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LECTURE NOTES

Astrophysical fluid dynamics

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These lecture notes and example problems are based on a course given at the University of Cambridge in Part III of the Mathematical Tripos. Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma. The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws are explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, are discussed. The spherical blast wave resulting from a supernova, and involving a strong shock, is a classic problem that can be solved analytically. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

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1. Introduction

1.1. *Areas of application*

Astrophysical fluid dynamics (AFD) is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets and also the interstellar medium, the intergalactic medium and cosmology itself. A fluid description is not applicable (i) in regions that are solidified, such as the rocky or icy cores of giant planets (under certain conditions) and the crusts of neutron stars, and (ii) in very tenuous regions where the medium is not sufficiently collisional (see § 2.9.3).

Important areas of application include:

(i) Instabilities in astrophysical fluids
(ii) Convection
(iii) Differential rotation and meridional flows in stars
(iv) Stellar oscillations driven by convection, instabilities or tidal forcing
(v) Astrophysical dynamos
(vi) Magnetospheres of stars, planets and black holes
(vii) Interacting binary stars and Roche-lobe overflow
(viii) Tidal disruption and stellar collisions
(ix) Supernovae
(x) Planetary nebulae
(xi) Jets and winds from stars and discs
(xii) Star formation and the physics of the interstellar medium

(xiii) Astrophysical discs, including protoplanetary discs, accretion discs in interacting binary stars and galactic nuclei, planetary rings, etc.
(xiv) Other accretion flows (Bondi, Bondi–Hoyle, etc.)
(xv) Processes related to planet formation and planet–disc interactions
(xvi) Planetary atmospheric dynamics
(xvii) Galaxy clusters and the physics of the intergalactic medium
(xviii) Cosmology and structure formation

1.2. *Theoretical varieties*

There are various flavours of AFD in common use. The basic model involves a compressible, inviscid fluid and is Newtonian (i.e. non-relativistic). This is known as hydrodynamics (HD) or gas dynamics (to distinguish it from incompressible hydrodynamics). The thermal physics of the fluid may be treated in different ways, either by assuming it to be isothermal or adiabatic, or by including radiative processes in varying levels of detail.

Magnetohydrodynamics (MHD) generalizes this theory by including the dynamical effects of a magnetic field. Often the fluid is assumed to be perfectly electrically conducting (ideal MHD). One can also include the dynamical (rather than thermal) effects of radiation, resulting in a theory of radiation (magneto)hydrodynamics. Dissipative effects such as viscosity and resistivity can be included. All these theories can also be formulated in a relativistic framework.

(i) HD: hydrodynamics
(ii) MHD: magnetohydrodynamics
(iii) RHD: radiation hydrodynamics
(iv) RMHD: radiation magnetohydrodynamics
(v) GRHD: general relativistic hydrodynamics
(vi) GRRMHD: general relativistic radiation magnetohydrodynamics, etc.

1.3. *Characteristic features*

AFD typically differs from 'laboratory' or 'engineering' fluid dynamics in the relative importance of certain effects. Compressibility and gravitation are often important in AFD, while magnetic fields, radiation forces and relativistic phenomena are important in some applications. Effects that are often unimportant in AFD include viscosity, surface tension and the presence of solid boundaries.

2. Ideal gas dynamics

2.1. *Fluid variables*

A fluid is characterized by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The specific volume (volume per unit mass) is $v = 1/\rho$.

We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionization and recombination.

2.2. *Eulerian and Lagrangian viewpoints*

In the Eulerian viewpoint we consider how fluid properties vary in time at a point that is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian

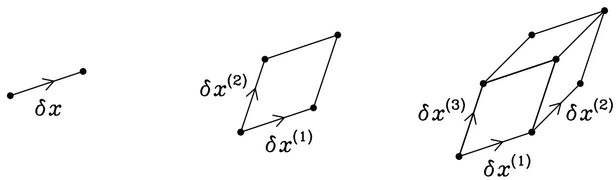

FIGURE 1. Examples of material line, surface and volume elements.

time-derivative is simply the partial differential operator

$$\frac{\partial}{\partial t}. \tag{2.1}$$

In the Lagrangian viewpoint we consider how fluid properties vary in time at a point that moves with the fluid at velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The Lagrangian time derivative is then

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla. \tag{2.2}$$

2.3. *Material points and structures*

A material point is an idealized fluid element, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of the fluid. (Note that the true particles of which the fluid is composed have in addition a random thermal motion.) Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are distorted by it.

An infinitesimal material line element $\delta\boldsymbol{x}$ (figure 1) evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u}. \tag{2.3}$$

It changes its length and/or orientation in the presence of a velocity gradient. (Since $\delta\boldsymbol{x}$ is only a time-dependent vector rather than a vector field, the time derivative could be written as an ordinary derivative $\mathrm{d}/\mathrm{d}t$. The notation $\mathrm{D}/\mathrm{D}t$ is used here to remind us that $\delta\boldsymbol{x}$ is a material structure that moves with the fluid.)

Infinitesimal material surface and volume elements can be defined from two or three material line elements according to the vector product and the triple scalar product (figure 1)

$$\delta\boldsymbol{S} = \delta\boldsymbol{x}^{(1)}\times\delta\boldsymbol{x}^{(2)}, \quad \delta V = \delta\boldsymbol{x}^{(1)}\cdot\delta\boldsymbol{x}^{(2)}\times\delta\boldsymbol{x}^{(3)}. \tag{2.4a,b}$$

They therefore evolve according to

$$\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} = (\nabla\cdot\boldsymbol{u})\,\delta\boldsymbol{S} - (\nabla\boldsymbol{u})\cdot\delta\boldsymbol{S}, \quad \frac{\mathrm{D}\delta V}{\mathrm{D}t} = (\nabla\cdot\boldsymbol{u})\,\delta V, \tag{2.5a,b}$$

as follows from the above equations (exercise). The second result is easier to understand: the volume element increases when the flow is divergent. These equations are most easily derived using Cartesian tensor notation. In this notation the equation for $\delta\boldsymbol{S}$ reads

$$\frac{\mathrm{D}\delta S_i}{\mathrm{D}t} = \frac{\partial u_j}{\partial x_j}\,\delta S_i - \frac{\partial u_j}{\partial x_i}\,\delta S_j. \tag{2.6}$$

2.4. *Equation of mass conservation*

The equation of mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2.7}$$

has the typical form of a conservation law: ρ is the mass density (mass per unit volume) and $\rho \boldsymbol{u}$ is the mass flux density (mass flux per unit area). An alternative form of the same equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}. \tag{2.8}$$

If $\delta m = \rho \delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0. \tag{2.9}$$

2.5. *Equation of motion*

The equation of motion,

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p, \tag{2.10}$$

derives from Newton's second law per unit volume with gravitational and pressure forces. $\Phi(\boldsymbol{x}, t)$ is the gravitational potential and $\boldsymbol{g} = -\nabla \Phi$ is the gravitational field. The force due to pressure acting on a volume V with bounding surface S is

$$-\int_S p \, \mathrm{d}\boldsymbol{S} = \int_V (-\nabla p) \, \mathrm{d}V. \tag{2.11}$$

Viscous forces are neglected in ideal gas dynamics.

2.6. *Poisson's equation*

The gravitational potential is related to the mass density by Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \rho, \tag{2.12}$$

where G is Newton's constant. The solution

$$\Phi(\boldsymbol{x}, t) = \Phi_{int} + \Phi_{ext} = -G \int_V \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, \mathrm{d}^3\boldsymbol{x}' - G \int_{\hat{V}} \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, \mathrm{d}^3\boldsymbol{x}' \tag{2.13}$$

generally involves contributions from both the fluid region V under consideration and the exterior region $\hat{V}$.

A non-self-gravitating fluid is one of negligible mass for which Φ_{int} can be neglected. More generally, the Cowling approximation[1] consists of treating Φ as being specified in advance, so that Poisson's equation is not coupled to the other equations.

[1] Thomas George Cowling (1906–1990), British.

2.7. *Thermal energy equation*

In the absence of non-adiabatic heating (e.g. by viscous dissipation or nuclear reactions) and cooling (e.g. by radiation or conduction),

$$\frac{\mathrm{D}s}{\mathrm{D}t}=0, \tag{2.14}$$

where s is the specific entropy (entropy per unit mass). Fluid elements undergo reversible thermodynamic changes and preserve their entropy.

This condition is violated in shocks (see § 6.3).

The thermal variables (T, s) can be related to the dynamical variables (p, ρ) via an equation of state and standard thermodynamic identities. The most important case is that of an ideal gas together with black-body radiation,

$$p=p_g+p_r=\frac{k\rho T}{\mu m_H}+\frac{4\sigma T^4}{3c}, \tag{2.15}$$

where k is Boltzmann's constant, m_H is the mass of the hydrogen atom, σ is Stefan's constant and c is the speed of light. μ is the mean molecular weight (the average mass of the particles in units of m_H), equal to 2.0 for molecular hydrogen, 1.0 for atomic hydrogen, 0.5 for fully ionized hydrogen and approximately 0.6 for ionized matter of typical cosmic abundances. Radiation pressure is usually negligible except in the centres of high-mass stars and in the immediate environments of neutron stars and black holes. The pressure of an ideal gas is often written in the form $\mathcal{R}\rho T/\mu$, where $\mathcal{R}=k/m_H$ is a version of the universal gas constant.

We define the first adiabatic exponent

$$\Gamma_1=\left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s, \tag{2.16}$$

which is related to the ratio of specific heat capacities

$$\gamma=\frac{c_p}{c_v}=\frac{T\left(\dfrac{\partial s}{\partial T}\right)_p}{T\left(\dfrac{\partial s}{\partial T}\right)_v} \tag{2.17}$$

by (exercise)

$$\Gamma_1=\chi_\rho\gamma, \tag{2.18}$$

where

$$\chi_\rho=\left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T \tag{2.19}$$

can be found from the equation of state. We can then rewrite the thermal energy equation as

$$\frac{\mathrm{D}p}{\mathrm{D}t}=\frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}=-\Gamma_1 p\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}. \tag{2.20}$$

For an ideal gas with negligible radiation pressure, $\chi_\rho=1$ and so $\Gamma_1=\gamma$. Adopting this very common assumption, we write

$$\frac{\mathrm{D}p}{\mathrm{D}t}=-\gamma p\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u}. \tag{2.21}$$

2.8. *Simplified models*

A perfect gas may be defined as an ideal gas with constant c_v, c_p, γ and μ. Equipartition of energy for a classical gas with n degrees of freedom per particle gives $\gamma = 1 + 2/n$. For a classical monatomic gas with $n = 3$ translational degrees of freedom, $\gamma = 5/3$. This is relevant for fully ionized matter. For a classical diatomic gas with two additional rotational degrees of freedom, $n = 5$ and $\gamma = 7/5$. This is relevant for molecular hydrogen. In reality Γ_1 is variable when the gas undergoes ionization or when the gas and radiation pressures are comparable. The specific internal energy (or thermal energy) of a perfect gas is

$$e = \frac{p}{(\gamma - 1)\rho} \left[= \frac{n}{\mu m_H} \frac{1}{2} kT \right]. \tag{2.22}$$

(Note that each particle has an internal energy of $kT/2$ per degree of freedom, and the number of particles per unit mass is $1/\mu m_H$.)

A barotropic fluid is an idealized situation in which the relation $p(\rho)$ is known in advance. We can then dispense with the thermal energy equation. e.g. if the gas is strictly isothermal and perfect, then $p = c_s^2 \rho$ with $c_s = \text{const.}$ being the isothermal sound speed. Alternatively, if the gas is strictly homentropic and perfect, then $p = K\rho^\gamma$ with $K = \text{const.}$

An incompressible fluid is an idealized situation in which $\mathrm{D}\rho/\mathrm{D}t = 0$, implying $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. This can be achieved formally by taking the limit $\gamma \to \infty$. The approximation of incompressibility eliminates acoustic phenomena from the dynamics.

The ideal gas law itself is not valid at very high densities or where quantum degeneracy is important.

2.9. *Microphysical basis*

It is useful to understand the way in which the fluid dynamical equations are derived from microphysical considerations. The simplest model involves identical neutral particles of mass m and negligible size with no internal degrees of freedom.

2.9.1. *The Boltzmann equation*

Between collisions, particles follow Hamiltonian trajectories in their six-dimensional $(\boldsymbol{x}, \boldsymbol{v})$ phase space:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = a_i = -\frac{\partial \Phi}{\partial x_i}. \tag{2.23a,b}$$

The distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ specifies the number density of particles in phase space. The velocity moments of f define the number density $n(\boldsymbol{x}, t)$ in real space, the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and the velocity dispersion $c(\boldsymbol{x}, t)$ according to

$$\int f \,\mathrm{d}^3\boldsymbol{v} = n, \qquad \int \boldsymbol{v} f \,\mathrm{d}^3\boldsymbol{v} = n\boldsymbol{u}, \qquad \int |\boldsymbol{v} - \boldsymbol{u}|^2 f \,\mathrm{d}^3\boldsymbol{v} = 3nc^2. \tag{2.24a–c}$$

Equivalently,

$$\int v^2 f \,\mathrm{d}^3\boldsymbol{v} = n(u^2 + 3c^2). \tag{2.25}$$

The relation between velocity dispersion and temperature is $kT = mc^2$.

In the absence of collisions, f is conserved following the Hamiltonian flow in phase space. This is because particles are conserved and the flow in phase space is incompressible (Liouville's theorem). More generally, f evolves according to Boltzmann's equation,

$$\frac{\partial f}{\partial t}+v_j\frac{\partial f}{\partial x_j}+a_j\frac{\partial f}{\partial v_j}=\left(\frac{\partial f}{\partial t}\right)_c. \tag{2.26}$$

The collision term on the right-hand side is a complicated integral operator but has three simple properties corresponding to the conservation of mass, momentum and energy in collisions:

$$\int m\left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v}=0,\quad \int m\boldsymbol{v}\left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v}=\boldsymbol{0},\quad \int \frac{1}{2}mv^2\left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3\boldsymbol{v}=0. \tag{2.27a–c}$$

The collision term is local in $\boldsymbol{x}$ (not even involving derivatives) although it does involve integrals over $\boldsymbol{v}$. The equation $(\partial f/\partial t)_c=0$ has the general solution

$$f=f_M=(2\pi c^2)^{-3/2}n\,\exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{u}|^2}{2c^2}\right), \tag{2.28}$$

with parameters n, $\boldsymbol{u}$ and c that may depend on $\boldsymbol{x}$. This is the Maxwellian distribution.

2.9.2. *Derivation of fluid equations*

A crude but illuminating model of the collision operator is the Bhatnagar–Gross–Krook (BGK) approximation

$$\left(\frac{\partial f}{\partial t}\right)_c\approx-\frac{1}{\tau}(f-f_M), \tag{2.29}$$

where f_M is a Maxwellian distribution with the same n, $\boldsymbol{u}$ and c as f and τ is the relaxation time. This can be identified approximately with the mean free flight time of particles between collisions. In other words the collisions attempt to restore a Maxwellian distribution on a characteristic time scale τ. They do this by randomizing the particle velocities in a way consistent with the conservation of momentum and energy.

If the characteristic time scale of the fluid flow is much greater than τ, then the collision term dominates the Boltzmann equation and f must be very close to f_M. This is the hydrodynamic limit.

The velocity moments of f_M can be determined from standard Gaussian integrals, in particular (exercise)

$$\int f_M\,\mathrm{d}^3\boldsymbol{v}=n,\quad \int v_i f_M\,\mathrm{d}^3\boldsymbol{v}=nu_i, \tag{2.30a,b}$$

$$\int v_iv_jf_M\,\mathrm{d}^3\boldsymbol{v}=n(u_iu_j+c^2\delta_{ij}),\quad \int v^2v_if_M\,\mathrm{d}^3\boldsymbol{v}=n(u^2+5c^2)u_i. \tag{2.31a,b}$$

We obtain equations for mass, momentum and energy by taking moments of the Boltzmann equation weighted by $(m, mv_i, mv^2/2)$. In each case the collision term integrates to zero because of its conservative properties, and the $\partial/\partial v_j$ term can be integrated by parts. We replace f with f_M when evaluating the left-hand sides and note that $mn=\rho$:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)=0, \tag{2.32}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left[\rho(u_i u_j + c^2\delta_{ij})\right] - \rho a_i = 0, \tag{2.33}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{3}{2}\rho c^2\right) + \frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho u^2 + \frac{5}{2}\rho c^2\right)u_i\right] - \rho u_i a_i = 0. \tag{2.34}$$

These are equivalent to the equations of ideal gas dynamics in conservative form (see § 4) for a monatomic ideal gas ($\gamma = 5/3$). The specific internal energy is $e = (3/2)c^2 = (3/2)kT/m$.

This approach can be generalized to deal with molecules with internal degrees of freedom and also to plasmas or partially ionized gases where there are various species of particle with different charges and masses. The equations of MHD can be derived by including the electromagnetic forces in Boltzmann's equation.

2.9.3. *Validity of a fluid approach*

The essential idea here is that deviations from the Maxwellian distribution are small when collisions are frequent compared to the characteristic time scale of the flow. In higher-order approximations these deviations can be estimated, leading to the equations of dissipative gas dynamics including transport effects (viscosity and heat conduction).

The fluid approach breaks down if the mean flight time τ is not much less than the characteristic time scale of the flow, or if the mean free path $\lambda \approx c\tau$ between collisions is not much less than the characteristic length scale of the flow. λ can be very long (measured in astronomical units or parsecs) in very tenuous gases such as the interstellar medium, but may still be smaller than the size of the system.

Some typical order-of-magnitude estimates:

Solar-type star: centre $\rho \sim 10^2$ g cm^{-3}, $T \sim 10^7$ K; photosphere $\rho \sim 10^{-7}$ g cm^{-3}, $T \sim 10^4$ K; corona $\rho \sim 10^{-15}$ g cm^{-3}, $T \sim 10^6$ K.

Interstellar medium: molecular clouds $n \sim 10^3$ cm^{-3}, $T \sim 10$ K; cold medium (neutral) $n \sim 10 - 100$ cm^{-3}, $T \sim 10^2$ K; warm medium (neutral/ionized) $n \sim 0.1 - 1$ cm^{-3}, $T \sim 10^4$ K; hot medium (ionized) $n \sim 10^{-3} - 10^{-2}$ cm^{-3}, $T \sim 10^6$ K.

The Coulomb cross-section for 'collisions' (i.e. large-angle scatterings) between charged particles (electrons or ions) is $\sigma \approx 1 \times 10^{-4}(T/\mathrm{K})^{-2}$ cm^2. The mean free path is $\lambda = 1/(n\sigma)$.

Related examples (see appendix A): A.1–A.4.

3. Ideal magnetohydrodynamics

3.1. *Elementary derivation of the MHD equations*

Magnetohydrodynamics (MHD) is the dynamics of an electrically conducting fluid (a fully or partially ionized gas or a liquid metal) containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

3.1.1. *Galilean electromagnetism*

The equations of Newtonian gas dynamics are invariant under the Galilean transformation to a frame of reference moving with uniform velocity $\boldsymbol{v}$,

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t, \quad t' = t. \tag{3.1a,b}$$

Under this change of frame, the fluid velocity transforms according to

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}, \tag{3.2}$$

while scalar variables such as p, ρ and Φ are invariant. The Lagrangian time derivative $\mathrm{D}/\mathrm{D}t$ is also invariant, because the partial derivatives transform according to

$$\nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{3.3a,b}$$

In Maxwell's electromagnetic theory the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are governed by the equations

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \quad \nabla \cdot \boldsymbol{E} = \frac{\rho_e}{\epsilon_0}, \tag{3.4a–d}$$

where μ_0 and ϵ_0 are the vacuum permeability and permittivity, $\boldsymbol{J}$ is the electric current density and ρ_e is the electric charge density. (In these notes we use rationalized (e.g. SI) units for electromagnetism. In astrophysics it is also common to use Gaussian units, which are discussed in appendix B.)

It is well known that Maxwell's equations are invariant under the Lorentz transformation of special relativity, with $c = (\mu_0 \epsilon_0)^{-1/2}$ being the speed of light. These equations cannot be consistently coupled with those of Newtonian gas dynamics, which are invariant under the Galilean transformation. To derive a consistent Newtonian theory of MHD, valid for situations in which the fluid motions are slow compared to the speed of light, we must use Maxwell's equations without the displacement current $\epsilon_0\ \partial \boldsymbol{E}/\partial t$,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \quad \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{3.5a–c}$$

(We will not require the fourth Maxwell equation, involving $\nabla \cdot \boldsymbol{E}$, because the charge density will be found to be unimportant.) It is easily verified (exercise) that these pre-Maxwell equations[2] are indeed invariant under the Galilean transformation, provided that the fields transform according to

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \quad \boldsymbol{B}' = \boldsymbol{B}, \quad \boldsymbol{J}' = \boldsymbol{J}. \tag{3.6a–c}$$

These relations correspond to the limit of the Lorentz transformation for electromagnetic fields[3] when $|\boldsymbol{v}| \ll c$ and $|\boldsymbol{E}| \ll c|\boldsymbol{B}|$.

Under the pre-Maxwell theory, the equation of charge conservation takes the simplified form $\nabla \cdot \boldsymbol{J} = 0$; this is analogous to the use of $\nabla \cdot \boldsymbol{u} = 0$ as the equation of mass conservation in the incompressible (highly subsonic) limit of gas dynamics. The equation of energy conservation takes the simplified form

$$\frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} \right) = 0, \tag{3.7}$$

in which the energy density, $B^2/2\mu_0$, is purely magnetic (because $|\boldsymbol{E}| \ll c|\boldsymbol{B}|$), while the energy flux density has the usual form of the Poynting vector $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$. We will verify the self-consistency of the approximations made in Newtonian MHD in § 3.1.4.

[2]It was by introducing the displacement current that Maxwell identified electromagnetic waves, so it is appropriate that a highly subluminal approximation should neglect this term.

[3]This was called the magnetic limit of Galilean electromagnetism by Le Bellac & Lévy-Leblond (1973).

3.1.2. *Induction equation*

In the ideal MHD approximation we regard the fluid as a perfect electrical conductor. The electric field in the rest frame of the fluid therefore vanishes, implying that

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} \tag{3.8}$$

in a frame in which the fluid velocity is $\boldsymbol{u}(\boldsymbol{x}, t)$. This condition can be regarded as the limit of a constitutive relation such as Ohm's law, in which the effects of resistivity (i.e. finite conductivity) are neglected.

From Maxwell's equations, we then obtain the ideal induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}). \tag{3.9}$$

This is an evolutionary equation for $\boldsymbol{B}$ alone, $\boldsymbol{E}$ and $\boldsymbol{J}$ having been eliminated. The divergence of the induction equation,

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0, \tag{3.10}$$

ensures that the solenoidal character of $\boldsymbol{B}$ is preserved.

3.1.3. *The Lorentz force*

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a bulk Lorentz force

$$\boldsymbol{F}_m = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \tag{3.11}$$

per unit volume. This can be understood as the sum of the Lorentz forces on individual particles of charge q and velocity $\boldsymbol{v}$,

$$\sum q\boldsymbol{v} \times \boldsymbol{B} = \left(\sum q\boldsymbol{v}\right) \times \boldsymbol{B}. \tag{3.12}$$

(The electrostatic force can be shown to be negligible in the limit relevant to Newtonian MHD; see § 3.1.4.)

In Cartesian coordinates

$$\begin{aligned}
(\mu_0 \boldsymbol{F}_m)_i &= \epsilon_{ijk}\left(\epsilon_{jlm}\frac{\partial B_m}{\partial x_l}\right) B_k \\
&= \left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right) B_k \\
&= B_k \frac{\partial B_i}{\partial x_k} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right).
\end{aligned} \tag{3.13}$$

Thus

$$\boldsymbol{F}_m = \frac{1}{\mu_0}\boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla\left(\frac{B^2}{2\mu_0}\right). \tag{3.14}$$

The first term can be interpreted as a curvature force due to a magnetic tension $T_m = B^2/\mu_0$ per unit area in the field lines; if the field is of constant magnitude then this term is equal to T_m times the curvature of the field lines, and is directed towards the centre of curvature. The second term is the gradient of an isotropic magnetic pressure

$$p_m = \frac{B^2}{2\mu_0}, \tag{3.15}$$

which is also equal to the energy density of the magnetic field.

The magnetic tension gives rise to Alfvén waves[4] (see later), which travel parallel to the magnetic field with characteristic speed

$$v_a = \left(\frac{T_m}{\rho}\right)^{1/2} = \frac{B}{(\mu_0\rho)^{1/2}}, \tag{3.16}$$

the Alfvén speed. This is often considered as a vector Alfvén velocity,

$$\boldsymbol{v}_a = \frac{\boldsymbol{B}}{(\mu_0\rho)^{1/2}}. \tag{3.17}$$

The magnetic pressure also affects the propagation of sound waves, which become magnetoacoustic waves (or magnetosonic waves; see later).

The combination

$$\varPi = p + \frac{B^2}{2\mu_0} \tag{3.18}$$

is often referred to as the total pressure, while the ratio

$$\beta = \frac{p}{B^2/2\mu_0} \tag{3.19}$$

is known as the plasma beta.

3.1.4. *Self-consistency of approximations*

Three effects neglected in a Newtonian theory of MHD are (i) the displacement current in Maxwell's equations (compared to the electric current), (ii) the bulk electrostatic force on the fluid (compared to the magnetic Lorentz force) and (iii) the electrostatic energy (compared to the magnetic energy). We can verify the self-consistency of these approximations by using order-of-magnitude estimates or scaling relations. If the fluid flow has a characteristic length scale L, time scale T, velocity $U \sim L/T$ and magnetic field B, then the electric field can be estimated from (3.8) as $E \sim UB$. The electric current density and charge density can be estimated from Maxwell's equations as $J \sim \mu_0^{-1}B/L$ and $\rho_e \sim \epsilon_0 E/L$. Hence the ratios of the three neglected effects to the terms that are retained in Newtonian MHD can be estimated as follows:

$$\frac{\epsilon_0|\partial\boldsymbol{E}/\partial t|}{|\boldsymbol{J}|} \sim \frac{\epsilon_0 UB/T}{\mu_0^{-1}B/L} \sim \frac{U^2}{c^2}, \tag{3.20}$$

$$\frac{|\rho_e\boldsymbol{E}|}{|\boldsymbol{J}\times\boldsymbol{B}|} \sim \frac{\epsilon_0 E^2/L}{\mu_0^{-1}B^2/L} \sim \frac{U^2}{c^2}, \tag{3.21}$$

$$\frac{\epsilon_0|\boldsymbol{E}|^2/2}{|\boldsymbol{B}|^2/2\mu_0} \sim \frac{U^2}{c^2}. \tag{3.22}$$

Therefore Newtonian MHD corresponds to a consistent approximation of relativistic MHD for highly subluminal flows that is correct to the leading order in the small parameter U^2/c^2.

