Cambridge Part III Maths

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Astrophysical Fluid Dynamics

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Contents

Lecture 1
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1. Introduction

1.1. Areas of application

Astrophysical fluid dynamics (AFD) is relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, the interstellar medium, the intergalactic medium, and cosmology itself. A fluid description is not applicable in regions that are solidified, such as the rocky or icy cores of giant planets and the crusts of neutron stars, and also in very genuous regions where the medium is not sufficiently collisional.

1.2. Theoretical varieties

Various flavours of AFD are in use. The basic models we will consider are:

Hydrodynamics (HD) / Newtonian gas dynamics: This model is non-relativistic, compressible, ideal (inviscid and adiabatic), self-gravitating, and usually assumes a perfect gas.

Magnetohydrodynamics (MHD): This model is the same as above, with the addition of a magnetic field. We will often use ideal MHD, which assumes a perfectly conducting fluid.

1.3. Characteristic features

The elements of theory often important in AFD are compressibility, gravitation, and thermal physics. Sometimes, magnetic fields, radiation forces, and relativity are important. Rarely important aspects are viscosity, surface tension, and solid boundaries.

1.4. Useful data

Some useful data for the course, in CGS (centimetres, grams, seconds) units:

Newton's constant	$G = 6.674 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$
Boltzmann's constant	$k = 1.381 \times 10^{-16}\,\mathrm{erg}\,\mathrm{K}^{-1}$
Stefan's constant	$\sigma = 5.670 \times 10^{-5}\,\mathrm{erg}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{K}^{-4}$
Speed of light	$c = 2.998 \times 10^{1-}\,\mathrm{cm}\,\mathrm{s}^{-1}$
Hydrogen mass	$m_H = 1.674 \times 10^{-24}\,\mathrm{g}$
Solar mass	$M_s = 1.988 \times 10^{33}\,\mathrm{g}$
Solar radius	$R_s = 6.957 \times 10^{10}\,\mathrm{cm}$
Solar luminosity	$L_s = 3.828 \times 10^{33}\,\mathrm{ergs}^{-1}$
Parsec	$pc = 3.086 \times 10^{18}\,\mathrm{cm}$
Astronomial unit (AU)	$au = 1.496 \times 10^{13}\,\mathrm{cm}$
Joule erg conversion	$1\mathrm{J} = 10^7\mathrm{erg}$

2. Ideal gas dynamics

2.1. Fluid variables

A fluid is characterised by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The *specific volume* (volume per unit mass) is $v = 1/\rho$.
We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionisation and recombination.

2.2. Eulerian and Lagrangian viewpoints

In the *Eulerian* viewpoint we consider how fluid properties vary in time at a point which is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian time derivative is simply ∂_t.
In the *Lagrangian* viewpoint we consider how fluid properties vary in time at a point which moves with the fluid at velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The Lagrangian time derivative (or material derivative) is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

2.3. Material points and structures

A material point is an idealised fluid element, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of the fluid. Note that the true particles of which the fluid is composed also have random thermal motion.

Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are deformed by it. An infinitesimal material line element $\delta\boldsymbol{x}$ evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u}$$

It changes its length and/or orientation in the presence of a velocity gradient.
Infinitesimal material surface or volume elements can be defined from two or three material line elements according to the vector product and the triple scalar product.

$$\begin{aligned}\delta\boldsymbol{S} &= \delta\boldsymbol{x}^{(1)}\times\delta\boldsymbol{x}^{(2)}\\ \delta V &= \delta\boldsymbol{x}^{(1)}\cdot\delta\boldsymbol{x}^{(2)}\times\delta\boldsymbol{x}^{(3)}\end{aligned}$$

They evolve according to

$$\begin{aligned}\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta\boldsymbol{S} - \nabla\boldsymbol{u}\cdot\delta\boldsymbol{S}\\ \frac{\mathrm{D}\delta V}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta V\end{aligned}$$

The second result is easier to understand: the volume element increases when the flow is divergent.

2.4. Equation of mass conservation

The equation of mass conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

has typical form of conservation law: rate of change of a density and divergence of a flux. Here, ρ is mass density and $\rho\boldsymbol{u}$ is mass flux density. An alternative form of the same equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u}$$

If $\delta m = \rho\delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0$$

2.5. Equation of motion

The equation of motion

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\nabla\Phi - \nabla p$$

derives from Newton's second law per unit volume with gravitational and pressure forces. The gravitational potential is $\Phi(\boldsymbol{x}, t)$ and $\boldsymbol{g} = -\nabla\Phi$ is the gravitational field.
The force due to pressure acting on a volume V with bounding surface S is

$$-\int_S p\,\mathrm{d}\boldsymbol{S} = \int_V (-\nabla p)\,\mathrm{d}V$$

Viscous forces are neglected in ideal gas dynamics.

2.6. Poisson's equation

The gravitational potential is related to the mass density by Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho$$

where G is Newton's constant. The solution

$$\Phi = \Phi_{\text{int}} + \Phi_{\text{ext}} = -G\int_V \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}' - G\int_{\hat{V}} \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}'$$

generally involves contributions form both the fluid region V under consideration and the exterior region $\hat{V}$. A *non-self-gravitating* fluid is one of negligible mass for which Φ_{int} can be neglected. More generally, the *Cowling approximation* consists of treating Φ as being specified in advance, so that Poisson's equation is not coupled to the other equations.

2.7. Thermal energy equation and equation of state

In the absence of non-adiabatic heating (e.g. by viscous dissipation or nuclear reactions) and cooling (e.g. by radiation or conduction),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0$$

where s is the *specific entropy* (entropy per unit mass). Fluid element undergo reversible thermodynamic changes and preserve their entropy (adiabatic flow). This condition is violated in shocks (see section 6).
The thermal variables (T, s) can be related to the dynamical variables (P, ρ) via an *equation of state* and standard thermodynamic identities. The most important case is that of an *ideal gas* with *blackbody radiation*

$$p = p_g + p_r = \frac{k\rho T}{\mu m_H} + \frac{4\sigma T^4}{3c}$$

where k is Boltzmann's constant, m_H is mass of a hydrogen atom, σ is Stefan's constant, c is the speed of light, and μ is the mean molecular weight, defined as the average mass of the particles in units of m_H, equal to

- 2.0 for molecular hydrogen
- 1.0 for atomic hydrogen
- 0.5 for fully ionised hydrogen
- about 0.6 for ionised matter of typical cosmic abundances.

The component p_g is the *gas pressure* and p_r is the *radiation pressure*. Radiation pressure is usually negligible except in the centres of high mass stars and in the immediate environments of neutron stars and black holes. The pressure of an ideal gas is often written in the form $\mathcal{R}\rho T/\mu$ where $\mathcal{R} = k/m_H$ is a version of the universal gas constant.
We define the *first adiabatic exponent*

$$\Gamma_1 = \left(\frac{\partial \log p}{\partial \log \rho}\right)_S$$

which is related to the ratio of specific heat capacities

$$\gamma = \frac{c_p}{c_v} = \frac{T\left(\frac{\partial s}{\partial T}\right)_p}{T\left(\frac{\partial s}{\partial T}\right)_V}$$

by $\Gamma_1 = \chi_\rho \gamma$ where

$$\chi_\rho = \left(\frac{\partial \log p}{\partial \log \rho}\right)_T$$

can be found from the equation of state. We can then rewrite the thermal energy equation as

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\Gamma_1 p \nabla \cdot \boldsymbol{u}$$

For an ideal gas with negligible radiation pressure, $\chi_\rho = 1$ and so $\Gamma_1 = \gamma$. Adopting this very common assumption, we write

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \boldsymbol{u}$$

2.8. Simplified models

A *perfect gas* may be defined as an ideal gas with constant c_v, c_p, γ and μ. Equipartition of energy for a classical gas with n degrees of freedom per particle gives

$$\gamma = 1 + \frac{2}{n}$$

For a classical monatomic gas with $n = 3$ translational degrees of freedom, $\gamma = 5/3$. This is relevant for fully ionised matter. For a classical diatomic gas with two addditional rotational degrees of freedom, $n = 5$ and $\gamma = 7/5$. This is relevant for molecular hydrogen. In reality, Γ_1 is variable when the gas undergoes ionisation or when the gas and radiation pressures are comparable.
The specific *internal energy* (or *thermal energy*) of a perfect gas is

$$e = \frac{p}{(\gamma - 1)\rho} = \frac{n}{\mu m_H}\frac{1}{2}kT$$

Note that each particle has an internal energy of $\frac{1}{2}kT$ per degree of freedom, and the number of particles per unit mass is $1/\mu m_H$.
A *barotropic* fluid is an idealised situation in which the relation $p(\rho)$ is known in advance. We can then dispense with the thermal energy equation. For example, if the gas is strictly isothermal and perfect, $p = c_s^2 \rho$ with the constant c_s being the isothermal sound speed. Alternatively, if the gas is strictly homentropic (constant s) and perfect, then $p = K\rho^\gamma$ with K constant.
An *incompressible* fluid is an idealised situation in which $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, implying $\nabla \cdot \boldsymbol{u} = 0$. This can be achieved formally by taking the limit $\gamma \to \infty$. The approximation of incompressibility eliminates acoustic phenomena from the dynamics. The ideal gas law itself is not valid at very high densities or where quantum degeneracy is important.

2.9. Microphysical basis

It is useful to understand the way in which the fluid-dynamical equations are derived from microphysical considerations. The simplest model involves identical neutral particles of mass m of negligible size with no internal degrees of freedom.

2.9.1. Boltzmann equation

Between collisions, particles follow Hamiltonian trajectories in their six dimensional $(\boldsymbol{x}, \boldsymbol{v})$ phase space:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = a_i - \frac{\partial \Phi}{\partial x_i}$$

The distribution funnction $f(\boldsymbol{x}, \boldsymbol{v}, t)$ specifies the number density of particles in phase space. The velocity moments of f define the number $n(\boldsymbol{x}, t)$ in real space, the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the velocity dispersion $c(\boldsymbol{x}, t)$ according to

$$\int f \, \mathrm{d}^3\boldsymbol{v} = n$$

$$\int \boldsymbol{v} f \, \mathrm{d}^3\boldsymbol{v} = n\boldsymbol{u}$$

$$\int |\boldsymbol{v} - \boldsymbol{u}|^2 f \, \mathrm{d}^3\boldsymbol{v} = 3nc^2$$

Equivalently,

$$\int v^2 f \, \mathrm{d}^3\boldsymbol{v} = n(\boldsymbol{u}^2 + 3c^2)$$

The relation between velocity dispersion and temperature is $kT = mc^2$. In the absence of collisions, f is conserved following the Hamiltonian flow in phase space. This is because particles are conserved and the flow in phase space is incompressible. More generally, f evolves according to *Boltzmann's equation*

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_c$$

The collision term on the RHS is a complicated integral operator but has 3 simple properties corresponding to the conservation of mass, momentum and energy in collisions.

$$\int m \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

$$\int m\boldsymbol{v} \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

$$\int \frac{1}{2} m\boldsymbol{v}^2 \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v} = 0$$

The collision term is local in $\boldsymbol{x}$ (not even involving derivatives) although it does involve integrals over $\boldsymbol{v}$. The equation $\left(\frac{\partial f}{\partial t}\right)_c = 0$ has the general solution

$$f = f_M = (2\pi c^2)^{-3/2} n \exp\left(-\frac{|\boldsymbol{v} - \boldsymbol{u}|^2}{2c^2}\right)$$

with parameters $n, \boldsymbol{u}$ and c that may depend on $\boldsymbol{x}$. This is the *Maxwellian distribution.*

2.9.2. Derivation of fluid equations

A crude but illuminating model of the collision operator is the *BGK approximation*

$$\left(\frac{\partial f}{\partial t}\right)_c \approx -\frac{1}{\tau}(f - f_M)$$

where f_M is a Maxwellian distribution with the same $n, \boldsymbol{u}$ and c as f, and τ is the *relaxation time*. This timescale τ can be identified approximately with the mean flight time of particles between collisions. The collisions attempt to restore a Maxwellian distribution on a characteristic timescale τ. They do this by randomising the particle velocities in a way consistent with the conservation of momentum and energy. If the characteristic timescale of the fluid flow is much greater than τ, then the collision term dominates the Boltzmann equation and f is very close to f_m. This is the *hydrodynamic limit.*
The velocity moments of f_M can be determined from standard Gaussian integrals, in particular

$$\begin{aligned}\int f_M \, \mathrm{d}^3\boldsymbol{v} &= n \\ \int v_i f_M \, \mathrm{d}^3\boldsymbol{v} &= n u_i \\ \int v_i v_j f_M \, \mathrm{d}^3\boldsymbol{v} &= n(u_i u_j + c^2 \delta_{ij}) \\ \int \boldsymbol{v}^2 v_i f_M \, \mathrm{d}^3\boldsymbol{v} &= n(\boldsymbol{u}^2 + 5c^2) u_i\end{aligned}$$

We obtain equations for mass, momentum and energy by taking moments of the Boltzmann equation weighted by $(m, mv_i, \frac{1}{2}m\boldsymbol{v}^2)$. In each case the collision term integrates to 0 because of its conservative properties, and the $\partial/\partial v_j$ term can be integrated by parts. We replace f with f_M when evaluating the LHS and note that $mn = \rho$:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) &= 0 \\ \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left[\rho(u_i u_j + c^2 \delta_{ij})\right] - \rho a_i &= 0 \\ \frac{\partial}{\partial t}\left[\frac{1}{2}\rho \boldsymbol{u}^2 + \frac{3}{2}\rho c^2\right] + \frac{\partial}{\partial x_i}\left[(\frac{1}{2}\rho \boldsymbol{u}^2 + \frac{5}{2}\rho c^2) u_i\right] - \rho u_i a_i &= 0\end{aligned}$$

These are equivalent to the equations of ideal gas dynamics in conservative form (see section 4) for a monatomic ideal gas ($\gamma = 5/3$). The specific internal energy is $\rho = \frac{3}{2}c^2 = \frac{3}{2}\frac{kT}{m}$.

2.10. Validity of fluid approach

Deviations from Maxwellian distribution are small when collisions are frequent compared to the characteristic timescale of the flow. In higher-order approximations these deviations can be estimated, leading to the equations of dissipative gas dynamics including transport effects (viscosity and heat conduction).
The fluid approach breaks down if the mean flight time τ is not much less than the characteristic timescale of the flow, or if the mean free path $\lambda \approx c\tau$ between collisions is not much less than the characteristic lengthscale of the flow.
The Coulomb cross-section for ‘collisions’ (i.e. large-angle scatterings) between charged particles (electrons or ions) is

$$\sigma \approx 1 \times 10^{-4} (T/K)^{-2} \mathrm{cm}^2$$

The mean free path is $\lambda = 1/n\sigma$.

3. Ideal magnetohydrodynamics

3.1. Elementary derivation of MHD equations

Magnetohydrodynamics (MHD) is the dynamics of an electrically conducting fluid (a full or partially ionised gas or a liquid metal) containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

3.1.1. Galilean electromagnetism

The equations of Newtonian gas dynamics are invariant under the Galilean transformation to a frame of reference moving with uniform velocity $\boldsymbol{v}$,

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t, \quad t' = t$$

Under this change of frame, the fluid velocity transforms according to

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}$$

while scalar variables such as p, ρ, and Φ are invariant. The Lagrangian time derivative $\mathrm{D}/\mathrm{D}t$ is also invariant, because the partial derivatives transform according to

$$\nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

To derive a consistent Newtonian theory of MHD, valid for situations in which the fluid motions are slow compared to the speed of light, we use Maxwell's equations without the displacement current:

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times \boldsymbol{E} \\ \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \end{aligned}$$

We will not require the fourth Maxwell equation, involving $\nabla \cdot \boldsymbol{E}$, because the charge density will be found to be unimportant. The equation of energy conservation is

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) = -\boldsymbol{E} \cdot \boldsymbol{J}$$

in which the energy density $B^2/2\mu_0$ is purely magnetic, while the energy flux density has the usual form of the *Poynting vector* $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$. These 'pre-Maxwell' equations are invariant under the Galilean transformation, provided that the fields transform according to

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \quad \boldsymbol{B}' = \boldsymbol{B}, \quad \boldsymbol{J}' = \boldsymbol{J}$$

3.1.2. Induction equation

In the *ideal MHD approximation* we regard the fluid as a perfect electrical conductor. The electric field in the rest frame of the fluid therefore vanishes, implying that

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B}$$

in a frame in which the fluid velocity is $\boldsymbol{u}(\boldsymbol{x}, t)$. This condition can be regarded as the limit of a constitutive relationship such as Ohm's law, in which the effects of resistivity (i.e. finite conductivity) are neglected. From Maxwell's equations, we then obtain the *ideal induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

This is an evolutionary equation for $\boldsymbol{B}$ alone, having elimnated $\boldsymbol{E}$ and $\boldsymbol{J}$. The divergence of the induction equation,

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$$

ensures that $\boldsymbol{B}$ remains solenoidal.

3.1.3. Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a Lorentz force

$$\boldsymbol{F}_m = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. This can be understood as the sum of the Lorentz forces on individual particles of charge q and velocity $\boldsymbol{v}$,

$$\sum q\boldsymbol{v} \times \boldsymbol{B} = \left(\sum q\boldsymbol{v}\right) \times \boldsymbol{B}$$

The electrostatic force can be shown to be negligible in the Newtonian limit. In Cartesian coordinates we have

$$\begin{aligned}
(\mu_0 \boldsymbol{F}_m)_i &= \varepsilon_{ijk}\varepsilon_{jlm}\frac{\partial B_m}{\partial x_l}B_k \\
&= \left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right)B_k \\
&= B_k\frac{\partial B_i}{\partial x_k} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right)
\end{aligned}$$

Thus the Lorentz force can be written as

$$\boldsymbol{F}_m = \frac{1}{\mu_0}\boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

The first term can be interpreted as a *curvature force* due to a *magnetic tension* $T_m = B^2/\mu_0$ per unit area in the field lines. The second term is the gradient of an isotropic *magnetic pressure* $p_m = B^2/2\mu_0$ which is also the magnetic energy density. The magnetic tension gives rise to *Alfvén waves* (see later), which travel parallel to the magnetic field with characteristic speed

$$v_a = \left(\frac{T_m}{\rho}\right)^{1/2} = \frac{B}{\sqrt{\mu_0 \rho}}$$

the *Alfvén speed*, or (vector) *Alfvén velocity*

$$\boldsymbol{v}_a = \frac{\boldsymbol{B}}{\sqrt{\mu_0 \rho}}$$

The magnetic pressure also affects the propagation of sound waves, which become *magnetoacoustic waves* (see later). The combination

$$\Pi = p + \frac{B^2}{2\mu_0}$$

is often referred to as the *total pressure*, while the ratio

$$\beta = \frac{p}{B^2/2\mu_0}$$

is known as the *plasma beta*.

3.1.4. Summary of the MHD equations

The full set of ideal MHD equations is

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) &= 0 \\
\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\
\frac{\mathrm{D}s}{\mathrm{D}t} &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\
\nabla \cdot \boldsymbol{B} &= 0
\end{aligned}$$

together with the equation of state, Poisson's equation, etc., as required. Most of these equations can be written in at least one other way that may be useful in different circumstances.

3.2. Physical interpretation of MHD

There are two aspects to MHD:

- Advection of $\boldsymbol{B}$ by $\boldsymbol{u}$ (induction equation)
- Dynamical back-reaction of $\boldsymbol{B}$ on $\boldsymbol{u}$.

3.2.1. Kinematics of the magnetic field

The ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

has a beautiful geometric interpretation: magnetic field lines are 'frozen in' to the fluid and can be identified with material curves. One way to show this result is to use the identity

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + \boldsymbol{u}(\nabla \cdot \boldsymbol{B})$$

to write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u})$$

and use the equation of mass conservation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right) \cdot \nabla \boldsymbol{u}$$

This is exactly the same equation satisfied by a material line element (section 2.3). Therefore a magnetic field line (an integral curve of $\boldsymbol{B}/\rho$) is advected and distorted by the fluid in the same way as a material curve. A complementary property is that the magnetic flux $\delta\Phi = \boldsymbol{B} \cdot \delta\boldsymbol{S}$ through a material surface element is conserved:

$$\begin{aligned}
\frac{\mathrm{D}\delta\Phi}{\mathrm{D}t} &= \frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} \cdot \delta\boldsymbol{S} + \boldsymbol{B} \cdot \frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} \\
&= \left(B_j \frac{\partial u_i}{\partial x_j} - B_i \frac{\partial u_j}{\partial x_j} \right) \delta S_i + B_i \left(\frac{\partial u_j}{\partial x_j} \delta S_i - \frac{\partial u_j}{\partial x_i} \delta S_j \right) \\
&= 0
\end{aligned}$$

By extension, we have conservation of the magnetic flux $\Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$ passing through any material surface.

3.2.2. (Partial) analogy with vorticity

In homentropic (uniform entropy) or barotropic ideal fluid dynamics in the absence of a magnetic field, the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ satisfies

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})$$

Vortex lines are then 'frozen in' to the fluid. Conserved quantity

$$\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}$$

is analogous to

$$\int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x}$$

However, $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ whereas in MHD $\boldsymbol{B}$ and $\boldsymbol{u}$ are indirectly related through the equation of motion and the Lorentz force, so the analogy between vorticity dynamics and MHD is limited in scope.

