Cambridge Part III Maths

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Fluid Dynamics of Climate

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Contents

Lecture 1
12/10/20

1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p - \rho \nabla \phi + \rho \boldsymbol{F}$$

The body forces are assumed to be conservative with potential ϕ, e.g. $\phi = gz$ for gravitational force. $\boldsymbol{F}$ is the frictional force.
Consider a reference frame rotating about the z-axis with constant angular velocity $\boldsymbol{\Omega}$. Axes in the inertial frame are denoted with a subscript $_I$ and axes in the rotating frame are denoted with a subscript $_R$.

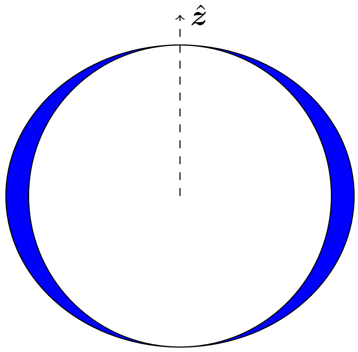
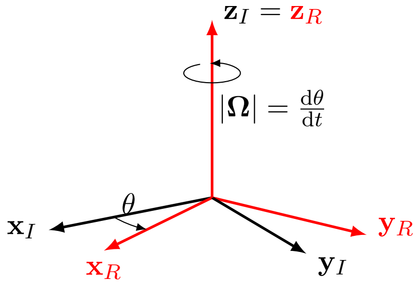


Figure 1: Geopotential ocean surface relative to a spherical Earth.



For a point with position vector $\boldsymbol{x}$ and velocity $\boldsymbol{u}_R = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R$ in the rotating reference frame

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{x}$$

or equivalently $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}$. Hence the acceleration is

$$\begin{aligned}\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}\right]\right)_R + \boldsymbol{\Omega} \times (\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x})_R \\ &= \left(\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t}\right)_R + 2\boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})\end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal accelertion.* Note that we can write the centrifugal acceleration in the form of a conservative force

$$\begin{aligned}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) = \nabla\phi_c \\ \phi_c = -\frac{1}{2}\left|\boldsymbol{\Omega} \times \boldsymbol{x}\right|^2\end{aligned}$$

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u}\right) = -\nabla p - \rho\nabla\left(\phi + \phi_c\right) + \rho\boldsymbol{F} \tag{1}$$

We group the potential terms into a *geopotential* $\Phi \equiv \phi + \phi_c$. The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.
Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

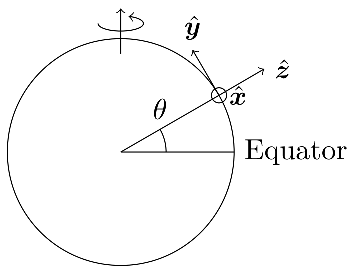


Figure 2: Local Cartesian coordinates

1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2).
In this coordinate system $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$. Hence if $\boldsymbol{u} = (u, v, w)$ then

$$\begin{aligned} 2\boldsymbol{\Omega} \times \boldsymbol{u} &= (2\Omega w\cos\theta - 2\Omega v\sin\theta, 2\Omega u\sin\theta, -2\Omega u\cos\theta) \\ &= (-fv + f^*w, fu - f^*u) \end{aligned}$$

where $f \equiv 2\Omega\sin\theta$ is the *Coriolis parameter* and $f^* \equiv 2\Omega\cos\theta$.

Example. In Cambridge, $\theta = 52.1°N$ so

$$\begin{aligned} f &= 2\Omega\sin\theta \\ &= 2 \cdot \frac{2\pi}{3600 \cdot 24} \cdot 0.79 s^{-1} \\ &\approx 1.14 \times 10^{-4} s^{-1} \end{aligned}$$

At mid-latitudes, $f \sim 10^{-4}$ is a good approximation.

We can simplify the Coriolis acceleration expression; often $f^*w \ll fv$ and $f^*u \ll g$. Hence

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \approx (-fv, fu, 0) = f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

This is the *traditional approximation.* This is *not* always a good approximation, particularly at intermediate scales.

