Cambridge Part III Maths

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Fluid Dynamics of Climate

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Contents

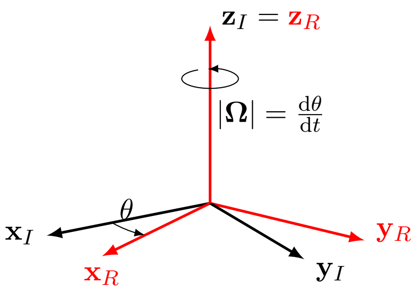
Lecture 1
12/10/20

1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p - \rho \nabla \phi + \rho \boldsymbol{F}$$

The body forces are assumed to be conservative with potential ϕ, e.g. $\phi = gz$ for gravitational force. $\boldsymbol{F}$ is the frictional force.
Consider a reference frame rotating about the z-axis with constant angular velocity $\boldsymbol{\Omega}$. Axes in the inertial frame are denoted with a subscript $_I$ and axes in the rotating frame are denoted with a subscript $_R$.



For a point with position vector $\boldsymbol{x}$ and velocity $\boldsymbol{u}_R = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R$ in the rotating reference frame

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{x}$$

or equivalently $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}$. Hence the acceleration is

$$\begin{aligned}
\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}\right]\right)_R + \boldsymbol{\Omega} \times (\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x})_R \\
&= \left(\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t}\right)_R + 2\boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})
\end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal accelertion*. Note that we can write the centrifugal acceleration in the form of a conservative force

$$\begin{aligned}
\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) = \nabla \phi_c \\
\phi_c = -\frac{1}{2} |\boldsymbol{\Omega} \times \boldsymbol{x}|^2
\end{aligned}$$

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u}\right) = -\nabla p - \rho \nabla (\phi + \phi_c) + \rho \boldsymbol{F} \tag{1}$$

We group the potential terms into a *geopotential* $\Phi \equiv \phi + \phi_c$. The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.
Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

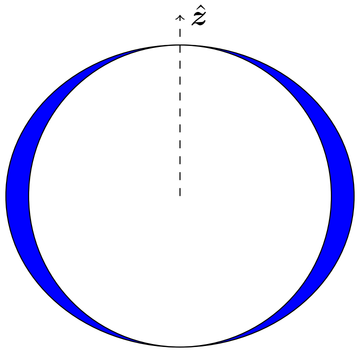


Figure 1: Geopotential ocean surface relative to a spherical Earth.

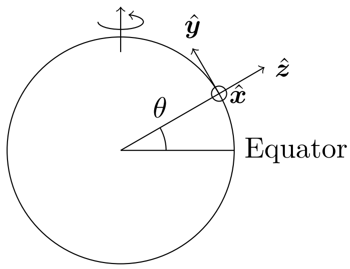


Figure 2: Local Cartesian coordinates

1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2). In this coordinate system $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$. Hence if $\boldsymbol{u} = (u, v, w)$ then

$$\begin{aligned} 2\boldsymbol{\Omega} \times \boldsymbol{u} &= (2\Omega w\cos\theta - 2\Omega v\sin\theta, 2\Omega u\sin\theta, -2\Omega u\cos\theta) \\ &= (-fv + f^*w, fu - f^*u) \end{aligned}$$

where $f \equiv 2\Omega\sin\theta$ is the *Coriolis parameter* and $f^* \equiv 2\Omega\cos\theta$.

Example. In Cambridge, $\theta = 52.1°N$ so

$$\begin{aligned} f &= 2\Omega\sin\theta \\ &= 2 \cdot \frac{2\pi}{3600 \cdot 24} \cdot 0.79s^{-1} \\ &\approx 1.14 \times 10^{-4}s^{-1} \end{aligned}$$

At mid-latitudes, $f \sim 10^{-4}$ is a good approximation.

We can simplify the Coriolis acceleration expression; often $f^*w \ll fv$ and $f^*u \ll g$. Hence

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \approx (-fv, fu, 0) = f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

This is the *traditional approximation.* This is *not* always a good approximation, particularly at intermediate scales.