3.2.3. Lorentz force

The Lorentz force per unit volume,

$$\boldsymbol{F}_m = \frac{1}{\mu_0} \boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla(B^2/2\mu_0)$$

can also be written as the divergence of the *Maxwell stress tensor*

$$\boldsymbol{F}_m = \nabla \cdot \mathsf{M}, \qquad \mathsf{M} = \frac{1}{\mu_0} \left(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2} \mathsf{I} \right)$$

where I is the identity tensor. In Cartesian components,

$$(\boldsymbol{F}_m)_i = \frac{\partial \mathsf{M}_{ji}}{\partial x_j}, \qquad \mathsf{M}_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{B^2}{2} \delta_{ij} \right)$$

If $\boldsymbol{B}$ is locally aligned with the x-axis, then

$$\mathsf{M} = \begin{pmatrix} T_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} p_m & 0 & 0 \\ 0 & p_m & 0 \\ 0 & 0 & p_m \end{pmatrix}$$

showing the magnetic tension and pressure. Combining the ideas of magnetic tension and a frozen-in field leads to the picture of field lines as elastic strings embedded in the fluid. There is a close analogy between MHD and the dynamics of polymer solutions. The magnetic field imparts elasticity to the fluid.

3.2.4. Differential rotation and torsional Alfvén waves

First consider the kinematic behaviour of a magnetic field in the presence of a prescribed velocity field involving differential rotation. In cylindrical polar coordinates (r, ϕ, z) let

$$\boldsymbol{u} = r\Omega(r, z)\hat{\boldsymbol{\phi}}$$

Consider an axisymmetric magnetic field, which we separate into *poloidal* (meridional: r and z) and *toroidal* (azimuthal: ϕ) parts:

$$\boldsymbol{B} = \boldsymbol{B}_p(r, z, t) + B_\phi(r, z, t)\hat{\boldsymbol{\phi}}$$

The ideal induction equation reduces to

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0, \qquad \frac{\partial B_\phi}{\partial t} = r\boldsymbol{B}_p \cdot \nabla\Omega$$

Differential rotation winds the poloidal field to generate a torsional field. To obtain a steady state without winding, we require the angular velocity to be constant along each magnetic field line:

$$\boldsymbol{B}_p \cdot \nabla\Omega = 0$$

a result known as *Ferraro's law of isorotation.*
There is an energetic cost to winding the field, as work is done against magnetic tension. In a dynamical situation a strong magnetic field tends to enforce isorotation along its length. Generalise the analysis to allow for axisymmetric torsional oscillations:

$$\boldsymbol{u} = r\Omega(r, z, t)\hat{\boldsymbol{\phi}}$$

The azimuthal component of the equation of motion is then

$$\rho r \frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0 r}\boldsymbol{B}_p \cdot \nabla(r\boldsymbol{B}_\phi)$$

Combine with the induction equation above to give

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\mu_0 \rho r^2}\boldsymbol{B}_p \cdot \nabla(r^2\boldsymbol{B}_p \cdot \nabla\Omega)$$

This equation describes *torsional Alfvén waves.* For example, if $\boldsymbol{B}_p = B_z\hat{\boldsymbol{z}}$ is vertical and uniform, then

$$\frac{\partial^2 \Omega}{\partial t^2} = v_a^2 \frac{\partial^2 \Omega}{\partial z^2}$$

This is not strictly an exact non-linear analysis because we have neglected the force balance (and indeed motion) in the meridional plane.

3.2.5. Force-free fields

In regions of low density, e.g. the solar corona, $\boldsymbol{B}$ may be dynamically dominant over inertia, gravity and gas pressure. We then have (approximately) a force-free magnetic field such that

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = 0$$

Since $\nabla \times \boldsymbol{B}$ must be parallel to $\boldsymbol{B}$ in this case, we may write

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B} \tag{1}$$

for some scalar field $\lambda(\boldsymbol{x})$. Taking the divergence of (1) gives

$$0 = \boldsymbol{B} \cdot \nabla \lambda$$

so λ is constant along each magnetic field line. In the special case where λ is constant, known as a *linear force-free magnetic field*, the curl of (1) results in the *Helmholtz equation*

$$-\nabla^2 \boldsymbol{B} = \lambda^2 \boldsymbol{B}$$

which admits a wide variety of solutions. A subset of force-free magnetic fields consists of *potential* or *current-free* magnetic fields for which

$$\nabla \times \boldsymbol{B} = 0$$

In a true vacuum, $\boldsymbol{B}$ must be potential. However, only an extremely low density of electrons is needed to make the force-free description more relevant.
An example of a force-free field in cylindrical polar coordinates (r, ϕ, z) is

$$\boldsymbol{B} = B_\phi(r)\hat{\boldsymbol{\phi}} + B_z(r)\hat{\boldsymbol{z}}$$
$$\nabla \times \boldsymbol{B} = -\frac{\mathrm{d}B_z}{\mathrm{d}r}\hat{\boldsymbol{\phi}} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\phi\right)\hat{\boldsymbol{z}}$$

Now $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ implies

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B_z}{\mathrm{d}r}\right) = \lambda^2 B_z$$

which is the z component of the Helmholtz equation. The solution regular at $r = 0$ is

$$B_z = B_0 J_0(\lambda r), \qquad B_\phi = B_0 J_1(\lambda r)$$

where J_n is the Bessel function of order n. Note that $J_0(x)$ satisfies

$$(xJ_0')' + xJ_0 = 0, \qquad J_1(x) = -J_0'(x)$$

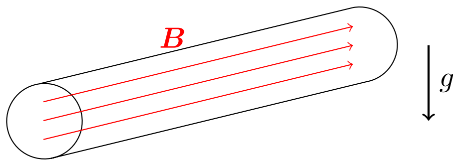
The helical nature of this field is typical of force-free fields with $\lambda \neq 0$.

3.2.6. Magnetostatic equilibrium and magnetic buoyancy

A *magnetostatic equilibrium* is a static solution ($\boldsymbol{u} = 0$) of the equation of motion, i.e. one satisfying

$$0 = -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

together with $\nabla \cdot \boldsymbol{B} = 0$. While solutions do exist, inhomogeneities in $\boldsymbol{B}$ often result in a lack of equilibrium. A magnetic flux tube is an idealised situation in which the magnetic field is localised to the interior of a tube and vanishes outside.



To balance the total pressure at the interface, the gas pressure must be lower inside. Unless the temperatures are different, the density is lower inside. In a gravitational field the tube experiences an upward buoyancy force and tends to rise.

4. Conservation laws & hyperbolic structure

4.1. Introduction

An equation in conservative form is written

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

where $q(\boldsymbol{x}, t)$ is the density of some property and $\boldsymbol{F}(\boldsymbol{x}, t)$ is the flux density of the same quantity. The total amount in a time-independent volume V is

$$Q = \int_V q \,\mathrm{d}V$$

and evolves due to the flux through the bounding surface

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_V \nabla \cdot \boldsymbol{F} \mathrm{d}V = -\int_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

The prototypical choice for this equation is mass conservation with $q = \rho, \boldsymbol{F} = \rho\boldsymbol{u}$.
A *material invariant* is a scalar field $f(\boldsymbol{x}, t)$ for which $\frac{\mathrm{D}f}{\mathrm{D}t} = 0$. Then f is constant for each fluid element, and is therefore conserved following the fluid motion. An example is specific entropy s in ideal fluid dynamics. Combined with mass conservation we obtain an equation in conservative form:

$$\frac{\partial}{\partial t}(\rho f) + \nabla \cdot (\rho f \boldsymbol{u}) = 0$$

4.2. Total energy equation

Starting from the ideal MHD equations, we construct the total energy equation piece by piece. First, consider *kinetic energy*:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}u^2\right) = \rho\boldsymbol{u} \cdot \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\boldsymbol{u} \cdot \nabla\Phi - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mu_0}\boldsymbol{u} \cdot (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

Gravitational energy (assuming a non-self-gravitating fluid and Φ independent of t):

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho\boldsymbol{u} \cdot \nabla\Phi$$

Internal (thermal) energy (using the fundamental thermodynamic identity $\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v$):

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \cancel{\rho T \frac{\mathrm{D}s}{\mathrm{D}t}} + p \frac{\mathrm{D}\log\rho}{\mathrm{D}t} = -p\nabla \cdot \boldsymbol{u}$$

Summing these three equations gives

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{u}^2 + \Phi + e\right) = -\nabla \cdot (p\boldsymbol{u}) + \frac{1}{\mu_0}\boldsymbol{u} \cdot (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

Rewrite the Lorentz force term:

$$\frac{1}{\mu_0}\boldsymbol{u} \cdot (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \cdot (-\boldsymbol{u} \times \boldsymbol{B}) = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E}$$

Using mass conservation:

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2 + \Phi + e)\right] + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + e) + p\boldsymbol{u}\right] = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \cdot \boldsymbol{E}$$

Finally, consider the magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0}\right) = \frac{1}{\mu_0}\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{\mu_0}\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E})$$

The total energy equation is then

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2 + \Phi + e) + \frac{\boldsymbol{B}^2}{2\mu_0}\right] + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right] = 0$$

where $h = e + p/\rho$ is the *specific enthalpy* and we have used the identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{B}$$

to include the *Poynting vector* (EM energy flux density) $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$. The total energy is therefore conserved.

For a self-gravitating system satisfying Poisson's equation, the gravitational energy can instead be regarded as $-g^2/8\pi G$:

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) = -\frac{1}{4\pi G}\nabla\Phi \cdot \frac{\partial \nabla\Phi}{\partial t}$$

using the fact $g = -\nabla\Phi$. We can 'integrate by parts' by writing

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla \cdot \left(\frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t}\right) = \frac{\Phi}{4\pi G}\frac{\partial \nabla^2\Phi}{\partial t} = \Phi\frac{\partial \rho}{\partial t} = -\Phi\nabla \cdot (\rho\boldsymbol{u})$$

Hence in conservative form we have

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right] = 0$$

The total energy equation is then

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2 + e) - \frac{g^2}{8\pi G} + \frac{\boldsymbol{B}^2}{2\mu_0}\right] + \nabla \cdot \left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\Phi}{4\pi G}\frac{\partial \nabla\Phi}{\partial t} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right] = 0$$

It is important to note that some of the gravitational and magnetic energy of an astrophysical body is stored in the exterior region, even if mass density vanishes there.

4.3. Helicity conservation

In ideal fluid dynamics there are geometrical or topological invariants:

- (potential) vorticity/circulation
- kinetic helicity $\boldsymbol{u} \cdot \boldsymbol{\omega}$

The Lorentz force breaks these conservation laws, but new topological invariants associated with $\boldsymbol{B}$ appear. The *magnetic helicity* in a volume V with bounding surface S is defined as

$$H_m = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Now

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla \Phi_e = \boldsymbol{u} \times \boldsymbol{B} - \nabla \Phi_e$$

where Φ_e is the *electrostatic potential* ('uncurl' of the induction equation). Thus

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) = -\boldsymbol{B} \cdot \nabla \Phi_e + \boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

So H_m is conserved in ideal MHD:

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) + \nabla \cdot [\Phi_e \boldsymbol{B} + \boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B})] = 0$$

Under a *gauge transformation*

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \chi$$
$$\Phi_e \to \Phi_e - \frac{\partial \chi}{\partial t}$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, but H_m changes by

$$\int_V \boldsymbol{B} \cdot \nabla \chi \, \mathrm{d}V = \int_V \nabla \cdot (\chi \boldsymbol{B}) \, \mathrm{d}V = \int_S \chi \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}S$$

So H_m is not uniquely defined unless $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on the surface S. The magnetic helicity H_m is a *pseudoscalar*, i.e. it changes sign under a reflection. Non-zero helicity $H_m \neq 0$ occurs only when $\boldsymbol{B}$ lacks reflection symmetry. H_m can be interpreted topologically in terms of the twistedness and knottedness of $\boldsymbol{B}$ (see ES2, Q4). Since $\boldsymbol{B}$ is 'frozen in' to the fluid and is deformed continuously by it, the topological properties of $\boldsymbol{B}$ are conserved.
The equivalent conserved quantity in homentropic or barotropic ideal gas dynamics (without $\boldsymbol{B}$) is the *kinetic helicity*

$$H_k = \int_V \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \, \mathrm{d}V$$

The *cross-helicity* in a volume V is

$$H_c = \int_V \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V$$

It is helpful to write the equation of motion in ideal MHD in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \nabla(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) = T \nabla s + \frac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{2}$$

using $\mathrm{d}h = T\mathrm{d}s + v\mathrm{d}p$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{B}) + \nabla\cdot\left[\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{B}) + (\frac{1}{2}\boldsymbol{u}^2 + \Phi + h)\boldsymbol{B}\right] = T\boldsymbol{B}\cdot\nabla s$$

So H_c is conserved in homentropic/barotropic flow, i.e. when $T\boldsymbol{B}\cdot\nabla s = 0$.
Bernoulli's theorem follows from the inner product of the equation of motion (2) with $\boldsymbol{u}$. In steady flow,

$$\boldsymbol{u}\cdot\nabla(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) = 0$$

i.e. the *Bernoulli function* $\frac{1}{2}\boldsymbol{u}^2 + \Phi + h$ is constant along streamlines, but only if $\boldsymbol{u}\cdot\boldsymbol{F}_m = 0$, i.e. if $\boldsymbol{B}$ does no work on the flow, for example if $\boldsymbol{u}$ is parallel to $\boldsymbol{B}$.

4.4. Symmetries

The equations of ideal gas dynamics and MHD have numerous symmetries. For an isolated, self-gravitating system:

- Translations of time and space, and rotations of space: related (via Noether's theorem) to the conservation of energy, momentum and angular momentum
- Reversal of time: related to the absence of dissipation
- Reflections of space (but note that $\boldsymbol{B}$ is a pseudovector and behaves oppositely to $\boldsymbol{u}$ under a reflection)
- Galilean transformations
- Reversal of the sign of $\boldsymbol{B}$
- Similarity transformations: if space and time are rescaled by independent factors λ and μ, i.e.

  $$\boldsymbol{x}\to\lambda\boldsymbol{x}, \qquad t\to\mu t$$

  then (for a perfect gas) we have $\boldsymbol{u}\to\lambda\mu^{-1}\boldsymbol{u}, \rho\to\mu^{-2}\rho, p\to\lambda^2\mu^{-4}p, \Phi\to\lambda^2\mu^{-2}\Phi, \boldsymbol{B}\to\lambda\mu^{-2}\boldsymbol{B}$.

For a non-isolated system with an external potential Φ_{ext}, these symmetries (other than $\boldsymbol{B}\to-\boldsymbol{B}$) apply only if Φ_{ext} has them. But for a non-self-gravitating perfect gas, the mass can be rescaled by any factor λ:

$$\rho\to\lambda\rho, \quad p\to\lambda p, \quad \boldsymbol{B}\to\lambda^{1/2}\boldsymbol{B}$$

4.5. Hyperbolic structure

The hyperbolic structure is one way of understanding wave modes and information propagation in a fluid. It is fundamental to the construction of some numerical methods. We neglect gravity here, because it involves instantaneous action at a distance, i.e. not a finite wave speed. We write the equations of ideal gas dynamics as

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u} &= 0\\ \frac{\partial p}{\partial t} + \boldsymbol{u}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{u} &= 0\\ \frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho}\nabla p &= 0\end{aligned}$$

and combine in the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0$$

where $\boldsymbol{U}$ is a 5D 'state vector'

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ p \\ u_x \\ u_y \\ u_z \end{pmatrix}$$

and $\mathsf{A}_x, \mathsf{A}_y, \mathsf{A}_z$ are 5×5 matrices. This works because every term in the equation involves a first derivative with respect to either time or space.

$$\mathsf{A}_x = \begin{pmatrix} u_x & & \rho & & \\ & u_x & \gamma p & & \\ & 1/\rho & u_x & & \\ & & & u_x & \\ & & & & u_x \end{pmatrix} \qquad \mathsf{A}_y = \begin{pmatrix} u_y & & & \rho & \\ & u_y & & \gamma p & \\ & & u_y & & \\ & 1/\rho & & u_y & \\ & & & & u_y \end{pmatrix}$$

$$\mathsf{A}_z = \begin{pmatrix} u_z & & & & \rho \\ & u_z & & & \gamma p \\ & & u_z & & \\ & & & u_z & \\ & 1/\rho & & & u_z \end{pmatrix}$$

The system of equations is hyperbolic if the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector $\boldsymbol{n}$ and if the eigenvectors span the 5D space. The eigenvalues can be identified as wave speeds, and the eigenvectors as wave modes, with $\boldsymbol{n}$ being the unit wavevector, locally normal to the wavefronts. Taking $\boldsymbol{n} = \hat{\boldsymbol{x}}$ WLOG, we find

$$\det(\mathsf{A}_x - v\mathsf{I}) = -(v - u_x)^3 \left[(v - u_x)^2 - v_s^2\right]$$

where $v_s = \sqrt{\gamma p/\rho}$ is the *adiabatic sound speed*. The wave speeds v are real and the system is indeed hyperbolic. Two modes are *sound waves* (acoustic waves) which have speed $u_x \pm v_s$ and propagate at the sound speed relative to the fluid. Their eigenvectors

$$\begin{pmatrix} \rho \\ \gamma p \\ \pm v_s \\ 0 \\ 0 \end{pmatrix}$$

involve perturbations of density, pressure and longitudinal velocity. The other 3 modes have $v = u_x$ and do not propagate relative to the fluid. Their eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The *entropy wave* perturbs the density but not the pressure. Since $s = s(\rho, p)$, the entropy is perturbed. The *vortical waves* perturb the tranverse velocity and therefore the vorticity. These waves propagate at the fluid velocity because entropy and vorticity are conserved.

To extend the analysis to ideal MHD, consider the induction equation in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) = 0$$

and include the Lorentz force in the equation of motion:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\rho}\left(p + \frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0 \rho}\boldsymbol{B} \cdot \nabla \boldsymbol{B} = 0$$

Every term still involves a first derivative, so the MHD equations can be written as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0$$

where $\boldsymbol{U}$ is now an 8D state vector and the A_i are three 8×8 matrices. The characteristic polynomial of A_x is

$$\det\left(\mathsf{A}_x - v\mathsf{I}\right) = (v - u_x)^2 \left((v - u_x)^2 - v_{ax}^2\right)\left((v - u_x)^4 - (v_s^2 + v_a^2)(v - u_x)^2 + v_s^2 v_{ax}^2\right)$$

The wave speeds v are real and the system is hyperbolic. The MHD wavemodes will be examined in section 5. In this representation, there are two modes with $v = u_x$ that do not propagate relative to the fluid. One is the entropy wave, which is physical and involves only a density perturbation. The other is the '$\nabla \cdot \boldsymbol{B}$' mode which is unphysical and involves a perturbation of $\nabla \cdot \boldsymbol{B}$ (i.e. of B_x, in the case $\boldsymbol{n} = \hat{\boldsymbol{x}}$). This must be eliminated by imposing the constraint $\nabla \cdot \boldsymbol{B} = 0$. The vortical waves are replaced by Alfvén waves with speeds $u_x \pm v_{ax}$.

4.6. Stress tensor

In the absence of external forces, the equation of motion of a fluid can usually be written as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \mathsf{T}$$

where T is the *stress tensor*, a symmetric second rank tensor field. Using mass conservation, we can relate this to the equation of momentum conservation:

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} - \mathsf{T}) = 0$$

So $-\mathsf{T}$ is the momentum flux density, excluding the advective flux. For a self-gravitating system in ideal MHD, the stress tensor is

$$\mathsf{T} = -p\mathsf{I} - \frac{1}{4\pi G}(\boldsymbol{g}\boldsymbol{g} - \frac{1}{2}g^2\mathsf{I}) + \frac{1}{\mu_0}(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\mathsf{I})$$

The gravitational stress tensor works for a self-gravitating system in which $\boldsymbol{g} = -\nabla\Phi$ and ρ are related through Poisson's equation

$$-\nabla \cdot \boldsymbol{g} = \nabla^2\Phi = 4\pi G\rho$$

For a general vector field $\boldsymbol{v}$, it can be shown that

$$\begin{aligned} \nabla \cdot (\boldsymbol{v}\boldsymbol{v} - \frac{1}{2}v^2\mathsf{I}) &= (\nabla \cdot \boldsymbol{v})\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla(\frac{1}{2}v^2) \\ &= (\nabla \cdot \boldsymbol{v})\boldsymbol{v} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} \end{aligned}$$

In the magnetic case ($\boldsymbol{v} = \boldsymbol{B}$) this simplifies to $(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$. In the gravitational case ($\boldsymbol{v} = \boldsymbol{g}$) it simplifies to $(\nabla \cdot \boldsymbol{g})\boldsymbol{g} = -4\pi G\rho\boldsymbol{g}$, which becomes the force per unit volume $\rho\boldsymbol{g}$ when divided by $-4\pi G$.