[4]Hannes Olof Gösta Alfvén (1908–1995), Swedish. Nobel Prize in Physics (1970) 'for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics'.

3.1.5. *Summary of the MHD equations*

The full set of ideal MHD equations is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{3.23}$$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{3.24}$$

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{3.25}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{3.26}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3.27}$$

together with the equation of state, Poisson's equation, etc., as required. Most of these equations can be written in at least one other way that may be useful in different circumstances.

These equations display the essential nonlinearity of MHD. When the velocity field is prescribed, an artifice known as the kinematic approximation, the induction equation is a relatively straightforward linear evolutionary equation for the magnetic field. However, a sufficiently strong magnetic field will modify the velocity field through its dynamical effect, the Lorentz force. This nonlinear coupling leads to a rich variety of behaviour. Of course, the purely hydrodynamic nonlinearity of the $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ term, which is responsible for much of the complexity of fluid dynamics, is still present.

3.2. *Physical interpretation of MHD*

There are two aspects to MHD: the advection of $\boldsymbol{B}$ by $\boldsymbol{u}$ (induction equation) and the dynamical back-reaction of $\boldsymbol{B}$ on $\boldsymbol{u}$ (Lorentz force).

3.2.1. *Kinematics of the magnetic field*

The ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \tag{3.28}$$

has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material curves. This is sometimes known as Alfvén's theorem.

One way to show this result is to use the identity

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) - \boldsymbol{u} \cdot \nabla \boldsymbol{B} + \boldsymbol{u}(\nabla \cdot \boldsymbol{B}) \tag{3.29}$$

to write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u}), \tag{3.30}$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}, \tag{3.31}$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right)=\left(\frac{\boldsymbol{B}}{\rho}\right)\boldsymbol{\cdot}\nabla\boldsymbol{u}. \tag{3.32}$$

This is exactly the same equation satisfied by a material line element $\delta\boldsymbol{x}$ (2.3). Therefore a magnetic field line (an integral curve of $\boldsymbol{B}/\rho$) is advected and distorted by the fluid in the same way as a material curve.

A complementary property is that the magnetic flux $\delta\Phi=\boldsymbol{B}\boldsymbol{\cdot}\delta\boldsymbol{S}$ through a material surface element is conserved:

$$\begin{aligned}\frac{\mathrm{D}\delta\Phi}{\mathrm{D}t} &= \frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t}\boldsymbol{\cdot}\delta\boldsymbol{S}+\boldsymbol{B}\boldsymbol{\cdot}\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t}\\ &= \left(B_j\frac{\partial u_i}{\partial x_j}-B_i\frac{\partial u_j}{\partial x_j}\right)\delta S_i+B_i\left(\frac{\partial u_j}{\partial x_j}\delta S_i-\frac{\partial u_j}{\partial x_i}\delta S_j\right)\\ &= 0.\end{aligned} \tag{3.33}$$

By extension, we have conservation of the magnetic flux passing through any material surface.

Precisely the same equation as the ideal induction equation,

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}), \tag{3.34}$$

is satisfied by the vorticity $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ in homentropic or barotropic ideal fluid dynamics in the absence of a magnetic field, in which case the vortex lines are 'frozen in' to the fluid (see Example A.2). The conserved quantity that is analogous to the magnetic flux through a material surface is the flux of vorticity through that surface, which, by Stokes's theorem, is equivalent to the circulation $\oint\boldsymbol{u}\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x}$ around the bounding curve. However, the fact that $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by the curl operation, whereas in MHD $\boldsymbol{B}$ and $\boldsymbol{u}$ are indirectly related through the equation of motion and the Lorentz force, means that the analogy between vorticity dynamics and MHD is limited in scope.

Related examples: A.5, A.6.

3.2.2. *The Lorentz force*

The Lorentz force per unit volume,

$$\boldsymbol{F}_m=\frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{\cdot}\nabla\boldsymbol{B}-\nabla\left(\frac{B^2}{2\mu_0}\right), \tag{3.35}$$

can also be written as the divergence of the Maxwell stress tensor:

$$\boldsymbol{F}_m=\nabla\boldsymbol{\cdot}\boldsymbol{\mathsf{M}},\quad \boldsymbol{\mathsf{M}}=\frac{1}{\mu_0}\left(\boldsymbol{B}\boldsymbol{B}-\frac{B^2}{2}\boldsymbol{\mathsf{I}}\right), \tag{3.36}$$

where $\boldsymbol{\mathsf{I}}$ is the identity tensor. (The electric part of the electromagnetic stress tensor is negligible in the limit relevant for Newtonian MHD, for the same reason that the electrostatic energy is negligible.) In Cartesian coordinates

$$(\boldsymbol{F}_m)_i=\frac{\partial M_{ji}}{\partial x_j},\quad M_{ij}=\frac{1}{\mu_0}\left(B_iB_j-\frac{B^2}{2}\delta_{ij}\right). \tag{3.37a,b}$$

 https://doi.org/10.1017/S0022377816000489

If the magnetic field is locally aligned with the x-axis, then

$$\boldsymbol{M} = \begin{bmatrix} T_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} p_m & 0 & 0 \\ 0 & p_m & 0 \\ 0 & 0 & p_m \end{bmatrix}, \tag{3.38}$$

showing the magnetic tension and pressure.

Combining the ideas of magnetic tension and a frozen-in field leads to the picture of field lines as elastic strings embedded in the fluid. Indeed there is a close analogy between MHD and the dynamics of dilute solutions of long-chain polymer molecules. The magnetic field imparts elasticity to the fluid.

3.2.3. *Differential rotation and torsional Alfvén waves*

We first consider the kinematic behaviour of a magnetic field in the presence of a prescribed velocity field involving differential rotation. In cylindrical polar coordinates (r, ϕ, z), let

$$\boldsymbol{u} = r\Omega(r, z)\,\boldsymbol{e}_\phi. \tag{3.39}$$

Consider an axisymmetric magnetic field, which we separate into poloidal (meridional: r and z) and toroidal (azimuthal: ϕ) parts:

$$\boldsymbol{B} = \boldsymbol{B}_p(r, z, t) + B_\phi(r, z, t)\,\boldsymbol{e}_\phi. \tag{3.40}$$

The ideal induction equation reduces to (exercise)

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0, \qquad \frac{\partial B_\phi}{\partial t} = r\boldsymbol{B}_p \cdot \nabla\Omega. \tag{3.41}$$

Differential rotation winds the poloidal field to generate a toroidal field. To obtain a steady state without winding, we require the angular velocity to be constant along each magnetic field line:

$$\boldsymbol{B}_p \cdot \nabla\Omega = 0, \tag{3.42}$$

a result known as Ferraro's law of isorotation[5].

There is an energetic cost to winding the field, as work is done against magnetic tension. In a dynamical situation a strong magnetic field tends to enforce isorotation along its length.

We now generalize the analysis to allow for axisymmetric torsional oscillations:

$$\boldsymbol{u} = r\Omega(r, z, t)\,\boldsymbol{e}_\phi. \tag{3.43}$$

The azimuthal component of the equation of motion is (exercise)

$$\rho r\frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0 r}\boldsymbol{B}_p \cdot \nabla(rB_\phi). \tag{3.44}$$

This combines with the induction equation to give

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\mu_0 \rho r^2}\boldsymbol{B}_p \cdot \nabla(r^2\boldsymbol{B}_p \cdot \nabla\Omega). \tag{3.45}$$

This equation describes torsional Alfvén waves. For example, if $\boldsymbol{B}_p = B_z\boldsymbol{e}_z$ is vertical and uniform, then

$$\frac{\partial^2 \Omega}{\partial t^2} = v_a^2\frac{\partial^2 \Omega}{\partial z^2}. \tag{3.46}$$

This is not strictly an exact nonlinear analysis because we have neglected the force balance (and indeed motion) in the meridional plane.

[5]Vincenzo Ferraro (1902–1974), British.

3.2.4. *Force-free fields*

In regions of low density, such as the solar corona, the magnetic field may be dynamically dominant over the effects of inertia, gravity and gas pressure. Under these circumstances we have (approximately) a force-free magnetic field such that

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{0}. \tag{3.47}$$

Vector fields $\boldsymbol{B}$ satisfying this equation are known in a wider mathematical context as Beltrami fields. Since $\nabla \times \boldsymbol{B}$ must be parallel to $\boldsymbol{B}$, we may write

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}, \tag{3.48}$$

for some scalar field $\lambda(\boldsymbol{x})$. The divergence of this equation is

$$0 = \boldsymbol{B} \boldsymbol{\cdot} \nabla \lambda, \tag{3.49}$$

so that λ is constant along each magnetic field line. In the special case $\lambda = \text{const.}$, known as a linear force-free magnetic field, the curl of (3.48) results in the Helmholtz equation

$$-\nabla^2 \boldsymbol{B} = \lambda^2 \boldsymbol{B}, \tag{3.50}$$

which admits a wide variety of non-trivial solutions.

A subset of force-free magnetic fields consists of potential or current-free magnetic fields for which

$$\nabla \times \boldsymbol{B} = \boldsymbol{0}. \tag{3.51}$$

In a true vacuum, the magnetic field must be potential. However, only an extremely low density of charge carriers (i.e. electrons) is needed to make the force-free description more relevant.

An example of a force-free field in cylindrical polar coordinates (r, ϕ, z) is

$$\left.\begin{aligned} \boldsymbol{B} &= B_\phi(r)\,\boldsymbol{e}_\phi + B_z(r)\,\boldsymbol{e}_z, \\ \nabla \times \boldsymbol{B} &= -\frac{\mathrm{d}B_z}{\mathrm{d}r}\,\boldsymbol{e}_\phi + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\phi)\,\boldsymbol{e}_z. \end{aligned}\right\} \tag{3.52}$$

Now $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ implies

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B_z}{\mathrm{d}r}\right) = \lambda^2 B_z, \tag{3.53}$$

which is the z component of the Helmholtz equation. The solution regular at $r = 0$ is

$$B_z = B_0 \mathrm{J}_0(\lambda r), \quad B_\phi = B_0 \mathrm{J}_1(\lambda r), \tag{3.54a,b}$$

where J_n is the Bessel function of order n (figure 2). (Note that $\mathrm{J}_0(x)$ satisfies $(x\mathrm{J}_0')' + x\mathrm{J}_0 = 0$ and $\mathrm{J}_1(x) = -\mathrm{J}_0'(x)$.) The helical nature of this field is typical of force-free fields with $\lambda \neq 0$.

When applied to a infinite cylinder (e.g. as a simplified model of a magnetized astrophysical jet), the solution could be extended from the axis to the first zero of J_1 and then matched to a uniform external axial field B_z. In this case the net axial current is zero. Alternatively the solution could be extended from the axis to the first zero of J_0 and matched to an external azimuthal field $B_\phi \propto r^{-1}$ generated by the net axial current.

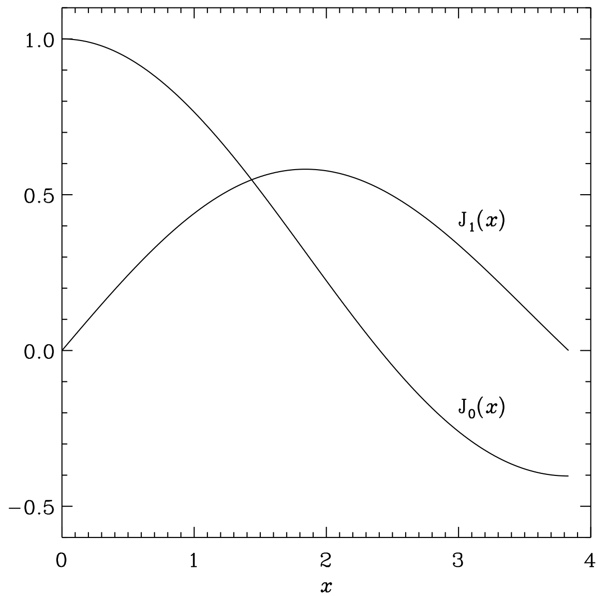

FIGURE 2. The Bessel functions $J_0(x)$ and $J_1(x)$ from the origin to the first zero of J_1.

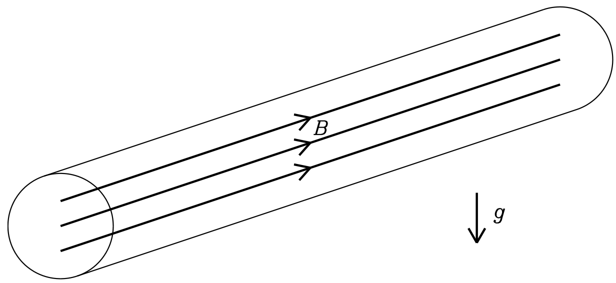

FIGURE 3. A buoyant magnetic flux tube.

3.2.5. *Magnetostatic equilibrium and magnetic buoyancy*

A magnetostatic equilibrium is a static solution ($\boldsymbol{u} = \boldsymbol{0}$) of the equation of motion, i.e. one satisfying

$$\boldsymbol{0} = -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{3.55}$$

together with $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$.

While it is possible to find solutions in which the forces balance in this way, inhomogeneities in the magnetic field typically result in a lack of equilibrium. A magnetic flux tube (figure 3) is an idealized situation in which the magnetic field is localized to the interior of a tube and vanishes outside. To balance the total pressure at the interface, the gas pressure must be lower inside. Unless the temperatures are different, the density is lower inside. In a gravitational field the tube therefore experiences an upward buoyancy force and tends to rise.

Related examples: A.7–A.9.

4. Conservation laws, symmetries and hyperbolic structure

4.1. *Introduction*

There are various ways in which a quantity can be said to be 'conserved' in fluid dynamics or MHD. If a quantity has a density (amount per unit volume) $q(\boldsymbol{x}, t)$ that satisfies an equation of the conservative form

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F} = 0, \tag{4.1}$$

then the vector field $\boldsymbol{F}(\boldsymbol{x}, t)$ can be identified as the flux density (flux per unit area) of the quantity. The rate of change of the total amount of the quantity in a time-independent volume V,

$$Q = \int_V q \, \mathrm{d}V, \tag{4.2}$$

is then equal to minus the flux of $\boldsymbol{F}$ through the bounding surface S:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_V (\nabla \cdot \boldsymbol{F}) \, \mathrm{d}V = -\int_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}. \tag{4.3}$$

If the boundary conditions on S are such that this flux vanishes, then Q is constant; otherwise, changes in Q can be accounted for by the flux of $\boldsymbol{F}$ through S. In this sense the quantity is said to be conserved. The prototype is mass, for which $q = \rho$ and $\boldsymbol{F} = \rho\boldsymbol{u}$.

A material invariant is a scalar field $f(\boldsymbol{x}, t)$ for which

$$\frac{\mathrm{D}f}{\mathrm{D}t} = 0, \tag{4.4}$$

which implies that f is constant for each fluid element, and is therefore conserved following the fluid motion. A simple example is the specific entropy in ideal fluid dynamics. When combined with mass conservation, this yields an equation in conservative form,

$$\frac{\partial}{\partial t}(\rho f) + \nabla \cdot (\rho f \boldsymbol{u}) = 0. \tag{4.5}$$

4.2. *Synthesis of the total energy equation*

Starting from the ideal MHD equations, we construct the total energy equation piece by piece.

Kinetic energy:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}u^2\right) = \rho\boldsymbol{u} \cdot \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\boldsymbol{u} \cdot \nabla\Phi - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mu_0}\boldsymbol{u} \cdot [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]. \tag{4.6}$$

Gravitational energy (assuming initially that the system is non-self-gravitating and that Φ is independent of t):

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho\boldsymbol{u} \cdot \nabla\Phi. \tag{4.7}$$

Internal (thermal) energy (using the fundamental thermodynamic identity $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}v$):

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \rho T \frac{\mathrm{D}s}{\mathrm{D}t} + p \frac{\mathrm{D}\ln\rho}{\mathrm{D}t} = -p\nabla \cdot \boldsymbol{u}. \tag{4.8}$$

Sum of these three:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}u^2 + \Phi + e\right) = -\nabla\cdot(p\boldsymbol{u}) + \frac{1}{\mu_0}\boldsymbol{u}\cdot[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]\,. \tag{4.9}$$

The last term can be rewritten as

$$\frac{1}{\mu_0}\boldsymbol{u}\cdot[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}] = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot(-\boldsymbol{u}\times\boldsymbol{B}) = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}. \tag{4.10}$$

Using mass conservation:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + \Phi + e\right)\right] + \nabla\cdot\left[\rho\boldsymbol{u}\left(\frac{1}{2}u^2 + \Phi + e\right) + p\boldsymbol{u}\right] = \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}. \tag{4.11}$$

Magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} = -\frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\times\boldsymbol{E}. \tag{4.12}$$

Total energy:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + \Phi + e\right) + \frac{B^2}{2\mu_0}\right] + \nabla\cdot\left[\rho\boldsymbol{u}\left(\frac{1}{2}u^2 + \Phi + h\right) + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0, \tag{4.13}$$

where $h = e + p/\rho$ is the specific enthalpy and we have used the identity $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\times\boldsymbol{E} - \boldsymbol{E}\cdot\nabla\times\boldsymbol{B}$. Note that $(\boldsymbol{E}\times\boldsymbol{B})/\mu_0$ is the Poynting vector, the electromagnetic energy flux density. The total energy is therefore conserved.

For a self-gravitating system satisfying Poisson's equation, the gravitational energy density can instead be regarded as $-g^2/8\pi G$:

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) = -\frac{1}{4\pi G}\nabla\Phi\cdot\frac{\partial\nabla\Phi}{\partial t} \tag{4.14}$$

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla\cdot\left(\frac{\Phi}{4\pi G}\frac{\partial\nabla\Phi}{\partial t}\right) = \frac{\Phi}{4\pi G}\frac{\partial\nabla^2\Phi}{\partial t} = \Phi\frac{\partial\rho}{\partial t} = -\Phi\nabla\cdot(\rho\boldsymbol{u}) \tag{4.15}$$

$$\frac{\partial}{\partial t}\left(-\frac{g^2}{8\pi G}\right) + \nabla\cdot\left(\rho\boldsymbol{u}\Phi + \frac{\Phi}{4\pi G}\frac{\partial\nabla\Phi}{\partial t}\right) = \rho\boldsymbol{u}\cdot\nabla\Phi. \tag{4.16}$$

The total energy equation is then

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}u^2 + e\right) - \frac{g^2}{8\pi G} + \frac{B^2}{2\mu_0}\right] \\ &\quad + \nabla\cdot\left[\rho\boldsymbol{u}\left(\frac{1}{2}u^2 + \Phi + h\right) + \frac{\Phi}{4\pi G}\frac{\partial\nabla\Phi}{\partial t} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0.\end{aligned} \tag{4.17}$$

It is important to note that some of the gravitational and magnetic energy of an astrophysical body is stored in the exterior region, even if the mass density vanishes there.

4.3. *Other conservation laws in ideal MHD*

In ideal fluid dynamics there are certain invariants with a geometrical or topological interpretation. In homentropic or barotropic flow, for example, vorticity (or, equivalently, circulation) and kinetic helicity are conserved, while, in non-barotropic flow, potential vorticity is conserved (see Example A.2). The Lorentz force breaks these conservation laws because the curl of the Lorentz force per unit mass does not vanish in general. However, some new topological invariants associated with the magnetic field appear.

The magnetic helicity in a volume V with bounding surface S is defined as

$$H_m = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V, \tag{4.18}$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. Now

$$\frac{\partial\boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\Phi_e = \boldsymbol{u}\times\boldsymbol{B} - \nabla\Phi_e, \tag{4.19}$$

where Φ_e is the electrostatic potential. This can be thought of as the 'uncurl' of the induction equation. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) = -\boldsymbol{B}\cdot\nabla\Phi_e + \boldsymbol{A}\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B}). \tag{4.20}$$

In ideal MHD, therefore, magnetic helicity is conserved:

$$\frac{\partial}{\partial t}(\boldsymbol{A}\cdot\boldsymbol{B}) + \nabla\cdot[\Phi_e\boldsymbol{B} + \boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})] = 0. \tag{4.21}$$

However, care is needed because $\boldsymbol{A}$ is not uniquely defined. Under a gauge transformation $\boldsymbol{A}\mapsto\boldsymbol{A}+\nabla\chi$, $\Phi_e\mapsto\Phi_e-\partial\chi/\partial t$, where $\chi(\boldsymbol{x}, t)$ is a scalar field, $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, but H_m changes by an amount

$$\int_V \boldsymbol{B}\cdot\nabla\chi\,\mathrm{d}V = \int_V \nabla\cdot(\chi\boldsymbol{B})\,\mathrm{d}V = \int_S \chi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{4.22}$$

Therefore H_m is not uniquely defined unless $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on the surface S.

Magnetic helicity is a pseudoscalar quantity: it changes sign under a reflection of the spatial coordinates. Indeed, it is non-zero only when the magnetic field lacks reflectional symmetry. It can also be interpreted topologically in terms of the twistedness and knottedness of the magnetic field (see Example A.10). Since the field is 'frozen in' to the fluid and deformed continuously by it, the topological properties of the field are conserved. The equivalent conserved quantity in homentropic or barotropic ideal gas dynamics (without a magnetic field) is the kinetic helicity

$$H_k = \int_V \boldsymbol{u}\cdot(\nabla\times\boldsymbol{u})\,\mathrm{d}V. \tag{4.23}$$

The cross-helicity in a volume V is

$$H_c = \int_V \boldsymbol{u}\cdot\boldsymbol{B}\,\mathrm{d}V. \tag{4.24}$$

It is helpful here to write the equation of motion in ideal MHD in the form

$$\frac{\partial \boldsymbol{u}}{\partial t}+(\nabla\times\boldsymbol{u})\times\boldsymbol{u}+\nabla\left(\frac{1}{2}u^2+\Phi+h\right)=T\nabla s+\frac{1}{\mu_0\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}, \tag{4.25}$$

using the relation $\mathrm{d}h=T\,\mathrm{d}s+v\,\mathrm{d}p$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{B})+\nabla\boldsymbol{\cdot}\left[\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{B})+\left(\frac{1}{2}u^2+\Phi+h\right)\boldsymbol{B}\right]=T\boldsymbol{B}\boldsymbol{\cdot}\nabla s, \tag{4.26}$$

and so cross-helicity is conserved in ideal MHD in homentropic or barotropic flow.

Bernoulli's theorem follows from the inner product of (4.25) with $\boldsymbol{u}$. In steady flow

$$\boldsymbol{u}\boldsymbol{\cdot}\nabla\left(\frac{1}{2}u^2+\Phi+h\right)=0, \tag{4.27}$$

which implies that the Bernoulli function $(1/2)u^2+\Phi+h$ is constant along streamlines, but only if $\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{F}_m=0$ (e.g. if $\boldsymbol{u}\parallel\boldsymbol{B}$), i.e. if the magnetic field does no work on the flow.

Related examples: A.10, A.11.

4.4. *Symmetries of the equations*

The equations of ideal gas dynamics and MHD have numerous symmetries. In the case of an isolated, self-gravitating system, these include:

(i) Translations of time and space, and rotations of space: related (via Noether's theorem) to the conservation of energy, momentum and angular momentum.
(ii) Reversal of time: related to the absence of dissipation.
(iii) Reflections of space (but note that $\boldsymbol{B}$ is a pseudovector and behaves oppositely to $\boldsymbol{u}$ under a reflection).
(iv) Galilean transformations.
(v) Reversal of the sign of $\boldsymbol{B}$.
(vi) Similarity transformations (exercise): if space and time are rescaled by independent factors λ and μ, i.e.

$$\boldsymbol{x}\mapsto\lambda\boldsymbol{x},\quad t\mapsto\mu\, t, \tag{4.28a,b}$$

then

$$\boldsymbol{u}\mapsto\lambda\mu^{-1}\boldsymbol{u},\quad \rho\mapsto\mu^{-2}\rho,\quad p\mapsto\lambda^2\mu^{-4}p,\quad \Phi\mapsto\lambda^2\mu^{-2}\Phi,\quad \boldsymbol{B}\mapsto\lambda\mu^{-2}\boldsymbol{B}. \tag{4.29a–e}$$

(This symmetry requires a perfect gas so that the thermodynamic relations are scale free.)

In the case of a non-isolated system with an external potential Φ_{ext}, these symmetries (other than $\boldsymbol{B}\mapsto-\boldsymbol{B}$) apply only if Φ_{ext} has them. However, in the approximation of a non-self-gravitating system, the mass can be rescaled by any factor λ such that

$$\rho\mapsto\lambda\rho,\quad p\mapsto\lambda p,\quad \boldsymbol{B}\mapsto\lambda^{1/2}\boldsymbol{B}. \tag{4.30a–c}$$

(This symmetry also requires a perfect gas.)

4.5. *Hyperbolic structure*

Analysing the so-called hyperbolic structure of the equations of AFD is one way of understanding the wave modes of the system and the way in which information propagates in the fluid. It is fundamental to the construction of some types of numerical method for solving the equations. We temporarily neglect the gravitational force here, because in a Newtonian theory it involves instantaneous action at a distance and is not associated with a finite wave speed.