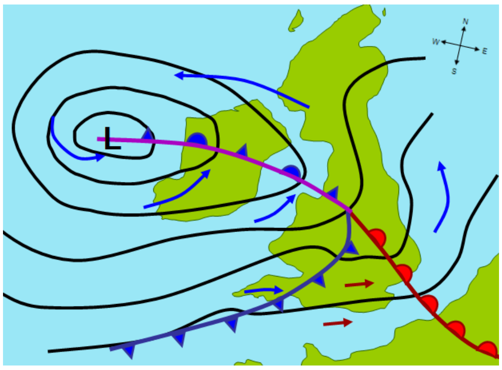


Figure 3: Lines of constant pressure p act as streamlines for the horizontal flow.

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1.2 Scale analysis.

Define characteristic scales for length L, time T, and velocity U. Non-dimensional variables are denoted with a superscript star: $\boldsymbol{u}^* = \boldsymbol{u}/U$, etc.
Using these scalings with $\boldsymbol{F} = \nu \nabla^2 \boldsymbol{u}$ we have

$$\frac{U}{T}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \frac{U^2}{L}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + fU\hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho}\nabla\left(p + \rho\Phi\right) + \frac{\nu U}{L^2}\nabla_*^2 \boldsymbol{u}^*$$

Dividing through by fU leaves the Coriolis acceleration term ord(1) with other terms scaled relatively.

$$\frac{1}{fT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \mathrm{Ro}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + \hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho f U}\nabla\left(p + \rho\Phi\right) + \mathrm{E}\nabla_*^2 \boldsymbol{u}^*$$

where $\mathrm{Ro} \equiv \frac{U}{fL}$ is the *Rossby number* and $\mathrm{E} \equiv \frac{\nu}{fL^2}$ is the *Ekman number*.

Example. For an atmospheric storm, $U \sim 10ms^{-1}, L \sim 1000km, f \sim 10^{-4}s^{-1}$. Thus $\mathrm{Ro} \sim 0.1, \mathrm{E} \sim 10^{-13}$.

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Further, if $T = L/U$, then $\mathrm{Ro} = U/fL = 1/fT$. For small Ro, E, on surfaces of constant Φ, $f\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla p$. This is *geostrophic balance*. In components, we have

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The equations of geostrophic balance can be arranged to give the horizontal velocity: $\boldsymbol{u}_H$

$$\boldsymbol{u}_H \equiv (u, v) = \frac{1}{\rho f}\hat{\boldsymbol{z}} \times \nabla p$$

Horizontal velocity is perpendicular to ∇p and hence parallel to isobars (lines of constant p), i.e. pressure acts like a streamfunction (see figure 3).
In the Northern Hemisphere, air moves clockwise around high p and anticlockwise around low p. A *cyclonic* rotation is in the same sense as $\boldsymbol{\Omega}$, *anticyclonic* in the opposite sense as $\boldsymbol{\Omega}$.

1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small Ro, E)

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0 \\ 2\boldsymbol{\Omega} \times \boldsymbol{u} &= -\frac{1}{\rho}\nabla p \end{aligned} \tag{2}$$

Taking the curl of (2) we have

$$\begin{aligned} \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\Omega_l u_m \\ &= \varepsilon_{kij}\varepsilon_{klm}\Omega_l \partial_j u_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\Omega_l \partial_j u_m \\ &= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i \end{aligned}$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0$$

For $\boldsymbol{\Omega} = (0, 0, \Omega)$, this implies $\frac{\partial w}{\partial z} = 0$. If $w = 0$ on some horizontal surface (e.g. ground) then $w = 0$ everywhere.
Also, $u_x + v_y = 0$, i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in 'columns' parallel to $\boldsymbol{\Omega}$, called *Taylor columns*.

2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density ρ_0 with angular velocity $\boldsymbol{\Omega} = (0, 0, f/2)$. Assume small ampltiude motions (i.e. $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$), i.e. neglect $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$. From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \tag{3}$$

$$v_t + fu = -\frac{p_y}{\rho_0} \tag{4}$$

$$w_t = -\frac{p_z}{\rho_0} \tag{5}$$

$$u_x + v_y + w_z = 0 \tag{6}$$

We will eliminate variables in favour of p.

$$\begin{aligned} \nabla \cdot ((3) - (5)) &\implies \nabla^2 p = \rho_0 f\,(v_x - u_y) \\ \partial_x(4) - \partial_y(3)\&(6) &\implies (v_x - u_y)_t = f w_z \end{aligned}$$

Combining these and using (5) we have

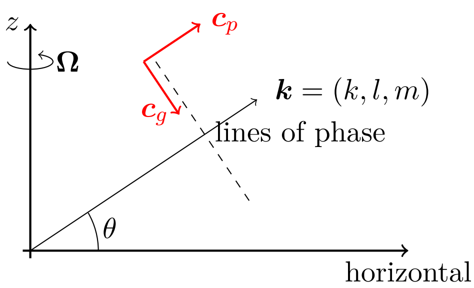
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for p. Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed $\boldsymbol{c}_p = w/\boldsymbol{k}$ and group velocity

$$\boldsymbol{c}_g = \frac{\partial w}{\partial \boldsymbol{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\boldsymbol{k}|^{3/2}}$$

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Note that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$. Also note $|\omega| \leq |f|$.

