Cambridge Part III Maths

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Fluid Dynamics of Environment

based on a course given by
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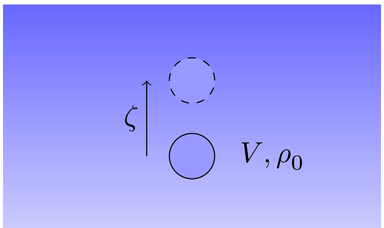
Notes created based on Josh Kirklin's LaTeX packages & classes. Please do not distribute these notes other than to fellow Part III students. Please send errors and suggestions to cwp29@cam.ac.uk.

Contents

Lecture 1
22/01/21

1 Internal waves

1.1 Intuitive version



Consider a fluid parcel of volume V and density ρ_0 in a fluid with density profile $\hat{\rho}(z)$. Suppose the parcel is moved upwards by ζ. The parcel experiences a *buoyancy force* $B = gV\zeta\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$. Newton's second law gives

$$\ddot{\zeta} + \left(-\frac{g}{\rho_0}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}\right)\zeta = 0$$

The *buoyancy frequency* (or Brunt-Väisälä frequency) is defined as

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$$

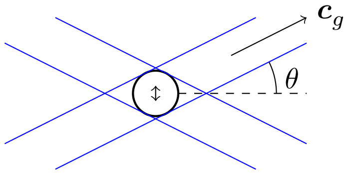
which has general solution

$$\zeta = A\cos Nt + B\sin Nt$$

If we instead consider a fluid slab inclined at angle θ with the vertical rather than a fluid parcel, the slab can fall in its plane much more easily than in the vertical. Hence in this situation we have

$$\ddot{\zeta} + N^2 \cos^2 \theta \zeta = 0$$

The dispersion relation is thus $\omega/N = \cos\theta$.
Now consider a sphere oscillating at frequency ω in the vertical in a stratified fluid with density $\rho(z)$. The fluid resonates in bands at angle θ satisfying the dispersion relation, provided $\omega < N$. Intuitively, the group velocity must be out of the beams as energy is radiated away.



At the leading edge of the rays, baroclinic vorticity is generated by the movement of fluid of different density to its surroundings. This provides the mechanism for the instability.

1.2 Rigorous derivation

Consider a fluid in which the mean pressure $p_0(z)$ and the mean density $\rho_0(z)$ are in hydrostatic balance when the fluid is at rest:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g$$

Assume that the vertical lengthscale for ρ_0 variation is L. Motion is governed by the Navier-Stokes equations (1) and (2) with $\nu = 0$, and mass conservation (3).

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p - \rho g \hat{\boldsymbol{z}} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \rho = 0 \tag{3}$$

Following the Boussinesq approximation, assume small perturbations to the mean state: $\rho = \rho_0(z) + \tilde{\rho}$ and $p = p_0(z) + \tilde{p}$ where $\tilde{p} \ll p_0, \tilde{\rho} \ll \rho_0$. Under this approximation, the momentum equation (2) becomes

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0} g \hat{\boldsymbol{z}} \\ &= -\frac{1}{\rho_0}\nabla(p + \rho_0 g z) - g' \hat{\boldsymbol{z}}\end{aligned}$$

where $g' = g(\rho - \rho_0)/\rho_0$ is the *reduced gravity*. We now linearise $\boldsymbol{u}$ about a state of rest, ignoring second order quantities in the velocity disturbance $\boldsymbol{u}'$. It is now further desirable to split the disturbance components into $\tilde{\rho} = \hat{\rho} + \rho', \tilde{p} = \hat{p} + p'$ where $\hat{\rho}, \hat{p}$ are in hydrostatic balance. We have

$$\begin{aligned}\nabla \cdot \boldsymbol{u}' &= 0 \\ \frac{\partial \rho'}{\partial t} + w' \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z} = \frac{\partial \rho'}{\partial t} - w' \frac{\rho_0}{g} N^2 &= 0 \\ \frac{\partial \boldsymbol{u}'}{\partial t} &= -\frac{1}{\rho_0}\nabla(p_0 + \hat{p}) - \frac{g\hat{\rho}}{\rho_0}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}\end{aligned}$$

Hydrostatic balance eliminates the first two RHS terms of the momentum equation; the hydrostatic pressure field is

$$p_0 + \hat{p} = -\int g\hat{\rho}\,\mathrm{d}z$$

Finally we have

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}$$

Define buoyancy $b = -g\rho'/\rho_0$. The governing equations are now

$$\begin{aligned}\frac{\partial b}{\partial t} &= -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}}\\ \frac{\partial \boldsymbol{u}'}{\partial t} &= -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}}\end{aligned}$$

To eliminate pressure, we take the curl of the momentum equation to get

$$\frac{\partial \boldsymbol{\zeta}'}{\partial t} = -\hat{\boldsymbol{z}} \times \nabla b$$

where $\boldsymbol{\zeta}' = \nabla \times \boldsymbol{u}'$ is the disturbance vorticity. Using the buoyancy equation we have

$$\left[\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2\right] w' = 0$$

where $\nabla_H = (\partial_x, \partial_y)$ is the horizontal gradient operator. This equation admits plane wave solutions

$$w'(\boldsymbol{x}, t) = \Re\left[\hat{w}(t)e^{i(k_x x + k_y y - \omega t)}\right]$$

where $\hat{w}$ satisfies

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)\hat{w} = 0$$

which has general solution

$$\begin{aligned}\hat{w} &= \Re\left[Ae^{-inz} + Be^{inz}\right]\\ n^2 &= (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)\end{aligned}$$

If $\omega > N$, n is imaginary and, defining $\gamma = \sqrt{1 - N^2/\omega^2}$, we have

$$w' = (Ae^{-\gamma kz} + Be^{\gamma kz})e^{i(k_x x + k_y y - \omega t)}$$

When $N = 0$, we get potential flow. If $0 < N < \omega$ then we have rescaled potential flow, with scaling γ. If $\omega < N$ then n is real and solutions are oscillatory with

$$n^2 = kz^2 = (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)$$

The wavenumber vector is $\boldsymbol{k} = (k_x, k_y, k_z) = (k, l, m)$. Hence

$$\frac{\omega^2}{N^2} = \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} = 1 - \frac{k_z^2}{|\boldsymbol{k}|^2} = \cos^2\theta$$

where θ is the angle between $\boldsymbol{k}$ and the horizontal plane.