4.7. Virial theorem

The *virial equations* are the spatial moments of the equation of motion. They provide integral measures of the balance of forces acting on the fluid. The first moments are generally most useful. Recall

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t}=\frac{\partial T_{ji}}{\partial x_j}$$

Consider

$$\begin{aligned}\rho\frac{\mathrm{D}^2}{\mathrm{D}t^2}(x_ix_j)&=\rho\frac{\mathrm{D}}{\mathrm{D}t}(u_ix_j+x_iu_j)\\&=2\rho u_iu_j+x_j\frac{\partial T_{ki}}{\partial x_k}+x_i\frac{\partial T_{kj}}{\partial x_k}\end{aligned}$$

Consider a material volume V bounded by a material surface S. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,\mathrm{d}m=\int_V\frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}m$$

where f is any function and $\mathrm{d}m=\rho\mathrm{d}V$ is the material-invariant mass element. Integrate over V to get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V x_ix_j\,\mathrm{d}m&=\int_V\left(2\rho u_iu_j+x_j\frac{\partial T_{ki}}{\partial x_k}+x_i\frac{\partial T_{kj}}{\partial x_k}\right)\mathrm{d}V\\&=\int_V\left(2\rho u_iu_j-T_{ji}-T_{ij}\right)\mathrm{d}V+\int_S\left(x_jT_{ki}+x_iT_{kj}\right)n_k\,\mathrm{d}S\end{aligned}$$

where we have integrated by parts and used the divergence theorem:

$$\begin{aligned}\int_V f\frac{\partial g_k}{\partial x_k}\,\mathrm{d}V&=\int_V\left[\frac{\partial}{\partial x_k}(fg_k)-g_k\frac{\partial f}{\partial x_k}\right]\mathrm{d}V\\&=\int_S fg_kn_k\,\mathrm{d}S-\int_V g_k\frac{\partial f}{\partial x_k}\,\mathrm{d}V\end{aligned}$$

For an isolated system with no external sources of gravity or magnetic field, $\boldsymbol{g}$ decays as $|\boldsymbol{x}|^{-2}$ at large distance, and $\boldsymbol{B}$ decays faster. Therefore T_{ij} decays as $|\boldsymbol{x}|^{-4}$ and the surface integral can be eliminated if we let V occupy the whole space. Divide by 2 to obtain the *tensor virial theorem*:

$$\frac{1}{2}\frac{\mathrm{d}^2I_{ij}}{\mathrm{d}t^2}=2K_{ij}-\mathcal{T}_{ij}$$

where

$$I_{ij}=\int x_ix_j\,\mathrm{d}m$$

is related to the inertia tensor of the system,

$$K_{ij}=\int\frac{1}{2}u_iu_j\,\mathrm{d}m$$

is a kinetic energy tensor and

$$\mathcal{T}_{ij}=\int T_{ij}\,\mathrm{d}V$$

is the integrated stress tensor. Note if the above conditions are not satisfied, there will be an additional contribution from the surface integral. The *scalar virial theorem* is the trace of this equation:

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K - \mathcal{T}$$

Note that K is the total KE, and

$$\begin{aligned}-\mathcal{T} &= \int \left(3p - \frac{g^2}{8\pi G} + \frac{B^2}{2\mu_0}\right)\,\mathrm{d}V \\ &= 3(\gamma - 1)U + W + M\end{aligned}$$

for a perfect gas with no external gravity, where U, W and M are the total internal, gravitational and magnetic energies. Thus

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + 3(\gamma - 1)U + W + M$$

On the right-hand side, only W is negative. For the system to be bound (i.e. to not fly apart) the kinetic, internal and magnetic energies are limited by

$$2K + 3(\gamma - 1)U + M \leq |W|$$

In fact, equality must hold, at least on average, unless the system is collapsing or contracting. The tensor virial theorem provides more specific information relating to the energies associated with individual directions. This is particularly relevant in cases where anisotropy is introduced by rotation or a magnetic field.

5. Linear waves in homogeneous media

In ideal MHD the density, pressure and magnetic field evolve according to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\boldsymbol{u}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{u} \\ \frac{\partial p}{\partial t} &= -\boldsymbol{u}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{u} \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{u}\times\boldsymbol{B})\end{aligned}$$

Consider a magnetostatic equilibrium in which

$$\rho = \rho_0(\boldsymbol{x}), \quad p = p_0(\boldsymbol{x}), \quad \boldsymbol{B} = \boldsymbol{B}_0(\boldsymbol{x}), \quad \boldsymbol{u} = 0$$

Now consider small perturbations from equilibrium, such that

$$\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}) + \delta\rho(\boldsymbol{x}, t)$$

with $|\delta\rho| \ll \rho_0$, etc. The linearised equations are

$$\begin{aligned}\frac{\partial \delta\rho}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla\rho_0 - \rho_0\nabla\cdot\delta\boldsymbol{u} \\ \frac{\partial \delta p}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla p_0 - \gamma p_0\nabla\cdot\delta\boldsymbol{u} \\ \frac{\partial \delta\boldsymbol{B}}{\partial t} &= \nabla\times(\delta\boldsymbol{u}\times\boldsymbol{B}_0)\end{aligned}$$

Introduce the *displacement* $\boldsymbol{\xi}(\boldsymbol{x}, t)$ such that $\delta \boldsymbol{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}$. Integrate to obtain

$$\begin{aligned}
\delta\rho &= -\boldsymbol{\xi}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{\xi} \\
\delta p &= -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi} \\
\delta\boldsymbol{B} &= \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \\
&= \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B}
\end{aligned}$$

We can now drop the subscript 0 without danger of confusion. Note arbitrary additive functions of $\boldsymbol{x}$ can be discarded if all variables have the same harmonic time dependence. The linearised equation of motion is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\nabla\delta\Phi - \delta\rho\nabla\Phi - \nabla\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\delta\boldsymbol{B}) \tag{3}$$

where the total pressure perturbation is

$$\begin{aligned}
\delta\Pi &= \delta p + \frac{1}{\mu_0}\boldsymbol{B}\cdot\delta\boldsymbol{B} \\
&= -\boldsymbol{\xi}\cdot\nabla\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi})
\end{aligned}$$

The gravitational potential perturbation satisfies

$$\nabla^2\delta\Phi = 4\pi G\delta\rho$$

Consider a basic state of uniform ρ, p and $\boldsymbol{B}$, in the absence of gravity. This system is homogeneous but anisotropic, because $\boldsymbol{B}$ distinguishes a particular direction. The problem simplifies to

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla\delta\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}))$$

with

$$\delta\Pi = -\left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi})$$

Since the basic state is independent of $\boldsymbol{x}$ and t, there are plane wave solutions of the form

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \Re\left[\tilde{\boldsymbol{\xi}}\exp(i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t))\right]$$

where ω and $\boldsymbol{k}$ are the frequency and wavevector, and $\tilde{\boldsymbol{\xi}}$ is a constant vector representing the amplitude of the wave. For such solutions (omitting the tilde)

$$\rho\omega^2\boldsymbol{\xi} = \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]\boldsymbol{k} + \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\left[(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{\xi} - \boldsymbol{B}(\boldsymbol{k}\cdot\boldsymbol{\xi})\right] \tag{4}$$

For transverse displacements $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ this simplifies to

$$\rho\omega^2\boldsymbol{\xi} = \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2\boldsymbol{\xi}$$

These solutions are Alfvén waves with dispersion relation

$$\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{v}_a)^2$$

Given the dispersion relation $\omega(\boldsymbol{k})$ of any wave, the *phase and group velocities* are

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}}, \qquad \boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \nabla_{\boldsymbol{k}}\omega$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. The phase velocity $\boldsymbol{v}_p$ is the velocity at which the phase of the wave travels, whilst the group velocity $\boldsymbol{v}_g$ is the velocity at which the energy of the wave (or the centre of a wavepacket) is transported. For Alfvén waves, $\omega = \pm \boldsymbol{k}\cdot\boldsymbol{v}_a$, hence:

$$\boldsymbol{v}_p = \pm v_a \cos\theta \hat{\boldsymbol{k}}, \qquad \boldsymbol{v}_g = \pm \boldsymbol{v}_a$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$. To find the other solutions, consider $\boldsymbol{k}\cdot(4)$ and $\boldsymbol{B}\cdot(4)$:

$$\begin{aligned}
\rho\omega^2 \boldsymbol{k}\cdot\boldsymbol{\xi} &= \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]k^2 \\
\rho\omega^2 \boldsymbol{B}\cdot\boldsymbol{\xi} &= \gamma p(\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{k}\cdot\boldsymbol{B}
\end{aligned}$$

Write these together as

$$\begin{pmatrix} \rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2 & \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})k^2 \\ -\gamma p(\boldsymbol{k}\cdot\boldsymbol{B}) & \rho\omega^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{k}\cdot\boldsymbol{\xi} \\ \boldsymbol{B}\cdot\boldsymbol{\xi} \end{pmatrix} = 0$$

The 'trivial solution' $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ corresponds to the Alfvén wave. The other solutions satisfy

$$\rho\omega^2\left[\rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2\right] + \gamma p k^2 \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2 = 0$$

which simplifies to (divide by $\rho^2 k^4$):

$$v_p^4 - (v_s^2 + v_a^2)v_p^2 + v_s^2 v_a^2 \cos^2\theta = 0$$

where$v_s^2 = \gamma p/\rho$ is the adiabatic sound speed. The two solutions

$$v_p^2 = \frac{1}{2}(v_s^2 + v_a^2) \pm \left[\frac{1}{4}(v_s^2 + v_a^2)^2 - v_s^2 v_a^2 \cos^2\theta\right]^{1/2}$$

are called *fast and slow magnetoacoustic waves* respectively. In the special case $\theta = 0$ (i.e. $\boldsymbol{k} \parallel \boldsymbol{B}$) we have

$$v_p^2 = v_s^2 \quad \text{or} \quad v_p^2 = v_a^2$$

together with $v_p^2 = v_a^2$ for the Alfvén wave. Note the fast wave could be either $v_p^2 = v_s^2$ or $v_p^2 = v_a^2$, whichever is greater. In the special case $\theta = \pi/2$ (i.e. $\boldsymbol{k}\perp\boldsymbol{B}$), we have

$$v_p^2 = v_s^2 + v_a^2 \quad \text{or} \quad v_p^2 = 0$$

together with $v_p^2 = 0$ for the Alfvén wave.
Magnetic tension gives rise to Alfvén waves, similar to waves on an elastic string. Magnetic pressure responds to compression, so modifies the propagation of acoustic waves. *Friedrichs diagrams* are parametric plots of $\boldsymbol{v}_p(\theta)$ and $\boldsymbol{v}_g(\theta)$ for all θ: see figure 1 and 2.

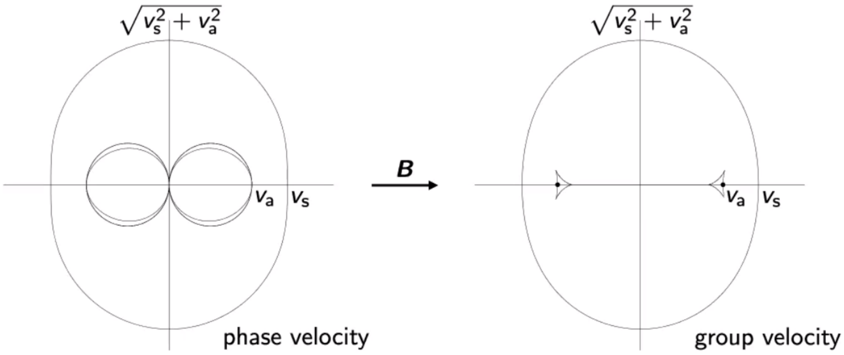


Figure 1: Friedrichs diagram for the case $v_a < v_s$ ($v_a = 0.7 v_s$)

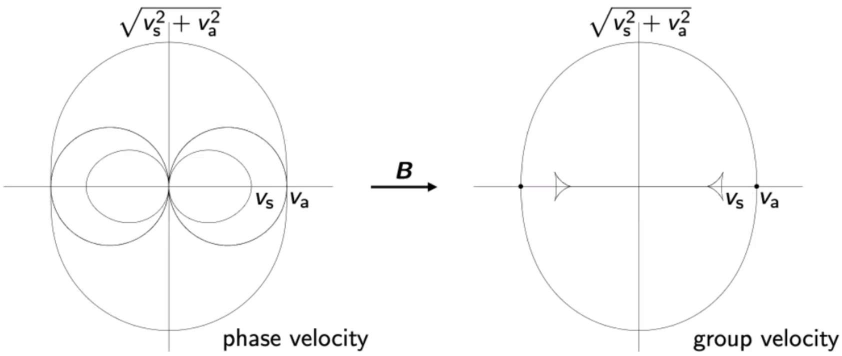


Figure 2: Friedrichs diagram for the case $v_a > v_s$ ($v_s = 0.7 v_a$)

Interpretation.

- The fast wave is a quasi-isotropic acoustic-type wave in which both gas and magnetic pressure contribute.
- The slow wave is an acoustic-type wave that is strongly guided by $\boldsymbol{B}$.
- The Alfvén wave is similar to a wave on an elastic string, propagating via magnetic tension and perfectly guided by $\boldsymbol{B}$.

6. Non-linear waves, shocks and discontinuities

6.1. 1D gas dynamics

The equations of mass conservation and motion in 1D are

$$\begin{aligned}\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}&=-\rho\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\end{aligned}$$

We assume the gas is homentropic (s constant) and perfect. This eliminates the entropy wave and leaves only the two sound waves. Then $p\propto\rho^\gamma$ and $v_s^2=\gamma p/\rho\propto\rho^{\gamma-1}$. We use v_s as a variable in place of ρ or p:

$$\mathrm{d}p=v_s^2\,\mathrm{d}\rho,\qquad \mathrm{d}\rho=\frac{\rho}{v_s}\frac{2\,\mathrm{d}v_s}{\gamma-1}$$

Then

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v_s\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right)&=0\\ \frac{\partial}{\partial t}\left(\frac{2v_s}{\gamma-1}\right)+u\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right)+v_s\frac{\partial u}{\partial x}&=0\end{aligned}$$

Add and subtract the two equations:

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(u+v_s)\frac{\partial}{\partial x}\right]\left(u+\frac{2v_s}{\gamma-1}\right)&=0\\ \left[\frac{\partial}{\partial t}+(u-v_s)\frac{\partial}{\partial x}\right]\left(u-\frac{2v_s}{\gamma-1}\right)&=0\end{aligned}$$

Define the two *Riemann invariants*

$$R_\pm=u\pm\frac{2v_s}{\gamma-1}$$

Then we deduce that $R_\pm=$ const. along a *characteristic (curve)* $C_\pm$ of gradient

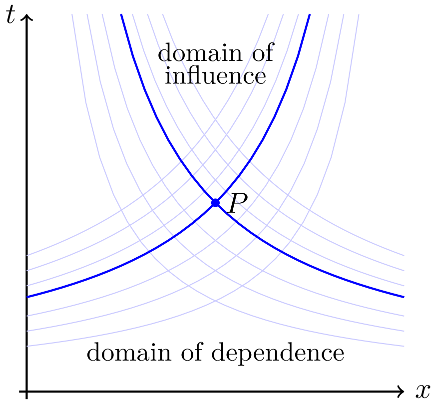
$$\frac{\mathrm{d}x}{\mathrm{d}t}=u\pm v_s$$

in the (x,t) plane. The $\pm$ characteristics form an interlocking web covering the space-time diagram. Both R_+ and R_- are needed to reconstruct the solution (u and v_s). Half the information is propagated along C_+ and half along C_-. In general $C_\pm$ are not known in advance but must be determined along with the solution. $C_\pm$ propagate at speed v_s to the right and left, *with respsect to the moving fluid.* This may be viewed as a non-linear generalisation of the solution of the classical wave equation $f(x-v_st)+g(x+v_st)$.

6.1.1. Method of characteristics

Sketch of a numerical method of solution:

1. Start with initial data (u and v_s) for all relevant x at $t = t_i$.
2. Determine characteristic slopes at t_i.
3. Propagate $R_\pm$ to $t = t_{i+1} = t_i + \delta t$, neglecting variation of characteristic slopes.
4. Combine $R_\pm$ to find u and v_s at each x at $t = t_{i+1}$.
5. Re-evaluate slopes and repeat.



The *domain of dependence* of a point P in the spacetime diagram is the region bounded by the $C_\pm$ through P and located in the past of P. The *domain of influence* of P is the region bounded by the $C_\pm$ through P and located in the future. The solution at P cannot depend on anything that occurs outside the domain of dependence. Similarly, the solution at P cannot influence anything outside the domain of influence.

6.1.2. A simple wave

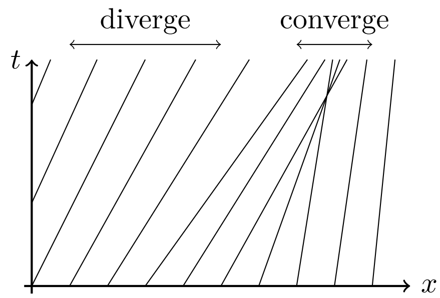
Suppose R_- is uniform: the same constant value on every C_- characteristic emanating from an undisturbed region to the right. Its value everywhere is that of the undisturbed region:

$$u - \frac{2v_s}{\gamma - 1} = u_0 - \frac{2v_{s0}}{\gamma - 1}$$

Then, along the C_+, both $R_\pm$ and so u and v_s, are constant. The C_+ therefore have constant slope $v = u + v_s$, so they are straight lines. The statement that the wavespeed v is constant along the family of straight lines $\frac{\mathrm{d}x}{\mathrm{d}t} = v$ is expressed by the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

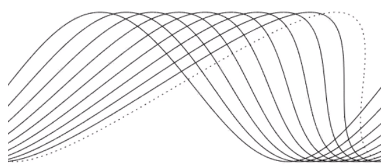
known as the *inviscid Burgers equation* or *non-linear advection equation*.
This equation has only one set of characteristics, with slope $\mathrm{d}x/\mathrm{d}t = v$, which is easily solved by the method of characteristics. The initial data define $v_0(x) = v(x, 0)$ and the characteristics are straight lines.



In regions where $\mathrm{d}v_0/\mathrm{d}x > 0$ the characteristics diverge in the future. In regions where $\mathrm{d}v_0/\mathrm{d}x < 0$ the characteristics converge and will form a *shock* at some point. Contradictory information arrives at the same event, leading to a breakdown of the solution.

6.1.3. Wave steepening

Another viewpoint of shock formation is *wave steepening*. Each point of the graph $v(x)$ moves at its wave speed v. The crest moves fastest and eventually overtakes the trough to the right of it. The



profile would become multiple-valued, but the wave breaks, forming a discontinuity. The formal solution of the inviscid Burgers equation is

$$v(x,t) = v_0(x_0)$$

with $x = x_0 + v_0(x_0)t$. By the chain rule,

$$\frac{\partial v}{\partial x} = \frac{v_0'}{1 + v_0' t}$$

which diverges first at the breaking time

$$t^* = \frac{1}{\max(-v_0')}$$

6.2. Simple non-linear waves

Recall the hyperbolic structure of the equations:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \mathsf{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = 0, \qquad \boldsymbol{U} = [\rho, p, \boldsymbol{u}, \boldsymbol{B}]^T$$

These are hyperbolic because the eigenvalues of $\mathsf{A}_i n_i$ are real for any unit vector n_i. The eigenvalues are wave speeds, eigenvectors are wave modes.
In a simple wave propagating in the x-direction, all physical quantities are functions of a single variable, the phase $\psi(x,t)$. Then $\boldsymbol{U} = \boldsymbol{U}(\psi)$ and so

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\psi}\frac{\partial \psi}{\partial t} + \mathsf{A}_x \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\psi}\frac{\partial \psi}{\partial x} = 0$$

This equation is satisfied if $\mathrm{d}\boldsymbol{U}/\mathrm{d}\psi$ is an eigenvector of the hyperbolic system and if

$$\frac{\partial \psi}{\partial t} + v\frac{\partial \psi}{\partial x} = 0$$

where v is the corresponding wave speed (eigenvalue). But since $v = v(\psi)$ we again find

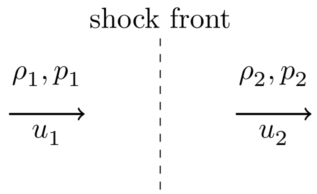
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

which is the inviscid Burgers equation. Steepening is therefore generic for simple waves, but waves do not always steepen in practice. For example, linear dispersion arising from Coriolis or buoyancy forces can counteract nonlinear wave steepening. Waves propagating on a non-uniform background are not simple waves. Waves may be damped by diffusive processes (viscosity, thermal conduction or resistivity) before they can steepen.
Even some simple waves do not steepen. This happens if the wave speed v does not depend on the variables that actually vary in the wave mode, for which there are two simple examples: the entropy wave ($v = u_x$) in which ρ varies but not p or u_x. Also, the Alfvén wave ($v = u_x \pm v_{ax}$) in which u_y, u_z, B_y, B_z vary but not ρ, p, u_x, B_x. In these cases the relevant solution of the inviscid Burgers equation is just $v = \text{const}$. The slow and fast magnetoacoustic waves, though, are ‘generally nonlinear’ and undergo steepening.