2.1 Inertial (free) oscillations

Assume $\nabla p = \mathbf{0}$. The x and y components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus $u = U \sin ft$ where f is the *inertial frequency*. Similarly, we have $v = U \cos ft$. For a particle with position (x_p, y_p) floating on an ocean surface $z = 0$ moving with the fluid velocity, we have

$$\begin{aligned} \frac{\mathrm{d}x_p}{\mathrm{d}t} &= u \implies x_p = -\frac{U}{f} \cos ft + x_0 \\ \frac{\mathrm{d}y_p}{\mathrm{d}t} &= v \implies y_p = -\frac{U}{f} \sin ft + y_0 \end{aligned}$$

Thus the motion of fluid particles describes describes *inertial circles* with radius $\frac{2U}{f}$.

2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress $\boldsymbol{\tau}_w$. Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e. $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e. $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e. $w(z = 0) = 0$
4. No flow in deep ocean, i.e. $\lim_{z \to -\infty} \boldsymbol{u} = \mathbf{0}$
5. Constant density ρ
6. Traditional approximation

Continuity (incompressibility) says $u_x + v_y + w_z = 0$. Assumptions 2 and 3 then imply $w = 0$ everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$
$$fu = \nu v_{zz} \tag{8}$$

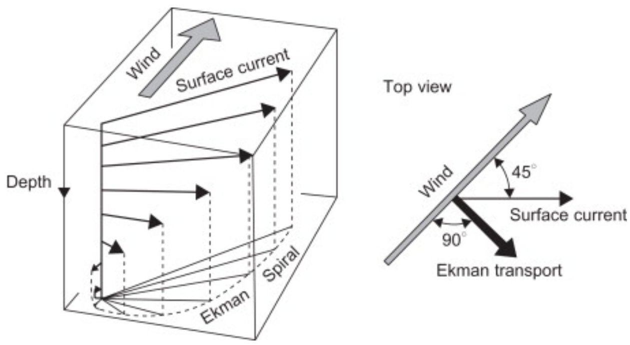

Figure 4: Ekman spiral.

Define the *complex velocity* $\mathcal{V} \equiv u + iv$. Then

$$\mathcal{V}_{zz} = \frac{if}{\nu}\mathcal{V} \tag{9}$$

Without loss of generality, assume $\boldsymbol{\tau}_w$ is aligned with the x-axis: $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$. Boundary conditions for (9) are

$$\mathcal{V}_z = \left(\frac{\tau_w}{\rho\nu}, 0\right) \quad \text{at } z = 0$$
$$\mathcal{V} = (0, 0) \quad \text{as } z \to -\infty$$

Thus $\mathcal{V} = Ae^{(1+i)z/\delta}$ where $\delta = \sqrt{\frac{2\nu}{f}}$, $A = \frac{\tau_w \delta(1-i)}{2\rho\nu}$. In terms of the velocity components, we have

$$u = \frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$
$$v = -\frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where $\nu\boldsymbol{u}_z \approx 0$ at $z = -h$. Since $\nu\boldsymbol{u}_z(z = 0) = (\tau_w/\rho, 0)$, the *Ekman transport* $\boldsymbol{U}_T$ is

$$U_T \equiv \int_{-h}^{0} u \,\mathrm{d}z = 0$$
$$V_T \equiv \int_{-h}^{0} v \,\mathrm{d}z = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented 90° to the right of the applied wind shear stress (in the Northern Hemisphere).

