Cambridge Part III Maths

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Fluid Dynamics of the Solid Earth

based on a course given by
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Contents

Lecture 1
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1 Introduction

The course will use the wealth of observations of the solid Earth to motivate mathematical models of the physical processes governing its evolution. The dynamic evolution is governed by a rich variety of physical processes occuring on a wide range of length and time scales.

- The Earth's core is formed by the solidifcation of a mixture of molten iron and various light elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field.
- On million year timescales, the solid mantle convects, and as it upwells to the surface it partially melts leading to the volcanism.
- At the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record (figure 1)
- In the Earth's surface, fluids flow through poroous rocks, for example groundwater aquifers which feed streams and rivers which erode the solid surface.
- On the Earth's surface, similarity physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice over land and ice shelves over the ocean.

Predominantly, the mathematics is of slow viscous flows. Topics include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic

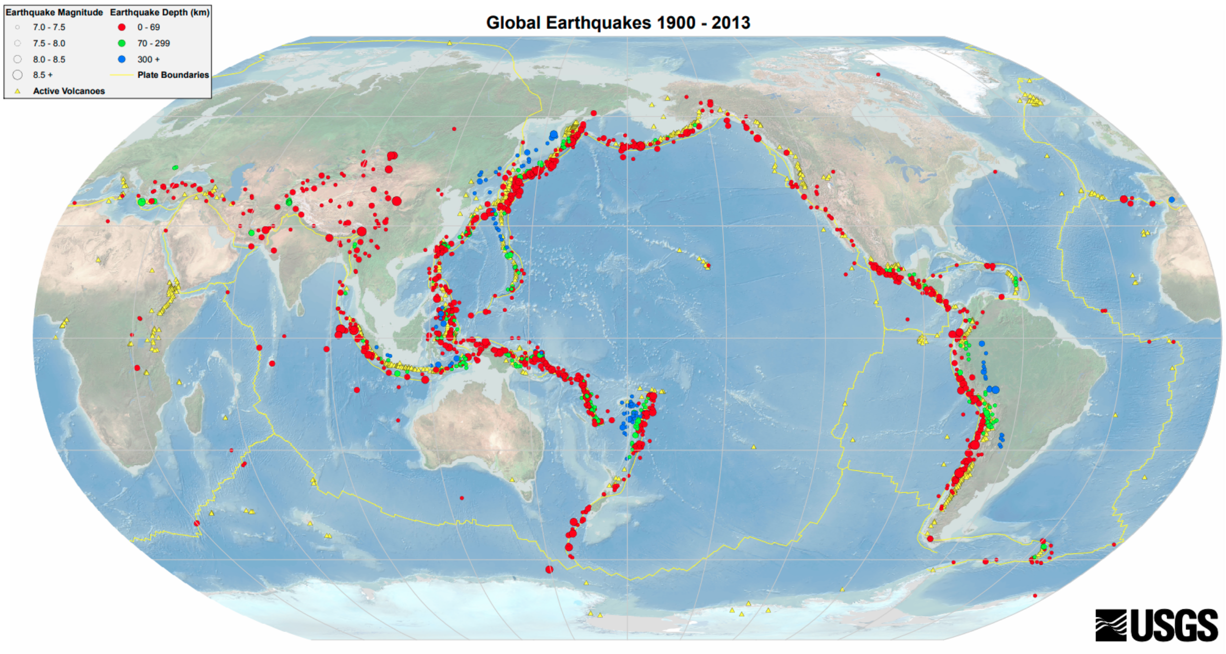

Figure 1: Map of global earthquakes, visibly localised to tectonic plate boundaries.

and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

2 Half-space cooling of the oceanic plates (mid-ocean ridges)

The bottom surface of Earth's oceans, particularly clear in the Atlantic ocean, has a large scale structure in which the middle of the ocean (the *mid-ocean ridge*) is shallower than regions closer to the continents. The mid-ocean ridge forms as a result of separating tectonic plates. We know that the plates move apart here due to magnetic anomalies forming 'stripes' of alternating polarity. The quasi-periodic flipping of the Earth's magnetic polarity allows dating of the stripes. The plates are driven apart by convection of the Earth's mantle.
We wish to form a model describing the depth of the ocean floor near mid-ocean ridges. First we estimate the temperature field. Consider an idealised model with a flat surface (for now), observed plate spreading rate U, surface temperature T_0, deep mantle temperature T_1. The temperature field is described by the advection-diffusion equation

$$\rho c_p \left(\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T \right) = \nabla \cdot (k \nabla T)$$

where c_p is specific heat capacity, k is thermal conductivity, ρ is density, all assumed constant. For simplicity, we combine these constants into the thermal diffusivity $\kappa = k/\rho c_p$. Then

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$

We wish to find the steady state profile with $\partial_t = 0, \boldsymbol{u} = U\hat{\boldsymbol{x}}$ where U is constant. Note that far from the ridge axis, the thickness of the plate is much smaller than the extent of the plate. Hence in terms of scalings, $z \ll x$ and we may neglect the ∂_x^2 component of ∇^2. We have

$$U \frac{\partial T}{\partial x} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \approx \kappa \frac{\partial^2 T}{\partial z^2} \tag{1}$$

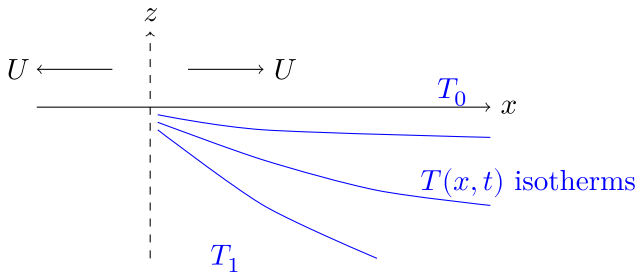

Figure 2: Schematic diagram of mid-ocean ridge spreading and mantle temperature isotherms.

The scaling given by this equation is $U\Delta T/x \sim \kappa \Delta T/z^2$ where $\Delta T = T_1 - T_0$ is the natural temperature scale. There is no natural lengthscale, so we use that given by the advection-diffusion equation:

$$z \sim \sqrt{\frac{\kappa x}{U}}$$

We can proceed by finding a self-similar solution with similarity variable

$$\eta = \frac{z}{2\sqrt{\frac{\kappa x}{U}}}$$

and seek solutions of the form

$$\theta = \frac{T - T_0}{T_1 - T_0} = \theta(\eta)$$

Using the variables η, θ, (1) becomes

$$-U\Delta T \frac{\eta}{2x}\theta_\eta = \frac{\kappa \Delta T}{4\frac{\kappa x}{U}}\theta_{\eta\eta}$$
$$\implies \theta_{\eta\eta} + 2\eta\theta_\eta = 0$$

We can integrate directly to get $\theta_\eta = ae^{-\eta^2}$, which gives

$$\theta = b + a\int_0^\eta e^{-y^2}\,\mathrm{d}y$$

The boundary conditions are $\theta(0) = 1$ and $\theta(\infty) = 1$ based on the definitions of T_0 and T_1. The thermal structure away from the ridge is then

$$T = T_0 + (T_1 - T_0)\,\mathrm{erf}\left(\frac{z}{2\sqrt{\frac{\kappa x}{U}}}\right)$$

where the error function erf and its complement erfc are defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,\mathrm{d}y$$
$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$