Cambridge Part III Maths

Lent 2020

Fluid Dynamics of the Solid Earth

based on a course given by
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Notes created based on Josh Kirklin's LaTeX packages & classes. Please do not distribute these notes other than to fellow Part III students. Please send errors and suggestions to cwp29@cam.ac.uk.

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Lecture 1
21/01/21

1 Introduction

The course will use the wealth of observations of the solid Earth to motivate mathematical models of the physical processes governing its evolution. The dynamic evolution is governed by a rich variety of physical processes occuring on a wide range of length and time scales.

- The Earth's core is formed by the solidifcation of a mixture of molten iron and various light elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field.
- On million year timescales, the solid mantle convects, and as it upwells to the surface it partially melts leading to the volcanism.
- At the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record (figure 1)
- In the Earth's surface, fluids flow through poroous rocks, for example groundwater aquifers which feed streams and rivers which erode the solid surface.

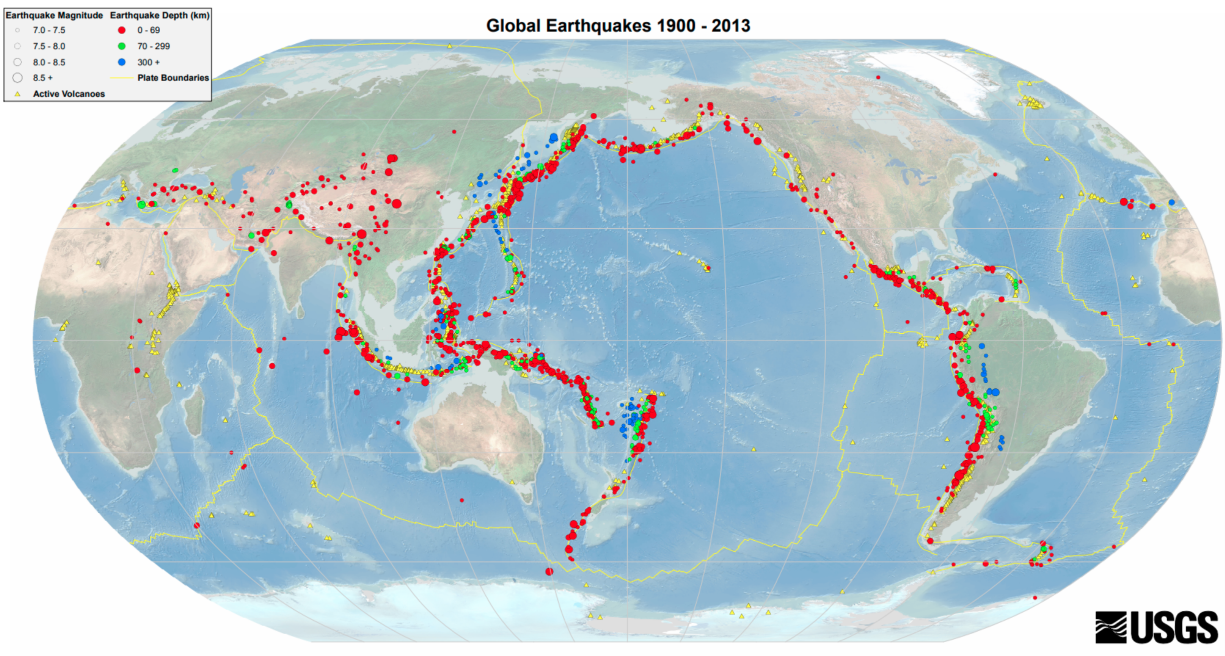


Figure 1: Map of global earthquakes, visibly localised to tectonic plate boundaries.

- On the Earth's surface, similarity physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice over land and ice shelves over the ocean.

Predominantly, the mathematics is of slow viscous flows. Topics include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

2 Plate cooling

Here we consider a half-space cooling model of the oceanic lithosphere (oceanic plates). The bottom surface of Earth's oceans, particularly clear in the Atlantic ocean, has a large scale structure in which the middle of the ocean (the *mid-ocean ridge*) is shallower than regions closer to the continents. The mid-ocean ridge forms as a result of separating tectonic plates. We know that the plates move apart here due to magnetic anomalies forming 'stripes' of alternating polarity. The quasi-periodic flipping of the Earth's magnetic polarity allows dating of the stripes. The plates are driven apart by convection of the Earth's mantle.