2.4 Ekman pumping

Consider a wind stress $\tau_w(y)$ that varies over large scales. Then from incompressibility

$$\int_{-h}^{0} w_z \,\mathrm{d}z = -\int_{-h}^{0} u_x \,\mathrm{d}z - \int_{-h}^{0} v_y \,\mathrm{d}z$$

Thus for h constant,

$$-w(z=-h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\tau_w}{\rho f}\right)$$

In general we have

$$w(z=-h) = \hat{\boldsymbol{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$

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3 Rotating shallow water equations

Consider a thin layer of fluid with constant density ρ. Define characteristic scales

- length $L =$ horiz., $H =$ vert.
- velocity U
- time T
- pressure P

such that $\partial_x, \partial_y \sim \frac{1}{L}, \partial_z \sim \frac{1}{H}$. Define the *aspect ratio* $\delta \equiv H/L$. We will assume $\delta \ll 1$. From continuity (incompressibility) we have

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \implies \frac{w}{H} &= \mathcal{O}(U/L) \\ \implies w &= \mathcal{O}(\delta U) \end{aligned}$$

Using the traditional approximation and assuming the fluid is inviscid, the x-momentum equation

$$\begin{array}{rcccccccc} & \dfrac{\partial u}{\partial t} & + u\dfrac{\partial u}{\partial x} & + v\dfrac{\partial u}{\partial y} & + w\dfrac{\partial u}{\partial z} & - fv & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \text{scaling:} & \dfrac{U}{T} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{wU}{H} & fU & = & \dfrac{P}{\rho L} \end{array} \tag{10}$$

Thus if p_x appears at leading order then

$$P \sim \rho U \max(L/T, U, fL)$$

Similarly the z-momentum equation and its scalings are

$$\begin{array}{rccccccc} & \dfrac{\partial w}{\partial t} & + u\dfrac{\partial w}{\partial x} & + v\dfrac{\partial w}{\partial y} & + w\dfrac{\partial w}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} - g \\ \text{scaling:} & \dfrac{w}{T} & \dfrac{Uw}{L} & \dfrac{Uw}{L} & \dfrac{w^2}{H} & = & \dfrac{P}{\rho H} \end{array} \tag{11}$$

Hence $\frac{\mathrm{D}w}{\mathrm{D}t} \sim \max(\frac{w}{T}, \frac{Uw}{L})$. Comparing with the pressure term, we have

$$\begin{aligned} \frac{\frac{\mathrm{D}w}{\mathrm{D}t}}{\frac{1}{\rho}\frac{\partial p}{\partial z}} &\sim \frac{\max(\frac{w}{T}, \frac{Uw}{L})}{\frac{U}{H}\max(\frac{L}{T}, \frac{U}{L}, f)} \\ &\sim \delta^2 \frac{\max(\frac{1}{T}, \frac{U}{L})}{\max(\frac{1}{T}, \frac{U}{L}, f)} \end{aligned}$$

Therefore to $\mathcal{O}(\delta^2)$ we have *hydrostatic balance*. To this order, (11) becomes

$$\frac{\partial p}{\partial z} - \rho g \implies p = \rho g(\eta - z)$$

assuming $p = 0$ at $z = \eta(x, y, t)$. Similarly, we have $\frac{1}{\rho}p_x = g\eta_x$ and $\frac{1}{\rho}p_y = g\eta_y$. Hence horizontal acceleration (i.e. the LHS of (10)) is independent of z. Motivated by this, we *assume* that horizontal velocity is also independent of z. For $Ro \ll 1$, this follows from the Tayor-Proudman theorem. Re-writing (10) with these results we have

$$u_t + uu_x + vu_y - fv = -g\eta_x \tag{12}$$
$$v_t + uv_x + vv_y + fu = -g\eta_y \tag{13}$$

since $u_z = v_z = 0$ by assumption. Integrating the continuity equation gives

$$w = -z(u_x + v_y) + A(x, y, t)$$

where A is to be determined by the boundary conditions. Requiring no normal flow at $z = -H_0 + h_b$ is imposed by $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ where $\boldsymbol{n} = \nabla(z - h_b)$. Thus

$$-u\frac{\partial h_b}{\partial x} - v\frac{\partial h_b}{\partial y} + w = 0$$

Hence

$$A(x, y, t) = u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$$

The kinematic boundary condition at $z = \eta$ is $\frac{\mathrm{D}\eta}{\mathrm{D}t} = w$ which may be written as

$$\eta_t + u\eta_x + v\eta_y - w = 0$$

where $w = -\eta(u_x + v_y) + u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$. Combining these boundary conditions gives

$$\eta_t + [(H_0 - h_b + \eta)u]_x + [(H_0 - h_b + \eta)v]_y = 0 \tag{14}$$

If $H \equiv H_0 - h_b + \eta$ is the total depth of the fluid, then since $H_t = \eta_t$,

$$H_t + (uH)_x + (vH)_y = 0 \tag{15}$$

which is a statement of the conservation of volume (equivalently mass, since ρ is constant). Equations (12), (13), and (14) are the *rotating shallow water* (SW) equations.

