Cambridge Part III Maths

Lent 2021

Hydrodynamic Stability

based on a course given by
Prof. Richard Kerswell

written up by
Charles Powell

Contents

1 Introduction

We are typically interested in whether a given flow solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{u}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t)$ and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2}\hat{\boldsymbol{u}}^2(\boldsymbol{x}, t)\,\mathrm{d}V$$

A solution is said to be stable if

$$\lim_{t\to\infty} \frac{E(t)}{E(0)} = 0$$

for all perturbations $\hat{\boldsymbol{u}}$. Conversely, if there exists $\hat{\boldsymbol{u}}$ such that $E(t) \nrightarrow 0$ then $\boldsymbol{u}$ is unstable. The nature of $E(0)$ determines the type of perturbation:

- If $E(0) \to 0$ we have an infinitesimal disturbance
- If $E(0) < \delta$ then we probe finite amplitude disturbances
- If $E(0) \to \infty$ this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}$$

If $\boldsymbol{U}(\boldsymbol{x})$ is a steady (basic) solution then

$$\boldsymbol{U}\cdot\nabla\boldsymbol{U} + \nabla P = \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{U}$$

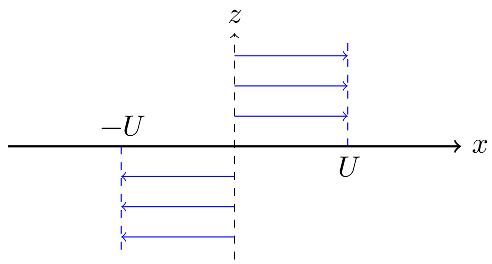
Let $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t), p = P + \hat{p}$. Then

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\hat{\boldsymbol{u}} + \cancel{\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}} + \nabla\hat{p} = \frac{1}{\mathrm{Re}}\nabla^2\hat{\boldsymbol{u}}$$

The term $\hat{u}\cdot\nabla\boldsymbol{U}$ is stabilising whilst the term $\nabla^2\hat{u}/\mathrm{Re}$ is destabilising. Therefore, we expect stability as $\mathrm{Re} \to 0$ the stabilising term dominates, and instability as $\mathrm{Re} \to \infty$ when the destablising term dominates. Thus there exists some value $\mathrm{Re}_{\mathrm{crit}}$ at which instability arises. We will ask what this value is, and what is the form of the initial instability/mode/pattern?

2 Kelvin-Helmholtz instability

See Drazin (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazin).



Consider a flow $\boldsymbol{u} = U(z)\hat{\boldsymbol{x}}$ where

$$U(z) = \begin{cases} U & z > 0 \\ -U & z < 0 \end{cases}$$

The linearised, *inviscid* equation for perturbation $\hat{\boldsymbol{u}}$ is

$$\begin{aligned} \frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \hat{w}U'\hat{\boldsymbol{x}} + U\frac{\partial \hat{\boldsymbol{u}}}{\partial x} + \nabla \hat{p} &= 0 \\ \nabla \cdot \hat{\boldsymbol{u}} &= 0 \end{aligned}$$

The boundary conditions are $\hat{\boldsymbol{u}} \to 0$ as $z \to \pm\infty$, i.e. no energy is radiated in from infinity. We will work in 2D with velocity components $(\hat{u}, \hat{w}) = (\psi_z, -\psi_x)$ and let $\psi(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$ where c is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U-c)\begin{pmatrix} \phi' \\ -i\alpha\phi \end{pmatrix} + \begin{pmatrix} -i\alpha U'\phi \\ 0 \end{pmatrix} + \begin{pmatrix} i\alpha p \\ \frac{\partial p}{\partial z} \end{pmatrix} = 0$$

We can eliminate p via $\partial_z(\text{top}) - i\alpha(\text{bottom})$ to get

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

with boundary conditions $\phi \to 0$ as $z \to \pm\infty$. This is *Rayleigh's equation.* Note that c is the crucial eigenvalue. We wish to know when $c_i = \Im(c) > 0$ as a function of $U(z)$, as c_i is the growth rate:

$$\hat{u} \propto e^{i\alpha(x-ct)} = e^{i\alpha(c-c_r t - ic_i t)} = e^{i\alpha(x-c_r t)+\alpha c_i t}$$

Note the following:

- There is a symmetry $\alpha \mapsto -\alpha$, so without loss of generality we consider $\alpha > 0$.
- The complex conjugate is also a solution with $c \mapsto c^*$. Hence an unstable mode has a damped partner, so we have stability only if all modes are 'neutral' i.e. $c_i = 0$.
- There is a possible singularity at y where $U(y) = c$, called the *critical layer*. If c is real, see later.

We now solve Rayleigh's equation with $U(z)$ defined as before. We solve above and below $z = 0$ and piece the solutions together. Since $U'' = 0$, we have

$$\phi'' = \alpha^2 \phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ Be^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at $z = 0$ are

1. Pressure $\hat{p}$ continuous at $z = 0$, with $\hat{p}$ given by:

$$\hat{p} = U'\phi - (U - c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z - \zeta(x,t)\right) = 0$$

where $z = \zeta(x,t)$ is the position of the surface. After linearising, we have

$$w - \frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} = 0$$

Inserting the form of w and U we require that

$$\zeta = -\frac{\phi}{U - c}$$

is continuous across $z = 0$.

Requiring p continuous gives

$$-(U - c)A(-\alpha) = -(-U - c)B(\alpha)$$

Requiring ζ continuous gives

$$\frac{A}{U - c} = \frac{B}{-U - c}$$

Hence we have

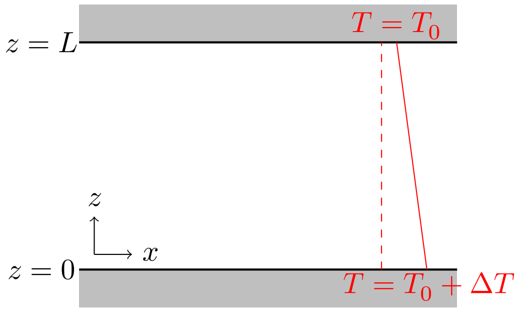
$$(U - c)^2 = -(U + c)^2$$

i.e. $c = \pm iU$ so the growth rate is αU. Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small α)
- by adding surface tension, which stabilises short wavelengths (large α), e.g. Drazin page 50 equation 3.21.

3 Thermal instabilities: Rayleigh-Benard convection

Consider two parallel plates separated by distance L with fluid subject to gravity and temperatue difference ΔT between the plates. The lower plate is heated to $T_0 + \Delta T$ whilst the upper plate is fixed at temperature T_0.



The basic state consists of no motion, with heat transfer by conduction only.

Governing equations. The governing equations are those of momentum, mass, and (thermal) energy conservation.

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \mu \nabla^2 \boldsymbol{u} + \rho g \hat{\boldsymbol{z}}$$
$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$
$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = 0$$

To close the set of equations we need a relationship between ρ and T. Most cases of interest have ΔT and $\Delta\rho$ small, i.e. $\Delta\rho \ll \rho_0, \Delta T \ll T_0$. Two consequences of this assumption are:

1. We can Taylor expand $\rho = \rho(T)$:

   $$\rho \approx \rho(T_0)\left[1 - \alpha(T - T_0)\right]$$

   where $\alpha > 0$ is the coefficient of thermal expansion, such that T increases when ρ decreases. We write $\rho_0 = \rho(T_0)$.

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term $\rho g \hat{\boldsymbol{z}}$. Importantly, we can assume the fluid is incompressible.

Define $\theta = T - T_0$. The governing equations are now

$$\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p = \mu \nabla^2 \boldsymbol{u} + \rho_0 (1 - \alpha\theta) g \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta$$
$$\nabla \cdot \boldsymbol{u} = 0$$

The basic state is $u = 0, \theta = \Delta T(1 - z/L)$ and

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_0 (1 - \alpha \Delta T (1 - z/L)) g$$

We now non-dimensionalise using scalings $t \sim L^2/\kappa, u \sim \kappa/L, \theta \sim \Delta T$, e.g. $\theta = \Delta T \theta^*$ where θ^* is the non-dimensionalised variable. We normalise the $\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}$ term, to get:

$$\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*} + \nabla^* p^* = \frac{\mu}{\rho_0 \kappa} \nabla^{*2} \boldsymbol{u}^* + \frac{\alpha g \Delta T L^3}{\kappa^2} \theta^* \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla^* \theta^* = \nabla^{*2} \theta^*$$

Define the *Prandtl number*

$$\sigma \equiv \frac{\nu}{\kappa} = \frac{\mu}{\rho_0 \kappa}$$

which is the ratio of viscous/momentum diffusion to thermal diffusion. Typical values are 0.72 in air, 7 in water, 10^5 in magma. We also define the *Rayleigh number*

$$\mathrm{Ra} \equiv \frac{\alpha \Delta T g L^3}{\kappa \nu}$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the * notation, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \sigma \nabla^2 \boldsymbol{u} + \sigma \mathrm{Ra} \theta \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \nabla^2 \theta$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Boundary conditions. There are three combinations of boundary condition available in this problem, with the choice fixed wall (no slip) or stress free (free slip).

$\theta = 0$, $z = 1$	Fixed wall: $\boldsymbol{u} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$, $z = 0$	$\boldsymbol{u} = 0$: Fixed wall	$\boldsymbol{u} = 0$: Fixed wall	$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$: Free slip

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

Basic state. In the basic state we have conductive profile $\boldsymbol{u}_0 = 0, \theta_0 = 1 - z$ and from integration $p_0 = \sigma \mathrm{Ra}(z - \frac{1}{2}z^2)$. We generate linearised equations for perturbations $\theta = \theta_0 + \theta', \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', p = p_0 + p'$. As usual with linear stability analysis, we assume $(\theta', \boldsymbol{u}', p')$ are small.

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \cancel{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} + \nabla p' = \sigma \nabla^2 \boldsymbol{u}' + \sigma \mathrm{Ra} \theta' \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta'}{\partial t} - w' + \cancel{\boldsymbol{u}' \cdot \nabla \theta'} = \nabla^2 \theta'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

Dropping the $'$ notation for clarity we have perturbation equations

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{u} + \nabla p = \sigma \mathrm{Ra}\theta\hat{\boldsymbol{z}} \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = w \tag{3}$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g. $\theta = \theta_0 + \theta' = 1$ at $z = 0$ combined with $\theta_0 = 1$ at $z = 0$ gives $\theta' = 0$. Similarly, $\theta' = 0$ at $z = 1$ and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.
From $\nabla\times$(1) we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\boldsymbol{\omega} = \sigma\mathrm{Ra}\nabla\times\theta\hat{\boldsymbol{z}}$$

Taking the curl again and using $\nabla\times\boldsymbol{\omega} = \nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$ we have

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2\boldsymbol{u}) = \sigma\mathrm{Ra}\nabla\times(\nabla\times\theta\hat{\boldsymbol{z}}) = \sigma\mathrm{Ra}\left(\nabla\frac{\partial\theta}{\partial z} - \hat{\boldsymbol{z}}\nabla^2\theta\right)$$

The z component is

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)(-\nabla^2 w) = \sigma\mathrm{Ra}\nabla_H^2\theta \tag{4}$$

where $\nabla_H^2 = \partial_x^2 + \partial_y^2$. Now (3) can be used to eliminate θ by applying the operator $(\partial_t - \nabla^2)$:

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \sigma\mathrm{Ra}\nabla_H^2 w \tag{5}$$

This is a 6^{th} order PDE for w, hence we need three boundary conditions at each wall $z = 0, 1$. We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad \text{at } \ z = 0, 1$$

The second set of conditions comes from incompressibility. Taking $\partial_z(\nabla\cdot\boldsymbol{u})$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right) + \frac{\partial^2 w}{\partial z^2} = 0 \implies w_{zz} = 0$$

The third and final set of conditions comes from requiring $\theta = 0$ at $z = 0, 1$. From (4), $\nabla_H^2\theta = 0$ implies

$$\left(\frac{\partial}{\partial t} - \sigma\nabla^2\right)\nabla^2 w = 0$$

We now have 6 boundary conditions to supplement the PDE.

Normal mode solution. Seek a solution $w(x, y, z, t) = W(z)e^{ik_1x + ik_2y + \lambda t}$ where k_1, k_2 are wavenumbers and $\lambda \in \mathbb{C}$ is the growth rate. Write $D = \mathrm{d}/\mathrm{d}z$ and $k = \sqrt{k_1^2 + k_2^2}$ since the problem is rotationally symmetric in the (x, y) plane. Substituting into (5) we have

$$(\lambda - [D^2 - k^2])(\lambda - \sigma[D^2 - k^2])(D^2 - k^2)W = -\sigma\mathrm{Ra}k^2 W$$

with boundary conditions at $z = 0, 1$:

$$\begin{aligned} & W(0) = W(1) = 0 \\ & D^2W(0) = D^2W(1) = 0 \\ \left[\lambda - \sigma(D^2 - k^2)\right]\left[D^2 - k^2\right] W = 0 \implies & D^4W(0) = D^4W(1) = 0 \end{aligned}$$

The objective is to find

$$\max_k \Re\{\lambda(k; \mathrm{Ra}, \sigma)\}$$

The onset of linear instability (for a given σ) at $\mathrm{Ra} = \mathrm{Ra}_{\mathrm{crit}}$ is defined by

$$\max_k \Re\{\lambda(k; \mathrm{Ra}_{\mathrm{crit}}, \sigma)\} = 0$$

In general, $\lambda \in \mathbb{C}$, but for this problem it can be proven that at marginality $\Im(\lambda) = 0$ as well as $\Re(\lambda) = 0$; a condition called the *principle of exchange of stabilities.* Hence setting $\lambda = 0$ in the above, we get

$$(D^2 - k^2)^3 W = -\mathrm{Ra}\, k^2 W \tag{6}$$

Note that σ drops out of the problem! It's easy to see $W(z) = \sin(n\pi z)$ solves (6) and satisfies the free-slip BCs. Hence

$$(n^2\pi^2 + k^2)^3 = \mathrm{Ra}\, k^2$$

Criticality is then given by

$$\mathrm{Ra}_{\mathrm{crit}} = \min_{n,k} \frac{(n^2\pi^2 + k^2)^3}{k^2}$$

We find the minimum in the usual way:

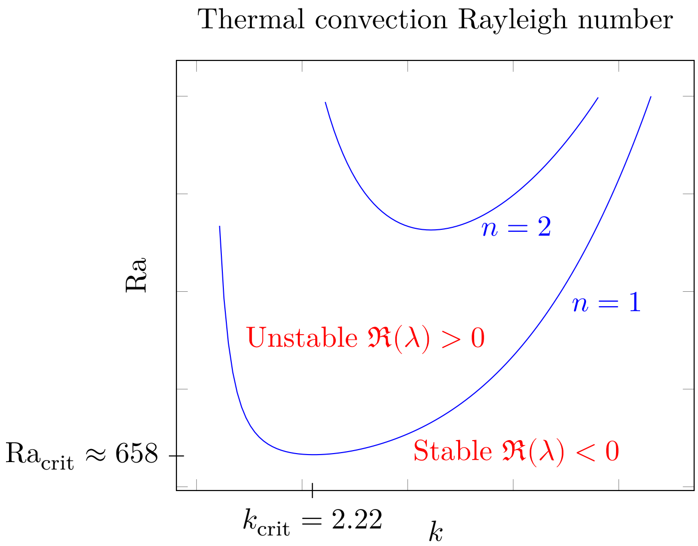
$$\begin{aligned} \frac{\partial \mathrm{Ra}}{\partial k} &= \frac{3(2k)(n^2\pi^2 + k^2)^2 k^2 - 2k(n^2\pi^2 + k^2)^3}{k^4} \\ &= \frac{2k(n^2\pi^2 + k^2)^2(3k^2 - (n^2\pi^2 + k^2))}{k^4} = 0 \\ \implies 2k^2 &= n^2\pi^2 \\ \implies k &= \frac{n\pi}{\sqrt{2}} \end{aligned}$$

Given $k = n\pi/\sqrt{2}$ the Rayleigh number is

$$\mathrm{Ra}(k = \frac{n\pi}{\sqrt{2}}) = \frac{(n^2\pi^2 + \frac{1}{2}n^2\pi^2)^3}{n^2\pi^2/2} = \frac{27}{4} n^4\pi^4$$

Clearly the critical Rayleigh number is given by $n = 1$, hence

$$\begin{aligned} \mathrm{Ra}_{\mathrm{crit}} &= \frac{27}{4}\pi^4 \sim 658 \\ k_{\mathrm{crit}} &= \frac{\pi}{\sqrt{2}} \sim 2.22 \end{aligned}$$



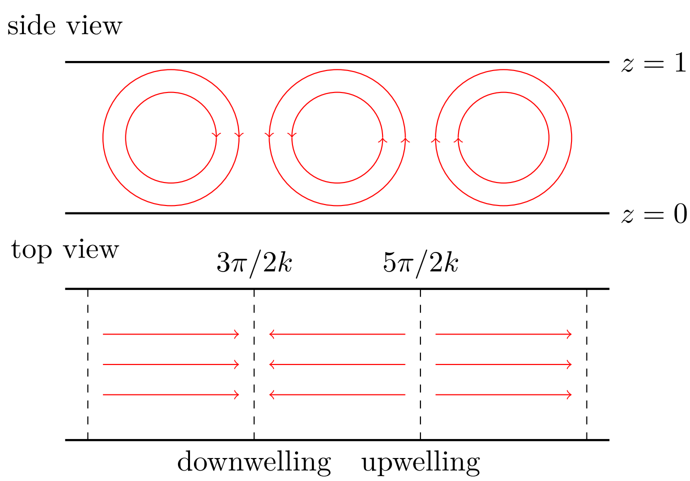
Results for other boundary conditions are:

- Free–rigid boundary: $\text{Ra}_{\text{crit}} \sim 1101, k_c = 2.68$
- Rigid–rigid boundary: $\text{Ra}_{\text{crit}} \sim 1708, k_c = 3.117$

Notice that at criticality only the size of k is specified, *not* its direction. Hence there are an infinite number of possibilities $\boldsymbol{k} = (k\cos\phi, k\sin\phi)$. Various different patterns which tesselate are as follows.

