

# Cambridge Part III Maths

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## Hydrodynamic Stability

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Notes created based on Josh Kirklin's L<sup>A</sup>T<sub>E</sub>X packages & classes. Please do not distribute these notes other than to fellow Part III students. Please send errors and suggestions to [cwp29@cam.ac.uk](mailto:cwp29@cam.ac.uk).

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### 1 Introduction

We are typically interested in whether a given flow solution  $\mathbf{u}(\mathbf{x}, t)$  is 'stable', certainly to small (infinitesimal) disturbances and perhaps to larger perturbations too. We perturb  $\mathbf{u}(\mathbf{x})$  to  $\mathbf{u}(\mathbf{x}) + \hat{\mathbf{u}}(\mathbf{x}, t)$  and define the *perturbation energy* as

$$E(t) \equiv \int \frac{1}{2} \hat{\mathbf{u}}^2(\mathbf{x}, t) \, dV$$

A solution is said to be stable if

$$\lim_{t \rightarrow \infty} \frac{E(t)}{E(0)} = 0$$

for all perturbations  $\hat{\mathbf{u}}$ . Conversely, if there exists  $\hat{\mathbf{u}}$  such that  $E(t) \nrightarrow 0$  then  $\mathbf{u}$  is unstable. The nature of  $E(0)$  determines the type of perturbation:

- If  $E(0) \rightarrow 0$  we have an infinitesimal disturbance
- If  $E(0) < \delta$  then we probe finite amplitude disturbances
- If  $E(0) \rightarrow \infty$  this probes the *global* stability

In the first 9 lectures we focus on the first situation, which is linear stability analysis. Consider the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{\text{Re}} \nabla^2 \mathbf{u}$$

If  $\mathbf{U}(\mathbf{x})$  is a steady (basic) solution then

$$\mathbf{U} \cdot \nabla \mathbf{U} + \nabla P = \frac{1}{\text{Re}} \nabla^2 \mathbf{U}$$

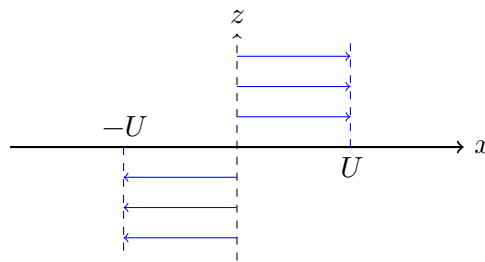
Let  $\mathbf{u} = \mathbf{U}(\mathbf{x}) + \hat{\mathbf{u}}(\mathbf{x}, t)$ ,  $p = P + \hat{p}$ . Then

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{U} + \mathbf{U} \cdot \nabla \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + \nabla \hat{p} = \frac{1}{\text{Re}} \nabla^2 \hat{\mathbf{u}}$$

The term  $\hat{\mathbf{u}} \cdot \nabla \mathbf{U}$  is stabilising whilst the term  $\nabla^2 \hat{\mathbf{u}} / \text{Re}$  is destabilising. Therefore, we expect stability as  $\text{Re} \rightarrow 0$  the stabilising term dominates, and instability as  $\text{Re} \rightarrow \infty$  when the destabilising term dominates. Thus there exists some value  $\text{Re}_{\text{crit}}$  at which instability arises. We will ask what this value is, and what is the form of the initial instability/mode/pattern?

## 2 Kelvin-Helmholtz instability

See Drazin (2002), section 3.3, pages 47–50. Here we take a different approach and derive Rayleigh's equation (example 8.3, page 151 of Drazin).



Consider a flow  $\mathbf{u} = U(z)\hat{\mathbf{x}}$  where

$$U(z) = \begin{cases} U & z > 0 \\ -U & z < 0 \end{cases}$$

The linearised, *inviscid* equation for perturbation  $\hat{\mathbf{u}}$  is

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{w} U' \hat{\mathbf{x}} + U \frac{\partial \hat{\mathbf{u}}}{\partial x} + \nabla \hat{p} &= 0 \\ \nabla \cdot \hat{\mathbf{u}} &= 0 \end{aligned}$$

The boundary conditions are  $\hat{\mathbf{u}} \rightarrow 0$  as  $z \rightarrow \pm\infty$ , i.e. no energy is radiated in from infinity. We will work in 2D with velocity components  $(\hat{u}, \hat{w}) = (\psi_z, -\psi_x)$  and let  $\psi(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$  where  $c$  is a complex eigenvalue, currently unknown. Formally, this is equivalent to taking a Fourier transform. We have

$$i\alpha(U - c) \begin{pmatrix} \phi' \\ -i\alpha\phi \end{pmatrix} + \begin{pmatrix} -i\alpha U' \phi \\ 0 \end{pmatrix} + \begin{pmatrix} i\alpha p \\ \frac{\partial p}{\partial z} \end{pmatrix} = 0$$

We can eliminate  $p$  via  $\partial_z(\text{top}) - i\alpha(\text{bottom})$  to get

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

with boundary conditions  $\phi \rightarrow 0$  as  $z \rightarrow \pm\infty$ . This is *Rayleigh's equation*. Note that  $c$  is the crucial eigenvalue. We wish to know when  $c_i = \Im(c) > 0$  as a function of  $U(z)$ , as  $c_i$  is the growth rate:

$$\hat{u} \propto e^{i\alpha(x-ct)} = e^{i\alpha(c-c_r t - i c_i t)} = e^{i\alpha(x-c_r t) + \alpha c_i t}$$

Note the following:

- There is a symmetry  $\alpha \mapsto -\alpha$ , so without loss of generality we consider  $\alpha > 0$ .
- The complex conjugate is also a solution with  $c \mapsto c^*$ . Hence an unstable mode has a damped partner, so we have stability only if all modes are ‘neutral’ i.e.  $c_i = 0$ .
- There is a possible singularity at  $y$  where  $U(y) = c$ , called the *critical layer*. If  $c$  is real, see later.