1.2 Scale analysis.

Define characteristic scales for length L, time T, and velocity U. Non-dimensional variables are denoted with a superscript star: $\boldsymbol{u}^* = \boldsymbol{u}/U$, etc.
Using these scalings with $\boldsymbol{F} = \nu\nabla^2\boldsymbol{u}$ we have

$$\frac{U}{T}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \frac{U^2}{L}\boldsymbol{u}^* \cdot \nabla^*\boldsymbol{u}^* + fU\hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho}\nabla\left(p + \rho\Phi\right) + \frac{\nu U}{L^2}\nabla_*^2\boldsymbol{u}^*$$

Dividing through by fU leaves the Coriolis acceleration term ord(1) with other terms scaled relatively.

$$\frac{1}{fT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \mathrm{Ro}\boldsymbol{u}^* \cdot \nabla^*\boldsymbol{u}^* + \hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho f U}\nabla\left(p + \rho\Phi\right) + \mathrm{E}\nabla_*^2\boldsymbol{u}^*$$

where $\mathrm{Ro} \equiv \frac{U}{fL}$ is the *Rossby number* and $\mathrm{E} \equiv \frac{\nu}{fL^2}$ is the *Ekman number.*

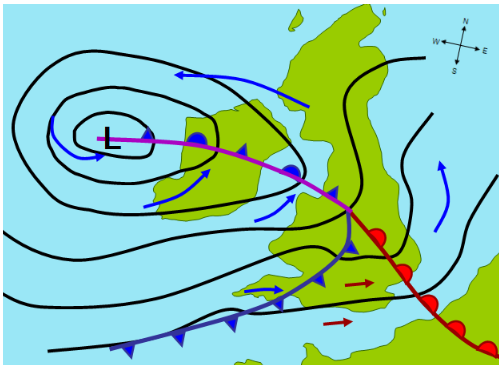


Figure 3: Lines of constant pressure p act as streamlines for the horizontal flow.

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> **Example.** For an atmospheric storm, $U \sim 10ms^{-1}, L \sim 1000km, f \sim 10^{-4}s^{-1}$. Thus $\mathrm{Ro} \sim 0.1, \mathrm{E} \sim 10^{-13}$.

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Further, if $T = L/U$, then $\mathrm{Ro} = U/fL = 1/fT$. For small Ro, E, on surfaces of constant Φ, $f\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla p$. This is *geostrophic balance*. In components, we have

$$
\begin{aligned}
-fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y}
\end{aligned}
$$

The equations of geostrophic balance can be arranged to give the horizontal velocity: $\boldsymbol{u}_H$

$$
\boldsymbol{u}_H \equiv (u, v) = \frac{1}{\rho f}\hat{\boldsymbol{z}} \times \nabla p
$$

Horizontal velocity is perpendicular to ∇p and hence parallel to isobars (lines of constant p), i.e. pressure acts like a streamfunction (see figure 3).
In the Northern Hemisphere, air moves clockwise around high p and anticlockwise around low p. A *cyclonic* rotation is in the same sense as $\boldsymbol{\Omega}$, *anticyclonic* in the opposite sense as $\boldsymbol{\Omega}$.

1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small Ro, E)

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &= 0 \\
2\boldsymbol{\Omega} \times \boldsymbol{u} &= -\frac{1}{\rho}\nabla p
\end{aligned}
\tag{2}
$$

Taking the curl of (2) we have

$$
\begin{aligned}
\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\Omega_l u_m \\
&= \varepsilon_{kij}\varepsilon_{klm}\Omega_l \partial_j u_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\Omega_l \partial_j u_m \\
&= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i
\end{aligned}
$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0$$

For $\boldsymbol{\Omega} = (0, 0, \Omega)$, this implies $\frac{\partial w}{\partial z} = 0$. If $w = 0$ on some horizontal surface (e.g. ground) then $w = 0$ everywhere.
Also, $u_x + v_y = 0$, i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in 'columns' parallel to $\boldsymbol{\Omega}$, called *Taylor columns*.

2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density ρ_0 with angular velocity $\boldsymbol{\Omega} = (0, 0, f/2)$. Assume small ampltiude motions (i.e. $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$), i.e. neglect $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$. From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \tag{3}$$
$$v_t + fu = -\frac{p_y}{\rho_0} \tag{4}$$
$$w_t = -\frac{p_z}{\rho_0} \tag{5}$$
$$u_x + v_y + w_z = 0 \tag{6}$$

We will eliminate variables in favour of p.

$$\nabla \cdot ((3) - (5)) \implies \nabla^2 p = \rho_0 f (v_x - u_y)$$
$$\partial_x(4) - \partial_y(3) \& (6) \implies (v_x - u_y)_t = f w_z$$

Combining these and using (5) we have

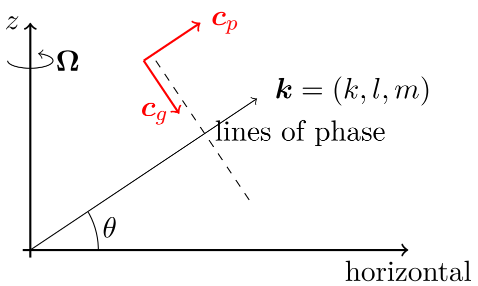
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for p. Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed $\boldsymbol{c}_p = w/\boldsymbol{k}$ and group velocity

$$\boldsymbol{c}_g = \frac{\partial w}{\partial \boldsymbol{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\boldsymbol{k}|^{3/2}}$$

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Note that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$. Also note $|\omega| \leq |f|$.