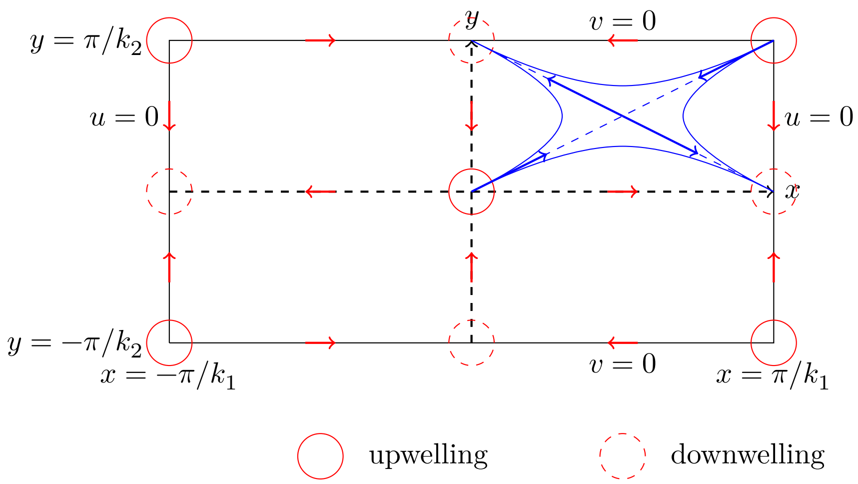
1. **2D rolls.** Orientate x-axis along k such that $k_2 = 0$. We have velocity components (w specified in problem, u follows from incompressibility)

$$w = W(z)\sin kx$$
$$v = 0$$
$$u = \frac{\pi\cos\pi z\cos kx}{k}$$



2. **Rectangles.** Velocity components are

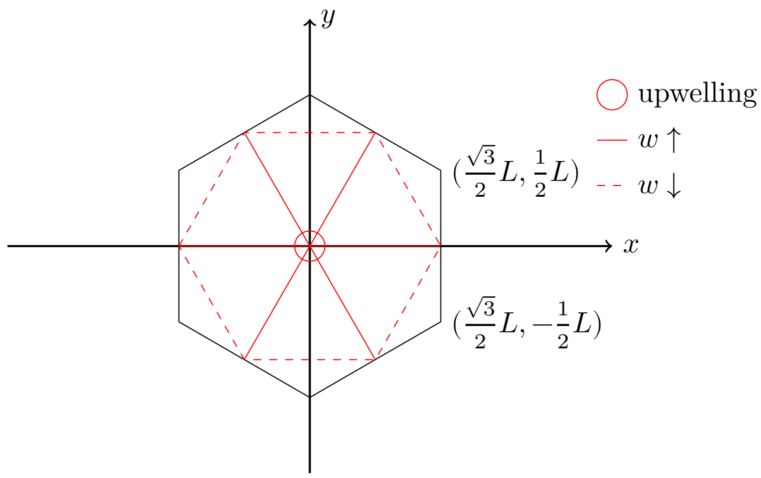
$$w = W(z)\cos k_1 x \cos k_2 y$$
$$v = -\frac{k_2}{k^2} W' \cos k_1 x \sin k_2 y$$
$$u = -\frac{k_1}{k^2} W' \sin k_1 x \cos k_2 y$$



3. **Hexagons.** Vertical velocity component

$$w = W(z)\left[\cos\frac{k}{2}(\sqrt{3}x + y) + \cos\frac{k}{2}(\sqrt{3}x - y) + \cos ky\right]$$

This is flow in a hexagon of side length $L = 4\pi/3k$.



4 Centrifugal instabilities

Flows with curved streamlines can be unstable due to centrifugal effects.

4.1 Rayleigh's criterion

We will concentrate on axisymmetric flows. Consider an azimuthal flow

$$\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}} = r\Omega(r)\hat{\boldsymbol{\theta}}$$

The inviscid, axisymmetric equations for a general flow $\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}}$ are

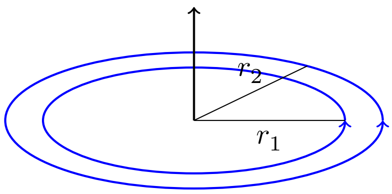
$$
\begin{aligned}
\frac{\partial u_r}{\partial t} + \boldsymbol{u}\cdot\nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\partial u_\theta}{\partial t} + \boldsymbol{u}\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} &= \cancel{-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}} \\
\frac{\partial u_z}{\partial t} + \boldsymbol{u}\cdot\nabla u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \\
\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned}
$$

where $\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial}{\partial\theta}} + u_z\frac{\partial}{\partial z}$. Cancelled terms are absent in the axisymmetric setting. The *centrifugal* term is $-u_\theta^2/r$ in the r-momentum equation. The θ-momentum equation can be rearranged, and multiplied by r to give a material conservation equation:

$$
\begin{aligned}
\frac{\partial}{\partial r}(r u_\theta) + r u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial}{\partial z}(r u_\theta) + r\left(\frac{u_r u_\theta}{r}\right) &= 0 \\
\implies \frac{\partial}{\partial t}(r u_\theta) + u_r\frac{\partial}{\partial r}(r u_\theta) + u_z\frac{\partial}{\partial z}(r u_\theta) &= 0 \\
\implies \frac{\mathrm{D}}{\mathrm{D}t}(r u_\theta) &= 0
\end{aligned}
$$

This expresses conservation of angular momentum: the angular momentum per unit mass is $I = r u_\theta$, hence $\frac{\mathrm{D}I}{\mathrm{D}t} = 0$. This result also follows from Kelvin's circulation theorem, using the circulation $\Gamma = 2\pi r u_\theta$ for an inviscid fluid. The statement says that if $\boldsymbol{u} = u_\theta(r)\hat{\boldsymbol{\theta}}$ (i.e. axisymmetric azimuthal flow) then $I = I(r)$ is a basic state.

What distributions of $I(r)$ could be stable? Rayleigh's argument considers 2 rings of fluid at radius r_1 and $r_2 (> r_1)$ respectively.



The kinetic energy is

$$E = \frac{1}{2}\rho\left(\frac{I_1^2}{r_1^2} + \frac{I_2^2}{r_2^2}\right)$$

Now suppose the rings swap places due to a perturbation, but they keep their angular momentum (since it is materially conserved). The new KE is

$$E_{\text{new}} = \frac{1}{2}\left(\frac{I_2^2}{r_1^2} + \frac{I_1^2}{r_2^2}\right)$$

Hence the swap has resulted in an energy change

$$\Delta E = (I_2^2 - I_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

We can expect instability if $\Delta E < 0$. Since $r_2 > r_1$, the second factor is positive hence

$$\Delta E < 0 \iff I_2^2 < I_1^2$$

Hence Rayleigh's criterion for stability is $I_2^2 \geq I_1^2$ or equivalently

$$\frac{\mathrm{d}I^2}{\mathrm{d}r} \geq 0$$

i.e. angular momentum does not increase outwards. Note that with $I = ru_\theta = r^2\Omega$ we have the condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) \geq 0$$

for stability. This is often written using the *Rayleigh determinant*

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right)$$

Hence stability is predicted if $\Phi \geq 0$.

4.2 Derivation via linear stability analysis

Consider Taylor-Couette geometry: cylindrical walls at r_1 and r_2 with an inviscid base state $\boldsymbol{u} = r\Omega(r)\widehat{\boldsymbol{\theta}}$, with axisymmetric perturbations $\boldsymbol{u}'$. We have incompressibility

$$\nabla \cdot \boldsymbol{u}' = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(ru_r') + \frac{\partial u_z'}{\partial z} = 0$$

The Euler equations for this perturbation are

$$\begin{aligned}
\frac{\partial u_r'}{\partial t} - \frac{2r\Omega u_\theta'}{r} &= -\frac{1}{\rho}\frac{\partial p'}{\partial r} \\
\frac{\partial u_\theta'}{\partial t} + u_r'\frac{\mathrm{d}}{\mathrm{d}r}(r\Omega) + \frac{u_r' r\Omega}{r} &= 0 \\
\frac{\partial u_z'}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial z}
\end{aligned}$$

Now specify normal mode decomposition

$$\begin{pmatrix} u_r' \\ u_\theta' \\ u_z' \\ p' \end{pmatrix} = \begin{pmatrix} \hat{u}_r(r) \\ \hat{u}_\theta(r) \\ \hat{u}_z(r) \\ \hat{p}(r) \end{pmatrix} e^{ikz+\sigma t}$$

Only axisymmetric perturbations are considered. The Euler equations become

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r) + ik\hat{u}_z &= 0 \\
\sigma\hat{u}_r - 2\Omega\hat{u}_\theta &= -\frac{1}{\rho}\frac{\mathrm{d}\hat{p}}{\mathrm{d}r} \\
\sigma\hat{u}_\theta + \hat{u}_r(\Omega + (r\Omega)_r) &= 0 \\
\sigma\hat{u}_z &= -\frac{1}{\rho}ik\hat{p}
\end{aligned}$$

We can reduce this system down to a single equation for $\hat{u}_r$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\right)\hat{u}_r - k^2\hat{u}_r - 2\frac{k^2}{\sigma^2}\Omega(2\Omega + r\Omega')\hat{u}_r = 0$$

This is a second order ODE for $\hat{u}_r$ with BCs $\hat{u}_r = 0$ at $r = r_1, r_2$. For this flow, Rayleigh's determinant is

$$\Phi \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = 4\Omega^2 + 2r\Omega'\Omega$$

Hence the ODE for $\hat{u}_r$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right) - k^2\hat{u}_r = \frac{k^2}{\sigma^2}\Phi(r)\hat{u}_r \tag{7}$$

Multiply (7) by $r\hat{u}_r^*$ (complex conjugate) and integrate from r_1 to r_2:

$$\int_{r_1}^{r_2} r\hat{u}_r^*\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term may be integrated by parts to give:

$$\cancel{\left[r\hat{u}_r^*\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\right]_{r_1}^{r_2}} - \int_{r_1}^{r_2}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r^*)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\hat{u}_r)\mathrm{d}r - k^2\int_{r_1}^{r_2} r|\hat{u}_r|^2\,\mathrm{d}r = \frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r$$

The first term vanishes since $\hat{u}_r = 0$ at $r = r_1, r_2$. Labelling the first integral as $H_1 > 0$ and the second as $H_2 > 0$, we have

$$\frac{k^2}{\sigma^2}\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r = -H_1 - k^2 H_2 < 0$$

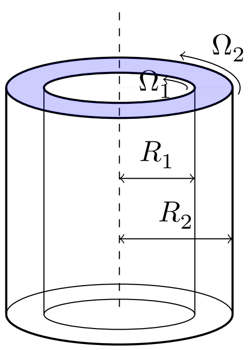
If $\Phi \geq 0$ then $\sigma^2 < 0$, i.e. σ is imaginary and we have stability. If instead $\Phi < 0$ somewhere in the domain, then potentially

$$\int_{r_1}^{r_2} r\Phi|\hat{u}_r|^2\,\mathrm{d}r < 0$$

in which case $\sigma^2 > 0$ and we have instability. Hence $\Phi < 0$ somewhere in the domain is *necessary* (but not sufficient) condition for instability. So this formal analysis confirms Rayleigh's heuristic criterion. Note, really we need to consider non-axisymmetric perturbations too.

4.3 Taylor vortices

Apply Rayleigh's criterion to Taylor-Couette flow.



When viscosity is present, the general solution with $\partial_\theta = \partial_z = 0$ is

$$u_\theta(r) = Ar + \frac{B}{r}$$

No-slip boundary conditions at $r = R_1, R_2$ give

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}}$$

Note this solves $(\nabla^2 - 1/r^2)u_\theta = 0$ where $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r)$. In this case $\Omega = u_\theta/r = A + B/r^2$ hence Rayleigh's determinant is

$$\Phi = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4\Omega^2\right) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4\left(A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}\right)\right] = 4A^2\left(1 + \frac{B}{Ar^2}\right)$$

For convenience we define $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2 < 1$. Then

$$\Phi = 4A^2\left[1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2}\right]$$

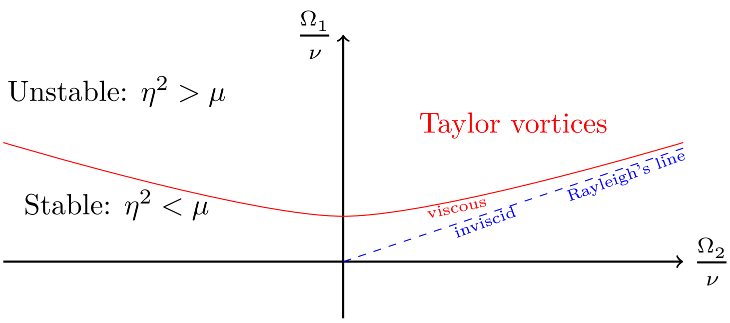
For stability, i.e. $\Phi \geq 0$ everywhere, we require for all $r \in [R_1, R_2]$

$$1 \geq \frac{(1-\mu)R_1^2}{(\eta^2-\mu)r^2} \geq \frac{1-\mu}{\eta^2-\mu}$$

where the last inequality follows since $R_1^2/r^2 \geq 1$ for all $r \in [R_1, R_2]$. There are now two cases:

- If $\eta^2 > \mu$ then
$$\eta^2 - \mu \geq 1 - \mu \implies \eta^2 \geq 1$$
This is a contradiction since $\eta < 1$.
- Otherwise $\eta^2 < \mu$, so
$$\eta^2 - \mu \leq 1 - \mu \implies \eta^2 \leq 1$$

Thus Rayleigh's criterion is $\eta^2 < \mu$ for stability.



For a fixed geometry (i.e. fixed η) we can plot a stability diagram, with Rayleigh's line $\eta^2 = \mu = \Omega_2/\Omega_1$ marking the stability heuristic. In Taylor-Couette geometry, the instability often manifests itself as *Taylor vortices*, though there are many different modes of instability depending on Ω_1, Ω_2, ν.

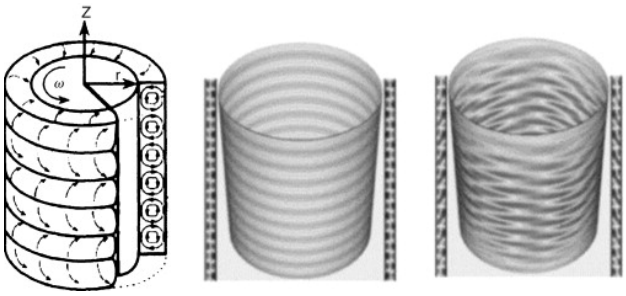
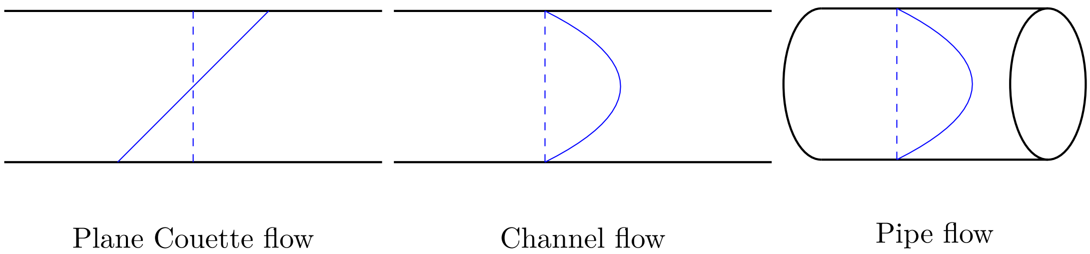


Figure 1: Taylor vortices, from Dutta and Ray, 2004.

5 Parallel shear flows

For some flows, inviscid analysis gives a good approximation to the stability properties of a viscous fluid (e.g. Kelvin-Helmholtz, Taylor-Couette flow) but for others, it does not (e.g. plane Couette flow, channel flow, pipe flow). In these flows, viscosity can be *de*stabilising.



5.1 Inviscid analysis

Consider a parallel shear flow $U(z)\hat{\boldsymbol{x}}$. The non-dimensionalised Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$$
$$\nabla \cdot \boldsymbol{u} = 0$$

with boundary conditions $\boldsymbol{u} \cdot \hat{\boldsymbol{z}} = 0$ at $z = z_1, z_2$. The basic flow is $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with P constant – any constant form of the pressure is valid. Add small perturbations

$$\boldsymbol{u} = U(z)\hat{\boldsymbol{x}} + \boldsymbol{u}', \quad p = P + p'$$

The Euler equations become

$$\frac{\partial \boldsymbol{u}'}{\partial t} + U\frac{\partial \boldsymbol{u}'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z}\hat{\boldsymbol{x}} = -\nabla p'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

with boundary conditions $w' = 0$ at $z = z_1, z_2$. All equations have coefficients independent of x, y, t so we can separate the variables by taking normal modes of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = \hat{\boldsymbol{u}}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$
$$p'(\boldsymbol{x}, t) = \hat{p}(z)e^{i(\alpha x + \beta y - \alpha c t)}$$

Note we have replaced the usual σ with $-i\alpha c$. It is understood that the physical fluid perturbation velocity $\boldsymbol{u}'$ is represented by the real part, e.g.

$$w' = \left[\Re(\hat{w})\cos(\alpha x + \beta y - \alpha c_r t) - \Im(\hat{w})\sin(\alpha x + \beta y - \alpha c_r t)\right] e^{\alpha c_i t}$$

This mode is a wave travelling with phase speed $\alpha c_r/\sqrt{\alpha^2+\beta^2}$ in the $(\alpha, \beta, 0)$ direction and it decays like $e^{\alpha c_i t}$ for $c_i < 0$, or grows if $c_i > 0$. The equations are now

$$i\alpha(U-c)\hat{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\alpha\hat{p} = 0 \tag{8}$$

$$i\alpha(U-c)\hat{v} + i\beta\hat{p} = 0 \tag{9}$$

$$i\alpha(U-c)\hat{w} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}z} = 0 \tag{10}$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0 \tag{11}$$

with boundary conditions $\hat{w} = 0$ at $z = z_1, z_2$. This is an eigenvalue problem in $c \in \mathbb{C}$. Instability corresponds to $c_i > 0$ and $c_i \leq 0$ for stability.

5.1.1 Squire's transformation (Squire, 1933)

Before attempting to solve (8)–(11), we consider Squire's transformation. Define the transformed variables

$$\tilde{\alpha} = \sqrt{\alpha^2+\beta^2}, \quad \tilde{u} = \frac{\alpha\hat{u} + \beta\hat{v}}{\tilde{\alpha}}, \quad \tilde{p} = \frac{\tilde{\alpha}\hat{p}}{\alpha}$$

Construct $(\alpha(8) + \beta(9))/\alpha$:

$$i\tilde{\alpha}(U-c)\tilde{u} + \frac{\mathrm{d}U}{\mathrm{d}z}\hat{w} + i\tilde{\alpha}\tilde{p} = 0 \tag{12}$$

Similarly $\tilde{\alpha}(10)/\alpha$:

$$i\tilde{\alpha}(U-c)\hat{w} + \frac{\mathrm{d}\tilde{p}}{\mathrm{d}z} = 0 \tag{13}$$

Incompressibility is now expressed as

$$i\tilde{\alpha}\tilde{u} + \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} = 0$$

The transformed system has the same form as (8)–(11) with $\beta = \hat{v} = 0$ and $\alpha \to \tilde{\alpha}, \hat{u} \to \tilde{u}, \hat{p} \to \tilde{p}$ but c unchanged. Thus the eigenvalue c depends on $\sqrt{\alpha^2+\beta^2}$ but the growth rate is αc_i. So the largest growth rate αc_i is given by $\beta = 0$ for all wavenumber pairs (α, β) with $\sqrt{\alpha^2+\beta^2}$ constant. Hence it is sufficient to consider $\beta = 0$ disturbances only. To any unstable 3D mode $\alpha \neq 0, \beta \neq 0$ there corresponds a more unstable 2D mode with $\beta = 0$.

5.1.2 Rayleigh's equation

Work in 2D (Squires). Use streamfunction ψ' such that

$$u' = \psi'_z, \ v' = 0, \ w' = -\psi'_x$$

Further, let $\psi'(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$ so that it is now clear that c_r is the phase speed in the x direction. Now $\hat{u} = \frac{\mathrm{d}\phi}{\mathrm{d}z}$ and $\hat{w} = -i\alpha\phi$ (notice the phase difference). Then (12) becomes

$$i\alpha(U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}(-i\alpha\phi) + i\alpha\hat{p} = 0$$

$$\implies \hat{p} = \frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}$$

Substituting into (13) gives

$$\begin{aligned} i\alpha(U-c)(-i\alpha\phi) + \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}U}{\mathrm{d}z}\phi - (U-c)\frac{\mathrm{d}\phi}{\mathrm{d}z}\right] = 0 \\ \implies (U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \end{aligned} \tag{14}$$

with boundary conditions $\phi = 0$ at $z = z_1, z_2$. This is *Rayleigh's equation (1880).*

Comments.

