\beta^{1/3}\frac{x-L}{\nu^{1/3}}} && \text{eastern boundary correction} \end{aligned}$$

Note that the Eastern boundary correction is $\mathcal{O}(\nu^{1/3})$ whilst the Western boundary correction is $\mathcal{O}(1)$.

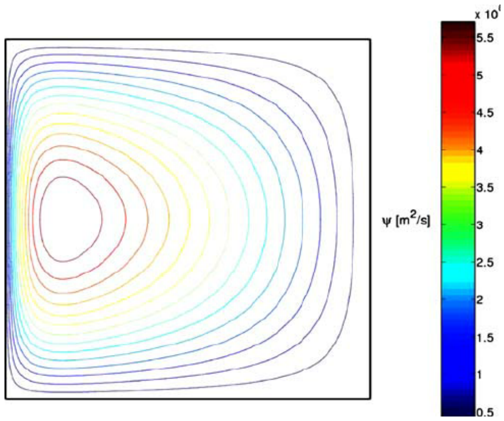
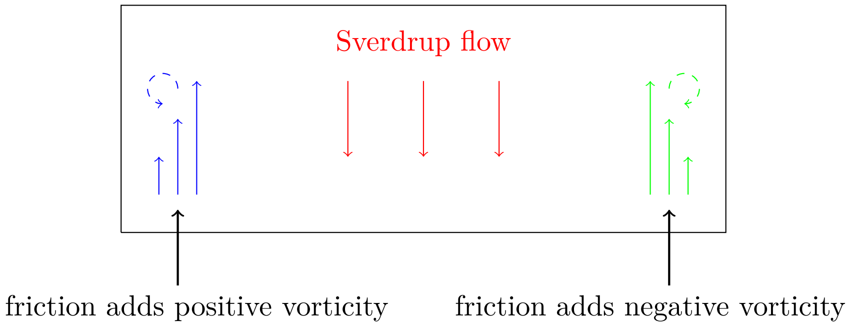


Figure 5: Streamlines of ψ demonstrating western boundary currents.

Physical explanation. The cause of western boundary currents can be physically explained by vorticity. The wind stress curl $w < 0$ inputs negative vorticity in the interior flow. The flow in the western boundary layer inputs positive vorticity to compensate.



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3 Three-dimensional Waves & Instabilities

3.1 Stratification

3.1.1 Boussinesq approximation

We will consider stably stratified flow under the *Boussinesq approximation*: we assume the density ρ may be split into two parts ρ_0 and ρ' with ρ_0 constant and $\rho'/\rho_0 \ll 1$. The pressure may then also be split into two parts, $p_0(z)$, such that $-p_0'(z) = \rho_0 g$, and the remainder p'. This says $p_0(z)$ is the hydrostatic pressure and p' is the excess, whilst ρ_0 is the density component in hydrostatic balance, and stratification arises from ρ'. The vertical component of the momentum equations may then be written

$$\begin{aligned}
\frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho_0 + \rho'}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0 + \rho'}\frac{\partial p'}{\partial z} - g \\
&= -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} + \frac{\partial p_0}{\partial z}\left[\frac{1}{\rho_0} - \frac{1}{\rho_0 + \rho'}\right] - \frac{\partial p'}{\partial z}\frac{1}{\rho_0 + \rho'} - g \\
&\approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g
\end{aligned}$$

where terms including ρ' are discarded unless multiplying g. The *Boussinesq equations* are therefore

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\boldsymbol{g}$$
$$\nabla \cdot \boldsymbol{u} = 0$$
$$\frac{\mathrm{D}\rho'}{\mathrm{D}t} = 0$$

where no assumption has been placed on the direction of $\boldsymbol{f}$.
These equations make clear the role of buoyancy: a light fluid parcel ($\rho' < 0$) experiences an upward force and a heavy fluid parcel ($\rho' > 0$) experiences a downward force. It is further useful to write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$ where $\rho_s(z)$ is a *background* or *reference* density and $\tilde{\rho}$ is the *disturbance density* which is zero for fluid at rest.
The stability of the reference density state is determined by the *buoyancy frequency* or *Brunt-Väisäla frequency* N defined by

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_s}{\mathrm{d}z}$$

3.1.2 Atmosphere & ocean stratification

In the ocean, the buoyancy frequency N is typically $10^{-2}s^{-1}$ in the upper ocean where stratification is strong, and $5 \times 10^{-4}s^{-1}$ in the deep ocean, where stratification is weak.
In the atmosphere, calculating N needs to take account of compressibility, because the density ρ is not conserved by a fluid parcel in reversible, dissipationless motion. The quantity that is instead conserved is the *potential temperature*

$$\theta = T\left(\frac{p}{p_0}\right)^{-2/7}$$

where T is temperature. The corresponding buoyancy frequency is

$$N^2 = -\frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

In this course we will use the Boussinesq approximation for the atmosphere and ocean, despite issues with compressibility.

3.1.3 Internal gravity waves

We linearise about a resting state with density structure represented by the buoyancy frequency N. For simplicity, background rotation is ignored for the time being. We define the *buoyancy* $\sigma = -\rho' g/\rho_0$ for convenience.

$$\tilde{\boldsymbol{u}}_t = -\frac{1}{\rho_0}\nabla\tilde{p} + \tilde{\sigma}\hat{\boldsymbol{z}}$$
$$\nabla \cdot \tilde{\boldsymbol{u}} = 0$$
$$\tilde{\sigma} + N^2\tilde{w} = 0$$

where the notation $\tilde{\boldsymbol{u}}$ denotes the disturbance quantities away from a state of rest. These can be combined into a single equation for $\tilde{w}$:

$$\nabla^2\tilde{w}_{tt} + N^2(\tilde{w}_{xx} + \tilde{w}_{yy}) = 0$$

Assuming N^2 is constant, seek plane wave solutions $\tilde{w} = \hat{w}e^{i(kx+ly+mz-\omega t)}$ (real part implicit). This gives a dispersion relation

$$\omega^2 = N^2 \frac{k^2+l^2}{k^2+l^2+m^2}$$

- If $N^2 > 0$ we get oscillatory motion, and if $N^2 < 0$ we get exponentially growing disturbances.
- We have $0 \leq |\omega| \leq N$ with the lower limit achieved in the limit $k^2 + l^2 \ll m^2$.
- Define $\theta = \tan^{-1}(m(k^2+l^2)^{-1/2})$, so that θ is the angle a surface of constant phase makes with the vertical, so that $\omega = \pm N \cos\theta$. Recall surfaces of constant phase are perpendicular to $\boldsymbol{k}$.
- Owing to incompressibility $\nabla \cdot \boldsymbol{u} = 0$, $\boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0$ i.e. the velocity vector is perpendicular to the wavevector, hence the velocity vector lies in surfaces of constant phase. Thus θ is also the angle that fluid parcel trajectories make with the vertical.
- Since $|\omega| \leq N$, only disturbances with a sufficiently low frequency can propagate as waves. Localised forcing with frequency greater than N will remain localised rather than propagating.
- In the limit $\theta \to 0$, surfaces of constant phase are vertical and the wavevector is horizontal with $\omega \approx \pm N$.
- In the limit $\theta \to \frac{\pi}{2}$, surfaces of constant phase are horizontal and the wavevector is vertical with $\omega \approx 0$.
- Note that the angle of the phase surfaces is independent of wave amplitude. Hence waves on a range of spatial scales all have phase surfaces oriented in the same direction.

Further, note the group velocity is

$$\boldsymbol{c}_g \equiv \frac{\partial \omega}{\partial \boldsymbol{k}} = \pm \frac{N}{(k^2+l^2)^{1/2}(k^2+l^2+m^2)^{3/2}}(km^2, lm^2, -m(k^2+l^2))$$

Lecture 10 4/11/20 which gives $\boldsymbol{c}_g \cdot \boldsymbol{k} = 0$, i.e. the group velocity lies in surfaces of constant phase.

3.2 3D quasi-geostrophic equations

3.2.1 Basic facts about rotation & stratification

1. Assuming the buoyancy frequency N is constant and $\boldsymbol{f} = f\hat{\boldsymbol{z}}$ is vertical, the dispersion relation for small amplitude waves is
$$\omega^2 = \frac{N^2k^2 + f^2m^2}{k^2+m^2}$$
where $\boldsymbol{k} = (k, l, m)$. Thus the relative strength of stratification vs. rotation is N/L vs. f/D, where L is the horizonal lengthscale and D is the vertical lengthscale.
2. Typically, $N \gg f$. In the deep ocean, $N \sim 10^{-3}s^{-1}$ and in the upper ocean and atmosphere $N \sim 10^{-2}s^{-1}$, whilst $f \sim 10^{-4}s^{-1}$.
3. Rotation is important only if $L \gg D$, which implies vertical velocities are much smaller than horizontal velocities. Hence the *hydrostatic approximation* is valid.

4. Given $L \gg D$, the Coriolis force may be neglected in the vertical momentum equation, and in the horizontal momentum equation only the part of the Coriolis force associated with the horizontal velocity is important. This can be seen as follows: let $\boldsymbol{f} = \boldsymbol{f}_h + \boldsymbol{f}_v$ and $\boldsymbol{u} = \boldsymbol{u}_h + \boldsymbol{u}_v$, where $_h$ and $_v$ denote horizontal and vertical components respectively. Then,

$$\boldsymbol{f} \times \boldsymbol{u} = \boldsymbol{f}_h \times \boldsymbol{u}_h + \boldsymbol{f}_v \times \boldsymbol{u}_h + \boldsymbol{f}_h \times \boldsymbol{u}_v + \cancel{\boldsymbol{f}_v \times \boldsymbol{u}_v} \approx \boldsymbol{f}_h \times \boldsymbol{u}_h + \boldsymbol{f}_v \times \boldsymbol{u}_h$$

where we have assumed $|\boldsymbol{u}_v| \ll |\boldsymbol{u}_h|$. At low latitude, this approximation fails since $\boldsymbol{f}_v \ll \boldsymbol{f}_h$. Following the traditional approximation, the horizontal component of the Coriolis force is negligible under the hydrostatic approximation compared to other terms in the vertical momentum equations. Hence only the $\boldsymbol{f}_v \times \boldsymbol{u}_h$ contribution is retained in the horizontal momentum equations. This is equivalent to replacing $\boldsymbol{f}$ with its vertical component only.

5. Given the above assumptions, as well as assuming the fluid layer is thin compared to the radius of the Earth, we get the *primitive equations*.

Further, we invoke the β-plane approximation $f = f_0 + \beta y$. The full $3D$ Boussinesq primitive equations on a β-plane are

$$\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} - (f_0 + \beta y)v &= -\frac{1}{\rho_0}p'_x \\
\frac{\mathrm{D}v}{\mathrm{D}t} + (f_0 + \beta y)u &= -\frac{1}{\rho_0}p'_y \\
p'_z &= -\rho' g \\
\frac{\mathrm{D}\rho'}{\mathrm{D}t} &= \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}$$

This set of equations is formed of three *prognostic* equations which can be used to evolve the five dependent variables, and two instantaneous constraints. There are strong similarities to the shallow water equations on a β-plane: at small Rossby number, the shallow water equations have ‘fast’ modes (e.g. Poincaré waves & Kelvin waves for the shallow-water equations, internal gravity waves for the primitive equations) and ‘slow’ modes of waves which are close to geostrophic balance.

3.2.2 Thermal wind equation

When the Rossby number is small, we expect the flow to be close to geostrophic balance, so that

$$\begin{aligned}
-fv &= -\frac{1}{\rho_0}p'_x \\
fu &= -\frac{1}{\rho_0}p'_y
\end{aligned}$$

Differentiating with respect to z and using the hydrostatic relation, we have the *thermal wind equations*

$$\begin{aligned}
fv_z &= -\frac{g}{\rho_0}\rho'_x \\
fu_z &= \frac{g}{\rho_0}\rho'_y
\end{aligned}$$

Here, the density perturbation ρ' can be viewed analogously to temperature. These equations relate vertical changes in velocity with horizontal changes in density.

3.2.3 Potential vorticity

In the shallow-water equations, the shallow-water potential vorticity was conserved. Under the Boussinesq primitive equations, instead the *Rossby-Ertal potential vorticity* P is conserved materially, where

$$P = \frac{1}{\rho_0}(\boldsymbol{f} + \boldsymbol{\zeta}) \cdot \nabla \rho'$$

In terms of velocities this is equivalent to

$$P = \frac{1}{\rho_0}\left[(f_v + v_x - u_y)\rho'_z + u_z \rho'_y - v_z \rho'_x\right]$$

Note that forcing and dissipation terms are not yet included. These will alter the material conservation of P, giving rise to features in the P field which can affect or drive the evolution of the flow.

3.2.4 3D quasi-geostrophic equations

Primitive equations. Following the same procedure as with the shallow-water equations, we aim to find a prognostic equations for the slow (close to geostrophic balance) motion from the Boussinesq primitive equations on a β-plane.
First, we write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$ where $\rho_s(z)$ represents density variation in a hydrostatically balanced state with no motion, and $\tilde{\rho}$ is associated with motion of the fluid when it is disturbed from said resting state. We also write $p'(x, y, z, t) = p_s(z) + \tilde{p}(x, y, z, t)$ where each term is in hydrostatic balance with the corresponding density term, i.e.

$$\frac{\mathrm{d}p_s}{\mathrm{d}z} = -\rho_s g$$
$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

The density equation $\frac{\mathrm{D}\rho'}{\mathrm{D}t} = 0$ thus becomes

$$\frac{\mathrm{D}\tilde{\rho}}{\mathrm{D}t} + w\frac{\mathrm{d}\rho_s}{\mathrm{d}z} = 0$$

The velocity vield is divided into a part which is in geostrophic balance with the pressure field (assuming constant Coriolis parameter $f = f_0$) and a remainder, the *ageostrophic velocity*:

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a \qquad \text{where } f_0 \boldsymbol{k} \times \boldsymbol{u}_g = -\frac{1}{\rho_0}\nabla_h \tilde{p}$$

Note that the vertical component of the geostrophic velocity $\boldsymbol{u}_g$ is zero, and $\nabla \cdot \boldsymbol{u}_g = 0$. We also assume that the latitudinal (y) lengthscale L_y is sufficiently small that $\beta L_y \ll f_0$. Then if $Ro \ll 1$, it follows that $|\boldsymbol{u}_a| \ll |\boldsymbol{u}_g|$. Thus we are domain limited in latitude.

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The primitive equations may now be written

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a) \cdot \nabla\right](u_g + u_a) - (f_0 + \beta y)(v_g + v_a) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} \tag{33}$$

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a) \cdot \nabla\right](v_g + v_a) + (f_0 + \beta y)(u_g + u_a) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} \tag{34}$$

$$-\frac{\partial \tilde{p}}{\partial z} - \tilde{\rho} g = 0$$

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a) \cdot \nabla\right]\tilde{\rho} + w_a \frac{\mathrm{d}\rho_s}{\mathrm{d}z} = 0$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} = 0$$

Approximation validity.