3.1 Potential vorticity (PV)

Denote the vertical vorticity by $\zeta = v_x - u_y$. Consider $\partial_x(13) - \partial_y(12)$, which gives

$$\zeta_t + u\zeta_x + v\zeta_y + vf_y = -(\zeta + f)(u_x + v_y)$$

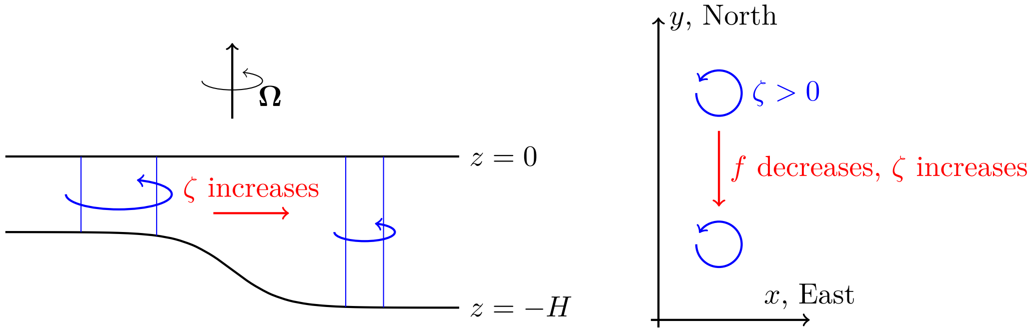
Now from conservation of volume (15),

$$u_x + v_y = -\frac{1}{H}\frac{\mathrm{D}H}{\mathrm{D}t}$$

Combining these relates the material derivative of ζ and H by

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\zeta + f}{H}\frac{\mathrm{D}H}{\mathrm{D}t} \implies \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{16}$$

Let $q \equiv \frac{\zeta+f}{H}$, the *shallow water potential vorticity* (SWPV). SWPV is conserved following fluid motion. We call ζ the *relative vorticity* and f the *planetary vorticity*. ζ and f will change as a fluid moves to conserve SWPV (changing f) and angular momentum (changing depth).



Lecture 5
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4 Small amplitude motions in rotating SW

Consider a stationary fluid with depth $H_s(x, y) = H_0 - h_b$. The fluid surface is then perturbed by $\eta(x, y, t)$ where $\eta \ll H_s$. The total depth is $H(x, y, t) = H_s + \eta$. For $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$, linearise the shallow water equations:

$$u_t - fv = -g\eta_x \tag{17}$$
$$v_t + fu = -g\eta_y \tag{18}$$
$$\eta_t + (uH_s)_x + (vH_s)_y = 0$$

Assuming f is constant, we have from $\partial_x(17) + \partial_y(18)$ and $\partial_y(17) - \partial_x(18)$:

$$\partial_t \left[\left(\partial_t^2 + f^2 \right) \eta - \nabla \cdot (gH_s \nabla \eta) \right] - fgJ(H_s, \eta) = 0 \tag{19}$$

where the Jacobian $J(a, b) = a_x b_y - a_y b_x$. For the velocity components we have

$$\left(\partial_t^2 + f^2 \right) u = -g \left(\eta_{xt} + f\eta_y \right) \tag{20}$$
$$\left(\partial_t^2 + f^2 \right) v = -g \left(\eta_{yt} + f\eta_x \right) \tag{21}$$

4.1 Steady flows

We now assume $\partial_t = 0$. From (20), (21),

$$u = -\frac{g}{f}\eta_y, \qquad v = \frac{g}{f}\eta_x$$

This is *shallow water geostrophic balance*: the surface displacement η acts as a streamfunction. Applying the steady assumption to (19) gives $J(H_s, \eta) = 0$ which implies $\eta = \eta(H_s(x, y))$. Hence linearised steady geostrophic flow in shallow water follows contours of constant depth. Steady PV conservation follows from (16) with $\partial_t = 0$ and assuming $\zeta \ll f$

$$\boldsymbol{u} \cdot \nabla \frac{f}{H_s} = 0$$

Thus when f varies, the flow follows contours of constant f/H_s.