- Rayleigh's equation involves α^2 only so need only consider $\alpha > 0$.
- If (ϕ, c) solves the problem then so does (ϕ^*, c^*). So if there exists a growing mode, there also exists a corresponding decaying mode. Hence stability means $c \in \mathbb{R}$ for all α.
- A singularity exists at $U(z_c) = c$ – this is called a critical layer and only occurs when $c \in \mathbb{R}$. Critical layers are important in solving IVPs and relating Rayleigh's equation to its viscous analogue, the Orr-Sommerfield equation (see later).
- There are two types of eigensolution:
  - Continuous spectrum $c \in [\min U, \max U]$ and ϕ has a discontinuous derivative at z_c. This type of solution is never unstable.
  - Discrete spectrum of complex conjugate pairs. This solution can be unstable.

5.1.3 Properties of Rayleigh's equation.

Inflection point criterion. Suppose $c_i > 0$, i.e. consider an unstable mode. Multiply Rayleigh's equation by ϕ^* and integrate from z_1 to z_2:

$$\int_{z_1}^{z_2} \left[\phi^*\phi'' - \alpha^2|\phi|^2 - \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0$$

Integrate the first term by parts and note $\phi = \phi^* = 0$ at z_1 and z_2. Hence

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + \alpha^2|\phi|^2 + \frac{U''}{U-c}|\phi|^2\right] \mathrm{d}z = 0 \tag{15}$$

Take imaginary part:

$$\begin{aligned} \Im\left[\int_{z_1}^{z_2} \frac{U''(U-c^*)}{|U-c|^2}|\phi|^2 \mathrm{d}z\right] = 0 \\ \implies -c_i \int_{z_1}^{z_2} \frac{U''}{|U-c|^2}|\phi|^2 \mathrm{d}z = 0 \end{aligned}$$

But $c_i > 0$ so we must have

$$\int_{z_1}^{z_2} \frac{U''}{|U-c|^2}|\phi|^2 \mathrm{d}z = 0$$

Now $|U-c|^2 > 0$ and $|\phi|^2 > 0$ so U'' must change sign somewhere in $[z_1, z_2]$. Thus $U'' = 0$ at least once is a necessary condition for inviscid instability, called the *inflection point criterion.*

Fjørtoft's condition. A stronger form of the inflection point criterion was obtained by Fjørtoft (1950): given a monotonic mean velocity profile $U(z)$, a necessary condition for instability is that $U''(U-U_s)<0$ for some $z\in[z_1,z_2]$ with $U_s=U(z_s)$ where $U''(z_2)=0$.
To see this, take the real part of (15) to get

$$\int_{z_1}^{z_2}\frac{U''(U-c_r)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

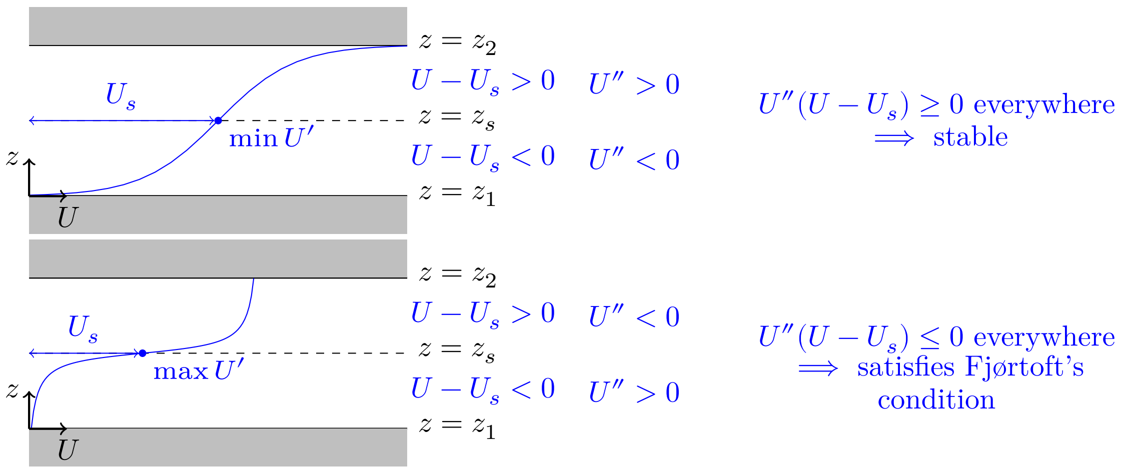
Add the term

$$(c_r-U_s)\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2\mathrm{d}z=0$$

which vanishes if $c_i>0$ by above. Then

$$\int_{z_1}^{z_2}\frac{U''(U-U_s)}{|U-c|^2}|\phi|^2\,\mathrm{d}z=-\int_{z_1}^{z_2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2+\alpha^2|\phi|^2\,\mathrm{d}z$$

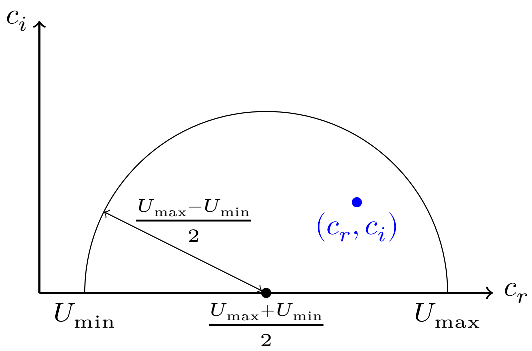
The RHS terms are negative definite, and $|\phi^2|>0$ as well as $|U-c|^2>0$. Hence $U''(U-U_s)<0$ somewhere in $[z_1,z_2]$. This means that the inflection point has to be a maximum (rather than a minimum) of the spanwise vorticity $U'(z)\hat{\boldsymbol{y}}$.



Howard's semicircle theorem Due to Howard (1961). The unstable eigenvalues of the Rayleigh equation satisfy

$$\left[c_r-\frac{1}{2}(U_{\max}+U_{\min})\right]^2+c_i^2\leq\left[\frac{1}{2}(U_{\max}-U_{\min})\right]^2$$

This is best viewed as a geometric condition: the unstable eigenvalues lie in a semicircle centred at $\frac{1}{2}(U_{\max}+U_{\min})$ of radius $\frac{1}{2}(U_{\max}-U_{\min})$.



Let $\Psi = \frac{\phi}{U-c}$. Rayleigh's equation (14) in terms of Ψ is

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(U-c)\Psi\right] - \alpha^2(U-c)\Psi\right) = U''(U-c)\Psi$$

Evaluating the derivative and simplifying gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(U-c)^2\frac{\mathrm{d}\Psi}{\mathrm{d}z}\right] = \alpha^2(U-c)^2\Psi$$

Multiply the equation by Ψ^* and integrate over $[z_1, z_2]$:

$$\int_{z_1}^{z_2} \Psi^*\left[(U-c)^2\Psi'\right]'\,\mathrm{d}z = \alpha^2\int_{z_1}^{z_2}(U-c)^2|\Psi|^2\,\mathrm{d}z$$

We then integrate by parts and note that $\Psi = \phi/(U-c) = 0$ on $z = z_1, z_2$. Hence

$$\int_{z_1}^{z_2}(U-c)^2\left[|\Psi'|^2 + \alpha^2|\Psi|^2\right]\mathrm{d}z = 0$$

Denote the $[\dots]$ factor by Q. We have $Q > 0$ and $c \in \mathbb{C}$. Taking real and imaginary parts gives

$$\int_{z_1}^{z_2}\left[(U-c_r)^2 - c_i^2\right]Q\,\mathrm{d}z = 0$$

$$-2c_i\int_{z_1}^{z_2}(U-c_r)Q\,\mathrm{d}z = 0$$

Since Q is strictly positive, $U - c_r$ has to change sign in $[z_1, z_2]$. Hence

$$U_{\min} < c_r < U_{\max}$$

Rewrite the imaginary part as

$$\int_{z_1}^{z_2} UQ\,\mathrm{d}z = c_r\int_{z_1}^{z_2}Q\,\mathrm{d}z \tag{16}$$

and the real part as

$$\begin{aligned}\int_{z_1}^{z_2}U^2Q\,\mathrm{d}z &= 2c_r\int_{z_1}^{z_2}UQ\,\mathrm{d}z + (-c_r^2 + c_i^2)\int_{z_1}^{z_2}Q\,\mathrm{d}z\\ &\overset{(16)}{=} 2c_r^2\int_{z_1}^{z_2}Q\,\mathrm{d}z + (c_i^2 - c_r^2)\int_{z_1}^{z_2}Q\,\mathrm{d}z\\ &= (c_r^2 + c_i^2)\int_{z_1}^{z_2}Q\,\mathrm{d}z \end{aligned}\tag{17}$$

Now 'notice' that

$$\int_{z_1}^{z_2}(U-U_{\min})(U-U_{\max})Q\,\mathrm{d}z \le 0$$

since the first factor is ≥ 0, the second is ≤ 0 and $Q > 0$. Expanding the terms we have

$$\int_{z_1}^{z_2}\left[U^2Q - (U_{\min}+U_{\max})UQ + U_{\min}U_{\max}Q\right]\mathrm{d}z \le 0$$

Now using (16) and (17) we can rewrite as

$$\int_{z_1}^{z_2} \left[(c_r^2 + c_i^2) - (U_{\min} + U_{\max})c_r + U_{\min}U_{\max}\right] Q \, \mathrm{d}z \leq 0$$

$$\implies \int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + U_{\min}U_{\max} - \left(\frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2\right] Q \, \mathrm{d}z \leq 0$$

$$\implies \int_{z_1}^{z_2} \left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2\right] Q \, \mathrm{d}z \leq 0$$

Equivalently we can write

$$\left[\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2\right] \int_{z_1}^{z_2} Q \, \mathrm{d}z \leq 0$$

But $\int_{z_1}^{z_2} Q \, \mathrm{d}z > 0$ so

$$\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2 \leq 0$$

which establishes the semicircle theorem.

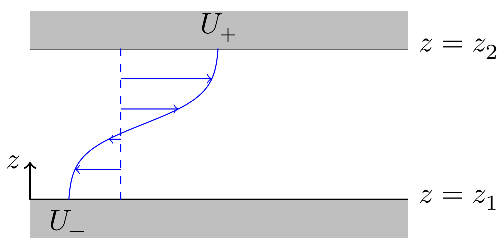
5.1.4 Predictions

For channel flow $\boldsymbol{U}(z) = (1 - z^2)\hat{\boldsymbol{x}}$ we have $U'' \neq 0$, i.e. no inflection points, so no inviscid instability predicted. However, channel flow is linearly unstable at sufficiently high Reynolds number. We must add viscosity to gain a more accurate stability heuristic.

5.2 Viscous analysis

Consider a basic state $\boldsymbol{U} = U(z)\hat{\boldsymbol{x}}$ with $P = p_0 - Gx$ and $U(z_1) = U_-, U(z_2) = U_+$. At leading order, Navier-Stokes gives

$$-G = \frac{1}{\mathrm{Re}} U''$$



Special cases are

- Plane Poiseuille flow (PPF) $U(z) = 1 - z^2$ in $[-1, 1]$ and $G = 2/\mathrm{Re}, U_+ = U_- = 0$.
- Plane Couette flow (PCF) with $U(z) = z$ in $[-1, 1]$ and $G = 0$, $U_+ = 1$, $U_- = -1$.

The linearised Navier-Stokes equations for a perturbation $\boldsymbol{u}', p'$ are

$$\begin{aligned}\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + w'\frac{\mathrm{d}U}{\mathrm{d}z} &= -\frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u' && (18)\\ \frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} &= -\frac{\partial p'}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v' && (19)\\ \frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} &= -\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}}\nabla^2 w' && (20)\\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0\end{aligned}$$

The divergence of the first three equations is

$$\nabla \cdot \begin{pmatrix}(18)\\(19)\\(20)\end{pmatrix} \implies \frac{\partial w'}{\partial x}\frac{\mathrm{d}U}{\mathrm{d}z} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x} = -\nabla^2 p'$$

Hence $\nabla^2 p' = -2U' w'_x$. Now consider ∇^2(20):

$$\nabla^2\left[\frac{\partial w'}{\partial z} + U\frac{\partial w'}{\partial x}\right] = -\frac{\partial}{\partial z}\nabla^2 p' + \frac{1}{\mathrm{Re}}\nabla^4 w'$$

Combining these results we have

$$\begin{aligned}\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\nabla^2 w' + U''\frac{\partial w'}{\partial x} + 2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial^2 w'}{\partial x\partial z} &= -\frac{\partial}{\partial z}\left(-2\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial x}\right)\\ \implies \left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right)\nabla^2 - U''\frac{\partial}{\partial x}\right] w' &= 0 && (21)\end{aligned}$$

with boundary conditions $w' = w'_z = 0$ on the boundaries. This is a fourth order PDE with 4 boundary conditions, so w' is fully determined. To close the problem we need another equation: first define the *normal vorticity*

$$\eta' \equiv \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} = \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$$

Now ∂_y(18) $- \partial_x$(19) gives

$$\begin{aligned}\frac{\partial \eta'}{\partial t} + U\frac{\partial \eta'}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} &= \frac{1}{\mathrm{Re}}\nabla^2\eta'\\ \implies \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^2\right]\eta' &= -\frac{\mathrm{d}U}{\mathrm{d}z}\frac{\partial w'}{\partial y} && (22)\end{aligned}$$

with boundary conditions $\eta' = 0$ on the boundaries since tangential velocities vanish at the boundaries. We have reduced $(u', v', w', p') \to (w', \eta')$. Given w' and η' determined from (21) and (22), we can generate v', w', p' from

$$\begin{aligned}u'_x + v'_y &= -w'_z\\ u'_y - v'_x &= \eta'\\ \nabla^2 p' &= -2U' w'_x\end{aligned}$$

5.2.1 Orr-Sommerfeld & Squire Equations

Introduce normal modes / wavelike disturbances / apply a Fourier transform:

$$(w', \eta')(x, y, z, t) = (\hat{w}(z), \hat{\eta}(z))e^{i(\alpha x + \beta y - \alpha c t)}$$

Let $k^2 = \alpha^2 + \beta^2$ be the total horizontal wavenumber. Then (21) and (22) become

$$\left[i\alpha(U-c)(D^2-k^2) - i\alpha U'' - \frac{1}{\text{Re}}(D^2-k^2)^2\right]\hat{w} = 0 \tag{23}$$

$$\left[i\alpha(U-c) - \frac{1}{\text{Re}}(D^2-k^2)\right]\hat{\eta} = -i\beta U'\hat{w} \tag{24}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}z}$ as usual. Equation (23) is the *Orr-Sommerfeld equation* (Orr 1907, Sommerfeld 1908) and equation (24) is the Squire equation (Squire 1933).

- The Orr-Sommerfeld (OS) equation is the viscous extension of the Rayleigh equation.
- System (23) and (24) has two types of solution:
  1. OS modes $(\hat{w}, \hat{\eta})$ where $\hat{w}$ solves (23) and $\hat{\eta}$ is the forced response in (24).
  2. Squire modes $(0, \hat{\eta})$ which are always damped. Consider (24)/$(-i\alpha)$:

     $$c\hat{\eta} = U\hat{\eta} + \frac{i}{\alpha\text{Re}}(D^2-k^2)\hat{\eta}$$

     Multiply by $\hat{\eta}^*$:

     $$c|\hat{\eta}|^2 = U|\hat{\eta}|^2 + \frac{i}{\alpha\text{Re}}\hat{\eta}^*(D^2-k^2)\hat{\eta}$$

     Take the imaginary part and integrate over $[z_1, z_2]$:

     $$\begin{aligned} c_i\int_{z_1}^{z_2}|\hat{\eta}|^2\,\mathrm{d}z &= \frac{1}{\alpha\text{Re}}\int_{z_1}^{z_2}\frac{i\hat{\eta}^*(D^2-k^2)\hat{\eta} - (-i)\hat{\eta}(D^2-k^2)\hat{\eta}^*}{2i}\,\mathrm{d}z \\ &= \frac{1}{\alpha\text{Re}}\int_{z_1}^{z_2}\frac{1}{2}(\hat{\eta}^*D^2\hat{\eta} + \hat{\eta}D^2\hat{\eta}^*) - k^2|\hat{\eta}|^2\,\mathrm{d}z \\ &= -\frac{1}{\alpha\text{Re}}\int_{z_1}^{z_2}|D\hat{\eta}|^2 + k^2|\hat{\eta}|^2\,\mathrm{d}z < 0 \end{aligned}$$

     Thus $c_i < 0$ so solutions are damped.

  Hence we just need to consider the OS equation to establish instability.
- Squire's theorem holds for the OS equation. The 3D version is

  $$(U-c)(D^2-k^2)\hat{w} - U''\hat{w} - \frac{1}{i\alpha\text{Re}}(D^2-k^2)^2\hat{w} = 0$$

  Compare with the 2D version

  $$(U-c)(D^2-\hat{\alpha}^2)\hat{w} - U''\hat{w} - \frac{1}{i\hat{\alpha}\hat{\text{Re}}}(D^2-\hat{\alpha}^2)^2\hat{w} = 0$$

  where $\hat{\alpha} = k^2 = \alpha^2 + \beta^2$ and

  $$\hat{\text{Re}} = \frac{\alpha\text{Re}_{3D}}{\hat{\alpha}} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\text{Re}_{3D} \le \text{Re}_{3D}$$

Thus each 3D OS mode corresponds to a 2D OS mode at a *lower* Re. Note this is a slightly different result from the inviscid case where 2D always had a larger growth rate. We can instead note that if the critical Reynolds number for linear stability is Re_c then

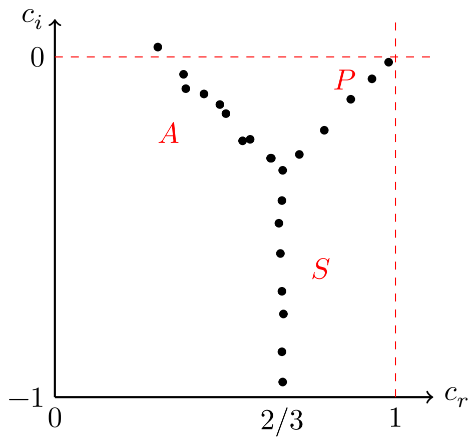
$$\mathrm{Re}_c = \min_{\alpha,\beta} \mathrm{Re}_c(\alpha,\beta) = \min_{\alpha} \mathrm{Re}_c(\alpha,0)$$

where the first equality defines Re_c and the second is Squire's theorem. This led to a focus on the 2D OS equation.

- What is the connection between Rayleigh and OS equations?
  - OS is non-singular and has a countably infinite number of eigenvalues and its eigenfunctions are complete (Scheisted 1960). Note if the interval of flow is unbounded, there is a continuous spectrum of neutrally stable eigenfunctions in addition to the discrete spectrum (Herron 1987).
  - OS equation is fourth order whilst Rayleigh's equation is second order. 2 OS modes approximate Rayleigh modes, the other 2 modes fix the boundary conditions at the walls (lots of work on this – see Drazin & Reid (1981)).
  - Today it is absolutely routine to numerically solve the OS eigenvalue problem for $\mathrm{Re} \leq 10^7$. Very famous paper by Orszag (1971) used spectral methods as opposed to shooting techniques or finite difference to predict Re_c in channel flow.