- Given that $Ro \ll 1$ we may approximate $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ by $\frac{\mathrm{D}_g \boldsymbol{u}_g}{\mathrm{D}t}$ where $\frac{\mathrm{D}_g}{\mathrm{D}t} = \partial_t + \boldsymbol{u}_g \cdot \nabla$
- Also $\beta y \boldsymbol{u}$ can be approximated by $\beta y \boldsymbol{u}_g$.
- Note that $f_0(u_a, v_a)$ and $\beta y(u_g, v_g)$ are of similar size. Hence the requirement $\beta L_y / f_0 \ll 1$ is better expressed by $\beta L_y / f_0 \sim Ro$.
- Also, $w_a \frac{\mathrm{d}\rho_s}{\mathrm{d}z}$ is retained, but $w_a \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}z}$ is not, which requires $|\mathrm{d}\tilde{\rho}/\mathrm{d}z| \ll |\mathrm{d}\rho_s/\mathrm{d}z|$. Denoting the horizontal scale as L and the vertical scale as D, the thermal wind equation gives $g\tilde{\rho}/L\rho_0 \sim f_0 U/D$ where U is the typical horizontal velocity scale. Hence we have

  $$\frac{\tilde{\rho}_z}{\rho_{s,z}} \sim \frac{f_0 U L \rho_0}{g D^2 \rho_{s,z}} = \frac{U}{f_0 L}\left(\frac{L f_0}{ND}\right)^2 = RoBu$$

  where $Bu \equiv (Lf_0/DN)^2$ is the *Burger number* and we have used $\frac{N^2 D}{g} \sim \rho_s/\rho_0$ which follows from the definition of N. For our approximation to be valid, we require $RoBu \ll 1$. If $Bu \sim 1$, then this is implied by $Ro \ll 1$.

Quasi-geostrophic potential vorticity. To reduce the primitive equations to a single prognostic equation, we eliminate $\boldsymbol{u}_a$ by taking the curl of horizontal momentum, i.e. $\partial_x(34) - \partial_y(33)$. The non-divergence of geostrophic velocity then gives a vorticity equation

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] + \beta v_g + f_0\left[\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right] = 0$$

Finally, we eliminate u_a, v_a and w_a using the remaining primitive equations to give

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] + \beta v_g + f_0 \frac{\partial}{\partial z}\left[\frac{\mathrm{D}_g \tilde{\rho}}{\mathrm{D}t} \Big/ \frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right] = 0$$

We now define a streamfunction $\psi = \frac{\tilde{p}}{\rho_0 f_0}$ analogous to the geostrophic streamfunction. Then $u = -\psi_y, v = \psi_x$ from hydrostatic balance $\tilde{\rho} = -\rho_0 f_0 \psi_z / g$. Our prognostic equation, called the *quasi-geostrophic potential vorticity equation* is then

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\psi_{xx} + \psi_{yy} + \left(\frac{f_0^2 \psi_z}{N^2}\right)_z\right] + \beta \psi_x = 0 \tag{35}$$

or equivalently $\frac{\mathrm{D}_g q}{\mathrm{D}t} = 0$ where q is the quasi-geostrophic potential vorticity

$$q = \psi_{xx} + \psi_{yy} + \left(\frac{f_0^2 \psi_z}{N^2}\right)_z + \beta y$$

and $N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_s}{\mathrm{d}z}$. In terms of the QGPV, the (geostrophic) material derivative is

$$\frac{\mathrm{D}_g}{\mathrm{D}t} = \frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}$$

Under the quasi-geostrophic approximation, q is conserved following the horizontal geostrophic flow. The QGPV equation is an approximation to the statemenet of material conservation of

Rossby-Ertel potential vorticity following the flow along ρ' surfaces in a Boussinesq flow, or θ surfaces in a compressible flow.
If q is known, then ψ can be calculated via the *potential vorticity inversion operator*

$$\psi = \left[\partial_x^2 + \partial_y^2 + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\right)\right]^{-1}(q - \beta y)$$

Application of this operator requires boundary conditions on ψ or its derivatives.

Boundary conditions.

- At rigid side boundaries, we require the normal component of $\boldsymbol{u}$ is zero (no flux condition), which requires ψ constant along the boundary.
- At rigid top or bottom boundaries, we require the kinematic boundary condition to be satisfied, i.e. $\frac{\mathrm{D}z}{\mathrm{D}t} = w = \frac{\mathrm{D}h}{\mathrm{D}t}$ on the boundary $z = z_b + h$ where z_b is constant and h is the topographic perturbation. We may express w_a in terms of other variables via the density equation to get
$$w \sim w_a = -\frac{\mathrm{D}_g\tilde{\rho}}{\mathrm{D}t}\left(\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right)^{-1} = \frac{\mathrm{D}h}{\mathrm{D}t} \approx \frac{\mathrm{D}_g h}{\mathrm{D}t}$$
Hence $h \sim \tilde{\rho}\left(\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right)^{-1} \sim D\left(\frac{U}{f_0 L}\right)\left(\frac{L^2 f_0^2}{N^2 D^2}\right) = RoBuD$, so we require $h \ll D$. The boundary condition may then be linearised, so that it can be applied at $z = z_b$. Writing $\tilde{\rho}$ in terms of ψ, we have
$$\frac{\mathrm{D}_g}{\mathrm{D}t}\psi_z = -\frac{N^2}{f_0}\frac{\mathrm{D}_g}{\mathrm{D}t}h$$
at $z = z_b$. This is a prognostic equation for ψ_z at the bottom surface which relates (physically) to material rate of change of density or temperature.

Physical interpretation of QGPV. The density or temperature at horizontal boundaries have similar importance to the QGPV in the interior of the flow. The physical interpretations of different contributions to the quasi-geostrophic potential vorticity q are

$$q = \underbrace{\psi_{xx} + \psi_{yy}}_{\text{relative vorticity}} + \underbrace{\left(\frac{f_0^2}{N^2}\psi_z\right)_z}_{\text{stretching}} + \underbrace{\beta y}_{\text{planetary vorticity}}$$

The stretching measures vertical gradients in density perturbations, i.e. the amount by which nearby density surfaces move apart or together. The ratio of the relativity vorticity term to the stretching term is $1/Bu$. If $Bu \ll 1$ then relative vorticity dominates, whilst if $Bu \gg 1$ then stretching dominates. If $Bu \sim 1$ the terms are comparable and the ratio of horizontal to vertical scales $L/D \sim N/f_0$ is implied by this condition as is called *Prandtl's ratio of scales.*
The 3D quasi-geostrophic equations have a structural similarity to the equations of 2D vortex dynamics, in that there is a non-local dependence of ψ on q. In 2D vortex dynamics, the non-locality is purely horizontal, whilst in the 3D quasi-geostrophic equations the non-locality is also in the vertical and the PV field in a localised region at a given level influences the ψ field outside that region on the same level *and* at other levels.
If N is constant in height, then the PV operator is isotropic in scales coordinates $x, y, Nz/f_0$. The evolution equations are not isotropic since the flow only has horizontal components. Therefore we expect solutions of the QG equations to tend towards isotropy in the coordinates above, but the isotropy is likely not exact.

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3.2.5 Quasi-geostrophic 'point vortex'

Here we will consider an illustrative simple 3D quasi-geostrophic flow calculation known as a 'point vortex', i.e. we assume

$$q = UL^2\delta(x, y, Nz/f_0)$$

where $\delta(\boldsymbol{x})$ is the Dirac delta function and solve

$$\psi_{xx} + \psi_{yy} + \left[\frac{f_0^2}{N^2}\psi_z\right]_z = UL^2\delta(x, y, z)$$

for $\psi(x, y, z)$. We assume that N is constant and that the βy term in QGPV can be neglected. U and L are respectively a constant velocity and a constant length, to give dimensional consistency. First, rescale z by defining $\bar{z} = Nz/f_0$. In Cartesians $(x, y, \bar{z})$, we now have a 3D Laplacian and we deduce the solution with $\psi \to 0$ as $|x|, |y|, |\bar{z}|$ tend to infinity is

$$\psi(x, y, z) = -\frac{UL^2}{4\pi}\frac{1}{\sqrt{x^2 + y^2 + \bar{z}^2}} = -\frac{1}{4\pi}\frac{1}{(x^2 + y^2 + N^2z^2/f_0^2)^{1/2}}$$

The horizontal velocity components are then

$$(u, v) = \frac{UL^2}{4\pi}\frac{(-y, x)}{(x^2 + y^2 + N^2z^2/f_0)^{3/2}}$$

and the density perturbation is given by

$$\tilde{\rho} = -\frac{f\rho_0}{g}\psi_z = -\frac{f\rho_0 UL^2}{4\pi g}\frac{z}{(x^2 + y^2 + N^2z^2/f_0^2)^{3/2}}$$

Far from the point vortex, the sum of the relative vorticity and the stretching term in q is zero, but, consistent with the fact the circulation (both velocity and density perturbations) extend away from the point vortex, they are individually non-zero.

3.3 Waves & instabilities in 3D QG

3.3.1 Framework

Consider the quasi-geostrophic potential vorticity equation (35) as a starting point and consider small-amplitude disturbances to a background (geostrophic) flow which satisfies these equations. The background flow is assumed to be only in the x-direction and to depend only on y, i.e.

$$(u_g, v_g) = (U(y, z), 0)$$

There is a corresponding background quasi-geostrophic stream function $\Psi(y, z)$ such that $U = -\Psi_y$ and a background quasi-geostrophic potential vorticity $Q(y, z) = \Psi_{yy} + (f_0\Psi_z/N^2)_z + \beta y$.
As opposed to deriving the Boussinesq approximation, we will now use the prime $()'$ notation to denote disturbance quantities. The quasi-geostrophic equations retaining only linear terms in disturbance quantities are

$$\left[\frac{\partial}{\partial t} + U(y, z)\frac{\partial}{\partial x}\right](\psi'_{xx} + \psi'_{yy} + (\frac{f_0^2}{N^2}\psi'_z)_z) + (\beta - U_{yy} - (\frac{f_0^2}{N^2}U_z)_z)\psi'_x = 0 \tag{36}$$

The boundary condition at any boundary for top and bottom is here illustrated for $z = 0$,

$$\left[\frac{\partial}{\partial t} + U(y, 0)\frac{\partial}{\partial x}\right]\psi'_z - U_z(y, 0)\psi'_x = -\frac{N^2}{f_0}\left[\frac{\partial}{\partial t} + U(y, 0)\frac{\partial}{\partial x}\right]h'$$

- The quantity $\beta - U_{yy} - (f_0^2 U_z/N^2)_z$ plays a potentially important role: this is the y-gradient of QGPV in the background state.
- The quantity $U_z(y,0)$ in the boundary condition, which is proportional to the y-gradient of density at $z=0$ may also be important.
- Note that the boundary condition is derived from the density equation $\frac{\mathrm{D}\rho'}{\mathrm{D}t}=0$, and the density distribution at the boundary has similar status in the equtions to the interior distribution of QGPV.

3.3.2 Rossby waves – vertical modes

Consider the $3D$ quasi-geostrophic equations in an oceanic configuration with a free surface at $z=0$ in the resting undisturbed state, and a flat bottom at $z=-H$ with buoyancy frequency $N(z)$ which is not assumed constant. Assume that in the undisturbed state the height of the free surface is

$$z=\eta'(x,y,t)$$

and assume disturbances are suitably small that we may estimate $p(x,y,0,t)=p_{\mathrm{atm}}+\rho_0 g\eta'$. Then at $z=0$ we have $\rho_0 g w = \frac{\mathrm{D}_g \tilde{p}(z=0)}{\mathrm{D}t}$. Using the expression for pressure and vertical velocity under the quasi-geostrophic approximation, the boundary condition at $z=0$ is

$$\frac{\mathrm{D}_g \psi'_z}{\mathrm{D}t}+\frac{N^2}{g}\frac{\mathrm{D}_g \psi'}{\mathrm{D}t}=0$$

Similarly, at $z=-H$ the boundary condition is

$$\frac{\mathrm{D}_g \psi'_z}{\mathrm{D}t}=0$$

The QGPV equation then reduces to

$$\left(\psi'_{xx}+\psi'_{yy}+(\frac{f_0^2}{N^2}\psi'_z)_z\right)_t+\beta\psi'_x=0$$

Seek solutions of the form $\psi'(x,y,z,t)=\phi(x,y,t)P(z)$ where P satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{N^2}\frac{\mathrm{d}P}{\mathrm{d}z}\right]=-\frac{1}{gh}P$$

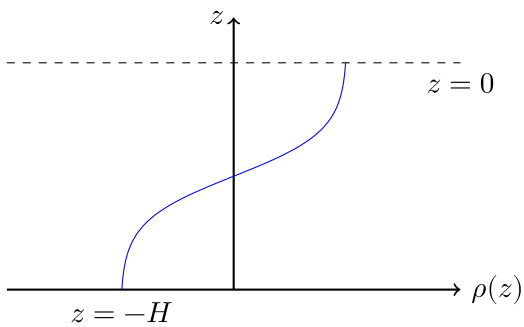
for consistency, where h is a suitably constant and at $z=0$ P satisfies $P_z+(N^2/g)P=0$ and at $z=-H$ P satisfies $P_z=0$. This is an eigenvalue for h, called the *vertical structure equation*, and we expect a countable sequence of possible values $h_1>h_2>\cdots>0$, with the maximum value h_1 corresponding to the simplest possible structure for $P(z)$.

- The height g/N^2 is typically large compared to the depth H (or the vertical length scale associated with variations in stratification). Thus the boundary condition at $z=0$ can be approximated by $P_z=0$. This is the *rigid lid approximation*, equivalent to imposing zero vertical velocity.
- Using the rigid lid approximation, the boundary condition gives P non-zero at $z=0$ and the solution may therefore be used to give a good first estimate of the pressure variation at $z=0$ and hence the variation in free-surface height.

- If $N = N_0$ constant then the largest value $h_1 = N_0^2 H^2/g\pi^2$, i.e.

$$(gh_1)^{1/2} = N_0 H/\pi$$

In this case, $P_1(z)$ has a single zero in the interior layer. The vertical displacement corresponds to $P_1'(z)$ which has a single maximum in the interior of the layer. This is the *first baroclinic mode*[1]. For realistic ocean stratification, we typically find $(gh_1)^{1/2} \approx 3ms^{-1}$ and the second baroclinic mode $(gh_2)^{1/2} \approx 1ms^{-1}$.



- Given h_i and $P_i(z)$ the corresponding equation for $\phi_i(x,y,t)$, describing the horizontal structure of the i^{th} mode is

$$\left(\phi_{ixx} + \phi_{iyy} - \frac{f_0^2}{gh_i}\phi_i\right)_t + \beta\phi_{ix} = 0$$

which is the quasi-geostrophic equation for shallow water of depth h_i.

- We have reduced the three-dimensional problem to an equivalent set of single layer problem, one for each mode, with the layer depths determined as eigenvalues of the vertical structure equation.

- For each vertical mode there is a corresponding Rossby radius of deformation given by

$$L_{iR} = \frac{(gh_i)1/2}{f_0}$$

THe first baroclinic mode has $L_{1R} \approx 30km$, and the second $L_{2R} \approx 10km$ at mid-latitutes.

- For scales much greater than L_R the single-layer dispersion relation implies that the phase and group velocities are westward and given by βL_R^2. The waves are non-dispersive in this limit. For the first baroclinic mode, this gives a Rossby wave speed of approximately $1.5 \times 10^{-2} ms^{-1}$ which is extremely small, and makes observation difficult.

- The wave speed increases towards the equator, but the formula breaks as the equator is approached and we must instead consider equatorial Rossby waves (later in course).

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[1]Baroclinic means 'with vertical structure', where P is not simply related to ρ, whereas barotropic means p is a function of ρ at all levels.