In ideal gas dynamics, the equation of mass conservation, the thermal energy equation and the equation of motion (omitting gravity) can be written as

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho+\rho\nabla\cdot\boldsymbol{u}=0, \\ \frac{\partial p}{\partial t}+\boldsymbol{u}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{u}=0, \\ \frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{1}{\rho}\nabla p=\boldsymbol{0}, \end{aligned}\right\} \tag{4.31}$$

and then combined into the form

$$\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{A}_i\frac{\partial \boldsymbol{U}}{\partial x_i}=\boldsymbol{0}, \tag{4.32}$$

where

$$\boldsymbol{U}=\begin{bmatrix} \rho \\ p \\ u_x \\ u_y \\ u_z \end{bmatrix} \tag{4.33}$$

is a five-dimensional 'state vector' and $\boldsymbol{A}_x$, $\boldsymbol{A}_y$ and $\boldsymbol{A}_z$ are the three 5×5 matrices

$$\begin{bmatrix} u_x & 0 & \rho & 0 & 0 \\ 0 & u_x & \gamma p & 0 & 0 \\ 0 & \frac{1}{\rho} & u_x & 0 & 0 \\ 0 & 0 & 0 & u_x & 0 \\ 0 & 0 & 0 & 0 & u_x \end{bmatrix}, \quad \begin{bmatrix} u_y & 0 & 0 & \rho & 0 \\ 0 & u_y & 0 & \gamma p & 0 \\ 0 & 0 & u_y & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & u_y & 0 \\ 0 & 0 & 0 & 0 & u_y \end{bmatrix}, \quad \begin{bmatrix} u_z & 0 & 0 & 0 & \rho \\ 0 & u_z & 0 & 0 & \gamma p \\ 0 & 0 & u_z & 0 & 0 \\ 0 & 0 & 0 & u_z & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 & u_z \end{bmatrix}. \tag{4.34a–c}$$

This works because every term in the equations involves a first derivative with respect to either time or space.

The system of equations is said to be hyperbolic if the eigenvalues of $\boldsymbol{A}_i n_i$ are real for any unit vector $\boldsymbol{n}$ and if the eigenvectors span the five-dimensional space. As will be seen in § 6.2, the eigenvalues can be identified as wave speeds, and the eigenvectors as wave modes, with $\boldsymbol{n}$ being the unit wavevector, locally normal to the wavefronts.

Taking $\boldsymbol{n}=\boldsymbol{e}_x$ without loss of generality, we find (exercise)

$$\det(\boldsymbol{A}_x-v\boldsymbol{I})=-(v-u_x)^3\left[(v-u_x)^2-v_s^2\right], \tag{4.35}$$

where

$$v_s=\left(\frac{\gamma p}{\rho}\right)^{1/2} \tag{4.36}$$

is the adiabatic sound speed. The wave speeds v are real and the system is indeed hyperbolic.

Two of the wave modes are sound waves (acoustic waves), which have wave speeds $v = u_x \pm v_s$ and therefore propagate at the sound speed relative to the moving fluid. Their eigenvectors are

$$\begin{bmatrix} \rho \\ \gamma p \\ \pm v_s \\ 0 \\ 0 \end{bmatrix} \tag{4.37}$$

and involve perturbations of density, pressure and longitudinal velocity.

The remaining three wave modes have wave speed $v = u_x$ and do not propagate relative to the fluid. Their eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.38a–c}$$

The first is the entropy wave, which involves only a density perturbation but no pressure perturbation. Since the entropy can be considered as a function of the density and pressure, this wave involves an entropy perturbation. It must therefore propagate at the fluid velocity because the entropy is a material invariant. The other two modes with $v = u_x$ are vortical waves, which involve perturbations of the transverse velocity components, and therefore of the vorticity. Conservation of vorticity implies that these waves propagate with the fluid velocity.

To extend the analysis to ideal MHD, we may consider the induction equation in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{B}(\nabla \cdot \boldsymbol{u}) = \boldsymbol{0}, \tag{4.39}$$

and include the Lorentz force in the equation of motion. Every new term involves a first derivative. So the equation of mass conservation, the thermal energy equation, the equation of motion and the induction equation can be written in the combined form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = \boldsymbol{0}, \tag{4.40}$$

where

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ p \\ u_x \\ u_y \\ u_z \\ B_x \\ B_y \\ B_z \end{bmatrix} \tag{4.41}$$

is now an eight-dimensional 'state vector' and the $\boldsymbol{A}_i$ are three 8×8 matrices, e.g.

$$\boldsymbol{A}_x = \begin{bmatrix} u_x & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u_x & \gamma p & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\rho} & u_x & 0 & 0 & 0 & \dfrac{B_y}{\mu_0 \rho} & \dfrac{B_z}{\mu_0 \rho} \\ 0 & 0 & 0 & u_x & 0 & 0 & -\dfrac{B_x}{\mu_0 \rho} & 0 \\ 0 & 0 & 0 & 0 & u_x & 0 & 0 & -\dfrac{B_x}{\mu_0 \rho} \\ 0 & 0 & 0 & 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & B_y & -B_x & 0 & 0 & u_x & 0 \\ 0 & 0 & B_z & 0 & -B_x & 0 & 0 & u_x \end{bmatrix}. \tag{4.42}$$

We now find, after some algebra,

$$\det(\boldsymbol{A}_x - v\boldsymbol{I}) = (v-u_x)^2 \left[(v-u_x)^2 - v_{ax}^2\right] \left[(v-u_x)^4 - (v_s^2+v_a^2)(v-u_x)^2 + v_s^2 v_{ax}^2\right]. \tag{4.43}$$

The wave speeds v are real and the system is indeed hyperbolic. The various MHD wave modes will be examined later (§ 5).

In this representation, there are two modes that have $v = u_x$ and do not propagate relative to the fluid. One is still the entropy wave, which is physical and involves only a density perturbation. The other is the 'div$\boldsymbol{B}$' mode, which is unphysical and involves a perturbation of $\nabla \cdot \boldsymbol{B}$ (i.e. of B_x, in the case $\boldsymbol{n} = \boldsymbol{e}_x$). This must be eliminated by imposing the constraint $\nabla \cdot \boldsymbol{B} = 0$. (In fact the equations in the form we have written them imply that $(\nabla \cdot \boldsymbol{B})/\rho$ is a material invariant and could be non-zero unless the initial condition $\nabla \cdot \boldsymbol{B} = 0$ is imposed.) The vortical waves are replaced by Alfvén waves with speeds $u_x \pm v_{ax}$.

4.6. *Stress tensor and virial theorem*

In the absence of external forces, the equation of motion of a fluid can usually be written in the form

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{T} \quad \text{or} \quad \rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial T_{ji}}{\partial x_j}, \tag{4.44a,b}$$

where $\boldsymbol{T}$ is the stress tensor, a symmetric second-rank tensor field. Using the equation of mass conservation, we can relate this to the conservative form of the momentum equation,

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} - \boldsymbol{T}) = \boldsymbol{0}, \tag{4.45}$$

which shows that $-\boldsymbol{T}$ is the momentum flux density excluding the advective flux of momentum.

For a self-gravitating system in ideal MHD, the stress tensor is

$$\boldsymbol{T} = -p\,\boldsymbol{I} - \frac{1}{4\pi G}\left(\boldsymbol{g}\boldsymbol{g} - \frac{1}{2}g^2\,\boldsymbol{I}\right) + \frac{1}{\mu_0}\left(\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\,\boldsymbol{I}\right), \tag{4.46}$$

or, in Cartesian components,

$$T_{ij} = -p\,\delta_{ij} - \frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}g^2\,\delta_{ij}\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\,\delta_{ij}\right). \tag{4.47}$$

We have already identified the Maxwell stress tensor associated with the magnetic field. The idea of a gravitational stress tensor works for a self-gravitating system in which the gravitational field $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$ and the density ρ are related through Poisson's equation $-\boldsymbol{\nabla}\cdot\boldsymbol{g} = \nabla^2\Phi = 4\pi G\rho$. In fact, for a general vector field $\boldsymbol{v}$, it can be shown that (exercise)

$$\boldsymbol{\nabla}\cdot(\boldsymbol{v}\boldsymbol{v} - \tfrac{1}{2}v^2\,\boldsymbol{I}) = (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2\right) = (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{v} + (\boldsymbol{\nabla}\times\boldsymbol{v})\times\boldsymbol{v}. \tag{4.48}$$

In the magnetic case ($\boldsymbol{v} = \boldsymbol{B}$) the first term in the final expression vanishes, while in the gravitational case ($\boldsymbol{v} = \boldsymbol{g}$) the second term vanishes, leaving $-4\pi G\rho\boldsymbol{g}$, which becomes the force per unit volume, $\rho\boldsymbol{g}$, when divided by $-4\pi G$.

The virial equations are the spatial moments of the equation of motion, and provide integral measures of the balance of forces acting on the fluid. The first moments are generally the most useful. Consider

$$\rho\frac{\mathrm{D}^2}{\mathrm{D}t^2}(x_i x_j) = \rho\frac{\mathrm{D}}{\mathrm{D}t}(u_i x_j + x_i u_j) = 2\rho u_i u_j + x_j\frac{\partial T_{ki}}{\partial x_k} + x_i\frac{\partial T_{kj}}{\partial x_k}. \tag{4.49}$$

Integrate this equation over a material volume V bounded by a surface S (with material invariant mass element $\mathrm{d}m = \rho\,\mathrm{d}V$):

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_V x_i x_j\,\mathrm{d}m &= \int_V\left(2\rho u_i u_j + x_j\frac{\partial T_{ki}}{\partial x_k} + x_i\frac{\partial T_{kj}}{\partial x_k}\right)\mathrm{d}V \\ &= \int_V(2\rho u_i u_j - T_{ji} - T_{ij})\,\mathrm{d}V + \int_S(x_j T_{ki} + x_i T_{kj})n_k\,\mathrm{d}S,\end{aligned} \tag{4.50}$$

where we have integrated by parts using the divergence theorem. In the case of an isolated system with no external sources of gravity or magnetic field, $\boldsymbol{g}$ decays proportional to $|\boldsymbol{x}|^{-2}$ at large distance, and $\boldsymbol{B}$ decays faster. Therefore T_{ij} decays proportional to $|\boldsymbol{x}|^{-4}$ and the surface integral can be eliminated if we let V occupy the whole of space. We then obtain (after division by 2) the tensor virial theorem

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = 2K_{ij} - \mathcal{T}_{ij}, \tag{4.51}$$

where

$$I_{ij} = \int x_i x_j\,\mathrm{d}m \tag{4.52}$$

is related to the inertia tensor of the system,

$$K_{ij} = \int \frac{1}{2}u_i u_j\,\mathrm{d}m \tag{4.53}$$

is a kinetic energy tensor and

$$\mathcal{T}_{ij} = \int T_{ij}\,\mathrm{d}V \tag{4.54}$$

is the integrated stress tensor. (If the conditions above are not satisfied, there will be an additional contribution from the surface integral.)

The scalar virial theorem is the trace of this tensor equation, which we write as

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K - \mathcal{T}. \tag{4.55}$$

Note that K is the total kinetic energy. Now

$$-\mathcal{T} = \int \left(3p - \frac{g^2}{8\pi G} + \frac{B^2}{2\mu_0}\right) \mathrm{d}V = 3(\gamma - 1)U + W + M, \tag{4.56}$$

for a perfect gas with no external gravitational field, where U, W and M are the total internal, gravitational and magnetic energies. Thus

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + 3(\gamma - 1)U + W + M. \tag{4.57}$$

On the right-hand side, only W is negative. For the system to be bound (i.e. not fly apart) the kinetic, internal and magnetic energies are limited by

$$2K + 3(\gamma - 1)U + M \leqslant |W|. \tag{4.58}$$

In fact equality must hold, at least on average, unless the system is collapsing or contracting.

The tensor virial theorem provides more specific information relating to the energies associated with individual directions, and is particularly relevant in cases where anisotropy is introduced by rotation or a magnetic field. It has been used in estimating the conditions required for gravitational collapse in molecular clouds. A higher-order tensor virial method was used by Chandrasekhar and Lebovitz to study the equilibrium and stability of rotating ellipsoidal bodies (Chandrasekhar 1969).

5. Linear waves in homogeneous media

In ideal MHD the density, pressure and magnetic field evolve according to

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &= -\boldsymbol{u}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{u}, \\ \frac{\partial p}{\partial t} &= -\boldsymbol{u}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{u}, \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla\times(\boldsymbol{u}\times\boldsymbol{B}). \end{aligned}\right\} \tag{5.1}$$

Consider a magnetostatic equilibrium in which the density, pressure and magnetic field are $\rho_0(\boldsymbol{x})$, $p_0(\boldsymbol{x})$ and $\boldsymbol{B}_0(\boldsymbol{x})$. The above equations are exactly satisfied in this basic state because $\boldsymbol{u} = \boldsymbol{0}$ and the time derivatives vanish. Now consider small perturbations from equilibrium, such that $\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}) + \delta\rho(\boldsymbol{x}, t)$ with $|\delta\rho| \ll \rho_0$, etc. The linearized equations are

$$\left.\begin{aligned} \frac{\partial \delta\rho}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla\rho_0 - \rho_0\nabla\cdot\delta\boldsymbol{u}, \\ \frac{\partial \delta p}{\partial t} &= -\delta\boldsymbol{u}\cdot\nabla p_0 - \gamma p_0\nabla\cdot\delta\boldsymbol{u}, \\ \frac{\partial \delta\boldsymbol{B}}{\partial t} &= \nabla\times(\delta\boldsymbol{u}\times\boldsymbol{B}_0). \end{aligned}\right\} \tag{5.2}$$

By introducing the displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ such that $\delta\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$, we can integrate these equations to obtain

$$\left.\begin{aligned} \delta\rho &= -\boldsymbol{\xi}\cdot\nabla\rho - \rho\nabla\cdot\boldsymbol{\xi}, \\ \delta p &= -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi}, \\ \delta\boldsymbol{B} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla\boldsymbol{B}. \end{aligned}\right\} \tag{5.3}$$

We have now dropped the subscript '0' without danger of confusion.

(The above relations allow some freedom to add arbitrary functions of $\boldsymbol{x}$. At least when studying wave-like solutions in which all variables have the same harmonic time dependence, such additional terms can be discarded.)

The linearized equation of motion is

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\nabla\delta\Phi - \delta\rho\nabla\Phi - \nabla\delta\Pi + \frac{1}{\mu_0}(\delta\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\delta\boldsymbol{B}), \tag{5.4}$$

where the perturbation of total pressure is

$$\delta\Pi = \delta p + \frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{\mu_0} = -\boldsymbol{\xi}\cdot\nabla\Pi - \left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi}). \tag{5.5}$$

The gravitational potential perturbation satisfies the linearized Poisson equation

$$\nabla^2\delta\Phi = 4\pi G\,\delta\rho. \tag{5.6}$$

We consider a basic state of uniform density, pressure and magnetic field, in the absence of gravity. Such a system is homogeneous but anisotropic, because the uniform field distinguishes a particular direction. The problem simplifies to

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla\delta\Pi + \frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\left[\boldsymbol{B}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{B}(\nabla\cdot\boldsymbol{\xi})\right], \tag{5.7}$$

with

$$\delta\Pi = -\left(\gamma p + \frac{B^2}{\mu_0}\right)\nabla\cdot\boldsymbol{\xi} + \frac{1}{\mu_0}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{\xi}). \tag{5.8}$$

Owing to the symmetries of the basic state, plane-wave solutions exist, of the form

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \mathrm{Re}[\tilde{\boldsymbol{\xi}}\,\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t)], \tag{5.9}$$

where ω and $\boldsymbol{k}$ are the frequency and wavevector, and $\tilde{\boldsymbol{\xi}}$ is a constant vector representing the amplitude of the wave. For such solutions, (5.7) gives

$$\rho\omega^2\boldsymbol{\xi} = \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]\boldsymbol{k} + \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\left[(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{\xi} - \boldsymbol{B}(\boldsymbol{k}\cdot\boldsymbol{\xi})\right], \tag{5.10}$$

where we have changed the sign and omitted the tilde.

For transverse displacements that are orthogonal to both the wavevector and the magnetic field, i.e. $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$, this equation simplifies to

$$\rho\omega^2\boldsymbol{\xi} = \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2\boldsymbol{\xi}. \tag{5.11}$$

Such solutions are called Alfvén waves. Their dispersion relation is

$$\omega^2 = (\boldsymbol{k}\cdot\boldsymbol{v}_a)^2. \tag{5.12}$$

Given the dispersion relation $\omega(\boldsymbol{k})$ of any wave mode, the phase and group velocities of the wave can be identified as

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}}, \quad \boldsymbol{v}_g = \frac{\partial\omega}{\partial\boldsymbol{k}} = \boldsymbol{\nabla}_k\omega, \tag{5.13a,b}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$. The phase velocity is that with which the phase of the wave travels, while the group velocity is that which the energy of the wave (or the centre of a wavepacket) is transported.

For Alfvén waves, therefore,

$$\boldsymbol{v}_p = \pm v_a\cos\theta\,\hat{\boldsymbol{k}}, \quad \boldsymbol{v}_g = \pm\boldsymbol{v}_a, \tag{5.14a,b}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$.

To find the other solutions, we take the inner product of (5.10) with $\boldsymbol{k}$ and then with $\boldsymbol{B}$ to obtain first

$$\rho\omega^2\boldsymbol{k}\cdot\boldsymbol{\xi} = \left[\left(\gamma p + \frac{B^2}{\mu_0}\right)\boldsymbol{k}\cdot\boldsymbol{\xi} - \frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{\xi}\right]k^2 \tag{5.15}$$

and then

$$\rho\omega^2\boldsymbol{B}\cdot\boldsymbol{\xi} = \gamma p(\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{k}\cdot\boldsymbol{B}. \tag{5.16}$$

These equations can be written in the form

$$\begin{bmatrix} \rho\omega^2 - \left(\gamma p + \dfrac{B^2}{\mu_0}\right)k^2 & \dfrac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})k^2 \\ -\gamma p(\boldsymbol{k}\cdot\boldsymbol{B}) & \rho\omega^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{k}\cdot\boldsymbol{\xi} \\ \boldsymbol{B}\cdot\boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.17}$$

The 'trivial solution' $\boldsymbol{k}\cdot\boldsymbol{\xi} = \boldsymbol{B}\cdot\boldsymbol{\xi} = 0$ corresponds to the Alfvén wave that we have already identified. The other solutions satisfy

$$\rho\omega^2\left[\rho\omega^2 - \left(\gamma p + \frac{B^2}{\mu_0}\right)k^2\right] + \gamma p k^2\frac{1}{\mu_0}(\boldsymbol{k}\cdot\boldsymbol{B})^2 = 0, \tag{5.18}$$

which simplifies to

$$v_p^4 - (v_s^2 + v_a^2)v_p^2 + v_s^2 v_a^2\cos^2\theta = 0. \tag{5.19}$$

The two solutions

$$v_p^2 = \tfrac{1}{2}(v_s^2 + v_a^2) \pm \left[\tfrac{1}{4}(v_s^2 + v_a^2)^2 - v_s^2 v_a^2\cos^2\theta\right]^{1/2} \tag{5.20}$$

are called fast and slow magnetoacoustic (or magnetosonic) waves, respectively.

In the special case $\theta = 0$ ($\boldsymbol{k}\parallel\boldsymbol{B}$), we have

$$v_p^2 = v_s^2 \quad \text{or} \quad v_a^2, \tag{5.21}$$

together with $v_p^2 = v_a^2$ for the Alfvén wave. Note that the fast wave could be either $v_p^2 = v_s^2$ or $v_p^2 = v_a^2$, whichever is greater.

 https://doi.org/10.1017/S0022377816000489

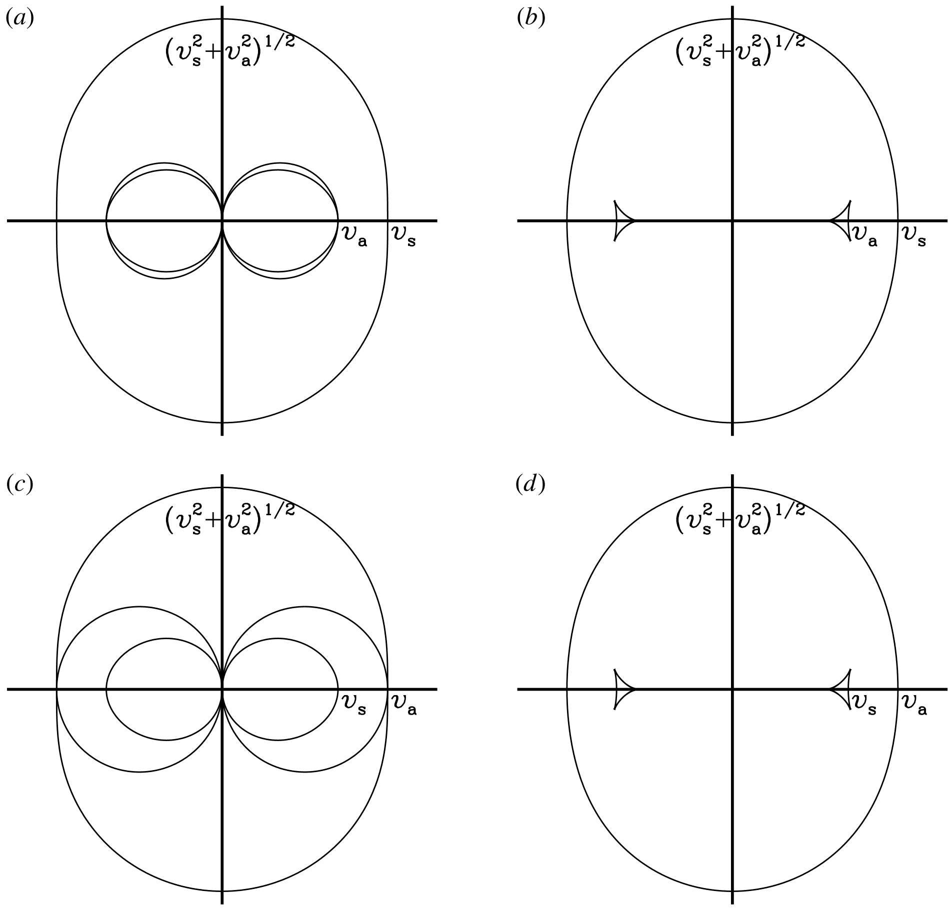

FIGURE 4. Polar plots of the phase velocity (*a*,*c*) and group velocity (*b*,*d*) of MHD waves for the cases $v_a = 0.7\, v_s$ (*a*,*b*) and $v_s = 0.7\, v_a$ (*c*,*d*) with a magnetic field in the horizontal direction. (The group velocity plot for the Alfvén wave consists of the two points $(\pm v_a, 0)$.)

In the special case $\theta = \pi/2$ ($\boldsymbol{k} \perp \boldsymbol{B}$), we have

$$v_p^2 = v_s^2 + v_a^2 \quad \text{or} \quad 0, \tag{5.22}$$

together with $v_p^2 = 0$ for the Alfvén wave.

The effects of the magnetic field on wave propagation can be understood as resulting from the two aspects of the Lorentz force. The magnetic tension gives rise to Alfvén waves, which are similar to waves on an elastic string, and are trivial in the absence of the magnetic field. In addition, the magnetic pressure affects the response of the fluid to compression, and therefore modifies the propagation of acoustic waves.

The phase and group velocity for the full range of θ are usually exhibited in Friedrichs diagrams[6] (figure 4).

[6] Kurt Otto Friedrichs (1901–1982), German–American.

We can interpret:

(i) the fast wave as a quasi-isotropic acoustic-type wave in which both gas and magnetic pressure contribute;
(ii) the slow wave as an acoustic-type wave that is strongly guided by the magnetic field;
(iii) the Alfvén wave as analogous to a wave on an elastic string, propagating by means of magnetic tension and perfectly guided by the magnetic field.

Related example: A.12.

6. Nonlinear waves, shocks and other discontinuities

6.1. *One-dimensional gas dynamics*

6.1.1. *Riemann's analysis*

The equations of mass conservation and motion in one dimension are

$$\left.\begin{aligned} \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} &= -\rho\frac{\partial u}{\partial x}, \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}. \end{aligned}\right\} \tag{6.1}$$

We assume the gas is homentropic ($s = \text{const.}$) and perfect. (This eliminates the entropy wave and leaves only the two sound waves.) Then $p \propto \rho^\gamma$ and $v_s^2 = \gamma p/\rho \propto \rho^{\gamma-1}$. It is convenient to use v_s as a variable in place of ρ or p:

$$\mathrm{d}p = v_s^2\,\mathrm{d}\rho, \quad \mathrm{d}\rho = \frac{\rho}{v_s}\left(\frac{2\,\mathrm{d}v_s}{\gamma-1}\right). \tag{6.2a,b}$$

Then

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v_s\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right) &= 0, \\ \frac{\partial}{\partial t}\left(\frac{2v_s}{\gamma-1}\right) + u\frac{\partial}{\partial x}\left(\frac{2v_s}{\gamma-1}\right) + v_s\frac{\partial u}{\partial x} &= 0. \end{aligned}\right\} \tag{6.3}$$

We add and subtract to obtain

$$\left[\frac{\partial}{\partial t} + (u+v_s)\frac{\partial}{\partial x}\right]\left(u + \frac{2v_s}{\gamma-1}\right) = 0, \tag{6.4}$$

$$\left[\frac{\partial}{\partial t} + (u-v_s)\frac{\partial}{\partial x}\right]\left(u - \frac{2v_s}{\gamma-1}\right) = 0. \tag{6.5}$$

Define the two Riemann invariants

$$R_\pm = u \pm \frac{2v_s}{\gamma-1}. \tag{6.6}$$

Then we deduce that $R_\pm = \text{const.}$ along a characteristic (curve) of gradient $\mathrm{d}x/\mathrm{d}t = u \pm v_s$ in the (x, t) plane. The $+$ and $-$ characteristics form an interlocking web covering the space–time diagram (figure 5).

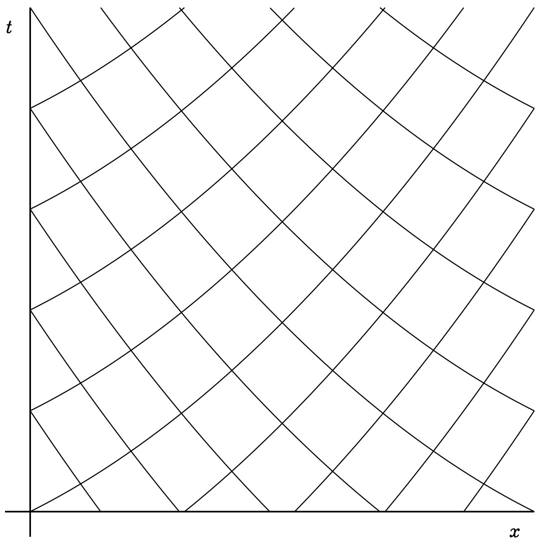

FIGURE 5. Characteristic curves in the space–time diagram.