5.2.2 Channel flow (PPF)

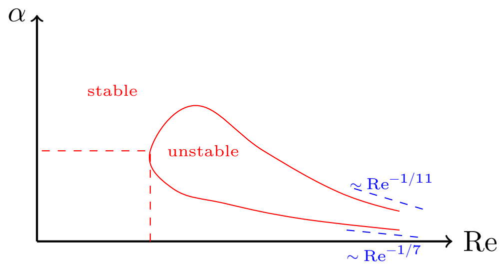
Thomas (1953) found $\mathrm{Re}_c = 5780$ at $\alpha_c = 1.026$ using finite differences (FD). Further FD estimates came from Nachtshen (1964) with $(\mathrm{Re}_c, \alpha_c) = (5767, 1.02)$ and Grosch & Salwen (1968) with $(\mathrm{Re}_c, \alpha_c) = (5750, 1.025)$. The accepted result now is from Orszag (1971) with $\mathrm{Re}_c = 5772.22$ at $\alpha_c = 1.02056$ using spectral methods.
Solving the Orrfield-Sommerfield equation with $\mathrm{Re} = 7000, \alpha = 1, \beta = 0$ gives the following plot.



Note the single unstable eigenvalue with $c_i > 0$. The eigenvalues arise in 3 families, denoted A for Airy, P for Pekeris, and S for Scheisted.

- A: $c_r \to 0$, wall modes, advected towards wall
- P: $c_r \to 1$, centre modes, advected towards centreline
- S: $c_r \approx 2/3$, identified by Mach (1976).

Note that a parabolic base state does not have an inflection point, but *does* have an unstable mode for large Re. Hence viscosity must be destabilising. The unstable mode is called a Tollmien-Schlichting mode/wave (Tollmien 1935, Schlichting 1933). Tollmien was the first to show the OS equation has instability for non-inflection point profiles.
However, as $\mathrm{Re} \to \infty$ we are left with the Rayleigh equation and stability, so somewhere in between they must match up. The stability diagram for the OS equation with finite Reynolds number appears as follows:



The neutral curve closes as there is no instability for $\mathrm{Re} \to \infty$.

5.2.3 Other flows

Type of flow	Profile	Stable?	$\mathrm{Re}_{\mathrm{crit}}$	α_{crit}	Proof?
Uniform	$U = \text{const.}$	Yes	∞	–	trivial
PCF	$U = z,\ z \in [-1, 1]$	Yes	∞	–	Romanov 1973
PPF	$U = 1 - z^2,\ z \in [-1, 1]$	No	5772	1.02	–
Blasius BL	$U = f'(z), z \geq 0*$	No	520	0.3	–
Shear layer	$U = \tanh z, -\infty < z < \infty$	No	0	0	–
Jet/wake	$U = \operatorname{sech}^2 z, -\infty < z < \infty$	No	4.02	0.17	–
HPF (pipe flow)	$\boldsymbol{U} = (1 - r^2)\hat{\boldsymbol{z}},\ 0 < r < 1$	Yes	∞	–	Chen et al., 2019?

Notes. Blasius boundary layer (BL) profile f solves $f''' + ff'' = 0$ subject to $f(0) = f'(0) = 0, f'(\infty) = 1$. This is boundary layer flow over an infinite plate.
HPF stands for Hagen-Poiseuille flow. Pipe flow observed to be unstable at $\mathrm{Re} \approx \mathcal{O}(2000)$ (Reynolds, 1883). The transition is at $\mathcal{O}(2000)$ with reasonable care, $\mathcal{O}(12,000)$ with a very careful experiment to minimise disturbances. The world record is $\mathrm{Re} \sim 10^5$ accredited to Pfenniger 1961. Conclusion: pipe flow is unstable to finite amplitude disturbances and threshold for instability *decreases* as $\mathrm{Re} \to \infty$.

6 Transient Growth & IVPs

So far in this course, the analysis has been 'modal' – identifying eigenfunctions and eigenvalues of linear operators around basic states. This can miss interesting features of the linearised dynamics over 'short' times. Need to consider initial value problems (IVPs).

6.1 Example of IVP analysis

Consider the initial value problem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ \eta \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix} + \begin{pmatrix} \eta^2 \\ -v\eta \end{pmatrix} \\ &= \mathsf{L}(\mathrm{Re})\begin{pmatrix} v \\ \eta \end{pmatrix} + \boldsymbol{N}(v,\eta)\end{aligned} \tag{25}$$

The first term is the linear part, and the second is the nonlinear part emulating the nonlinearities of the dynamical equations. The eigenfunctions of L are $-1/\mathrm{Re}$ and $-2/\mathrm{Re}$ so a basic state

$$\begin{pmatrix} v \\ \eta \end{pmatrix} = \mathbf{0}$$

is linearly stable. Then we must ask if all disturbances decay exponentially? Certainly asymptotically ($t \to \infty$) *but* not over short times.
We can solve (25) linearised:

$$\begin{aligned} \dot{v} &= -\frac{1}{\mathrm{Re}}v && \implies v(t) = v_0 e^{-t/\mathrm{Re}} \\ \dot{\eta} &= v - \frac{2}{\mathrm{Re}}\eta && \implies (\eta e^{\frac{2t}{\mathrm{Re}}})_t = v_0 e^{t/\mathrm{Re}} \\ & && \implies \eta = \mathrm{Re}v_0 e^{-t/\mathrm{Re}} + (\eta_0 - \mathrm{Re}v_0)e^{-2t/\mathrm{Re}} \end{aligned}$$

Hence the solution is

$$\begin{aligned}\begin{pmatrix} v \\ \eta \end{pmatrix} &= v_0 \begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix} e^{-t/\mathrm{Re}} + (\eta_0 - \mathrm{Re}v_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t/\mathrm{Re}} \\ &= \begin{pmatrix} v_0\left(1 - \frac{t}{\mathrm{Re}}\right) + \mathcal{O}(t^2) \\ \eta_0 + t\left(v_0 - \frac{2\eta_0}{\mathrm{Re}}\right) + \mathcal{O}(t^2) \end{pmatrix}\end{aligned}$$

The linearised solution demonstrates the possibility for short term *algebraic* growth of η provided $v_0 - 2\eta_0/\mathrm{Re} > 0$. To make this more specific, define a norm

$$E \equiv \frac{1}{2}\left(v^2 + \eta^2\right)$$

and assume $\eta_0 = v_0 = 1$. Then

$$\begin{aligned} E(t) &= \frac{1}{2}\left((1 - t/\mathrm{Re} + \dots)^2 + (1 + t(1 - 2/\mathrm{Re}) + \dots)^2\right) \\ &= 1 + \left(1 - \frac{3}{\mathrm{Re}}\right)t + \mathcal{O}(t^2) \end{aligned}$$

So there is energy grwoth at least initially for $\mathrm{Re} > 3$. What is going on? The eigenvectors of L are

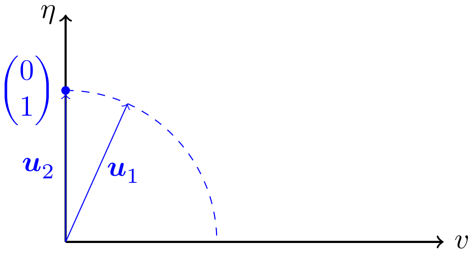
$$\begin{aligned} (\lambda_1, \boldsymbol{u}_1) &= (-\frac{1}{\mathrm{Re}}, \frac{1}{\sqrt{1+\mathrm{Re}^2}}\begin{pmatrix} 1 \\ \mathrm{Re} \end{pmatrix}) \\ (\lambda_2, \boldsymbol{u}_2) &= (-\frac{2}{\mathrm{Re}}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \end{aligned}$$

Note that these eigenvectors *overlap*. They satisfy $\boldsymbol{u}_1^T \cdot \boldsymbol{u}_1 = \boldsymbol{u}_2^T \cdot \boldsymbol{u}_2 = 1$ and also

$$\boldsymbol{u}_1^T \cdot \boldsymbol{u}_2 = \frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}} \to 1 \;\text{ as}\, \mathrm{Re} \to \infty$$

Hence the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is very inefficient in representing disturbances directed along v. For example,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{1+\mathrm{Re}^2}\boldsymbol{u}_1 - \mathrm{Re}\boldsymbol{u}_2$$



Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ decay at different rates, 'growth' appears as large coefficients $\sqrt{1+\mathrm{Re}^2}, \mathrm{Re}$ no longer largely cancel.

6.2 Key points

Matrix L is non-normal.

> **Definition.** A matrix L is non-normal if $L^T L \neq L L^T$, otherwise L is normal.

Note this definition is extensible to operators, i.e. an operator L is normal if

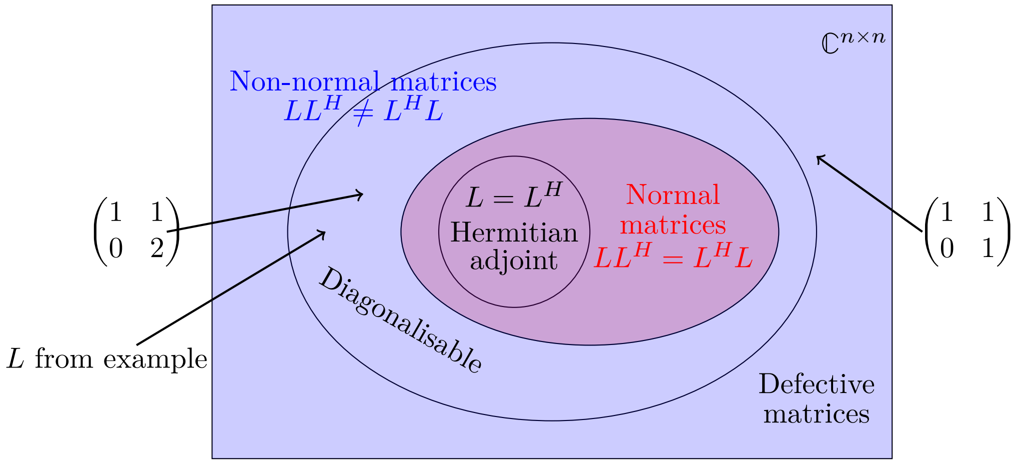
$$\langle u, Lv \rangle = \langle L^T u, v \rangle \implies \langle u, LL^T v \rangle = \langle u, L^T L v \rangle$$

For the matrix L defined in (25), we have

$$L^T L = \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{2}{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mathrm{Re}^2} + 1 & -\frac{2}{\mathrm{Re}} \\ -\frac{2}{\mathrm{Re}} & \frac{4}{\mathrm{Re}^2} \end{pmatrix}$$
$$L L^T = \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{2}{\mathrm{Re}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{2}{\mathrm{Re}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathrm{Re}^2} & -\frac{1}{\mathrm{Re}} \\ -\frac{1}{\mathrm{Re}} & \frac{4}{\mathrm{Re}^2} + 1 \end{pmatrix}$$

So L is non-normal.

Normality implies complete set of orthonormal eigenvectors. Hence a non-normal matrix has non-orthogonal eigenvectors. Consider all complex square matrices $\mathbb{C}^{n \times n}$ and denote the Hermitian conjugate by superscript ^{H}. The space may be split into normal and non-normal categories as follows.



Choice of norm is important. Consider a general matrix L which is diagonalisable, i.e. $\exists Q$ such that $Q^{-1}LQ = \Lambda$, a diagonal matrix. Then there is no growth in the norm

$$E' \equiv \boldsymbol{x}^H (Q^{-1})^H Q^{-1}\boldsymbol{x} = \boldsymbol{x}^H W \boldsymbol{x}$$

where $W = (Q^{-1})^H Q^{-1}$ is the *weight*.
Proof: let $\boldsymbol{y} = Q^{-1}\boldsymbol{x}$ and consider

$$\begin{aligned}
\frac{\mathrm{d}E'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{y}^H \cdot \boldsymbol{y}\right) &= \boldsymbol{y}^H \cdot \dot{\boldsymbol{y}} + \dot{\boldsymbol{y}}^H \cdot \dot{\boldsymbol{y}} \\
&= (Q^{-1}\dot{\boldsymbol{x}})^H Q^{-1}\boldsymbol{x} + (Q^{-1}\boldsymbol{x})^H Q^{-1}\dot{\boldsymbol{x}} \qquad \text{assuming } \dot{Q} = 0 \\
&= \boldsymbol{x}^H L^H (Q^{-1})^H Q^{-1}\boldsymbol{x} + \boldsymbol{x}^H (Q^{-1})^H Q^{-1} L\boldsymbol{x} \qquad \text{since } \dot{\boldsymbol{x}} = L\boldsymbol{x} \\
&= \boldsymbol{x}^H Q^{-H} Q^H L^H Q^{-H} Q^{-1}\boldsymbol{x} + \boldsymbol{x}^H Q^{-H} Q^{-1} L Q Q^{-1}\boldsymbol{x} \\
&= \boldsymbol{y}^H \Lambda^* \boldsymbol{y} + \boldsymbol{y}^H \Lambda \boldsymbol{y} \qquad \text{since } \Lambda^* = Q^H L^H Q^{-H} \\
&= \boldsymbol{y}^H (\Lambda^* + \Lambda)\boldsymbol{y} \\
&= 2\boldsymbol{y}^H \Re\left[\Lambda\right]\boldsymbol{y}
\end{aligned}$$

Hence $\dot{E}'$ is negative if all eigenvalues of $\Re\left[\Lambda\right]$ are negative. The key here is that $\boldsymbol{y} = Q^{-1}\boldsymbol{x}$ transforms $\boldsymbol{x}$ into a basis of eigenvectors. For example, for L in the previous example we have

$$Q = \begin{pmatrix} \frac{1}{\sqrt{1+\mathrm{Re}^2}} & 0 \\ \frac{\mathrm{Re}}{\sqrt{1+\mathrm{Re}^2}} & 1 \end{pmatrix} \implies Q^{-1} = \begin{pmatrix} \sqrt{1+\mathrm{Re}^2} & 0 \\ -\mathrm{Re} & 1 \end{pmatrix}$$

Hence in the basis of eigenvectors $(v \;\; \eta)^T$ is

$$Q^{-1}\boldsymbol{x} = Q^{-1}\begin{pmatrix} v \\ \eta \end{pmatrix} = \begin{pmatrix} v\sqrt{1+\mathrm{Re}^2} \\ \eta - \mathrm{Re}v \end{pmatrix}$$

So above result indicates

$$(1+\mathrm{Re}^2)_v^2 + (\eta - \mathrm{Re}v)^2 = E'$$

decays. We can calculate explicitly:

$$\begin{aligned}
\frac{\mathrm{d}E'}{\mathrm{d}t} &= 2(1+\mathrm{Re}^2)v\dot{v} + 2(\eta - \mathrm{Re}v)(\dot{\eta} - \mathrm{Re}\dot{v}) \\
&= 2(1+\mathrm{Re}^2)\left(\frac{-1}{\mathrm{Re}^2}v^2\right) + 2(\eta - \mathrm{Re}v)\left(2v - \frac{2\eta}{\mathrm{Re}}\right) \\
&= -\frac{2(1+\mathrm{Re}^2)}{\mathrm{Re}}v^2 - \frac{4(\eta-\mathrm{Re}v)^2}{\mathrm{Re}} \\
&< 0 \ \text{ for } v \neq 0, \eta \neq 0
\end{aligned}$$

Non-normality is necessary but not sufficient for transient growth. Non-normality does not imply transient growth, but transient growth does imply non-normality. For example, consider the norm E defined in section 6.1, $E = (v^2 + \eta^2)/2$. We have

$$\begin{aligned}
\dot{E} &= (v \quad \eta)\begin{pmatrix}\dot{v} \\ \dot{\eta}\end{pmatrix} \\
&= -\frac{1}{\mathrm{Re}}v^2 - \frac{2}{\mathrm{Re}}\eta^2 + v\eta \\
&= -\left(\frac{1}{\sqrt{\mathrm{Re}}}v - \frac{\sqrt{\mathrm{Re}}}{2}\eta\right)^2 + \left(\frac{\mathrm{Re}^2 - 8}{4\mathrm{Re}}\right)\eta^2
\end{aligned} \tag{26}$$

So provided $\mathrm{Re}^2 < 8$, no growth is possible even though L is non-normal. Note from (26) we can see that if $\mathrm{Re}^2 > 8$ the maximum *initial* growth is obtained for $v_0 = \frac{\mathrm{Re}}{2}\eta_0$ so if $E_0 = 1$, then

$$(v_0, \eta_0) = \left(\frac{\mathrm{Re}\sqrt{2}}{\sqrt{4+\mathrm{Re}^2}}, \frac{\sqrt{8}}{\sqrt{4+\mathrm{Re}^2}}\right)$$

maximises the initial growth.
We can do the same analysis for the whole system

$$\begin{aligned}
\dot{v} &= -\frac{1}{\mathrm{Re}}v + \eta^2 \\
\dot{\eta} &= v - \frac{2}{\mathrm{Re}}\eta - v\eta
\end{aligned}$$

as the non-linearity is energy preserving (exactly as in the Navier-Stokes equations). We have

$$(v, \eta)\begin{pmatrix}\dot{v} \\ \dot{\eta}\end{pmatrix} = (v, \eta)L\begin{pmatrix}v \\ \eta\end{pmatrix}$$

as before. Thus the nonlinear terms satisfy

$$(v, \eta)N(v, \eta) = 0 = v(\eta^2) + \eta(-v\eta)$$

Hence *any* initial condition will decay monotonically for $\mathrm{Re} < \sqrt{8}$, i.e. $(v, \eta) = \mathbf{0}$ is then a global attractor.
Exactly the same type of analysis (energy stability analysis) can be done for the Navier-Stokes equations (hope to revisit later).

How to find max growth and optimal ICs. We can find the optimal initial conditions as a function of T, a chosen time. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ \eta \end{pmatrix} = L\begin{pmatrix} v \\ \eta \end{pmatrix} \implies \begin{pmatrix} v \\ \eta \end{pmatrix} = e^{Lt}\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}$$

Let $A \equiv e^{Lt}$. By direct calculation, we have

$$A = \begin{pmatrix} e^{-t/\mathrm{Re}} & 0 \\ \mathrm{Re}(e^{-t/\mathrm{Re}} - e^{-2t/\mathrm{Re}}) & e^{-2t/\mathrm{Re}} \end{pmatrix}$$

or more generally

$$e^{\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix} t} = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}^n t^n = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} & e^{\lambda_2 t} \end{pmatrix}$$

Define the *energy gain* at time T

$$\begin{aligned} G(T;\mathrm{Re}) &\equiv \max_{v_0,\eta_0} \frac{E(T)}{E(0)} \\ &= \max_{v_0,\eta_0} \frac{(v(T)\ \ \eta(T))\begin{pmatrix} v(T) \\ \eta(T) \end{pmatrix}}{(v_0\ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\ &= \max_{v_0,\eta_0} \frac{\left[A(T)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}\right]^T \left[A(T)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}\right]}{(v_0\ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\ &= \max_{v_0,\eta_0} \frac{(v_0\ \ \eta_0)A^T A\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}}{(v_0\ \ \eta_0)\begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix}} \\ &= \|A\|_2^2 \end{aligned}$$

The maximum over v_0, η_0 is equivalent to the maximum eigenvalue of $A^T A$. Since $A^T A$ is real and symmetric, it is normal, so all eigenvalues are real. $A^T A$ is also positive definite, so all eigenvalues are positive. The norm

$$\|A\|_2^2$$

is usually computed using singular value decomposition (SVD). From the above, we have

$$G = \sigma_1^2 = \text{largest eigenvalue of } A^T A$$

and σ_1 is the largest singular value of A. The corresponding eigenvector is the optimal initial condition.