3.3.3 Baroclinic instability

Here we consider a flow for which disturbances that are small-amplitude initially may grow substantially in ampltiude. In the atmosphere and ocean, an important instability mechanism is associated with sloping density surfaces, i.e. horizontal density gradients, present due to vertical shear. Here we consider a problem due to Eady. Consider a basic state on an f-plane, with constant buoyancy frequency N and flow in the x-direction $U = \Lambda z$, so that $\psi_0 = -\Lambda zy$, $\Lambda > 0$. We use the rigid lid approximation with horizontal rigid boundaries at $z = 0$ and $z = H$.
The y-gradient of the QGPV is

$$\beta - U_y y - (\frac{f_0^2}{N^2} U_z)_z = -(\frac{f_0^2}{N^2}\Lambda)_z = 0$$

Hence the linearised QGPV equation reduces to

$$q'_t + \Lambda z q'_x = 0$$

where $q'(x, y, z, t) = q'(x - \Lambda zt, y, z, 0)$. Physically, this says the q' field at any level is advected by the horizontal flow. Thus the q' field is of no consequence in the analysis of possible instability and we choose $q'(x, y, z, 0) = 0$ for convenience. The boundary conditions are

$$\begin{aligned} \psi'_{zt} - \Lambda \psi'_x &= 0 && \text{at } z = 0 \\ \psi'_{zt} + \Lambda H \psi'_{zx} - \Lambda \psi'_x &= 0 && \text{at } z = H \end{aligned}$$

and the interior flow satisfies

$$\psi'_{xx} + \psi'_{yy} + \frac{f_0^2}{N^2}\psi'_{zz} = 0$$

Seek plane wave solutions of the form

$$\psi' = \Re\left[\hat{\psi}(z) e^{i(kx+ly-kct)}\right]$$

with this form, c is the complex phase speed in the x-direction. There will be an instability if $k\Im[c] > 0$. Note wlog we can take $k > 0$. Substituting into the interior equation we find

$$\begin{aligned} \hat{\psi}_{zz} - \mu^2 \hat{\psi} &= 0 \\ \mu &= \frac{N_0}{f_0}(l^2 + k^2)^{1/2} \\ \implies \hat{\psi} &= Ae^{\mu z} + Be^{-\mu z} \end{aligned}$$

Substituting into the boundary conditions and solving for c we find

$$c = \frac{1}{2}\Lambda H\left[1 \pm \sqrt{F(\mu H)}\right]$$

where

$$F(\mu H) = 1 - 4\frac{\coth \mu H}{\mu H} + \frac{4}{\mu^2 H^2}$$

It can be shown that F is an increasing function with $F(\mu H) \to 1$ as $\mu H \to \infty$. Hence there is an instability if $\mu H < \hat{\mu}_c = 0.2399$ where $F(\hat{\mu}_c) = 0$. The growth rate is given by

$$k\Im[c] = \frac{f_0 \Lambda}{N}\frac{k}{(k^2 + l^2)^{1/2}}\left[\frac{1}{2}\mu H\sqrt{-F(\mu H)}\right]$$

Comments.

- The formula for the growth rate shows that for given μ the growth rate is maximised when l is zero. Hence the maximum growth rate will occur when $\frac{1}{2}\mu H\sqrt{-F(\mu H)}$ is maximised over μ, and for this value of μ, when $l = 0$. The maximum value is in fact 0.31 when $\mu H = 1.61$.
- Hence the maximum growth rate in the Eady problem is $0.31\frac{f_0\Lambda}{N}$, i.e. proportional to the vertical shear Λ. In this case, the x-wavenumber k is $1.61f_0/NH$ and $l = 0$.
- In the limit $\mu H \to \infty$, $c/\Lambda H \to 1/\mu H$ or $c/\Lambda H \to 1 - 1/\mathrm{M}H$. In the first case, $B \gg A$ and the wave is *bottom trapped.* In the second case $A \sim B$ and since $\mu H \gg 1$, the wave is *top trapped.* The top wave propagates in the negative x-direction (against the flow) and the bottom wave propagates with the flow.
- If there is only a lower boundary, we find dispersion relation $c = \Lambda/\mu$ hence we have an eastward travelling wave trapped at the lower boundary (for positive vertical shear).
- The propagation mechanism may be understood in terms of circulation induced by a surface density change. Anomalies in the boundary density influence the interior ψ, which generates a flow which supports the anomalies.
- In the case of two boundaries, the instability mechanism results from the phase locking of the trapped top and bottom waves. The fact the instability occurs only for sufficiently small μH is because at larger μH the interaction between the boundary waves is too weak.
- If there were instead side boundaries at $y = 0$ and $y = L$ then $l = \pm n\pi/L$ would be quantised with $n = 1, 2, \dots$.

Density transport. Consider the density transport by the growing wave. The density flux in the y-direction is $\overline{\rho' v'}$ where the overline represents an x-average. In terms of the QG streamfunction, we have

$$\begin{aligned}\overline{\rho' v'} &= -\frac{\rho_0 f_0}{g}\overline{\psi'_x \psi'_z} \\ &= \dots \\ &= \frac{2k\mu^2}{f_0}\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\frac{|A|^2\Lambda\Im[c]}{|c^*\mu - \Lambda|^2} < 0\end{aligned}$$

Hence the density flux is negative, i.e. light fluid is transported in the positive y-direction and heavy fluid in the negative y-direction, tending to weaken the y-gradient of density and hence release some of the potential energy of the background state.
Thus a growing disturbance on a basic state with a positive density gradient equtor to pole produces a poleward flux of light fluid, and an equatorward flux of heavy fluid. This is a *baroclinic instability.*

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3.4 Fronts

A *front* is an elongated region with a large horizontal density gradient. These are associated with

- large vertical motion
- severe weather
- exchange of water between ocean surface and interior

The process by which fronts develop is called *frontogenesis.*

3.4.1 Frontogenesis function

Consider the QG buoyancy equation with $|\boldsymbol{u}_a| \ll |\boldsymbol{u}_g|$ and $b \equiv = -\frac{g\rho}{\rho_0}$.

$$\frac{\partial b}{\partial t} + \boldsymbol{u}_g \cdot \nabla_H b + w_a N^2 = 0 \tag{37}$$

Using index notation, ∇_H(37) gives

$$\partial_t \partial_i b + (\partial_i u_{g,j})(\partial_j b) + u_{g,j} \partial j \partial i b + N^2 \partial_i w_a = 0$$

The buoyancy equation is supplemented by boundary conditions $w_a = 0$ at $z = 0, H$. At these boundaries, we therefore have

$$\frac{\mathrm{D}_g}{\mathrm{D}t} \nabla_H b = \boldsymbol{Q}$$

where $\frac{\mathrm{D}_g}{\mathrm{D}t} = \partial_t + \boldsymbol{u}_g \cdot \nabla_H$ as usual and the *Q-vector* is

$$\boldsymbol{Q} \equiv (-u_{g,x} b_x - g_{g,x} b_y, -u_{g,y} b_x - v_{g,y} b_y)$$

The gradient of the equation applicable at the boundaries $z = 0, H$ is then

$$\frac{\mathrm{D}_g}{\mathrm{D}t} |\nabla_H b|^2 = 2\boldsymbol{Q} \cdot \nabla_H b \tag{38}$$

where the right hand side is the *frontogenesis function*. This equation says that the gradients in buoyancy are advected by $\boldsymbol{Q}$.

3.4.2 Vertical velocity in 3D QG equations

Recall the 3D quasi-geostrophic equations can be written

$$\frac{\mathrm{D}_g}{\mathrm{D}t} \boldsymbol{u}_g + f\hat{\boldsymbol{z}} \times \boldsymbol{u}_g = 0 \tag{39}$$

$$\frac{\mathrm{D}_g}{\mathrm{D}t} b + w_a N^2 = 0 \tag{40}$$

$$\nabla_H \cdot \boldsymbol{u}_g = 0$$

$$\nabla_H \cdot \boldsymbol{u}_a + w_{a,z} = 0 \tag{41}$$

Recall also the thermal wind equations relating horizontal velocity gradients and buoyancy gradients

$$f u_{g,z} = -b_y, \qquad f v_{g,z} = b_x$$

Our aim is to eliminate the geostrophic velocities to give a diagnostic equation relating the Q-vector and the vertical ageostrophic velocity. Using thermal wind balance and the x component of ∂_z(39) we have

$$-\frac{\mathrm{D}_g}{\mathrm{D}t} b_y - b_y u_{g,x} + b_x u_{g,y} - f^2 v_{a,z} = 0 \tag{42}$$

We form a similar equation via ∂_y(40):

$$\frac{\mathrm{D}_g}{\mathrm{D}t} b_y + u_{g,y} b_x + v_{g,y} b_y + w_{a,y} N^2 = 0 \tag{43}$$

Now combining (42) and (43) and from continuity, $v_{g,y} = -u_{g,x}$, gives

$$N^2 w_{a,y} - f^2 v_{a,z} = -2u_{g,y} b_x - 2U_{g,x} b_y \tag{44}$$

Applying the same process with the y component of ∂_z(39) gives

$$N^2 w_{a,x} - f^2 u_{a,z} = -2v_{g,y} b_x - 2v_{g,x} b_y \tag{45}$$

Finally we eliminate $\boldsymbol{u}_g$ in favour of $\boldsymbol{Q}$ via ∂_x(45) + ∂_y(44) and using (41) to give the *QG Omega equation*.

$$N^2 \nabla_H^2 w_a + f^2 w_{a,zz} = 2\nabla_H \cdot \boldsymbol{Q}$$

This is a diagnostic elliptic equation, which can be solved subject to boundary conditions on w. Note that we have eliminated any time derivatives in this equation, making it diagnostic.

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3.4.3 Ageostrophic secondary circulation (ASC)

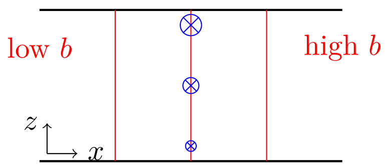
Consider a front $\frac{\partial b}{\partial x} > 0$ in thermal wind balance

$$\frac{\partial \boldsymbol{u}_g}{\partial z} = \frac{1}{f}\frac{\partial b}{\partial x}\hat{\boldsymbol{y}}$$

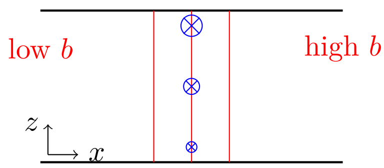
subject to horizontal strain $\boldsymbol{u}_g$ such that

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} = -\alpha$$

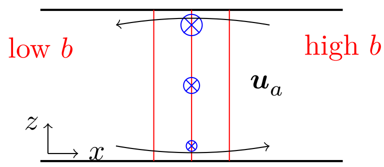
with $\alpha > 0$ constant. Note that to be fully clear, the velocity should be written $\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_s$ where $\boldsymbol{u}_g$ is the thermal wind and $\boldsymbol{u}_s$ is the straining velocity. The Q-vector is then $\boldsymbol{Q} = (\alpha b_x, 0)$ so that $\boldsymbol{Q} \cdot \nabla b > 0$ for $\alpha > 0$. The influence of this front is as follows. Note that each of these influences happen simultaneously, but for clarity we consider the consequences one by one. Our initial set-up is shown below with red lines representing buoyancy surfaces.



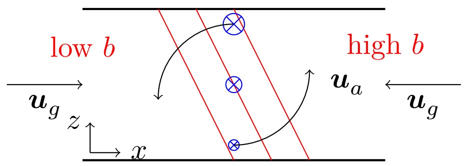
1. A convergent strain $\alpha > 0$ increases $\frac{\partial b}{\partial x}$, by (38).



2. The increase in $\frac{\partial b}{\partial x}$ results in an increase in the hydrostatic pressure gradient in the x-direction. The unbalanced pressure gradient drives an *ageostrophic secondary circulation* $\boldsymbol{u}_a$.



3. The ageostrophic secondary circulation causes the front to 'slump'.



The ageostrophic velocity $\boldsymbol{u}_a$ can be found from the *Sawyer-Eliassen equation*. Ageostrophic strain adds to the background strain.

3.4.4 PV conservation in the Boussinesq equations

Recall the Boussinesq primitive equations. The Navier-Stokes equations under Boussinesq approximations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -\nabla p + \boldsymbol{F}$$

where $\boldsymbol{F}$ is a generic frictional forcing, and p is redefined to absorb $\frac{1}{\rho_0}$. In addition, we have the buoyancy conservation equation

$$\frac{\partial b}{\partial t} + \boldsymbol{u} \cdot \nabla b = D$$

where D is the *diabatic heating/cooling* term. FInally, we have incompressibility $\nabla \cdot \boldsymbol{u} = 0$. We will assume f is constant in line with the traditional approximation. Taking the curl of Boussinesq NS we have

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = f\frac{\partial u}{\partial z} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nabla \times \boldsymbol{F}$$

Let $\boldsymbol{\omega}_a = \boldsymbol{\omega} + f\hat{\boldsymbol{k}}$, called *absolute vorticity*. Then

$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = \boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} + \nabla \times \boldsymbol{F} \tag{46}$$

Note that

$$(\boldsymbol{\omega}_a \cdot \nabla)\frac{\mathrm{D}b}{\mathrm{D}t} = \boldsymbol{\omega}_a \cdot \frac{\mathrm{D}\nabla b}{\mathrm{D}t} + (\boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u}) \cdot \nabla b$$

Now adding $\nabla b \cdot$ (46) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}_a \cdot \nabla b) = \boldsymbol{\omega}_a \cdot \nabla D + \nabla b \cdot \nabla \times \boldsymbol{F}$$

In the absence of diabatic and frictional forcing, $\boldsymbol{\omega}_a \cdot \nabla b$ is conserved. We call $\boldsymbol{\omega}_a \cdot \nabla b$ the *Ertel PV*.

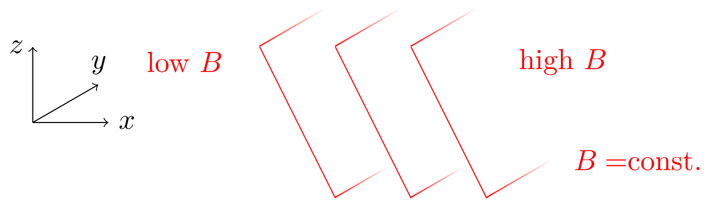
3.5 Internal waves & instabilities in fronts

Consider an unbounded fluid with uniform horizontal and vertical buoyancy gradients:

$$B = M^2 x + N^2 z$$

where M and N are constant. This is the basic state buoyancy field. We assume this buoyancy field is in thermal wind balance, so that the basic state velocity is

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}z} = \frac{M^2}{f}$$



Consider the evolution of small perturbations to this basic state, denoted $\boldsymbol{u}, b, p$. We assume that these perturbations are independent of y, the along-front direction. We also assume that the perturbations are hydrostatic, that f is constant, and we use the traditional approximation. The linearised equations are

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$$u_t - fv = -p_x \tag{47a}$$
$$v_t + w\frac{M^2}{f} + fu = 0 \tag{47b}$$
$$p_z = b \tag{47c}$$
$$b_t + wN^2 + uM^2 = 0 \tag{47d}$$
$$u_x + w_z = 0 \tag{47e}$$

We can eliminate variables in favour of w. Eliminate v via ∂_t(47a) and (47b):

$$u_{tt} + wM^2 + uf^2 = -p_{xt} \tag{48}$$

Eliminate u via ∂_x(48) and (47e):

$$-(\partial_t^2 + f^2)w_z + w_x M^2 = -p_{xxt} \tag{49}$$

Eliminate pressure via ∂_z(49) and (47c):

$$-(\partial_t^2 + f^2)w_{zz} + w_{xz}M^2 = -p_{xxzt} = -b_{xxt}$$

Now ∂_x(47d) and (47e) gives

$$b_{xt} + w_x N^2 - w_z M^2 = 0$$

Hence we have

$$(\partial_t^2 + f^2)w_{zz} + N^2 w_{xx} - 2M^2 w_{xz} = 0$$

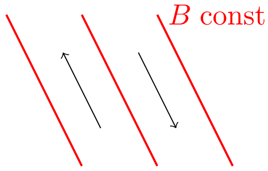
We now seek plane wave solutions with $w = \hat{w}e^{i(kx+mz-\omega t)}$ which gives dispersion relation

$$\omega^2 = f^2 + \frac{k^2}{m^2}N^2 - 2\frac{k}{m}M^2 \tag{50}$$

This is the dispersion relation for hydrostatic rotating stratified internal waves modified by $M^2 = \frac{\partial B}{\partial x}$. Note: instability is possible when $\omega^2 < 0$ if M^2 is large enough. Motivated by this, let $\sigma \equiv -i\omega$ be the growth rate of the internal waves, then (50) gives

$$\sigma^2 = -f^2 - N^2\left(\frac{k}{m} - \frac{M^2}{N^2}\right)^2 + \frac{M^4}{N^2}$$

The most unstable modes have $\frac{k}{m} = \frac{M^2}{N^2}$, in which case the waves are aligned with surfaces of constant B.