6.3. Shocks & discontinuities

6.3.1. Jump conditions

Discontinuities are resolved by diffusive processes (viscosity, thermal conduction or resistivity) that become more important on smaller length scales. We could solve the non-ideal (M)HD equations to resolve the internal structure of a shock. This internal solution could be matched onto external solutions in which diffusion is neglected. However, the matching conditions can be determined from conservation laws without resolving internal structure.
Consider a shock front at rest at $x = 0$ (make a Galilean transformation if necessary). Look for a stationary, 1D solution (equivalent to assuming a separation of scales) in which gas flows from left to right.



On the left is upstream, pre-shock material (ρ_1, p_1, etc.) and on the right is downstream, post-shock material (ρ_2, p_2, etc.) Consider any equation in conservative form

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

For a stationary, 1D solution, F_x is constant. Write the matching condition as

$$[F_x]_1^2 = F_{x,2} - F_{x,1} = 0$$

Including non-ideal effects gives rise to additional diffusive fluxes (not of mass but of momentum, energy, magnetic flux, etc.) The diffusive fluxes are negligible outside the shock, so they do not affect the jump conditions. This is valid as long as the new physics does not introduce any source terms in the equations. So the total energy is a properly conserved quantity, *but not the entropy* (see later).

Consider the conservative form of the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot \left(\rho u_i \boldsymbol{u} + \Pi \boldsymbol{e}_i - \frac{B_i \boldsymbol{B}}{\mu_0} \right) = 0$$

Including gravity makes no difference to the jump conditions because Φ is continuous (it satisfies $\nabla^2 \Phi = 4\pi G \rho$). From mass conservation, we have

$$[\rho u_x]_1^2 = 0$$

From momentum conservation:

$$\left[\rho u_x^2 + \Pi - \frac{B_x^2}{\mu_0} \right]_1^2 = 0$$

$$\left[\rho u_x u_y + \Pi - \frac{B_x B_y}{\mu_0} \right]_1^2 = 0$$

$$\left[\rho u_x u_z + \Pi - \frac{B_x B_z}{\mu_0} \right]_1^2 = 0$$

From the solenoidal condition $\nabla \cdot \boldsymbol{B} = 0$:

$$[B_x]_1^2 = 0$$

From the induction equation $\boldsymbol{B}_t + \nabla \times \boldsymbol{E} = 0$:

$$\left[u_x B_y - u_y B_x \right]_1^2 = - [E_z]_1^2 = 0$$

$$[u_x B_z - u_z B_x]_1^2 = \left[E_y \right]_1^2 = 0$$

These are the standard electromagnetic relations at an interface: continuity of $\perp$ component of $\boldsymbol{B}$ and $\|$ components of $\boldsymbol{E}$. From total energy conservation:

$$\left[\rho u_x \left(\frac{1}{2} u^2 + h \right) + \frac{1}{\mu_0} (E_y B_z - E_z B_y) \right]_1^2 = 0$$

Although the entropy in ideal MHD satisfies an equation of conservative form,

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \boldsymbol{u}) = 0$$

the dissipation of energy within the shock provides a source term for entropy. The entropy flux is *not* continuous across the shock.

6.3.2. Non-magnetic shocks

First consider a *normal shock* ($u_y = u_z = 0$) with no magnetic field. We obtain the *Rankine-Hugoniot relations*

$$[\rho u_x]_1^2 = 0$$

$$[\rho u_x^2 + p]_1^2 = 0$$

$$\left[\rho u_x \left(\frac{1}{2} u_x^2 + h \right) \right]_1^2 = 0$$

The specific enthalpy of a perfect gas is

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

The analytical solution is derived in example sheet 3. Introduce the upstream Mach number (the *shock Mach number*):

$$\mathcal{M}_1 = \frac{u_{x1}}{v_{s1}} > 0$$

where v_{s1} is the upstream adiabatic sound speed. Then we find

$$\begin{aligned}
\frac{u_{x1}}{u_{x2}} = \frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2} \\
\frac{p_2}{p_1} &= \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1} \\
\mathcal{M}_2^2 &= \frac{2 + (\gamma-1)\mathcal{M}_1^2}{2\gamma M_1^2 - (\gamma-1)}
\end{aligned}$$

Note that ρ_2/ρ_1 and p_2/p_1 are increasing functions of $\mathcal{M}_1$. The case $\mathcal{M}_1 = \mathcal{M}_2 = 1$ is trivial (no shock): $\rho_2/\rho_1 = 1$, $p_2/p_1 = 1$. Two cases exist in general:

- *Compression shock*

$$\mathcal{M}_1 > 1, \qquad \mathcal{M}_2 < 1, \qquad \rho_2 > \rho_1, \qquad p_2 > p_1$$

- *Rarefaction shock*

$$\mathcal{M}_1 < 1, \qquad \mathcal{M}_2 > 1, \qquad \rho_2 < \rho_1, \qquad p_2 < p_1$$

The entropy change $[s]_1^2$ in passing through the shock is positive for compression shocks and negative for rarefaction shocks (see ES3, Q1). Therefore only compression shocks are physically realisable; rarefaction shocks are excluded by the second law of thermodynamics. All shocks involve dissipation and irreversibility. The fact that $\mathcal{M}_1 > 1$ while $\mathcal{M}_2 < 1$ means the shock travels supersonically relative to the upstream gas and subsonically relative to the downstream gas.

- In the *weak shock* limit $\mathcal{M}_1 - 1 \ll 1$ the relative velocity of the fluid and the shock is close to the sound speed on both sides.

- In the *strong shock* limit $\mathcal{M}_1 \gg 1$, common in astrophysical applications,

$$\begin{aligned}
\frac{u_{x1}}{u_{x2}} = \frac{\rho_2}{\rho_1} &\to \frac{\gamma+1}{\gamma-1} \\
\frac{p_2}{p_1} &\gg 1 \\
\mathcal{M}_2^2 &\to \frac{\gamma-1}{2\gamma}
\end{aligned}$$

Note that the compression ratio ρ_2/ρ_1 is finite (and equal to 4 when $\gamma = 5/3$). In the rest frame of the undisturbed (upstream) gas the *shock speed* is $u_{\mathrm{sh}} = -u_{x1}$. The downstream density, velocity (in that frame) and pressure in the limit of a strong shock are (to be used in section 7)

$$\rho_2 = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1$$
$$u_{x2} - u_{x1} = \frac{2u_{\mathrm{sh}}}{\gamma+1}$$
$$p_2 = \frac{2\rho_1 u_{\mathrm{sh}}^2}{\gamma+1}$$

A lot of thermal energy is generated out of kinetic energy by a strong shock:

$$e_2 = \frac{2u_{\mathrm{sh}}^2}{(\gamma+1)^2}$$

6.3.3. Oblique shocks

When u_y or u_z is non-zero, we also require

$$\left[\rho u_x u_y\right]_1^2 = \left[\rho u_x u_z\right]_1^2 = 0$$

Since ρu_x is continuous across the shock (and non-zero), we deduce that $\left[u_y\right]_1^2 = [u_z]_1^2 = 0$. Momentum and energy conservation apply as before, and we recover the Rankine-Hugoniot relations.

6.3.4. Other discontinuities

The discontinuity is not called a shock if there is no normal flow ($u_x = 0$). In this case we can deduce only that $[p]_1^2 = 0$. Arbitrary discontinuities are allowed in ρ, u_y and u_z. These are related to the entropy and vortical waves. A jump in ρ is a *contact discontinuity* and a jump in u_y or u_z is a *tangential discontinuity* or *vortex sheet* (vorticity proportional to $\delta(x)$). These discontinuities are not produced naturally by wave steepening, because the entropy and vortical waves do not steepen. However they do appear in the Riemann problem (see later) and other situations with discontinuous initial conditions.

6.3.5. MHD shocks and discontinuities

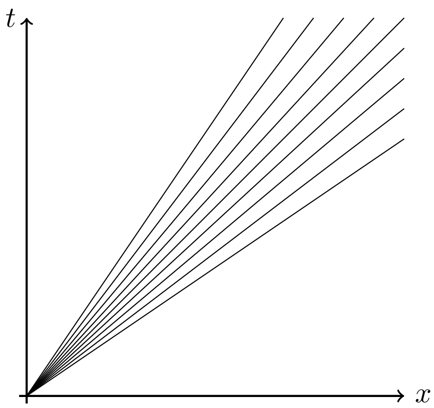
When a magnetic field is included, the jump conditions allow a wider variety of solutions. There are different types of discontinuity associated with the three MHD waves (Alfvén, slow and fast), which we will not discuss here. Since the parallel components of $\boldsymbol{B}$ need not be continuous, it is pssible for them to 'switch on' or 'switch off' on passage through a shock.
A *current sheet* is a tangential discontinuity in the magnetic field. For example, suppose B_y changes sign across the interface, with $B_x = 0$. Then the current density $J_z \propto \delta(x)$.

6.3.6. The Riemann problem

The Riemann problem is a fundamental initial value problem for a hyperbolic system and plays a central role in some numerical methods for solving the equations of AFD.
The initial condition at $t = 0$ consists of two uniform states separated by a discontinuity at $x = 0$. In the case of 1D gas dynamics, we have

$$\rho = \begin{cases} \rho_L & x < 0 \\ \rho_R & x > 0 \end{cases} \qquad p = \begin{cases} p_L & x < 0 \\ p_R & x > 0 \end{cases} \qquad u_x = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

An example is the 'shock tube' problem in which gas at different pressure is at rest either side of a partition which is released at $t = 0$. It can be shown that the initial discontinuity resolves generically into three simple waves. The inner one is a contact discontinuity. The other ones are shocks or rarefaction waves (see below).
The initial data define no natural length-scale, but they do allow a characteristic velocity scale c to be defined (although not uniquely). The result is a *similarity solution* in which variables depend on x and t only through the dimensionless combination $\xi = x/ct$.
Unlike the physical rarefaction shock, the *rarefaction wave* (or *expansion wave*) is a non-dissipative, homentropic, continuous simple wave in which $\nabla \cdot \boldsymbol{u} > 0$. If we seek a similarity solution $v = v(\xi)$ of the inviscid Burgers equation $v_t + vv_x = 0$ we find $v = x/t$ (or the trivial solution $v = \text{const}$). The characteristics form an *expansion fan*.



The '+' rarefaction wave has

$$u + v_s = \frac{x}{t}, \qquad R_- = u - \frac{2v_s}{\gamma - 1} = \text{const.}$$

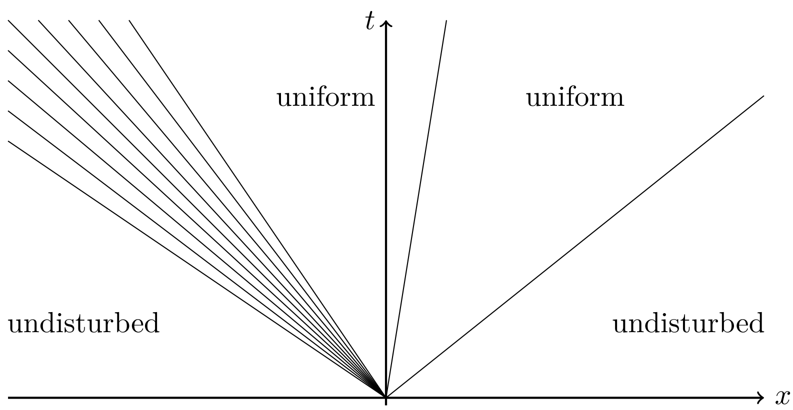
determined by the undisturbed right-hand state. The '-' rarefaction wave has

$$u - v_s = \frac{x}{t}, \qquad R_+ = u + \frac{2v_s}{\gamma - 1} = \text{const.}$$

determined by the undisturbed left-hand state. In each case u and v_s are linear functions of x/t and

$$\nabla \cdot \boldsymbol{u} = \left(\frac{2}{\gamma + 1}\right)\frac{1}{t} > 0$$

A typical outcome of a shock-tube problem consists of (from left to right): undisturbed region, rarefaction wave, uniform region, contact discontinuity, uniform region, shock, undisturbed region.



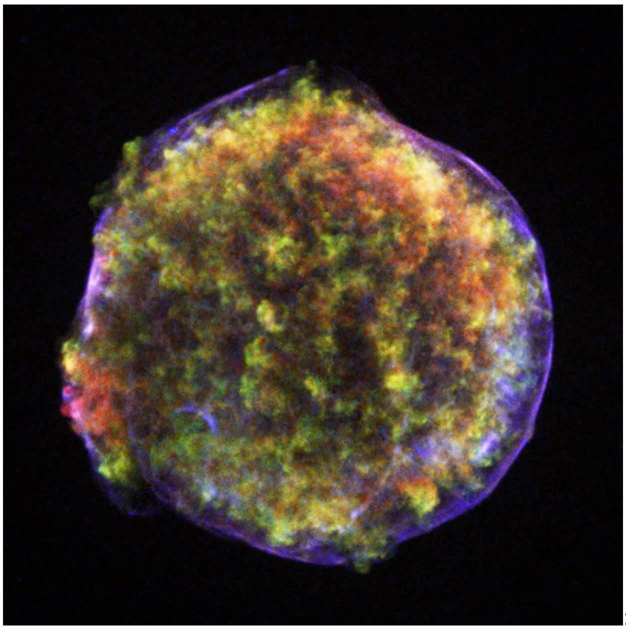
;

Figure 3: Tycho's supernova, SN1572. Source: Wikipedia.

In Godunov's method and related algorithms, the equations of AFD are advanced in time by solving (either exactly or approximately) a Riemann problem at each cell boundary.

7. Spherical blast waves: supernovae

7.1. Introduction

In this section (r, θ, ϕ) are spherical polar coordinates. In a supernova, an energy of order 10^{51} erg (10^{44} J) is released into the interstellar medium. An expanding spherical blast wave is formed as the explosion sweeps up the surrounding gas. Several good examples of these supernova remnants are observed in the galaxy, e.g. Tycho's supernova of 1572 and Kepler's supernova of 1604.
The effect is similar to a bomb. Photographs of the first atomic bomb test in New Mexico in 1945 allowed both Sedov in the Soviet Union and Taylor in the UK to work out the bomb's energy (20 kt of TNT $\sim 10^{14}$ J).
Suppose an energy E is released at $t = 0, r = 0$ and that the explosion is spherically symmetric. The external medium has density ρ_0 and pressure p_0. In the *Sedov-Taylor phase* of the explosion, the pressure $p \gg p_0$. Then a strong shock is formed and the external pressure p_0 can be neglected (formally set to zero). Gravity is also negligible in the dynamics.

7.2. Governing equations

A spherically symmetric flow of a perfect gas with purely radial velocity $u_r = u(r, t)$ satisfies

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\rho = -\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) \tag{5}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)u = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{6}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln(p\rho^{-\gamma}) = 0 \tag{7}$$

which are the equations of mass continuity, momentum conservation and energy conservation. These imply the total energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{2}\rho u^2+\frac{\gamma p}{\gamma-1}\right)u\right]=0 \tag{8}$$

The shock is at $r=R(t)$ so the shock speed is $\dot{R}$. The equations are to be solved in $0<r<R$ with the strong shock conditions at $r=R$:

$$\rho=\frac{\gamma+1}{\gamma-1}\rho_0 \tag{9}$$

$$u=\frac{2\dot{R}}{\gamma+1} \tag{10}$$

$$p=\frac{2\rho_0\dot{R}^2}{\gamma+1} \tag{11}$$

which are the Rankine-Hugoniot conditions in the limit of a strong shock. The total energy of the explosion is

$$E=\int_0^R\left(\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\right)4\pi r^2\,\mathrm{d}r=\text{const.}$$

The thermal energy of the external medium is negligible.

7.3. Dimensional analysis

The dimensional parameteres of the problem on which the solution might depend are E and ρ_0. Their dimensions are

$$[E]=ML^2T^{-2},\qquad [\rho_0]=ML^{-3}$$

Together, they do not define a characteristic lengthscale, so the explosion is 'scale-free' or 'self-similar'. If the dimensional analysis includes the time t since the explosion, however, we can find a time-dependent characteristic lengthscale. The radius of the shock must be

$$R=\alpha\left(\frac{Et^2}{\rho_0}\right)^{1/5}$$

where α is a dimensionless constant to be determined.

7.4. Similarity solution

Self-similarity is expressed using the dimensionless *similarity variable* $\xi=r/R(t)$. The solution has the form

$$\rho=\rho_0\tilde{\rho}(\xi),\qquad u=\dot{R}\tilde{u}(\xi),\qquad p=\rho_0\dot{R}^2\tilde{p}(\xi)$$

where $\tilde{\rho},\tilde{u}$ and $\tilde{p}$ are dimensionless functions of ξ to be determined.
Meaning: the graph of u versus r, for example, has a constant shape but both axes of the graph are rescaled as time proceeds and the shock expands.

7.5. Dimensionless equations

Substitute the known forms $\rho = \rho_0 \tilde{\rho}, u = \dot{R}\tilde{u}, p = \rho_0 \dot{R}^2 \tilde{p}$ into (5), (6), (7) and cancel dimensionless factors to get

$$(\tilde{u} - \xi)\tilde{\rho}' = -\frac{\tilde{\rho}}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2 \tilde{u}) \tag{12}$$

$$(\tilde{u} - \xi)\tilde{u}' - \frac{3}{2}\tilde{u} = -\frac{\tilde{p}'}{\tilde{\rho}} \tag{13}$$

$$(\tilde{u} - \xi)\left(\frac{\tilde{p}'}{\tilde{p}} - \frac{\gamma\tilde{\rho}'}{\tilde{\rho}}\right) - 3 = 0 \tag{14}$$

This uses $\dot{R} \propto t^{-3/5}, \xi \propto Rt^{-2/5}$. The strong shock conditions (9), (10), (11) at $r = R$ imply

$$\tilde{\rho} = \frac{\gamma+1}{\gamma-1}, \qquad \tilde{u} = \frac{2}{\gamma+1}, \qquad \tilde{p} = \frac{2}{\gamma+1} \tag{15}$$

at $\xi = 1$. The total energy integral becomes a normalisation condition

$$1 = \frac{16\pi}{25}\alpha^5 \int_0^1 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right)\xi^2 \,\mathrm{d}\xi$$

that will determine the value of α.

7.6. First integral

The total energy equation (8) is in 1D conservative form

$$\frac{\partial q}{\partial t} = \frac{\partial F}{\partial r} = 0 \tag{16}$$

with

$$q = r^2\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right)$$
$$F = r^2\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1}\right)u$$

For the similarity solution, we have

$$q = \rho_0 R^2 \dot{R}^2 \tilde{q}(\xi), \qquad F = \rho_0 R^2 \dot{R}^3 \tilde{F}(\xi) \tag{17}$$

with

$$\tilde{q} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right) \tag{18}$$

$$\tilde{F} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma\tilde{p}}{\gamma-1}\right)\tilde{u} \tag{19}$$

Substitute (17) into (16), noting that $R^2\dot{R}^2 \propto t^{4/5}t^{-6/5} \propto t^{-2/5} \propto R^{-1}$:

$$\rho_0 R^2 \dot{R}^2\left[-\frac{\dot{R}}{R}\tilde{q} + \frac{\mathrm{d}\tilde{q}}{\mathrm{d}\xi}\left(-\frac{\dot{R}}{R}\xi\right) + \frac{\dot{R}}{R}\frac{\mathrm{d}\tilde{F}}{\mathrm{d}\xi}\right] = 0$$

$$\implies -\tilde{q} - \xi\frac{\mathrm{d}\tilde{q}}{\mathrm{d}\xi} + \frac{\mathrm{d}\tilde{F}}{\mathrm{d}\xi} = 0$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\tilde{F} - \xi\tilde{q}\right) = 0$$

Hence $\tilde{F} - \xi\tilde{q} = \text{const.} = 0$ for a solution that is finite at $\xi = 0$. Using (18) and (19), solve $\tilde{F} - \xi\tilde{q} = 0$ for $\tilde{p}$:

$$\tilde{p} = \frac{(\gamma-1)\tilde{\rho}\tilde{u}^2(\xi-\tilde{u})}{2(\gamma\tilde{u}-\xi)}$$

This is compatible with the boundary conditions at $\xi = 1$ expressed by (15). Having found a first integral, we can dispense with the thermal energy equation (14). Let $\tilde{u} = v\xi$. We now have from (12) and (13):

$$(v-1)\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}\ln\xi} = -\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} - 3v$$

$$(v-1)\left(\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} + v\right) + \frac{1}{\tilde{\rho}\xi^2}\frac{\mathrm{d}}{\mathrm{d}\ln\xi}\left[\frac{(\gamma-1)\tilde{\rho}\xi^2v^2(1-v)}{2(\gamma v-1)}\right] = \frac{3}{2}v$$

Eliminate $\tilde{\rho}$:

$$\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} = \frac{v(\gamma v-1)\left[5-(3\gamma-1)v\right]}{\gamma(\gamma+1)v^2 - 2(\gamma+1)v + 2}$$

Invert and split into partial fractions:

$$\frac{\mathrm{d}\ln\xi}{\mathrm{d}v} = -\frac{2}{5v} + \frac{\gamma(\gamma-1)}{(2\gamma+1)(\gamma v-1)} + \frac{13\gamma^2-7\gamma+12}{5(2\gamma+1)\left[5-(3\gamma-1)v\right]}$$

The solution is

$$\xi \propto v^{-2/5}(\gamma v-1)^{(\gamma-1)/(2\gamma+1)}\left[5-(3\gamma-1)v\right]^{-(13\gamma^2-7\gamma+12)/5(2\gamma+1)(3\gamma-1)}$$

Then

$$\begin{aligned}\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}v} &= -\frac{1}{v-1} - \frac{3v}{v-1}\frac{\mathrm{d}\ln\xi}{\mathrm{d}v}\\ &= \frac{2}{(2-\gamma)(1-v)} + \frac{3\gamma}{(2\gamma+1)(\gamma v-1)} - \frac{13\gamma^2-7\gamma+12}{(2-\gamma)(2\gamma+1)\left[5-(3\gamma-1)v\right]}\end{aligned}$$

The solution is

$$\tilde{\rho} \propto (1-v)^{-2/(2-\gamma)}(\gamma v-1)^{3/(2\gamma+1)}\left[5-(3\gamma-1)v\right]^{(13\gamma^2-7\gamma+12)/(2-\gamma)(2\gamma+1)(3\gamma-1)}$$

For example, with $\gamma = 5/3$:

$$\xi \propto v^{-2/5}\left(\frac{5v}{3}-1\right)^{2/13}(5-4v)^{-82/195}$$

$$\tilde{\rho} \propto (1-v)^{-6}\left(\frac{5v}{3}-1\right)^{9/13}(5-4v)^{82/13}$$

To satisfy $v = 2/(\gamma+1) = 3/4$ and $\tilde{\rho} = (\gamma+1)/(\gamma-1) = 4$ at $\xi = 1$ we have

$$\xi = \left(\frac{4v}{3}\right)^{-2/5}\left(\frac{20v}{3}-4\right)^{2/13}\left(\frac{5}{2}-2v\right)^{-82/195}$$

$$\tilde{\rho} = 4(4-4v)^{-6}\left(\frac{20v}{3}-4\right)^{9/13}(\frac{5}{2}-2v)^{82/13}$$

Then, from the first integral

$$\tilde{p} = \frac{3}{4}\left(\frac{4v}{3}\right)^{6/5}(4-4v)^{-5}\left(\frac{5}{2}-2v\right)^{82/15}$$

In this solution, ξ ranges from 0 to 1, and v from $3/5$ to $3/4$. The normalisation integral (numerically) gives $\alpha \approx 1.152$.