6.3 Singular value decomposition

In the previous section we refer to singular value decomposition; here we demonstrate existence and uniqueness of SVD.

Theorem. Every matrix $A \in \mathbb{C}^{m\times n}$ has a SVD

$$A = U\Sigma V^H$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma = \text{diag}(\sigma_i) \in \mathbb{R}^{m\times n}$ with $\sigma_1 < \sigma_2 < \dots$. Furthermore, the singular values $\{\sigma_j\}$ are uniquely determined and if A is square ($m = n$) and the σ_j are distinct then the left and right singular vectors $\{u_j\}$, $\{v_j\}$ are uniquely determined up to complex signs. Note in particular that $\sigma_j \in \mathbb{R}$.
How can we relate eigenvalues of A^TA to the singular values of A? Given $A = U\Sigma V^H$ where $U^HU = I, V^HV = I$ and Σ is diagonal, we have

$$AV = U\Sigma$$

so A maps columns of V onto columns of U with 'scale' factors in Σ, i.e. the singular values. For example,

$$A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$$

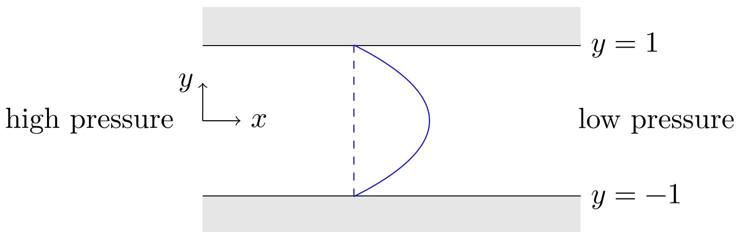
Similarly, $A^T = V\Sigma U^H$ so $A^TU = V\Sigma$ and A^T maps columns of U back to columns of V, e.g. $A^T\boldsymbol{u}_i = \sigma_i\boldsymbol{v}_i$. Hence

$$\begin{aligned}\sigma_i\boldsymbol{v}_i = A^T\boldsymbol{u}_i &= A^T\left(\frac{1}{\sigma_1}A\boldsymbol{v}_i\right)\\ \implies A^TA\boldsymbol{v}_i &= \sigma_i^2\boldsymbol{v}_i\end{aligned}$$

Thus σ_i^2 are eigenvalues of A^TA. Since σ_i are ordered so that $\sigma_1 > \sigma_2 > \dots$, $G = \sigma_1^2$ is the square of the largest singular value of A.
Note if A is normal then singular values are eigenvalues. If A is non-normal, this does not hold.

6.4 Energy growth in viscous channel flows

This section is largely based on research work by Reddy & Henningson, JFM, **252**, pp. 209–238 (1993). Consider pressure driven flow through a channel.



The basic state is $\boldsymbol{U} = U(y)\hat{\boldsymbol{x}}$ with $U(y) = 1 - y^2$ and $\nabla P = -\frac{2}{\text{Re}}\hat{\boldsymbol{x}}$. Consider a perturbation $(\boldsymbol{u}, p)$:

$$\begin{aligned}\boldsymbol{u}_{\text{total}} &= U(y)\hat{\boldsymbol{x}} + \boldsymbol{u}\\ p_{\text{total}} &= P + p\end{aligned}$$

The linearised Navier-Stokes are (as usual)

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\hat{\boldsymbol{u}} + \nabla p &= \frac{1}{\text{Re}}\nabla^2\boldsymbol{u} \qquad (27)\\ \nabla\cdot\boldsymbol{u} &= 0\end{aligned}$$

First, consider $\hat{\boldsymbol{y}} \cdot \nabla \times$ (27):

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta + \frac{\partial v}{\partial z}\frac{\mathrm{d}U}{\mathrm{d}y} = \frac{1}{\mathrm{Re}}\nabla^2\eta$$

where $\eta = \hat{\boldsymbol{y}} \cdot \nabla \times \boldsymbol{u} = \partial_z u - \partial_x w$ (using $\boldsymbol{u} = (u, v, w)$ as usual). Consider also $\hat{\boldsymbol{y}} \cdot \nabla \times (\nabla \times (27))$ which gives

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - \frac{\partial v}{\partial x}\frac{\mathrm{d}^2U}{\mathrm{d}y^2} = \frac{1}{\mathrm{Re}}\nabla^4 v$$

The no-slip boundary conditions $u = v = w = 0$ on $y = \pm 1$ translates to BCs on η and v:

$$\begin{aligned} \eta(y = \pm 1) &= 0 \\ v(y = \pm 1) &= 0 \\ \left.\frac{\partial v}{\partial y}\right|_{y=\pm 1} &= 0 \end{aligned}$$

where the last condition follows from $\nabla \cdot \boldsymbol{u} = 0$ and the fact tangential derivatives also vanish on $y = \pm 1$. We have reduced the second order system for 3 variables (u, v, w) supplemented by $\nabla \cdot \boldsymbol{u}$ and p into a second and fourth order system for v and η.

We now take a Fourier transform in x and z, i.e. write

$$[v(\boldsymbol{x}, t), \eta(\boldsymbol{x}, t)] = [\hat{v}(y, t), \hat{\eta}(y, t)]\, e^{i(\alpha x + \beta z)}$$

Then write

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{v} \\ \hat{\eta}\end{pmatrix} = \begin{pmatrix}\mathcal{L}_{os} & 0 \\ \mathcal{L}_c & \mathcal{L}_{sq}\end{pmatrix}\begin{pmatrix}\hat{v} \\ \hat{\eta}\end{pmatrix} = \mathcal{L}\begin{pmatrix}\hat{v} \\ \hat{\eta}\end{pmatrix} \tag{28}$$

where

$$\begin{aligned} \mathcal{L}_{os} &= \frac{1}{D^2 - k^2}\left[\frac{1}{\mathrm{Re}}(D^2 - k^2)^2 - i\alpha U(D^2 - k^2) + i\alpha D^2 U\right] \\ \mathcal{L}_c &= -i\beta DU \\ \mathcal{L}_{sq} &= \frac{1}{\mathrm{Re}}(D^2 - k^2) - i\alpha U \end{aligned}$$

with $D \equiv \mathrm{d}/\mathrm{d}y$ and $k^2 = \alpha^2 + \beta^2$. The operators $\mathcal{L}_{os}$ and $\mathcal{L}_c$ are the Orr-Sommerfeld (Orr 1907 & Sommerfeld 1908) and Squire (Squire 1933) operators, and $\mathcal{L}_c$ is the coupling operator. Notice that the structure of (28) is reminiscent of the introductory model in section 6.1.

We can write the solution to the IVP (28) in terms of an eigenfunction expansion since they form a complete set (Diprima & Habetler 1969) using the norm (31) defined later. Let $\{\lambda_j\}$ and $\{\mu_j\}$ be the eigenvalues of $\mathcal{L}_{os}$ and $\mathcal{L}_{sq}$ respectively. Then

$$\hat{\boldsymbol{v}} = \begin{pmatrix}\hat{v}(y, t) \\ \hat{\eta}(y, t)\end{pmatrix} = \sum_j A_j e^{\lambda_j t}\begin{pmatrix}\tilde{v}_j(y) \\ \tilde{\eta}_j^p(y)\end{pmatrix} + \sum_j B_j e^{\mu_j t}\begin{pmatrix}0 \\ \tilde{\eta}_j(y)\end{pmatrix} \tag{29}$$

The first sum is of OS modes, where $\{\tilde{v}_j\}$ are the eigenfunctions of $\mathcal{L}_{os}$, and $\{\tilde{\eta}_j^p\}$ are the forced normal vorticity (η) functions corresponding to the velocity functions. The second sum is of Squire modes, where $\{\tilde{\eta}_j\}$ are the eigenfunctions of $\mathcal{L}_{sq}$. The coefficients $\{A_j\}$ and $\{B_j\}$ are set by the initial conditions.

We now combine the eigenvalues λ_j and μ_j into a set

$$\Lambda = \{\lambda_j, \mu_j\}$$

with Λ_j ordered by decreasing real part. Further, define the corresponding eigenfunctions to Λ_j as

$$\tilde{\boldsymbol{q}}_j(y) = \begin{pmatrix} \tilde{v}_j(y) \\ \tilde{\eta}_j(y) \end{pmatrix}$$

The solution may then be written

$$\hat{\boldsymbol{v}}(y,t) = \sum_j a_j \tilde{\boldsymbol{q}}_j(y) e^{\Lambda_j t} \tag{30}$$

Typically, we use the (kinetic) energy norm $E = \iint_{\alpha,\beta} E(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta$ where

$$E(\alpha,\beta) = \frac{1}{2}\int_{-1}^{1} \{|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2\}\mathrm{d}y$$

To determine u and w in terms of v and η, consider $\eta = u_z - w_x = i\beta u - i\alpha w$. Hence

$$\nabla\cdot\boldsymbol{u} = 0 \implies -Dv = i\alpha u + i\beta w$$

In matrix form we have

$$\begin{pmatrix} i\alpha & i\beta \\ i\beta & -i\alpha \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -Dv \\ \eta \end{pmatrix}$$

Finally, inverting this equation gives

$$\begin{pmatrix} u \\ w \end{pmatrix} = \frac{1}{k^2}\begin{pmatrix} -i\alpha & -i\beta \\ -i\beta & i\alpha \end{pmatrix}\begin{pmatrix} -Dv \\ \eta \end{pmatrix}$$

Hence u and w can be written

$$u = \frac{i\alpha Dv - i\beta\eta}{k^2}$$
$$w = \frac{i\beta Dv + i\alpha\eta}{k^2}$$

The energy norm may then be written as

$$E(\alpha,\beta) = \|\hat{\boldsymbol{v}}\|^2 = \frac{1}{2k^2}\int_{-1}^{1} |D\hat{v}|^2 + k^2|\hat{v}|^2 + |\hat{\eta}|^2 \,\mathrm{d}y \tag{31}$$

We now wish to determine the energy growth gain, defined by

$$G(\alpha,\beta;\mathrm{Re},t) = \sup_{\hat{\boldsymbol{v}}(y,0)} \frac{\|\hat{\boldsymbol{v}}(y,t)\|^2}{\|\hat{\boldsymbol{v}}(y,0)\|^2}$$

Using the solution (30) (from (29)) derived above, we have (note that supremum is now equivalently taken over initial condition coefficients $\boldsymbol{a}$)

$$\begin{aligned} G &= \sup_{\boldsymbol{a}} \frac{\left\|\sum_j a_j \tilde{\boldsymbol{q}}_j e^{\Lambda_j t}\right\|^2}{\left\|\sum_j a_j \tilde{\boldsymbol{q}}_j\right\|^2} \\ &= \sup_{\boldsymbol{a}} \frac{\sum_i^N \sum_j^N a_i^* e^{\Lambda_i^* t}\langle\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{q}}_j\rangle a_j e^{\Lambda_j t}}{\sum_i^N \sum_j^N a_i^* \langle\tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{q}}_j\rangle a_j} \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product

$$\langle \tilde{\boldsymbol{q}}_j, \tilde{\boldsymbol{q}}_l \rangle = \frac{1}{2k^2} \int_{-1}^{1} (D\hat{v}_j)^* D\hat{v}_l + k^2 \hat{v}_j^* \hat{v}_l + \hat{\eta}_j^* \hat{\eta}_l \, \mathrm{d}y$$

Let $A_{ij} = \langle \tilde{\boldsymbol{q}}_i, \tilde{\boldsymbol{q}}_l \rangle$. Then $A \in \mathbb{C}^{N \times N}$ is Hermitian ($A_{ij}^* = A_{ji}$) and positive definite (the norm is positive definite). Then there exists a matrix F such that $A = F^H F$ (Cholesky decomposition). Using the notation

$$e^{\Lambda t} = \mathrm{diag}(e^{\Lambda_1 t}, e^{\Lambda_2 t}, \dots)$$

we can write the growth gain as

$$\begin{aligned} G &= \sup_{\boldsymbol{a}} \frac{\sum \sum a_i^* e^{\Lambda_i^* t} (F^H F)_{ij} a_j e^{\Lambda_j t}}{\sum \sum a_i^* (F^H F)_{ij} a_j} \\ &= \sup_{\boldsymbol{a}} \frac{(F e^{\Lambda t} \boldsymbol{a})^H (F e^{\Lambda t} \boldsymbol{a})}{(F\boldsymbol{a})^H (F\boldsymbol{a})} \end{aligned}$$

Let $\boldsymbol{x} = F\boldsymbol{a}$. Then we can take the supremum over $\boldsymbol{x}$ and write

$$\begin{aligned} G &= \sup_{\boldsymbol{x}} \frac{(F e^{\Lambda t} F^{-1} \boldsymbol{x})^H (F e^{\Lambda t} F^{-1} \boldsymbol{x})}{\boldsymbol{x}^H \boldsymbol{x}} \\ &= \sup_{\boldsymbol{x}} \frac{\left\| F e^{\Lambda t} F^{-1} \boldsymbol{x} \right\|^2}{\left\| \boldsymbol{x} \right\|^2} \\ &= \left\| F e^{\Lambda t} F^{-1} \right\|_2^2 \end{aligned}$$

where $\left\| \cdot \right\|_2^2$ is the matrix norm defined by

$$\left\| A \right\|_p \equiv \sup_{\boldsymbol{x} \neq 0} \frac{\left\| A\boldsymbol{x} \right\|_p}{\left\| \boldsymbol{x} \right\|_p}$$

and the p-norm is

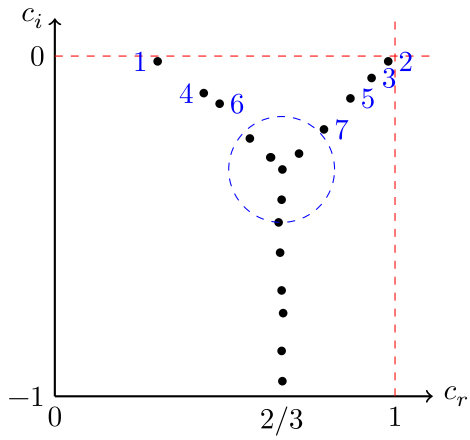
$$\left\| \boldsymbol{x} \right\|_p = \left(\sum_i x_i^p \right)^{1/p}$$

Note if $\tilde{\boldsymbol{q}}_j$ are orthogonal (i.e. $\mathcal{L}$ is normal) then A is diagonal, so F is diagonal. Hence

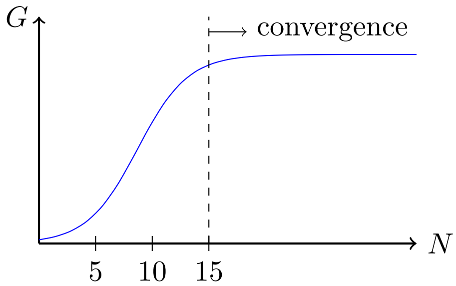
$$\begin{aligned} \left\| F e^{\Lambda t} F^{-1} \right\|^2 &= \left\| e^{\Lambda t} \right\|^2 \\ &= \max_{\lambda_j} \left| e^{\lambda_j t} \right|^2 \\ &= \max_{\lambda_j} e^{2\Re(\lambda_j) t} \end{aligned}$$

so the norm is determined by eigenvalues of A in this case. As mentioned previously, the 2-norm of any matrix can be computed by SVD.

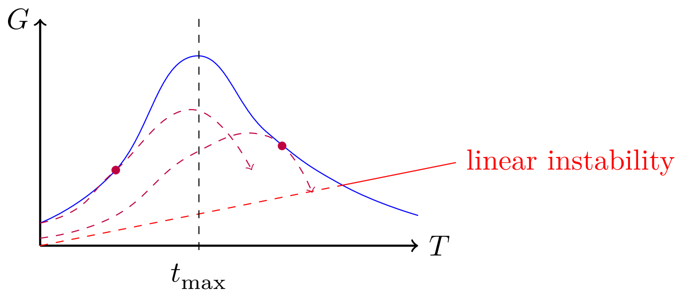
How is G estimated? Recall eigenvalues of $\mathcal{L}_{os}$ generically form a Y-shape when plotted in the complex plane:



Once again, note that eigenvalue with imaginary part close to 0. At Re = 5772, this eigenvalue becomes unstable. Numbering the eigenvalues by descending imaginary part, we can ask how many eigenvalues are required for an accurate estimate of G? It turns out we just need to include the 'neck' of the Y-shape, circled in blue. This corresponds to the first 15 (roughly) eigenvalues.



A plot of max growth rate G against time T demonstrates an *envelope* – there is not necessarily a mode which matches the maximum growth rate at all times T, but instead numerous different modes which may yield the maximum growth rate at a *specific* time T. The envelope decays for large T as only linear instability modes remain. The key point is that the optimal initial conditions depend on the time T at which G should be maximised.



Optimal growth in wall-bounded shear flows. The optimal growth gains G and the time $t_{\max}$ at which this maximum is reached is shown in the table below for a some canonical wall-bounded shear flows.

Notice in particular that the maximum growth gain $G \sim \mathrm{Re}^2$ and $t_{\max} \sim \mathrm{Re}$. Also, the maximum growth has β non-zero (2D spanwise variation) in contrast to Squire's theorem (2D streamwise variation). References:

	G	$t_{\max}$	α	β	ref.
PPF	$2 \times 10^{-4}\mathrm{Re}^2$	$0.076\mathrm{Re}$	0	2.04	(1)
PCF	$1.2 \times 10^{-3}\mathrm{Re}^2$	$0.117\mathrm{Re}$	$\frac{35}{\mathrm{Re}}$	1.6	(1)
Pipe flow	$7 \times 10^{-5}\mathrm{Re}^2$	$0.048\mathrm{Re}$	0	1	(2)
Blasius BL	$1.5 \times 10^{-3}\mathrm{Re}^2$	$0.778\mathrm{Re}$	0	0.65	(3)

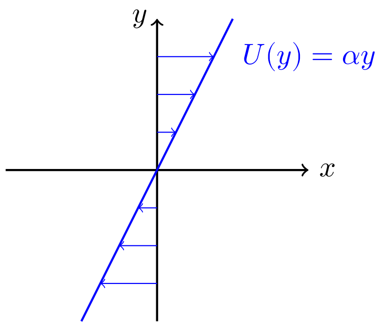
1. Trefethen et al. (1993)
2. Schmid & Henningson (1994)
3. Butler & Farrel (1992)

6.5 Mechanisms of transient growth

Take the (v, η) equations

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta + \frac{\partial v}{\partial z}\frac{\mathrm{d}U}{\mathrm{d}y} = \frac{1}{\mathrm{Re}}\nabla^2\eta$$
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - \frac{\partial v}{\partial x}\frac{\mathrm{d}^2U}{\mathrm{d}y^2} = \frac{1}{\mathrm{Re}}\nabla^4 v$$

and consider infinite shear $U(y) = \alpha y$ in an unbounded domain.