This is called *symmetric instability* (SI). For $k/m = M^2/N^2$, instability occurs if $\sigma^2 > 0$ which requires

$$\frac{M^4}{N^2} - f^2 > 0$$

or $\mathrm{Ri} = N^2/(\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}z})^2 < 1$ where Ri is the *gradient Richardson number*.
The Ertel PV of the basic state is

$$q \equiv (f\hat{\boldsymbol{k}} + \boldsymbol{\omega}) \cdot \nabla b = fN^2 - \frac{M^4}{f}$$

since $\boldsymbol{\omega} = -M^2/f\hat{\boldsymbol{x}}$. Symmetric instability develops if $fq < 0$. This is more general and holds for an unbalanced basic state too. This raises a paradox: symmetric instability requires $fq < 0$ to develop, but q is conserved so remains negative. However, we know observationally that the instability does not grow indefinitely, so there must be some mechanism by which SI equilibriates.

How does $fq < 0$ develop? Recall the Omega equation

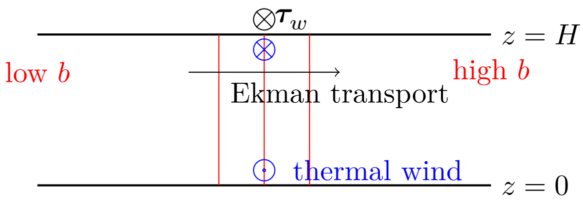
$$\frac{\mathrm{D}q}{\mathrm{D}t} = \boldsymbol{\omega}_a \cdot \nabla D + \nabla b \cdot \nabla \times \boldsymbol{F} = \nabla \cdot (\boldsymbol{\omega}_a D - \nabla b \times \boldsymbol{F})$$

since $\nabla \cdot \boldsymbol{\omega}_a = 0$ and div curl vanishes. Consider the region $0 \le z \le H$ and neglect horizontal fluxes. Enforce boundary conditions $w = 0$ on $z = 0, H$. Then integrating the Omega equation we have

$$\frac{\partial}{\partial t}\int_0^H \overline{q}\,\mathrm{d}z = \left[\overline{(\boldsymbol{\omega}_a \cdot \hat{\boldsymbol{k}})D} - \overline{\nabla b \times \boldsymbol{F}} \cdot \hat{\boldsymbol{z}}\right]_0^H$$

where $\bar{\cdot}$ denotes the horizontal average. For $\boldsymbol{\omega}_a \cdot \hat{\boldsymbol{k}} > 0$, $\overline{q}$ decreases if

- Convective forcing:
  - $D|_{z=H} < 0$ buoyancy loss due to cooling
  - $D|_{z=0} > 0$ buoyancy gain due to heating
- Frictional forcing:
  - $\nabla b \times \boldsymbol{F} \cdot \hat{\boldsymbol{z}}|_{z=H} > 0$ frictional stress aligned with thermal wind
  - $\nabla b \times \boldsymbol{F} \cdot \hat{\boldsymbol{z}}|_{z=H} < 0$ frictional stress anti-aligned with thermal wind



4 Mean Flows

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4.1 Wave mean-flow interaction

4.1.1 Definitions

We consider averages taken over x, and use the notation $\overline{(.)}$ to denote the x-average, for example

$$\overline{\chi} = \frac{1}{L}\int_0^L \chi \, \mathrm{d}x$$

where the flow extends over $0 \le x \le L$. We typically assume periodicity in x, motivated by atmospheric or circumpolar ocean geometry, with x corresponding to longitude. Then we define $\chi' = \chi - \overline{\chi}$ with χ' often being alled the *wave* or *eddy* component.

- If the mean state is the background state then this matches the use of $(.)'$ in the derivation of the Boussinesq primitive equations. However, here the mean state may evolve in time whilst in the Boussinesq derivation the background state is typically time-independent.
- If periodicity in x is assumed, then $\overline{\chi_x} = 0$. Hence $\overline{\theta_x \psi} = \overline{(\theta\psi)_x} - \overline{\theta\psi_x} = -\overline{\theta\psi_x}$.
- Note that $\overline{(.)}$ is an Eulerian average, i.e. it is taken at fixed values of y and z.

For convenience, henceforth we will drop the $\tilde{}$ notation for ρ and p and these quantities will be interpreted respectively as the variation of density and pressure away from the state with reference density $\rho_s(z)$.

4.1.2 Wave propagation and wave activity

Dispersion relations. For plane waves, i.e. waves with sinusoidal structure in the spatial coordinates x, y, z, or some subset of those coordinates, the dispersion relation gives the frequency ω as a function of the spatial wavenumber $\boldsymbol{k}$. The phase velocity $\boldsymbol{c}_p$ with components ω/k_i and the group velocity $\boldsymbol{c}_g$ with components $\partial\omega/\partial k_i$ follow from the dispersion relation. Use of these various quantities requires a *scale separation* between the waves and the background state, with the length scale of variation of the background state being much larger than the wavelength.

Wave activity conservation relation. The wave activity conservation relation is an equation of the typical conservation form

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \boldsymbol{F} = \mathcal{D}$$

where $\mathcal{A}$ is the *wave-activity density*, $\boldsymbol{F}$ is a flux and $\mathcal{D}$ is a term representing dissipation of wave activity, which is associated with physical processes that are dissipative or otherwise non-conservative.

We may derive this relation from the QG equations, starting from the QGPV equation (36) linearised about a basic state flow in the x direction. Denote the x component of the velocity $\overline{u}$ and the corresponding gradient of QGPV in the y direction $\overline{q}_y$. This implies that the instantaneous x-average can be considered a basic state, whereas in (36) we have assumed the basic state is a self-consistent steady solution of the QG equations. This inconsistency is resolved by assuming that any time evolution of $\overline{u}, \overline{q}$ is slow (cf. WKB approximation).

We may re-write (36) as

$$\frac{\partial q'}{\partial t} + \overline{u}\frac{\partial q'}{\partial x} + v'\overline{q}_y = \mathcal{D}'$$

where $\mathcal{D}'$ represents the effect of dissipation on q'. Multiplying by $q'/\overline{q}_y$, assuming that $\overline{q}_y$ varies slowly in time, and taking the x-average gives

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\overline{q'^2}}{\overline{q}_y}\right] + \overline{v'q'} = \frac{\overline{q'\mathcal{D}'}}{\overline{q}_y} \tag{51}$$

We now use the *generalised Taylor identity* which follows from substituting in the QGPV streamfunction for v', q':

$$\begin{aligned} v'q' &= \psi'_x\left(\psi'_{xx} + \psi'_{yy} + \left(\psi'_z\frac{f_0^2}{N^2}\right)_z\right) \\ &= \left(\frac{1}{2}(\psi'_x)^2\right)_x + \left(\psi'_x\psi'_y\right)_y - \left(\frac{1}{2}(\psi'_y)^2\right)_x + \left(\psi'_x\psi'_z\frac{f_0^2}{N^2}\right)_z - \left(\frac{1}{2}(\psi'_z)^2\frac{f_0^2}{N^2}\right)_x \end{aligned}$$

Taking the x-average of the above (noting x-averages of x-derivatives vanish) and subtituting into (51) gives

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\overline{q'^2}}{\overline{q}_y}\right] + \frac{\partial}{\partial y}\left[-\overline{u'v'}\right] + \frac{\partial}{\partial z}\left[-\frac{gf_0}{\rho_0 N^2}\overline{v'\rho'}\right] = \frac{\overline{q'\mathcal{D}'}}{\overline{q}_y}$$

Note that the terms within the y and z derivatives have been re-expressed in terms of u', v' and ρ'. This equation has the structure

$$\frac{\partial\overline{\mathcal{A}}}{\partial t} + \frac{\partial\overline{F^{(y)}}}{\partial y} + \frac{\partial\overline{F^{(z)}}}{\partial z} = \overline{\mathcal{D}_{\mathcal{A}}}$$

which expresses the fact that $\overline{\mathcal{A}}$ is the density of a quantity that can be transported by a flux with components $\overline{F^{(y)}}$ and $\overline{F^{(z)}}$ in the y and z directions and destroyed or created at a rate $\overline{\mathcal{D}_{\mathcal{A}}}$ per unit volume. Here, $\overline{\mathcal{A}}$ is the *Eliassen-Palm wave activity*

$$\overline{\mathcal{A}} = \frac{1}{2}\frac{\overline{q'^2}}{\overline{q}_y}$$

and $\overline{\boldsymbol{F}}$ is the *Eliassen-Palm flux* with components

$$(\overline{\boldsymbol{F}^{(y)}}, \overline{\boldsymbol{F}^{(z)}}) = (-\overline{u'v'}, -f_0\frac{\overline{v'\rho'}}{\frac{\mathrm{d}\rho_s}{/}\mathrm{d}z})$$

This conservation relation need not require a scale-separation assumption, but it is helpful if the conservation relation is consistent with the cases where there is scale-separation, in the sense that is then satisfies the *group-velocity* property $\langle\boldsymbol{F}\rangle = \langle\mathcal{A}\rangle\boldsymbol{c}_g$ where $\langle\cdot\rangle$ denotes a phase average. The Eliassen-Palm wave activity and flux satisfy this property.

Eddy fluxes and wave propagation. The flux $\boldsymbol{F}$ associates correlations between different wave or eddy quantities with directions of wave propagation. The Eliassen-Palm flux y-component $\overline{F^{(y)}} = -\overline{u'v'}$ and the z-component $\overline{F^{(z)}} = -gf_0\overline{v'\rho'}/N^2\rho_0$ imply that the eddy flux in the y direction of the x-component of momentum $\overline{u'v'}$ satisfies

$$\overline{u'v'}\begin{cases} < 0 & \text{for Northward group propagation} \\ > 0 & \text{for Southward group propagation} \end{cases}$$

and that the eddy flux in the y direction of the density $\overline{v'\rho'}$ satisfies

$$\overline{v'\rho'} \begin{cases} < 0 & \text{for upward group propagation} \\ > 0 & \text{for downard group propagation} \end{cases}$$

Note that these results hold for Rossby waves under the small Ro assumption and also assuming that the y-gradient of PV in the basic state is positive, as is always the case in the real atmosphere or ocean if this gradient is dominated by β. For other waves (e.g. Poincaré waves, internal waves or equatorial waves) the results are different.

4.1.3 Mean-flow evolution equations

Using the division into 'mean' and 'eddy' parts, we may apply the averaging operator to the Boussinesq β-plane primitive equations to give

$$\begin{aligned} \overline{u}_t + (\overline{uv})_y + (\overline{uw})_z - \overline{v}(f_0 + \beta y) &= 0 \\ \overline{v}_t + (\overline{v^2})_y + (\overline{wv})_z + (f_0 + \beta y)\overline{u} &= -\frac{\overline{p}_y}{\rho_0} \\ -\overline{p}z - \overline{\rho}g &= 0 \\ \overline{v}_y + \overline{w}_z &= 0 \\ \overline{\rho}_t + (\overline{\rho v})_y + (\overline{\rho w})_z &= 0 \end{aligned} \tag{52}$$

Note that to derive the above it is necessary to use the non-divergence of the velocity field. The simplest approach is to re-write $\frac{\mathrm{D}u}{\mathrm{D}t}$ as $u_t + (u^2)_x + (uv)_y + (uw)_z$ and similarly for $\frac{\mathrm{D}v}{\mathrm{D}t}$ and $\frac{\mathrm{D}\rho}{\mathrm{D}t}$. The definition of the averaging operator as well as (52) gives

$$(\overline{uv})_y + (\overline{uw})_z = (\overline{uv})y + (\overline{uv})_z + (\overline{u'v'})_y + (\overline{u'w'})_z = \overline{v}\overline{u}_y + \overline{w}\overline{u}_z + (\overline{u'v'})_y + (\overline{u'w'})_z$$

Note the primes denote disturbances from the x-average. Now applying the small Ro scaling, dividing horizontal velocities into geostrophic and ageostrophic components and noting that $\overline{v}_g$ is zero, we have

$$\overline{u}_t - f_0\overline{v}_a = -(\overline{u'v'})_y \tag{53}$$

$$f_0\overline{u} = -\frac{\overline{p}_y}{\rho_0} \tag{54}$$

$$-\overline{p}_z - \overline{\rho}g = 0 \tag{55}$$

$$\overline{v}_{ay} + \overline{w}_{az} = 0 \tag{56}$$

$$\overline{\rho}_t + \overline{w}_a\frac{\mathrm{d}\rho_s}{\mathrm{d}z} = -(\overline{\rho'v'})_y \tag{57}$$

This is a coupled set of equations for the five Eulerian-mean quantities $\overline{u}_t, \overline{\rho}_t, \overline{p}, \overline{v}_a, \overline{w}_a$, where the lack of subscript on u, v implies the geostrophic components.
There are two *eddy forcing* terms $-(\overline{u'v'})_y$ and $-(\overline{\rho'v'})_y$, determined respectively by the eddy momentum flux and the eddy density (or heat) flux.

4.1.4 Transformed Eulerian mean equations

We now make a transformation $\overline{w}_a \to \overline{w}_a^*$ defined by

$$\overline{w}_a^* = \overline{w}_a + \frac{(\overline{\rho'v'})_y}{\mathrm{d}\rho_s/\mathrm{d}z}$$

and define $\overline{v}_a^*$ such that

$$\overline{w}_{az}^* + \overline{v}_{ay}^* = 0 \tag{58}$$

Hence we have

$$\overline{v}_a^* = \overline{v}_a - \frac{\partial}{\partial z}\left[\frac{\overline{\rho' v'}}{\mathrm{d}\rho_s/\mathrm{d}z}\right]$$

The x-momentum and density equations then become

$$\overline{u}_t - f_0 \overline{v}_a^* = -(\overline{u'v'})_y + \left(\frac{f_0 \overline{\rho' v'}}{\mathrm{d}\rho_s/\mathrm{d}z}\right)_z = \nabla \cdot \bar{\boldsymbol{F}} \tag{59}$$

$$\overline{\rho}_t + \overline{w}_a^* \frac{\mathrm{d}\rho_s}{\mathrm{d}z} = 0 \tag{60}$$

These two equations combined with (54), (55), (58) are the *transformed Eulerian mean equations.* The transformation has combined the two separate eddy forcing terms into a single term $\nabla \cdot \bar{\boldsymbol{F}}$ in the x-momentum equation and has removed the eddy forcing terms in the density equation. Note $\bar{\boldsymbol{F}}$ is the Eliassen-Palm flux vector and here $\nabla \cdot \bar{\boldsymbol{F}}$ is interpreted as the force acting on the mean flow due to the eddies.