We now solve Rayleigh's equation with  $U(z)$  defined as before. We solve above and below  $z = 0$  and piece the solutions together. Since  $U'' = 0$ , we have

$$\phi'' = \alpha^2\phi$$

which admits a solution satisfying the boundary conditions:

$$\phi = \begin{cases} A^{-\alpha z} & z > 0 \\ B e^{\alpha z} & z < 0 \end{cases}$$

The matching conditions at  $z = 0$  are

1. Pressure  $\hat{p}$  continuous at  $z = 0$ , with  $\hat{p}$  given by:

$$\hat{p} = U'\phi - (U - c)\phi'$$

2. Kinematic condition at the surface:

$$\frac{D}{Dt}(z - \zeta(x, t)) = 0$$

where  $z = \zeta(x, t)$  is the position of the surface. After linearising, we have

$$w - \frac{\partial \zeta}{\partial t} - U \frac{\partial \zeta}{\partial x} = 0$$

Inserting the form of  $w$  and  $U$  we require that

$$\zeta = -\frac{\phi}{U - c}$$

is continuous across  $z = 0$ .

Requiring  $p$  continuous gives

$$-(U - c)A(-\alpha) = -(-U - c)B(\alpha)$$

Requiring  $\zeta$  continuous gives

$$\frac{A}{U - c} = \frac{B}{-U - c}$$

Hence we have

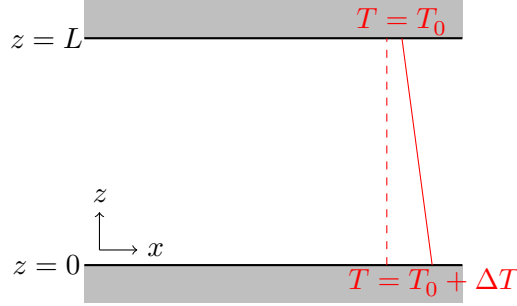
$$(U - c)^2 = -(U + c)^2$$

i.e.  $c = \pm iU$  so the growth rate is  $\alpha U$ . Thus the flow is unstable to waves of all wavelengths. The instability may be remedied

- by adding a density stratification, which stabilises long wavelengths (small  $\alpha$ )
- by adding surface tension, which stabilises short wavelengths (large  $\alpha$ ), e.g. Drazin page 50 equation 3.21.

### 3 Thermal instabilities: Rayleigh-Bernard convection

Consider two parallel plates separated by distance  $L$  with fluid subject to gravity and temperature difference  $\Delta T$  between the plates. The lower plate is heated to  $T_0 + \Delta T$  whilst the upper plate is fixed at temperature  $T_0$ .



The basic state consists of no motion, with heat transfer by conduction only.

**Governing equations.** The governing equations are those of momentum, mass, and (thermal) energy conservation.

$$\begin{aligned} \rho \frac{D\mathbf{u}}{Dt} + \nabla p &= \mu \nabla^2 \mathbf{u} + \rho g \hat{\mathbf{z}} \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T &= \kappa \nabla^2 T \\ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

To close the set of equations we need a relationship between  $\rho$  and  $T$ . Most cases of interest have  $\Delta T$  and  $\Delta\rho$  small, i.e.  $\Delta\rho \ll \rho_0$ ,  $\Delta T \ll T_0$ . Two consequences of this assumption are:

1. We can Taylor expand  $\rho = \rho(T)$ :

$$\rho \approx \rho(T_0) [1 - \alpha(T - T_0)]$$

where  $\alpha > 0$  is the coefficient of thermal expansion, such that  $T$  increases when  $\rho$  decreases. We write  $\rho_0 = \rho(T_0)$ .

2. We can adopt a Boussinesq approximation: acknowledge density changes only in the buoyancy term  $\rho g \hat{\mathbf{z}}$ . Importantly, we can assume the fluid is incompressible.

Define  $\theta = T - T_0$ . The governing equations are now

$$\begin{aligned}\rho_0 \frac{D\mathbf{u}}{Dt} + \nabla p &= \mu \nabla^2 \mathbf{u} + \rho_0 (1 - \alpha \theta) g \hat{\mathbf{z}} \\ \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= \kappa \nabla^2 \theta \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

The basic state is  $u = 0, \theta = \Delta T(1 - z/L)$  and

$$\frac{dp}{dz} = -\rho_0 (1 - \alpha \Delta T(1 - z/L))g$$

We now non-dimensionalise using scalings  $t \sim L^2/\kappa, u \sim \kappa/L, \theta \sim \Delta T$ , e.g.  $\theta = \Delta T \theta^*$  where  $\theta^*$  is the non-dimensionalised variable. We normalise the  $\frac{D\mathbf{u}^*}{Dt^*}$  term, to get:

$$\begin{aligned}\frac{D\mathbf{u}^*}{Dt^*} + \nabla^* p^* &= \frac{\mu}{\rho_0 \kappa} \nabla^{*2} \mathbf{u}^* + \frac{\alpha g \Delta T L^3}{\kappa^2} \theta^* \hat{\mathbf{z}} \\ \frac{\partial \theta^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \theta^* &= \nabla^{*2} \theta^*\end{aligned}$$

Define the *Prandtl number*

$$\sigma \equiv \frac{\nu}{\kappa} = \frac{\mu}{\rho_0 \kappa}$$

which is the ratio of viscous/momentum diffusion to thermal diffusion. Typical values are 0.72 in air, 7 in water,  $10^5$  in magma. We also define the *Rayleigh number*

$$\text{Ra} \equiv \frac{\alpha \Delta T g L^3}{\kappa \nu}$$

which is the ratio of destabilising buoyancy to stabilising diffusion. Dropping the  $*$  notation, we have

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \sigma \nabla^2 \mathbf{u} + \sigma \text{Ra} \theta \hat{\mathbf{z}} \\ \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= \nabla^2 \theta \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

**Boundary conditions.** There are three combinations of boundary condition available in this problem, with the choice fixed wall (no slip) or stress free (free slip).