Note $[\alpha] = T^{-1}, [\nu] = L^2T^{-1}$ are the only parameters in the problem so we cannot define an intrinsic lengthscale which is independent of Re (i.e. if we used ν, α to form a lengthscale, $\mathrm{Re} \sim 1$ necessarily.) Hence we return to units:

$$\left(\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial x}\right)\nabla^2 v - \nu\nabla^4 v = 0 \tag{32}$$
$$\left(\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial x}\right)\eta - \nu\nabla^2\eta = -\alpha\frac{\partial v}{\partial z} \tag{33}$$

Recall $\eta \equiv \hat{\boldsymbol{y}} \cdot \nabla \times \boldsymbol{u} = u_z - w_x$. Look for *Kelvin modes* of the form

$$[v, \eta](\boldsymbol{x}, t) = [\hat{v}, \hat{\eta}](t)e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}$$

From (32) we have

$$\left(\frac{\partial}{\partial t} + \alpha y\frac{\partial}{\partial x}\right)\left(-\boldsymbol{k}^2(t)\hat{v}(t)e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}\right) = \nu\boldsymbol{k}^4\hat{v}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$-(\boldsymbol{k}^2\hat{v})_t e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \boldsymbol{k}^2\hat{v}(i\boldsymbol{k}_t\cdot\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \alpha\boldsymbol{k}^2 y\hat{v}(ik_1)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \nu\boldsymbol{k}^4\hat{v}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$-(\boldsymbol{k}^2\hat{v})_t - i\boldsymbol{k}_t\cdot\boldsymbol{x}\boldsymbol{k}^2\hat{v} - \alpha\boldsymbol{k}^2 y\hat{v}ik_1 = \nu\boldsymbol{k}^4\hat{v}$$

Separating into components which are $\boldsymbol{x}$-dependent and $\boldsymbol{x}$-independent gives

$$-(\boldsymbol{k}^2\hat{v})_t = \nu\boldsymbol{k}^4\hat{v} \tag{34}$$

$$-i\boldsymbol{k}_t\cdot\boldsymbol{x} - \alpha k_1 y = 0 \tag{35}$$

The second equation (35) can also be written

$$\dot{k}_1 x + \dot{k}_2 y + \dot{k}_3 z + \alpha k_1 y = 0$$

Hence $\dot{k}_1 = \dot{k}_3 = 0$ and $\dot{k}_2 = -\alpha k_1$ so

$$\begin{aligned} k_1 &= k_{01} \\ k_2 &= k_{02} - \alpha k_{01} t \\ k_3 &= k_{03} \end{aligned}$$

Now (34) can be written

$$\begin{aligned} (\boldsymbol{k}^2\hat{v})_t &= -\nu\boldsymbol{k}^2(\boldsymbol{k}^2\hat{v}) \\ \implies \boldsymbol{k}^2\hat{v} &= \boldsymbol{k}_0^2\hat{v}(0)e^{-\nu\int_0^t \boldsymbol{k}^2(\tau)\,\mathrm{d}\tau} \\ \implies \hat{v}(t) &= \frac{\boldsymbol{k}_0^2}{\boldsymbol{k}^2(t)}\hat{v}(0)e^{-\nu\int_0^t \boldsymbol{k}^2(\tau)\,\mathrm{d}\tau} \end{aligned}$$

Then from (33)

$$\hat{\eta}_t + \nu\boldsymbol{k}^2(t)\hat{\eta} = -i\alpha k_3\hat{v}(t)$$

It is useful to look at two 2D special cases.

2D with $\partial_z = 0$ (no spanwise variation). If there is no spanwise variation then $k_3 = 0$ and all the 'action' is in the $\hat{v}$ equation, as the $\hat{\eta}$ equation is unforced. We have

$$\hat{v}(t) = \hat{v}(0)\frac{k_{01}^2 + k_{02}^2}{k_{01}^2 + (k_{02} - \alpha k_{01} t)^2}e^{-\nu\int_0^t \boldsymbol{k}^2(\tau)\,\mathrm{d}\tau}$$

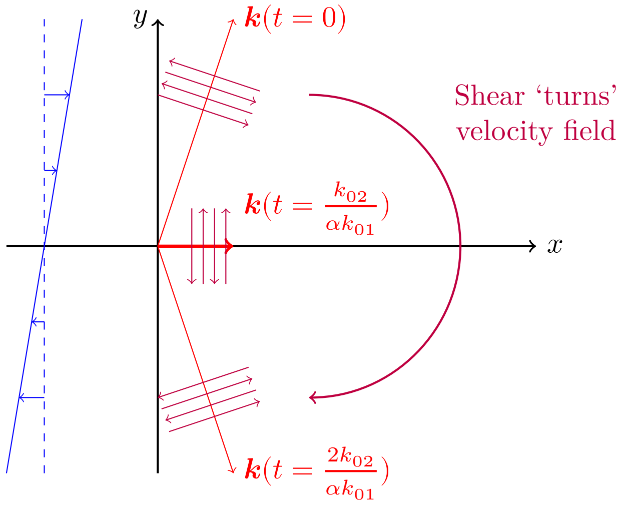
If $k_{02} \gg k_{01}$, we expect large growth on 'fast' times $\mathcal{O}(1)$, as opposed to $\mathcal{O}(1/\nu)$ times. We see this as follows: denote the $\boldsymbol{k}_0^2/\boldsymbol{k}^2$ factor as A. Then

- At $t = 0$, $A \sim 1$.
- At $t = k_{02}/\alpha k_{01}$, $A \to \frac{k_{02}^2}{k_{01}^2} \gg 1$.
- At $t = 2k_{02}/\alpha k_{01}$, $A \to 1$.

Incompressibility combined with $\boldsymbol{u} = \hat{u}(t)\exp(i\boldsymbol{k}(t)\cdot\boldsymbol{x})$ gives

$$ik_1\hat{u} = ik_2\hat{v} = 0$$

i.e. $\boldsymbol{k}$ is perpendicular to (u, v).



This is the *Orr mechanism* (1907). The energy is maximised at $t = k_{02}/\alpha k_{01}$. The initial conditions point into the shear, with the energy growth arising as the shear 'rotates' the velocity field. It is a 2D mechanism, inviscid (happens quickly), and growth is $\mathcal{O}(\mathrm{Re}^0)$.

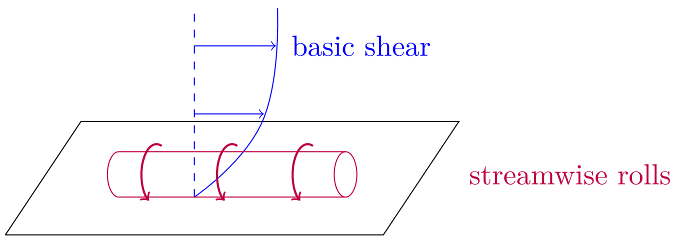
2D with $\partial_x = 0$ (no streamwise variation). If there is no streamwise variation then $k_1 = 0$, hence k is constant. Then

$$\begin{aligned} \hat{v}(t) &= \hat{v}(0)e^{-\nu \boldsymbol{k}^2 t} \\ \hat{\eta} + \nu \boldsymbol{k}^2 \hat{\eta} &= -i\alpha k_3 \hat{v} = -i\alpha k_3 \hat{v}(0) e^{-\nu \boldsymbol{k}^2 t} \\ \implies \left(\hat{\eta} e^{\nu \boldsymbol{k}^2 t}\right)_t &= -i\alpha k_3 \hat{v}(0) \\ \implies \hat{\eta}(t) &= \hat{\eta}_0 e^{-\nu \boldsymbol{k}^2 t} - i\alpha k_3 t \hat{v}(0) e^{-\nu \boldsymbol{k}^2 t} \end{aligned}$$

The exponential factors damp $\hat{\eta}$, but the t factor in the second term provides the possibility of algebraic growth. In the case of no streamwise variation we have $\eta = ik_3 u$, so

$$\hat{u}(t) = \left[\hat{u}(0) - \alpha t \hat{v}(0)\right] e^{-\nu \boldsymbol{k}^2 t}$$

Hence we expect algebraic growth in $\hat{u}$ of amplitude $\mathcal{O}(1/\nu)$, so growth in energy of order $\mathcal{O}(1/\nu^2) = \mathcal{O}(\mathrm{Re}^2)$. This explains the previous observations that $G_{\max} = \mathcal{O}(\mathrm{Re}^2)$ and $t_{\max} = \mathcal{O}(\mathrm{Re})$. This is the *lift-up mechanism* whereby small, slowly decaying velocities advect fluid across the shear.



Streamwise rolls advect slower flowing fluid away from the boundary and faster flowing fluid towards the boundary. This causes a large anomaly in the streamwise flow component. Streamwise rolls decay over a $\mathcal{O}(\mathrm{Re})$ timescale, which gives $\mathcal{O}(\mathrm{Re}^2)$ growth.

6.6 Orr-Sommerfeld operator is non-normal

The adjoint operator is dependent on the choice of the inner product. Recall $D \equiv \mathrm{d}/\mathrm{d}y$ and define the inner product here as

$$\begin{aligned}\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle &\equiv \int_{-1}^{1} (D\boldsymbol{v}_1)^*(D\boldsymbol{v}_2) + k^2 \boldsymbol{v}_1^* \boldsymbol{v}_2 \,\mathrm{d}y \\ &= \int_{-1}^{1} \boldsymbol{v}_1^* \left[(k^2 - D^2)\boldsymbol{v}_2\right] \mathrm{d}y\end{aligned}$$

Hence we have

$$\begin{aligned}\langle \boldsymbol{v}_1, \mathcal{L}_{os}\boldsymbol{v}_2 \rangle &= \int_{-1}^{1} \boldsymbol{v}_1^*(k^2 - D^2)\{\frac{1}{D^2 - k^2}\left[\frac{1}{\mathrm{Re}}(D^2 - k^2)^2 - i\alpha U(D^2 - k^2) + i\alpha D^2 U\right]\}\boldsymbol{v}_2 \,\mathrm{d}y \\ &= \langle \mathcal{L}_{os}^\dagger \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle\end{aligned}$$

by definition of $\mathcal{L}_{os}^\dagger$. Hence the adjoint may be written

$$\mathcal{L}_{os}^\dagger = \frac{1}{D^2 - k^2}\left[i\alpha U(D^2 - k^2) + 2i\alpha DUD + \frac{1}{\mathrm{Re}}(D^2 - k^2)^2\right]$$

It can be shown that $\langle \boldsymbol{v}_1, \mathcal{L}_{os}^\dagger \mathcal{L}_{os}\boldsymbol{v}_2 \rangle \neq \langle \boldsymbol{v}_1, \mathcal{L}_{os}\mathcal{L}_{os}^\dagger \boldsymbol{v}_2 \rangle$ for all $\boldsymbol{v}_1, \boldsymbol{v}_2$. Hence $\mathcal{L}_{os}$ is a non-normal operator, so we can expect energy growth even if the eigenvalues of $\mathcal{L}_{os}$ indicate damping.

6.7 Energy stability analysis

Consider the Navier-Stokes equations with a specified velocity boundary condition $\boldsymbol{u} = \boldsymbol{U}_b$ on ∂V. Suppose there is a steady basic flow $\boldsymbol{U}(\boldsymbol{x})$. Consider disturbances to the basic state $\boldsymbol{u} = \boldsymbol{U} + \hat{\boldsymbol{u}}, p = P + \hat{p}$. The Navier-Stokes equations become

$$\nabla \cdot \hat{\boldsymbol{u}} = 0$$

$$\frac{\partial \hat{\boldsymbol{u}}}{\partial t} + \boldsymbol{U} \cdot \nabla \hat{\boldsymbol{u}} + \hat{\boldsymbol{u}} \cdot \nabla \boldsymbol{U} + \nabla \hat{p} - \frac{1}{\mathrm{Re}}\nabla^2 \hat{\boldsymbol{u}} = -\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}} \tag{36}$$

$$\implies \frac{\partial \hat{\boldsymbol{u}}}{\partial t} - \mathcal{L}\hat{\boldsymbol{u}} = -\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}} \tag{37}$$

where $\mathcal{L}$ is defined by (36). The boundary conditions on $\hat{\boldsymbol{u}}$ are $\hat{\boldsymbol{u}} = 0$ on ∂V. Using energy norm $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int \boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}V$, apply to disturbance equation (37) by taking $\langle \hat{\boldsymbol{u}}, (37) \rangle$ to get

$$\frac{\partial}{\partial t}\langle \frac{1}{2}\hat{\boldsymbol{u}}^2 \rangle = \langle \hat{\boldsymbol{u}}, \mathcal{L}\hat{\boldsymbol{u}} \rangle - \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}} \rangle$$

The last term vanishes since

$$\begin{aligned}\int \hat{\boldsymbol{u}} \cdot (\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}}) \,\mathrm{d}V &= \int \hat{\boldsymbol{u}} \cdot \nabla(\frac{1}{2}\hat{\boldsymbol{u}}^2) \,\mathrm{d}V \\ &= \int \nabla \cdot (\frac{1}{2}\hat{\boldsymbol{u}}^2 \hat{\boldsymbol{u}}) - \cancel{\frac{1}{2}\hat{\boldsymbol{u}}^2 \nabla \cdot} \hat{\boldsymbol{u}} \,\mathrm{d}V \\ &= \oint_{\partial V} \frac{1}{2}\hat{\boldsymbol{u}}^2 \hat{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{S} = 0\end{aligned}$$

We now split the operator $\mathcal{L}$ into symmetric and anti-symmetric parts:

$$\frac{\partial}{\partial t}\langle \frac{1}{2}\hat{\boldsymbol{u}}^2 \rangle = \langle \hat{\boldsymbol{u}}, \frac{1}{2}(\mathcal{L} + \mathcal{L}^\dagger)\hat{\boldsymbol{u}} \rangle + \langle \hat{\boldsymbol{u}}, \frac{1}{2}(\mathcal{L} - \mathcal{L}^\dagger)\hat{\boldsymbol{u}} \rangle$$

Once again the last term vanishes since

$$\langle \hat{\boldsymbol{u}}, \frac{1}{2}(\mathcal{L} - \mathcal{L}^\dagger)\hat{\boldsymbol{u}} \rangle = \frac{1}{2}\langle \hat{\boldsymbol{u}}, \mathcal{L}\hat{\boldsymbol{u}} \rangle - \frac{1}{2}\langle \mathcal{L}\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}} \rangle = 0$$

by definition of the adjoint $\mathcal{L}^\dagger$ and symmetry of the inner product. Thus we have

$$\frac{\partial}{\partial t}\langle \frac{1}{2}\hat{\boldsymbol{u}}^2 \rangle = \langle \hat{\boldsymbol{u}}, \frac{1}{2}(\mathcal{L} + \mathcal{L}^\dagger)\hat{\boldsymbol{u}} \rangle$$

Note that $\mathcal{L} + \mathcal{L}^\dagger$ is a self-adjoint operator: $(\mathcal{L} + \mathcal{L}^\dagger)^\dagger = \mathcal{L}^\dagger + \mathcal{L}$. Hence it is normal with real eigenvalues. For energy growth (LHS > 0) we need an eigenvalue of $\mathcal{L} + \mathcal{L}^\dagger$ to be positive. Recall, for linear instability, we need an eigenvalue of $\mathcal{L}$ to have a positive real part.
Let Re_E be the first Reynold's number for energy growth and Re_L the linear instability threshold. Then generically

$$0 \leq \mathrm{Re}_E \leq \mathrm{Re}_L$$

The first inequality is usually strict (Serrin, 1959) and the region $\mathrm{Re}_E \leq \mathrm{Re} \leq \mathrm{Re}_L$ is the region where energy growth is possible without eigenvalue instability.

Example. For Plane Couette Flow (PCF), $\mathrm{Re}_E = 20.67$ and $\mathrm{Re}_L = \infty$. For $0 \leq \mathrm{Re} < \mathrm{Re}_E$ no growth is possible at all, hence the basic flow $\boldsymbol{U}$ is a global attractor: all initial conditions will decay back to the basic flow.

Example. For a uniformly rotating flow, in the rotating frame the Navier-Stokes equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u}$$

as well as $\nabla \cdot \boldsymbol{u} = 0$ and we use boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on ∂V which is the boundary of some region under consideration (e.g. for an object in the uniformly rotating flow, ∂V is the objects surface). Energy stability analysis gives

$$\frac{\partial}{\partial t}\langle \frac{1}{2}\boldsymbol{u}^2 \rangle = -\frac{1}{\mathrm{Re}}\langle |\nabla \boldsymbol{u}|^2 \rangle < 0$$

so the flow is absolutely stable.

6.8 Time stepping to growth

The matrix/SVD approach to calculating transient growth is only really feasible for 1D or possibly 2D problems because of the size of the matrices. A better approach is a matrix-free method which is extendable to time-dependent problems, different norms, and *adding non-linearity.*

6.8.1 Constrained optimisation (linear)

Consider the energy growth of small disturbances on top of a base flow $\boldsymbol{u}_{\text{lam}}$ by forming the Lagrangian

$$
\begin{aligned}
G &= G(\boldsymbol{u}, p, \lambda, \boldsymbol{\nu}, \pi; T) \\
&= \langle \frac{1}{2} |\boldsymbol{u}(\boldsymbol{x}, T)|^2 \rangle && \text{objective functional} \\
&+ \lambda(\boldsymbol{x}, t) \left[\langle \frac{1}{2} |\boldsymbol{u}(\boldsymbol{x}, 0)|^2 \rangle - 1 \right] && \text{initial amplitude condition} \\
&+ \int_0^T \langle \boldsymbol{\nu}(\boldsymbol{x}, t) \cdot \left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}_{\text{lam}} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_{\text{lam}} + \nabla p - \frac{1}{\text{Re}} \nabla^2 \boldsymbol{u} \right] \rangle \, \mathrm{d}t && \text{satisfies NS equations} \\
&+ \int_0^T \langle \pi(\boldsymbol{x}, t) \nabla \cdot \boldsymbol{u} \rangle \, \mathrm{d}t && \text{satisfies incompressibility}
\end{aligned}
$$

where $\langle \chi \rangle \equiv \int \chi \, \mathrm{d}V$. The first term is the energy functional to be optimisied, and the remaining terms enforce unit initial amplitude, Navier-Stokes, and incompressibility via Lagrange multipliers $\lambda, \boldsymbol{\nu}, \pi$ respectively. The Euler-Lagrange equations to 'stationarise' G follow by expanding perturbations $\boldsymbol{u} + \delta\boldsymbol{u}, p + \delta p$, etc. and requiring the *first variations* $\frac{\delta G}{\delta \boldsymbol{u}}, \frac{\delta G}{\delta p}$, etc. vanish. For variations of p we have

$$
\begin{aligned}
0 = \int_0^T \langle \frac{\delta G}{\delta p} \delta p \rangle \, \mathrm{d}t &\equiv \int_0^T \langle \boldsymbol{\nu} \cdot \nabla \delta p \rangle \, \mathrm{d}t \\
&= \int_0^T \langle \nabla \cdot (\boldsymbol{\nu} p) - \delta p \nabla \cdot \boldsymbol{\nu} \rangle \, \mathrm{d}t \\
&= \int_0^T \oint \boldsymbol{\nu} \delta p \cdot \mathrm{d}\boldsymbol{S} - \langle \delta p \nabla \cdot \boldsymbol{\nu} \rangle \, \mathrm{d}t
\end{aligned}
$$

Hence we require $\boldsymbol{\nu} = \mathbf{0}$ on ∂V and $\nabla \cdot \boldsymbol{\nu} = 0$ throughout V. Now consider variations of $\boldsymbol{u}$ (with $\delta \boldsymbol{u} = \mathbf{0}$ on ∂V)

$$
\begin{aligned}
0 = \int_0^T \langle \frac{\delta G}{\delta \boldsymbol{u}} \cdot \delta \boldsymbol{u} \rangle \, \mathrm{d}t &= \langle \boldsymbol{u}(\boldsymbol{x}, T) \delta \boldsymbol{u}(\boldsymbol{x}, T) \rangle + \lambda \langle \boldsymbol{u}(\boldsymbol{x}, 0) \cdot \delta \boldsymbol{u}(\boldsymbol{x}, 0) \rangle + \int_0^T \langle \pi \nabla \cdot \delta \boldsymbol{u} \rangle \, \mathrm{d}t \\
&+ \int_0^T \langle \boldsymbol{\nu} \cdot \left[\frac{\partial \delta \boldsymbol{u}}{\partial t} + \boldsymbol{u}_{\text{lam}} \cdot \nabla \delta \boldsymbol{u} + \delta \boldsymbol{u} \cdot \nabla \boldsymbol{u}_{\text{lam}} - \frac{1}{\text{Re}} \nabla^2 \delta \boldsymbol{u} \right] \rangle \, \mathrm{d}t \\
&= \langle \delta \boldsymbol{u}(\boldsymbol{x}, T) \cdot [\boldsymbol{u}(\boldsymbol{x}, T) + \boldsymbol{\nu}(\boldsymbol{x}, T)] \rangle + \langle \delta \boldsymbol{u}(\boldsymbol{x}, 0) \cdot [\lambda \boldsymbol{u}(\boldsymbol{x}, 0) - \boldsymbol{\nu}(\boldsymbol{x}, 0)] \rangle \\
&+ \int_0^T \langle \delta \boldsymbol{u} \cdot \left[-\frac{\partial \boldsymbol{\nu}}{\partial t} - \boldsymbol{u}_{\text{lam}} \cdot \nabla \boldsymbol{\nu} + \boldsymbol{\nu} \cdot (\nabla \boldsymbol{u}_{\text{lam}})^T - \nabla \pi - \frac{1}{\text{Re}} \nabla^2 \boldsymbol{\nu} \right] \rangle \, \mathrm{d}t
\end{aligned}
$$

The expression [...] in the final integral is called the *dual (linearised) Navier-Stokes equation.* Hence for $\frac{\delta G}{\delta \boldsymbol{u}} = 0$ we have

$$
\boldsymbol{u}(\boldsymbol{x}, T) + \boldsymbol{\nu}(\boldsymbol{x}, T) = 0 \tag{38}
$$

$$
\lambda \boldsymbol{u}(\boldsymbol{x}, 0) - \boldsymbol{\nu}(\boldsymbol{x}, 0) = 0 \tag{39}
$$

$$
\frac{\partial \boldsymbol{\nu}}{\partial t} + \boldsymbol{u}_{\text{lam}} \cdot \nabla \boldsymbol{\nu} - \boldsymbol{\nu} \cdot (\nabla \boldsymbol{u}_{\text{lam}})^T + \nabla \pi + \frac{1}{\text{Re}} \nabla^2 \boldsymbol{\nu} = 0 \quad \forall t \in [0, T]
$$

These together with equations imposed by the Lagrange multipliers must all be solved. The system of equations are solved iteratively as follows.