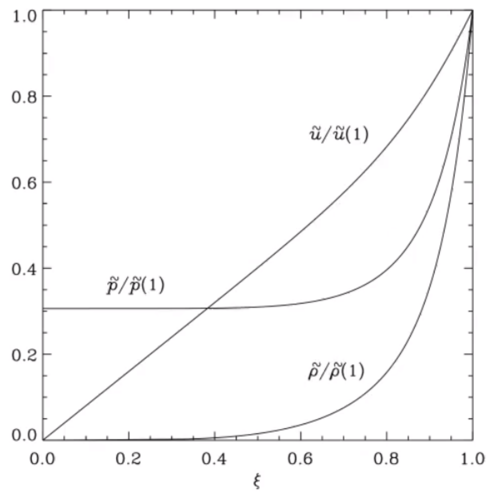


Figure 4: Normalised solutions for $0 \leq \xi \leq 1$ with $\gamma = 5/3$.

7.7. Applications

Some rough estimates from the above analysis:

- For a supernova, $E \sim 10^{51}$ erg and $\rho_0 \sim 10^{-24}\mathrm{g} \cdot \mathrm{cm}^{-3}$. Then $R \approx 6\mathrm{pc}$ and $\dot{R} \approx 2000\mathrm{km} \cdot \mathrm{s}^{-1}$ at $t = 1000\mathrm{yr}$.
- For the 1945 New Mexico explosion, $E \approx 8 \times 10^{20}$ erg, $\rho_0 \approx 1.2 \times 10^{-3}\mathrm{g} \cdot \mathrm{cm}^{-3}$. Then $R \approx 100\mathrm{m}$ and $\dot{R} \approx 4\mathrm{km} \cdot \mathrm{s}^{-1}$ at $t = 0.01\mathrm{s}$.

The similarity method is useful in a wide range of nonlinear problems. In this case it reduced PDEs to integrable ODEs.

8. Spherically symmetric steady flows

8.1. Introduction

In this section (r, θ, ϕ) are spherical polar coordinates. We will consider spherically symmetric steady flows in the context of stellar winds and accretion. Many stars, including the Sun, lose mass through a *stellar wind*. The gas must be sufficiently hot to escape from the star's gravitational field. Gravitating bodies can also accrete gas from the interstellar medium. The simplest models neglect rotation and magnetic fields and involve a steady, spherically symmetric flow.

8.2. Basic equations

Consider the purely radial flow of a perfect gas, either away from or towards a body of mass M. The gas is non-self-gravitating, so $\Phi = -GM/r$. The fluid variables are functions of r only. The only velocity component is $u_r = u(r)$. Mass conservation implies

$$4\pi r^2 \rho u = -\dot{M} = \text{const.}$$

If $u > 0$ (a stellar wind), $-\dot{M}$ is the mass-loss rate. If $u < 0$, $\dot{M}$ is the mass accretion rate. Ignore the slow change in mass M. The thermal energy equation (assuming $\boldsymbol{u} \neq 0$) implies homentropic flow:

$$p = K\rho^\gamma, \qquad K = \text{const.}$$

This is also referred to as polytropic. The equation of motion (only one component) is

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r}$$

or we may use the integral form (Bernoulli's equation):

$$\frac{1}{2}u^2 + \Phi + h = B = \text{const.}$$

Recall for a perfect gas,

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{v_s^2}{\gamma - 1}$$

In highly subsonic flow, the $\frac{1}{2}u^2$ term is negligible and the gas is *quasi-hydrostatic* with gravitational and pressure gradient forces balancing. In highly supersonic flow, the h term is negligible and the flow is *quasi-ballistic* (free falling). We are usually interested in transonic solutions that pass smoothly from subsonic to supersonic.
The aim is to solve for $u(r)$ and determine $\dot{M}$. At what rate does a star lose mass through a wind, or a black hole accrete mass from the surrounding medium? Assume $B > 0$; otherwise the flow cannot reach infinity.

8.3. First treatment

Use the equation of motion

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r} \tag{20}$$

Rewrite the pressure gradient as

$$\begin{aligned} -\frac{\mathrm{d}p}{\mathrm{d}r} = -p\frac{\mathrm{d}\ln p}{\mathrm{d}r} &= -\gamma p \frac{\mathrm{d}\ln \rho}{\mathrm{d}r} \\ &= \rho v_s^2 \left(\frac{2}{r} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right) \end{aligned}$$

Multiply (20) by u/ρ:

$$(u^2 - v_s^2)\frac{\mathrm{d}u}{\mathrm{d}r} = u\left(\frac{2v_s^2}{r} - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

A critical point (*sonic point*) occurs at any radius $r = r_s$ where $|u| = v_s$. For the flow to pass smoothly form subsonic to supersonic, the right-hand side must vanish at the sonic point:

$$\frac{2v_{ss}^2}{r_s} - \frac{GM}{r_s^2} = 0$$

Evaluate Bernoulli's equation at the sonic point:

$$\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right)v_{ss}^2 - \frac{GM}{r_s} = B$$

Deduce that

$$v_{ss}^2 = \frac{2(\gamma - 1)}{5 - 3\gamma}B, \qquad r_s = \frac{5 - 3\gamma}{4(\gamma - 1)}\frac{GM}{B}$$

Assuming that $B > 0$ (see earlier), there is a unique transonic solution for $1 \leq \gamma < 5/3$. The case $\gamma = 1$ can be treated separately or by taking a limit. Now evaluate $\dot{M}$ at the sonic point:

$$|\dot{M}| = 4\pi r_s^2 \rho_s v_{ss}$$

8.4. Second treatment

Use Bernoulli's equation

$$\frac{1}{2}u^2 - \frac{GM}{r} + \frac{v_s^2}{\gamma - 1} = B \tag{21}$$

Introduce the local Mach number $\mathcal{M} = |u|/v_s$. Then

$$4\pi r^2 \rho v_s \mathcal{M} = |\dot{M}|, \qquad v_s^2 = \gamma K \rho^{\gamma-1}$$

Eliminate ρ:

$$v_s^{\gamma+1} = \gamma K \left(\frac{|\dot{M}|}{4\pi r^2 \mathcal{M}} \right)^{\gamma-1}$$

Then (21) becomes

$$\frac{1}{2}v_s^2 \mathcal{M}^2 - \frac{GM}{r} + \frac{v_s^2}{\gamma - 1} = B$$

Substitute for v_s and separate variables:

$$f(\mathcal{M}) = g(r)$$

where

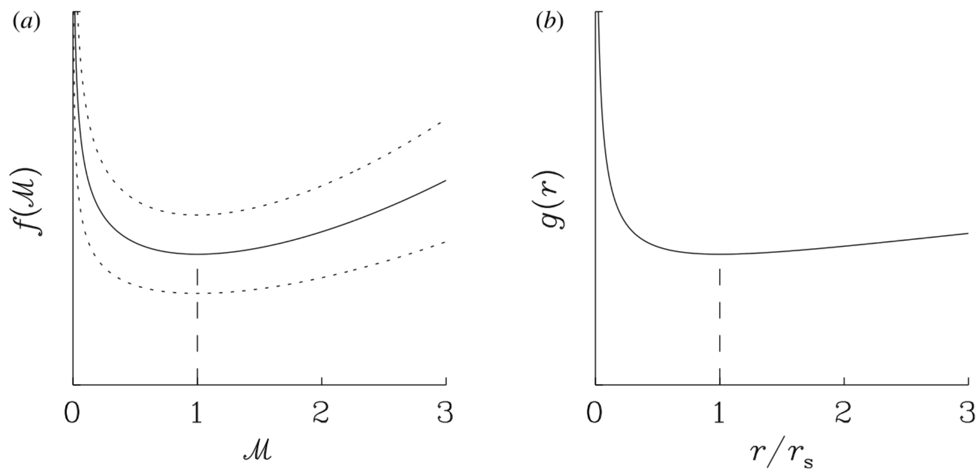
$$f(\mathcal{M}) = (\gamma K)^{\frac{2}{\gamma+1}} \left(\frac{|\dot{M}|}{4\pi} \right)^{\frac{2(\gamma-1)}{\gamma+1}} \left[\frac{\mathcal{M}^{\frac{4}{\gamma+1}}}{2} + \frac{\mathcal{M}^{-\frac{2(\gamma-1)}{\gamma+1}}}{\gamma - 1} \right]$$

$$g(r) = Br^{\frac{4(\gamma-1)}{\gamma+1}} + GMr^{-\frac{5-3\gamma}{\gamma+1}}$$

Assume that $1 < \gamma < 5/3$ and $B > 0$. Each of f and g is the sum of a positive power and a negative power, with positive coefficients. $f(\mathcal{M})$ has a minimum at $\mathcal{M} = 1$, and $g(r)$ has a minimum at

$$r = \frac{5 - 3\gamma}{4(\gamma - 1)} \frac{GM}{B}$$

which is the sonic radius identified previously. A smooth passage through the sonic point is possible only if $|\dot{M}|$ has a special value, so that the minima of f and g are equal.



If $|\dot{M}|$ is too large then the solution does not work for all r. If $|\dot{M}|$ is too small then the solution remains subsonic (or supersonic) for all r, which may not agree with the boundary conditions. The $(r, \mathcal{M})$ plane shows an X-type critical point at $(r_s, 1)$ (figure 5). For $r \ll r_s$ the subsonic solution is close to a hydrostatic atmosphere. The supersonic solution is close to free fall. For $r \gg r_s$ the subsonic solution approaches a uniform state (p =const., ρ =const.) The supersonic solution is close to u =const., so $\rho \propto r^{-2}$.

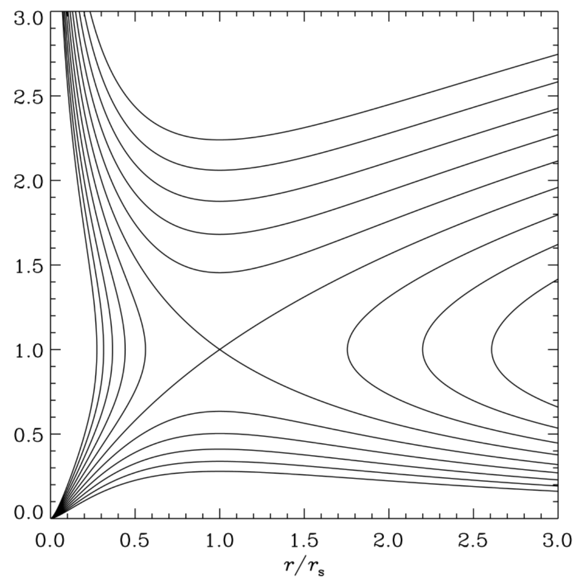


Figure 5: Solution curves for a stellar wind or accretion flow in the case $\gamma = 4/3$.

8.5. Stellar wind

For a stellar wind the appropriate solution is subsonic (quasi-hydrostatic) at small r, supersonic (coasting) at large r. Parker (1958) first presented this model of the solar wind. The mass loss rate $\dot{M}$ can be determined from the quasi-hydrostatic region, e.g. ρ and T at the base of the corona. A completely hydrostatic solution is unacceptable unless the external medium can provide a significant non-zero pressure. Subsonic solutions with $|\dot{M}|$ less than the critical value are usually unacceptable for similar reasons. The interstellar medium arrests the supersonic solar wind in a *termination shock* well beyond Pluto's orbit.

8.6. Accretion

In spherical or Bondi (1952) accretion, we consider a gas which is uniform and at rest at infinity (pressure p_0, density ρ_0):

$$B = \frac{v_{s0}^2}{\gamma - 1} \implies v_{ss}^2 = \frac{2v_{s0}^2}{5 - 3\gamma}$$

Appropriate solutions are subsonic (uniform) at large r, supersonic (free falling) at small r. If the accreting object has a dense surface (a star rather than a black hole), the flow will be arrested by a shock above the surface.

The accretion rate of the transonic solution is

$$\begin{aligned}\dot{M} &= 4\pi r_s^2 \rho_s v_{ss} \\ &= 4\pi r_s^2 \rho_0 v_{s0} \left(\frac{v_{ss}}{v_{s0}}\right)^{\frac{\gamma+1}{\gamma-1}} \\ &= f(\gamma) \dot{M}_B\end{aligned}$$

where

$$\dot{M}_B = \frac{\pi G^2 M^2 \rho_0}{v_{s0}^3} = 4\pi r_a^2 \rho_0 v_{s0}$$

is the characteristic *Bondi accretion rate* and

$$f(\gamma) = \left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}$$

is a dimensionless factor. Here,

$$r_a = \frac{GM}{2v_{s0}^2}$$

is the nominal *accretion radius*, roughly the radius within which the mass M captures the surrounding medium into a supersonic inflow. Note $f(\gamma) \to e^{3/2}$ as $\gamma \to 1$ and $f(\gamma) \to 1$ as $\gamma \to 5/3$. The case $\gamma = 1$ admits a sonic point, but the important case $\gamma = 5/3$ does not.

At different times in its life, a star may gain mass from, or lose mass to, its environment. Currently the sun is losing mass at $-\dot{M} \approx 2 \times 10^{-4}\, M_{\text{sun}}\, \text{yr}^{-1}$. If it were not, it could theoretically accrete at $\dot{M}_B \approx 3 \times 10^{-15}\, M_{\text{sun}}\, \text{yr}^{-1}$ from the interstellar medium.

9. Axisymmetric rotating magnetised flows

9.1. Introduction

In this section (r, ϕ, z) are cylindrical polar coordinates. Stellar winds and jets from accretion discs are examples of outflows in which rotation and magnetic fields are often important or essential. We consider steady ($\partial_t = 0$), axisymmetric ($\partial_\phi = 0$) models based on ideal MHD.

9.2. Axisymmetric representation

The solenoidal condition for an axisymmetric magnetic field is

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

We may write

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

where $\psi(r, z)$ is the *magnetic flux function*. This is related to the vector potential by $\psi = rA_\phi$. The magnetic flux contained inside the circle ($r =$const., $z =$const.) is

$$\int_0^r B_z(r', z) 2\pi r' \, \mathrm{d}r' = 2\pi \psi(r, z)$$

plus a constant that can be set to 0: see figure 6.

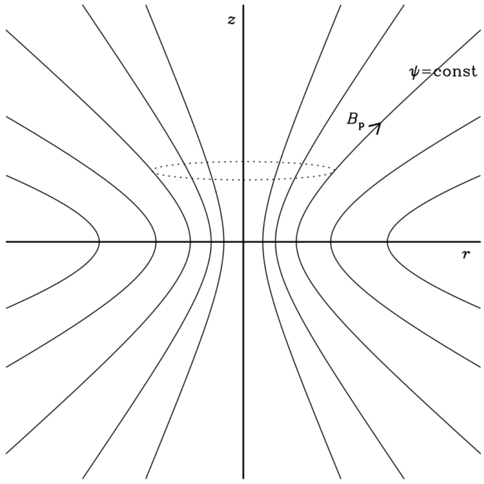


Figure 6: Magnetic flux function and poloidal magnetic field.

Since $\boldsymbol{B}\cdot\nabla\psi = 0$, ψ labels the magnetic field lines or their surfaces of revolution, known as *magnetic surfaces*. We have

$$\begin{aligned}\boldsymbol{B} &= \nabla\psi\times\nabla\phi + B_\phi\hat{\boldsymbol{\phi}}\\ &= -\frac{1}{r}\hat{\boldsymbol{\phi}}\times\nabla\psi + B_\phi\hat{\boldsymbol{\phi}}\\ &= \boldsymbol{B}_p + B_\phi\hat{\boldsymbol{\phi}}\end{aligned}$$

where $\boldsymbol{B}_p$ is the poloidal (meridional) part and B_ϕ is the toroidal (azimuthal) part. Note that $\nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{B}_p = 0$. We can also write the velocity as

$$\boldsymbol{u} = \boldsymbol{u}_p + u_\phi\boldsymbol{e}_\phi$$

although $\nabla\cdot\boldsymbol{u}_p \neq 0$ in general.

9.3. Mass loading & angular velocity

The steady induction equation in ideal MHD,

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0$$

implies

$$\boldsymbol{u}\times\boldsymbol{B} = -\boldsymbol{E} = \nabla\Phi_e$$

where Φ_e is the electrostatic potential. Now

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{B} &= (\boldsymbol{u}_p + u_\phi\boldsymbol{e}_\phi)\times(\boldsymbol{B}_p + B_\phi\boldsymbol{e}_\phi)\\ &= \underbrace{\left[\boldsymbol{e}_\phi\times(u_\phi\boldsymbol{B}_p - B_\phi\boldsymbol{u}_p)\right]}_{\text{poloidal}} + \underbrace{\left[\boldsymbol{u}_p\times\boldsymbol{B}_p\right]}_{\text{toroidal}}\end{aligned}$$

For an axisymmetric solution with $\partial_\phi\Phi_e = 0$,

$$\boldsymbol{u}_p\times\boldsymbol{B}_p = 0$$

i.e. the poloidal velocity is parallel to the poloidal magnetic field. Let $\rho\boldsymbol{u}_p = k\boldsymbol{B}_p$ where k is the *mass loading*, i.e. the ratio of mass flux to magnetic flux.
The steady equation of mass conservation is

$$\begin{aligned} 0 &= \nabla\cdot(\rho\boldsymbol{u}) \\ &= \nabla\cdot(\rho\boldsymbol{u}_p) \\ &= \nabla\cdot(k\boldsymbol{B}_p) \\ &= \boldsymbol{B}_p\cdot\nabla k \end{aligned}$$

Therefore $k = k(\psi)$, i.e. k is a *surface function*, constant on each magnetic surface. We now have

$$\begin{aligned} \boldsymbol{u}\times\boldsymbol{B} &= \boldsymbol{e}_\phi\times(u_\phi\boldsymbol{B}_p - B_\phi\boldsymbol{u}_p) \\ &= \left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho}\right)\nabla\psi \end{aligned}$$

Take the curl:

$$0 = \nabla\left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho}\right)\times\nabla\psi$$

Therefore

$$\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} = \omega$$

where $\omega(\psi)$ is another surface function, known as the *angular velocity of the magnetic surface.* The complete velocity field is then

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega\boldsymbol{e}_\phi$$

i.e. total velocity is parallel to the total magnetic field in a frame rotating with angular velocity ω (for each individual magnetic surface). It is useful to think of the fluid being constrained to move along the field line like a bead on a rotating wire.

9.4. Entropy

The steady thermal energy equation

$$\boldsymbol{u}\cdot\nabla s = 0$$

implies $\boldsymbol{u}_p\cdot\nabla s = 0$, so $\boldsymbol{B}_p\cdot\nabla s = 0$, i.e.

$$s = s(\psi)$$

is another surface function.