$\theta = 0$ $z = 1$	<div style="border-top: 2px solid black; width: 100%;"></div> <b>Fixed wall</b> $\mathbf{u} = 0$	<div style="border-top: 2px solid black; width: 100%;"></div> <b>Free slip</b> $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$	<div style="border-top: 2px solid black; width: 100%;"></div> <b>Free slip</b> $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
$\theta = 1$ $z = 0$	<div style="border-bottom: 2px solid black; width: 100%;"></div> $\mathbf{u} = 0$ <b>Fixed wall</b>	<div style="border-bottom: 2px solid black; width: 100%;"></div> $\mathbf{u} = 0$ <b>Fixed wall</b>	<div style="border-bottom: 2px solid black; width: 100%;"></div> $w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ <b>Free slip</b>

The double fixed wall case is easiest to replicate in a lab, whilst the double free slip case is the easiest analytically, which we shall use.

**Basic state.** In the basic state we have conductive profile  $\mathbf{u}_0 = 0, \theta_0 = 1 - z$  and from integration  $p_0 = \sigma \text{Ra}(z - \frac{1}{2}z^2)$ . We generate linearised equations for perturbations  $\theta = \theta_0 + \theta', \mathbf{u} = \mathbf{u}_0 + \mathbf{u}', p = p_0 + p'$ . As usual with linear stability analysis, we assume  $(\theta', \mathbf{u}', p')$  are small.

$$\begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \mathbf{u}' + \nabla p' &= \sigma \nabla^2 \mathbf{u}' + \sigma \text{Ra} \theta' \hat{\mathbf{z}} \\ \frac{\partial \theta'}{\partial t} - w' + \mathbf{u}' \cdot \nabla \theta' &= \nabla^2 \theta' \\ \nabla \cdot \mathbf{u}' &= 0 \end{aligned}$$

Dropping the  $'$  notation for clarity we have perturbation equations

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) \mathbf{u} + \nabla p = \sigma \text{Ra} \theta \hat{\mathbf{z}} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

$$\left( \frac{\partial}{\partial t} - \nabla^2 \right) \theta = w \quad (3)$$

The perturbation boundary conditions also follow by inserting variables into the total boundary conditions, e.g.  $\theta = \theta_0 + \theta' = 1$  at  $z = 0$  combined with  $\theta_0 = 1$  at  $z = 0$  gives  $\theta' = 0$ . Similarly,  $\theta' = 0$  at  $z = 1$  and in fact all boundary conditions are homogeneous. To proceed further, we need to reduce the equations (1),(2) and (3) into a single equation.

From  $\nabla \times (1)$  we have

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) \boldsymbol{\omega} = \sigma \text{Ra} \nabla \times \theta \hat{\mathbf{z}}$$

Taking the curl again and using  $\nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$  we have

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) (-\nabla^2 \mathbf{u}) = \sigma \text{Ra} \nabla \times (\nabla \times \theta \hat{\mathbf{z}}) = \sigma \text{Ra} \left( \nabla \frac{\partial \theta}{\partial z} - \hat{\mathbf{z}} \nabla^2 \theta \right)$$

The  $z$  component is

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) (-\nabla^2 w) = \sigma \text{Ra} \nabla_H^2 \theta \quad (4)$$

where  $\nabla_H^2 = \partial_x^2 + \partial_y^2$ . Now (3) can be used to eliminate  $\theta$  by applying the operator  $(\partial_t - \nabla^2)$ :

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) \left( \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w = \sigma \text{Ra} \nabla_H^2 w \quad (5)$$

This is a 6<sup>th</sup> order PDE for  $w$ , hence we need three boundary conditions at each wall  $z = 0, 1$ . We use stress-free (i.e. free slip) at both walls to simplify analysis. Thus we have

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at } z = 0, 1$$

The second set of conditions comes from incompressibility. Taking  $\partial_z(\nabla \cdot \mathbf{u})$  we have

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial z} \right) + \frac{\partial^2 w}{\partial z^2} = 0 \implies w_{zz} = 0$$

The third and final set of conditions comes from requiring  $\theta = 0$  at  $z = 0, 1$ . From (4),  $\nabla_H^2 \theta = 0$  implies

$$\left( \frac{\partial}{\partial t} - \sigma \nabla^2 \right) \nabla^2 w = 0$$

We now have 6 boundary conditions to supplement the PDE.

**Normal mode solution.** Seek a solution  $w(x, y, z, t) = W(z)e^{ik_1x + ik_2y + \lambda t}$  where  $k_1, k_2$  are wavenumbers and  $\lambda \in \mathbb{C}$  is the growth rate. Write  $D = d/dz$  and  $k = \sqrt{k_1^2 + k_2^2}$  since the problem is rotationally symmetric in the  $(x, y)$  plane. Substituting into (5) we have

$$(\lambda - [D^2 - k^2])(\lambda - \sigma [D^2 - k^2])(D^2 - k^2)W = -\sigma \text{Ra} k^2 W$$

with boundary conditions at  $z = 0, 1$ :

$$\begin{aligned} W(0) &= W(1) = 0 \\ D^2 W(0) &= D^2 W(1) = 0 \\ [\lambda - \sigma(D^2 - k^2)] [D^2 - k^2] W &= 0 \implies D^4 W(0) = D^4 W(1) = 0 \end{aligned}$$

The objective is to find

$$\max_k \Re\{\lambda(k; \text{Ra}, \sigma)\}$$

The onset of linear instability (for a given  $\sigma$ ) at  $\text{Ra} = \text{Ra}_{\text{crit}}$  is defined by

$$\max_k \Re\{\lambda(k; \text{Ra}_{\text{crit}}, \sigma)\} = 0$$

In general,  $\lambda \in \mathbb{C}$ , but for this problem it can be proven that at marginality  $\Im(\lambda) = 0$  as well as  $\Re(\lambda) = 0$ ; a condition called the *principle of exchange of stabilities*. Hence setting  $\lambda = 0$  in the above, we get

$$(D^2 - k^2)^3 W = -\text{Ra} k^2 W \quad (6)$$

Note that  $\sigma$  drops out of the problem! It's easy to see  $W(z) = \sin(n\pi z)$  solves (6) and satisfies the free-slip BCs. Hence

$$(n^2\pi^2 + k^2)^3 = \text{Ra} k^2$$

Criticality is then given by

$$\text{Ra}_{\text{crit}} = \min_{n,k} \frac{(n^2\pi^2 + k^2)^3}{k^2}$$

We find the minimum in the usual way:

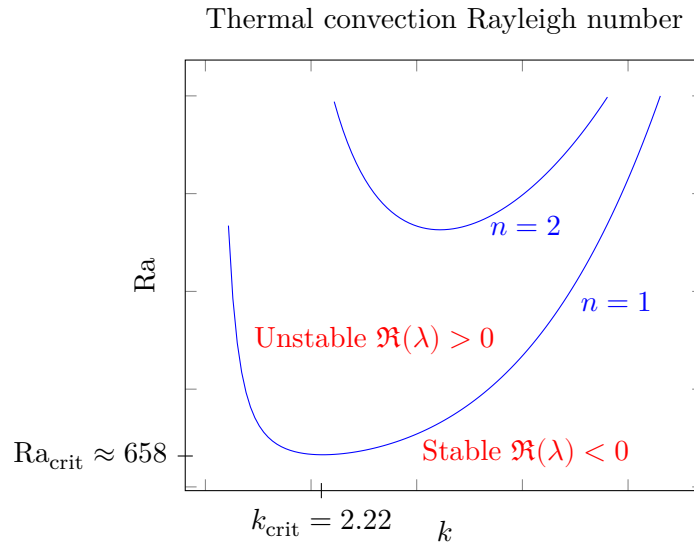
$$\begin{aligned} \frac{\partial \text{Ra}}{\partial k} &= \frac{3(2k)(n^2\pi^2 + k^2)^2 k^2 - 2k(n^2\pi^2 + k^2)^3}{k^4} \\ &= \frac{2k(n^2\pi^2 + k^2)^2(3k^2 - (n^2\pi^2 + k^2))}{k^4} = 0 \\ \implies 2k^2 &= n^2\pi^2 \\ \implies k &= \frac{n\pi}{\sqrt{2}} \end{aligned}$$

Given  $k = n\pi/\sqrt{2}$  the Rayleigh number is

$$\text{Ra}(k = \frac{n\pi}{\sqrt{2}}) = \frac{(n^2\pi^2 + \frac{1}{2}n^2\pi^2)^3}{n^2\pi^2/2} = \frac{27}{4}n^4\pi^4$$

Clearly the critical Rayleigh number is given by  $n = 1$ , hence

$$\begin{aligned} \text{Ra}_{\text{crit}} &= \frac{27}{4}\pi^4 \sim 658 \\ k_{\text{crit}} &= \frac{\pi}{\sqrt{2}} \sim 2.22 \end{aligned}$$



Results for other boundary conditions are:

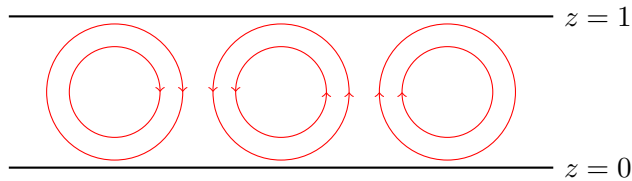
- Free–rigid boundary:  $Ra_{\text{crit}} \sim 1101, k_c = 2.68$
- Rigid–rigid boundary:  $Ra_{\text{crit}} \sim 1708, k_c = 3.117$

Notice that at criticality only the size of  $k$  is specified, *not* its direction. Hence there are an infinite number of possibilities  $\mathbf{k} = (k \cos \phi, k \sin \phi)$ . Various different patterns which tessellate are as follows.

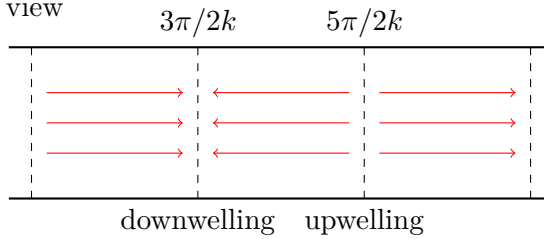
1. **2D rolls.** Orientate  $x$ -axis along  $k$  such that  $k_2 = 0$ . We have velocity components ( $w$  specified in problem,  $u$  follows from incompressibility)

$$\begin{aligned} w &= W(z) \sin kx \\ v &= 0 \\ u &= \frac{\pi \cos \pi z \cos kx}{k} \end{aligned}$$