Note that both Riemann invariants are needed to reconstruct the solution (u and v_s). Half of the information is propagated along one set of characteristics and half along the other.

In general the characteristics are not known in advance but must be determined along with the solution. The + and − characteristics propagate at the speed of sound to the right and left, respectively, with respect to the motion of the fluid.

This concept generalizes to nonlinear waves the solution of the classical wave equation for acoustic waves on a uniform and static background, which is of the form $f(x - v_s t) + g(x + v_s t)$.

6.1.2. *Method of characteristics*

A numerical method of solution can be based on the following idea:

(i) Start with the initial data (u and v_s) for all relevant x at $t = 0$.
(ii) Determine the characteristic slopes at $t = 0$.
(iii) Propagate the $R_\pm$ information for a small increment of time, neglecting the variation of the characteristic slopes.
(iv) Combine the $R_\pm$ information to find u and v_s at each x at the new value of t.
(v) Re-evaluate the slopes and repeat.

The domain of dependence of a point P in the space–time diagram is that region of the diagram bounded by the ± characteristics through P and located in the past of P. The solution at P cannot depend on anything that occurs outside the domain of dependence. Similarly, the domain of influence of P is the region in the future of P bounded by the characteristics through P (figure 6).

6.1.3. *A simple wave*

Suppose that R_- is uniform, having the same constant value on every characteristic emanating from an undisturbed region to the right. Its value everywhere is that of the

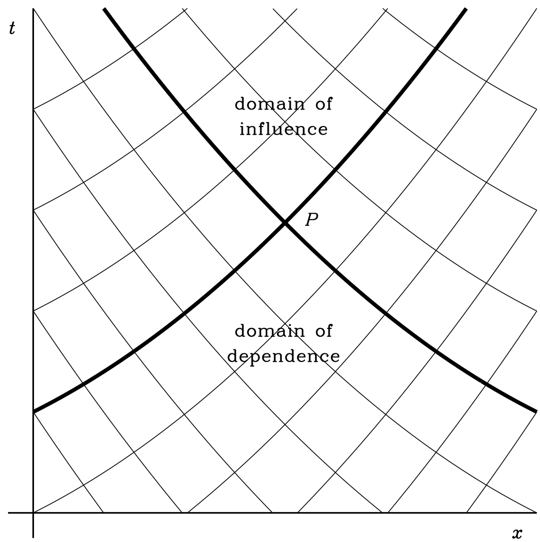

FIGURE 6. Domains of dependence and of influence.

undisturbed region:

$$u - \frac{2v_s}{\gamma - 1} = u_0 - \frac{2v_{s0}}{\gamma - 1}. \tag{6.7}$$

Then, along the + characteristics, both R_+ and R_-, and therefore u and v_s, must be constant. The + characteristics therefore have constant slope $v = u + v_s$, so they are straight lines.

The statement that the wave speed v is constant on the family of straight lines $\mathrm{d}x/\mathrm{d}t = v$ is expressed by the equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0. \tag{6.8}$$

This is known as the inviscid Burgers equation[7] or the nonlinear advection equation.

The inviscid Burgers equation has only one set of characteristics, with slope $\mathrm{d}x/\mathrm{d}t = v$. It is easily solved by the method of characteristics. The initial data define $v_0(x) = v(x, 0)$ and the characteristics are straight lines. In regions where $\mathrm{d}v_0/\mathrm{d}x > 0$ the characteristics diverge in the future. In regions where $\mathrm{d}v_0/\mathrm{d}x < 0$ the characteristics converge and will form a shock at some point. Contradictory information arrives at the same point in the space–time diagram, leading to a breakdown of the solution (figure 7).

Another viewpoint is that of wave steepening. The graph of v versus x evolves in time by moving each point at its wave speed v. The crest of the wave moves fastest and eventually overtakes the trough to the right of it. The profile would become multiple valued, but this is physically meaningless and the wave breaks, forming a discontinuity (figure 8).

Indeed, the formal solution of the inviscid Burgers equation is

$$v(x, t) = v_0(x_0) \quad \text{with } x = x_0 + v_0(x_0)t. \tag{6.9}$$

[7]Johannes (Jan) Martinus Burgers (1895–1981), Dutch.

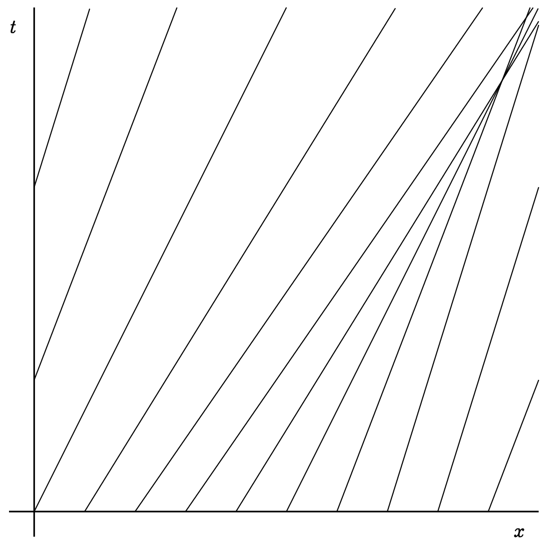

FIGURE 7. Formation of a shock from intersecting characteristics.

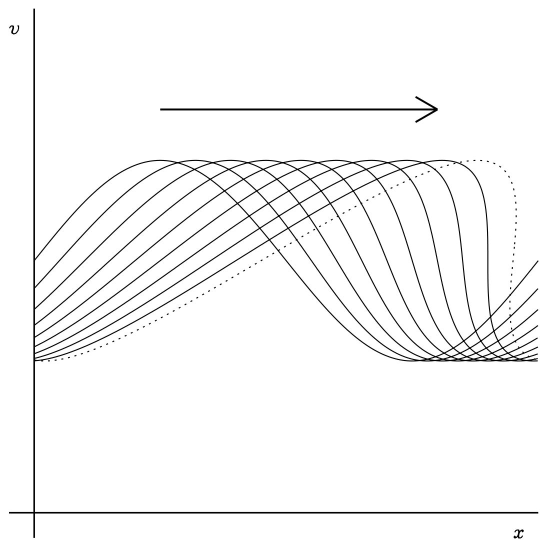

FIGURE 8. Wave steepening and shock formation. The dotted profile is multiple valued and is replaced in practice with a discontinuous profile including a shock.

By the chain rule, $\partial v/\partial x = v_0'/(1 + v_0' t)$, which diverges first at the breaking time $t = 1/\max(-v_0')$.

The crest of a sound wave moves faster than the trough for two reasons. It is partly because the crest is denser and hotter, so the sound speed is higher (unless the gas is isothermal), but it is also because of the self-advection of the wave (recall that the wave speed is $u + v_s$). The breaking time depends on the amplitude and wavelength of the wave.

6.2. *General analysis of simple nonlinear waves*

Recall the hyperbolic structure of the equations of AFD (§ 4.5):

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} = \boldsymbol{0}, \quad \boldsymbol{U} = [\rho, p, \boldsymbol{u}, \boldsymbol{B}]^{\mathrm{T}}. \tag{6.10}$$

The system is hyperbolic because the eigenvalues of $\boldsymbol{A}_i n_i$ are real for any unit vector n_i. The eigenvalues are identified as wave speeds, and the corresponding eigenvectors as wave modes.

In a simple wave propagating in the x-direction, all physical quantities are functions of a single variable, the phase $\varphi(x, t)$. Then $\boldsymbol{U} = \boldsymbol{U}(\varphi)$ and so

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\varphi}\frac{\partial \varphi}{\partial t} + \boldsymbol{A}_x \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}\varphi}\frac{\partial \varphi}{\partial x} = 0. \tag{6.11}$$

This equation is satisfied if $\mathrm{d}\boldsymbol{U}/\mathrm{d}\varphi$ is an eigenvector of the hyperbolic system and if

$$\frac{\partial \varphi}{\partial t} + v\frac{\partial \varphi}{\partial x} = 0, \tag{6.12}$$

where v is the corresponding wave speed. But since $v = v(\varphi)$ we again find

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0, \tag{6.13}$$

the inviscid Burgers equation.

Wave steepening is therefore generic for simple waves. However, waves do not always steepen in practice. For example, linear dispersion arising from Coriolis or buoyancy forces (see § 11) can counteract nonlinear wave steepening. Waves propagating on a non-uniform background are not simple waves. In addition, waves may be damped by diffusive processes (viscosity, thermal conduction or resistivity) before they can steepen.

Furthermore, even some simple waves do not undergo steepening, in spite of the above argument. This happens if the wave speed v does not depend on the variables that actually vary in the wave mode. One example is the entropy wave in hydrodynamics, in which the density varies but not the pressure or the velocity. The wave speed is the fluid velocity, which does not vary in this wave; therefore the relevant solution of the inviscid Burgers equation is just $v = \text{const}$. Another example is the Alfvén wave, which involves variations in transverse velocity and magnetic field components, but whose speed depends on the longitudinal components and the density. The slow and fast magnetoacoustic waves, though, are 'genuinely nonlinear' and undergo steepening.

6.3. *Shocks and other discontinuities*

6.3.1. *Jump conditions*

Discontinuities are resolved in reality by diffusive processes (viscosity, thermal conduction or resistivity) that become more important on smaller length scales. Properly, we should solve an enhanced set of equations to resolve the internal structure of a shock. This internal solution would then be matched on to the external solution in which diffusion can be neglected. However, the matching conditions can in fact be determined from general principles without resolving the internal structure.

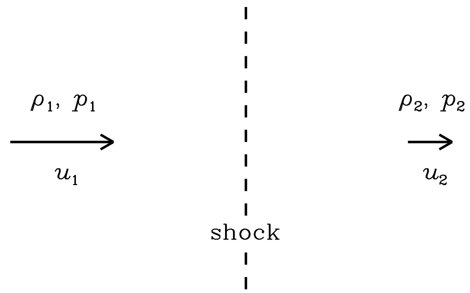

FIGURE 9. A shock front in its rest frame.

Without loss of generality, we consider a shock front at rest at $x = 0$ (making a Galilean transformation if necessary). We look for a stationary, one-dimensional solution in which gas flows from left to right. On the left is upstream, pre-shock material (ρ_1, p_1, etc.). On the right is downstream, post-shock material (ρ_2, p_2, etc.) (figure 9).

Consider any equation in conservative form

$$\frac{\partial q}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F} = 0. \tag{6.14}$$

For a stationary solution in one dimension,

$$\frac{\mathrm{d}F_x}{\mathrm{d}x} = 0, \tag{6.15}$$

which implies that the flux density F_x has the same value on each side of the shock. We write the matching condition as

$$[F_x]_1^2 = F_{x2} - F_{x1} = 0. \tag{6.16}$$

Including additional physics means that additional diffusive fluxes (not of mass but of momentum, energy, magnetic flux, etc.) are present. These fluxes are negligible outside the shock, so they do not affect the jump conditions. This approach is permissible provided that the new physics does not introduce any source terms in the equations. So the total energy is a properly conserved quantity but not the entropy (see later).

From mass conservation:

$$[\rho u_x]_1^2 = 0. \tag{6.17}$$

From momentum conservation:

$$\left[\rho u_x^2 + \Pi - \frac{B_x^2}{\mu_0}\right]_1^2 = 0, \tag{6.18}$$

$$\left[\rho u_x u_y - \frac{B_x B_y}{\mu_0}\right]_1^2 = 0, \tag{6.19}$$

$$\left[\rho u_x u_z - \frac{B_x B_z}{\mu_0}\right]_1^2 = 0. \tag{6.20}$$

From $\nabla \cdot \boldsymbol{B} = 0$:

$$[B_x]_1^2 = 0. \tag{6.21}$$

From $\partial \boldsymbol{B}/\partial t + \nabla \times \boldsymbol{E} = \boldsymbol{0}$:

$$[u_x B_y - u_y B_x]_1^2 = -[E_z]_1^2 = 0, \tag{6.22}$$

$$[u_x B_z - u_z B_x]_1^2 = [E_y]_1^2 = 0. \tag{6.23}$$

(These are the standard electromagnetic conditions at an interface: the normal component of $\boldsymbol{B}$ and the tangential components of $\boldsymbol{E}$ are continuous.) From total energy conservation:

$$\left[\rho u_x \left(\frac{1}{2}u^2 + h\right) + \frac{1}{\mu_0}(E_y B_z - E_z B_y)\right]_1^2 = 0. \tag{6.24}$$

Note that the conservative form of the momentum equation used above is (cf. 4.45)

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla \cdot \left(\rho u_i \boldsymbol{u} + \Pi\, \boldsymbol{e}_i - \frac{B_i \boldsymbol{B}}{\mu_0}\right) = 0. \tag{6.25}$$

Including gravity makes no difference to the shock relations because Φ is always continuous (it satisfies $\nabla^2 \Phi = 4\pi G \rho$).

Although the entropy in ideal MHD satisfies an equation of conservative form,

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \boldsymbol{u}) = 0, \tag{6.26}$$

the dissipation of energy within the shock provides a source term for entropy. Therefore the entropy flux is not continuous across the shock.

6.3.2. *Non-magnetic shocks*

First consider a normal shock ($u_y = u_z = 0$) with no magnetic field. We obtain the Rankine–Hugoniot relations[8]

$$[\rho u_x]_1^2 = 0, \quad [\rho u_x^2 + p]_1^2 = 0, \quad \left[\rho u_x \left(\tfrac{1}{2}u_x^2 + h\right)\right]_1^2 = 0. \tag{6.27a–c}$$

The specific enthalpy of a perfect gas is

$$h = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho} \tag{6.28}$$

and these equations can be solved algebraically (see Example A.13). Introduce the upstream Mach number (the shock Mach number)

$$\mathcal{M}_1 = \frac{u_{x1}}{v_{s1}} > 0. \tag{6.29}$$

Then we find

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}, \quad \frac{p_2}{p_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{(\gamma+1)}, \tag{6.30a,b}$$

[8]William John Macquorn Rankine (1820–1872), British. Pierre-Henri Hugoniot (1851–1887), French.

and

$$\mathcal{M}_2^2 = \frac{2 + (\gamma - 1)\mathcal{M}_1^2}{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}. \tag{6.31}$$

Note that ρ_2/ρ_1 and p_2/p_1 are increasing functions of $\mathcal{M}_1$. The case $\mathcal{M}_1 = 1$ is trivial as it corresponds to $\rho_2/\rho_1 = p_2/p_1 = 1$. The other two cases are the compression shock ($\mathcal{M}_1 > 1$, $\mathcal{M}_2 < 1$, $\rho_2 > \rho_1$, $p_2 > p_1$) and the rarefaction shock ($\mathcal{M}_1 < 1$, $\mathcal{M}_2 > 1$, $\rho_2 < \rho_1$, $p_2 < p_1$).

It is shown in Example A.13 that the entropy change in passing through the shock is positive for compression shocks and negative for rarefaction shocks. Therefore only compression shocks are physically realizable. Rarefaction shocks are excluded by the second law of thermodynamics. All shocks involve dissipation and irreversibility.

The fact that $\mathcal{M}_1 > 1$ while $\mathcal{M}_2 < 1$ means that the shock travels supersonically relative to the upstream gas and subsonically relative to the downstream gas.

In the weak shock limit $\mathcal{M}_1 - 1 \ll 1$ the relative velocity of the fluid and the shock is close to the sound speed on both sides.

In the strong shock limit $\mathcal{M}_1 \gg 1$, common in astrophysical applications, we have

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} \to \frac{\gamma+1}{\gamma-1}, \quad \frac{p_2}{p_1} \gg 1, \quad \mathcal{M}_2^2 \to \frac{\gamma-1}{2\gamma}. \tag{6.32a–c}$$

Note that the compression ratio ρ_2/ρ_1 is finite (and equal to 4 when $\gamma = 5/3$). In the rest frame of the undisturbed (upstream) gas the shock speed is $u_{sh} = -u_{x1}$. The downstream density, velocity (in that frame) and pressure in the limit of a strong shock are (as will be used in § 7)

$$\rho_2 = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1, \quad u_{x2} - u_{x1} = \frac{2u_{sh}}{\gamma+1}, \quad p_2 = \frac{2\rho_1 u_{sh}^2}{\gamma+1}. \tag{6.33a–c}$$

A significant amount of thermal energy is generated out of kinetic energy by the passage of a strong shock:

$$e_2 = \frac{2u_{sh}^2}{(\gamma+1)^2}. \tag{6.34}$$

6.3.3. *Oblique shocks*

When u_y or u_z is non-zero, we have the additional relations

$$[\rho u_x u_y]_1^2 = [\rho u_x u_z]_1^2 = 0. \tag{6.35}$$

Since ρu_x is continuous across the shock (and non-zero), we deduce that $[u_y]_1^2 = [u_z]_1^2 = 0$. Momentum and energy conservation apply as before, and we recover the Rankine–Hugoniot relations. (See Example A.14.)

6.3.4. *Other discontinuities*

The discontinuity is not called a shock if there is no normal flow ($u_x = 0$). In this case we can deduce only that $[p]_1^2 = 0$. Arbitrary discontinuities are allowed in ρ, u_y and u_z. These are related to the entropy and vortical waves. If there is a jump in ρ we have a contact discontinuity. If there is a jump in u_y or u_z we have a tangential discontinuity or vortex sheet (the vorticity being proportional to $\delta(x)$). Note that these discontinuities are not produced naturally by wave steepening, because the entropy and vortical waves do not steepen. However they do appear in the Riemann problem (§ 6.3.6) and other situations with discontinuous initial conditions.

6.3.5. *MHD shocks and discontinuities*

When a magnetic field is included, the jump conditions allow a wider variety of solutions. There are different types of discontinuity associated with the three MHD waves (Alfvén, slow and fast), which we will not discuss here. Since the parallel components of $\boldsymbol{B}$ need not be continuous, it is possible for them to 'switch on' or 'switch off' on passage through a shock.

A current sheet is a tangential discontinuity in the magnetic field. A classic case would be where B_y, say, changes sign across the interface, with $B_x = 0$. The current density is then proportional to $\delta(x)$.

6.3.6. *The Riemann problem*

The Riemann problem is a fundamental initial value problem for a hyperbolic system and plays a central role in some numerical methods for solving the equations of AFD.

The initial condition at $t = 0$ consists of two uniform states separated by a discontinuity at $x = 0$. In the case of one-dimensional gas dynamics, we have

$$\rho = \begin{cases} \rho_L, & x < 0 \\ \rho_R, & x > 0 \end{cases}, \quad p = \begin{cases} p_L, & x < 0 \\ p_R, & x > 0 \end{cases}, \quad u_x = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}, \tag{6.36a–c}$$

where 'L' and 'R' denote the left and right states. A simple example is a 'shock-tube' problem in which gas at different pressures is at rest on either side of a partition, which is released at $t = 0$.

It can be shown that the initial discontinuity resolves generically into three simple waves. The inner one is a contact discontinuity while the outer ones are shocks or rarefaction waves (see below).

The initial data define no natural length scale for the Riemann problem, but they do allow a characteristic velocity scale c to be defined (although not uniquely). The result is a similarity solution in which variables depend on x and t only through the dimensionless combination $\xi = x/ct$.

Unlike the unphysical rarefaction shock, the rarefaction wave (or expansion wave) is a non-dissipative, homentropic, continuous simple wave in which $\nabla \cdot \boldsymbol{u} > 0$. If we seek a similarity solution $v = v(\xi)$ of the inviscid Burgers equation $v_t + v v_x = 0$ we find $v = x/t$ (or the trivial solution $v = \text{const.}$). The characteristics form an expansion fan (figure 10).

The '+' rarefaction wave has $u + v_s = x/t$ and $R_- = u - 2v_s/(\gamma - 1) = \text{const.}$, determined by the undisturbed right-hand state. The '−' rarefaction wave has $u - v_s = x/t$ and $R_+ = u + 2v_s/(\gamma - 1) = \text{const.}$, determined by the undisturbed left-hand state. In each case u and v_s are linear functions of x/t and $\nabla \cdot \boldsymbol{u} = 2t^{-1}/(\gamma + 1) > 0$.

A typical outcome of a shock-tube problem consists of (from left to right): undisturbed region, rarefaction wave, uniform region, contact discontinuity, uniform region, shock, undisturbed region (figure 11).

In Godunov's method and related computational algorithms, the equations of AFD are advanced in time by solving (either exactly or approximately) a Riemann problem at each cell boundary.

Related examples: A.13–A.16.

7. Spherical blast waves: supernovae

Note: in this section (r, θ, ϕ) are spherical polar coordinates.

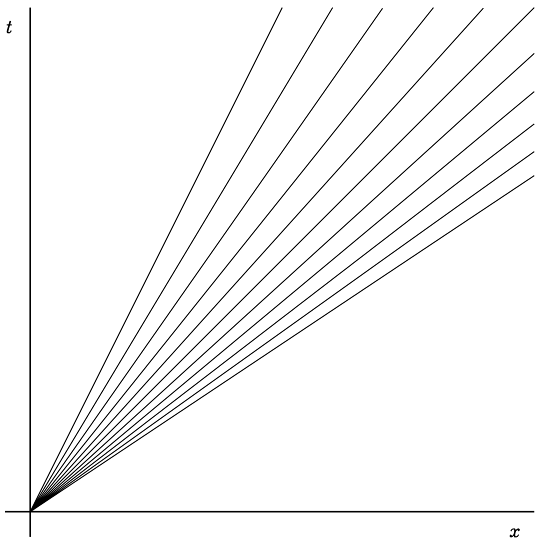

FIGURE 10. Expansion fan of characteristics in a rarefaction wave.

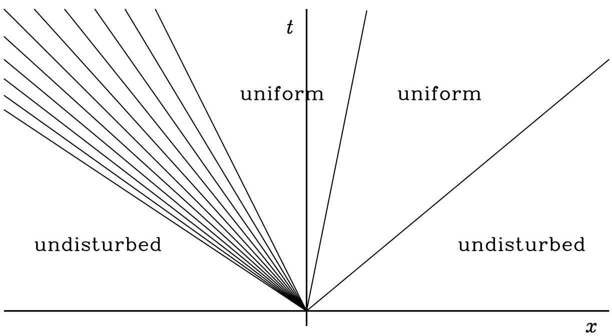

FIGURE 11. Typical outcome of a shock-tube problem. The two uniform regions are separated by a contact discontinuity. The other discontinuity is a shock.

7.1. *Introduction*

In a supernova, an energy of order 10^{51} erg (10^{44} J) is released into the interstellar medium. An expanding spherical blast wave is formed as the explosion sweeps up the surrounding gas. Several good examples of these supernova remnants are observed in the Galaxy, e.g. Tycho's supernova of 1572 and Kepler's supernova of 1604[9].

The effect is similar to a bomb. When photographs[10] (complete with length and time scales) were released of the first atomic bomb test in New Mexico in 1945, both Sedov[11] in the Soviet Union and Taylor[12] in the UK were able to work out the energy of the bomb (equivalent to approximately 20 kilotons of TNT), which was supposed to be a secret.

[9]See http://en.wikipedia.org/wiki/Supernova_remnant
[10]See http://www.atomicarchive.com/Photos/Trinity
[11]Leonid Ivanovitch Sedov (1907–1999), Russian.
[12]Sir Geoffrey Ingram Taylor (1886–1975), British.

We suppose that an energy E is released at $t=0$, $r=0$ and that the explosion is spherically symmetric. The external medium has density ρ_0 and pressure p_0. In the Sedov–Taylor phase of the explosion, the pressure $p \gg p_0$. Then a strong shock is formed and the external pressure p_0 can be neglected (formally set to zero). Gravity is also negligible in the dynamics.

7.2. *Governing equations*

The equations governing the spherically symmetric flow of a perfect gas, with radial velocity $u_r = u(r, t)$, may be written as

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\rho &= -\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u), \\ \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)u &= -\frac{1}{\rho}\frac{\partial p}{\partial r}, \\ \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln(p\rho^{-\gamma}) &= 0. \end{aligned}\right\} \tag{7.1}$$

These imply the total energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1}\right)u\right] = 0. \tag{7.2}$$

The shock is at $r = R(t)$, and the shock speed is $\dot{R}$. The equations are to be solved in $0 < r < R$ with the strong shock conditions (6.33) at $r = R$:

$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0, \quad u = \frac{2\dot{R}}{\gamma+1}, \quad p = \frac{2\rho_0\dot{R}^2}{\gamma+1}. \tag{7.3a–c}$$

The total energy of the explosion is

$$E = \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) 4\pi r^2\,\mathrm{d}r = \text{const.}, \tag{7.4}$$

the thermal energy of the external medium being negligible.

7.3. *Dimensional analysis*

The dimensional parameters of the problem on which the solution might depend are E and ρ_0. Their dimensions are

$$[E] = ML^2T^{-2}, \quad [\rho_0] = ML^{-3}. \tag{7.5a,b}$$

Together, they do not define a characteristic length scale, so the explosion is 'scale free' or 'self-similar'. If the dimensional analysis includes the time t since the explosion, however, we can find a time-dependent characteristic length scale. The radius of the shock must be

$$R = \alpha\left(\frac{Et^2}{\rho_0}\right)^{1/5}, \tag{7.6}$$

where α is a dimensionless constant to be determined.

7.4. *Similarity solution*

The self-similarity of the explosion is expressed using the dimensionless similarity variable $\xi = r/R(t)$. The solution has the form

$$\rho = \rho_0\, \tilde{\rho}(\xi), \quad u = \dot{R}\, \tilde{u}(\xi), \quad p = \rho_0 \dot{R}^2\, \tilde{p}(\xi), \tag{7.7a–c}$$

where $\tilde{\rho}(\xi)$, $\tilde{u}(\xi)$ and $\tilde{p}(\xi)$ are dimensionless functions to be determined. The meaning of this type of solution is that the graph of u versus r, for example, has a constant shape but both axes of the graph are rescaled as time proceeds and the shock expands.

