Cambridge Part III Maths

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Hydrodynamic Stability

based on a course given by
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Contents

1 Introduction

We are typically interested in whether a given flow solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{u}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}, t)$ and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2}\hat{\boldsymbol{u}}^2(\boldsymbol{x}, t)\,\mathrm{d}V$$

A solution is said to be stable if

$$\lim_{t\to\infty} \frac{E(t)}{E(0)} = 0$$

for all perturbations $\hat{\boldsymbol{u}}$. Conversely, if there exists $\hat{\boldsymbol{u}}$ such that $E(t) \nrightarrow 0$ then $\boldsymbol{u}$ is unstable. The nature of $E(0)$ determines the type of perturbation:

- If $E(0) \to 0$ we have an infinitesimal disturbance
- If $E(0) < \delta$ then we probe finite amplitude disturbances
- If $E(0) \to \infty$ this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}$$

If $\boldsymbol{U}(\boldsymbol{x})$ is a steady (basic) solution then

$$\boldsymbol{U}\cdot\nabla\boldsymbol{U}+\nabla P=\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{U}$$

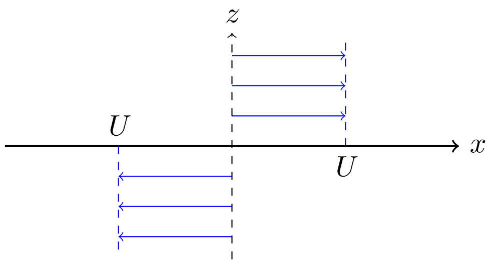
Let $\boldsymbol{u}=\boldsymbol{U}(\boldsymbol{x})+\hat{\boldsymbol{u}}(\boldsymbol{x},t), p=P+\hat{p}$. Then

$$\frac{\partial\hat{\boldsymbol{u}}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{U}+\boldsymbol{U}\cdot\nabla\hat{\boldsymbol{u}}+\cancel{\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}}+\nabla\hat{p}=\frac{1}{\mathrm{Re}}\nabla^2\hat{\boldsymbol{u}}$$

The term $\hat{u}\cdot\nabla\boldsymbol{U}$ is stabilising whilst the term $\nabla^2\hat{u}/\mathrm{Re}$ is stabilising. Therefore, we expect stability as $\mathrm{Re}\to 0$ as this term dominates, and instability as $\mathrm{Re}\to\infty$. Thus there exists some value $\mathrm{Re}_{\mathrm{crit}}$ at which instability arises. We will ask what this value is, and what is the form of initial instability/mode/pattern?

2 Kelvin-Helmholtz instability

See Drazen (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazen).



Consider a flow $\boldsymbol{u}=U(z)\hat{\boldsymbol{x}}$ where

$$U(z)=\begin{cases}U & z>0\\ -U & z<0\end{cases}$$

The linearised, *inviscid* equation for perturbation $\hat{\boldsymbol{u}}$ is

$$\frac{\partial\hat{\boldsymbol{u}}}{\partial t}+\hat{w}U'\hat{\boldsymbol{x}}+U\frac{\partial\hat{\boldsymbol{u}}}{\partial x}+\nabla\hat{p}=0$$
$$\nabla\cdot\hat{\boldsymbol{u}}=0$$

The boundary conditions are $\hat{\boldsymbol{u}}\to 0$ as $z\to\pm\infty$, i.e. no energy radiated in from infinity. We will work in 2D $(\hat{u},\hat{w})=(\psi_z,-\psi_x)$ and let $\psi(x,z,t)=\phi(z)e^{i\alpha(x-ct)}$ where c is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U-c)\begin{pmatrix}\phi'\\ -i\alpha\phi\end{pmatrix}+\begin{pmatrix}-i\alpha U'\phi\\ 0\end{pmatrix}+\begin{pmatrix}i\alpha p\\ \frac{\partial p}{\partial z}\end{pmatrix}=0$$

We can eliminate p via $\partial_z(\text{top})-i\alpha(\text{bottom})$ to get

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi=0$$

with boundary conditions $\phi\to 0$ as $z\to\pm\infty$. This is *Rayleigh's equation.* Note that c is the crucal eigenvalue. We wish to know when $c_i=\Im(c)>0$ as a function of $U(z)$, as c_i is the growth rate:

$$\hat{u}\propto e^{i\alpha(x-ct)}=e^{i\alpha(c-c_r t-ic_i t)}=e^{i\alpha(x-c_r t)+\alpha c_i t}$$

Note the following:

- There is a symmetry $\alpha \mapsto -\alpha$, so without loss of generality we consider $\alpha > 0$.
- The complex conjugate is also a solution with $c \mapsto c^*$. Hence an unstable mode has a damped partner, so we have stability only if all modes are 'neutral' i.e. $c_i = 0$.
- There is a possible singularity at y where $U(y) = c$, called the *critical layer*. If c is real, see later.