The transformation does not change the response of $\overline{u}_t$ or $\overline{\rho}_t$, but the eddy density flux appears to play a different role. In the standard Eulerian-mean formalism the density flux $\overline{\rho' v'}$ appears as a forcing in the density equation, whilst in the transformed formalism it is part of the force, and in fact $f_0 \overline{\rho' v'}/(\mathrm{d}\rho_s/\mathrm{d}z)$ appears to act as a vertical momentum flux. Hence in the transformed Eulerian-mean formalism, just as horizontally propagating Rossby waves transfer momentum in the horizontal, vertically propagating Rossby waves can be considered to transfer momentum in the vertical.

Note that the Eulerian-mean circulation and the transformed Eulerian-mean circulation satisfy different boundary conditions.

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4.1.5 Non-acceleration conditions

The divergence of the mean Eliassen-Palm flux $\nabla \cdot \bar{\boldsymbol{F}}$ appears as the complete eddy forcing on the mean flow. We also have the Eliassen-Palm wave activity conservation relation

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \boldsymbol{F} = \mathcal{D}$$

from which we deduce

$$\nabla \cdot \bar{\boldsymbol{F}} = \overline{\mathcal{D}} - \frac{\partial \overline{\mathcal{A}}}{\partial t}$$

It follows that if $\overline{\mathcal{A}}_t = 0$, i.e. the waves are steady, and $\mathcal{D} = 0$, i.e. there are no dissipative or other non-conservative effects, then $\nabla \cdot \bar{\boldsymbol{F}} = 0$ and hence $\overline{u}_t = 0$, i.e. there is no acceleration of the mean flow. This is called a *non-acceleration theorem.* These theorems focus attention on what is needed for there to be a mean flow acceleration.

Note that what is referred to as wave dissipation here may correspond to a range of physical effects, including explicitly dissipative processes such as viscosity or some other frictional effect, thermal or molecular or density diffusion, or other thermal damping acting on their own. It may be that the waves *break*, i.e. they become strongly non-linear, generating turbulence and hence adding to the dissipative processes that would have otherwise been weak.

Under geostrophic scaling, we can show that

$$\nabla \cdot \bar{\boldsymbol{F}} = \overline{v' q'}$$

where the right-hand quantity is the northward latitudinal flux of QGPV. Similarly, we could quantity the effect of eddies on the mean flow via the QGPV equation, now taking the form

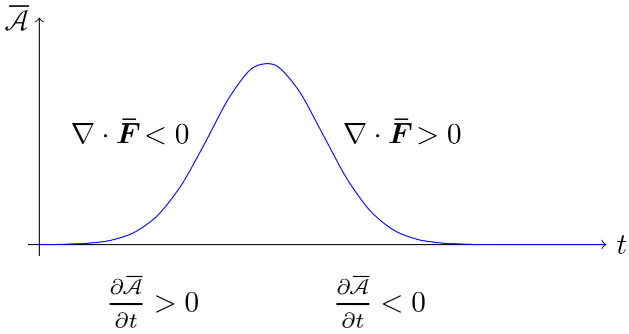
$$\overline{q}_t + (\overline{v'q'})_y = 0$$

The mean acceleration $\overline{u}_t$ and the rate of change of mean density $\overline{\rho}_t$ could then be deduced by using the appropriate inversion operator. An advantaghe of using the transformed Eulerian-mean equations is the response in the mean circulation $(\overline{v}_a^*, \overline{w}_a^*)$ is explicitly visible.

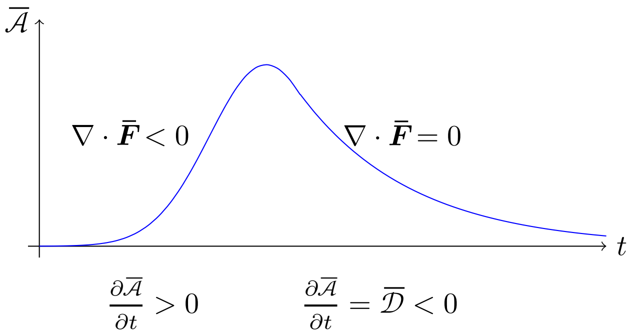
4.1.6 Wave dissipation

Consider two-dimensional flow on a β-plane, governed by the QGPV equation for a single-layer with the deformation radius $L_d \to \infty$, with waves propagating in the y-direction. The *absolute vorticity* ζ_a is the sum of the relative vorticity ζ and the βy term which is materially conserved in the absence of dissipation.

- In the case of no dissipation, i.e. $\overline{\mathcal{D}} = 0$, then as a wavepacket propagates through a region, $\frac{\partial \overline{\mathcal{A}}}{\partial t} > 0$ when it arrives, so $\nabla \cdot \overline{\boldsymbol{F}} > 0$. As the wavepacket leaves the region, $\frac{\partial \overline{\mathcal{A}}}{\partial t} < 0$ hence $\nabla \cdot \overline{\boldsymbol{F}} < 0$. The force on the mean flow is first negative, then positive, so the time-integrated force is zero. Hence the waves have no net effect on the mean flow.



- When dissipation is present, the wave arrives with $\frac{\partial \overline{\mathcal{A}}}{\partial t} > 0$ and hence $\nabla \cdot \overline{\boldsymbol{F}} < 0$ as before. As the wave dissipates, $\frac{\partial \overline{\mathcal{A}}}{\partial t} = \overline{\mathcal{D}} < 0$ and hence $\nabla \cdot \overline{\boldsymbol{F}} = 0$. Therefore a net negative force is applied to the mean flow.

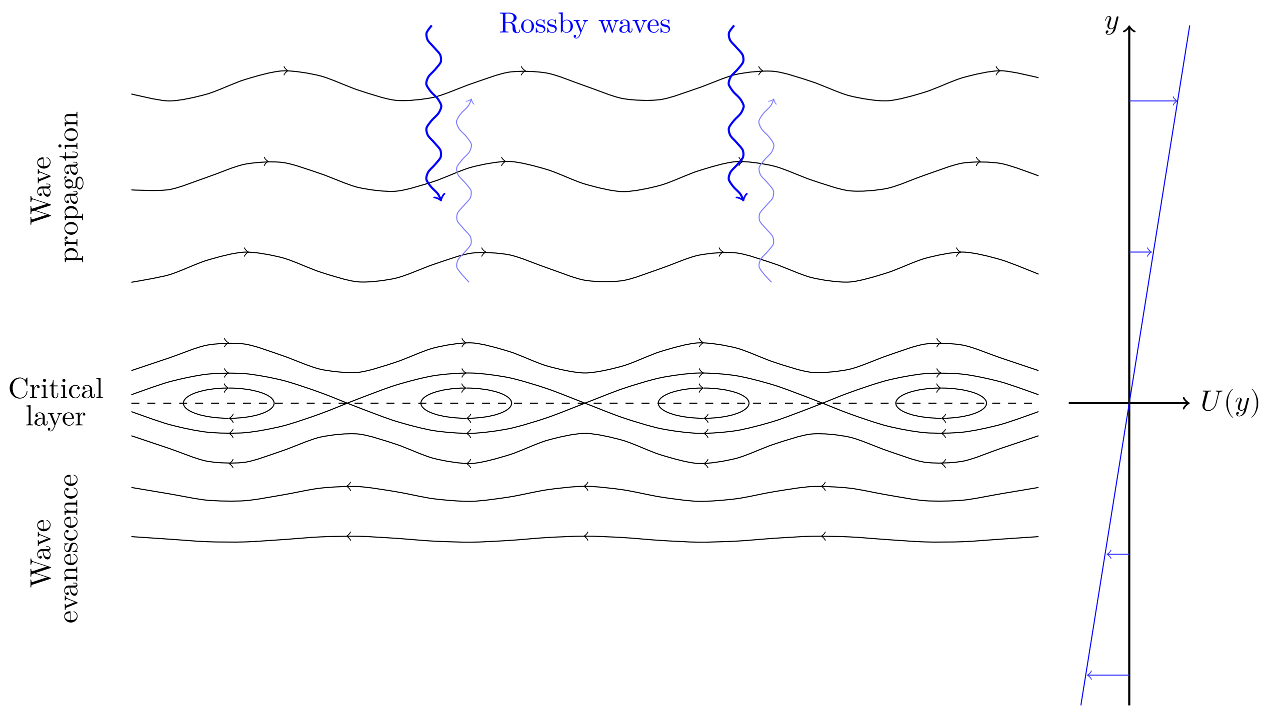


Hence for there to be a net effect on the mean flow in some region, the waves must arrive but not leave, e.g. through dissipation.

Rossby wave critical layer. Consider a forced Rossby wave on a shear flow $U(y)$. The equation describing the Rossby waves is (36) and we assume a plane-wave form $\psi' = \hat{\psi}(y)e^{ik(x-ct)}$. The equation for $\hat{\psi}(y)$ is then

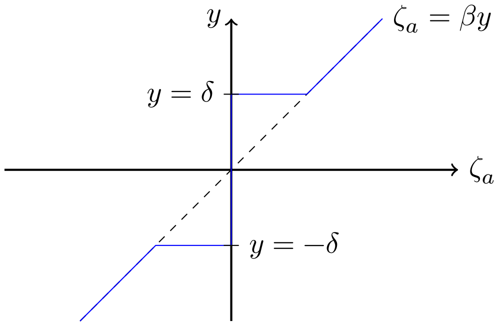
$$(U-c)(\hat{\psi}_{yy} - k^2\hat{\psi}) + (\beta - U_{yy})\hat{\psi} = 0$$

The regions where $U = c$ are called *critical lines*. The sign of $\hat{\psi}_{yy}/\hat{\psi}$ changes from one side of the critical line to the other, so that on one side waves propagate and on the other the waves are evanescent. The equation is singular at $U = c$ hence there is no possible steady linear non-dissipative balance and our governing equation is insufficient. This small but finite region about the critical line is the *critical layer*. In a configuarion where the waves are generated away from the critical line and propagate towards it, they dissipate near the critical layer which implies a systematic force is applied to the flow in the critical layer. There may be some reflection by this critical layer, but this can only be determined by considering the detailed dynamics in this layer.



Rearrangement of absolute vorticity. When non-linearity is an important part of the processes in the critical layer then the effect may be understood in terms of rearrangement of the pre-existing absolute vorticity profile, which may be understood as 'wave breaking'. In the model problem above, the rearrangement occurs in a *non-linear critical layer* centred about $y = 0$ through advection around the 'cat's eye' streamlines.

A simple model is to suppose that before the waves arrive, the absolute vorticity is $\zeta_a = \beta y$ everywhere and after the waves have arrived the absolute vorticity is *mixed* within some region $|y| < \delta$ to be equal to zero, the average value before the waves arrived.



Hence the change in relative vorticity $\zeta = -\beta y$ in $|y| < \delta$ and, assuming that the change in the x-component of velocity Δu is zero outside of $|y| < \delta$, then we have

$$\Delta u = \frac{\beta}{2}(y^2 - \delta^2)$$

in $|y| < \delta$. The change in the x-momentum within this region is thus

$$\int_{-\delta}^{\delta} \Delta u \, \mathrm{d}y = -\frac{2}{3}\beta\delta^3$$

Note the rearrangement of absolute vorticity does not *locally* conserved momentum – an external force is required – but this force is suppled by the transport of momentum by the waves.

4.1.7 Summary

Two important general principles on wave propagation and wave mean-flow interaction are:

1. There can be long-range transfer of momentum. Propagating waves transfer momentum from the region where they are generated to the region where they dissipate or break.

2. Dissipating or breaking waves change the potential vorticity distribution in the region where dissipation of breaking occurs. This change is not usually consistent with local conservation of momentum, but is consistent with the long-range transfer of momentum by the waves into or out of the region, i.e. total conservation of momentum.

Mean-flow interaction problem subtleties. We have established that $\overline{u}_t = -(\overline{u'v'})_y$ (and corresponding expressions in the 3D case), but the most difficult part of the wave mean-flow interaction problem is to predict the dependence of $\overline{u'v'}$ on $\overline{u}$. Ideally, $\overline{u'v'} = \mathcal{F}[\overline{u}]$ for some $\mathcal{F}$ which we could then solve for the evolution of $\overline{u}$. The functional $\mathcal{F}$ must incorporate the effects of flow-dependent wave propagation and flow-dependent trajnsport of potential vorticity. For example, down-gradient diffusion of momentum is a poor model for $\mathcal{F}$ since Rossby waves generated where the flow is strong and positive and dissipating where the flow is weak imply a wave activity flux from the strong-flow region to the weak-flow region. Hence there is a momentum flux from the weak-flow region to the strong-flow region, i.e. momentum transport *up* the gradient.

Mean-flow & baroclinic instability. The choice of $\mathcal{F}$ is particularly important in understanding the effect of baroclinic instability on the mean flow. From our previous discussion of the Eady problem, we know $\overline{\rho'v'} < 0$ for the growing wave, and $\overline{u'v'} = 0$. Hence the Eliassen-Palm flux is purely upward. In more complicated basic states with eastward jet-like structure in the y direction, $\overline{u'v'} \neq 0$ and the pattern of $\overline{u'v'}$ implies wave-activity flux *out* of the jet and hence flux of eastward horizontal momentum *into* the jet. Again, the momentum flux is up-gradient. This is because the growing instability corresponds to a *source* of wave activity in the jet and hence wave propagation out of the jet.

4.2 Mean meridional circulations

Lecture 19 25/11/20

4.2.1 Introduction

We will examine in more detail the mean response of a fluid to wave forcing, comparing the Eulerian-mean viewpoint expressed by the coupled set of equations (53), (54), (55), (56), (57), and the transformed Eulerian-mean viewpoint expressed by the coupled set of equations (54), (55), (58), (59), (60).

Each of the velocity fields $(\overline{v}_a, \overline{w}_a)$ and $(\overline{v}_a^*, \overline{w}_a^*)$ represents a circulation in the *meridional* (y, z) plane. Since each field is non-divergent we may define streamfunctions $\overline{\chi}_a, \overline{\chi}_a^*$ such that

$$(\overline{v}_a, \overline{w}_a) = (\overline{\chi}_{az}, -\overline{\chi}_{ay})$$
$$(\overline{v}_a^*, \overline{w}_a^*) = (\overline{\chi}_{az}^*, -\overline{\chi}_{ay}^*)$$

It may be shown from the Eulerian-mean and transformed Eulerian-mean equations that

$$f_0^2\overline{\chi}_{azz} + N^2\overline{\chi}_{ayy} = f_0(\overline{u'v'})_{yz} - \frac{g}{\rho_0}(\overline{\rho'v'})_{yy} = -f_0\overline{F^{(y)}}_{yz} + \frac{N^2}{f_0}\overline{F^{(z)}}_{yy} \tag{61}$$

$$f_0^2\overline{\chi}_{azz}^* + N^2\overline{\chi}_{ayy}^* = f_0(\overline{u'v'})_{yz} - f_0^2\left(\frac{\overline{\rho'v'}}{\mathrm{d}\rho_s/\mathrm{d}z}\right)_{zz} = -f_0(\nabla\cdot\bar{\boldsymbol{F}})_z \tag{62}$$

where $\bar{\boldsymbol{F}} = (0, \overline{F^{(y)}}, \overline{F^{(z)}})$ is the x-averaged Eliassen-Palm flux. These equations express the forcing on the mean meridional circulation by the eddy fluxes of momentum and density (or, equivalently, by the EP flux divergence).