4.2 Waves in an unbounded domain

Assume H_s is constant. From (19), we have

$$\left(\partial_t^2 + f^2\right)\eta - gH_s\nabla^2\eta = 0$$

Seek plane wave solutions to this wave equation with ansatz $\eta = \eta_0 \exp(i(kx + ly - \omega t))$. The dispersion relation is then

$$\omega^2 = f^2 + gH_s(k^2 + l^2) \tag{22}$$

If $f = 0$, i.e. no rotation, then the frequency is $\omega = \pm\sqrt{gH_s}|\boldsymbol{k}| = \omega_0$ and the phase speed is $|c_p| = \frac{|\omega|}{|\boldsymbol{k}|} = \sqrt{gH_s} = c_0$. For $f \neq 0$, we get *Poincaré* waves with

$$\omega^2 > \omega_0^2, \qquad |c_p| > c_0$$

i.e. rotation increases the frequency and phase speed. Define the *Rossby deformation scale* $R_D \equiv \frac{c_0}{f}$. From (22),

$$\frac{\omega^2}{f^2} = 1 + R_D^2|\boldsymbol{k}|^2$$

Without loss of generality, let $l = 0$, by reorienting x and y. If $\eta = \eta_0\cos(kx - \omega t)$ then (20), (21) imply the fluid velocity is

$$u = \frac{\omega_0\eta_0}{kH_s}\cos(kx - \omega t)$$
$$v = \frac{f\eta_0}{kH_s}$$

Thus the motion is an ellipse, also known as a *tidal ellipse*, which reduces to intertial circles if $\omega_0 = f$:

$$u^2 + \frac{\omega_0^2}{f^2}v^2 = \frac{\omega_0^2\eta_0^2}{k^2H_s^2}$$

Since $\omega > f$, the fluid moves anticylonically. The Rossby deformation scale R_D is the length scale for which rotation becomes important. Consider short and long waves:

- Short waves: $|\boldsymbol{k}|R_D \gg 1$. We have $\omega^2 \to gH_s|\boldsymbol{k}|^2$ i.e. non-rotating shallow water gravity waves.
- Long waves: $|\boldsymbol{k}|R_D \ll 1$. We have $\omega^2 \to f^2$ i.e. inertial waves where fluid moves in inertial circles. Gravity is not involved.

Lecture 6
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5 Geostrophic adjustment

Consider the response of rotating shallow water to an initial state *not* in geostrophic balance. Here, we consider $\eta(x, y,) = \eta_0\text{sgn}(x)$, $\boldsymbol{u}(x, y, 0) = \boldsymbol{0}$, so the initial PV is 0.
Assume f is constant, the perturbation is small $\eta_0 \ll H$, the PV is small $\zeta \ll f$, and the bottom is flat $H_s = H_0$. Linearise the shallow water PV:

$$q = \frac{f + \zeta}{H_0 + \eta} = \frac{f}{H_0}\left(1 + \frac{\zeta}{f} + \dots\right)\left(1 - \frac{\eta}{H_0} + \dots\right) \approx \frac{f}{H_0}\left(1 + \frac{\zeta}{f} - \frac{\eta}{H_0}\right)$$

Since PV is conserved, we have

$$\frac{\zeta}{f} - \frac{\eta}{H_0} = -\frac{\eta_0}{H_0}\mathrm{sgn}(x) \qquad \forall t \tag{23}$$

By symmetry, $\partial_y \equiv 0$ so the PV is $\zeta = v_x$. The linearised shallow water equations in this case

$$\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= 0 \\ \eta_t + H_0 u_x &= 0 \end{aligned}$$

Using these equations we have

$$\zeta = v_x = \frac{u_{xt} + g\eta_{xx}}{f} = -\frac{1}{fH_0}\eta_{tt} + \frac{g}{f}\eta_{xx}$$

Now conservation of potential vorticity (23) gives

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = f^2\eta_0\mathrm{sgn}(x)$$

where $c^2 \equiv gH_0$. This is a *Klein-Gordon equation* where the $f^2\eta$ term adds elasticity to the waves.