side view



top view



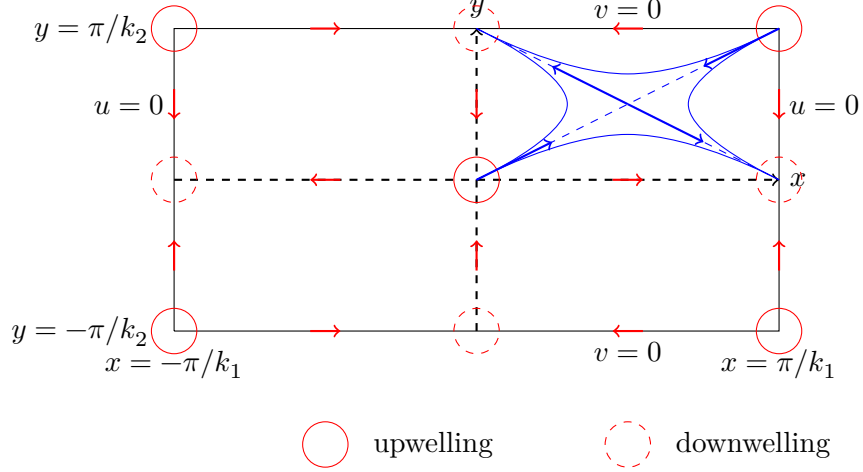


2. **Rectangles.** Velocity components are

$$w = W(z) \cos k_1 x \cos k_2 y$$

$$v = -\frac{k_2}{k^2} W' \cos k_1 x \sin k_2 y$$

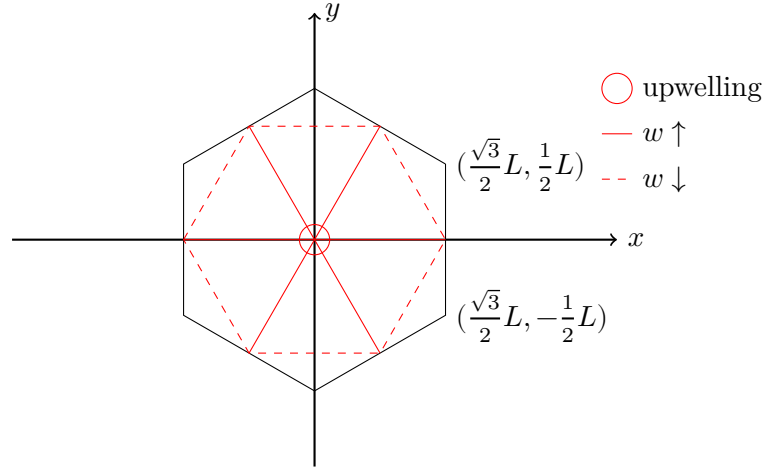
$$u = -\frac{k_1}{k^2} W' \sin k_1 x \cos k_2 y$$



3. **Hexagons.** Vertical velocity component

$$w = W(z) \left[ \cos \frac{k}{2} (\sqrt{3}x + y) + \cos \frac{k}{2} (\sqrt{3}x - y) + \cos ky \right]$$

This is flow in a hexagon of side length  $L = 4\pi/3k$ .



## 4 Centrifugal instabilities

Flows with curved streamlines can be unstable due to centrifugal effects.

### 4.1 Rayleigh's criterion

We will concentrate on axisymmetric flows. Consider an azimuthal flow

$$\mathbf{u} = u_\theta(r) \hat{\boldsymbol{\theta}} = r\Omega(r) \hat{\boldsymbol{\theta}}$$

The inviscid, axisymmetric equations for a general flow  $\mathbf{u} = u_r \hat{\mathbf{r}} + u_\theta \hat{\boldsymbol{\theta}} + u_z \hat{\mathbf{z}}$  are

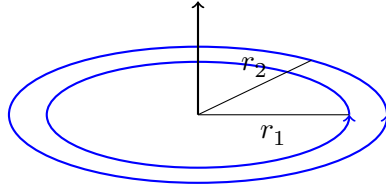
$$\begin{aligned}\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ \frac{\partial u_\theta}{\partial t} + \mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho} \frac{1}{r} \frac{\partial p}{\partial \theta} \\ \frac{\partial u_z}{\partial t} + \mathbf{u} \cdot \nabla u_z &= -\frac{1}{\rho} \frac{\partial p}{\partial z} \\ \frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} &= 0\end{aligned}$$

where  $\mathbf{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$ . Cancelled terms are absent in the axisymmetric setting. The *centrifugal* term is  $-u_\theta^2/r$  in the  $r$ -momentum equation. The  $\theta$ -momentum equation can be rearranged, and multiplied by  $r$  to give a material conservation equation:

$$\begin{aligned}\frac{\partial}{\partial r}(r u_\theta) + r u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial}{\partial z}(r u_\theta) + r \left( \frac{u_r u_\theta}{r} \right) &= 0 \\ \Rightarrow \frac{\partial}{\partial t}(r u_\theta) + u_r \frac{\partial}{\partial r}(r u_\theta) + u_z \frac{\partial}{\partial z}(r u_\theta) &= 0 \\ \Rightarrow \frac{D}{Dt}(r u_\theta) &= 0\end{aligned}$$

This expresses conservation of angular momentum: the angular momentum per unit mass is  $I = r u_\theta$ , hence  $\frac{DI}{Dt} = 0$ . This result also follows from Kelvin's circulation theorem, using the circulation  $\Gamma = 2\pi r u_\theta$  for an inviscid fluid. The statement says that if  $\mathbf{u} = u_\theta(r) \hat{\boldsymbol{\theta}}$  (i.e. axisymmetric azimuthal flow) then  $I = I(r)$  is a basic state.

**What distributions of  $I(r)$  could be stable?** Rayleigh's argument considers 2 rings of fluid at radius  $r_1$  and  $r_2(> r_1)$  respectively.



The kinetic energy is

$$E = \frac{1}{2} \rho \left( \frac{I_1^2}{r_1^2} + \frac{I_2^2}{r_2^2} \right)$$

Now suppose the rings swap places due to a perturbation, but they keep their angular momentum (since it is materially conserved). The new KE is

$$E_{\text{new}} = \frac{1}{2} \left( \frac{I_2^2}{r_1^2} + \frac{I_1^2}{r_2^2} \right)$$

Hence the swap has resulted in an energy change

$$\Delta E = (I_2^2 - I_1^2) \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right)$$

We can expect instability if  $\Delta E < 0$ . Since  $r_2 > r_1$ , the second factor is positive hence

$$\Delta E < 0 \iff I_2^2 < I_1^2$$

Hence Rayleigh's criterion for stability is  $I_2^2 \geq I_1^2$  or equivalently

$$\frac{dI^2}{dr} \geq 0$$

i.e. angular momentum does not increase outwards. Note that with  $I = ru_\theta = r^2\Omega$  we have the condition

$$\frac{d}{dr}(r^4\Omega^2) \geq 0$$

for stability. This is often written using the *Rayleigh determinant*

$$\Phi \equiv \frac{1}{r} \frac{d}{dr}(r^4\Omega^2)$$

Hence stability is predicted if  $\Phi \geq 0$ .