4.2.2 Boundary conditions

The equations (61) and (62) are supplemented by boundary conditiosn on $\overline{\chi}_a$ or $\overline{\chi}_a^*$. The side boundary condition is straightforward: e.g. $\overline{\chi}_{az} = \overline{\chi}_{az}^* = 0$ at y-boundaries. The bottom boundary condition requires care when deriving, particularly if the bottom is not flat. Consider the case of topographic forcing at the lower boundary $z = h$ so that the full non-linear boundary condition is

$$w = \frac{\mathrm{D}h}{\mathrm{D}t}$$

at $z = h(x, y, t)$ where h is the *topographic height*. If the topography is small amplitude, we may use a Taylor series expansion to re-express the full boundary condition in terms of quantities at $z = 0$ rather than $z = h$:

$$w + hw_z = h_t + uh_x + vh_y + u_zhh_x + v_zhh_y + \mathcal{O}(h^3)$$

where all RHS quantities are now evaluated at $z = 0$. Consider now the continuity equation $u_x + v_y + w_z = 0$ from which we can use to write the LHS as

$$hw_z = -hu_x - hv_y = -(hu)_x - (hv)_y + h_xu + h_yv$$

and substituting into the previous equation we have

$$w = h_t + u_zhh_x + v_zhh_y + (hu)_x + (hv)_y + \mathcal{O}(h^3)$$

We now assume that in the absence of topography the velocity is purely longitudinal (i.e. x-direction), so that $v = \mathcal{O}(h)$ and $\overline{h} = 0$ for all t. We may also replace $\overline{w}$ by $\overline{w}_a$ in line with quasi-geostrophic scaling. Hence, taking x-averages, at leading order we have

$$\overline{w}_a(y, 0, t) = (\overline{h'v'})_y$$

Hence the Eulerian mean velocity does not necessarily vanish at $z = 0$. The corresponding boundary condition on the transformed Eulerian mean circulation is

$$\overline{w}_a^*(y, 0, t) = (\overline{h'v'})_y + \frac{(\overline{\rho'v'})_y}{\mathrm{d}\rho_s/\mathrm{d}z}$$

Assuming the waves are steady and there is no density dissipation at the lower boundary, we may use $\frac{\mathrm{D}}{\mathrm{D}t}\rho(x, y, h, t) = 0$ to re-write the lower boundary condition on ρ as

$$\overline{u}h'_x \frac{\mathrm{d}\rho_s}{\mathrm{d}z} = -\overline{u}\rho'_x - \psi'_x \overline{\rho}_y$$

Now multiplying by ψ' and averaging, it follows that $f_0\overline{h'\psi'_x} = -\overline{F^{(z)}}$ and hence the lower boundary condition on $\overline{w}_a$ and $\overline{w}_a^*$ may be written as

$$\overline{w}_a = -\frac{1}{f_0}(\overline{F^{(z)}})_y \quad \text{and} \quad \overline{w}_a^* = 0 \quad \text{on } z = 0$$

Note that it is the transformed Eulerian mean vertical velocity $\overline{w}_a^*$ which vanishes at the lower boundary, and not $\overline{w}_a$.

4.2.3 Model problem

Consider a flow at small Ro confined to a β-plane longitudinal channel with rigid walls at $y = 0, L$. Waves are forced by topographic perturbations of the lower boundary of the form

$$h = \Re\left[h_0 e^{ikx} \sin\frac{\pi y}{L}\right]$$

The basic state flow is assumed to be in the x-direction, and the velocity a function of height so $\boldsymbol{u} = u_0(z)\hat{\boldsymbol{x}}$. The vertical variation of the waves depends on $u_0(z)$ and on the buoyancy frequency N. Assuming N is a function of z only, as in quasi-geostrophic theory, then we may write ψ' as

$$\psi' = \Re\left[\hat{\psi}(z)e^{ikx}\sin\frac{\pi y}{L}\right]$$

It follows that $\overline{\psi'_x\psi'_y} = 0$ so that the Eliassen-Palm flux is purely vertical with

$$\overline{F^{(z)}} = \frac{f_0^2}{N^2}\overline{\psi'_x\psi'_z} = \frac{f_0^2}{N^2}\Im\left[k\hat{\psi}(z)^*\hat{\psi}'(z)\right]\sin^2\frac{\pi y}{L} = F_0\Theta(z)\sin^2\frac{\pi y}{L}$$

where $F_0 = f_0^2/N^2$ and $\Theta(z) = \Im\left[k\hat{\psi}^*\hat{\psi}'\right]$. To be consistent with the basic properties of Rossby waves, for upward propagation we require $F_0\Theta(z) > 0$. We can determine $\Theta(z)$ by solving the equation for $\hat{\psi}(z)$ given the z-variation of the basic state and of any dissipative processes. To avoid this complication, we assume a simple step function form for $\Theta(z)$:

$$\Theta(z) = \begin{cases} 1 & z < H_d \\ 0 & z > H_d \end{cases}$$

This is a simple representation of a situation where waves are generated far below $z = H_d$, propagate upwards, and dissipate in a thin critical layer localised about $z = H_d$. The problem is then reduced to solving (61) or (62) with a given forcing term and with specified boundary conditions. The assumed form for $\overline{F^{(z)}}$ gives a forcing term in (61) which is non-zero as $z \to -\infty$, but a forcing term in (62) which tends to zero as $z \to \pm\infty$. Thus it is more straightforward to solve (62) and seek a solution $\overline{\chi}^*$ that tends to zero as $z \to \pm\infty$. Once $\overline{\chi}^*$ is known, $\overline{\chi}$ may be deduced using the inverse transformation.
The rigid wall boundary conditions imply $\overline{\chi}_{az} = \overline{\chi}^*_{az}$ at $y = 0, L$. It is therefore natural to expand the forcing and solution in a sine Fourier series:

$$\overline{F^{(z)}} = F_0\Theta(z)\sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{L}$$

$$\overline{\chi}^*_a = \sum_{n=1}^{\infty} \overline{\chi}^*_n(z)\sin\frac{n\pi y}{L}$$

where the c_n are the coefficients in the Fourier series for $\sin^2\frac{\pi y}{L}$. The governing ODE for $\overline{\chi}^*_n(z)$ follows from subtitution into (62):

$$f_0^2\overline{\chi}^{*\prime\prime}_n(z) - \frac{N^2\pi^2 n^2}{L^2}\overline{\chi}^*_n(z) = f_0F_0c_n\delta'(z - H_d)$$

The solution satisfying the boundary conditions $\overline{\chi}^*_n \to 0$ as $z \to \pm\infty$ is

$$\overline{\chi}^*_n(z) = \frac{F_0c_n}{2f_0}\exp\left[-\frac{N\pi n}{Lf_0}|z - H_d|\right]\mathrm{sgn}(z - H_D)$$

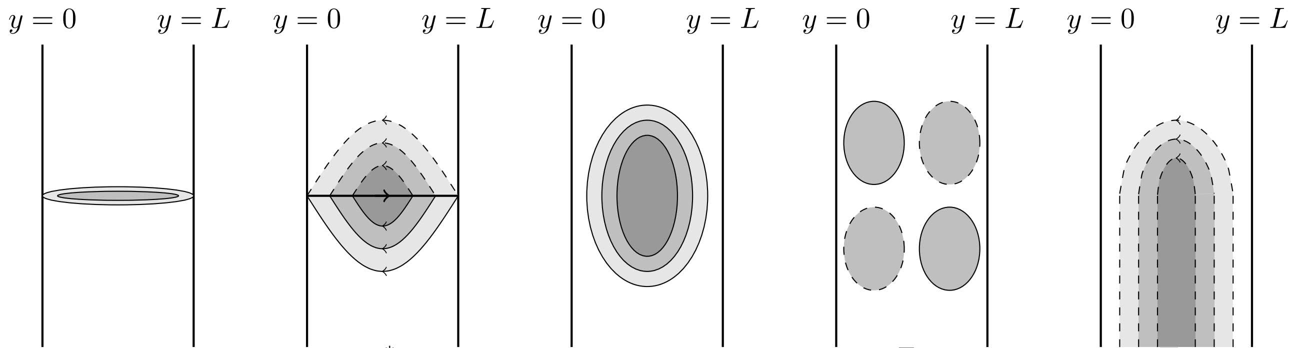
All flow variables can be deduced from the above solution. Using $(\cdot)_n$ to denote the n^{th} coefficient in the Fourier series, it follows that

$$(\overline{v}^*_a)_n = -\frac{F_0c_n}{2f_0}\frac{N\pi n}{Lf_0}\exp\left[-\frac{N\pi n}{Lf_0}|z - H_d|\right] + \frac{F_0c_n}{f_0}\delta(z - H_d)$$

and, noting tmyol[4] $\nabla\cdot\bar{\boldsymbol{F}} = -F_0c_n\delta(z - H_d)$;

$$(\overline{u}_t)_n = -\frac{F_0c_n}{2}\frac{N\pi n}{Lf_0}\exp\left[-\frac{N\pi n}{Lf_0}|z - H_d|\right]$$

Similar expressions may be derived for the Fourier coefficients of $\overline{w}^*_a$ and $\overline{\rho}_t$. In particular, note $\overline{\chi}_a = \overline{\chi}^*_a + \overline{F^{(z)}}/f_0$. We can derive a schematic view of the response of various quantities to the eddy forcing by considering only the $n = 1$ Fourier coefficients. Gray shading indicates negative values within solid lines, and positive values within dashed lines.



$\nabla\cdot\bar{\boldsymbol{F}}$ Shallow negative region | $\overline{\chi}^*_a$ Transformed Eulerian mean circulation | $\overline{u}_t$ Mean acceleration | $\overline{\rho}_t$ Mean rate of change of density | $\overline{\chi}_a$ Eulerian mean circulation

Note the contrast between the responses in the transformed Eulerian-mean circulation and the Eulerian-mean circulation and the implications for the balance in the x-momentum and density equations.

Transformed Eulerian-mean view: in the *wave propagation* region $z < H_d$ the vertical velocity $\overline{w}_a^*$ is zero and there is vertical transport of momentum via $\overline{F^{(z)}}$. In the *wave dissipation* region centred on $z = H_d$ there is localised wave force $\nabla \cdot \bar{\boldsymbol{F}}$ which is redistributed in the vertical by the meridional circulation $(\overline{v}_a^*, \overline{w}_a^*)$.

Eulerian-mean view: in the *wave propagation* region $z < H_d$ the vertical velocity $\overline{w}_a$ is non-zero. The effect of $\overline{w}_a$ on the mean density field is cancelled by the effect of the eddy flux $\overline{\rho' v'}$. There is vertical transport of planetary angular momentum (because the fluid moving upwards on one side of the channel has a different value of angular momentum to that moving downwards on the other side of the channel). In the *wave dissipation* region there is a latitudinal velocity $\overline{v}_a$ which provides a Coriolis force and hence leads to acceleration.
The transformed Eulerian-mean view is arguably simpler because it removes the cancellation between the effect of vertical advection by the mean flow, and the effect of eddy density fluxes. Also, it combines eddy fluxes into a single forcing term as noted previously. Additionally it may be shown that the transformed Eulerian-mean flow is more relevant to the transport of tracers. Vertical motion in the Eulerian-mean circulation does not imply corresponding vertical motion of tracers, e.g. the upward motion at high latitudes in the 'Ferrel cell' does not imply that tracers are transported upwards.
The transformed Eulerian-mean formalism can be interpreted as an approximation to taking averages not at fixed z, but over very thin layers between neighbouring density surfaces. The fact that the thickness and the z-position of the layer are both variable affects the calculated average. Momentum can be exchanged between neighbouring layers by pressure forces acting on their boundaries.

4.2.4 Dependence of response on vertical scale of $\nabla \cdot \bar{\boldsymbol{F}}$

In the model problem above, $\nabla \cdot \bar{\boldsymbol{F}}$ is non-zero only in a layer with very small vertical scale. Suppose instead that the 'forcing layer' has vertical scale D. Then (62) gives

$$\max\left[\frac{f_0^2\overline{\chi}_a^*}{D^2}, \frac{N^2\overline{\chi}_a^*}{L^2}\right] \sim \frac{f_0 F_0}{D}$$

The *shallow forcing regime* is when $ND/f_0L \ll 1$. Then $f_0^2\overline{\chi}_a^*/D^2 \sim f_0F_0/D$, hence $\overline{\chi}_a^* \sim F_0D/f_0$ and $\overline{v}_a^* \sim F_0/f_0$. The dominant balance in the momentum equations within the forcing layer is therefore that most of $\nabla \cdot \bar{\boldsymbol{F}}$ is balanced by the Coriolis force. The mean meridional circulation redistributes in the vertical the effect of $\nabla \cdot \bar{\boldsymbol{F}}$ and the resulting acceleration occurs over a region that is much deeper than the forcing layer.
The *deep forcing regime* is when $ND/f_0L \gg 1$. Then $N^2\overline{\chi}_a^*/L^2 \sim f_0F_0/D$, hence $\overline{\chi}_a^* \sim (f_0L/ND)^2F_0D/f_0$ and $\overline{v}_a^* \sim (f_0L/ND)^2F_0/f_0$. The Coriolis force therefore plays only a minor role in the momentum equation within the forcing layer and at each level $\nabla \cdot \bar{\boldsymbol{F}}$ is balanced by the x-component of the mean acceleration.
The different regimes could be illustrated by replacing $\Theta(z)$ with a simple function that varies from 1 to 0 over some finite vertical scale.

Physical interpretation of mean meridional circulation. The mean meridional circulation may be regarded as arising in order to maintain, under the effect of eddy forcing, the constraints of geostrophic and hydrostatic balance. Thus if a force is applied to a rotating system, the response cannot appear purely as an acceleration; there must be an accompanying change in the density field. If a force is *deep*, in the sense given above, most of the response will appear as acceleration. If the force is *shallow*, then most of the response will appear as a meridional circulation and hence a density change. Similarly, if an applied heating field is shallow, most of the response appears as a change in temperature or density, but if it is deep, then most will appear as meridional circulation, and hence as a change in velocity.

4.3 Equatorial waves

Lecture 20 27/11/20

Previously, we have considered wave motion when rotation and stratification co-exist under the shallow-water model. The analysis gave Poincaré waves and boundary Kelvin waves when $f = f_0$ is constant, and in the β-plane approximation with $f = f_0 + \beta y$, Rossby waves. Poincaré and Kelvin waves are 'fast' waves whilst Rossby waves are 'slow' waves since their phase speed scale as $f_0 L_D, \beta L_D^2$ respectively. At low latitudes when $\beta L_d \sim f_0$, this distinction is no longer clear. Dynamics at low latitudes require a different analysis.