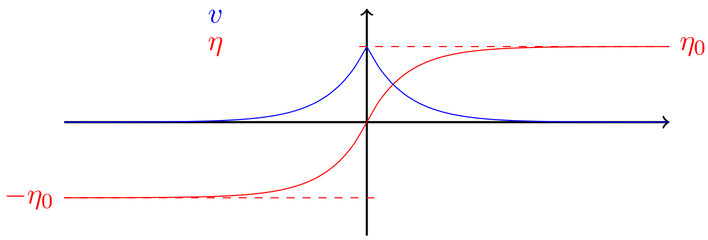
5.1 Steady solutions

Consider steady solutions. Owing to the step forcing, our BCs are to match η_x and η at $x = 0$. We find

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R_d} & x > 0 \\ -1 + e^{x/R_d} & x < 0 \end{cases} \tag{24}$$

where $R_d \equiv \sqrt{gH_0}/f$ is the *deformation radius*. From the equations of geotrophic balance we have the velocity components

$$u = 0, \qquad v = \frac{g\eta_0}{fR_d}e^{-|x|/R_d}$$



5.2 Transients

The steady solution (24) solves the geostrophic adjustment equation, but it does not match the initial conditions. We add this particular solution to a solution to the homogeneous equation

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = 0$$

with initial condition

$$\eta = \eta_0\mathrm{sgn}(x) - \eta_{\text{steady}} = \eta_0 e^{-|x|/R_d}\mathrm{sgn}(x)$$

We seek solutions of plane wave form

$$\eta = \hat{\eta}e^{i(kx-\omega t)}$$

with $\omega^2 = f^2 + c^2k^2$. These are Poincaré waves.

5.3 Energetics

The change in potential energy per unit length in the y direction is

$$\begin{aligned}
PE_{\text{initial}} - PE_{\text{final}} &= \int_{-\infty}^{\infty}\int_0^{\eta_i} \rho_0 g z \,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty} \rho_0 g z \,\mathrm{d}z\,\mathrm{d}x \\
&= 2\rho_0 g \left[\int_0^{\infty} \frac{\eta_i^2}{2}\,\mathrm{d}x - \int_0^{\infty} \frac{\eta_f^2}{2}\,\mathrm{d}x\right] \\
&= \rho_0 g \eta_0^2 \int_0^{\infty} \left[1 - (1 - e^{-x/R_d})^2\right] \mathrm{d}x \\
&= \frac{3}{2}\rho_0 g \eta_0^2 R_d
\end{aligned}$$

The change in kinetic energy per unit length in the y direction is

$$\begin{aligned}
KE_{\text{initial}} - KE_{\text{final}} &= \int_{-\infty}^{\infty}\int_{-H}^{\eta_i} \frac{1}{2}\rho_0 v_i^2 \,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty}\int_{-H}^{\eta_f} \frac{1}{2}\rho_0 v_f^2 \,\mathrm{d}z\,\mathrm{d}x \\
&\approx 0 - \frac{1}{2}\rho_0 \int_{-\infty}^{\infty} H_s v_f^2 \,\mathrm{d}x \\
&= -\rho_0 H_s \int_0^{\infty} \frac{g^2\eta_0^2}{f^2 R_d^2} e^{-2x/R_d}\,\mathrm{d}x \\
&= -\rho_0 \frac{R_d^2 g \eta_0^2}{R_d^2} \cdot -\frac{R_d}{2} \cdot \left[e^{-2x/R_d}\right]_0^{\infty} \\
&= -\rho_0 g \eta_0^2 \frac{R_d}{2}
\end{aligned}$$

Only $\frac{1}{3}$ of the potential energy released is converted into kinetic energy of the geostrophic flow.
Lecture 7 26/10/20 The remainder is radiated away by Poincaré waves.

6 Quasi-geostrophic equations

Large scale motions in the ocean and atmosphere are associated with small Rossby number $Ro \equiv \frac{U}{fL} \ll 1$. In this limit, the rotating shallow water equations are approximated by the SW quasi-geostrophic (SW QG) equation. Start from the SW PV equation:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{25}$$

Assumption 1: $Ro \ll 1$ Assuming a small Rossby number implies the flow is close to geostrophic balance with

$$f\hat{\boldsymbol{k}} \times \boldsymbol{u} \approx -g\nabla\eta$$

where $\hat{\boldsymbol{k}}$ is the vertical unit vector. Define the *geostrophic streamfunction* $\psi \equiv \frac{g\eta}{f}$. In terms of this streamfunction we have

$$\begin{aligned}
\boldsymbol{u} &\approx -\nabla \times (\psi\hat{\boldsymbol{k}}) \\
\zeta &= (\nabla \times \boldsymbol{u})\hat{\boldsymbol{k}} \approx \nabla^2\psi
\end{aligned}$$