## 4.2 Derivation via linear stability analysis

Consider Taylor-Couette geometry: cylindrical walls at  $r_1$  and  $r_2$  with an inviscid base state  $\mathbf{u} = r\Omega(r)\hat{\boldsymbol{\theta}}$ , with axisymmetric perturbations  $\mathbf{u}'$ . We have incompressibility

$$\nabla \cdot \mathbf{u}' = 0 \implies \frac{1}{r} \frac{\partial}{\partial r}(ru'_r) + \frac{\partial u'_z}{\partial z} = 0$$

The Euler equations for this perturbation are

$$\begin{aligned} \frac{\partial u'_r}{\partial t} - \frac{2r\Omega u'_\theta}{r} &= -\frac{1}{\rho} \frac{\partial p'}{\partial r} \\ \frac{\partial u'_\theta}{\partial t} + u'_r \frac{d}{dr}(r\Omega) + \frac{u'_r r \Omega}{r} &= 0 \\ \frac{\partial u'_z}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial z} \end{aligned}$$

Now specify normal mode decomposition

$$\begin{pmatrix} u'_r \\ u'_\theta \\ u'_z \\ p' \end{pmatrix} = \begin{pmatrix} \hat{u}_r(r) \\ \hat{u}_\theta(r) \\ \hat{u}_z(r) \\ \hat{p}(r) \end{pmatrix} e^{ikz + \sigma t}$$

Only axisymmetric perturbations are considered. The Euler equations become

$$\begin{aligned} \frac{1}{r} \frac{d}{dr}(r\hat{u}_r) + ik\hat{u}_z &= 0 \\ \sigma\hat{u}_r - 2\Omega\hat{u}_\theta &= -\frac{1}{\rho} \frac{d\hat{p}}{dr} \\ \sigma\hat{u}_\theta + \hat{u}_r(\Omega + (r\Omega)_r) &= 0 \\ \sigma\hat{u}_z &= -\frac{1}{\rho} ik\hat{p} \end{aligned}$$

We can reduce this system down to a single equation for  $\hat{u}_r$ :

$$\frac{d}{dr} \left( \frac{d}{dr} + \frac{1}{r} \right) \hat{u}_r - k^2 \hat{u}_r - 2 \frac{k^2}{\sigma^2} \Omega (2\Omega + r\Omega') \hat{u}_r = 0$$

This is a second order ODE for  $\hat{u}_r$  with BCs  $\hat{u}_r = 0$  at  $r = r_1, r_2$ . For this flow, Rayleigh's determinant is

$$\Phi \equiv \frac{1}{r} \frac{d}{dr} (r^4 \Omega^2) = 4\Omega^2 + 2r\Omega'\Omega$$

Hence the ODE for  $\hat{u}_r$  may be written as

$$\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r \hat{u}_r) \right) - k^2 \hat{u}_r = \frac{k^2}{\sigma^2} \Phi(r) \hat{u}_r \quad (7)$$

Multiply (7) by  $r\hat{u}_r^*$  (complex conjugate) and integrate from  $r_1$  to  $r_2$ :

$$\int_{r_1}^{r_2} r \hat{u}_r^* \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r \hat{u}_r) \right) dr - k^2 \int_{r_1}^{r_2} r |\hat{u}_r|^2 dr = \frac{k^2}{\sigma^2} \int_{r_1}^{r_2} r \Phi |\hat{u}_r|^2 dr$$

The first term may be integrated by parts to give:

$$\left[ r \hat{u}_r^* \frac{1}{r} \frac{d}{dr} (r \hat{u}_r) \right]_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{d}{dr} (r \hat{u}_r^*) \frac{1}{r} \frac{d}{dr} (r \hat{u}_r) dr - k^2 \int_{r_1}^{r_2} r |\hat{u}_r|^2 dr = \frac{k^2}{\sigma^2} \int_{r_1}^{r_2} r \Phi |\hat{u}_r|^2 dr$$

The first term vanishes since  $\hat{u}_r = 0$  at  $r = r_1, r_2$ . Labelling the first integral as  $H_1 > 0$  and the second as  $H_2 > 0$ , we have

$$\frac{k^2}{\sigma^2} \int_{r_1}^{r_2} r \Phi |\hat{u}_r|^2 dr = -H_1 - k^2 H_2 < 0$$

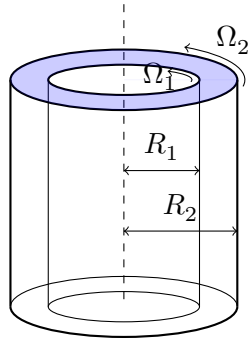
If  $\Phi \geq 0$  then  $\sigma^2 < 0$ , i.e.  $\sigma$  is imaginary and we have stability. If instead  $\Phi < 0$  somewhere in the domain, then potentially

$$\int_{r_1}^{r_2} r \Phi |\hat{u}_r|^2 dr < 0$$

in which case  $\sigma^2 > 0$  and we have instability. Hence  $\Phi < 0$  somewhere in the domain is *necessary* (but not sufficient) condition for instability. So this formal analysis confirms Rayleigh's heuristic criterion. Note, really we need to consider non-axisymmetric perturbations too.

### 4.3 Taylor vortices

Apply Rayleigh's criterion to Taylor-Couette flow.



When viscosity is present, the general solution with  $\partial_\theta = \partial_z = 0$  is

$$u_\theta(r) = Ar + \frac{B}{r}$$

No-slip boundary conditions at  $r = R_1, R_2$  give

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{\Omega_1 - \Omega_2}{R_1^{-2} - R_2^{-2}}$$

Note this solves  $(\nabla^2 - 1/r^2)u_\theta = 0$  where  $\nabla^2 = \frac{1}{r}\partial_r(r\partial_r)$ . In this case  $\Omega = u_\theta/r = A + B/r^2$  hence Rayleigh's determinant is

$$\Phi = \frac{1}{r^3} \frac{d}{dr} (r^4 \Omega^2) = \frac{1}{r^3} \frac{d}{dr} \left[ r^4 \left( A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4} \right) \right] = 4A^2 \left( 1 + \frac{B}{Ar^2} \right)$$

For convenience we define  $\mu = \Omega_2/\Omega_1$  and  $\eta = R_1/R_2 < 1$ . Then

$$\Phi = 4A^2 \left[ 1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2} \right]$$

For stability, i.e.  $\Phi \geq 0$  everywhere, we require for all  $r \in [R_1, R_2]$

$$1 \geq \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2} \geq \frac{1-\mu}{\eta^2 - \mu}$$

where the last inequality follows since  $R_1^2/r^2 \geq 1$  for all  $r \in [R_1, R_2]$ . There are now two cases:

- If  $\eta^2 > \mu$  then

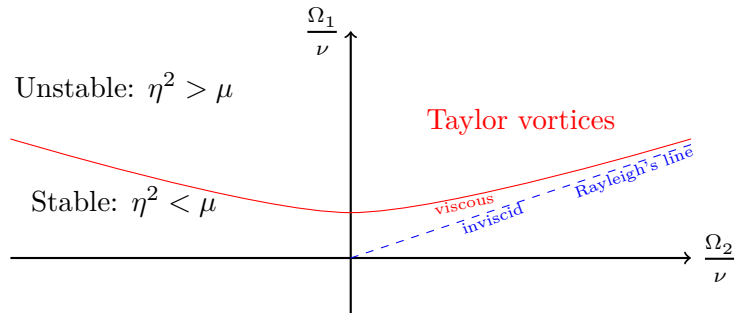
$$\eta^2 - \mu \geq 1 - \mu \implies \eta^2 \geq 1$$

This is a contradiction since  $\eta < 1$ .

- Otherwise  $\eta^2 < \mu$ , so

$$\eta^2 - \mu \leq 1 - \mu \implies \eta^2 \leq 1$$

Thus Rayleigh's criterion is  $\eta^2 < \mu$  for stability.



For a fixed geometry (i.e. fixed  $\eta$ ) we can plot a stability diagram, with Rayleigh's line  $\eta^2 = \mu = \Omega_2/\Omega_1$  marking the stability heuristic. In Taylor-Couette geometry, the instability often manifests itself as *Taylor vortices*, though there are many different modes of instability depending on  $\Omega_1, \Omega_2, \nu$ .

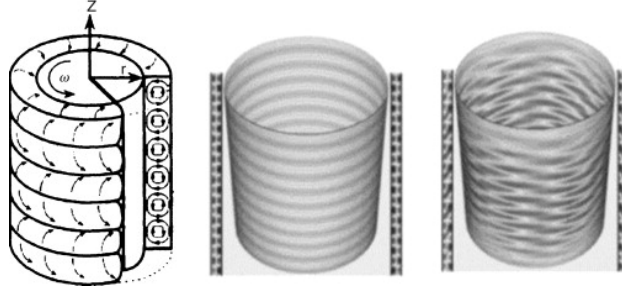
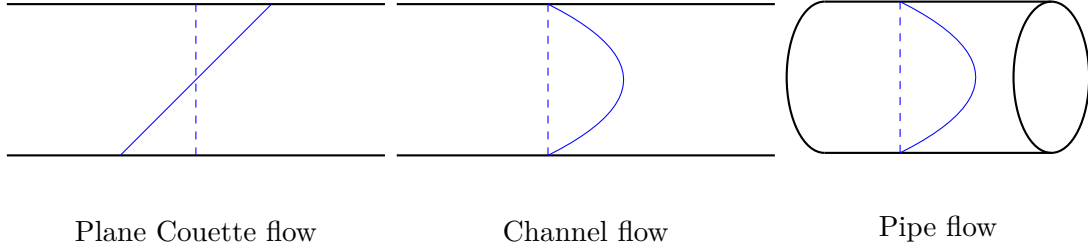


Figure 1: Taylor vortices, from Dutta and Ray, 2004.

## 5 Parallel shear flows

For some flows, inviscid analysis gives a good approximation to the stability properties of a viscous fluid (e.g. Kelvin-Helmholtz, Taylor-Couette flow) but for others, it does not (e.g. plane Couette flow, channel flow, pipe flow). In these flows, viscosity can be *destabilising*.



### 5.1 Inviscid analysis

Consider a parallel shear flow  $U(z)\hat{\mathbf{x}}$ . The non-dimensionalised Euler equations are

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p \\ \nabla \cdot \mathbf{u} &= 0\end{aligned}$$

with boundary conditions  $\mathbf{u} \cdot \hat{\mathbf{z}} = 0$  at  $z = z_1, z_2$ . The basic flow is  $\mathbf{U} = U(z)\hat{\mathbf{x}}$  with  $P$  constant – any constant form of the pressure is valid. Add small perturbations

$$\mathbf{u} = U(z)\hat{\mathbf{x}} + \mathbf{u}', \quad p = P + p'$$

The Euler equations become

$$\begin{aligned}\frac{\partial \mathbf{u}'}{\partial t} + U \frac{\partial \mathbf{u}'}{\partial x} + w' \frac{dU}{dz} \hat{\mathbf{x}} &= -\nabla p' \\ \nabla \cdot \mathbf{u}' &= 0\end{aligned}$$

with boundary conditions  $w' = 0$  at  $z = z_1, z_2$ . All equations have coefficients independent of  $x, y, t$  so we can separate the variables by taking normal modes of the form

$$\begin{aligned}\mathbf{u}'(\mathbf{x}, t) &= \hat{\mathbf{u}}(z) e^{i(\alpha x + \beta y - \alpha c t)} \\ p'(\mathbf{x}, t) &= \hat{p}(z) e^{i(\alpha x + \beta y - \alpha c t)}\end{aligned}$$

Note we have replaced the usual  $\sigma$  with  $-i\alpha c$ . It is understood that the physical fluid perturbation velocity  $\mathbf{u}'$  is represented by the real part, e.g.

$$\mathbf{u}' = [\Re(\hat{w}) \cos(\alpha x + \beta y - \alpha c_r t) - \Im(\hat{w}) \sin(\alpha x + \beta y - \alpha c_r t)] e^{\alpha c_i t}$$

This mode is a wave travelling with phase speed  $\alpha c_r / \sqrt{\alpha^2 + \beta^2}$  in the  $(\alpha, \beta, 0)$  direction and it decays like  $e^{\alpha c_i t}$  for  $c_i < 0$ , or grows if  $c_i > 0$ . The equations are now

$$i\alpha(U - c)\hat{u} + \frac{dU}{dz}\hat{w} + i\alpha\hat{p} = 0 \quad (8)$$

$$i\alpha(U - c)\hat{v} + i\beta\hat{p} = 0 \quad (9)$$

$$i\alpha(U - c)\hat{w} + \frac{d\hat{p}}{dz} = 0 \quad (10)$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{d\hat{w}}{dz} = 0 \quad (11)$$

with boundary conditions  $\hat{w} = 0$  at  $z = z_1, z_2$ . This is an eigenvalue problem in  $c \in \mathbb{C}$ . Instability corresponds to  $c_i > 0$  and  $c_i \leq 0$  for stability.