7.5. *Dimensionless equations*

We substitute these forms into (7.1) and cancel the dimensional factors to obtain

$$\left.\begin{aligned}
(\tilde{u} - \xi)\tilde{\rho}' &= -\frac{\tilde{\rho}}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2\tilde{u}), \\
(\tilde{u} - \xi)\tilde{u}' - \frac{3}{2}\tilde{u} &= -\frac{\tilde{p}'}{\tilde{\rho}}, \\
(\tilde{u} - \xi)\left(\frac{\tilde{p}'}{\tilde{p}} - \frac{\gamma\tilde{\rho}'}{\tilde{\rho}}\right) - 3 &= 0.
\end{aligned}\right\} \tag{7.8}$$

Similarly, the strong shock conditions are that

$$\tilde{\rho} = \frac{\gamma+1}{\gamma-1}, \quad \tilde{u} = \frac{2}{\gamma+1}, \quad \tilde{p} = \frac{2}{\gamma+1} \tag{7.9a–c}$$

at $\xi = 1$, while the total energy integral provides a normalization condition,

$$1 = \frac{16\pi}{25}\alpha^5 \int_0^1 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right)\xi^2\, \mathrm{d}\xi, \tag{7.10}$$

that will ultimately determine the value of α.

7.6. *First integral*

The one-dimensional conservative form of the total energy equation (7.2) is

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial r} = 0, \tag{7.11}$$

where

$$q = r^2\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right), \quad F = r^2\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1}\right)u. \tag{7.12a,b}$$

In dimensionless form,

$$q = \rho_0 R^2 \dot{R}^2\, \tilde{q}(\xi), \quad F = \rho_0 R^2 \dot{R}^3\, \tilde{F}(\xi), \tag{7.13a,b}$$

with

$$\tilde{q} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right), \quad \tilde{F} = \xi^2\left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma\tilde{p}}{\gamma-1}\right)\tilde{u}. \tag{7.14a,b}$$

We substitute the forms (7.13) into the energy equation (7.11) to find

$$-\xi\tilde{q}' - \tilde{q} + \tilde{F}' = 0, \tag{7.15}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\tilde{F} - \xi\tilde{q}) = 0. \tag{7.16}$$

Thus

$$\tilde{F} - \xi\tilde{q} = \text{const.} = 0 \tag{7.17}$$

for a solution that is finite at $\xi = 0$. This equation can be solved for $\tilde{p}$:

$$\tilde{p} = \frac{(\gamma - 1)\tilde{\rho}\tilde{u}^2(\xi - \tilde{u})}{2(\gamma\tilde{u} - \xi)}. \tag{7.18}$$

Note that this solution is compatible with the shock boundary conditions. Having found a first integral, we can now dispense with (e.g.) the thermal energy equation.

Let $\tilde{u} = v\xi$. We now have

$$(v - 1)\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}\ln\xi} = -\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} - 3v, \tag{7.19}$$

$$(v - 1)\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} + \frac{1}{\tilde{\rho}\xi^2}\frac{\mathrm{d}}{\mathrm{d}\ln\xi}\left[\frac{(\gamma - 1)\tilde{\rho}\xi^2 v^2(1 - v)}{2(\gamma v - 1)}\right] = \frac{3}{2}v. \tag{7.20}$$

Eliminate $\tilde{\rho}$:

$$\frac{\mathrm{d}v}{\mathrm{d}\ln\xi} = \frac{v(\gamma v - 1)[5 - (3\gamma - 1)v]}{\gamma(\gamma + 1)v^2 - 2(\gamma + 1)v + 2}. \tag{7.21}$$

Invert and split into partial fractions:

$$\frac{\mathrm{d}\ln\xi}{\mathrm{d}v} = -\frac{2}{5v} + \frac{\gamma(\gamma - 1)}{(2\gamma + 1)(\gamma v - 1)} + \frac{13\gamma^2 - 7\gamma + 12}{5(2\gamma + 1)[5 - (3\gamma - 1)v]}. \tag{7.22}$$

The solution is

$$\xi \propto v^{-2/5}(\gamma v - 1)^{(\gamma-1)/(2\gamma+1)}[5 - (3\gamma - 1)v]^{-(13\gamma^2-7\gamma+12)/5(2\gamma+1)(3\gamma-1)}. \tag{7.23}$$

Now

$$\begin{aligned}\frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}v} &= -\frac{1}{v - 1} - \frac{3v}{v - 1}\frac{\mathrm{d}\ln\xi}{\mathrm{d}v} \\ &= \frac{2}{(2 - \gamma)(1 - v)} + \frac{3\gamma}{(2\gamma + 1)(\gamma v - 1)} - \frac{13\gamma^2 - 7\gamma + 12}{(2 - \gamma)(2\gamma + 1)[5 - (3\gamma - 1)v]}.\end{aligned} \tag{7.24}$$

The solution is

$$\tilde{\rho} \propto (1 - v)^{-2/(2-\gamma)}(\gamma v - 1)^{3/(2\gamma+1)}[5 - (3\gamma - 1)v]^{(13\gamma^2-7\gamma+12)/(2-\gamma)(2\gamma+1)(3\gamma-1)}. \tag{7.25}$$

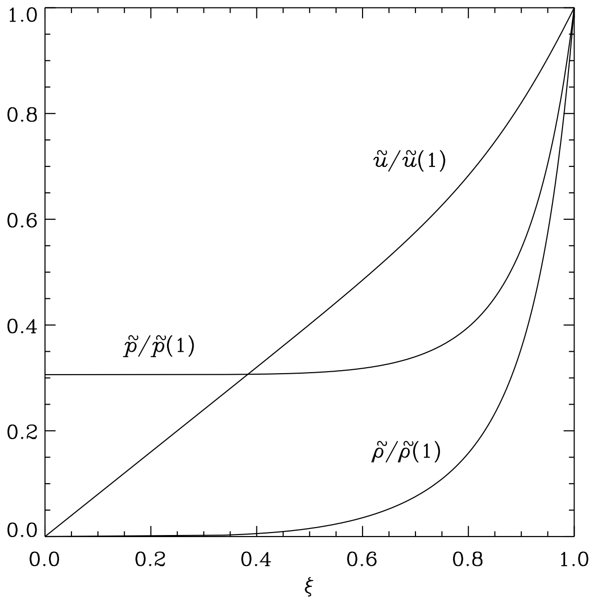

FIGURE 12. Sedov's solution for a spherical blast wave in the case $\gamma = 5/3$.

For example, in the case $\gamma = 5/3$:

$$\xi \propto v^{-2/5}\left(\frac{5v}{3}-1\right)^{2/13}(5-4v)^{-82/195}, \tag{7.26}$$

$$\tilde{\rho} \propto (1-v)^{-6}\left(\frac{5v}{3}-1\right)^{9/13}(5-4v)^{82/13}. \tag{7.27}$$

To satisfy $v = 2/(\gamma+1) = 3/4$ and $\tilde{\rho} = (\gamma+1)/(\gamma-1) = 4$ at $\xi = 1$:

$$\xi = \left(\frac{4v}{3}\right)^{-2/5}\left(\frac{20v}{3}-4\right)^{2/13}\left(\frac{5}{2}-2v\right)^{-82/195}, \tag{7.28}$$

$$\tilde{\rho} = 4\,(4-4v)^{-6}\left(\frac{20v}{3}-4\right)^{9/13}\left(\frac{5}{2}-2v\right)^{82/13}. \tag{7.29}$$

Then, from the first integral,

$$\tilde{p} = \frac{3}{4}\left(\frac{4v}{3}\right)^{6/5}(4-4v)^{-5}\left(\frac{5}{2}-2v\right)^{82/15}. \tag{7.30}$$

In this solution (figure 12), ξ ranges from 0 to 1, and v from 3/5 to 3/4. The normalization integral (numerically) yields $\alpha \approx 1.152$.

7.7. *Applications*

Some rough estimates are as follows:

(i) For a supernova: $E \sim 10^{51}$ erg, $\rho_0 \sim 10^{-24}$ g cm^{-3}. Then $R \approx 6$ pc and $\dot{R} \approx$ 2000 km s^{-1} at $t = 1000$ year.

(ii) For the 1945 New Mexico explosion: $E \approx 8 \times 10^{20}$ erg, $\rho_0 \approx 1.2 \times 10^{-3}$ g cm^{-3}. Then $R \approx 100$ m and $\dot{R} \approx 4$ km s^{-1} at $t = 0.01$ s.

The similarity method is useful in a very wide range of nonlinear problems. In this case it reduced partial differential equations to integrable ordinary differential equations.

Related example: A.17.

8. Spherically symmetric steady flows: stellar winds and accretion

Note: in this section (r, θ, ϕ) are spherical polar coordinates.

8.1. *Introduction*

Many stars, including the Sun, lose mass through a stellar wind. The gas must be sufficiently hot to escape from the star's gravitational field. Gravitating bodies can also accrete gas from the interstellar medium. The simplest models of these processes neglect the effects of rotation or magnetic fields and involve a steady, spherically symmetric flow.

8.2. *Basic equations*

We consider a purely radial flow, either away from or towards a body of mass M. The gas is perfect and non-self-gravitating, so $\Phi = -GM/r$. The fluid variables are functions of r only, and the only velocity component is $u_r = u(r)$.

Mass conservation for such a flow implies that the mass flux

$$4\pi r^2 \rho u = -\dot{M} = \text{const.} \tag{8.1}$$

If $u > 0$ (a stellar wind), $-\dot{M}$ is the mass loss rate. If $u < 0$ (an accretion flow), $\dot{M}$ is the mass accretion rate. We ignore the secular change in the mass M, which would otherwise violate the steady nature of the flow.

The thermal energy equation (assuming $u \neq 0$) implies homentropic flow:

$$p = K\rho^\gamma, \qquad K = \text{const.} \tag{8.2}$$

The equation of motion has only one component:

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r}. \tag{8.3}$$

Alternatively, we can use the integral form (Bernoulli's equation):

$$\frac{1}{2}u^2 + \Phi + h = B = \text{const.}, \qquad h = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho} = \frac{v_s^2}{\gamma - 1}. \tag{8.4}$$

In highly subsonic flow the $u^2/2$ term on the left-hand side of Bernoulli's equation is negligible and the gas is quasi-hydrostatic. In highly supersonic flow the h term is negligible and the flow is quasi-ballistic (freely falling). As discussed below, we are usually interested in transonic solutions that pass smoothly from subsonic to supersonic flow.

Our aim is to solve for $u(r)$, and to determine $\dot{M}$ if possible. At what rate does a star lose mass through a wind, or a black hole accrete mass from the surrounding medium?

8.3. *First treatment*

We first use the differential form of the equation of motion. Rewrite the pressure gradient using the other two equations:

$$-\frac{\mathrm{d}p}{\mathrm{d}r} = -p\frac{\mathrm{d}\ln p}{\mathrm{d}r} = -\gamma p\frac{\mathrm{d}\ln \rho}{\mathrm{d}r} = \rho v_s^2\left(\frac{2}{r} + \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right). \tag{8.5}$$

Equation (8.3), multiplied by u/ρ, becomes

$$(u^2 - v_s^2)\frac{\mathrm{d}u}{\mathrm{d}r} = u\left(\frac{2v_s^2}{r} - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right). \tag{8.6}$$

A critical point (sonic point) occurs at any radius $r = r_s$ where $|u| = v_s$. For the flow to pass smoothly from subsonic to supersonic, the right-hand side must vanish at the sonic point:

$$\frac{2v_{ss}^2}{r_s} - \frac{GM}{r_s^2} = 0. \tag{8.7}$$

Evaluate Bernoulli's equation (8.4) at the sonic point:

$$\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right) v_{ss}^2 - \frac{GM}{r_s} = B. \tag{8.8}$$

We deduce that

$$v_{ss}^2 = \frac{2(\gamma-1)}{(5-3\gamma)}B, \qquad r_s = \frac{(5-3\gamma)}{4(\gamma-1)}\frac{GM}{B}. \tag{8.9a,b}$$

There is a unique transonic solution, which exists only for $1 \leqslant \gamma < 5/3$. (The case $\gamma = 1$ can be treated separately or by taking a limit.)

Now evaluate $\dot{M}$ at the sonic point:

$$|\dot{M}| = 4\pi r_s^2 \rho_s v_{ss}. \tag{8.10}$$

8.4. *Second treatment*

We now use Bernoulli's equation instead of the equation of motion.

Introduce the local Mach number $\mathcal{M} = |u|/v_s$. Then

$$4\pi r^2 \rho v_s \mathcal{M} = |\dot{M}|, \qquad v_s^2 = \gamma K \rho^{\gamma-1}. \tag{8.11a,b}$$

Eliminate ρ to obtain

$$v_s = (\gamma K)^{1/(\gamma+1)}\left(\frac{|\dot{M}|}{4\pi r^2 \mathcal{M}}\right)^{(\gamma-1)/(\gamma+1)}. \tag{8.12}$$

Bernoulli's equation (8.4) is

$$\frac{1}{2}v_s^2\mathcal{M}^2 - \frac{GM}{r} + \frac{v_s^2}{\gamma-1} = B. \tag{8.13}$$

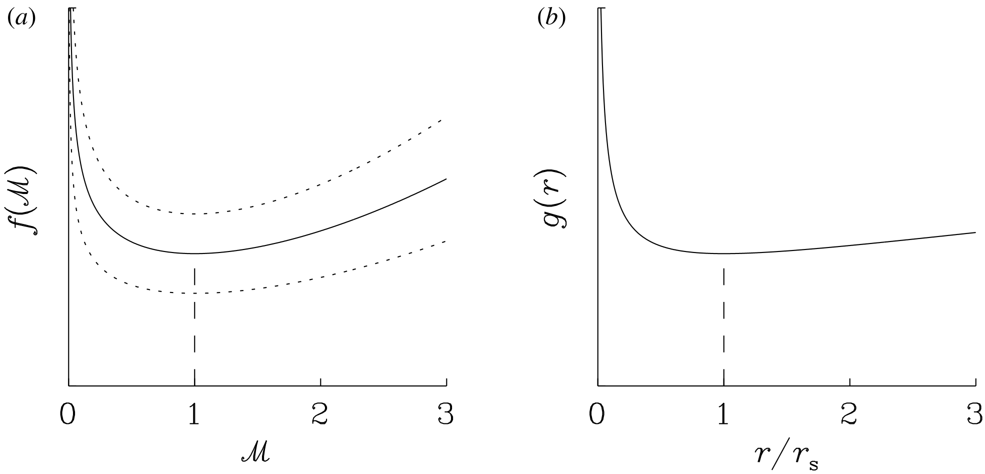

FIGURE 13. Shapes of the functions $f(\mathcal{M})$ and $g(r)$ for the case $\gamma = 4/3$. Only if $\dot{M}$ is equal to the critical value at which the minima of f and g coincide (solid line, left panel) does a smooth transonic solution exist.

Substitute for v_s and separate the variables:

$$(\gamma K)^{2/(\gamma+1)} \left(\frac{|\dot{M}|}{4\pi} \right)^{2(\gamma-1)/(\gamma+1)} \left[\frac{\mathcal{M}^{4/(\gamma+1)}}{2} + \frac{\mathcal{M}^{-2(\gamma-1)/(\gamma+1)}}{\gamma - 1} \right]$$
$$= Br^{4(\gamma-1)/(\gamma+1)} + GMr^{-(5-3\gamma)/(\gamma+1)}. \tag{8.14}$$

This equation is of the form $f(\mathcal{M}) = g(r)$. Assume that $1 < \gamma < 5/3$ and $B > 0$. (If $B < 0$ then the flow cannot reach infinity.) Then each of f and g is the sum of a positive power and a negative power, with positive coefficients. $f(\mathcal{M})$ has a minimum at $\mathcal{M} = 1$, while $g(r)$ has a minimum at

$$r = \frac{(5 - 3\gamma)}{4(\gamma - 1)} \frac{GM}{B}, \tag{8.15}$$

which is the sonic radius r_s identified previously. A smooth passage through the sonic point is possible only if $|\dot{M}|$ has a special value, so that the minima of f and g are equal. If $|\dot{M}|$ is too large then the solution does not work for all r. If it is too small then the solution remains subsonic (or supersonic) for all r, which may not agree with the boundary conditions (figure 13).

The $(r, \mathcal{M})$ plane shows an X-type critical point at $(r_s, 1)$ (figure 14).

For $r \ll r_s$ the subsonic solution is close to a hydrostatic atmosphere. The supersonic solution is close to free fall.

For $r \gg r_s$ the subsonic solution approaches a uniform state ($p =$ const., $\rho =$ const.). The supersonic solution is close to $u =$ const. (so $\rho \propto r^{-2}$).

8.5. *Stellar wind*

For a stellar wind the appropriate solution is subsonic (quasi-hydrostatic) at small r and supersonic (coasting) at large r. Parker (1958)[13] first presented this simplified

[13]Eugene Newman Parker (1927–), American.

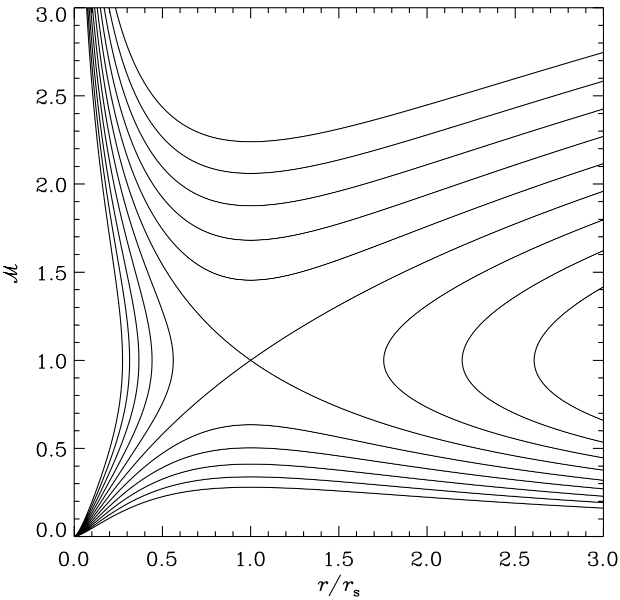

FIGURE 14. Solution curves for a stellar wind or accretion flow in the case $\gamma = 4/3$, showing an X-type critical point at the sonic radius and at Mach number $\mathcal{M} = 1$.

model for the solar wind. The mass loss rate can be determined from the properties of the quasi-hydrostatic part of the solution, e.g. the density and temperature at the base of the solar corona. A completely hydrostatic solution is unacceptable unless the external medium can provide a significant non-zero pressure. Subsonic solutions with $|\dot{M}|$ less than the critical value are usually unacceptable for similar reasons. (In fact the interstellar medium does arrest the supersonic solar wind in a termination shock well beyond Pluto's orbit.)

8.6. *Accretion*

In spherical or Bondi (1952)[14] accretion we consider a gas that is uniform and at rest at infinity (with pressure p_0 and density ρ_0). Then $B = v_{s0}^2/(\gamma - 1)$ and $v_{ss}^2 = 2v_{s0}^2/(5-3\gamma)$. The appropriate solution is subsonic (uniform) at large r and supersonic (freely falling) at small r. If the accreting object has a dense surface (a star rather than a black hole) then the accretion flow will be arrested by a shock above the surface.

The accretion rate of the critical solution is

$$\dot{M} = 4\pi r_s^2 \rho_s v_{ss} = 4\pi r_s^2 \rho_0 v_{s0} \left(\frac{v_{ss}}{v_{s0}}\right)^{(\gamma+1)/(\gamma-1)} = f(\gamma)\dot{M}_B, \tag{8.16}$$

where

$$\dot{M}_B = \frac{\pi G^2 M^2 \rho_0}{v_{s0}^3} = 4\pi r_a^2 \rho_0 v_{s0} \tag{8.17}$$

is the characteristic Bondi accretion rate and

$$f(\gamma) = \left(\frac{2}{5-3\gamma}\right)^{(5-3\gamma)/2(\gamma-1)} \tag{8.18}$$

[14] Sir Hermann Bondi (1919–2005), Austrian–British.

is a dimensionless factor. Here

$$r_a = \frac{GM}{2v_{s0}^2} \tag{8.19}$$

is the nominal accretion radius, roughly the radius within which the mass M captures the surrounding medium into a supersonic inflow.

Exercise: show that

$$\lim_{\gamma\to 1} f(\gamma) = \mathrm{e}^{3/2}, \qquad \lim_{\gamma\to 5/3} f(\gamma) = 1. \tag{8.20a,b}$$

(However, although the case $\gamma = 1$ admits a sonic point, the important case $\gamma = 5/3$ does not.)

At different times in its life a star may gain mass from, or lose mass to, its environment. Currently the Sun is losing mass at an average rate of approximately $2\times 10^{-14}\ M_\odot\ \mathrm{yr}^{-1}$. If it were not doing so, it could theoretically accrete at the Bondi rate of approximately $3\times 10^{-15}\ M_\odot\ \mathrm{yr}^{-1}$ from the interstellar medium.

Related examples: A.17–A.19.

9. Axisymmetric rotating magnetized flows: astrophysical jets

Note: in this section (r, ϕ, z) are cylindrical polar coordinates.

9.1. *Introduction*

Stellar winds and jets from accretion discs are examples of outflows in which rotation and magnetic fields have important or essential roles. Using cylindrical polar coordinates (r, ϕ, z), we examine steady $(\partial/\partial t = 0)$, axisymmetric $(\partial/\partial\phi = 0)$ models based on the equations of ideal MHD.

9.2. *Representation of an axisymmetric magnetic field*

The solenoidal condition for an axisymmetric magnetic field is

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0. \tag{9.1}$$

We may write

$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad B_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{9.2a,b}$$

where $\psi(r, z)$ is the magnetic flux function (figure 15). This is related to the magnetic vector potential by $\psi = rA_\phi$. The magnetic flux contained inside the circle ($r = \text{const.}$, $z = \text{const.}$) is

$$\int_0^r B_z(r', z)\, 2\pi r'\, \mathrm{d}r' = 2\pi\psi(r, z), \tag{9.3}$$

plus an arbitrary constant that can be set to zero.

Since $\boldsymbol{B}\cdot\nabla\psi = 0$, ψ labels the magnetic field lines or their surfaces of revolution, known as magnetic surfaces. The magnetic field may be written in the form

$$\boldsymbol{B} = \nabla\psi\times\nabla\phi + B_\phi\,\boldsymbol{e}_\phi = \left[-\frac{1}{r}\boldsymbol{e}_\phi\times\nabla\psi\right] + \left[B_\phi\,\boldsymbol{e}_\phi\right]. \tag{9.4}$$

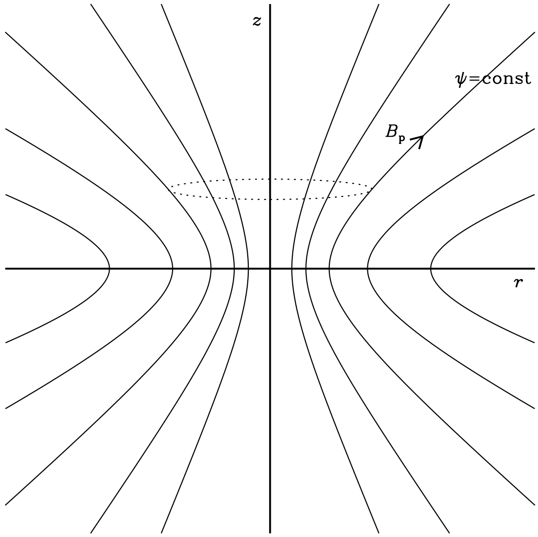

FIGURE 15. Magnetic flux function and poloidal magnetic field.

The two square brackets represent the poloidal (meridional) and toroidal (azimuthal) parts of the magnetic field:

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \, \boldsymbol{e}_\phi. \tag{9.5}$$

Note that

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{B}_p = 0. \tag{9.6}$$

Similarly, one can write the velocity in the form

$$\boldsymbol{u} = \boldsymbol{u}_p + u_\phi \, \boldsymbol{e}_\phi, \tag{9.7}$$

although $\nabla \cdot \boldsymbol{u}_p \neq 0$ in general.

9.3. *Mass loading and angular velocity*

The steady induction equation in ideal MHD,

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0}, \tag{9.8}$$

implies

$$\boldsymbol{u} \times \boldsymbol{B} = -\boldsymbol{E} = \nabla \varPhi_e, \tag{9.9}$$

where $\varPhi_e$ is the electrostatic potential. Now

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{B} &= (\boldsymbol{u}_p + u_\phi \, \boldsymbol{e}_\phi) \times (\boldsymbol{B}_p + B_\phi \, \boldsymbol{e}_\phi) \\ &= \left[\boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_p - B_\phi \boldsymbol{u}_p)\right] + \left[\boldsymbol{u}_p \times \boldsymbol{B}_p\right]. \end{aligned} \tag{9.10}$$

For an axisymmetric solution with $\partial \varPhi_e / \partial \phi = 0$, we have

$$\boldsymbol{u}_p \times \boldsymbol{B}_p = \boldsymbol{0}, \tag{9.11}$$

i.e. the poloidal velocity is parallel to the poloidal magnetic field[15]. Let

$$\rho \boldsymbol{u}_p = k\boldsymbol{B}_p, \tag{9.12}$$

where k is the mass loading, i.e. the ratio of mass flux to magnetic flux.

The steady equation of mass conservation is

$$0 = \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot (\rho \boldsymbol{u}_p) = \nabla \cdot (k\boldsymbol{B}_p) = \boldsymbol{B}_p \cdot \nabla k. \tag{9.13}$$

Therefore

$$k = k(\psi), \tag{9.14}$$

i.e. k is a surface function, constant on each magnetic surface.

We now have

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{e}_\phi \times (u_\phi \boldsymbol{B}_p - B_\phi \boldsymbol{u}_p) = \left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} \right) \nabla\psi. \tag{9.15}$$

Taking the curl of this equation, we find

$$\boldsymbol{0} = \nabla \left(\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} \right) \times \nabla\psi. \tag{9.16}$$

Therefore

$$\frac{u_\phi}{r} - \frac{kB_\phi}{r\rho} = \omega, \tag{9.17}$$

where $\omega(\psi)$ is another surface function, known as the angular velocity of the magnetic surface.

The complete velocity field may be written in the form

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega \boldsymbol{e}_\phi, \tag{9.18}$$

i.e. the total velocity is parallel to the total magnetic field in a frame of reference rotating with angular velocity ω. It is useful to think of the fluid being constrained to move along the field line like a bead on a rotating wire.

9.4. *Entropy*

The steady thermal energy equation,

$$\boldsymbol{u} \cdot \nabla s = 0, \tag{9.19}$$

implies that $\boldsymbol{B}_p \cdot \nabla s = 0$ and so

$$s = s(\psi) \tag{9.20}$$

is another surface function.