9.5. Angular momentum

The ϕ component of the equation of motion is

$$\begin{aligned} \rho\left(\boldsymbol{u}_p\cdot\nabla u_\phi + \frac{u_r u_\phi}{r}\right) &= \frac{1}{\mu_0}\left(\boldsymbol{B}_p\cdot\nabla B_\phi + \frac{B_r B_\phi}{r}\right) \\ \implies \frac{1}{r}\rho\boldsymbol{u}_p\cdot\nabla(ru_\phi) - \frac{1}{\mu_0 r}\boldsymbol{B}_p\cdot\nabla(rB_\phi) &= 0 \\ \implies \frac{1}{r}\boldsymbol{B}_p\cdot\nabla\left(kru_\phi - \frac{rB_\phi}{\mu_0}\right) &= 0 \end{aligned}$$

and so

$$ru_\phi = \frac{rB_\phi}{\mu_0 k} + l$$

where $l = l(\psi)$ is another surface function, the *angular momentum invariant.* l is the angular momentum removed in the outflow per unit mass, although part of the torque is carried by $\boldsymbol{B}$.

9.6. The Alfvén surface

Define the *poloidal Alfvén number* (cf. Mach number)

$$A = \frac{u_p}{v_{ap}}$$

Then

$$A^2 = \frac{\mu_0 \rho u_p^2}{B_p^2} = \frac{\mu_0 k^2}{\rho}$$

and so $A \propto \rho^{-1/2}$ on each magnetic surface. Consider the two equations

$$\frac{u_\phi}{r} = \frac{kB_\phi}{r\rho} + \omega, \qquad ru_\phi = \frac{rB_\phi}{\mu_0 k} + l$$

Eliminate B_ϕ to obtain

$$\begin{aligned} u_\phi &= \frac{r^2\omega - A^2 l}{r(1 - A^2)} \\ &= \left(\frac{1}{1 - A^2}\right) r\omega + \left(\frac{A^2}{A^2 - 1}\right) \frac{l}{r} \end{aligned}$$

For $A \ll 1$ we have

$$u_\phi \approx r\omega$$

i.e. co-rotation with the magnetic surface. For $A \gg 1$ we have

$$u_\phi \approx \frac{l}{r}$$

i.e. conservation of angular momentum. The point $r = r_a(\psi)$ where $A = 1$ is called the *Alfvén point.* The locus of Alfvén points for different magnetic surfaces is called the *Alfvén surface.* To avoid a singularity there we require

$$l = r_a^2 \omega$$

Typically the outflow will start at low velocity in high-density material, where $A \ll 1$. We can therefore identify ω as the angular velocity $u_\phi / r = \Omega_0$ of the footpoint $r = r_0$ of the magnetic field line at the source of the outflow. If mass is lost at a rate $\dot{M}$ in the outflow, angular momentum is lost at a rate $\dot{M} l = \dot{M} r_a^2 \Omega_0$. In contrast, in a hydrodynamic outflow, angular momentum is conserved by fluid elements and is therefore lost at a rate $\dot{M} r_0^2 \Omega_0$. A highly efficient removal of angular momentum occurs if $r_a / r_0 \gg 1$ (the *magnetic lever arm*). The loss of angular momentum through a stellar wind is called *magnetic braking.* For the Sun, r_a is between $20R_{\text{sun}}$ and $30R_{\text{sun}}$.

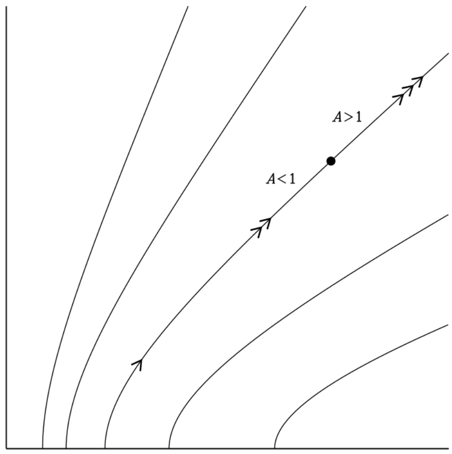


Figure 7: Acceleration through an Alfvén point along a poloidal magnetic field line, leading to angular momentum loss and magnetic braking.

9.7. The Bernoulli function

The total energy equation for a steady flow is

$$\nabla \cdot \left[\rho \boldsymbol{u} (\frac{1}{2} u^2 + \Phi + h) + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu} \right] = 0$$

Now since

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega \boldsymbol{e}_\phi$$

we have

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{u} \times \boldsymbol{B} \\ &= -r\omega \boldsymbol{e}_\phi \times \boldsymbol{B} \\ &= -r\omega \boldsymbol{e}_\phi \times \boldsymbol{B}_p \end{aligned}$$

which is purely poloidal. Thus

$$(\boldsymbol{E} \times \boldsymbol{B})_p = \boldsymbol{E} \times (B_\phi \boldsymbol{e}_\phi) = -r\omega B_\phi \boldsymbol{B}_p$$

The total energy equation is therefore

$$\begin{aligned} \nabla \cdot \left[k\boldsymbol{B}_p (\frac{1}{2} \boldsymbol{u}^2 + \Phi + h) - \frac{r\omega B_\phi}{\mu_0} \boldsymbol{B}_p \right] &= 0 \\ \boldsymbol{B}_p \cdot \nabla \left[k \left(\frac{1}{2} \boldsymbol{u}^2 + \Phi + h - \frac{r\omega B_\phi}{\mu_0 k} \right) \right] &= 0 \\ \frac{1}{2} \boldsymbol{u}^2 + \Phi + h - \frac{r\omega B_\phi}{\mu_0 k} &= \varepsilon \end{aligned}$$

where $\varepsilon = \varepsilon(\psi)$ is another surface function, the *energy invariant*. An alternative invariant is

$$\begin{aligned}\tilde{\varepsilon} &= \varepsilon - l\omega \\ &= \frac{1}{2}\boldsymbol{u}^2 + \Phi + h - \frac{r\omega B_\phi}{\mu_0 k} - \left(ru_\phi - \frac{rB_\phi}{\mu_0 k}\right)\omega \\ &= \frac{1}{2}\boldsymbol{u}^2 + \Phi + h - ru_\phi\omega \\ &= \frac{1}{2}u_p^2 + \frac{1}{2}(u_\phi - r\omega)^2 + \Phi_{cg} + h\end{aligned}$$

where

$$\Phi_{cg} = \Phi - \frac{1}{2}\omega^2 r^2$$

is the centrifugal-gravitational potential associated with the magnetic surface. $\tilde{\varepsilon}$ is the Bernoulli function of the flow in the frame rotating with angular velocity ω. In this frame the flow is strictly parallel to $\boldsymbol{B}$, so the Lorentz force does no work:

$$(\boldsymbol{J} \times \boldsymbol{B}) \perp \boldsymbol{B} \implies (\boldsymbol{J} \times \boldsymbol{B}) \perp (\boldsymbol{u} - r\omega\boldsymbol{e}_\phi)$$

9.8. Summary

We integrated almost all of the MHD equations, reducing them to algebraic equations on each magnetic surface. If $\boldsymbol{B}_p$ (or equivalently ψ) is known, these equations determine the solution on each magnetic surface. We must also

- specify the initial conditions at the source
- ensure the solution passes smoothly through critical points where $|\boldsymbol{u}_p|$ matches the speeds of slow and fast magnetoacoustic waves (ES3, Q5)

The component of the equation of motion perpendicular to $\boldsymbol{B}_p$ remains to be solved. This *transfield* or *Grad-Shafranov* equation determines the shape of the magnetic surfaces. It is a complicated nonlinear PDE for $\psi(r, z)$.

9.9. Outflow from an accretion disc

Consider the launching of an outflow from a thin accretion disc. The angular velocity of the disc corresponds (approximately) to circular Keplerian orbital motion around a central mass M:

$$\Omega(r) = \left(\frac{GM}{r^3}\right)^{1/2}$$

For a field line with footpoint at $r = r_0$, the angular velocity of the field line is

$$\omega = \Omega(r_0)$$

Then

$$\Phi_{cg} = -GM(r^2 + z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2$$

In the sub-Alfvénic region we have

$$\tilde{\varepsilon} \approx \frac{1}{2}\boldsymbol{u}_p^2 + \Phi_{cg} + h$$

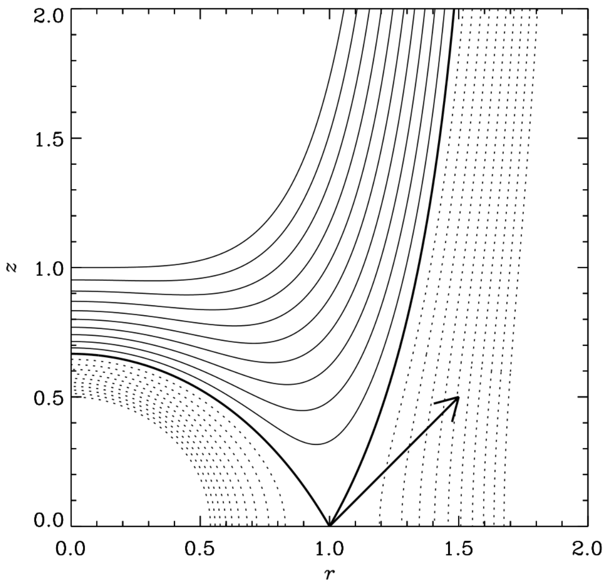


Figure 8: Contours of Φ_{cg}, in units such that $r_0 = 1$. The downhill directions are indicated by dotted contours. If the inclination of the poloidal magnetic field to the vertical direction at the surface of the disc exceeds 30°, gas is accelerated along the field lines away from the disc.

If the gas is very hot ($h \sim GM/r_0$) an outflow can be driven thermally (as in a stellar wind). Of more interest is the possibility of a dynamically driven outflow. For a 'cold' wind, the enthalpy h makes a negligible contribution in the Bernoulli equation. Whether the flow accelerates or not along the field line depends on the variation of Φ_{cg} along the field line.
In units such that $r_0 = 1$, the equation of the equipotential passing through the footpoint $(r_0, 0)$ is

$$(r^2 + z^2)^{-1/2} + \frac{r^2}{2} = \frac{3}{2}$$

Rearrange to get

$$z^2 = \frac{(4 - r^2)(r^2 - 1)^2}{(3 - r^2)^2} = \frac{(4 - r^2)(r + 1)^2(r - 1)^2}{(3 - r^2)^2}$$

Now close to the footpoint (1,0) we have

$$z^2 \approx 3(r - 1)^2 \implies z \approx \pm\sqrt{3}(r - 1)$$

The footpoint lies at a saddle point of Φ_{cg}. If the inclination i of the field line to the vertical exceeds 30°, the flow is acccelerated without thermal assistance. This is *magnetocentrifugal accceleration*. The critical equipotential has an asymptote at $r = r_0\sqrt{3}$. The field line must continue to expand sufficiently in the radial direction to sustain the acceleration.

9.10. Magnetically driven accretion

To allow a quantity of mass ΔM_{acc} to be accreted from radius r_0, its orbital angular momentum $r_0^2\Omega_0\Delta M_{\text{acc}}$ must be removed. The angular momentum removed by a quantity of mass ΔM_{jet} flowing out in a magnetized jet from radius r_0 is $l\Delta M_{\text{jet}} = r_a^2\Omega_0\Delta M_{\text{jet}}$. Therefore accretion can in principle be driven by an outflow, with

$$\frac{\dot{M}_{\text{acc}}}{\dot{M}_{\text{jet}}} \approx \frac{r_a^2}{r_0^2}$$

The magnetic lever arm allows an efficient removal of angular momentum if $r_a^2 \gg r_0^2$.

10. Waves and instabilities in stratified rotating astrophysical bodies

10.1. The energy principle

From section 5: the linearised equation of motion for small displacements $\boldsymbol{\xi}(\boldsymbol{x}, t)$ from a general static equilibrium in ideal MHD is

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho \nabla \delta \Phi - \delta\rho \nabla \Phi - \nabla \delta \Pi + \frac{1}{\mu_0}(\delta \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \boldsymbol{B} \cdot \nabla \delta \boldsymbol{B})$$

where the Eulerian perturbations are given by

$$\begin{aligned}
\delta\rho &= -\boldsymbol{\xi} \cdot \nabla \rho - \rho \nabla \cdot \boldsymbol{\xi} \\
\delta p &= -\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi} \\
\delta \boldsymbol{B} &= \boldsymbol{B} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{B}(\nabla \cdot \boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \nabla \boldsymbol{B} \\
\delta \Pi &= \delta p + \frac{\boldsymbol{B} \cdot \delta \boldsymbol{B}}{\mu_0} \\
&= -\boldsymbol{\xi} \cdot \nabla \Pi - \left(\gamma p + \frac{\boldsymbol{B}^2}{\mu_0}\right) \nabla \cdot \boldsymbol{\xi} + \frac{1}{\mu_0} \boldsymbol{B} \cdot (\boldsymbol{B} \cdot \nabla \boldsymbol{\xi})
\end{aligned}$$

and we allow self-gravitation via

$$\nabla^2 \delta \Phi = 4\pi G \delta\rho = -4\pi G \nabla \cdot (\rho \boldsymbol{\xi})$$

The equation of motion can be written as

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = -\rho \frac{\partial \delta \Phi}{\partial x_i} - \rho \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_j}\left(V_{ijkl} \frac{\partial \xi_k}{\partial x_l}\right)$$

where

$$V_{ijkl} = \left[(\gamma - 1)p + \frac{\boldsymbol{B}^2}{2\mu_0}\right] \delta_{ij}\delta_{kl} + \left(p + \frac{\boldsymbol{B}^2}{2\mu_0}\right) \delta_{il}\delta_{jk} + \frac{1}{\mu_0} B_j B_l \delta_{ik} - \frac{1}{\mu_0}(B_i B_j \delta_{kl} + B_k B_l \delta_{ij})$$

is a tensor with the symmetry $V_{ijkl} = V_{klij}$. This is of the form

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathcal{F} \boldsymbol{\xi} \tag{22}$$

where $\mathcal{F}$ is a linear differential operator (or integro-differential, if self-gravitation is included). The force operator $\mathcal{F}$ is self-adjoint with respect to the inner product

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = \int \rho \boldsymbol{\eta}^* \cdot \boldsymbol{\xi} \, \mathrm{d}V$$

if appropriate boundary conditions apply to the vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Let $\delta\Psi$ be the gravitational potential perturbation associated with the displacement $\boldsymbol{\eta}$, so

$$\nabla^2 \delta \Psi = -4\pi G \nabla \cdot (\rho \boldsymbol{\eta})$$

Then (integrals over all space)

$$
\begin{aligned}
\langle \boldsymbol{\eta}, \mathcal{F}\boldsymbol{\xi}\rangle &= \int \left[-\rho\eta_i^* \frac{\partial \delta\Phi}{\partial x_i} - \rho\eta_i^*\xi_j \frac{\partial^2\Phi}{\partial x_i \partial x_j} + \eta_i^* \frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial \xi_k}{\partial x_l}\right)\right] \mathrm{d}V \\
&= \int \left[-\delta\Phi \frac{\nabla^2\delta\Psi^*}{4\pi G} - \rho\xi_i\eta_j^* \frac{\partial^2\Phi}{\partial x_i \partial x_j} - V_{ijkl}\frac{\partial \xi_k}{\partial x_l}\frac{\partial \eta_i^*}{\partial x_j}\right] \mathrm{d}V \\
&= \int \left[\frac{\nabla(\delta\Phi)\cdot\nabla(\delta\Psi^*)}{4\pi G} - \rho\xi_i\eta_j^* \frac{\partial^2\Phi}{\partial x_i \partial x_j} + \xi_k \frac{\partial}{\partial x_l}\left(V_{ijkl}\frac{\partial \eta_i^*}{\partial x_j}\right)\right] \mathrm{d}V \\
&= \int \left[-\delta\Psi^* \frac{\nabla^2\delta\Phi}{4\pi G} - \rho\xi_i\eta_j^* \frac{\partial^2\Phi}{\partial x_i \partial x_j} + \xi_i \frac{\partial}{\partial x_j}\left(V_{klij}\frac{\partial \eta_k^*}{\partial x_l}\right)\right] \mathrm{d}V \\
&= \int \left[-\rho\xi_i \frac{\partial \delta\Psi^*}{\partial x_i} - \rho\xi_i\eta_j^* \frac{\partial^2\Phi}{\partial x_i \partial x_j} + \xi_i \frac{\partial}{\partial x_j}\left(V_{ijkl}\frac{\partial \eta_k^*}{\partial x_l}\right)\right] \mathrm{d}V \\
&= \langle \mathcal{F}\boldsymbol{\eta}, \boldsymbol{\xi}\rangle
\end{aligned}
$$

We assumed:

- the exterior is a medium of zero density in which the force-free limit of MHD holds;
- $\boldsymbol{B}$ decays sufficiently fast as $|\boldsymbol{x}| \to \infty$ to allow integration by parts (using the divergence theorem) ignoring surface terms;
- the body is isolated and self-gravitating, so $\delta\Phi = \mathcal{O}(r^{-1})$ or in fact $\mathcal{O}(r^{-2})$ if $\delta M = 0$.

We used the symmetries of $\partial^2\Phi/\partial x_i \partial x_j$ and V_{ijkl}. The functional

$$
\begin{aligned}
W[\boldsymbol{\xi}] &= -\frac{1}{2}\langle \boldsymbol{\xi}, \mathcal{F}\boldsymbol{\xi}\rangle \\
&= \frac{1}{2}\int \left(-\frac{|\nabla\delta\Phi|^2}{4\pi G} + \rho\xi_i^*\xi_j \frac{\partial^2\Phi}{\partial x_i}\partial x_j + V_{ijkl}\frac{\partial \xi_i^*}{\partial x_j}\frac{\partial \xi_k}{\partial x_l}\right) \mathrm{d}V
\end{aligned}
$$

is therefore real and represents the change in potential energy associated with $\boldsymbol{\xi}$.
Assuming the basic state is static, consider normal-mode solutions to (22) of the form

$$
\boldsymbol{\xi} = \Re\left[\tilde{\boldsymbol{\xi}}(\boldsymbol{x}) e^{-i\omega t}\right]
$$

for which we obtain

$$
-\omega^2 \tilde{\boldsymbol{\xi}} = \mathcal{F}\tilde{\boldsymbol{\xi}}
$$

and

$$
\omega^2 = -\frac{\langle \tilde{\boldsymbol{\xi}}, \mathcal{F}\tilde{\boldsymbol{\xi}}\rangle}{\langle \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\rangle} = \frac{2W\left[\tilde{\boldsymbol{\xi}}\right]}{\langle \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\rangle}
$$

Therefore ω^2 is real and we have either oscillations ($\omega^2 > 0$) or instability ($\omega^2 < 0$). The expression

$$
\omega^2 = \Lambda\left[\boldsymbol{\xi}\right] = \frac{2W\left[\tilde{\boldsymbol{\xi}}\right]}{\langle \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\rangle}
$$

satisfies the usual Rayleigh-Ritz variational principle for self-adjoint eigenvalue problems. The eigenvalues ω^2 are the stationary values of $\Lambda\left[\boldsymbol{\xi}\right]$ among trial displacements $\boldsymbol{\xi}$ satisfying the boundary conditions. The lowest eigenvalue is the global minimum of $\Lambda\left[\boldsymbol{\xi}\right]$. Therefore the equilibrium is unstable iff $W[\boldsymbol{\xi}] < 0$ for some trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, as this then implies that the global minimum is negative so there is *at least* one negative eigenvalue ω^2. This is the *energy principle*.

This discussion is incomplete because it assumes that the eigenfunctions form a complete set. In general a continuous spectrum, not associated with square-integrable modes, is also present. However, a necessary and sufficient condition for instability is that $W[\boldsymbol{\xi}] < 0$ as described above. To see this, consider (twice) the energy of the perturbation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}]\Big) &= \langle\ddot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}},\ddot{\boldsymbol{\xi}}\rangle - \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\boldsymbol{\xi},\mathcal{F}\dot{\boldsymbol{\xi}}\rangle \\ &= \langle\mathcal{F}\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\dot{\boldsymbol{\xi}},\mathcal{F}\boldsymbol{\xi}\rangle - \langle\mathcal{F}\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle \\ &= 0\end{aligned}$$

Therefore

$$\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = \text{const.}$$

where E is determined by the initial data $\boldsymbol{\xi}_0$ and $\dot{\boldsymbol{\xi}}_0$. If W is positive definite then the equilibrium is stable because $\boldsymbol{\xi}$ is limited by the constraint $W[\boldsymbol{\xi}] \leq E$.

Suppose that a (real) trial displacement $\boldsymbol{\eta}$ can be found for which

$$\frac{2W[\boldsymbol{\eta}]}{\langle\boldsymbol{\eta},\boldsymbol{\eta}\rangle} = -\gamma^2$$

where $\gamma > 0$. Choose initial conditions $\boldsymbol{\xi}_0 = \boldsymbol{\eta}$ and $\dot{\boldsymbol{\xi}}_0 = \gamma\boldsymbol{\eta}$. Then

$$\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = 0$$

Let

$$a(t) = \ln\left(\frac{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{\langle\boldsymbol{\eta},\boldsymbol{\eta}\rangle}\right)$$

so that

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2a}{\mathrm{d}t^2} &= \frac{2(\langle\boldsymbol{\xi},\mathcal{F}\boldsymbol{\xi}\rangle + \langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle)\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - 4\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2} \\ &= \frac{2(-2W[\boldsymbol{\xi}] + \langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle)\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - 4\langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2} \\ &= \frac{4(\langle\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}\rangle\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle - \langle\boldsymbol{\xi},\dot{\boldsymbol{\xi}}\rangle^2)}{\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle^2} \\ &\geq 0\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &\geq \dot{a}_0 = 2\gamma \\ \implies a &\geq 2\gamma t + a_0 = 2\gamma t\end{aligned}$$

Therefore the disturbance with these initial conditions grows at least as fast as $\exp(\gamma t)$ and the equilibrium is unstable.