2.1 Thermal problem

We wish to form a model describing the depth of the ocean floor near mid-ocean ridges. First we estimate the temperature field. Consider an idealised model with a flat surface (for now), observed plate spreading rate U, surface temperature T_0, deep mantle temperature T_1. The temperature field is described by the advection-diffusion equation

$$\rho c_p \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) = \nabla \cdot (k \nabla T)$$

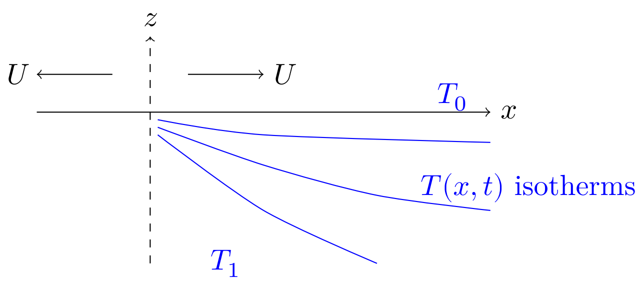


Figure 2: Schematic diagram of mid-ocean ridge spreading and mantle temperature isotherms.

where c_p is specific heat capacity, k is thermal conductivity, ρ is density, all assumed constant. For simplicity, we combine these constants into the thermal diffusivity $\kappa = k/\rho c_p$. Then

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$

We wish to find the steady state profile with $\partial_t = 0, \boldsymbol{u} = U\hat{\boldsymbol{x}}$ where U is constant. Note that far from the ridge axis, the thickness of the plate is much smaller than the extent of the plate. Hence in terms of scalings, $z \ll x$ and we may neglect the ∂_x^2 component of ∇^2. We have

$$U\frac{\partial T}{\partial x} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \approx \kappa \frac{\partial^2 T}{\partial z^2} \tag{1}$$

The scaling given by this equation is $U\Delta T/x \sim \kappa \Delta T/z^2$ where $\Delta T = T_1 - T_0$ is the natural temperature scale. There is no natural lengthscale, so we use that given by the advection-diffusion equation:

$$z \sim \sqrt{\frac{\kappa x}{U}}$$

We can proceed by finding a self-similar solution with similarity variable

$$\eta = \frac{z}{2\sqrt{\frac{\kappa x}{U}}}$$

and seek solutions of the form

$$\theta = \frac{T - T_0}{T_1 - T_0} = \theta(\eta)$$

Using the variables η, θ, (1) becomes

$$-U\Delta T \frac{\eta}{2x} \theta_\eta = \frac{\kappa \Delta T}{4\frac{\kappa x}{U}} \theta_{\eta\eta}$$

$$\implies \theta_{\eta\eta} + 2\eta\theta_\eta = 0$$

We can integrate directly to get $\theta_\eta = ae^{-\eta^2}$, which gives

$$\theta = b + a \int_0^\eta e^{-y^2} \, \mathrm{d}y$$

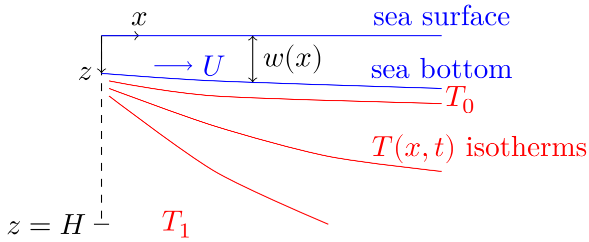


Figure 3: Schematic diagram of ocean depth and crust temperature surfaces.

The boundary conditions are $\theta(0) = 1$ and $\theta(\infty) = 1$ based on the definitions of T_0 and T_1. The thermal structure away from the ridge is then

$$T = T_0 + (T_1 - T_0)\,\mathrm{erf}\left(\frac{z}{2\sqrt{\frac{\kappa x}{U}}}\right) \tag{2}$$

where the error function erf and its complement erfc are defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,\mathrm{d}y$$
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

2.2 Ocean depth away from mid-ocean ridge

We now consider the depth of the ocean following from the temperature field derived above. We choose axes with z increasing downwards, placing the sea surface at $z = 0$ and the 'bottom' of the mantle at $z = H$. The coordinate x increases away from the mid-ocean ridge, with ocean depth $w(x)$ at a given x.
First, consider *isostacy*: Archimedean buoyancy applied to Earth's crust. This indicates the depth at which an object/fluid parcel of some density lies in a fluid of different density.