[15]It is possible to consider a more general situation in which rE_ϕ is equal to a non-zero constant. In this case there is a steady drift across the field lines and a steady transport of poloidal magnetic flux. However, such a possibility is best considered in the context of non-ideal MHD, which allows both advective and diffusive transport of magnetic flux and angular momentum.

9.5. *Angular momentum*

The azimuthal component of the equation of motion is

$$\left.\begin{aligned} \rho\left(\boldsymbol{u}_p\cdot\nabla u_\phi+\frac{u_r u_\phi}{r}\right)=\frac{1}{\mu_0}\left(\boldsymbol{B}_p\cdot\nabla B_\phi+\frac{B_r B_\phi}{r}\right)\\ \frac{1}{r}\rho\boldsymbol{u}_p\cdot\nabla(ru_\phi)-\frac{1}{\mu_0 r}\boldsymbol{B}_p\cdot\nabla(rB_\phi)=0\\ \frac{1}{r}\boldsymbol{B}_p\cdot\nabla\left(kru_\phi-\frac{rB_\phi}{\mu_0}\right)=0, \end{aligned}\right\} \tag{9.21}$$

and so

$$ru_\phi=\frac{rB_\phi}{\mu_0 k}+\ell, \tag{9.22}$$

where

$$\ell=\ell(\psi) \tag{9.23}$$

is another surface function, the angular momentum invariant. This is the angular momentum removed in the outflow per unit mass, although part of the torque is carried by the magnetic field.

9.6. *The Alfvén surface*

Define the poloidal Alfvén number (cf. the Mach number)

$$A=\frac{u_p}{v_{ap}}. \tag{9.24}$$

Then

$$A^2=\frac{\mu_0\rho u_p^2}{B_p^2}=\frac{\mu_0 k^2}{\rho}, \tag{9.25}$$

and so $A\propto\rho^{-1/2}$ on each magnetic surface.

Consider the two equations

$$\frac{u_\phi}{r}=\frac{kB_\phi}{r\rho}+\omega,\qquad ru_\phi=\frac{rB_\phi}{\mu_0 k}+\ell. \tag{9.26a,b}$$

Eliminate B_ϕ to obtain

$$u_\phi=\frac{r^2\omega-A^2\ell}{r(1-A^2)}=\left(\frac{1}{1-A^2}\right)r\omega+\left(\frac{A^2}{A^2-1}\right)\frac{\ell}{r}. \tag{9.27}$$

For $A\ll 1$ we have

$$u_\phi\approx r\omega, \tag{9.28}$$

i.e. the fluid is in uniform rotation, corotating with the magnetic surface. For $A\gg 1$ we have

$$u_\phi\approx\frac{\ell}{r}, \tag{9.29}$$

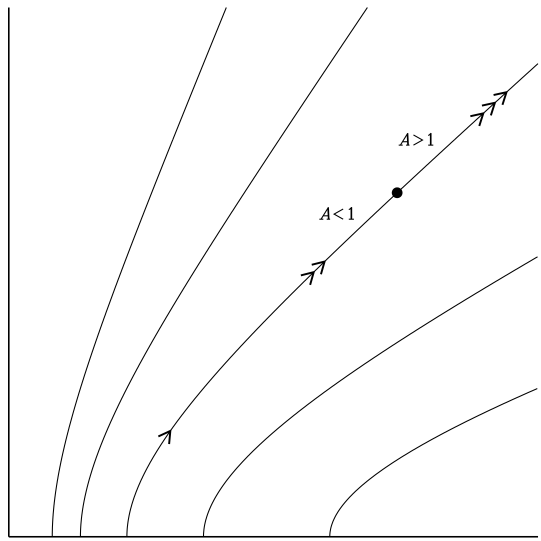

FIGURE 16. Acceleration through an Alfvén point along a poloidal magnetic field line, leading to angular momentum loss and magnetic braking.

i.e. the fluid conserves its specific angular momentum. The point $r = r_a(\psi)$ where $A = 1$ is the Alfvén point. The locus of Alfvén points for different magnetic surfaces forms the Alfvén surface. To avoid a singularity there we require

$$\ell = r_a^2 \omega. \tag{9.30}$$

Typically the outflow will start at low velocity in high-density material, where $A \ll 1$. We can therefore identify ω as the angular velocity $u_\phi/r = \Omega_0$ of the footpoint $r = r_0$ of the magnetic field line at the source of the outflow. It will then accelerate smoothly through an Alfvén surface and become super-Alfvénic ($A > 1$). If mass is lost at a rate $\dot{M}$ in the outflow, angular momentum is lost at a rate $\dot{M}\ell = \dot{M} r_a^2 \Omega_0$. In contrast, in a hydrodynamic outflow, angular momentum is conserved by fluid elements and is therefore lost at a rate $\dot{M} r_0^2 \Omega_0$. A highly efficient removal of angular momentum occurs if the Alfvén radius is large compared to the footpoint radius. This effect is the magnetic lever arm. The loss of angular momentum through a stellar wind is called magnetic braking (figure 16). In the case of the Sun, the Alfvén radius is approximately between 20 and $30\,R_\odot$.

9.7. *The Bernoulli function*

The total energy equation for a steady flow is

$$\nabla \cdot \left[\rho \boldsymbol{u} \left(\frac{1}{2} u^2 + \Phi + h \right) + \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} \right] = 0. \tag{9.31}$$

Now since

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega \boldsymbol{e}_\phi, \tag{9.32}$$

we have

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} = -r\omega \boldsymbol{e}_\phi \times \boldsymbol{B} = -r\omega \boldsymbol{e}_\phi \times \boldsymbol{B}_p, \tag{9.33}$$

which is purely poloidal. Thus

$$(\boldsymbol{E}\times\boldsymbol{B})_p=\boldsymbol{E}\times(B_\phi\,\boldsymbol{e}_\phi)=-r\omega B_\phi\boldsymbol{B}_p. \tag{9.34}$$

The total energy equation is therefore

$$\left.\begin{aligned}\nabla\cdot\left[k\boldsymbol{B}_p\left(\frac{1}{2}u^2+\varPhi+h\right)-\frac{r\omega B_\phi}{\mu_0}\boldsymbol{B}_p\right]=0\\ \boldsymbol{B}_p\cdot\nabla\left[k\left(\frac{1}{2}u^2+\varPhi+h-\frac{r\omega B_\phi}{\mu_0 k}\right)\right]=0\\ \frac{1}{2}u^2+\varPhi+h-\frac{r\omega B_\phi}{\mu_0 k}=\varepsilon,\end{aligned}\right\} \tag{9.35}$$

where

$$\varepsilon=\varepsilon(\psi) \tag{9.36}$$

is another surface function, the energy invariant.

An alternative invariant is

$$\begin{aligned}\tilde{\varepsilon}&=\varepsilon-\ell\omega\\ &=\frac{1}{2}u^2+\varPhi+h-\frac{r\omega B_\phi}{\mu_0 k}-\left(ru_\phi-\frac{rB_\phi}{\mu_0 k}\right)\omega\\ &=\frac{1}{2}u^2+\varPhi+h-ru_\phi\omega\\ &=\frac{1}{2}u_p^2+\frac{1}{2}(u_\phi-r\omega)^2+\varPhi_{cg}+h,\end{aligned} \tag{9.37}$$

where

$$\varPhi_{cg}=\varPhi-\tfrac{1}{2}\omega^2r^2 \tag{9.38}$$

is the centrifugal–gravitational potential associated with the magnetic surface. One can then see that $\tilde{\varepsilon}$ is the Bernoulli function of the flow in the frame rotating with angular velocity ω. In this frame the flow is strictly parallel to the magnetic field and the field therefore does no work because $\boldsymbol{J}\times\boldsymbol{B}\perp\boldsymbol{B}$ and so $\boldsymbol{J}\times\boldsymbol{B}\perp(\boldsymbol{u}-r\omega\boldsymbol{e}_\phi)$.

9.8. *Summary*

We have been able to integrate almost all of the MHD equations, reducing them to a set of algebraic relations on each magnetic surface. If the poloidal magnetic field $\boldsymbol{B}_p$ (or, equivalently, the flux function ψ) is specified in advance, these algebraic equations are sufficient to determine the complete solution on each magnetic surface separately, although we must also (i) specify the initial conditions at the source of the outflow and (ii) ensure that the solution passes smoothly through critical points where the flow speed matches the speeds of slow and fast magnetoacoustic waves (see Example A.21).

The component of the equation of motion perpendicular to the magnetic surfaces is the only piece of information not yet used. This ‘transfield’ or ‘Grad–Shafranov’ equation ultimately determines the equilibrium shape of the magnetic surfaces. It is a very complicated nonlinear partial differential equation for $\psi(r,z)$ and cannot be reduced to simple terms. We do not consider it here.

9.9. *Acceleration from the surface of an accretion disc*

We now consider the launching of an outflow from a thin accretion disc. The angular velocity $\Omega(r)$ of the disc corresponds approximately to circular Keplerian orbital motion around a central mass M:

$$\Omega \approx \left(\frac{GM}{r^3}\right)^{1/2}. \tag{9.39}$$

If the flow starts essentially from rest in high-density material ($A \ll 1$), we have

$$\omega \approx \Omega, \tag{9.40}$$

i.e. the angular velocity of the magnetic surface is the angular velocity of the disc at the footpoint of the field line. In the sub-Alfvénic region we have

$$\tilde{\varepsilon} \approx \tfrac{1}{2}u_p^2 + \Phi_{cg} + h. \tag{9.41}$$

As in the case of stellar winds, if the gas is hot (comparable to the escape temperature) an outflow can be driven by thermal pressure. Of more interest here is the possibility of a dynamically driven outflow. For a 'cold' wind the enthalpy makes a negligible contribution in this equation. Whether the flow accelerates or not above the disc then depends on the variation of the centrifugal–gravitational potential along the field line.

Consider a Keplerian disc in a point-mass potential. Let the footpoint of the field line be at $r = r_0$, and let the angular velocity of the field line be

$$\omega = \Omega_0 = \left(\frac{GM}{r_0^3}\right)^{1/2}, \tag{9.42}$$

as argued above. Then

$$\Phi_{cg} = -GM(r^2 + z^2)^{-1/2} - \frac{1}{2}\frac{GM}{r_0^3}r^2. \tag{9.43}$$

In units such that $r_0 = 1$, the equation of the equipotential passing through the footpoint (r_0, z) is

$$(r^2 + z^2)^{-1/2} + \frac{r^2}{2} = \frac{3}{2}. \tag{9.44}$$

This can be rearranged into the form

$$z^2 = \frac{(2-r)(r-1)^2(r+1)^2(r+2)}{(3-r^2)^2}. \tag{9.45}$$

Close to the footpoint $(1, 0)$ we have

$$z^2 \approx 3(r-1)^2 \quad \Rightarrow \quad z \approx \pm\sqrt{3}(r-1). \tag{9.46}$$

The footpoint lies at a saddle point of Φ_{cg} (figure 17). If the inclination of the field line to the vertical, i, at the surface of the disc exceeds 30°, the flow is accelerated without thermal assistance[16]. This is magnetocentrifugal acceleration.

The critical equipotential has an asymptote at $r = r_0\sqrt{3}$. The field line must continue to expand sufficiently in the radial direction in order to sustain the magnetocentrifugal acceleration.

[16] A more detailed investigation (Ogilvie & Livio 1998) shows that the Keplerian rotation of the disc is modified by the Lorentz force. There is then a potential barrier $\propto B^4$ to be overcome by the outflow, even when $i > 30°$, which means that some thermal assistance is required, especially when the disc is strongly magnetized.

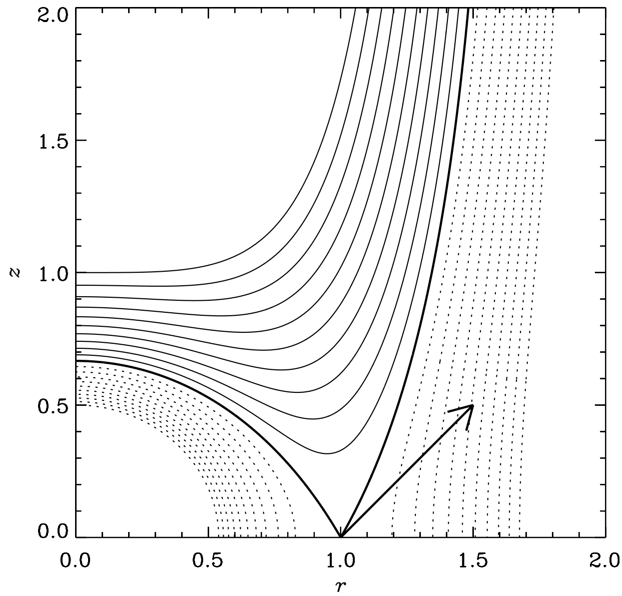

FIGURE 17. Contours of Φ_{cg}, in units such that $r_0 = 1$. The downhill directions are indicated by dotted contours. If the inclination of the poloidal magnetic field to the vertical direction at the surface of the disc exceeds 30°, gas is accelerated along the field lines away from the disc.

9.10. *Magnetically driven accretion*

To allow a quantity of mass ΔM_{acc} to be accreted from radius r_0, its orbital angular momentum $r_0^2 \Omega_0 \, \Delta M_{acc}$ must be removed. The angular momentum removed by a quantity of mass ΔM_{jet} flowing out in a magnetized jet from radius r_0 is $\ell \, \Delta M_{jet} = r_a^2 \Omega_0 \, \Delta M_{jet}$. Therefore accretion can in principle be driven by an outflow, with

$$\frac{\dot{M}_{acc}}{\dot{M}_{jet}} \approx \frac{r_a^2}{r_0^2}. \tag{9.47}$$

The magnetic lever arm allows an efficient removal of angular momentum if the Alfvén radius is large compared to the footpoint radius.

Related examples: A.20, A.21.

10. Lagrangian formulation of ideal MHD

10.1. *The Lagrangian viewpoint*

In § 11 we will discuss waves and instabilities in differentially rotating astrophysical bodies. Here we develop a general theory of disturbances to fluid flows that makes use of the conserved quantities in ideal fluids and takes a Lagrangian approach.

The flow of a fluid can be considered as a time-dependent map,

$$\boldsymbol{a} \mapsto \boldsymbol{x}(\boldsymbol{a}, t), \tag{10.1}$$

where $\boldsymbol{a}$ is the position vector of a fluid element at some initial time t_0, and $\boldsymbol{x}$ is its position vector at time t. The Cartesian components of $\boldsymbol{a}$ are examples of Lagrangian variables, labelling the fluid element. The components of $\boldsymbol{x}$ are Eulerian variables, labelling a fixed point in space. Any fluid property X (scalar, vector or tensor) can be regarded as a function of either Lagrangian or Eulerian variables:

$$X = X^L(\boldsymbol{a}, t) = X^E(\boldsymbol{x}, t). \tag{10.2}$$

The Lagrangian time-derivative is simply

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{a}}, \tag{10.3}$$

and the velocity of the fluid is

$$\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} = \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{a}}. \tag{10.4}$$

The aim of a Lagrangian formulation of ideal MHD is to derive a nonlinear evolutionary equation for the function $\boldsymbol{x}(\boldsymbol{a}, t)$. The dynamics is Hamiltonian in character and can be derived from a Lagrangian function or action principle. There are many similarities with classical field theories.

10.2. *The deformation tensor*

Introduce the deformation tensor of the flow,

$$F_{ij} = \frac{\partial x_i}{\partial a_j}, \tag{10.5}$$

and its determinant

$$F = \det(F_{ij}) \tag{10.6}$$

and inverse

$$G_{ij} = \frac{\partial a_i}{\partial x_j}. \tag{10.7}$$

We note the following properties. First, the derivative

$$\frac{\partial F}{\partial F_{ij}} = C_{ij} = F G_{ji} = \frac{1}{2}\epsilon_{ik\ell}\epsilon_{jmn}F_{km}F_{\ell n} \tag{10.8}$$

is equal to the cofactor C_{ij} of the matrix element F_{ij}. (This follows from the fact that the determinant can be expanded as the sum of the products of any row's elements with their cofactors, which do not depend on that row's elements.) Second, the matrix of cofactors has zero divergence on its second index:

$$\frac{\partial C_{ij}}{\partial a_j} = \frac{\partial}{\partial a_j}\left(\frac{1}{2}\epsilon_{ik\ell}\epsilon_{jmn}\frac{\partial x_k}{\partial a_m}\frac{\partial x_\ell}{\partial a_n}\right) = 0. \tag{10.9}$$

(This follows because the resulting derivative involves the contraction of the antisymmetric tensor ϵ_{jmn} with expressions that are symmetric in either jm or jn.)

Now

$$\frac{\mathrm{D}F_{ij}}{\mathrm{D}t} = \frac{\partial u_i}{\partial a_j} \tag{10.10}$$

and, according to (10.8),

$$\frac{\mathrm{D}\ln F}{\mathrm{D}t} = G_{ji}\frac{\mathrm{D}F_{ij}}{\mathrm{D}t} = \frac{\partial a_j}{\partial x_i}\frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial x_i} = \boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{10.11}$$

10.3. *Geometrical conservation laws*

The equations of ideal MHD comprise the equation of motion and three ‘geometrical’ conservation laws. These are the conservation of specific entropy (thermal energy equation),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{10.12}$$

the conservation of mass,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u}, \tag{10.13}$$

and the conservation of magnetic flux (induction equation),

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right)\cdot\nabla\boldsymbol{u}. \tag{10.14}$$

These equations describe the pure advection of fluid properties in a manner equivalent to the advection of various geometrical objects. The specific entropy is advected as a simple scalar, so that its numerical value is conserved by material points. The specific volume $v = 1/\rho$ is advected in the same way as a material volume element $\mathrm{d}V$. The quantity $\boldsymbol{B}/\rho$ is advected in the same way as a material line element $\delta\boldsymbol{x}$. Equivalently, the mass element $\delta m = \rho\delta V$ and the magnetic flux element $\delta\Phi = \boldsymbol{B}\cdot\delta\boldsymbol{S}$ satisfy $\mathrm{D}\delta m/\mathrm{D}t = 0$ and $\mathrm{D}\delta\Phi/\mathrm{D}t = 0$, where $\delta\boldsymbol{S}$ is a material surface element. All three conservation laws can be integrated exactly in Lagrangian variables.

The exact solutions of (10.12)–(10.14) are then

$$s^L(\boldsymbol{a},t) = s_0(\boldsymbol{a}), \quad \rho^L(\boldsymbol{a},t) = F^{-1}\rho_0(\boldsymbol{a}), \quad B_i^L(\boldsymbol{a},t) = F^{-1}F_{ij}B_{j0}(\boldsymbol{a}), \tag{10.15a--c}$$

where s_0, ρ_0, and $\boldsymbol{B}_0$ are the initial values at time t_0. The verification of (10.14) is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{B_i}{\rho}\right) = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{F_{ij}B_{j0}}{\rho_0}\right) = \frac{\partial u_i}{\partial a_j}\frac{B_{j0}}{\rho_0} = \frac{\partial u_i}{\partial x_k}F_{kj}\frac{B_{j0}}{\rho_0} = \left(\frac{B_k}{\rho}\right)\frac{\partial u_i}{\partial x_k}. \tag{10.16}$$

Note that the advected quantities at time t depend only on the initial values and on the instantaneous mapping $\boldsymbol{a}\mapsto\boldsymbol{x}$, not on the intermediate history of the flow. The ‘memory’ of an ideal fluid is perfect.

10.4. *The Lagrangian of ideal MHD*

Newtonian dynamics can be formulated using Hamilton’s principle of stationary action,

$$\delta\int L\,\mathrm{d}t = 0, \tag{10.17}$$

where the Lagrangian L is the difference between the kinetic energy and the potential energy of the system. By analogy, we may expect the Lagrangian of ideal MHD to take the form

$$L = \int \mathcal{L}\,\mathrm{d}V, \tag{10.18}$$

where (for a non-self-gravitating fluid)

$$\mathcal{L} = \rho\left(\frac{1}{2}u^2 - \Phi - e - \frac{B^2}{2\mu_0\rho}\right) \tag{10.19}$$

is the Lagrangian density.

To verify this, we assume that the equation of state can be written in the form $e = e(v, s)$, where $v = \rho^{-1}$ is the specific volume. Since $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}v$, we have

$$\left(\frac{\partial e}{\partial v}\right)_s = -p, \quad \left(\frac{\partial^2 e}{\partial v^2}\right)_s = \frac{\gamma p}{v} \tag{10.20a,b}$$

(strictly, γ should be Γ_1 here).

We then write the action using Lagrangian variables,

$$S[\boldsymbol{x}] = \iint \tilde{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{F})\,\mathrm{d}^3\boldsymbol{a}\,\mathrm{d}t, \tag{10.21}$$

with

$$\tilde{\mathcal{L}} = \rho_0 \left[\frac{1}{2}u^2 - \Phi(\boldsymbol{x}) - e(F\rho_0^{-1}, s_0) - \frac{F^{-1}F_{ij}B_{j0}F_{ik}B_{k0}}{2\mu_0\rho_0}\right]. \tag{10.22}$$

This uses the fact that F is the Jacobian determinant of the transformation $\boldsymbol{a} \mapsto \boldsymbol{x}$, or, equivalently, that $\rho\,\mathrm{d}^3\boldsymbol{x} = \rho_0\,\mathrm{d}^3\boldsymbol{a} = \mathrm{d}m$ is an invariant mass measure. $\tilde{\mathcal{L}}$ is now expressed in terms of the function $\boldsymbol{x}(\boldsymbol{a}, t)$ and its derivatives with respect to time ($\boldsymbol{u}$) and space ($\boldsymbol{F}$). The Euler–Lagrange equation for the variational principle $\delta S = 0$ is

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial \tilde{\mathcal{L}}}{\partial u_i} + \frac{\partial}{\partial a_j}\frac{\partial \tilde{\mathcal{L}}}{\partial F_{ij}} - \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = 0. \tag{10.23}$$

The straightforward terms are

$$\frac{\partial \tilde{\mathcal{L}}}{\partial u_i} = \rho_0 u_i, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial x_i} = -\rho_0 \frac{\partial \Phi}{\partial x_i}. \tag{10.24a,b}$$

Now

$$\begin{aligned}
\frac{\partial \tilde{\mathcal{L}}}{\partial F_{ij}} &= \left(p + \frac{B^2}{2\mu_0}\right)\frac{\partial F}{\partial F_{ij}} - \frac{F^{-1}B_{j0}F_{ik}B_{k0}}{\mu_0} \\
&= C_{ij}\left(p + \frac{B^2}{2\mu_0}\right) - \frac{1}{\mu_0}C_{kj}B_iB_k \\
&= -C_{kj}V_{ik},
\end{aligned} \tag{10.25}$$

where

$$V_{ik} = -\left(p + \frac{B^2}{2\mu_0}\right)\delta_{ik} + \frac{B_iB_k}{\mu_0} \tag{10.26}$$

is the stress tensor due to pressure and the magnetic field.

The Euler–Lagrange equation is therefore

$$\rho_0 \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\rho_0 \frac{\partial \Phi}{\partial x_i} + \frac{\partial}{\partial a_j}(C_{kj}V_{ik}). \tag{10.27}$$

Using (10.9) we note that

$$\frac{\partial}{\partial a_j}(C_{kj}V_{ik}) = C_{kj}\frac{\partial V_{ik}}{\partial a_j} = FG_{jk}\frac{\partial V_{ik}}{\partial a_j} = F\frac{\partial V_{ik}}{\partial x_k}. \tag{10.28}$$

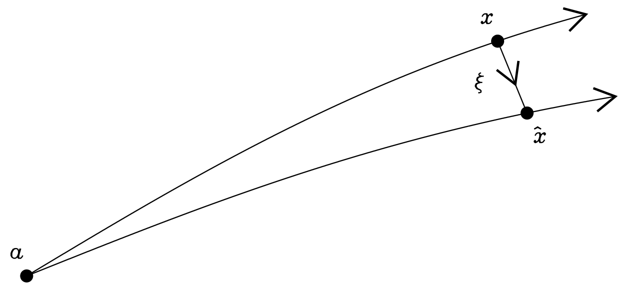

FIGURE 18. The Lagrangian displacement of a fluid element.

On dividing through by F, the Euler–Lagrange equation becomes the desired equation of motion,

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\nabla\Phi + \nabla\cdot\boldsymbol{V}. \tag{10.29}$$

In this construction, the fluid flow is viewed as a field $\boldsymbol{x}(\boldsymbol{a}, t)$ on the initial state space. Ideal MHD is seen as a nonlinear field theory derived from an action principle. When considering stability problems, it is useful to generalize this concept and to view a perturbed flow as a field on an unperturbed flow.