10.2. Spherically symmetric star

Consider the simplest model of a star which is spherically symmetric, and we neglect rotation and magnetic fields. Further, assume the star is in hydrostatic equilibrium so that $p(r)$ and $\rho(r)$ satisfy

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g$$

with

$$g(r) = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2}\int_0^r \rho(r')4\pi r'^2\,\mathrm{d}r'$$

The stratification induced by gravity provides a non-uniform background for wave propagation. In this case the linearised equation of motion is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\nabla\delta\Phi - \delta\rho\nabla\Phi - \nabla\delta p$$

with

$$\begin{aligned}\delta\rho &= -\nabla\cdot(\rho\boldsymbol{\xi})\\ \delta p &= -\gamma p\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p\\ \nabla^2\delta\Phi &= 4\pi G\delta\rho\end{aligned}$$

For normal modes $\propto \exp(-i\omega t)$ the equation of motion becomes

$$\rho\omega^2\boldsymbol{\xi} = \rho\nabla\delta\Phi + \delta\rho\nabla\Phi + \nabla\delta p$$

Taking the dot product with $\boldsymbol{\xi}^*$ and integrating over the volume V of the star gives

$$\omega^2\int_V \rho|\boldsymbol{\xi}|^2\,\mathrm{d}V = \int_V \boldsymbol{\xi}^*\cdot(\rho\nabla\delta\Phi + \delta\rho\nabla\Phi + \nabla\delta p)\,\mathrm{d}V$$

where V is the volume of the star. At the surface $S = \partial V$ of the star, we assume that ρ and p vanish. Then δp also vanishes on S (assuming that $\boldsymbol{\xi}$ and its derivatives are bounded). Integrate the δp term by parts:

$$\begin{aligned}\int_V \boldsymbol{\xi}^*\cdot\nabla\delta p\,\mathrm{d}V &= -\int_V (\nabla\cdot\boldsymbol{\xi})^*\delta p\,\mathrm{d}V\\ &= \int_V \frac{1}{\gamma p}(\delta p + \boldsymbol{\xi}\cdot\nabla p)^*\delta p\,\mathrm{d}V\\ &= \int_V\left[\frac{|\delta p|^2}{\gamma p} + \frac{1}{\gamma p}(\boldsymbol{\xi}^*\cdot\nabla p)(-\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi})\right]\mathrm{d}V\end{aligned}$$

The $\delta\rho$ term partially cancels with this:

$$\begin{aligned}\int_V \boldsymbol{\xi}^*\cdot(\delta\rho\nabla\Phi)\,\mathrm{d}V &= \int_V (-\boldsymbol{\xi}^*\cdot\nabla p)\frac{\delta\rho}{\rho}\,\mathrm{d}V\\ &= \int_V (\boldsymbol{\xi}^*\cdot\nabla p)(\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla\ln\rho)\,\mathrm{d}V\end{aligned}$$

The $\delta\Phi$ term can be transformed as in the previous section to give

$$\int_V \rho\boldsymbol{\xi}^*\cdot\nabla\delta\Phi\,\mathrm{d}V = -\int_\infty \frac{|\nabla\delta\Phi|^2}{4\pi G}\,\mathrm{d}V$$

(integral over all space). Thus

$$\begin{aligned}\omega^2 \int_V \rho|\boldsymbol{\xi}|^2\,\mathrm{d}V &= -\int_\infty \frac{|\nabla\delta\Phi|^2}{4\pi G}\,\mathrm{d}V + \int_V \left[\frac{|\delta p|^2}{\gamma p} - (\boldsymbol{\xi}^*\cdot\nabla p)(\frac{1}{\gamma}\boldsymbol{\xi}\cdot\nabla\ln p - \boldsymbol{\xi}\cdot\nabla\ln\rho)\right]\mathrm{d}V \\ &= -\int_\infty \frac{|\nabla\delta\Phi|^2}{4\pi G}\,\mathrm{d}V + \int_V \left(\frac{|\delta p|^2}{\gamma p} + \rho N^2|\xi_r|^2\right)\mathrm{d}V\end{aligned}$$

where $N(r)$ is the Brunt-Väisälä frequency (or *buoyancy frequency*) given by

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) \propto g\frac{\mathrm{d}s}{\mathrm{d}r}$$

N is the frequency of oscillation of a fluid element that is displaced vertically in a stably stratified atmosphere if it maintains pressure equilibrium with its surroundings. Stratification is stable if s increases outwards. The expression

$$\omega^2 \int_V \rho|\boldsymbol{\xi}|^2\,\mathrm{d}V = -\int_\infty \frac{|\nabla\delta\Phi|^2}{4\pi G}\,\mathrm{d}V + \int_V \left(\frac{|\delta p|^2}{\gamma p} + \rho N^2|\xi_r|^2\right)\mathrm{d}V$$

satisfies the energy principle. There are three contributions to ω^2: the self-gravitational term (destabilising), the acoustic term (stabilising) and the buoyancy term (stabilising if $N^2 > 0$). If $N^2 < 0$ for any interval of r, a trial displacement can be found such that $\omega^2 < 0$. This is done by localising ξ_r in that interval and arranging the other components of $\boldsymbol{\xi}$ such that $\delta p = 0$. Therefore the star is unstable if $\partial s/\partial r < 0$ anywhere. This is *Schwarzschild's criterion* for convective instability.

10.3. Modes of an incompressible sphere

In this subsection (r, θ, ϕ) are spherical polar coordinates. Analytical solutions can be obtained for a homogeneous incompressible 'star' of mass M and radius R with density profile

$$\rho = \left(\frac{3M}{4\pi R^3}\right) H(R - r)$$

where H is the Heaviside step function. For $r \le R$ we have

$$g = \frac{GMr}{R^3}, \qquad p = \frac{3GM^2(R^2 - r^2)}{8\pi R^6}$$

For an incompressible fluid,

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi} &= 0 \\ \delta\rho = -\boldsymbol{\xi}\cdot\nabla\rho &= \xi_r\left(\frac{3M}{4\pi R^3}\right)\delta(r - R)\end{aligned}$$

and

$$\nabla^2\delta\Phi = 4\pi G\delta\rho = \xi_r\left(\frac{3GM}{R^3}\right)\delta(r - R) \tag{23}$$

Here δp is indeterminate and is independent of $\boldsymbol{\xi}$. The linearised equation of motion is

$$-\rho\omega^2\boldsymbol{\xi} = -\rho\nabla\delta\Phi - \nabla\delta p$$

So we have potential flow where

$$\boldsymbol{\xi} = \nabla U, \qquad \nabla^2 U = 0 \ \ (r \leq R)$$

and U satisfies

$$-\rho\omega^2 U = -\rho\delta\Phi - \delta p$$

Appropriate solutions of Laplace's equation which are regular at $r = 0$ are the *solid spherical harmonics* (with arbitrary normalisation)

$$U = r^l Y_l^m(\theta, \phi)$$

where l and m are integers with $l \geq |m|$. From (23) we have

$$\delta\Phi = \begin{cases} A r^l Y_l^m & r < R \\ B r^{-l-1} Y_l^m & r > R \end{cases}$$

where A and B are constants to be determined. Matching conditions at $r = R$ are

$$[\delta\Phi]_-^+ = 0, \qquad \left[\frac{\partial\delta\Phi}{\partial r}\right]_-^+ = \xi_r \left(\frac{3GM}{R^3}\right)$$

Thus

$$BR^{-l-1} - AR^l = 0$$
$$-(l+1)BR^{-l-2} - lAR^{l-1} = lR^{l-1}\left(\frac{3GM}{R^3}\right)$$

which has solution

$$A = -\frac{l}{2l+1}\left(\frac{3GM}{R^3}\right), \qquad B = AR^{2l+1}$$

At $r = R$ the Lagrangian pressure perturbation should vanish:

$$\Delta p = \delta p + \boldsymbol{\xi} \cdot \nabla p = 0$$
$$\implies \left(\frac{3M}{4\pi R^3}\right)\left[\omega^2 R^l + \frac{l}{2l+1}\left(\frac{3GM}{R^3}\right)R^l\right] - \frac{3GM^2}{4\pi R^5} l R^{l-1} = 0$$

Hence the frequency is

$$\begin{aligned} \omega^2 &= \left(l - \frac{3l}{2l+1}\right)\frac{GM}{R^3} \\ &= \frac{2l(l-1)}{2l+1}\frac{GM}{R^3} \end{aligned} \tag{24}$$

This result was obtained by Lord Kelvin. Since $\omega^2 \geq 0$, the star is stable. Note that $l = 0$ corresponds to $\boldsymbol{\xi} = 0$ (trivial solution) and $l = 1$ corresponds to $\boldsymbol{\xi} = \text{const.}$ (translational mode of zero frequency). The remaining modes are non-trivial and are called *f modes* (fundamental modes). They are surface gravity waves, related to ocean waves for which $\omega^2 = gk$. In expression (24) the first term in brackets derives from surface gravity and the second derives from self-gravity.

10.4. The plane-parallel atmosphere

The local dynamics of a stellar atmosphere can be studied in a Cartesian approximation. Take $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$ to be constant. For hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g$$

10.4.1. Isothermal atmosphere

Take $p = c_s^2 \rho$ in equilibrium, with c_s constant:

$$\rho = \rho_0 e^{-z/H}, \qquad p = p_0 e^{-z/H}$$

where $H = c_s^2/g$ is the *isothermal scale-height.* In this case the buoyancy frequency is

$$\begin{aligned} N^2 &= g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}z} - \frac{\mathrm{d}\ln \rho}{\mathrm{d}z}\right) \\ &= \left(1 - \frac{1}{\gamma}\right)\frac{g}{H} \\ &= \text{const.} \end{aligned}$$

Hence an isothermal atmosphere is

- *stably (subadiabatically)*[1] *stratified* ($N^2 > 0$) if $\gamma > 1$
- *neutrally (adiabatically) stratified* ($N^2 = 0$) if $\gamma = 1$

10.4.2. Polytropic atmosphere

Take $p \propto \rho^{1+1/m}$ in equilibrium, where m is a positive constant. In general $1 + 1/m \neq \gamma$. For hydrostatic equilibrium,

$$\rho^{1/m}\frac{\mathrm{d}\rho}{\mathrm{d}z} \propto -\rho g \implies \rho^{1/m} \propto -z$$

if the top of the atmosphere is at $z = 0$, with vacuum above. Let

$$\rho = \rho_0 \left(-\frac{z}{H}\right)^m$$

for $z < 0$, where ρ_0 and H are constants. Then

$$p = p_0 \left(-\frac{z}{H}\right)^{m+1}$$

where

$$p_0 = \frac{\rho_0 g H}{m+1}$$

to satisfy $\mathrm{d}p/\mathrm{d}z = -\rho g$. In this case the buoyancy frequency is

$$N^2 = \left(m - \frac{m+1}{\gamma}\right)\frac{g}{-z}$$

A polytropic atmosphere is

- *stably (subadiabatically) stratified* ($N^2 > 0$) if $\gamma > 1 + 1/m$
- *neutrally (adiabatically) stratified* ($N^2 = 0$) if $\gamma = 1 + 1/m$
- *unstably (superadiabatically) stratified* ($N^2 < 0$) if $\gamma < 1 + 1/m$

[1] In adiabatic stratification the vertical temperature gradient is that of a fluid parcel which does not gain or lose any heat to the environment. In subadiabatic stratification the vertical temperature gradient (or *lapse rate*) is smaller than adiabatic.

10.4.3. Astroseismology

Return to the linearised equations (3), looking for solutions of the form

$$\boldsymbol{\xi} = \Re\left[\tilde{\boldsymbol{\xi}}(z)\exp(-i\omega t + i\boldsymbol{k}_h\cdot\boldsymbol{x})\right], \quad \text{etc.}$$

where h denotes a horizontal vector with only x and y components. Then

$$\begin{aligned}
-\rho\omega^2\boldsymbol{\xi}_h &= -i\boldsymbol{k}_h\delta p \\
-\rho\omega^2\xi_z &= -g\delta\rho - \frac{\mathrm{d}\delta p}{\mathrm{d}z} \\
\delta\rho &= -\xi_z\frac{\mathrm{d}\rho}{\mathrm{d}z} - \rho\Delta \\
\delta p &= -\xi_z\frac{\mathrm{d}p}{\mathrm{d}z} - \gamma p\Delta
\end{aligned}$$

where

$$\Delta = \nabla\cdot\boldsymbol{\xi} = i\boldsymbol{k}_h\cdot\boldsymbol{\xi}_h + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}$$

Note we have neglected self-gravity in the atmosphere so $\delta\Phi = 0$, which amounts to the Cowling approximation whereby Φ is specified. Only two z-derivatives of perturbations arise: $\mathrm{d}\delta p/\mathrm{d}z$ and $\mathrm{d}\xi_z/\mathrm{d}z$. This is a second-order system of ODEs, combined with algebraic equations (e.g. Δ). Eliminate $\boldsymbol{\xi}_h$:

$$\Delta = -\frac{k_h^2}{\rho\omega^2}\delta p + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}$$

where $k_h = |\boldsymbol{k}_h|$. Eliminate $\delta\rho$:

$$-\rho\omega^2\xi_z = g\xi_z\frac{\mathrm{d}\rho}{\mathrm{d}z} = \rho g\Delta - \frac{\mathrm{d}\delta p}{\mathrm{d}z}$$

Consider these two ODEs in combination with the remaining algebraic equation

$$\delta p = \rho g\xi_z - \gamma p\Delta$$

First approach: solve the algebraic equation for Δ and substitute to obtain two coupled ODEs

$$\begin{aligned}
\frac{\mathrm{d}\xi_z}{\mathrm{d}z} &= \frac{g}{v_s^2}\xi_z + \frac{1}{\rho v_s^2}\left(\frac{v_s^2k_h^2}{\omega^2} - 1\right)\delta p \\
\frac{\mathrm{d}\delta p}{\mathrm{d}z} &= \rho(\omega^2 - N^2)\xi_z - \frac{g}{v_s^2}\delta p
\end{aligned}$$

where $v_s^2k_h^2$ is the square of the *Lamb frequency*, i.e. the (z-dependent) frequency of a horizontal sound wave of wavenumber k_h. In a short-wavelength (WKB) approximation, where $\xi_z \propto \exp\left[i\int k_z(z)\,\mathrm{d}z\right]$ with $k_z \gg g/v_s^2$, the local dispersion relation derived from these ODEs is

$$v_s^2k_z^2 = (\omega^2 - N^2)\left(1 - \frac{v_s^2k_h^2}{\omega^2}\right)$$

Propagating waves ($k_z^2 > 0$) are possible when

- $\omega^2 > \max(v_s^2k_h^2, N^2)$: these are *p modes*, acoustic waves ('p' for pressure)
- $0 < \omega^2 < \min(v_s^2k_h^2, N^2)$: these are *g modes*, internal gravity waves ('g' for gravity)

There is a special incompressible solution in which $\Delta = 0$, i.e. $\delta p = \rho g \xi_z$:

$$\frac{\mathrm{d}\xi_z}{\mathrm{d}z} = \frac{g k_h^2}{\omega^2}\xi_z, \qquad \frac{\mathrm{d}\xi_z}{\mathrm{d}z} = \frac{\omega^2}{g}\xi_z$$

Hence for compatibility we require

$$\frac{g k_h^2}{\omega^2} = \frac{\omega^2}{g} \implies \omega^2 = \pm g k_h$$

The acceptable solution in which ξ_z decays with depth is

$$\omega^2 = g k_h, \qquad \xi_z \propto \exp(k_h z)$$

This is a *surface gravity wave* known in stellar oscillations as the *f mode* (fundamental mode). It is vertically evanescent.

The other wave solutions (p and g modes) can be found analytically in the case of a polytropic atmosphere. Eliminate variables in favour of Δ (algebra omitted: see official notes) to obtain

$$z\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2} + (m+2)\frac{\mathrm{d}\Delta}{\mathrm{d}z} - (A + k_h z)k_h\Delta = 0$$

where

$$A = \frac{m+1}{\gamma}\frac{\omega^2}{g k_h} + \left(m - \frac{m+1}{\gamma}\right)\frac{g k_h}{\omega^2}$$

is a dimensionless constant. Now let $\Delta = w(z)e^{k_h z}$ to obtain

$$z\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (m + 2 + 2k_h z)\frac{\mathrm{d}w}{\mathrm{d}z} - (A - m - 2)k_h w = 0$$

This is related to the *confluent hypergeometric equation.* There is a regular singular point at $z = 0$. Using the method of Frobenius, seek power-series solutions

$$w = \sum_{r=0}^{\infty} a_r z^{\sigma + r}$$

where σ is to be determined and $a_0 \neq 0$. The resulting indicial equation is

$$\sigma(\sigma + m + 1) = 0$$

The regular solution has $\sigma = 0$ with recurrence relation

$$\frac{a_{r+1}}{a_r} = \frac{(A - m - 2 - 2r)k_h}{(r+1)(r+m+2)}$$

In the case of an infinite series,

$$\frac{a_{r+1}}{a_r} \sim -\frac{2k_h}{r} \quad \text{as } r \to \infty$$

so w behaves like $e^{-2k_h z}$ as $z \to -\infty$ and Δ diverges like $e^{-k_h z}$ (recall z is increasingly negative moving down through the atmosphere). Solutions in which Δ decays with depth are those for which the series terminates and w is a polynomial. For a polynomial of degree $n-1$ ($n \geq 1$) we require $a_n = 0$ hence

$$A = 2n + m$$

This gives a quadratic equation for ω^2:

$$\frac{m+1}{\gamma}\left(\frac{\omega^2}{gk_h}\right)^2-(2n+m)\left(\frac{\omega^2}{gk_h}\right)+\left(m-\frac{m+1}{\gamma}\right)=0$$

A negative root for ω^2 (i.e. instability) exists if and only if

$$m-\frac{m+1}{\gamma}<0$$

or equivalently $N^2<0$, as expected from Schwarzschild's criterion. For $n\gg 1$, the large root is

$$\frac{\omega^2}{gk_h}\sim\frac{2n\gamma}{m+1}$$

which are p modes with $\omega^2\propto v_s^2$ and the small root is

$$\frac{\omega^2}{gk_h}\sim\frac{1}{2n}\left(m-\frac{m+1}{\gamma}\right)$$

which are g modes with $\omega^2\propto N^2$. The f mode is the 'trivial' solution $\Delta=0$. p (pressure) modes are acoustic waves which rely on compressibility. g (gravity) modes are gravity waves, which rely on buoyancy. Typical branches of the dispersion relation are shown in figure 9.

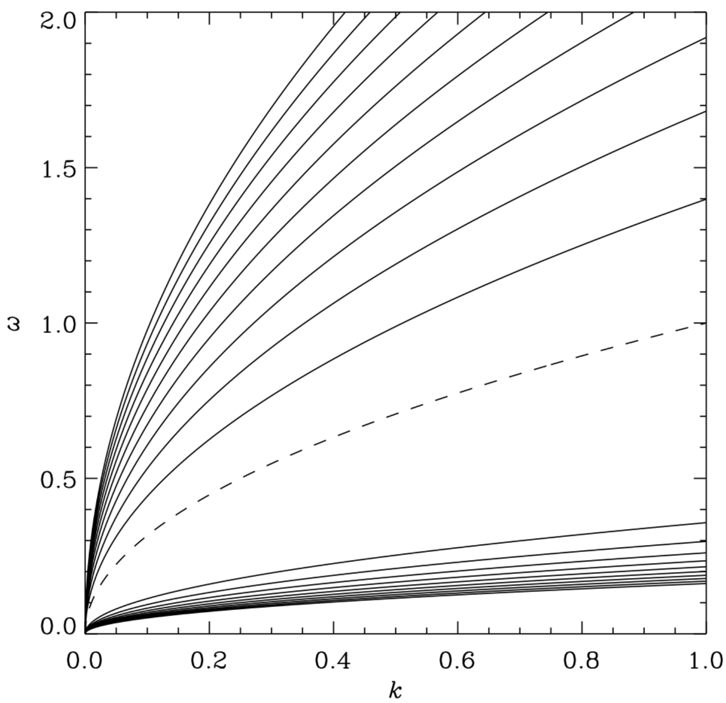


Figure 9: Dispersion relation, in arbitrary units, for a stably stratified plane-parallel polytropic atmosphere with $m=3$ and $\gamma=5/3$. The dashed line is the f mode. Above it are the first ten p modes and below it are the first ten g modes. Each curve is a parabola.