### 5.1.1 Squire's transformation (Squire, 1933)

Before attempting to solve (8)–(11), we consider Squire's transformation. Define the transformed variables

$$\tilde{\alpha} = \sqrt{\alpha^2 + \beta^2}, \quad \tilde{u} = \frac{\alpha\hat{u} + \beta\hat{v}}{\tilde{\alpha}}, \quad \tilde{p} = \frac{\tilde{\alpha}\hat{p}}{\alpha}$$

Construct  $(\alpha(8) + \beta(9))/\alpha$ :

$$i\tilde{\alpha}(U - c)\tilde{u} + \frac{dU}{dz}\hat{w} + i\tilde{\alpha}\tilde{p} = 0 \quad (12)$$

Similarly  $\tilde{\alpha}(10)/\alpha$ :

$$i\tilde{\alpha}(U - c)\hat{w} + \frac{d\tilde{p}}{dz} = 0 \quad (13)$$

Incompressibility is now expressed as

$$i\tilde{\alpha}\tilde{u} + \frac{d\hat{w}}{dz} = 0$$

The transformed system has the same form as (8)–(11) with  $\beta = \hat{v} = 0$  and  $\alpha \rightarrow \tilde{\alpha}$ ,  $\hat{u} \rightarrow \tilde{u}$ ,  $\hat{p} \rightarrow \tilde{p}$  but  $c$  unchanged. Thus the eigenvalue  $c$  depends on  $\sqrt{\alpha^2 + \beta^2}$  but the growth rate is  $\alpha c_i$ . So the largest growth rate  $\alpha c_i$  is given by  $\beta = 0$  for all wavenumber pairs  $(\alpha, \beta)$  with  $\sqrt{\alpha^2 + \beta^2}$  constant. Hence it is sufficient to consider  $\beta = 0$  disturbances only. To any unstable 3D mode  $\alpha \neq 0, \beta \neq 0$  there corresponds a more unstable 2D mode with  $\beta = 0$ .

### 5.1.2 Rayleigh's equation

Work in 2D (Squires). Use streamfunction  $\psi'$  such that

$$u' = \psi'_z, \quad v' = 0, \quad w' = -\psi'_x$$

Further, let  $\psi'(x, z, t) = \phi(z)e^{i\alpha(x-ct)}$  so that it is now clear that  $c_r$  is the phase speed in the  $x$  direction. Now  $\hat{u} = \frac{d\phi}{dz}$  and  $\hat{w} = -i\alpha\phi$  (notice the phase difference). Then (12) becomes

$$\begin{aligned} i\alpha(U - c)\frac{d\phi}{dz} + \frac{dU}{dz}(-i\alpha\phi) + i\alpha\hat{p} &= 0 \\ \implies \hat{p} &= \frac{dU}{dz}\phi - (U - c)\frac{d\phi}{dz} \end{aligned}$$

Substituting into (13) gives

$$\begin{aligned} i\alpha(U - c)(-i\alpha\phi) + \frac{d}{dz} \left[ \frac{dU}{dz}\phi - (U - c)\frac{d\phi}{dz} \right] &= 0 \\ \implies (U - c)(\phi'' - \alpha^2\phi) - U''\phi &= 0 \end{aligned}$$

with boundary conditions  $\phi = 0$  at  $z = z_1, z_2$ . This is *Rayleigh's equation (1880)*.

### Comments.

- Rayleigh's equation involves  $\alpha^2$  only so need only consider  $\alpha > 0$ .
- If  $(\phi, c)$  solves the problem then so does  $(\phi^*, c^*)$ . So if there exists a growing mode, there also exists a corresponding decaying mode. Hence stability means  $c \in \mathbb{R}$  for all  $\alpha$ .
- A singularity exists at  $U(z_c) = c$  – this is called a critical layer and only occurs when  $c \in \mathbb{R}$ . Critical layers are important in solving IVPs and relating Rayleigh's equation to its viscous analogue, the Orr-Sommerfeld equation (see later).
- There are two types of eigensolution:
  - Continuous spectrum  $c \in [\min U, \max U]$  and  $\phi$  has a discontinuous derivative at  $z_c$ . This type of solution is never unstable.
  - Discrete spectrum of complex conjugate pairs. This solution can be unstable.

### 5.1.3 Properties of Rayleigh's equation.

**Inflection point criterion.** Suppose  $c_i > 0$ , i.e. consider an unstable mode. Multiply Rayleigh's equation by  $\phi^*$  and integrate from  $z_1$  to  $z_2$ :

$$\int_{z_1}^{z_2} \left[ \phi^* \phi'' - \alpha^2 |\phi|^2 - \frac{U''}{U - c} |\phi|^2 \right] dz = 0$$

Integrate the first term by parts and note  $\phi = \phi^* = 0$  at  $z_1$  and  $z_2$ . Hence

$$\int_{z_1}^{z_2} \left[ |\phi'|^2 + \alpha^2 |\phi|^2 + \frac{U''}{U - c} |\phi|^2 \right] dz = 0$$

Take imaginary part:

$$\begin{aligned} \Im \left[ \int_{z_1}^{z_2} \frac{U''(U - c^*)}{|U - c|^2} |\phi|^2 dz \right] &= 0 \\ \implies -c_i \int_{z_1}^{z_2} \frac{U''}{|U - c|^2} |\phi|^2 dz &= 0 \end{aligned}$$

But  $c_i > 0$  so we must have

$$\int_{z_1}^{z_2} \frac{U''}{|U - c|^2} |\phi|^2 dz = 0$$

Now  $|U - c|^2 > 0$  and  $|\phi|^2 > 0$  so  $U''$  must change sign somewhere in  $[z_1, z_2]$ . Thus  $U'' = 0$  at least once is a necessary condition for inviscid instability, called the *inflection point criterion*.