10.5. *The Lagrangian displacement*

Now consider two different flows, $\boldsymbol{x}(\boldsymbol{a}, t)$ and $\hat{\boldsymbol{x}}(\boldsymbol{a}, t)$, for which the initial values of the advected quantities, s_0, ρ_0 and $\boldsymbol{B}_0$, are the same. The two deformation tensors are related by the chain rule,

$$\hat{F}_{ij} = J_{ik}F_{kj}, \tag{10.30}$$

where

$$J_{ik} = \frac{\partial\hat{x}_i}{\partial x_k} \tag{10.31}$$

is the Jacobian matrix of the map $\boldsymbol{x} \mapsto \hat{\boldsymbol{x}}$. Similarly,

$$\hat{F} = JF, \tag{10.32}$$

where

$$J = \det(J_{ij}) \tag{10.33}$$

is the Jacobian determinant. The advected quantities in the two flows are therefore related by the composition of maps,

$$\left.\begin{aligned} \hat{s}^L(\boldsymbol{a}, t) &= s^L(\boldsymbol{a}, t), \\ \hat{\rho}^L(\boldsymbol{a}, t) &= J^{-1}\rho^L(\boldsymbol{a}, t), \\ \hat{B}_i^L(\boldsymbol{a}, t) &= J^{-1}J_{ij}B_j^L(\boldsymbol{a}, t). \end{aligned}\right\} \tag{10.34}$$

The Lagrangian displacement (figure 18) is the relative displacement of the fluid element in the two flows,

$$\boldsymbol{\xi} = \hat{\boldsymbol{x}} - \boldsymbol{x}. \tag{10.35}$$

Thus (with $\xi_{i,j}=\partial\xi_i/\partial x_j$)

$$
\begin{aligned}
J_{ij} &= \delta_{ij}+\xi_{i,j},\\
J &= \tfrac{1}{6}\epsilon_{ijk}\epsilon_{lmn}J_{il}J_{jm}J_{kn}\\
&= \tfrac{1}{6}\epsilon_{ijk}\epsilon_{lmn}(\delta_{il}+\xi_{i,l})(\delta_{jm}+\xi_{j,m})(\delta_{kn}+\xi_{k,n})\\
&= \tfrac{1}{6}\epsilon_{ijk}\epsilon_{ijk}+\tfrac{1}{2}\epsilon_{ijk}\epsilon_{ijn}\xi_{k,n}+\tfrac{1}{2}\epsilon_{ijk}\epsilon_{imn}\xi_{j,m}\xi_{k,n}+O(\xi^3)\\
&= 1+\xi_{k,k}+\tfrac{1}{2}(\xi_{j,j}\xi_{k,k}-\xi_{j,k}\xi_{k,j})+O(\xi^3).
\end{aligned}
\tag{10.36}
$$

From the binomial theorem,

$$
J^{-1}=1-\xi_{k,k}+\tfrac{1}{2}(\xi_{j,j}\xi_{k,k}+\xi_{j,k}\xi_{k,j})+O(\xi^3).
\tag{10.37}
$$

10.6. *The Lagrangian for a perturbed flow*

We now use the action principle to construct a theory for the displacement as a field on the unperturbed flow: $\boldsymbol{\xi}=\boldsymbol{\xi}(\boldsymbol{x},t)$. The action for the perturbed flow is

$$
\hat{S}[\boldsymbol{\xi}]=\iint\hat{\mathcal{L}}\left(\boldsymbol{\xi},\frac{\partial\boldsymbol{\xi}}{\partial t},\boldsymbol{\nabla}\boldsymbol{\xi}\right)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t,
\tag{10.38}
$$

where

$$
\begin{aligned}
\hat{\mathcal{L}} &= \rho\left(\frac{1}{2}\hat{u}^2-\hat{\Phi}-\hat{e}-\frac{\hat{B}^2}{2\mu_0\hat{\rho}}\right)\\
&= \rho\left[\frac{1}{2}\hat{u}^2-\Phi(\boldsymbol{x}+\boldsymbol{\xi})-e(J\rho^{-1},s)-\frac{J^{-1}J_{ij}B_jJ_{ik}B_k}{2\mu_0\rho}\right],
\end{aligned}
\tag{10.39}
$$

with

$$
\hat{\boldsymbol{u}}=\frac{\mathrm{D}\hat{\boldsymbol{x}}}{\mathrm{D}t}=\boldsymbol{u}+\frac{\partial\boldsymbol{\xi}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}.
\tag{10.40}
$$

The Euler–Lagrange equation for the variational principle $\delta\hat{S}=0$ is

$$
\frac{\partial}{\partial t}\frac{\partial\hat{\mathcal{L}}}{\partial\dot{\xi}_i}+\frac{\partial}{\partial x_j}\frac{\partial\hat{\mathcal{L}}}{\partial\xi_{i,j}}-\frac{\partial\hat{\mathcal{L}}}{\partial\xi_i}=0,
\tag{10.41}
$$

where $\dot{\xi}_i=\partial\xi_i/\partial t$. We expand the various terms of $\hat{\mathcal{L}}$ in powers of $\boldsymbol{\xi}$:

$$
\begin{aligned}
\frac{1}{2}\rho\hat{u}^2 &= \frac{1}{2}\rho u^2+\rho u_i\left(\frac{\partial\xi_i}{\partial t}+u_j\frac{\partial\xi_i}{\partial x_j}\right)+\frac{1}{2}\rho\left(\frac{\partial\xi_i}{\partial t}+u_j\frac{\partial\xi_i}{\partial x_j}\right)\left(\frac{\partial\xi_i}{\partial t}+u_k\frac{\partial\xi_i}{\partial x_k}\right),\\
-\rho\Phi(\boldsymbol{x}+\boldsymbol{\xi}) &= -\rho\left[\Phi(\boldsymbol{x})+\xi_i\frac{\partial\Phi}{\partial x_i}+\frac{1}{2}\xi_i\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j}+O(\xi^3)\right],\\
-\rho\left(\hat{e}+\frac{\hat{B}^2}{2\mu_0\hat{\rho}}\right) &= -\left(\rho e+\frac{B^2}{2\mu_0}\right)-V_{ij}\frac{\partial\xi_i}{\partial x_j}-\frac{1}{2}V_{ijk\ell}\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_k}{\partial x_\ell}+O(\xi^3).
\end{aligned}
\tag{10.42}
$$

This last expression uses the fact that $\hat{e}$, $\hat{\boldsymbol{B}}$ and $\hat{\rho}$ depend only on $\nabla\boldsymbol{\xi}$ (through J and J_{ij}) and can therefore be expanded in a Taylor series in this quantity. A short calculation of this expansion gives

$$V_{ij} = -\left(p + \frac{B^2}{2\mu_0}\right)\delta_{ij} + \frac{B_i B_j}{\mu_0}, \tag{10.43}$$

which is the stress tensor used above, and

$$V_{ijk\ell} = \left[(\gamma-1)p + \frac{B^2}{2\mu_0}\right]\delta_{ij}\delta_{k\ell} + \left(p + \frac{B^2}{2\mu_0}\right)\delta_{i\ell}\delta_{jk} + \frac{1}{\mu_0}B_j B_\ell \delta_{ik} - \frac{1}{\mu_0}(B_i B_j \delta_{k\ell} + B_k B_\ell \delta_{ij}), \tag{10.44}$$

which has the symmetry

$$V_{ijk\ell} = V_{k\ell ij} \tag{10.45}$$

necessitated by its function in the Taylor series. We now have

$$\left.\begin{aligned} \frac{\partial\hat{\mathcal{L}}}{\partial\dot{\xi}_i} &= \rho u_i + \rho\frac{\mathrm{D}\xi_i}{\mathrm{D}t}, \\ \frac{\partial\hat{\mathcal{L}}}{\partial\xi_{i,j}} &= \rho u_i u_j + \rho u_j \frac{\mathrm{D}\xi_i}{\mathrm{D}t} - V_{ij} - V_{ijk\ell}\frac{\partial\xi_k}{\partial x_\ell} + O(\xi^2), \\ \frac{\partial\hat{\mathcal{L}}}{\partial\xi_i} &= -\rho\frac{\partial\Phi}{\partial x_i} - \rho\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + O(\xi^2). \end{aligned}\right\} \tag{10.46}$$

Since $\boldsymbol{\xi} = \boldsymbol{0}$ must be a solution of the Euler–Lagrange equation, it is no surprise that the terms independent of $\boldsymbol{\xi}$ cancel by virtue of the equation of motion of the unperturbed flow,

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j - V_{ij}) + \rho\frac{\partial\Phi}{\partial x_i} = 0. \tag{10.47}$$

The remaining terms are

$$\frac{\partial}{\partial t}\left(\rho\frac{\mathrm{D}\xi_i}{\mathrm{D}t}\right) + \frac{\partial}{\partial x_j}\left(\rho u_j\frac{\mathrm{D}\xi_i}{\mathrm{D}t} - V_{ijk\ell}\frac{\partial\xi_k}{\partial x_\ell}\right) + \rho\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + O(\xi^2) = 0, \tag{10.48}$$

or (making use of the equation of mass conservation)

$$\rho\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2} = \frac{\partial}{\partial x_j}\left(V_{ijk\ell}\frac{\partial\xi_k}{\partial x_\ell}\right) - \rho\xi_j\frac{\partial^2\Phi}{\partial x_i\partial x_j} + O(\xi^2). \tag{10.49}$$

This equation, which can be extended to any order in $\boldsymbol{\xi}$, provides the basis for a nonlinear perturbation theory for any flow in ideal MHD. In a linear theory we would neglect terms of $O(\xi^2)$.

10.7. *Notes on linear perturbations*

The Lagrangian perturbation ΔX of a quantity X is the difference in the values of the quantity in the two flows for the same fluid element,

$$\Delta X = \hat{X}^L(\boldsymbol{a}, t) - X^L(\boldsymbol{a}, t). \tag{10.50}$$

It follows that

$$\Delta s = 0, \quad \Delta \rho = -\rho \frac{\partial \xi_i}{\partial x_i} + O(\xi^2), \quad \Delta B_i = B_j \frac{\partial \xi_i}{\partial x_j} - B_i \frac{\partial \xi_j}{\partial x_j} + O(\xi^2), \tag{10.51a–c}$$

and

$$\Delta u_i = \frac{\mathrm{D}\xi_i}{\mathrm{D}t}. \tag{10.52}$$

In linear theory, $\nabla \boldsymbol{\xi}$ is small and terms higher than the first order are neglected. Thus

$$\Delta s = 0, \quad \Delta \rho = -\rho \nabla \cdot \boldsymbol{\xi}, \quad \Delta \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{\xi} - (\nabla \cdot \boldsymbol{\xi}) \boldsymbol{B}. \tag{10.53a–c}$$

In linear theory, $\Delta s = 0$ implies

$$\Delta p = \frac{\gamma p}{\rho} \Delta \rho = -\gamma p \nabla \cdot \boldsymbol{\xi}. \tag{10.54}$$

The Eulerian perturbation δX of a quantity X is the difference in the values of the quantity in the two flows at the same point in space,

$$\delta X = \hat{X}^E(\boldsymbol{x}, t) - X^E(\boldsymbol{x}, t). \tag{10.55}$$

By Taylor's theorem,

$$\Delta X = \delta X + \boldsymbol{\xi} \cdot \nabla X + O(\xi^2), \tag{10.56}$$

and so, in linear theory,

$$\delta X = \Delta X - \boldsymbol{\xi} \cdot \nabla X. \tag{10.57}$$

Thus

$$\left. \begin{aligned} \delta \rho &= -\rho \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \rho, \\ \delta p &= -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p, \\ \delta \boldsymbol{B} &= \boldsymbol{B} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{B} - (\nabla \cdot \boldsymbol{\xi}) \boldsymbol{B}, \end{aligned} \right\} \tag{10.58}$$

exactly as was obtained in § 5 for perturbations of magnetostatic equilibria.

The relation

$$\delta \boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} \tag{10.59}$$

can be used to introduce the Lagrangian displacement into a linear theory derived using Eulerian perturbations. Only in the case of a static basic state, $\boldsymbol{u} = \boldsymbol{0}$, does this reduce to the simple relation $\delta \boldsymbol{u} = \partial \boldsymbol{\xi} / \partial t$.

11. Waves and instabilities in stratified rotating astrophysical bodies

11.1. *The energy principle*

For linear perturbations to a static equilibrium ($\boldsymbol{u}=\mathbf{0}$), the displacement satisfies

$$\rho \frac{\partial^2 \xi_i}{\partial t^2} = -\rho \frac{\partial \delta \Phi}{\partial x_i} - \rho \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \frac{\partial}{\partial x_j}\left(V_{ijk\ell} \frac{\partial \xi_k}{\partial x_\ell} \right), \tag{11.1}$$

where we now allow for self-gravitation through

$$\nabla^2 \delta\Phi = 4\pi G\, \delta\rho = -4\pi G \nabla \cdot (\rho \boldsymbol{\xi}). \tag{11.2}$$

We may write (11.1) in the form

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathcal{F}\boldsymbol{\xi}, \tag{11.3}$$

where $\mathcal{F}$ is a linear differential operator (or integro-differential if self-gravitation is taken into account). The force operator $\mathcal{F}$ can be shown to be self-adjoint with respect to the inner product,

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle = \int \rho \boldsymbol{\eta}^* \cdot \boldsymbol{\xi}\, \mathrm{d}V \tag{11.4}$$

if appropriate boundary conditions apply to the vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Let $\delta\Psi$ be the gravitational potential perturbation associated with the displacement $\boldsymbol{\eta}$, so $\nabla^2\delta\Psi = -4\pi G \nabla \cdot (\rho \boldsymbol{\eta})$. Then

$$\begin{aligned}
\langle \boldsymbol{\eta}, \mathcal{F}\boldsymbol{\xi} \rangle &= \int \left[-\rho \eta_i^* \frac{\partial \delta\Phi}{\partial x_i} - \rho \eta_i^* \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \eta_i^* \frac{\partial}{\partial x_j}\left(V_{ijk\ell} \frac{\partial \xi_k}{\partial x_\ell}\right)\right] \mathrm{d}V \\
&= \int \left[-\delta\Phi \frac{\nabla^2 \delta\Psi^*}{4\pi G} - \rho \xi_i \eta_j^* \frac{\partial^2 \Phi}{\partial x_i \partial x_j} - V_{ijk\ell} \frac{\partial \xi_k}{\partial x_\ell} \frac{\partial \eta_i^*}{\partial x_j}\right] \mathrm{d}V \\
&= \int \left[\frac{\nabla(\delta\Phi) \cdot \nabla(\delta\Psi^*)}{4\pi G} - \rho \xi_i \eta_j^* \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \xi_k \frac{\partial}{\partial x_\ell}\left(V_{ijk\ell} \frac{\partial \eta_i^*}{\partial x_j}\right)\right] \mathrm{d}V \\
&= \int \left[-\delta\Psi^* \frac{\nabla^2 \delta\Phi}{4\pi G} - \rho \xi_i \eta_j^* \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \xi_i \frac{\partial}{\partial x_j}\left(V_{k\ell ij} \frac{\partial \eta_k^*}{\partial x_\ell}\right)\right] \mathrm{d}V \\
&= \int \left[-\rho \xi_i \frac{\partial \delta\Psi^*}{\partial x_i} - \rho \xi_i \eta_j^* \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \xi_i \frac{\partial}{\partial x_j}\left(V_{ijk\ell} \frac{\partial \eta_k^*}{\partial x_\ell}\right)\right] \mathrm{d}V \\
&= \langle \mathcal{F}\boldsymbol{\eta}, \boldsymbol{\xi} \rangle.
\end{aligned} \tag{11.5}$$

Here the integrals are over all space. We assume that the exterior of the body is a medium of zero density in which the force-free limit of MHD holds and $\boldsymbol{B}$ decays sufficiently fast as $|\boldsymbol{x}| \to \infty$ that we may integrate freely by parts (using the divergence theorem) and ignore surface terms. We also assume that the body is isolated and self-gravitating, so that $\delta\Phi = O(r^{-1})$, or in fact $O(r^{-2})$ if $\delta M = 0$. We have used the symmetry properties of $\partial^2\Phi/\partial x_i \partial x_j$ and $V_{ijk\ell}$.

The functional

$$W[\boldsymbol{\xi}] = -\frac{1}{2}\langle \boldsymbol{\xi}, \mathcal{F}\boldsymbol{\xi} \rangle = \frac{1}{2}\int \left(-\frac{|\nabla \delta\Phi|^2}{4\pi G} + \rho \xi_i^* \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j} + V_{ijk\ell} \frac{\partial \xi_i^*}{\partial x_j} \frac{\partial \xi_k}{\partial x_\ell} \right) \mathrm{d}V \tag{11.6}$$

is therefore real and represents the change in potential energy associated with the displacement $\boldsymbol{\xi}$.

If the basic state is static, we may consider normal mode solutions of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}(\boldsymbol{x})\exp(-\mathrm{i}\omega t)\right], \tag{11.7}$$

for which we obtain

$$-\omega^2\tilde{\boldsymbol{\xi}} = \mathcal{F}\tilde{\boldsymbol{\xi}} \tag{11.8}$$

and

$$\omega^2 = -\frac{\langle\tilde{\boldsymbol{\xi}}, \mathcal{F}\tilde{\boldsymbol{\xi}}\rangle}{\langle\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\rangle} = \frac{2W[\tilde{\boldsymbol{\xi}}]}{\langle\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\rangle}. \tag{11.9}$$

Therefore ω^2 is real and we have either oscillations ($\omega^2 > 0$) or instability ($\omega^2 < 0$).

The above expression for ω^2 satisfies the usual Rayleigh–Ritz variational principle for self-adjoint eigenvalue problems. The eigenvalues ω^2 are the stationary values of $2W[\boldsymbol{\xi}]/\langle\boldsymbol{\xi}, \boldsymbol{\xi}\rangle$ among trial displacements $\boldsymbol{\xi}$ satisfying the boundary conditions. In particular, the lowest eigenvalue is the global minimum value of $2W[\boldsymbol{\xi}]/\langle\boldsymbol{\xi}, \boldsymbol{\xi}\rangle$. Therefore the equilibrium is unstable if and only if $W[\boldsymbol{\xi}]$ can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions. This is called the energy principle.

This discussion is incomplete because it assumes that the eigenfunctions form a complete set. In general a continuous spectrum, not associated with square-integrable modes, is also present. However, it can be shown that a necessary and sufficient condition for instability is that $W[\boldsymbol{\xi}]$ can be made negative as described above. Consider the equation for twice the energy of the perturbation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}]\right) &= \langle\ddot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}}, \ddot{\boldsymbol{\xi}}\rangle - \langle\dot{\boldsymbol{\xi}}, \mathcal{F}\boldsymbol{\xi}\rangle - \langle\boldsymbol{\xi}, \mathcal{F}\dot{\boldsymbol{\xi}}\rangle \\ &= \langle\mathcal{F}\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\rangle + \langle\dot{\boldsymbol{\xi}}, \mathcal{F}\boldsymbol{\xi}\rangle - \langle\dot{\boldsymbol{\xi}}, \mathcal{F}\boldsymbol{\xi}\rangle - \langle\mathcal{F}\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\rangle \\ &= 0.\end{aligned} \tag{11.10}$$

Therefore

$$\langle\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = \text{const.}, \tag{11.11}$$

where E is determined by the initial data $\boldsymbol{\xi}_0$ and $\dot{\boldsymbol{\xi}}_0$. If W is positive definite then the equilibrium is stable because $\boldsymbol{\xi}$ is limited by the constraint $W[\boldsymbol{\xi}] \leqslant E$.

Suppose that a (real) trial displacement $\boldsymbol{\eta}$ can be found for which

$$\frac{2W[\boldsymbol{\eta}]}{\langle\boldsymbol{\eta}, \boldsymbol{\eta}\rangle} = -\gamma^2, \tag{11.12}$$

where $\gamma > 0$. Then let the initial conditions be $\boldsymbol{\xi}_0 = \boldsymbol{\eta}$ and $\dot{\boldsymbol{\xi}}_0 = \gamma\boldsymbol{\eta}$ so that

$$\langle\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\rangle + 2W[\boldsymbol{\xi}] = 2E = 0. \tag{11.13}$$

Now let

$$a(t) = \ln\left(\frac{\langle\boldsymbol{\xi}, \boldsymbol{\xi}\rangle}{\langle\boldsymbol{\eta}, \boldsymbol{\eta}\rangle}\right) \tag{11.14}$$

so that

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2\langle \boldsymbol{\xi}, \dot{\boldsymbol{\xi}} \rangle}{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle} \tag{11.15}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2 a}{\mathrm{d}t^2} &= \frac{2(\langle \boldsymbol{\xi}, \mathcal{F}\boldsymbol{\xi} \rangle + \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle)\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle - 4\langle \boldsymbol{\xi}, \dot{\boldsymbol{\xi}} \rangle^2}{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^2} \\ &= \frac{2(-2W[\boldsymbol{\xi}] + \langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle)\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle - 4\langle \boldsymbol{\xi}, \dot{\boldsymbol{\xi}} \rangle^2}{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^2} \\ &= \frac{4(\langle \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} \rangle \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle - \langle \boldsymbol{\xi}, \dot{\boldsymbol{\xi}} \rangle^2)}{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^2} \\ &\geqslant 0 \end{aligned} \tag{11.16}$$

by the Cauchy–Schwarz inequality. Thus

$$\left.\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} \geqslant \dot{a}_0 = 2\gamma \\ a \geqslant 2\gamma t + a_0 = 2\gamma t. \end{aligned}\right\} \tag{11.17}$$

Therefore the disturbance with these initial conditions grows at least as fast as $\exp(\gamma t)$ and the equilibrium is unstable.

11.2. *Spherically symmetric star*

The simplest model of a star neglects rotation and magnetic fields and assumes a spherically symmetric hydrostatic equilibrium in which $\rho(r)$ and $p(r)$ satisfy

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\rho g, \tag{11.18}$$

with inward radial gravitational acceleration

$$g(r) = \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2} \int_0^r \rho(r')\, 4\pi r'^2 \, \mathrm{d}r'. \tag{11.19}$$

The stratification induced by gravity provides a non-uniform background for wave propagation.

In this case the linearized equation of motion is (cf. § 5)

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho \boldsymbol{\nabla} \delta\Phi - \delta\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} \delta p, \tag{11.20}$$

with $\delta\rho = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{\xi})$, $\nabla^2 \delta\Phi = 4\pi G\, \delta\rho$ and $\delta p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p$. For normal modes $\propto \exp(-\mathrm{i}\omega t)$,

$$\left.\begin{aligned} \rho\omega^2 \boldsymbol{\xi} = \rho \boldsymbol{\nabla} \delta\Phi + \delta\rho \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \delta p \\ \omega^2 \int_V \rho |\boldsymbol{\xi}|^2 \, \mathrm{d}V = \int_V \boldsymbol{\xi}^* \cdot (\rho \boldsymbol{\nabla} \delta\Phi + \delta\rho \boldsymbol{\nabla} \Phi + \boldsymbol{\nabla} \delta p) \, \mathrm{d}V, \end{aligned}\right\} \tag{11.21}$$

where V is the volume of the star. At the surface S of the star, we assume that ρ and p vanish. Then δp also vanishes on S (assuming that $\boldsymbol{\xi}$ and its derivatives are bounded).

The δp term can be integrated by parts as follows:

$$\begin{aligned}\int_V \boldsymbol{\xi}^* \cdot \nabla \delta p \,\mathrm{d}V &= -\int_V (\nabla \cdot \boldsymbol{\xi})^* \delta p \,\mathrm{d}V \\ &= \int_V \frac{1}{\gamma p} (\delta p + \boldsymbol{\xi} \cdot \nabla p)^* \delta p \,\mathrm{d}V \\ &= \int_V \left[\frac{|\delta p|^2}{\gamma p} + \frac{1}{\gamma p} (\boldsymbol{\xi}^* \cdot \nabla p)(-\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}) \right] \mathrm{d}V. \end{aligned} \tag{11.22}$$

The $\delta \rho$ term partially cancels with the above:

$$\begin{aligned}\int_V \boldsymbol{\xi}^* \cdot (\delta\rho \nabla \Phi) &= \int_V (-\boldsymbol{\xi}^* \cdot \nabla p) \frac{\delta \rho}{\rho} \,\mathrm{d}V \\ &= \int_V (\boldsymbol{\xi}^* \cdot \nabla p)(\nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla \ln \rho) \,\mathrm{d}V. \end{aligned} \tag{11.23}$$

Finally, the $\delta \Phi$ term can be transformed as in § 11.1 to give

$$\int_V \rho \boldsymbol{\xi}^* \cdot \nabla \delta \Phi \,\mathrm{d}V = -\int_\infty \frac{|\nabla \delta \Phi|^2}{4\pi G} \,\mathrm{d}V, \tag{11.24}$$

where the integral on the right-hand side is over all space. Thus

$$\begin{aligned}\omega^2 \int_V \rho |\boldsymbol{\xi}|^2 \,\mathrm{d}V &= -\int_\infty \frac{|\nabla \delta \Phi|^2}{4\pi G} \,\mathrm{d}V \\ &\quad + \int_V \left[\frac{|\delta p|^2}{\gamma p} - (\boldsymbol{\xi}^* \cdot \nabla p) \cdot \left(\frac{1}{\gamma} \boldsymbol{\xi} \cdot \nabla \ln p - \boldsymbol{\xi} \cdot \nabla \ln \rho \right) \right] \mathrm{d}V \\ &= -\int_\infty \frac{|\nabla \delta \Phi|^2}{4\pi G} \,\mathrm{d}V + \int_V \left(\frac{|\delta p|^2}{\gamma p} + \rho N^2 |\xi_r|^2 \right) \mathrm{d}V, \end{aligned} \tag{11.25}$$

where $N(r)$ is the Brunt–Väisälä frequency[17] (or buoyancy frequency) given by

$$N^2 = g \left(\frac{1}{\gamma} \frac{\mathrm{d} \ln p}{\mathrm{d}r} - \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} \right) \propto g \frac{\mathrm{d}s}{\mathrm{d}r}. \tag{11.26}$$

N is the frequency of oscillation of a fluid element that is displaced vertically in a stably stratified atmosphere if it maintains pressure equilibrium with its surroundings. The stratification is stable if the specific entropy increases outwards.

The integral expression for ω^2 satisfies the energy principle. There are three contributions to ω^2: the self-gravitational term (destabilizing), the acoustic term (stabilizing) and the buoyancy term (stabilizing if $N^2 > 0$).

If $N^2 < 0$ for any interval of r, a trial displacement can always be found such that $\omega^2 < 0$. This is done by localizing ξ_r in that interval and arranging the other components of $\boldsymbol{\xi}$ such that $\delta p = 0$. Therefore the star is unstable if $\partial s / \partial r < 0$ anywhere. This is Schwarzschild's criterion[18] for convective instability.

[17] Sir David Brunt (1886–1965), British. Vilho Väisälä (1889–1969), Finnish.

[18] Karl Schwarzschild (1873–1916), German.

11.3. *Modes of an incompressible sphere*

Note: in this subsection (r, θ, ϕ) are spherical polar coordinates.