4.3.1 Horizontal structure and propagation

The shallow-water equations are still a suitable model for analysis at low latitudes. The equatorial β-plane approximation is $f = \beta y$, i.e. $f_0 = 0$ in the Coriolis parameter. Then the shallow-water equations linearised about a state of rest are

$$u_t - \beta y v = -g\eta_x \tag{63}$$
$$v_t + \beta y u = -g\eta_y \tag{64}$$
$$\eta_t + H(u_x + v_y) = 0 \tag{65}$$

where u and v the usual horizontal velocity components, η is the free surface displacement, and H is the layer depth in the resting state.
There are two dimensional parameters present in these equations, β and $c = \sqrt{gH}$. These can be used to form time and lengthscales $T_{\text{eq}} = (c\beta)^{-1/2}$ and $L_{\text{eq}} = (c/\beta)^{1/2}$. L_{eq} is referred to as the *equatorial deformation radius*, the analogue of the extratropical deformation radius $L_D = c/f_0$.
Taking $\partial_x(64) - \partial_y(63) - \beta y(65)$ gives

$$\frac{\partial}{\partial t}(v_x - u_y - \frac{\beta y \eta}{H}) + \beta v = \frac{\partial}{\partial t}(\zeta - \frac{\beta y \eta}{H}) + \beta v = 0 \tag{66}$$

where $\zeta = v_x - u_y$ is the relative vorticity. This expresses conservation of PV in the linearised approximation.
Now consider $\partial_t(64) - \beta y(63)$ which gives

$$v_{tt} + \beta^2 y^2 v = -g\eta_{yt} + \beta y g \eta_x$$

Substitute for η_t from (65) to get

$$v_{tt} + \beta^2 y^2 v = gH(u_{xy} + v_{yy}) + \beta y g \eta_x = c^2(u_y - v_x + \frac{\beta y \eta}{H})_x + c^2(v_{xx} + v_{yy})$$

The first term includes the potential vorticity associated with the disturbance, hence differentiate with respect to time and use (66) to get

$$v_{ttt} + \beta^2 y^2 v_t - c^2(v_{xx} + v_{yy})_t - \beta c^2 v_x = 0$$

This equation is structurally similar to that obtained for the extratropics; with βy replaced by f_0 and the additional β term eliminated we obtain the equation for Poincaré waves in the extratropics. Seek plane wave solutions of the form

$$v = \Re\left[\hat{v}(y)e^{i(kx-\omega t)}\right]$$

with k and ω the constant x-wavenumber and frequency respectively. We require $\hat{v}(y)$ bounded as $|y| \to \infty$. We find the ODE for $\hat{v}$ as

$$\left[\frac{\omega^2}{c^2} - \frac{\beta^2 y^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right]\hat{v} + \hat{v}_{yy} = 0$$

This equation along with the boundary condition as $|y| \to \infty$ defines an eigenvalue problem, yielding the eigenvalue condition

$$\omega^2 - c^2 k^2 - \frac{\beta k c^2}{\omega} = (2n+1)\beta c$$

for $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions

$$\hat{v}_n(y) = H_n(y\sqrt{\beta/c})e^{-\frac{\beta y^2}{2c}}$$

where H_n are the *Hermite polynomials* with $H_0(s) = 1, H_1(s) = 2s, H_2(s) = 4s^2 - 2$, etc. Non-dimensionalise the eigenvalue condition using $\omega T_{\text{eq}} = \hat{\omega}$ and $kL_{\text{eq}} = \hat{k}$. Then

$$\hat{\omega}^2 - \hat{k}^2 - \frac{\hat{k}}{\hat{\omega}} = 2n+1$$

This is a quadratic for $\hat{k}$ given $\hat{\omega}$ with roots

$$\hat{k} = -\frac{1}{2\hat{\omega}} \pm \sqrt{\hat{\omega}^2 + \frac{1}{4}\hat{\omega}^2 - (2n+1)}$$

Note that for any pair $(\hat{k}, \hat{\omega})$, there is a corresponding pair $(-\hat{k}, -\hat{\omega})$ which represents the same wave mode. Hence it is necessary to consider only half of the $(\hat{k}, \hat{\omega})$ plane and by convention we choose $\hat{\omega} > 0, -\infty < \hat{k} < \infty$.

- The $n = 0$ mode gives $\hat{k} = \hat{\omega} - 1/\hat{\omega}$ or $\hat{k} = -\hat{\omega}$. The second root is unphysical, so the $n = 0$ mode has dispersion relation
$$\hat{k} = \hat{\omega} - \frac{1}{\hat{\omega}}$$
For $\hat{\omega} > 1$ we have $\hat{k} > 0$ and for $0 < \hat{\omega} < 1$ we have $\hat{k} < 0$.
- The $n = 1, 2, \ldots$ modes have real roots only if $\hat{\omega}^4 - (2n+1)\hat{\omega}^2 + \frac{1}{4\hat{\omega}^2} > 0$, implying either $\hat{\omega}^2 < n + \frac{1}{2} - \sqrt{n(n+1)}$ or $\hat{\omega}^2 > \sqrt{n(n+1)} + n + \frac{1}{2}$. Hence there is a frequency gap, the size of which increases with n.

This covers all solutions with $v \neq 0$. There are more solutions with $v = 0$ missed by the above approach. Setting $v = 0$ in (63), (64), (65) which yields $\eta_{tt} - c^2\eta_{xx} = 0$. Requiring exponential decrease in y only (otherwise unphysical) yields dispersion relation $\omega = ck$ for plane wave solutions with

$$\hat{\eta}(y) \propto e^{-\frac{\beta y^2}{2c}}$$

This is an *equatorial Kelvin wave* analogous to the boundary Kelvin wave on an f-plane. There is no latitudinal velocity and the balance in the latitudinal momentum equations is geostrophic between the Coriolis force associated with u and the pressure gradient associated with η_y. Note that the relation $\omega = kc$ is equivalent to $\hat{\omega} = \hat{k}$ and corresponds to the $n = -1$ mode of our previous dispersion relation.

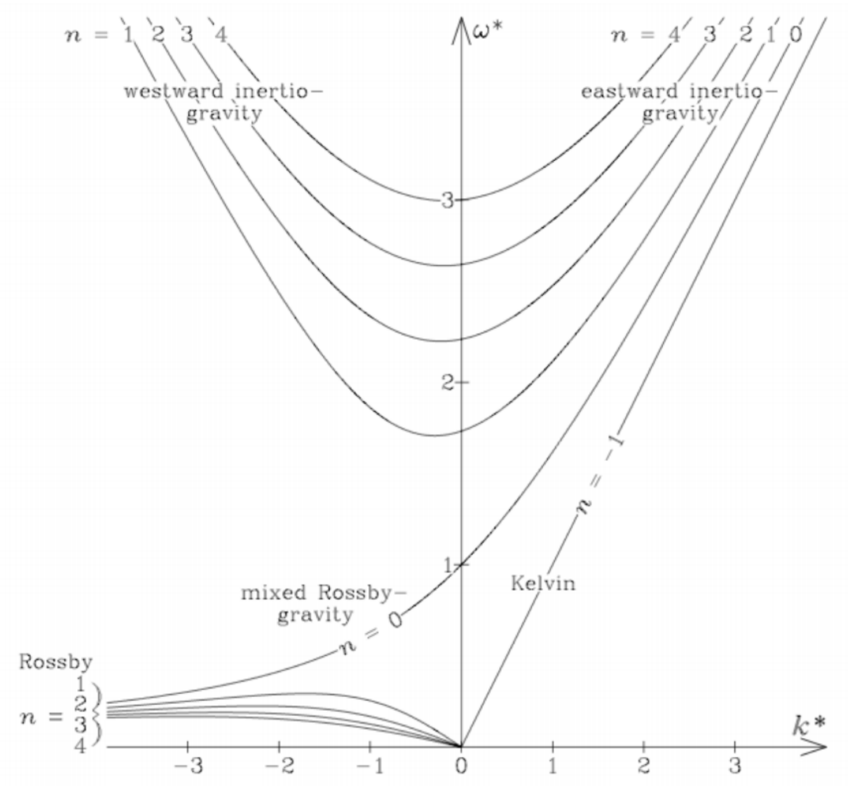


Figure 6: Equatorial wave dispersion relations

Summary The equatorial wave modes can be split into the following types.

- For $n = 1, 2, \dots$, for each $\omega > \sqrt{n(n+1)} + n + \frac{1}{2}$ there are two values of $\hat{k}$. These are high frequency *Inertio-gravity / Poincaré waves*. Note in the large $\hat{k}$ regime these may be reffered to as gravity waves, whilst in the small $\hat{k}$ regime they may be reffered to as inertial waves.
- For $n = 1, 2, \dots$, there are also two values of $\hat{k}$ for each $\omega < n + \frac{1}{2} - \sqrt{n(n+1)}$ which are low frequency *equatorial Rossby waves*.
- For $n = 0$ the dispersion relation $\hat{\omega} - 1/\hat{\omega} = \hat{k}$ is gravity-like $\hat{\omega} \sim \hat{k}$ as $\hat{k} \to \infty$ and Rossby-like $\hat{\omega} \sim 1/\hat{k}$ as $\hat{k} \to -\infty$. Hence the $n = 0$ mode is referred to as a (mixed) *Rossby-gravity wave*.
- The '$n = -1$' mode with dispersion relation $\hat{\omega} = \hat{k}$ is an *equatorial Kelvin wave*.

4.3.2 Horizontal propagation in the atmosphere or ocean

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The above detailed analysis applies for a *single-layer* fluid. As noted in section 3.3 for Rossby waves, the theory may be applied to a continuously stratified fluid by assuming a vertical structure of the disturbances such that

$$
\begin{aligned}
u(x, y, z, t) &= \tilde{u}(x, y, t)P(z) \\
v(x, y, z, t) &= \tilde{v}(x, y, t)P(z) \\
p(x, y, z, t) &= \frac{\rho_0}{g}\tilde{\eta}(x, y, t)P(z)
\end{aligned}
$$

where P satisfies the eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{N^2(z)}\frac{\mathrm{d}P}{\mathrm{d}z}\right] = -\frac{1}{gh}P = -\frac{1}{c^2}P \tag{67}$$

where $gh = c^2$ is an eigenvalue, possible values of which are determined by the equation together with the vertical boundary conditions. The horizontal structure and propagation characteristics of the disturbances are the same as in the shallow-water model in each 'layer' of *equivalent depth h*.

Baroclinic modes. In 'ocean-like' or 'troposphere-like' configurations there is a discrete set of eigenvalues, each with corresponding speed c and a different vertical structure, referred to as baroclinic modes and originally discussed in section 3.3. The same ideas would be relevant to the atmosphere if the tropopause is viewed as a rigid lid on the basis that the buoyancy frequency is much larger in the stratosphere compared with the troposhere. This is not valid if vertical propagation of waves from the troposphere to the stratosphere is of interest.

- For the equatorial ocean the first baroclinic mode has $c \sim 2ms^{-1}$, equivalent to $h \sim 0.4m$. The corresponding equatorial deformation scale is

  $$L_{\mathrm{eq}} = \sqrt{\frac{c}{\beta}} = \sqrt{\frac{2}{2.3 \times 10^{-11}}}m \sim 3 \times 10^5 m \sim 3^\circ \text{ of latitude}$$

  The corresponding timescale is $T_{\mathrm{eq}} \sim 1.5$ days. Note that this timescale is not particularly useful, given the variety of different wave frequencies and hence propagation speeds.

- For the tropical troposphere, observations suggest $c \sim 25ms^{-1}$ implying the deformation scale is $L_{\mathrm{eq}} \sim 10^6 m \sim 10^\circ$ of latitude, with corresponding timescale $T_{\mathrm{eq}} \sim 0.4$ days.

Influence of moisutre. Estimates of c based on *dry* dynamics and the depth of the tropical troposphere would be about $50ms^{-1}$, i.e. much larger than the value suggested by observations. This is likely due to the role of moisture in tropical dynamics, e.g. the fact that upward motion tends to lead to condensation and hence to internal heating, which can be argued to reduce the effective static stability from the 'dry' value. There is ongoing research on the subject of *convectively coupled equatorial waves.*

4.3.3 Matsuno-Gill forcing model

We will now consider the response of the equatorial atmosphere or ocean to a specified forcing. The problem was first motivated by the observed longitudinal structure of the tropical atmosphere and considers the response of the atmosphere to a specified heating, though the method can be applied more generally.

Following the previous section, we start with the shallow-water equations on an equatorial β-plane, expecting that the layer thickness H can be chosen appropriately for the tropical atmosphere. Heating is parameterised as a source term $\frac{1}{g}Q(x, y, t)$ on the RHS of the mass continuity equation. The pre-factor is included for algebraic convenience. We use the *long-wave approximation* which assumes that the lengthscale of x-variation is much larger than L_{eq} and hence the v_t term may be neglected so that the y-momentum equation simply expresses geostrophic balance in the y-direction.

We have

$$u_t - \beta y v = -g\eta_x \tag{68}$$
$$\beta y u = -g\eta_y \tag{69}$$
$$\eta_t + H(u_x + v_y) = \frac{1}{g}Q(x,y,t)$$

The unforced form of these equations allow the waves described by the full equtorial dispersion relation, confined to small values of $\hat{k}$ and $\hat{\omega}$, i.e. Rossby waves and Kelvin waves, but *not* mixed Rossby-gravity waves or inertio-gravity waves. To proceed, we introduce new variables q and r defined by

$$q = \frac{g}{c}\eta + u$$
$$r = \frac{g}{c}\eta - u$$

In terms of these variables, we have

$$q_t + cq_x + cv_y - \beta y v = \frac{Q}{c}$$
$$r_t - cr_x + cv_y + \beta y v = \frac{Q}{c}$$
$$cq_y + \beta y q = \beta y r - cr_y$$

We now separate out the latitudinal component by writing q, r, v and the forcing Q as sums of the latitudinal eigenfunctions derived in section 4.3.1, i.e.

$$[v, q, r, Q](x,y,t) = \sum_{n=0}^{\infty} [v_n, q_n, r_n, Q_n](x,t)\tilde{D}_n(Y)$$

where

$$\tilde{D}_n(Y) = H_n(Y)e^{-\frac{1}{2}Y^2}, \qquad Y = y\sqrt{\frac{\beta}{c}}$$

are the latitudinal eigenfunctions with $\tilde{D}'_n + Y\tilde{D}_n = 2n\tilde{D}_{n-1}$ and $\tilde{D}_n - Y\tilde{D}_n = -\tilde{D}_{n+1}$. Substituting these series into the governing equations yields

$$\frac{\partial q_0}{\partial t} + c\frac{\partial q_0}{\partial x} = \frac{Q_0}{c} \tag{70}$$
$$q_1 = 0$$

and for $n = 1, 2, ...$

$$\frac{\partial q_{n+1}}{\partial t} + c\frac{\partial q_{n+1}}{\partial x} - (c\beta)^{1/2}v_n = \frac{Q_{n+1}}{c} \tag{71}$$
$$\frac{\partial r_{n-1}}{\partial t} - c\frac{\partial r_{n-1}}{\partial x} + 2n(c\beta)^{1/2}v_n = \frac{Q_{n-1}}{c}$$
$$2(n+1)q_{n+1} = r_{n-1} \tag{72}$$

We may eliminate v_n and r_{n-1} from the last three equations to give

$$(2n+1)\frac{\partial q_{n+1}}{\partial t} - c\frac{\partial q_{n+1}}{\partial x} = \frac{1}{c}\left(nQ_{n+1} + \frac{1}{2}Q_{n-1}\right) \tag{73}$$

From (70) we see that q_0 is associated with the Kelvin wave, as this equation allows a wave travelling in the positive x-direction with speed c. The variables q_{n+1}, v_n, r_{n-1} for $n = 1, 2, ...$ are associated with the n^{th} Rossby wave since (73) allows a wave travelling in the negative x-direction with speed $c/(2n+1)$. The response to a forcing $Q(x, y)$ that is non-zero in a localised region and 'switched on' at $t = 0$ will therefore consist (if $Q_0 \neq 0$) of a Kelvin wave propagating eastward away from the forcing region and a set of Rossby waves propagating westward away from the forcing region.