We now solve Rayleigh's equation with $U(z)$ defined as before. We solve above and below $z = 0$ and piece the solutions together. Since $U'' = 0$, we have

$$\phi'' = \alpha^2 \phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ Be^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at $z = 0$ are

1. Pressure $\hat{p}$ continuous at $z = 0$, with $\hat{p}$ given by:

$$\hat{p} = U'\phi - (U - c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z - \zeta(x,t)\right) = 0$$

where $z = \zeta(x,t)$ is the position of the surface. After linearising, we have

$$w - \frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} = 0$$

Inserting the form of w and U we require that

$$\zeta = -\frac{\phi}{U - c}$$

is continuous across $z = 0$.

Requiring p continuous gives

$$-(U - c)A(-\alpha) = -(-U - c)B(\alpha)$$

Requiring ζ continuous gives

$$\frac{A}{U - c} = \frac{B}{-U - c}$$

Hence we have

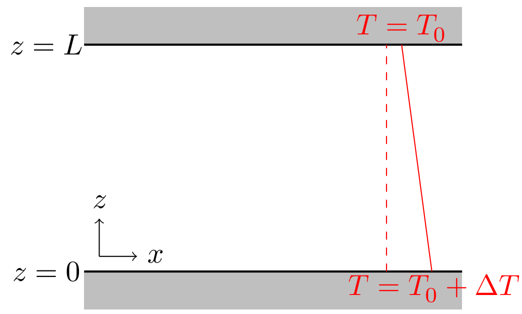
$$(U - c)^2 = -(U + c)^2$$

i.e. $c = \pm iU$ so the growth rate is αU. Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small α)
- by adding surface tension, which stabilises short wavelengths (large α), e.g. Drazen page 50 equation 3.21.

3 Thermal instabilities: convection

Consider two parallel plates separated by distance L with fluid subject to gravity and temperatue difference ΔT between the plates. The lower plate is heated.



The basic state consists of no motion, with heat transfer by conduction only.

Governing equations. The governing equations are

$$\begin{aligned} \rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p &= \mu \nabla^2 \boldsymbol{u} + \rho g \hat{\boldsymbol{z}} \\ \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T &= \kappa \nabla^2 T \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

We need a relationship between ρ and T. Most cases of interest have ΔT and $\Delta\rho$ small, i.e. $\Delta\rho \ll \rho_0, \Delta T \ll T_0$. Two consequences of this assumption are:

1. We can Taylor expand $\rho = \rho(T)$:

   $$\rho \approx \rho(T_0)\left[1 - \alpha(T - T_0)\right]$$

   where $\alpha > 0$ is the coefficient of thermal expansion, such that T increases when ρ decreases. We write $\rho_0 = \rho(T_0)$.

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term $\rho g \hat{\boldsymbol{z}}$. Importantly, we can assume the fluid is incompressible.

Define $\theta = T - T_0$. The governing equations are now

$$\begin{aligned} \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p &= \mu \nabla^2 \boldsymbol{u} + \rho_0(1 - \alpha\theta) g \hat{\boldsymbol{z}} \\ \frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta &= \kappa \nabla^2 \theta \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

The basic state is $u = 0, \theta = \Delta T(1 - z/L)$ and

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_0(1 - \alpha \Delta T(1 - z/L))g$$

We now non-dimensionalise using scalings $t \sim L^2/\kappa, u \sim \kappa/L, \theta \sim \Delta T$, e.g. $\theta = \Delta T \theta^*$ where θ^* is the non-dimensionalised variable. We normalise the $\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*}$ term, to get:

$$\frac{\mathrm{D}\boldsymbol{u}^*}{\mathrm{D}t^*} + \nabla^* p^* = \frac{\mu}{\rho_0 \kappa}\nabla^{*2}\boldsymbol{u}^* + \frac{\alpha g \Delta T L^3}{\kappa^2}\theta^* \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla^* \theta^* = \nabla^{*2}\theta^*$$