Comparison with observational data from the Sun demonstrates the parabolic dispersion relation: see figure 10. In solar-type stars, the inner part (radiative zone) is convectively stable ($N^2>0$) and the outer part (convective zone) is unstable ($N^2<0$): see figure 11. However, the convection is so efficient that only a very small entropy gradient is required to sustain the convective heat flux, so N^2 is very small and negative in the convective zone. Although g modes propagate in the radiative zone at frequencies smaller than N, they cannot reach the surface. Only f and p modes (excited by convection) are observed at the solar surface.

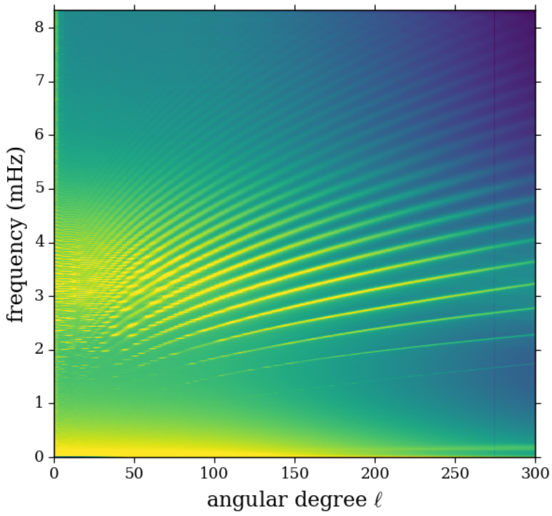


Figure 10: Power spectrum of medium angular degree solar oscillations, computed for 144 days of data from the MDI instrument aboard SOHO. Credit: E. J. Rhodes et al.

In more massive stars, the situation is reversed. Then f, p and g modes can be observed, in principle, at the surface. g modes are particularly well observed in certain classes of white dwarf.

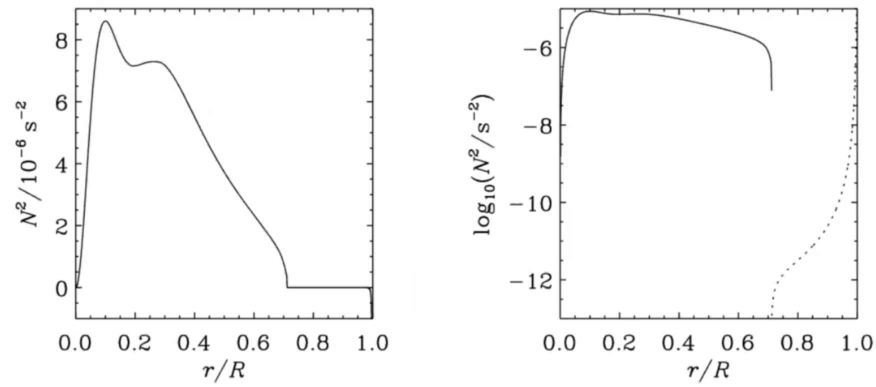


Figure 11: A standard model of the present Sun, up to the photosphere. Squared buoyancy frequency is plotted versus fractional radius. In the convective region of the log plot where $N^2 < 0$, the dotted line shows the logarithm of $-N^2$ instead.

10.5. Rotating fluid bodies

In this subsection (r, ϕ, z) are cylindrical polar coordinates.

10.5.1. Equilibrium

The equations of ideal gas dynamics in cylindrical polars are

$$\begin{aligned}
\frac{\mathrm{D}u_r}{\mathrm{D}t} - \frac{u_\phi^2}{r} &= -\frac{\partial\Phi}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} \\
\frac{\mathrm{D}u_\phi}{\mathrm{D}t} + \frac{u_r u_\phi}{r} &= -\frac{1}{r}\frac{\partial\Phi}{\partial\phi} - \frac{1}{\rho r}\frac{\partial p}{\partial\phi} \\
\frac{\mathrm{D}u_z}{\mathrm{D}t} &= -\frac{\partial\Phi}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \\
\frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\nabla\cdot\boldsymbol{u} \\
\frac{\mathrm{D}p}{\mathrm{D}t} &= -\gamma p\nabla\cdot\boldsymbol{u}
\end{aligned}$$

where

$$\begin{aligned}
\nabla\cdot\boldsymbol{u} &= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} \\
\frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\phi}{r}\frac{\partial}{\partial\phi} + u_z\frac{\partial}{\partial z}
\end{aligned}$$

Consider a steady, axisymmetric basic state with density $\rho(r,z)$, pressure $p(r,z)$, gravitational potential $\Phi(r,z)$ and with differential rotation

$$\boldsymbol{u} = r\Omega(r,z)\hat{\boldsymbol{\phi}}$$

For equilibrium we require

$$-r\Omega^2\hat{\boldsymbol{r}} = -\nabla\Phi - \frac{1}{\rho}\nabla p$$

Take the curl to obtain

$$\begin{aligned}
-r\frac{\partial\Omega^2}{\partial z}\hat{\boldsymbol{\phi}} &= \nabla p\times\nabla\frac{1}{\rho} \\
&= \nabla T\times\nabla s
\end{aligned} \tag{25}$$

This is the vorticity equation in a steady state, sometimes called the *thermal wind equation.* The equilibrium is *barotropic* if ∇p is parallel to $\nabla\rho$, i.e. $p = p(\rho)$, otherwise it is *baroclinic.* In a barotropic state, Ω is independent of z: $\Omega = \Omega(r)$ (a version of the Taylor-Proudman theorem). We can also write

$$\frac{1}{\rho}\nabla p = \boldsymbol{g} = -\nabla\Phi + r\Omega^2\hat{\boldsymbol{r}}$$

where $\boldsymbol{g}$ is the *effective gravity*, including the centrifugal force associated with the (non-uniform) rotation. In a barotropic state with $\Omega(r)$ we can write

$$\begin{aligned}
\boldsymbol{g} &= -\nabla\Phi_{cg} \\
\Phi_{cg} &= \Phi(r,z) + \Psi(r) \\
\Psi &= -\int r\Omega^2\,\mathrm{d}r
\end{aligned}$$

where the arbitrary constant from the indefinite integral is to be determined. Since $p = p(\rho)$ in the equilibrium state, we can define the *pseudo-enthalpy* $\tilde{h}(\rho)$ such that $\mathrm{d}\tilde{h} = \mathrm{d}p/\rho$. An example is a polytropic model for which

$$p = K\rho^{1+1/m}, \qquad \tilde{h} = (m+1)K\rho^{1/m}$$

where m is the polytropic index ($\tilde{h}$ is the true enthalpy only if the equilibrium is homentropic). The equilibrium condition then reduces to

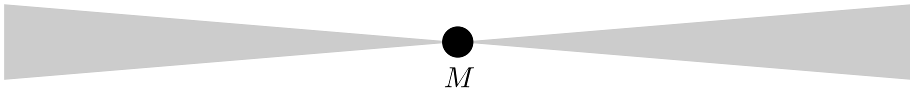
$$0 = -\nabla\Phi_{cg} - \nabla\tilde{h}$$

or equivalently

$$\Phi + \Psi + \tilde{h} = C = \text{const.} \tag{26}$$

Accretion disc around a body. Take for example an accretion disc around a central body of mass M. For a non-self-gravitating disc,

$$\Phi = -GM(r^2 + z^2)^{-1/2}$$



Assume the disc is barotropic. Define the additive constant in $\tilde{h}$ such that $\tilde{h} = 0$ at the surfaces $z = \pm H(r)$ of the disc where $\rho = p = 0$. Then (26) evaluated at the disc surface is

$$-GM(r^2 + H^2)^{-1/2} + \Psi(r) = C$$

from which

$$r\Omega^2 = -\frac{\mathrm{d}}{\mathrm{d}r}\left[GM(r^2 + H^2)^{-1/2}\right]$$

For example, if $H = \varepsilon r$ with ε the constant aspect ratio of the disc, then

$$\Omega^2 = (1 + \varepsilon^2)^{-1/2}\frac{GM}{r^3}$$

Hence the thinner the disc is, the closer it is to Keplerian rotation. This occurs since a thicker disc has a large outwards pressure gradient which compensates for some of the inward gravitational force, i.e. the centrifugal acceleration is smaller (sub-Keplerian). Having related $\Omega(r)$ and $H(r)$, we can find $\tilde{h}$ (and so ρ and p) from (26).

10.5.2. Linear perturbations

Since the basic state is independent of t and ϕ, consider linear perturbations of the form

$$\Re\left[\delta u_r(r, z)\exp(-i\omega t + im\phi)\right]$$

and similar for other variables, where $m \in \mathbb{Z}$ is the azimuthal wavenumber. The linearised equations in the Cowling approximation ($\delta\Phi = 0$) are

$$\begin{aligned}
-i\hat{\omega}\delta u_r - 2\Omega\delta u_\phi &= -\frac{1}{\rho}\frac{\partial\delta p}{\partial r} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial r} \\
-i\hat{\omega}\delta u_\phi + \frac{1}{r}\delta\boldsymbol{u}\cdot\nabla(r^2\Omega) &= -\frac{im\delta p}{\rho r} \\
-i\hat{\omega}\delta u_z &= -\frac{1}{\rho}\frac{\partial\delta p}{\partial z} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial z} \\
-i\hat{\omega}\delta\rho + \delta\boldsymbol{u}\cdot\nabla\rho &= -\rho\nabla\cdot\delta\boldsymbol{u} \\
-i\hat{\omega}\delta p + \delta\boldsymbol{u}\cdot\nabla p &= -\gamma p\nabla\cdot\delta\boldsymbol{u} \\
\nabla\cdot\delta\boldsymbol{u} &= \frac{1}{r}\frac{\partial}{\partial r}(r\delta u_r) + \frac{im\delta u_\phi}{r} + \frac{\partial\delta u_z}{\partial z}
\end{aligned}$$

where

$$\hat{\omega} = \omega - m\Omega$$

is the intrinsic frequency, i.e. the angular frequency of the wave measured in a frame of reference rotating with the local angular velocity of the fluid.
Eliminate δu_ϕ and $\delta\rho$ to get

$$\begin{aligned}(\hat{\omega}^2 - A)\delta u_r - B\delta u_z &= -\frac{i\hat{\omega}}{\rho}\left(\frac{\partial\delta p}{\partial r} - \frac{\partial p}{\partial r}\frac{\delta p}{\gamma p}\right) + 2\Omega\frac{im\delta p}{\rho r} \\ -C\delta u_r + (\hat{\omega}^2 - D)\delta u_z &= -\frac{i\hat{\omega}}{\rho}\left(\frac{\partial\delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right)\end{aligned}$$

where

$$\begin{aligned}A &= \frac{2\Omega}{r}\frac{\partial}{\partial r}(r^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial r}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial\rho}{\partial r}\right) \\ B &= \frac{2\Omega}{r}\frac{\partial}{\partial z}(r^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial r}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z}\right) \\ C &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial\rho}{\partial r}\right) \\ D &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z}\right)\end{aligned}$$

Note that A, B, C and D involve r and z derivatives of the specific angular momentum $r^2\Omega$ and the specific entropy s. The thermal wind equation (25) implies $B = C$, so the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$$

is symmetric.

10.5.3. The Høiland criteria

Introduce the Lagrangian displacement $\boldsymbol{\xi}$ via

$$\Delta\boldsymbol{u} = \delta\boldsymbol{u} + \boldsymbol{\xi}\cdot\nabla\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t}$$

i.e.

$$\begin{aligned}\delta u_r &= -i\hat{\omega}\xi_r \\ \delta u_\phi &= -i\hat{\omega}\xi_\phi - r\boldsymbol{\xi}\cdot\nabla\Omega \\ \delta u_z &= -i\hat{\omega}\xi_z\end{aligned}$$

Note that

$$\frac{1}{r}\frac{\partial}{\partial r}(r\delta u_r) + \frac{im\delta u_\phi}{r} + \frac{\partial\delta u_z}{\partial z} = -i\hat{\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{im\xi_\phi}{r} + \frac{\partial\xi_z}{\partial z}\right]$$

The linearised equations are an eigenvalue problem for ω but it is not self-adjoint except when $m = 0$ (axisymmetric perturbations). In this case,

$$\begin{aligned}(\omega^2 - A)\xi_r - B\xi_z &= \frac{1}{\rho}\left(\frac{\partial\delta p}{\partial r} - \frac{\partial p}{\partial r}\frac{\delta p}{\gamma p}\right) \\ -B\xi_r + (\omega^2 - D)\xi_z &= \frac{1}{\rho}\left(\frac{\partial\delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right)\end{aligned}$$

with

$$\delta p = -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p$$

Multiply the first equation by $\rho \xi_r^*$ and the second by $\rho \xi_z^*$ and integrate over the fluid volume V (with boundary conditions $\delta p = 0$)

$$\begin{aligned}
\omega^2 \int_V \rho \left(|\xi_r|^2 + |\xi_z|^2 \right) \mathrm{d}V &= \int_V \left[\rho Q(\boldsymbol{\xi}) + \boldsymbol{\xi}^* \cdot \nabla \delta p - \frac{\delta p}{\gamma p} \boldsymbol{\xi}^* \cdot \nabla p \right] \mathrm{d}V \\
&= \int_V \left[\rho Q(\boldsymbol{\xi}) - \frac{\delta p}{\gamma p} (\gamma p \nabla \cdot \boldsymbol{\xi}^* + \boldsymbol{\xi}^* \cdot \nabla p) \right] \mathrm{d}V \\
&= \int_V \left(\rho Q(\boldsymbol{\xi}) + \frac{|\delta p|^2}{\gamma p} \right) \mathrm{d}V
\end{aligned}$$

where

$$\begin{aligned}
Q(\boldsymbol{\xi}) &= A|\xi_r|^2 + B(\xi_r^* \xi_z + \xi_z^* \xi_r) + D|\xi_z|^2 \\
&= (\xi_r^*, \xi_z^*) \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} \xi_r \\ \xi_z \end{pmatrix}
\end{aligned}$$

is the (real) Hermitian form associated with M. Note that the integral does not involve ξ_ϕ. If we had included self-gravity, there would be another, negative contribution to ω^2.
The integral relation above shows that ω^2 is real. A variational property ensures that instability to *axisymmetric perturbations* occurs if and only if

$$\int_V \left(\rho Q(\boldsymbol{\xi}) + \frac{|\delta p|^2}{\gamma p} \right) \mathrm{d}V$$

can be made negative by a trial displacement. If Q is positive definite then $\omega^2 > 0$ and we have stability. Now the characteristic equation of M is

$$\lambda^2 - (A + D)\lambda + AD - B^2 = 0$$

The eigenvalues $\lambda_\pm$ are both positive if and only if

$$A + D > 0, \qquad AD - B^2 > 0 \tag{27}$$

If these conditions are satisfied throughout the fluid then $Q > 0$, which implies $\omega^2 > 0$, so that fluid is stable to axisymmetric perturbations (neglecting self-gravitation).
The conditions (27) are also necessary for stability. If one of the eigenvalues is negative in some region, then a trial displacement can be found which is localised in that region, has $\delta p = 0$ and $Q < 0$, implying instability. (By choosing $\boldsymbol{\xi}$ in the correct direction and tuning $\nabla \cdot \boldsymbol{\xi}$ appropriately, it is possible to arrange for $\delta p = -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p$ to vanish.)
Using $l = r^2 \Omega$ (specific angular momentum) and $s = c_p(\gamma^{-1} \ln p - \ln \rho) + \text{const.}$ (specific entropy of a perfect gas), we have

$$\begin{aligned}
A &= \frac{1}{r^3} \frac{\partial l^2}{\partial r} - \frac{g_r}{c_p} \frac{\partial s}{\partial r} \\
B &= \frac{1}{r^3} \frac{\partial l^2}{\partial z} - \frac{g_r}{c_p} \frac{\partial s}{\partial z} = -\frac{g_z}{c_p} \frac{\partial s}{\partial r} \\
D &= -\frac{g_z}{c_p} \frac{\partial s}{\partial z}
\end{aligned}$$

so the two conditions (27) become

$$\frac{1}{r^3}\frac{\partial l^2}{\partial r} - \frac{1}{c_p}\boldsymbol{g}\cdot\nabla s > 0$$

and

$$-g_z\left(\frac{\partial l^2}{\partial r}\frac{\partial s}{\partial z} - \frac{\partial l^2}{\partial z}\frac{\partial s}{\partial r}\right) > 0$$

These are the *Høiland stability criteria.* If the criteria are marginally satisfied, a further investigation may be required.

In the homentropic case $s = \text{const.}$ (a barotropic model) they reduce to the *Rayleigh criterion* for centrifugal stability,

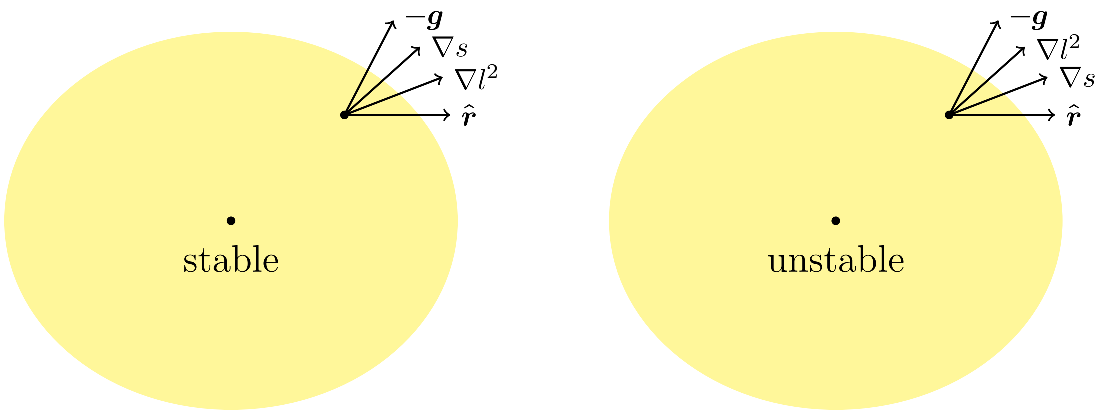
$$\frac{\mathrm{d}l^2}{\mathrm{d}r} > 0$$

where the derivative is now total since $l = l(r)$ in a barotropic model. This states that the specific angular momentum should increase with radius for stability.

The second Høiland criterion is equivalent to

$$(\hat{\boldsymbol{r}} \times (-\boldsymbol{g}))\cdot(\nabla l^2 \times \nabla s) > 0$$

In other words, the vector $\hat{\boldsymbol{r}} \times (-\boldsymbol{g})$ and $\nabla l^2 \times \nabla s$ should be parallel (rather than antiparallel). In a rotating star, for stability we require that the specific angular momentum should increase with r on each surface of constant entropy.



A. Useful Equation Sets

A.1. Ideal gas dynamics

The equations of *ideal gas dynamics* (defn) are

$$\begin{aligned}
\frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\nabla\cdot\boldsymbol{u} && \text{Mass conservation}\\
\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= -\rho\nabla\Phi - \nabla p && \text{Momentum conservation}\\
\nabla^2\Phi &= 4\pi G\rho && \text{Poisson's equation}\\
\frac{\mathrm{D}s}{\mathrm{D}t} &= 0 && \text{Thermal energy equation}\\
\frac{\mathrm{D}p}{\mathrm{D}t} &= -\gamma p\nabla\cdot\boldsymbol{u} && \text{Thermal energy equation (w/o radiation pressure)}
\end{aligned}$$

For a perfect gas, specific entropy may be expressed as

$$s = c_v \ln\frac{p}{\rho^\gamma} + \text{const.}$$

A.2. Ideal magnetohydrodynamics

The equations of *ideal MHD* are

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= 0 && \text{Mass conservation}\\
\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= -\rho\nabla\Phi - \nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} && \text{Momentum conservation}\\
\frac{\mathrm{D}s}{\mathrm{D}t} &= 0 && \text{Thermal energy equation}\\
\frac{\partial\boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) && \text{Ideal induction equation}\\
\nabla\cdot\boldsymbol{B} &= 0 && \text{Solenoidal condition}
\end{aligned}$$

supplemented by an equation of state, Poisson's equation, etc., as required. Equations can be combined to form the *total energy equation*:

$$\frac{\partial}{\partial t}\left[\rho(\frac{1}{2}\boldsymbol{u}^2 + \Phi + \rho) + \frac{\boldsymbol{B}^2}{2\mu_0}\right] + \nabla\cdot\left[\rho\boldsymbol{u}(\frac{1}{2}\boldsymbol{u}^2 + \Phi + h) + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0$$

where $h = e + p/\rho$ is the *specific enthalpy.*

A.3. Perturbation equations

The linearised equation of motion for a displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ from a general static equilibrium is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\nabla\delta\Phi - \delta\rho\nabla\Phi - \nabla\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\delta\boldsymbol{B})$$

where

$$\begin{aligned}
\delta\rho &= -\boldsymbol{\xi}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{\xi}\\
\delta p &= -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi}\\
\delta\boldsymbol{B} &= \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})\\
&= \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B}
\end{aligned}$$

and the total pressure perturbation is

$$\begin{aligned}\delta\Pi &= \delta p + \frac{1}{\mu_0}\boldsymbol{B}\cdot\delta\boldsymbol{B} \\ &= -\boldsymbol{\xi}\cdot\nabla\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi})\end{aligned}$$

The gravitational potential perturbation satisfies

$$\nabla^2\delta\Phi = 4\pi G\delta\rho$$