Assumption 2: small changes in f Recall the Coriolis parameter $f = 2\Omega \sin\theta$ where θ is latitude. Expand in a Taylor series about $\theta = \theta_0$ to get

$$f = f_0 + y\frac{\mathrm{d}f}{\mathrm{d}y}|_{\theta_0} + \cdots \approx f_0 + \beta y$$

where y is in the direction of local North, $f_0 = 2\Omega \sin\theta_0$ and β is defined as

$$\beta = \frac{1}{R}\frac{\mathrm{d}f}{\mathrm{d}\theta}|_{\theta_0} = \frac{2\Omega}{R}\cos\theta_0$$

with R the radius of Earth. For characteristic length scale L, assume $\frac{\beta L}{f_0} \ll 1$. This is the *$\beta$-plane approximation.*

Assumption 3: small changes in fluid height. This is consistent with small Rossby number: from geostrophic balance, we know $\eta \sim \frac{fUL}{g}$ and $\frac{\eta}{H_0} \sim \frac{fUL}{gH_0} = \frac{U}{fL}\frac{L^2}{R_D^2}$. Therefore $\eta/H_0 \ll 1$ if $Ro \ll \frac{R_D^2}{L^2}$. For $L \sim R_D$, $Ro \ll 1$ implies $\eta/H_0 \ll 1$. Further, we assume $h_b/H_0 \ll 1$.

Quasi-geostrophic equations. With these assumptions, SWPV becomes

$$\begin{aligned}
\frac{\zeta + f}{H_0 - h_b + \eta} &\approx \frac{f_0}{H_0}\frac{1 + \frac{\beta y}{f_0} + \frac{\zeta}{f_0}}{1 - \frac{h_b}{H_0} + \frac{\eta}{H_0}} \\
&\approx \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\nabla^2\psi}{f_0} + \frac{h_b}{H_0} - \frac{f_0\psi}{gH_0}\right) \\
&= \frac{f_0}{H_0}P_g
\end{aligned}$$

where P_g is the *quasi-geostrophic potential vorticity* and $\zeta = \nabla^2\psi, \eta = \frac{f_0\psi}{g}$. Hence from SWPV conservation (25),

$$\frac{\partial P_g}{\partial t} + \boldsymbol{u}\cdot\nabla P_g \approx 0$$

Using $\boldsymbol{u} \approx -\nabla \times (\psi\hat{\boldsymbol{k}})$, $\boldsymbol{u} = -\psi_y, v = \psi_x$ so

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) \approx 0 \tag{26}$$

This is the *shallow water Quasi-geostrophic* (SWQG) equation, which is one equaiton for one unknown ψ, as opposed to SWPV with 2 unknowns ζ, η.

6.1 Waves in QG

Assume a flat bottom $h_b = 0$. Linearise (26) about a state of rest (i.e. neglect terms $\mathcal{O}(\psi^2)$). Then

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{f_0^2}{gH_0}\psi\right) + \frac{\partial\psi}{\partial x}\beta = 0$$

Seek plane wave solutions of the form

$$\psi = \psi_0 e^{i(kx+ly-\omega t)}$$

with dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2 + R_D{-2}}, \qquad R_D \equiv \frac{\sqrt{gH_0}}{f_0}$$

This is the *Rossby wave dispersion relation.* Note $\omega = 0$ (i.e. no waves) if $\beta = 0$. Also, if $h_b = 0$ and $\beta = 0$ there are no wave solutions unlike rotating SW. Thus the QG system 'filters' out Poincaré waves. Note that $\beta = \frac{2\Omega}{R}\cos\theta \geq 0$, hence $c_p = \frac{\omega}{k} \leq 0$. Rossby wave speed is always directed to the *west.*

Consider the size of the dynamic terms in P_g, specifically the ratio of relative vorticity to surface height

$$\frac{\nabla^2\psi}{-\frac{f_0^2\psi}{gH_0}} \sim \frac{R_D^2}{L^2}$$

Hence relativity vorticity dominates at scales small compared to R_D whilst surface height dominates at scales large compared to R_D.