Denoting the density of the crust and mantle as ρ_c, ρ_m respectively, the crust of thicknes h sits at a depth b in the mantle. Hydrostatic balance gives $\rho_c g h = \rho_m g b$. Equivalently, we can consider a force balance between the weight of the curst and the buoyancy force:

$$\rho_c(h - b)g = (\rho_m - \rho_c)gb$$

Within the oceanic lithosphere we have a denssity field

$$\rho = \rho_m\,(1 - \alpha(T - T_1))$$

where $T = T(x, z)$ is the thermal model derived above, given by (2). Isostatic balance gives the following, which balances water weight, mantle weight, with water and mantle buoyancy. The ocean density is denoted by ρ_w.

$$
\begin{aligned}
\rho_w w_0 + \rho_m (H - w_0) &= \rho_w w(x) + \int_w^H \rho(T)\,\mathrm{d}z \\
&= \rho_w w + \rho_m (H - w) - \rho_m \alpha \int_w^H (T - T_1)\,\mathrm{d}z \\
\implies (\rho_m - \rho_w)(w - w_0) &= -\rho_m \alpha \int_w^H (T - T_1)\,\mathrm{d}z \\
&\approx -\rho_m \alpha (T_1 - T_0) \int_0^\infty \mathrm{erfc}(\eta) \cdot 2\sqrt{\frac{\kappa x}{U}}\,\mathrm{d}\eta
\end{aligned}
$$

where the last approximate equality follows from taking $w \to 0$ and $H \to \infty$, approximating the fact the mantle is much deeper than the ocean. The ocean depth is therefore

$$
w - w_0 = \frac{\rho_m}{\rho_m - \rho_w} \alpha (T_1 - T_0) \frac{2}{\sqrt{\pi}} \left(\frac{\kappa x}{U} \right)^{1/2}
$$

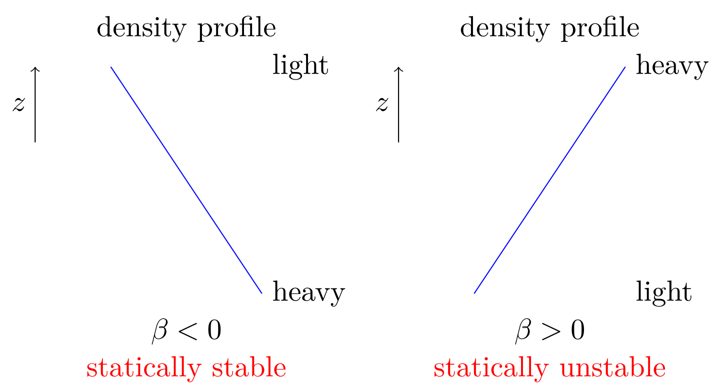
This model fits the data well near to the mid-ocean ridge, with crust age up to 75 million years. However, away from the ridge, the model breaks down as w tends to a constant as $x \to \infty$. The breakdown of the model is due to convection: the infinite depth curst approximation breaks down and convection dynamics become important.

3 Natural convection

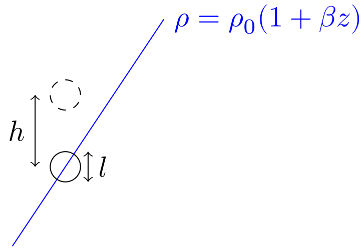
Natural convection arises in flows driven by density differences in a gravitational field, e.g. due to temperature or composition.

3.1 Static stability

Consider the case of no fluid motion $\boldsymbol{u} = 0$ and initial stratification $\rho = \rho_0(1 + \beta z)$. If $\beta < 0$, the dynamics are statically stable. If $\beta > 0$, the dynamics are statically unstable but dynamically could be stable.



Scaling analysis. Consider a fluid parcel of characteristic size l in unstable density profile $\rho = \rho_0(1 + \beta z), \beta > 0$. Suppose the parcel moves up a distance h in time τ.