Define the *Prandtl number*

$$\sigma \equiv \frac{\nu}{\kappa} = \frac{\mu}{\rho_0 \kappa}$$

which is the ratio of viscous/momentum to thermal diffusivity. Typical values are 0.72 in air, 7 in water, 10^5 in magma. We also define the *Rayleigh number*

$$\mathrm{Ra} \equiv \frac{\alpha \Delta T g L^3}{\kappa \nu}$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the * notation, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \sigma \nabla^2 \boldsymbol{u} + \sigma \mathrm{Ra} \theta \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \nabla^2 \theta$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Boundary conditions. There are three combinations of boundary condition available in this problem.

$\theta = 0$, $z = 1$	Fixed wall: $\boldsymbol{u} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	Free slip: $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$, $z = 0$	$\boldsymbol{u} = 0$: Fixed wall	$\boldsymbol{u} = 0$: Fixed wall	$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$: Free slip

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

Basic state. In the basic state we have conductive profile $\boldsymbol{u}_0 = 0, \theta_0 = 1 - z$ and from integration $p_0 = \sigma \mathrm{Ra}(z - \frac{1}{2}z^2)$. We generate linearised equations for perturbations $\theta = \theta_0 + \theta', \boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}', p = p_0 + p'$. As usual with linear stability analysis, we assume $(\theta, \boldsymbol{u}', p')$ are small.

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \cancel{\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'} + \nabla p' = \sigma \nabla^2 \boldsymbol{u}' + \sigma \mathrm{Ra} \theta' \hat{\boldsymbol{z}}$$
$$\frac{\partial \theta'}{\partial t} - w' + \cancel{\boldsymbol{u}' \cdot \nabla \theta'} = \nabla^2 \theta'$$
$$\nabla \cdot \boldsymbol{u}' = 0$$

Dropping the $'$ notation for clarity we have perturbation equations

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)\boldsymbol{u}+\nabla p=\sigma\mathrm{Ra}\theta\hat{\boldsymbol{z}} \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0 \tag{2}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\theta=w \tag{3}$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g. $\theta=\theta_0+\theta'=1$ at $z=0$ combined with $\theta_0=1$ at $z=0$ gives $\theta'=0$. Similarly, $\theta'=0$ at $z=1$ and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.
From $\nabla\times$(1) we have

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)\boldsymbol{\omega}=\sigma\mathrm{Ra}\nabla\times\theta\hat{\boldsymbol{z}}$$

Taking the curl again and using $\nabla\times\boldsymbol{\omega}=\nabla\times(\nabla\times\boldsymbol{u})=\nabla(\nabla\cdot\boldsymbol{u})-\nabla^2\boldsymbol{u}$ we have

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)(-\nabla^2\boldsymbol{u})=\sigma\mathrm{Ra}\nabla\times(\nabla\times\theta\hat{\boldsymbol{z}})=\sigma\mathrm{Ra}\left(\nabla\frac{\partial\theta}{\partial z}-\hat{\boldsymbol{z}}\nabla^2\theta\right)$$

The z component is

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)(-\nabla^2 w)=\sigma\mathrm{Ra}\nabla_H^2\theta \tag{4}$$

where $\nabla_H^2=\partial_x^2+\partial_y^2$. Now (3) can be used to eliminate θ by applying the operator $(\partial_t-\nabla^2)$:

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w=\sigma\mathrm{Ra}\nabla_H^2 w$$

This is a 6^{th} order PDE for w, hence we need three boundary conditions at each wall $z=0,1$. We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=0 \qquad \text{at } z=0,1$$

The second set of conditions comes from incompressibility. Taking $\partial_z(\nabla\cdot\boldsymbol{u})$ we have

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z}\right)+\frac{\partial^2 w}{\partial z^2}=0 \implies w_{zz}=0$$

The third and final set of conditions comes from requiring $\theta=0$ at $z=0,1$. From (3), $\nabla_H^2\theta=0$ implies

$$\left(\frac{\partial}{\partial t}-\sigma\nabla^2\right)\nabla^2 w=0$$

We now have 6 boundary conditions to supplement the PDE.