1. Choose $\boldsymbol{u}^{(0)}(\boldsymbol{x},0)$ such that
$$\langle \frac{1}{2}\left|\boldsymbol{u}^{(0)}(\boldsymbol{x},0)\right|^2 \rangle = 1$$
Then construct $\boldsymbol{u}^{(n+1)}(\boldsymbol{x},0)$ from $\boldsymbol{u}^{(n)}(\boldsymbol{x},0)$ as follows.

2. Solve the linearised Navier-Stokes equations to time step from $t=0$ to $t=T$ to get $\boldsymbol{u}^{(n)}(\boldsymbol{x},0)$.

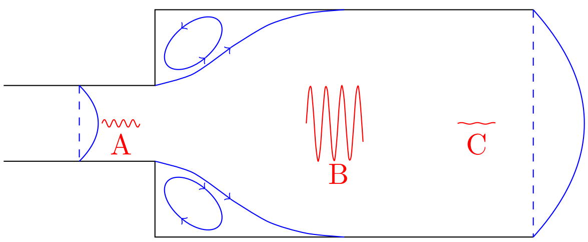
3. Use constraint (38) to find $\boldsymbol{\nu}^{(n)}(\boldsymbol{x},T) = -\boldsymbol{u}^{(n)}(\boldsymbol{x},T)$.

4. Solve the lineariseed dual Navier-Stokes equations to time step backwards from $t=T$ to $t=0$ to get $\boldsymbol{\nu}^{(n)}(\boldsymbol{x},0)$. Note time-stepping backwards is natural as $\boldsymbol{\nu}$ satisfies an 'anti-diffusion' equation of the form $\boldsymbol{\nu}_t \sim -\nabla^2\boldsymbol{\nu}$.

5. Use the result (prior to imposing (39))
$$\frac{\partial G}{\partial \delta \boldsymbol{u}(\boldsymbol{x},0)} = \lambda \boldsymbol{u}^{(n)}(\boldsymbol{x},0) - \boldsymbol{\nu}(\boldsymbol{x},0)$$
to step $\boldsymbol{u}$ to increase G. This may be achieved via *steepest ascent*:
$$\boldsymbol{u}^{(n+1)}(\boldsymbol{x},0) = \boldsymbol{u}^{(n)}(\boldsymbol{x},0) + \varepsilon\left[\frac{\delta G}{\delta \boldsymbol{u}^{(n)}(\boldsymbol{x},0)}\right]$$
i.e. take a small step ε in the direction where G increases. Then
$$\boldsymbol{u}^{(n+1)}(\boldsymbol{x},0) = \boldsymbol{u}^{(n)}(\boldsymbol{x},0) + \varepsilon(\lambda \boldsymbol{u}^{(n)}(\boldsymbol{x},0) - \boldsymbol{\nu}^{(n)}(\boldsymbol{x},0))$$
with λ chosen such that
$$1 = \frac{1}{2}\langle\left|\boldsymbol{u}^{(n+1)}(\boldsymbol{x},0)\right|^2\rangle$$
Here, ε is a small adjustable stepping parameter.

$$\begin{tikzcd}
\boldsymbol{u}^{(n)}(\boldsymbol{x},0) \arrow[r, "t=0 \qquad\qquad t=T", "\text{time-step linearised NS forwards}"'] & \boldsymbol{u}^{(n)}(\boldsymbol{x},T) \arrow[d, "\text{use (38)}"] \\
\boldsymbol{\nu}^{(n)}(\boldsymbol{x},0) \arrow[u, "\substack{\text{steepest ascent}\\ \text{on } \boldsymbol{u}^{(n)}(\boldsymbol{x},0) \text{ to}\\ \text{obtain } \boldsymbol{u}^{(n+1)}(\boldsymbol{x},0)}"] & \boldsymbol{\nu}^{(n)}(\boldsymbol{x},T) \arrow[l, "t=0 \qquad\qquad t=T"', "\text{time-step dual NS backwards}"]
\end{tikzcd}$$

Application. Consider flow through a rapid pipe expansion. The flow has a parabolic profile typical of pipe flow upstream and far downstream of the expansion inlet. Noise is introduced at the inlet A. Transient growth can mean this noise is magnified significantly at B but ultimately decays downstream at C if linearly stable. See Cantwell et al., Phys Fluids, **22** (2010) 034101.



Comments.

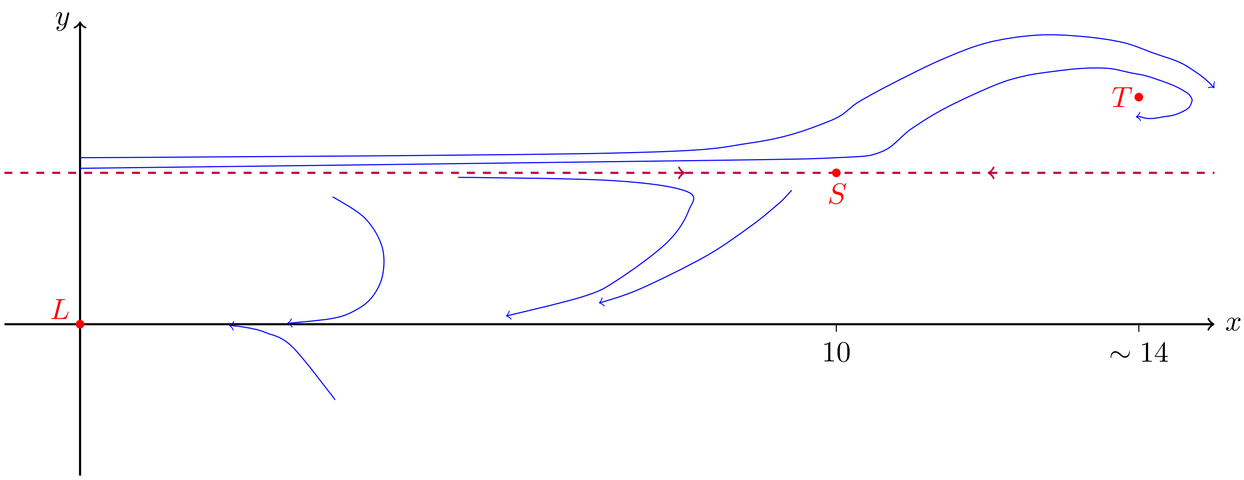
- It is easy to change the objective functional in this method: all difficulties lie in handling the governing equation rather than the objective functional.
- It is easy to handle a basic state $\boldsymbol{u}_{\text{lam}}(\boldsymbol{x}, t)$ which is time dependent. The matrix approach assumes a steady basic state.
- The variational approach can be extended to nonlinear disturbances easily.

6.8.2 Constrained optimisation (nonlinear)

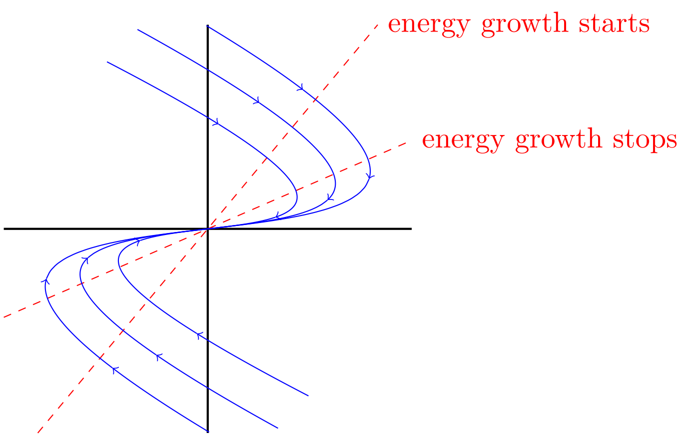
Consider as an example the nonlinear system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{pmatrix} -x + 10y \\ y(10e^{-x^2/100} - y)(y-1) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{x}) \tag{40}$$

See Kerswell et al., Prog. Rep. Phys **77** (2014) 085901 for a more in-depth discussion. A phase portrait of this dynamical system may be plotted with sample trajectories as follows. Red points indicate a fixed point.



The fixed points are labelled L for 'laminar' state, S for the saddle on the basin boundary, and T for 'turbulent' state. The basin of attraction of the fixed point L is $y < 1$. Transient growth appears in a region close to the origin: consider a 'zoomed in' view of the fixed point L. The trajectories temporarily increase distance from the origin, hence there is brief energy growth.



The key question is how to identify the important perturbation of lowest ampltiude which triggers *transition*, i.e. which puts the system in a different basin of attraction. The initial conditions of such a perturbation are referred to as a *minimal seed*. The approach considered here is constrained nonlinear optimisation in the sense that we use the full Navier-Stokes equations as constraints. The growth function is then a function of T *and* the initial energy E_0, since we can no longer assume it is infinitesimally small. New terms arise in the first variation, due to the non-linear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in the Navier-Stokes equations:

$$\begin{aligned}\int_0^T \langle\frac{\delta G}{\delta \boldsymbol{u}}\cdot\delta\boldsymbol{u}\rangle\,\mathrm{d}t &= \cdots + \int_0^T \langle\boldsymbol{\nu}\cdot[\delta\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\delta\boldsymbol{u}]\rangle\,\mathrm{d}t \\ &= \cdots + \int_0^T \langle\delta\boldsymbol{u}\cdot\left[(\nabla\boldsymbol{u})^T\cdot\boldsymbol{\nu}-\boldsymbol{u}\cdot\nabla\boldsymbol{\nu}\right]\rangle\,\mathrm{d}t\end{aligned}$$

The dual Navier-Stokes equation is then

$$-\frac{\partial\boldsymbol{\nu}}{\partial t}+[\nabla(\boldsymbol{u}+\boldsymbol{u}_{\mathrm{lam}})]^T\cdot\boldsymbol{\nu}-(\boldsymbol{u}+\boldsymbol{u}_{\mathrm{lam}})\cdot\nabla\boldsymbol{\nu}-\nabla\pi-\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{\nu}=0$$

Notice the following:

- We now need to integrate the *full* Navier-Stokes equations forward in time.
- Dual Navier-Stokes remains linear in $\boldsymbol{\nu}$ *but* it now depends on $\boldsymbol{u}$. A question of implementation is then whether to store the velocity field, or recalculate on the fly.
- G depends on the initial energy E_0 as well as time T.

Returning to the example (40), we have

$$G = x^2(T)+y^2(T)-\lambda\left[E_0-x^2(0)-y^2(0)\right]+\int_0^T \boldsymbol{\nu}(t)\cdot\left[\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}-\boldsymbol{f}(\boldsymbol{x})\right]\,\mathrm{d}t$$

Consider the $\boldsymbol{\nu}\cdot\boldsymbol{f}$ term:

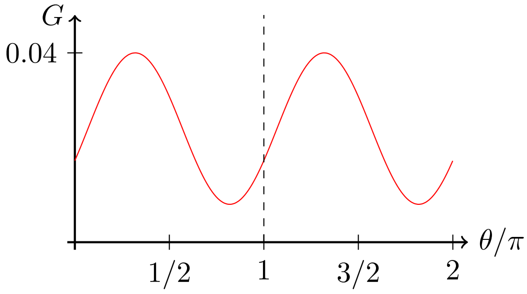
$$\begin{aligned}-\boldsymbol{\nu}\cdot\boldsymbol{f} &= -\nu_1(-x+10y)-\nu_2(y^2-y)(10e^{-x^2/100}-y)\\ \delta_{\boldsymbol{x}}(-\boldsymbol{\nu}\cdot\boldsymbol{f}) &= \nu_1\delta x-10\nu_1\delta y-\nu_2(2y-1)\delta y(10e^{-x^2/100}-y)-\nu_2(y^2-y)(-\frac{x}{5}e^{-x^2/100}\delta x-\delta y)\\ &= \delta\boldsymbol{x}\cdot\begin{pmatrix}\nu_1+\frac{x}{5}\nu_2(y^2-y)e^{-x^2/100}\\ -10\nu_1-\nu_2(2y-1)(10e^{-x^2/100}-y)+\nu_2(y^2-y)\end{pmatrix}\end{aligned}$$

Hence the Euler-Lagrange equations are

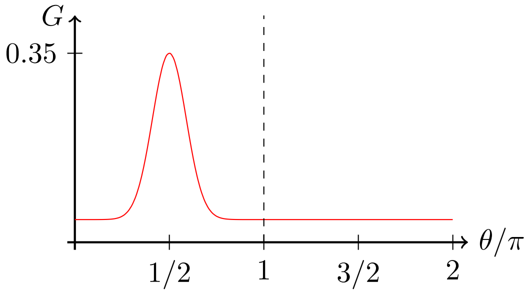
$$\begin{aligned}\frac{\delta G}{\delta\boldsymbol{x}} &= -\dot{\boldsymbol{\nu}}+\begin{pmatrix}\nu_1+\frac{x}{5}\nu_2(y^2-y)e^{-x^2/100}\\ -10\nu_1-\nu_2(2y-1)(10e^{-x^2/100}-y)+\nu_2(y^2-y)\end{pmatrix}=0\\ \frac{\delta G}{\delta\boldsymbol{\nu}} &= \dot{\boldsymbol{x}}-\boldsymbol{f}(\boldsymbol{x})=0\\ \frac{\delta G}{\delta\boldsymbol{x}(T)} &= 2\boldsymbol{x}(T)+\boldsymbol{\nu}(T)=0\\ \frac{\delta G}{\delta\boldsymbol{x}(0)} &= 2\lambda\boldsymbol{x}(0)-\boldsymbol{\nu}(0)=0\end{aligned}$$

Note that for constant E_0 the initial conditions lie on a circle in the (x,y) plane. Hence parametrise the initial conditions for fixed E_0 by angle θ from the x-axis. Choose $T=2$ (arbitrary, but illustrative choice).

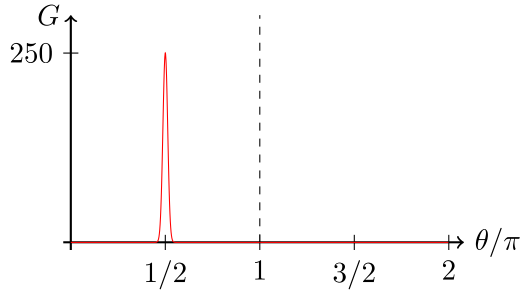
- For initial energy $E_0 = 10^{-8}$ we find G very small, an approximately linear picture (since small energy) and symmetry $\theta \to \theta + \pi$.



- Initial energy $E_0 = 0.9$. We now find asymmetry in θ and a peak near $\theta = \pi/2$, indicating some preference in initial conditions. G is also slightly larger.

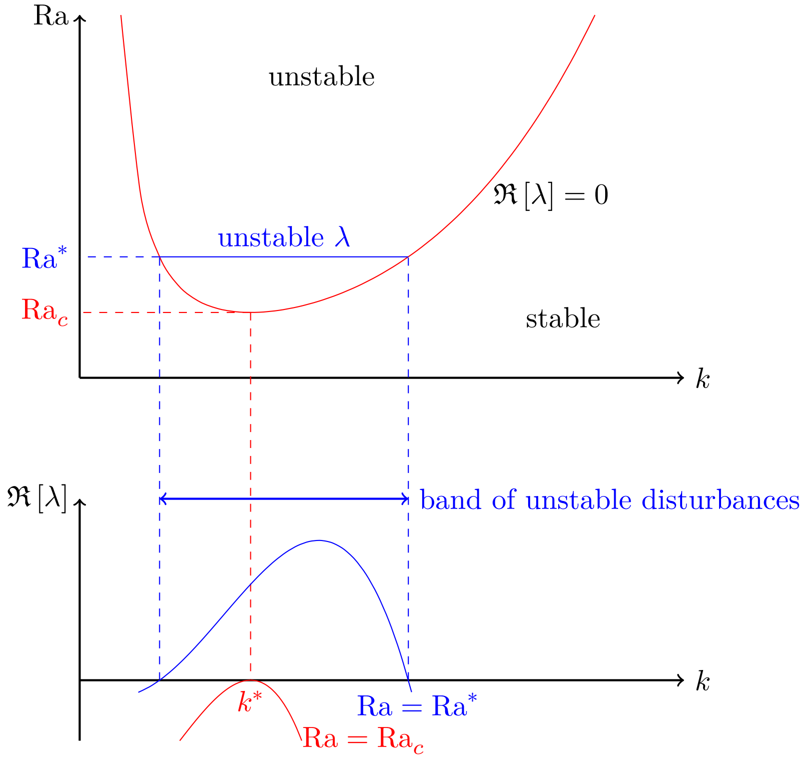


- Initial energy $E_0 = 1.0001$. A *minimal seed* is clearly identified for transition, i.e. the first time an initial condition exists which will move to another basin of attraction. Note that G is significantly larger despite a small increase in E_0: this is indicate of the fact the trajectory moves far from the origin, approaching the T fixed point and consequently the energy (distance from origin) is much larger.