2.1 Inertial (free) oscillations

Assume $\nabla p = \mathbf{0}$. The x and y components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus $u = U \sin ft$ where f is the *inertial frequency*. Similarly, we have $v = U \cos ft$. For a particle with position (x_p, y_p) floating on an ocean surface $z = 0$ moving with the fluid velocity, we have

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u \implies x_p = -\frac{U}{f}\cos ft + x_0$$
$$\frac{\mathrm{d}y_p}{\mathrm{d}t} = v \implies y_p = -\frac{U}{f}\sin ft + y_0$$

Thus the motion of fluid particles describes describes *inertial circles* with radius $\frac{2U}{f}$.

2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress $\boldsymbol{\tau}_w$. Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e. $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e. $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e. $w(z=0) = 0$
4. No flow in deep ocean, i.e. $\lim_{z\to-\infty} \boldsymbol{u} = \mathbf{0}$
5. Constant density ρ
6. Traditional approximation

Continuity (incompressibility) says $u_x + v_y + w_z = 0$. Assumptions 2 and 3 then imply $w = 0$ everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$
$$fu = \nu v_{zz} \tag{8}$$

Define the *complex velocity* $\mathcal{V} \equiv u + iv$. Then

$$\mathcal{V}_{zz} = \frac{if}{\nu}\mathcal{V} \tag{9}$$

Without loss of generality, assume $\boldsymbol{\tau}_w$ is aligned with the x-axis: $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$. Boundary conditions for (9) are

$$\mathcal{V}_z = \left(\frac{\tau_w}{\rho\nu}, 0\right) \quad \text{at } z = 0$$
$$\mathcal{V} = (0,0) \quad \text{as } z \to -\infty$$

Thus $\mathcal{V} = Ae^{(1+i)z/\delta}$ where $\delta = \sqrt{\frac{2\nu}{f}}$, $A = \frac{\tau_w\delta(1-i)}{2\rho\nu}$. In terms of the velocity components, we have

$$u = \frac{\tau_w}{\rho\sqrt{\nu f}}e^{z/\delta}\cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$
$$v = -\frac{\tau_w}{\rho\sqrt{\nu f}}e^{z/\delta}\sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

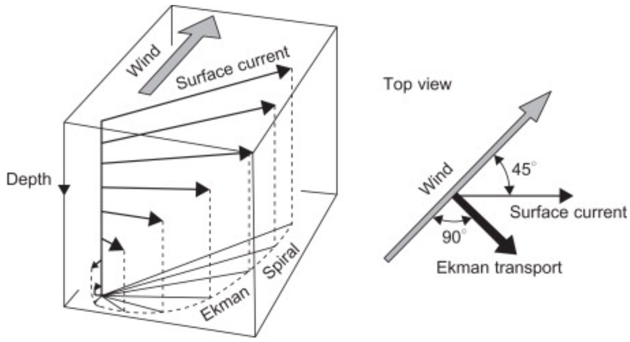

Figure 4: Ekman spiral.

2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where $\nu \boldsymbol{u}_z \approx 0$ at $z = -h$. Since $\nu \boldsymbol{u}_z(z=0) = (\tau_w/\rho, 0)$, the *Ekman transport* $\boldsymbol{U}_T$ is

$$U_T \equiv \int_{-h}^{0} u \,\mathrm{d}z = 0$$
$$V_T \equiv \int_{-h}^{0} v \,\mathrm{d}z = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented 90° to the right of the applied wind shear stress (in the Northern Hemisphere).

2.4 Ekman pumping

Consider a wind stress $\tau_w(y)$ that varies over large scales. Then from incompressibility

$$\int_{-h}^{0} w_z \,\mathrm{d}z = -\int_{-h}^{0} u_x \,\mathrm{d}z - \int_{-h}^{0} v_y \,\mathrm{d}z$$

Thus for h constant,

$$-w(z=-h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\tau_w}{\rho f}\right)$$

In general we have

$$w(z=-h) = \hat{\boldsymbol{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$