Steady state behaviour. As currently defined, the solution will not tend to a steady state. A simple way to allow a steady state solution is to add an identical linear damping to each of (68) and (69), i.e. replace ∂_t with $\partial_t + \alpha$. The q_n then satisfy **Lecture 22** 2/12/20

$$\alpha q_0 + c\frac{\mathrm{d}q_0}{\mathrm{d}x} = \frac{Q_0}{c} \tag{74}$$

$$(2n+1)\alpha q_{n+1} - c\frac{\mathrm{d}q_{n+1}}{\mathrm{d}x} = \frac{1}{c}\left(nQ_{n+1} + \frac{1}{2}Q_{n-1}\right) \tag{75}$$

All other information about the solution can be obtained from the q_n, the r_n from (72), and the v_n from the steady form of (71).
The addition of damping means that the wave response to a localised forcing $Q(x, y)$ decays with distance away from the forcing. The solution to (74) represents a Kelvin wave which appears only to the east of the forcing region and decays in amplitude with distance away from it, with decay scale c/α. Note that this part of the response is excited only by Q_0, i.e. by the part of $Q(x, y)$ that projects onto $\tilde{D}_0$. The solution to (75) represents a set of Rossby waves which appear only to the west of the forcing region and decay with distance away from it. The larger thev alue of n, the smaller the propagation speed and the more rapid the decay. The $n = 1$ wave gives a propagation speed of $c/3$, hence the fastest Rossby wave decays to the west on a scale $c/3\alpha$, i.e. one third the decay scale of the Kelvin wave to the west. Note that the n^{th} Rossby wave is excited by Q_{n-1} and Q_{n+1}. A forcing for which Q_0 is the only component therefore excites both a Kelvin wave and the $n = 1$ Rossby wave.

Interpretation. The Matsuno-Gill model is considered a classical model of the first baroclinic mode of the tropical troposphere, i.e. a circulation driven by a heating maximising in the mid-troposphere in which there is ascent or descent in the mid-troposphere, and the upper-level horizontal flow is in the opposite direction to the low-level horizontal flow. If the horizontal velocity in the shallow-water system is interpreted as representing the low-level horizontal flow, then the mid-tropospheric vertical velocity corresponds to the convergence of the low-level horizontal flow, i.e. in the steady-state system described above, to $-u_x - v_y = \alpha\eta + Q$.
The solution for a localised positive forcing with Q_0 component only shows eastward low-level flow to the west of the forcing region (Rossby-wave) and a westward low-level flow to the east (Kelvin-wave). The Rossby-wave low level flow has cyclonic flow on either side of the equator (anti-clockwise to the north, clockwise to the south). Correspondingly, at upper levels there are anticyclones on either side of the equator.
For the tropical troposphere the heating is primarily latent heating associated with precipitation. There is relatively large precipitation over the Indonesian and West Pacific regions and the above model therefore provides a simple representation of the latitude-longitude structure of the flow in this region.
Whilst this model is important and often-cited, some aspects are difficult to justify physically, for example the assumption of constant damping rate α, acting on horizontal velocities and on η, which may be regarded was representing temperature. Damping of temperature is a simple representation

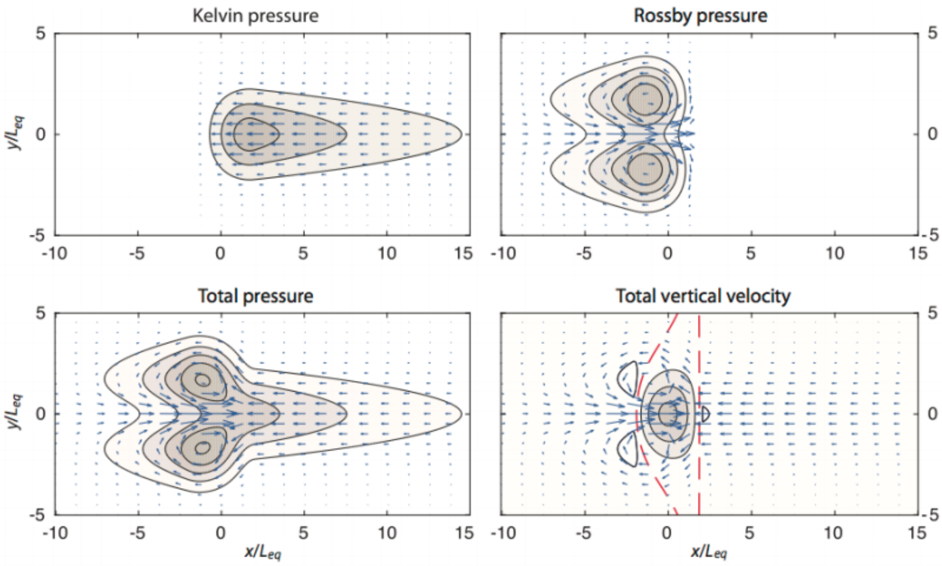

Figure 7: Matsuno-Gill model latitude-longitude structure.

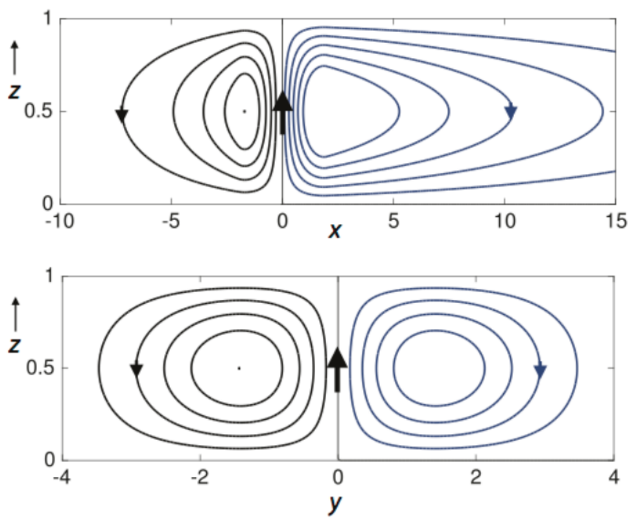

Figure 8: Matsuno-Gill model circulation in vertical plane.

of radiative effects, damping of horizontal velocities is a simple representation of frictional effects, which are plausible near the surface but less plausible at upper levels.

4.3.4 Vertical propagation

To consider vertical propagation, assume that the vertical structure is oscillatory and for convenience, assume the buoyancy frequency N is independent of z. The vertical structure is then defined by a constant vertical wavenumber m. The vertical structure equation (67) suggests the horizontal structure implied by the single-layer models will apply provided that $gh = c^2 = N^2/m^2$. Hence the dispersion relation becomes

$$\omega^2 - \frac{N^2k^2}{m^2} - \frac{\beta k N^2}{\omega m^2} = (2n+1)\frac{\beta N}{|m|} \quad \text{for } \; n = -1, 0, 1, 2, ... \tag{76}$$

where the previous analysis of Kelvin waves allows the $n = -1$ case, and as noted previously only one of the roots is admissible in each of the $n = -1$ and $n = 0$ cases. We require $|m|$ on the RHS since we assumed c is positive. We can non-dimensionalise the x and z wavenumbers by $\hat{k} = k\omega/\beta$ and $\hat{m} = m\omega^2/\beta N$, to get

$$\begin{aligned} |\hat{m}|^2 - (2n+1)|\hat{m}| - \hat{k}^2 - \hat{k} &= 0 \qquad && n = 1, 2, ... \\ |\hat{m}| - \hat{k} - 1 &= 0 && n = 0 \\ |\hat{m}| - \hat{k} &= 0 && n = -1 \end{aligned}$$

These equations define curves in the $(\hat{k}, \hat{m})$ plane on which ω may be determined in terms of k and m. The curves are symmetric about $\hat{m} = 0$, i.e. if $(\hat{k}, \hat{m})$ is a solution, so is $(\hat{k}, -\hat{m})$. The $n = 1, 2, ...$ case has solutions

$$\hat{m} = M_{\pm}(\hat{k}) = (n + \frac{1}{2}) \pm \left[(\hat{k} + \frac{1}{2})^2 + n(n+1)\right]^{1/2}$$

where the M_- solution is considered only for $n \geq 1$. The requirement $|\hat{m}| > 0$ implies the negative root is relevant only when $-1 < \hat{k} < 0$ for $n \geq 1$.
The group velocity can be deduced from the relation between $\hat{m}$ and $\hat{k}$. For simplicity, assume $m > 0$, for which

$$\frac{m\omega^2}{\beta N} = M_{\pm}(\frac{k\omega}{\beta})$$

Detailed analysis shows that for $\hat{m} > 0$, the horizontal group velocity component $\partial\omega/\partial k$ has the same sign as $M_{\pm}(k)$ and the vertical component $\partial\omega/\partial m$ has the opposite sign to $\hat{m}$. It is straightforward to show that for $\hat{m} < 0$ the sign of $\partial\omega/\partial k$ remains the same and the sign of $\partial\omega/\partial m$ reverses. The shape of the curves $M(\hat{k})$ imply the direction of the group velocity, indicated in the figure below. Note that the vertical wavenumber m has the opposite sign to the vertical component of the group velocity, i.e. upward for $m < 0$ and downward for $m > 0$.

Example.

- **Seasonal variation in the equatorial ocean.** There is seasonal variation of winds at the surface of the ocean, providing forcing at the annual frequency. Consider the $n = 2$ Rossby wave response. It is convenient to return to the dimensional form of the dispesrion relation (76). Consider the sizes of the c^2k^2 and $\beta kc^2/\omega$ terms on the LHS. The ratio of

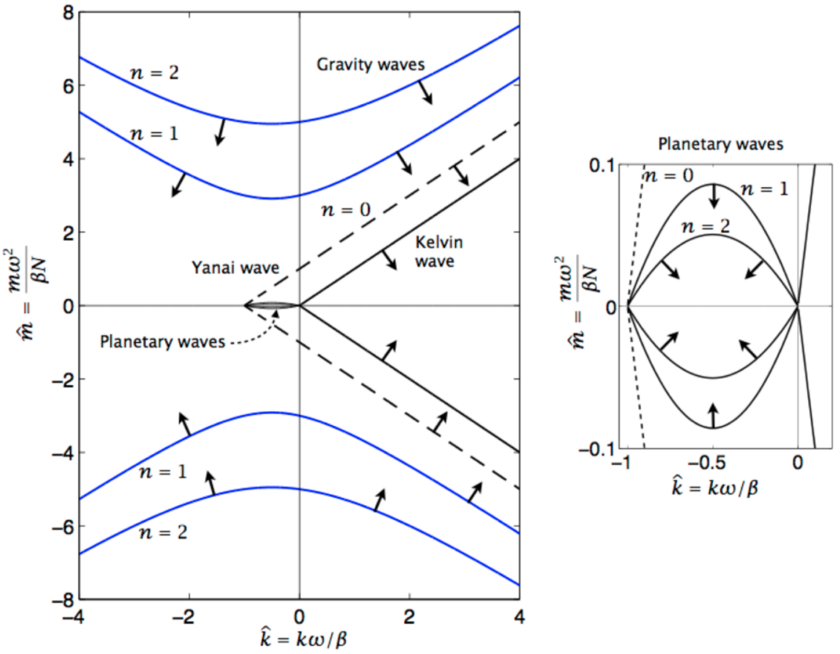

Figure 9: Dispersion diagram in $(\hat{k}, \hat{m})$ space for vertically propagating equatorial waves.

the first to the second is $k\omega/\beta$. For the annual frequency, $\omega/\beta \sim (6400/700)km \sim 10km$ and is the spatial scale is much larger than this, we can neglect the first term. The ω^2 term can be neglected on the basis that the frequency is small. Note neglecting this term also requires the assumption $m \ll Nk/\omega$, i.e. the vertical scale is not *too* small. The dispersion relation for $n = 2$ then reduces to

$$\omega = -\frac{kN}{3|m|} > 0$$

implying that $\partial\omega/\partial k = -\frac{1}{3}N/m = \omega/k$ and $\partial\omega/\partial m = \frac{1}{3}Nk\mathrm{sgn}(m)/|m|^2 = -\omega/m = \omega^2\mathrm{sgn}(m)/Nk$.

Consistent with the general considerations above, the vertical group velocity has the opposite sign to m, therefore for the oceanic response to surface forcing, take $m > 0$ and for consistecy with the dispersion relation take $k < 0$. Then $\partial\omega/\partial k$ is negative and group propagation is *to the west*. Since each of the physical fields is proportional to $\exp(i(kx + mz - \omega t))$, the phase speed is negative in the x direction and positive in the z direction. In (x, z) space, phase lines therefore propagate westward and upward. Note for a given frequency, the vertical component of the group velocity increases as N reduces, i.e. it will increase with depth. These features are all visible in the observed structure of the seasonal cycle in the qeuatorial Pacific.

- **Atmospheric Kelvin waves.** The dimensional form of the dispersion relation for Kelvin waves is

$$\omega = \frac{Nk}{|m|}$$

with $k > 0$. The vertical group velocity is $-\omega/m$, so $m < 0$ for upward propagation. Noting the same form of the physical fields as above, we see the waves with upward group velocity have downward phase propagation. This is clearly visible in time-height records of, for example, temperature perturbations in the lower stratosphere assocaited with Kelvin waves that are forced by convection in the troposphere.

- **Quasi-Biennial Oscillation.** Just as horizontally propagating Rossby waves can transport momentum in the horizontal, vertically propagating equatorial Kelvin waves can transport momentum in the vertical through the momentum flux $\overline{u'w'}$. Note this term is neglected in the QG approximation in the extratropics, but it is leading order for equatorial waves. Recall that there is a solution of the 3D equations with

$$u(x,y,z,t) = \tilde{u}(x,y,t)P(z)$$
$$p(x,y,z,t) = \rho_0 g\tilde{\eta}(x,y,t)P(z)$$

where $P(z)$ is a solution of the vertical structure equation (67) and $\tilde{u}, \tilde{\eta}$ are solutions of the single-layer shallow water equations. Hence $\rho(x,y,z,t) = -\rho_0 P'(z)\tilde{\eta}$ from hydrostatic balance and $w(x,y,z,t) = -(g/N^2)P'(z)\tilde{\eta}_t$ from the density equation. Choose $P(z) = e^{imz}$ representing a wave with vertical wavenumber m, i.e. $c = N/|m|$. Combining this with the (x,y,t) structure of the single-layer Kelvin wave solution, we have

$$u(x,y,z,t) = \Re\left[\hat{u}e^{i(kx+mz-kct)}\tilde{D}_0(Y)\right]$$
$$w(x,y,z,t) = \frac{g}{N^2}\Re\left[m\omega\hat{\eta}e^{i(kx+mz-kct)}\tilde{D}_0(Y)\right] = \Re\left[-\frac{k}{m}\hat{u}e^{i(kx-kct)}D_0(Y)\right]$$

Hence the momentum flux is

$$\overline{u'w'} = -\frac{k}{2m}|\hat{u}|^2 e^{-\frac{\beta y^2}{c}}$$

Equatorial Kelvin waves therefore transport eastward momentum in the direction of their group propagation. This is in contrast to Rossby waves, which transport westward momentum in the direction of their group propagation. Correspondingly, where Kelvin waves dissipate there is an eastward force exerted on the mean flow. This is part of the mechanism behind the Quasi-Biennial oscillation. Different waves drive the flow to the east or west. Whether or not these different waves can propagate in the vertical depends on the winds of the mean flow, hence there is a two-way feedback leading to an oscillation.