7 Weakly nonlinear analysis

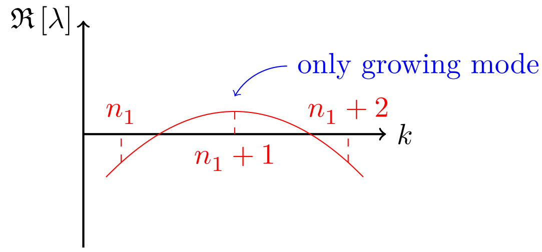
We now allow disturbances to have finite amplitude. In this section we ask what happens ‘beyond’ a linear instability. Consider for example the marginal stability curve for Rayleigh-Benard convection. When $\mathrm{Ra} = \mathrm{Ra}^* > \mathrm{Ra}_c$ we find a band of unstable eigenvalues corresponding to disturbances which (initially) grow exponentially.



Unstable disturbances cannot keep growing exponentially. Eventually, dispersion and more importantly nonlinear effects exert their influences to limit growth. In general, there are two situations:

- **Small systems**: allowable wavenumbers k are discretised and so if Ra (or an equivalently parameter, e.g. Re) is sufficiently close to (but larger than) Ra_c, only *one* normal mode is unstable. In this case, there is no need to worry about dispersion; only nonlinear effects are important and the amplitude of disturbances is governed by the *Landau equation* to be derived later.

  For example, consider a periodic flow $u(x) = u(x + L)$: e^{ikx} is a solution if $e^{ikL} = 1$, i.e. $k = \pm\frac{2\pi n}{L}$ for $n \in \mathbb{N}$. Hence wavenumbers are discretised. For small L, only one of these wavenumbers will yield a growing mode.

- **Spatially-extended systems**: here, there really is an unstable band of wavenumbers and we must include dispersion as well. The amplitude of disturbances is governed by the *Ginsburg-Landau equation*, derived later.

Analysis for both cases can be severe. In the following section we illustrate ideas using a model problem.

7.1 Weakly nonlinear theory: small systems

Consider the following problem as a simplifed proxy to the Navier-Stokes equations, retaining nonlinear terms.

$$\begin{gathered}\frac{\partial u}{\partial t} - \sin u = \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial z^2} \\ u = u(z,t) = 0 \ \text{ at } \ z = 0, \pi\end{gathered} \tag{41}$$

The base state is $u = 0$. Consider small disturbances: the linearised perturbation u' satisfies

$$\begin{gathered}u'_t - u' = \frac{1}{\mathrm{Re}}u'_{zz} \\ u' = 0 \ \text{ at } \ z = 0, \pi\end{gathered}$$

Solve using Fourier series: assume u' is of the form

$$u'(z,t) = \sum_{n=1}^{\infty} A_n(t)\sin nz$$

which implicitly satisfies $u'(0) = 0$. Then

$$\begin{aligned}\frac{\mathrm{d}A_n}{\mathrm{d}t} - A_n &= -\frac{n^2}{\mathrm{Re}}A_n \\ \frac{\mathrm{d}A_n}{\mathrm{d}t} &= \left(1 - \frac{n^2}{\mathrm{Re}}\right)A_n \\ \implies A_n(t) &= A_n(0)e^{\left(1-\frac{n^2}{\mathrm{Re}}\right)t}\end{aligned}$$

From this solution we see instability arises if $1 - n^2/\mathrm{Re} > 0$, i.e. $\mathrm{Re} > n^2$. Hence the critical Reynolds number is $\mathrm{Re}_c = 1$ and for $1 < \mathrm{Re} < 4$ there exists precisely one unstable mode. The key idea is that A_1 will grow *until nonlinear terms become important.* Thus to make progress in the analysis we wish to determine when nonlinear terms reach an order comparable to linear terms. Let u' have an amplitude ε, i.e

$$u' = \varepsilon A(t)\sin z$$

where we assume $A = \mathcal{O}(1)$. Return to the full problem (41) with $\mathrm{Re} = \mathrm{Re}_c + \Delta\mathrm{Re}$ where $\Delta\mathrm{Re} \ll \mathrm{Re}_c = 1$. We have

$$\begin{gathered}\varepsilon\frac{\partial A}{\partial t}\sin z - \left[\varepsilon A\sin z - \frac{\varepsilon^3 A^3 \sin^3 z}{3!} + ...\right] = \frac{1}{\mathrm{Re}_c + \Delta\mathrm{Re}}(-\varepsilon A\sin z) \\ \implies \varepsilon\frac{\partial A}{\partial t}\sin z - \varepsilon A\sin z + \frac{\varepsilon^3}{6}A^3\sin^3 z + ... = -\frac{\varepsilon A\sin z}{\mathrm{Re}_c}\left[1 - \frac{\Delta\mathrm{Re}}{\mathrm{Re}_c} + ...\right]\end{gathered}$$

The leading balance is thus (noting $A_t = 0$ at marginal stability)

$$-\varepsilon A\sin z = -\frac{\varepsilon A\sin z}{\mathrm{Re}_c}$$

leaving next order terms

$$\varepsilon\frac{\partial A}{\partial t}\sin z + \frac{\varepsilon^3}{6}A^3\sin^3 z = \frac{\Delta\mathrm{Re}}{\mathrm{Re}_c^2}\varepsilon A\sin z$$

The first term is $\mathcal{O}(\varepsilon A_t)$, the second term is $\mathcal{O}(\varepsilon^3)$ and the last term is $\mathcal{O}(\varepsilon \Delta \mathrm{Re})$. We wish to balance terms, hence scalings are set by

$$\frac{\partial A}{\partial t} = \mathcal{O}(\varepsilon^2) = \Delta \mathrm{Re}$$

Given this 'rough' analysis, we now construct a formal expansion. Let

$$u = \varepsilon u_1 + \varepsilon^3 u_3 + ...$$

where $u_1 = A(T)\sin z$ and $T = \varepsilon^2 t$ (cf. method of multiple scales, Perturbation Methods) hence $\partial_t = \varepsilon^2 \partial_T$. Further, let

$$\mathrm{Re} - \mathrm{Re}_c = \varepsilon^2 \mathrm{Re}_2 = \Delta \mathrm{Re}$$

where $\mathrm{Re}_2 = \pm 1$ depending on whether $\mathrm{Re} > \mathrm{Re}_c$ or $\mathrm{Re} < \mathrm{Re}_c$ respectively. We assume $\mathrm{Re} > \mathrm{Re}_c$, hence $\varepsilon = \sqrt{\mathrm{Re} - \mathrm{Re}_c}$. Substitute into (41) to get

$$\varepsilon^2 \frac{\partial}{\partial T}\left[\varepsilon u_1 + \varepsilon^3 u_3 + ...\right] - \sin\left(\varepsilon u_1 + \varepsilon^3 u_3 + ...\right) = \frac{1}{\mathrm{Re}_c + \varepsilon^2} \frac{\partial^2}{\partial z^2}\left[\varepsilon u_1 + \varepsilon^3 u_3 + ...\right]$$

$$\varepsilon^2 \frac{\partial}{\partial T}\left[\varepsilon u_1 + \varepsilon^3 u_3 + ...\right] - (\varepsilon u_1 + \varepsilon^3 u_3) + \frac{(\varepsilon u_1 + \varepsilon^3 u_3)^3}{3!} + ... = \frac{1}{\mathrm{Re}_c}\left(1 - \frac{\varepsilon}{\mathrm{Re}_c}\right)\frac{\partial^2}{\partial z^2}\left[\varepsilon u_1 + \varepsilon^3 u_3 + ...\right]$$

Equate terms of equal power in ε:

$$\mathcal{O}(\varepsilon): \qquad -u_1 = \frac{1}{\mathrm{Re}_c}\frac{\partial^2 u_1}{\partial z^2} \tag{42}$$

$$\mathcal{O}(\varepsilon^3): \qquad \frac{\partial u_1}{\partial T} - u_3 + \frac{1}{6}u_1^3 = \frac{1}{\mathrm{Re}_c}\frac{\partial^2 u_3}{\partial z^2} - \frac{1}{\mathrm{Re}_c^2}\frac{\partial^2 u_1}{\partial z^2} \tag{43}$$

At first order, (42) is satisfied by $u_1 = A(T)\sin z$ with $\mathrm{Re}_c = 1$ as expected. Rewrite (43):

$$\left[\frac{1}{\mathrm{Re}_c}\frac{\partial^2}{\partial z^2} + 1\right] u_3 = \frac{\partial u_1}{\partial T} + \frac{1}{6}u_1^3 + \frac{1}{\mathrm{Re}_c^2}\frac{\partial^2 u_1}{\partial z^2} \tag{44}$$

with $u_3 = 0$ at $z = 0, \pi$. This is a forced equation for u_3 with homogeneous boundary conditions.

Fredholm alternative. To proceed, we must import some operator theory which will provide a condition for solutions of (44) to exist and ultimately yield the Landau equation. For a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$,

1. if $A\boldsymbol{x} = 0$ has no non-trivial solution, A is invertible hence

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

   is the unique solution.

2. if $\det A = 0$ then either $\boldsymbol{b} \in \mathrm{range}(A)$ and there are an infinite number of solutions, or $\boldsymbol{b} \notin \mathrm{range}(A)$ and there are no solutions. Existence of a solution can be deduced without determining $\mathrm{range}(A)$: if $\det A = 0$ then there exists $\boldsymbol{y}$ such that $\boldsymbol{y}^T A = 0$. Hence taking the dot product of the matrix equation with $\boldsymbol{y}^T$ (on the left) gives

$$\boldsymbol{y}^T A \boldsymbol{x} = 0 = \boldsymbol{y}^T \boldsymbol{b}$$

   for a solution. This is a *solvability condition.*

The *Fredholm alternative* is the analogous operator version of the above. The PDE (44) has either

- a unique solution if
$$\left[\frac{1}{\mathrm{Re}_c}\frac{\partial^2}{\partial z^2}+1\right]u_3=0$$
has no (non-trivial) solutions (so operator invertible);

- no solution or an infinite number of solutions if
$$\left[\frac{1}{\mathrm{Re}_c}\frac{\partial^2}{\partial z^2}+1\right]u_3=0$$
has a solution.

We already know that
$$\left[\frac{1}{\mathrm{Re}_c}\frac{\partial^2}{\partial z^2}+1\right]u_1=0$$
so a solvability condition must be satisfied for u_3 to exist, i.e. for our expansion to make sense. This leads to the Landau equation without solving for u_3!
Multiply (44) by u_1 (equivalently multiply by $\sin z$) and integrate over the domain $z\in[0,\pi]$:

$$\int_0^\pi \sin z\left[\frac{1}{\mathrm{Re}_c}\frac{\partial^2}{\partial z^2}+1\right]u_3\,\mathrm{d}z=\int_0^\pi \sin z\left(\frac{\partial u_1}{\partial T}+\frac{1}{6}u_1^3+\frac{1}{\mathrm{Re}_c^2}\frac{\partial^2 u_1}{\partial z^2}\right)\mathrm{d}z$$

Denote the operator applied to u_3 in the integrand by $\mathcal{L}$ (note $\mathcal{L}u_1=0$). By noting that $\mathcal{L}$ is self-adjoint, or equivalently integrating by parts, we have

$$\int_0^\pi u_3\cancel{\mathcal{L}\sin z}\,\mathrm{d}z=\int_0^\pi \sin z\left(\frac{\partial u_1}{\partial T}+\frac{1}{6}u_1^3+\frac{1}{\mathrm{Re}_c^2}\frac{\partial^2 u_1}{\partial z^2}\right)\mathrm{d}z$$
$$\implies \int_0^\pi \sin z\left(\frac{\partial u_1}{\partial T}+\frac{1}{6}u_1^3+\frac{1}{\mathrm{Re}_c^2}\frac{\partial^2 u_1}{\partial z^2}\right)\mathrm{d}z=0$$
$$\implies \int_0^\pi \sin z\left[\sin z\frac{\mathrm{d}A}{\mathrm{d}T}+\frac{1}{6}A^3\sin^3 z-\frac{A}{\mathrm{Re}_c^2}\sin z\right]\mathrm{d}z=0$$

Evaluating the integral yields the *Landau equation* governing A:

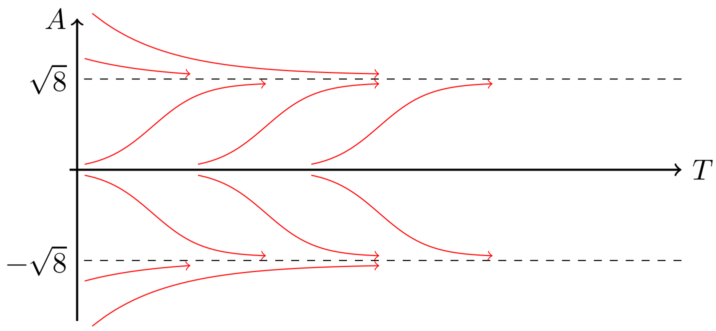
$$\frac{\mathrm{d}A}{\mathrm{d}T}=\frac{1}{\mathrm{Re}_c^2}A-\frac{1}{8}A^3=A-\frac{1}{8}A^3$$

If A^3 is small, we get exponential growth on slow timescale $\varepsilon^2 t$. When A becomes large enough, A_T becomes negative and growth stops. Note the following:

- As $\mathrm{Re}\to\mathrm{Re}_c$ the leading spatial character of the flow is determined by the unstable eigenfunction of the linear problem, and temporal evolution is given by the Landau equation.

- Notice the absence of an A^2 term: it is missing due to reflectional syummetry of the original problem (if u is a solution, so is $-u$), a property we expect to carry over to the Landau equation.

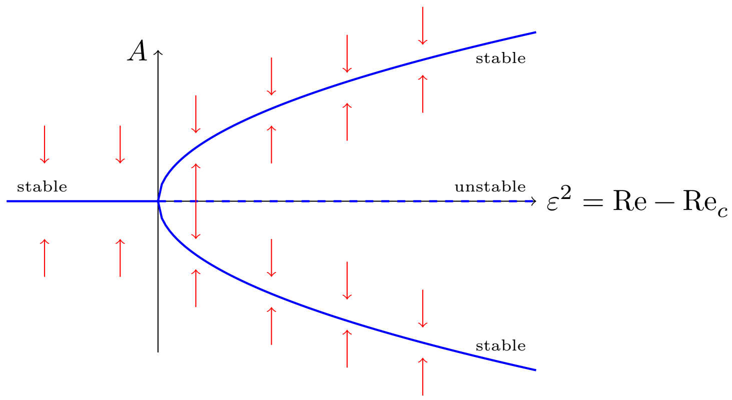
- By plotting sample trajectories and considering the sign of $\mathrm{d}A/\mathrm{d}T$ we can determine the long-term behaviour of solutions to the Landau equation.



If $A(0) > 0$, then $A(T) \to \sqrt{8}$ as $T \to \infty$. If $A(0) < 0$, then $A(T) \to -\sqrt{8}$ as $T \to \infty$. Hence

$$u \to \pm\varepsilon\sqrt{8}\sin z = \pm\sqrt{8(\mathrm{Re} - \mathrm{Re}_c)}\sin z$$

- Plotting A against ε^2 shows the stability of steady solutions to the Landau equation. Solid lines are stable, dashed lines are unstable.



  Arrows indicate the direction solutions move through the (A, ε^2) phase space, based on the sign of A_T. Eventually, all solutions end up on stable branches. The situation demonstrated here is referred to as a *supercritical pitchfork bifurcation* at $T = 0$.

- The Landau equation can be extended, e.g.

$$A_T = \alpha_1 A + \alpha_2 A^3 + \alpha_3 A^5 + ...$$

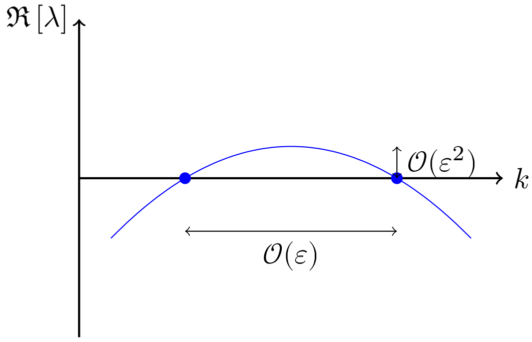
  An A^5 term would further complicate the phase plot given above, potentially adding a saddle bifurcation to the system.

- A *subcritical pitchfork bifurcation* requires $\alpha_2 > 0$.

- If the frequency were non-zero at the bifurcation point $T = 0$ we would instead find a *Hopf bifurcation*. This does not occur in this model problem or convection in general, but does arise in shear flows.

7.2 Weakly nonlinear theory: extended systems

In the case of spatially extended systems, it is no longer valid to assume only one mode is unstable for $\varepsilon^2 > 0$. We now need to include the fact that dispersion will occur, meaning energy moves between wavenumbers.



By assumption, we have that the unstable of range of wavenumbers is $\mathcal{O}(\varepsilon)$ wide, so $\Re[\lambda] = \mathcal{O}(\varepsilon^2)$ since a maximum is locally parabolic. Hence look for a solution

$$u = \varepsilon \left(A(X,T)e^{ikx} + A^*(X,T)e^{-ikx} \right) + \text{h.o.t.}$$

where $T = \varepsilon^2 t$ and $X = \varepsilon x$. This gives rise to an additional term of the form A_{XX} in the Landau equation, which now becomes

$$A_T = \alpha A + \beta A_{XX} \pm |A|^2 A$$

and in general A is complex. This is the *Ginsburg-Landau equation.* This equation describes, for example, the slow modulation of rolls just above criticality in Rayleigh-Benard convection.

7.2.1 Spatially periodic solutions

Consider solutions to the Ginsburg-Landau equation of the form $A(X,T) = ae^{i\kappa X}$. Then

$$a(\alpha - \beta\kappa^2) - a^3 = 0$$
$$\implies a = \sqrt{\alpha - \beta\kappa^2}$$

where the negative sign has been chosen to yield a balance. This corresponds to a total solution

$$u = \varepsilon\sqrt{\alpha - \beta K^2}e^{i(k+\varepsilon\kappa)x}$$

The maximum amplitude occurs at $\kappa = 0$, and the amplitude tends to 0 as $\kappa \to \sqrt{\alpha/\beta}$. Analysis of the stability of these states yields *Eckhaus instability.*

7.2.2 Localised solutions

Stable flows may have localised solutions as the nonlinearity is destabilising, so that in some parts of the domain a large amplitude disturbance may equilibriate only in that region. If bifurcation is subcritical and $\alpha < 0$, i.e. $\mathrm{Re} < \mathrm{Re}_c$, then

$$A_T = \alpha A + \beta A_{XX} + A^3$$

Look for a steady, real solution, hence

$$A_{XX} = \lambda A - \mu A^3$$

where $\lambda = -\alpha/\beta > 0$ and $\mu = 1/\beta > 0$. Rearrange to get

$$\frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{1}{2}A_X^2\right) = \lambda A - \mu A^3$$
$$A_X = A\sqrt{\lambda - \frac{1}{2}\mu A^2}$$
$$\implies \int \frac{\mathrm{d}A}{A\sqrt{\lambda - \frac{1}{2}\mu A^2}} = X$$

Note that the constant of integration has been disregarded for convenience (simply translates the solution). The solution is then

$$A = \sqrt{\frac{2\lambda}{\mu}}\,\mathrm{sech}\big(\sqrt{\lambda}X\big)$$

which corresponds to a total solution

$$u = \varepsilon\sqrt{\frac{2\lambda}{\mu}}\,\mathrm{sech}\big(\varepsilon\sqrt{\lambda}x\big)e^{ikx}$$

The sech factor localises the solution whilst the e^{ikx} factor provides global oscillation.