The rise of the fluid parcel relasess potential energy E which scales as

$$E \sim (\Delta\rho l^3)gh \sim (\rho_0\beta h l^3)gh \sim \rho_0\beta g h^2 l^3$$

The timescale for the rise is limited by diffusion buoyancy $\tau \sim l^2/\kappa$ where κ is thermal diffusivity. Viscous dissipation over timescale τ scales as the shear stress times the distance travelled:

$$\mathcal{D} \sim \frac{\mu U}{l} l^2 h \sim \mu \frac{h/\tau}{l} l^2 h \sim \frac{\mu\kappa h^2}{l}$$

Instability arises if $E \gtrapprox \mathcal{D}$, which we can write as

$$\rho_0\beta g h^2 l^3 \gtrapprox \frac{\mu\kappa h^2}{l}$$

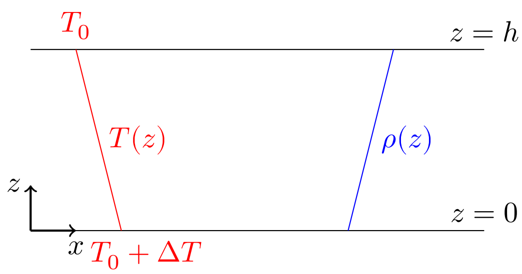
Define the *Rayleigh number*

$$\mathrm{Ra} = \frac{\rho_0\beta g l^4}{\kappa\mu}$$

Then there is instability if $\mathrm{Ra} \gtrapprox \mathcal{O}(1)$.

3.2 Onset of convection

Consider a fluid of depth h between $z = 0$ and $z = h$ with density profile $\rho(z)$ and temperature difference ΔT across the depth.



The governing equations are conservation of mass (3), conservation of momentum (4), and conservation of energy (5).

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} - \rho g\hat{\boldsymbol{z}} \tag{4}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T\right) = \nabla\cdot(k\nabla T) \tag{5}$$

For steady solutions with $\boldsymbol{u} = 0$ and $\partial_t = 0$ we find

$$
\begin{aligned}
T &= T_0 + \Delta T\left(1 - \frac{z}{h}\right) \\
\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} &= 0 \\
\frac{\partial p}{\partial z} &= -\rho_0 g\left(1 - \alpha(T - T_0)\right)
\end{aligned}
$$

We assess the stability by examining small perturbations to a steady base state:

$$
\begin{aligned}
&\boldsymbol{u} = 0 + \boldsymbol{u}'(\boldsymbol{x}, t) \\
&T = T_0 + \Delta T\left(1 - \frac{z}{h}\right) + T'(\boldsymbol{x}, t) \\
&p = p_0(z) + p'(\boldsymbol{x}, t)
\end{aligned}
$$

The linearied perturbation equations are thus

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u}' &= 0 \\
\rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} &= -\nabla p' + \mu \nabla^2 \boldsymbol{u}' + \rho_0 g \alpha T' \hat{\boldsymbol{z}} \\
\frac{\partial T'}{\partial t} - \frac{\Delta T}{h} w' &= \kappa \nabla^2 T'
\end{aligned}
$$

We wish to non-dimensionalise these equations. There are two intrinsic scales, temperature $\sim \Delta T$ and lengths $\sim h$. We form velocity and time characteristic scales via *diffusive scaling.* From the thermal equation:

$$
\frac{\Delta T}{t} \sim \frac{\Delta T U}{h} \sim \frac{\kappa \Delta T}{h^2}
$$

From the second relation we have $U \sim \kappa/h$, then from the first we have $t \sim h^2/\kappa$. The non-dimensionalised equations (dropping $'$ henceforth) are

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &= 0 \\
\frac{1}{\mathrm{Pr}} \frac{\partial \boldsymbol{u}}{\partial t} &= -\nabla p + \nabla^2 \boldsymbol{u} + \mathrm{Ra} T \hat{\boldsymbol{z}} \\
\frac{\partial T}{\partial t} - w &= \nabla^2 T
\end{aligned}
$$

where the *Prandtl number* $\mathrm{Pr} = \frac{\mu/\rho_0}{\kappa}$ which quantifies the importance of viscous diffusion versus thermal diffusion.