Analytical solutions can be obtained in the case of a homogeneous incompressible 'star' of mass M and radius R which has

$$\rho = \left(\frac{3M}{4\pi R^3}\right) H(R - r), \tag{11.27}$$

where H is the Heaviside step function. For $r \leqslant R$ we have

$$g = \frac{GMr}{R^3}, \quad p = \frac{3GM^2(R^2 - r^2)}{8\pi R^6}. \tag{11.28a,b}$$

For an incompressible fluid,

$$\nabla \cdot \boldsymbol{\xi} = 0, \tag{11.29}$$

$$\delta\rho = -\boldsymbol{\xi} \cdot \nabla\rho = \xi_r \left(\frac{3M}{4\pi R^3}\right) \delta(r - R) \tag{11.30}$$

and

$$\nabla^2 \delta\Phi = 4\pi G\, \delta\rho = \xi_r \left(\frac{3GM}{R^3}\right) \delta(r - R), \tag{11.31}$$

while δp is indeterminate and is a variable independent of $\boldsymbol{\xi}$. The linearized equation of motion is

$$-\rho\omega^2\boldsymbol{\xi} = -\rho\nabla\delta\Phi - \nabla\delta p. \tag{11.32}$$

Therefore we have potential flow: $\boldsymbol{\xi} = \nabla U$, with $\nabla^2 U = 0$ and $-\rho\omega^2 U = -\rho\delta\Phi - \delta p$ in $r \leqslant R$. Appropriate solutions of Laplace's equation regular at $r = 0$ are the solid spherical harmonics (with arbitrary normalization)

$$U = r^\ell Y_\ell^m(\theta, \phi), \tag{11.33}$$

where ℓ and m are integers with $\ell \geqslant |m|$. Equation (11.31) also implies

$$\delta\Phi = \begin{cases} Ar^\ell Y_\ell^m, & r < R, \\ Br^{-\ell-1} Y_\ell^m, & r > R, \end{cases} \tag{11.34}$$

where A and B are constants to be determined. The matching conditions from (11.31) at $r = R$ are

$$[\delta\Phi] = 0, \quad \left[\frac{\partial\delta\Phi}{\partial r}\right] = \xi_r \left(\frac{3GM}{R^3}\right). \tag{11.35a,b}$$

Thus

$$BR^{-\ell-1} - AR^\ell = 0, \quad -(\ell + 1)BR^{-\ell-2} - \ell AR^{\ell-1} = \ell R^{\ell-1} \left(\frac{3GM}{R^3}\right), \tag{11.36a,b}$$

with solution

$$A = -\frac{\ell}{2\ell + 1} \left(\frac{3GM}{R^3}\right), \quad B = AR^{2\ell+1}. \tag{11.37a,b}$$

At $r=R$ the Lagrangian pressure perturbation should vanish:

$$\left.\begin{aligned}\Delta p=\delta p+\boldsymbol{\xi}\cdot\nabla p=0\\ \left(\frac{3M}{4\pi R^3}\right)\left[\omega^2R^\ell+\left(\frac{\ell}{2\ell+1}\right)\left(\frac{3GM}{R^3}\right)R^\ell\right]-\frac{3GM^2}{4\pi R^5}\ell R^{\ell-1}=0\\ \omega^2=\left(\ell-\frac{3\ell}{2\ell+1}\right)\frac{GM}{R^2}=\frac{2\ell(\ell-1)}{2\ell+1}\frac{GM}{R^3}.\end{aligned}\right\} \tag{11.38}$$

This result was obtained by Lord Kelvin. Since $\omega^2\geqslant 0$ the star is stable. Note that $\ell=0$ corresponds to $\boldsymbol{\xi}=\mathbf{0}$ and $\ell=1$ corresponds to $\boldsymbol{\xi}=\text{const.}$, which is a translational mode of zero frequency. The remaining modes are non-trivial and are called f modes (fundamental modes). These can be thought of as surface gravity waves, related to ocean waves for which $\omega^2=gk$. In the first expression for ω^2 above, the first term in brackets derives from surface gravity, while the second derives from self-gravity.

11.4. *The plane-parallel atmosphere*

The local dynamics of a stellar atmosphere can be studied in a Cartesian ('plane-parallel') approximation. The gravitational acceleration is taken to be constant (appropriate to an atmosphere) and in the $-z$ direction. For hydrostatic equilibrium,

$$\frac{\mathrm{d}p}{\mathrm{d}z}=-\rho g. \tag{11.39}$$

A simple example is an isothermal atmosphere in which $p=c_s^2\rho$ with $c_s=\text{const.}$:

$$\rho=\rho_0\,\mathrm{e}^{-z/H},\quad p=p_0\,\mathrm{e}^{-z/H}. \tag{11.40a,b}$$

$H=c_s^2/g$ is the isothermal scale height. The Brunt–Väisälä frequency in an isothermal atmosphere is given by

$$N^2=g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}z}-\frac{\mathrm{d}\ln\rho}{\mathrm{d}z}\right)=\left(1-\frac{1}{\gamma}\right)\frac{g}{H}, \tag{11.41}$$

which is constant and is positive for $\gamma>1$. An isothermal atmosphere is stably (subadiabatically) stratified if $\gamma>1$ and neutrally (adiabatically) stratified if $\gamma=1$.

A further example is a polytropic atmosphere in which $p\propto\rho^{1+1/m}$ in the undisturbed state, where m is a positive constant. In general $1+1/m$ differs from the adiabatic exponent γ of the gas. For hydrostatic equilibrium,

$$\rho^{1/m}\frac{\mathrm{d}\rho}{\mathrm{d}z}\propto-\rho g\quad\Rightarrow\quad\rho^{1/m}\propto-z, \tag{11.42}$$

if the top of the atmosphere is located at $z=0$, with vacuum above. Let

$$\rho=\rho_0\left(-\frac{z}{H}\right)^m, \tag{11.43}$$

for $z<0$, where ρ_0 and H are constants. Then

$$p=p_0\left(-\frac{z}{H}\right)^{m+1}, \tag{11.44}$$

where

$$p_0 = \frac{\rho_0 g H}{m+1} \tag{11.45}$$

to satisfy $\mathrm{d}p/\mathrm{d}z = -\rho g$. In this case

$$N^2 = \left(m - \frac{m+1}{\gamma}\right)\frac{g}{-z}. \tag{11.46}$$

We return to the linearized equations, looking for solutions of the form

$$\boldsymbol{\xi} = \mathrm{Re}\left[\tilde{\boldsymbol{\xi}}(z)\exp(-\mathrm{i}\omega t + \mathrm{i}\boldsymbol{k}_h \cdot \boldsymbol{x})\right], \quad \text{etc.,} \tag{11.47}$$

where 'h' denotes a horizontal vector (having only x and y components). Then

$$\left.\begin{aligned} -\rho\omega^2\boldsymbol{\xi}_h &= -\mathrm{i}\boldsymbol{k}_h\,\delta p, \\ -\rho\omega^2\xi_z &= -g\,\delta\rho - \frac{\mathrm{d}\,\delta p}{\mathrm{d}z}, \\ \delta\rho &= -\xi_z\frac{\mathrm{d}\rho}{\mathrm{d}z} - \rho\varDelta, \\ \delta p &= -\xi_z\frac{\mathrm{d}p}{\mathrm{d}z} - \gamma p\varDelta, \end{aligned}\right\} \tag{11.48}$$

where

$$\varDelta = \nabla\cdot\boldsymbol{\xi} = \mathrm{i}\boldsymbol{k}_h\cdot\boldsymbol{\xi}_h + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}. \tag{11.49}$$

The self-gravitation of the perturbation is neglected in the atmosphere: $\delta\varPhi = 0$ (the Cowling approximation). Note that only two z-derivatives of perturbation quantities occur: $\mathrm{d}\,\delta p/\mathrm{d}z$ and $\mathrm{d}\xi_z/\mathrm{d}z$. This is a second-order system of ordinary differential equations (ODEs), combined with algebraic equations.

We can easily eliminate $\boldsymbol{\xi}_h$ to obtain

$$\varDelta = -\frac{k_h^2}{\rho\omega^2}\,\delta p + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}, \tag{11.50}$$

where $k_h = |\boldsymbol{k}_h|$, and eliminate $\delta\rho$ to obtain

$$-\rho\omega^2\xi_z = g\xi_z\frac{\mathrm{d}\rho}{\mathrm{d}z} + \rho g\varDelta - \frac{\mathrm{d}\,\delta p}{\mathrm{d}z}. \tag{11.51}$$

We consider these two differential equations in combination with the remaining algebraic equation

$$\delta p = \rho g\xi_z - \gamma p\varDelta. \tag{11.52}$$

A first approach is to solve the algebraic equation for $\varDelta$ and substitute to obtain the two coupled ODEs

$$\left.\begin{aligned} \frac{\mathrm{d}\xi_z}{\mathrm{d}z} &= \frac{g}{v_s^2}\xi_z + \frac{1}{\rho v_s^2}\left(\frac{v_s^2 k_h^2}{\omega^2} - 1\right)\delta p, \\ \frac{\mathrm{d}\,\delta p}{\mathrm{d}z} &= \rho(\omega^2 - N^2)\xi_z - \frac{g}{v_s^2}\delta p. \end{aligned}\right\} \tag{11.53}$$

Note that $v_s^2 k_h^2$ is the square of the 'Lamb frequency', i.e. the (z-dependent) frequency of a horizontal sound wave of wavenumber k_h. In a short-wavelength (WKB) approximation, where $\xi_z \propto \exp\left[\mathrm{i}\int k_z(z)\,\mathrm{d}z\right]$ with $k_z \gg g/v_s^2$, the local dispersion relation derived from these ODEs is

$$v_s^2 k_z^2 = (\omega^2 - N^2)\left(1 - \frac{v_s^2 k_h^2}{\omega^2}\right). \tag{11.54}$$

Propagating waves ($k_z^2 > 0$) are possible when

$$\text{either } \omega^2 > \max(v_s^2 k_h^2, N^2) \quad \text{or} \quad 0 < \omega^2 < \min(v_s^2 k_h^2, N^2). \tag{11.55a,b}$$

The high-frequency branch describes p modes (acoustic waves: 'p' for pressure) while the low-frequency branch describes g modes (internal gravity waves: 'g' for gravity).

There is a special incompressible solution in which $\Delta = 0$, i.e. $\delta p = \rho g \xi_z$. This satisfies

$$\frac{\mathrm{d}\xi_z}{\mathrm{d}z} = \frac{g k_h^2}{\omega^2}\xi_z, \quad \frac{\mathrm{d}\xi_z}{\mathrm{d}z} = \frac{\omega^2}{g}\xi_z. \tag{11.56a,b}$$

For compatibility of these equations,

$$\frac{g k_h^2}{\omega^2} = \frac{\omega^2}{g} \quad \Rightarrow \quad \omega^2 = \pm g k_h. \tag{11.57}$$

The acceptable solution in which ξ_z decays with depth is

$$\omega^2 = g k_h, \quad \xi_z \propto \exp(k_h z). \tag{11.58a,b}$$

This is a surface gravity wave known in stellar oscillations as the f mode (fundamental mode). It is vertically evanescent.

The other wave solutions (p and g modes) can be found analytically in the case of a polytropic atmosphere.[19] We now eliminate variables in favour of Δ. First use the algebraic relation to eliminate δp:

$$\Delta = -\frac{g k_h^2}{\omega^2}\xi_z + \frac{v_s^2 k_h^2}{\omega^2}\Delta + \frac{\mathrm{d}\xi_z}{\mathrm{d}z}, \tag{11.59}$$

$$-\rho\omega^2\xi_z = \rho g \Delta - \rho g \frac{\mathrm{d}\xi_z}{\mathrm{d}z} + \frac{\mathrm{d}(\gamma p \Delta)}{\mathrm{d}z}. \tag{11.60}$$

Then eliminate $\mathrm{d}\xi_z/\mathrm{d}z$:

$$-\rho\omega^2\xi_z = -\rho g \left(\frac{g k_h^2}{\omega^2}\xi_z - \frac{v_s^2 k_h^2}{\omega^2}\Delta\right) + \frac{\mathrm{d}(\gamma p \Delta)}{\mathrm{d}z}. \tag{11.61}$$

Thus we have

$$\frac{\mathrm{d}\xi_z}{\mathrm{d}z} - \frac{g k_h^2}{\omega^2}\xi_z = \left(1 - \frac{v_s^2 k_h^2}{\omega^2}\right)\Delta, \tag{11.62}$$

$$-\omega^2\left(1 - \frac{g^2 k_h^2}{\omega^4}\right)\xi_z = \frac{1}{\rho}\frac{\mathrm{d}(\rho v_s^2 \Delta)}{\mathrm{d}z} + v_s^2 \frac{g k_h^2}{\omega^2}\Delta. \tag{11.63}$$

[19]Lamb (1932), Art. 312.

Combine, eliminating ξ_z:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}-\frac{gk_h^2}{\omega^2}\right)\left[v_s^2\frac{\mathrm{d}\Delta}{\mathrm{d}z}+\frac{1}{\rho}\frac{\mathrm{d}(\rho v_s^2)}{\mathrm{d}z}\Delta+v_s^2\frac{gk_h^2}{\omega^2}\Delta\right]+\omega^2\left(1-\frac{g^2k_h^2}{\omega^4}\right)\left(1-\frac{v_s^2k_h^2}{\omega^2}\right)\Delta=0. \tag{11.64}$$

Expand out:

$$v_s^2\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2}+\left[\frac{\mathrm{d}v_s^2}{\mathrm{d}z}+\frac{1}{\rho}\frac{\mathrm{d}(\rho v_s^2)}{\mathrm{d}z}\right]\frac{\mathrm{d}\Delta}{\mathrm{d}z}+\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\rho}\frac{\mathrm{d}(\rho v_s^2)}{\mathrm{d}z}\right)+\frac{\mathrm{d}v_s^2}{\mathrm{d}z}\frac{gk_h^2}{\omega^2}-\frac{gk_h^2}{\omega^2}\frac{1}{\rho}\frac{\mathrm{d}(\rho v_s^2)}{\mathrm{d}z}\right.$$
$$\left.+\omega^2\left(1-\frac{g^2k_h^2}{\omega^4}-\frac{v_s^2k_h^2}{\omega^2}\right)\right]\Delta=0. \tag{11.65}$$

In the case of a polytropic atmosphere, $v_s^2\propto z$ and $\rho v_s^2\propto z^{m+1}$:

$$v_s^2\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2}+(m+2)\frac{v_s^2}{z}\frac{\mathrm{d}\Delta}{\mathrm{d}z}+\left[-m\frac{gk_h^2}{\omega^2}\frac{v_s^2}{z}+\omega^2\left(1-\frac{g^2k_h^2}{\omega^4}-\frac{v_s^2k_h^2}{\omega^2}\right)\right]\Delta=0. \tag{11.66}$$

In fact, $v_s^2/z=-\gamma g/(m+1)$. Divide through by this factor:

$$z\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2}+(m+2)\frac{\mathrm{d}\Delta}{\mathrm{d}z}-\left[m\frac{gk_h^2}{\omega^2}+\frac{(m+1)}{\gamma g}\omega^2\left(1-\frac{g^2k_h^2}{\omega^4}\right)+k_h^2z\right]\Delta=0. \tag{11.67}$$

Finally,

$$z\frac{\mathrm{d}^2\Delta}{\mathrm{d}z^2}+(m+2)\frac{\mathrm{d}\Delta}{\mathrm{d}z}-(A+k_hz)k_h\Delta=0, \tag{11.68}$$

where

$$A=\frac{(m+1)}{\gamma}\frac{\omega^2}{gk_h}+\left(m-\frac{m+1}{\gamma}\right)\frac{gk_h}{\omega^2} \tag{11.69}$$

is a dimensionless constant. Let $\Delta=w(z)\,\mathrm{e}^{k_hz}$:

$$z\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(m+2+2k_hz)\frac{\mathrm{d}w}{\mathrm{d}z}-(A-m-2)k_hw=0. \tag{11.70}$$

This is related to the confluent hypergeometric equation and has a regular singular point at $z=0$. Using the method of Frobenius, we seek power-series solutions

$$w=\sum_{r=0}^{\infty}a_rz^{\sigma+r}, \tag{11.71}$$

where σ is a number to be determined and $a_0\neq0$. The indicial equation is

$$\sigma(\sigma+m+1)=0 \tag{11.72}$$

and the regular solution has $\sigma=0$. The recurrence relation is then

$$\frac{a_{r+1}}{a_r}=\frac{(A-m-2-2r)k_h}{(r+1)(r+m+2)}. \tag{11.73}$$

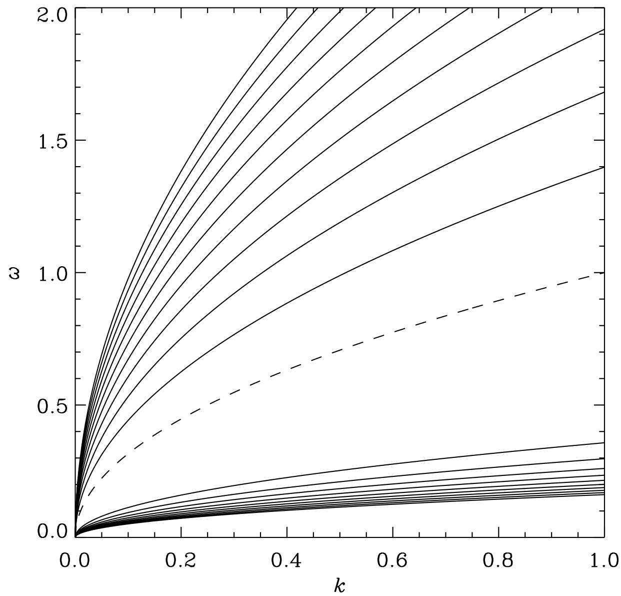

FIGURE 19. Dispersion relation, in arbitrary units, for a stably stratified plane-parallel polytropic atmosphere with $m = 3$ and $\gamma = 5/3$. The dashed line is the f mode. Above it are the first ten p modes and below it are the first ten g modes. Each curve is a parabola.

In the case of an infinite series, $a_{r+1}/a_r \sim -2k_h/r$ as $r \to \infty$, so w behaves like $\mathrm{e}^{-2k_h z}$ and $\varDelta$ diverges like $\mathrm{e}^{-k_h z}$ as $z \to -\infty$. Solutions in which $\varDelta$ decays with depth are those for which the series terminates and w is a polynomial. For a polynomial of degree $n-1$ ($n \geqslant 1$),

$$A = 2n + m. \tag{11.74}$$

This gives a quadratic equation for ω^2:

$$\frac{(m+1)}{\gamma}\left(\frac{\omega^2}{gk_h}\right)^2 - (2n+m)\left(\frac{\omega^2}{gk_h}\right) + \left(m - \frac{m+1}{\gamma}\right) = 0. \tag{11.75}$$

A negative root for ω^2 exists if and only if $m - (m+1)/\gamma < 0$, i.e. $N^2 < 0$, as expected from Schwarzschild's criterion for stability.

For $n \gg 1$, the large root is

$$\frac{\omega^2}{gk_h} \sim \frac{2n\gamma}{m+1} \quad (p \text{ modes}, \omega^2 \propto v_s^2) \tag{11.76}$$

and the small root is

$$\frac{\omega^2}{gk_h} \sim \frac{1}{2n}\left(m - \frac{m+1}{\gamma}\right) \quad (g \text{ modes}, \omega^2 \propto N^2). \tag{11.77}$$

The f mode is the 'trivial' solution $\varDelta = 0$. p modes ('p' for pressure) are acoustic waves, which rely on compressibility. g modes are gravity waves, which rely on buoyancy. Typical branches of the dispersion relation are illustrated in figure 19.

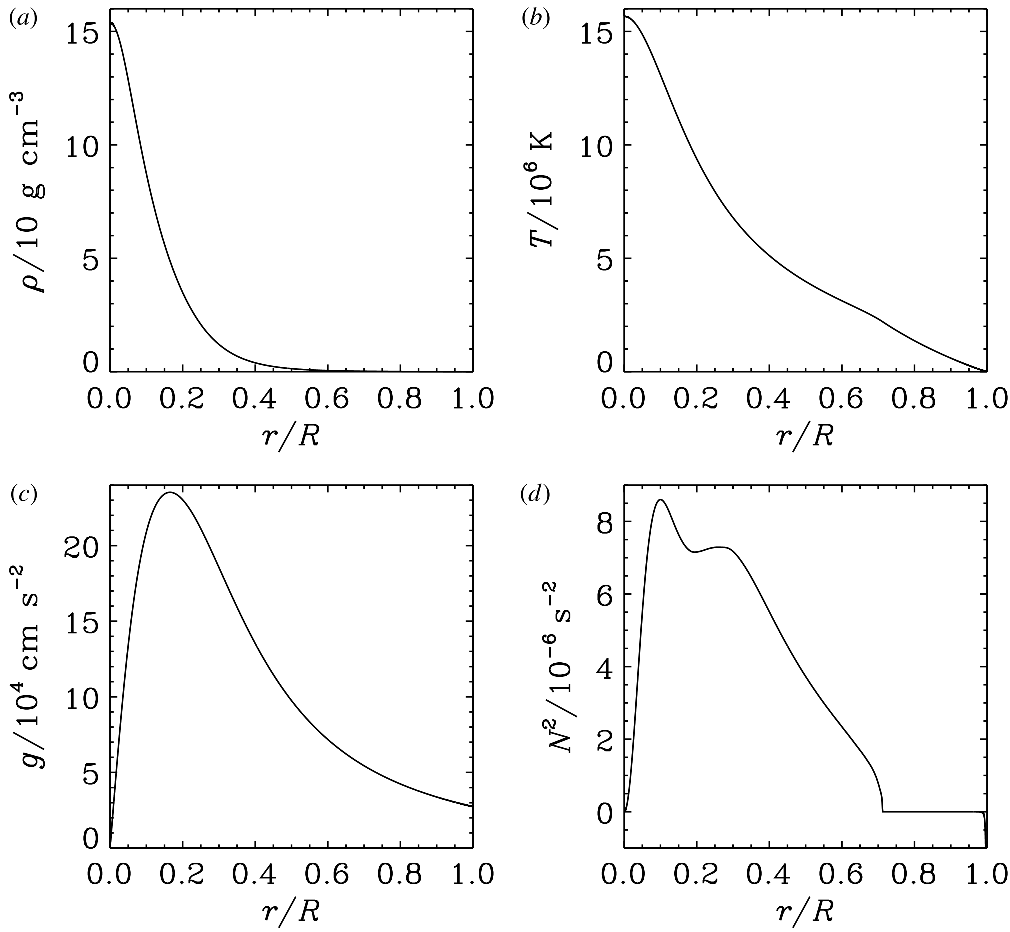

FIGURE 20. A standard model of the present Sun, up to the photosphere. Density, temperature, gravity and squared buoyancy frequency are plotted versus fractional radius.

In solar-type stars (see figures 20 and 21) the inner part (radiative zone) is convectively stable ($N^2 > 0$) and the outer part (convective zone) is unstable ($N^2 < 0$). However, the convection is so efficient that only a very small entropy gradient is required to sustain the convective heat flux, so N^2 is very small and negative in the convective zone. Although g modes propagate in the radiative zone at frequencies smaller than N, they cannot reach the surface. Only f and p modes (excited by convection) are observed at the solar surface.

In more massive stars the situation is reversed. Then f, p and g modes can be observed, in principle, at the surface. g modes are particularly well observed in certain classes of white dwarf.

Related examples: A.22–A.25.

11.5. *Tidally forced oscillations*

When astrophysical fluid bodies such as stars and planets orbit sufficiently close to one another, they deform each other in ways that can cause irreversible evolution of

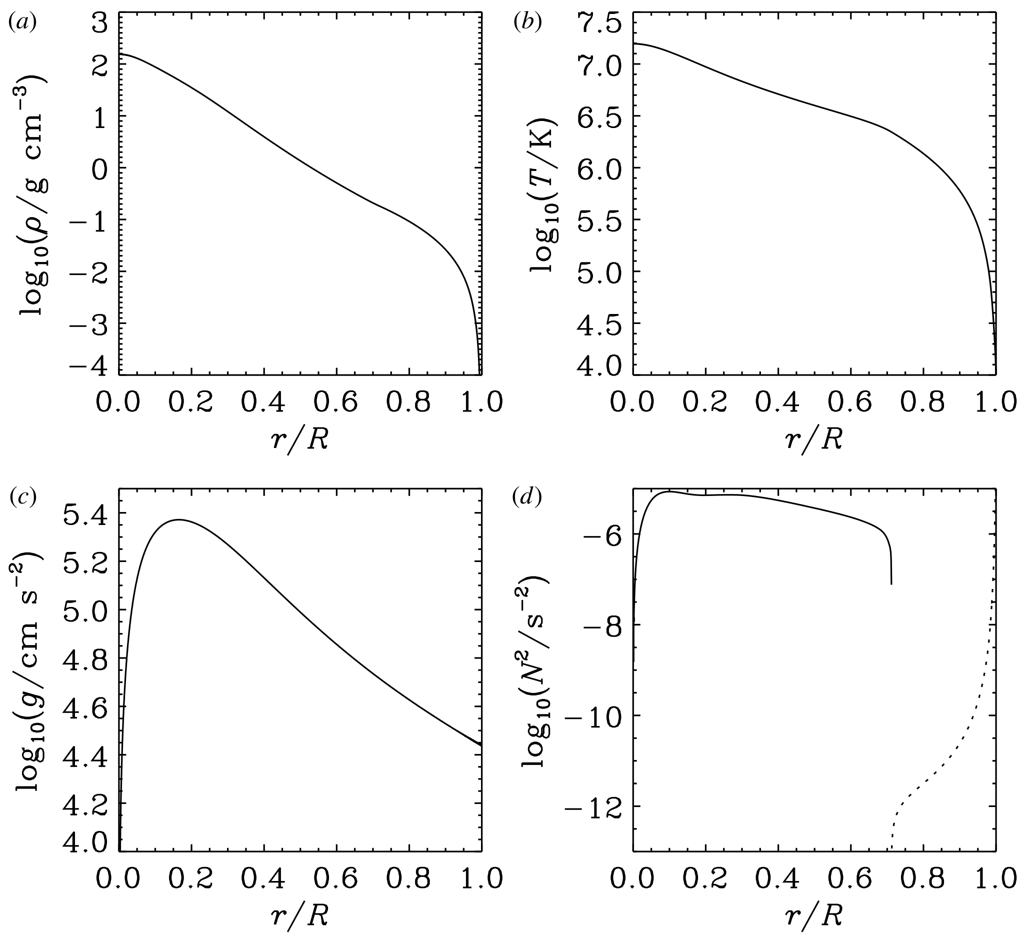

FIGURE 21. The same model plotted on a logarithmic scale. In the convective region where $N^2 < 0$, the dotted line shows $-N^2$ instead.

their spin and orbital motion over astronomical time scales. We consider here some of the simplest aspects of this problem.

Consider a binary star (or star–planet or planet–moon system, etc.) with a circular orbit. Let the orbital separation be a and the orbital (angular) frequency

$$\Omega_o = \left(\frac{GM}{a^3}\right)^{1/2}, \tag{11.78}$$

where $M = M_1 + M_2$ is the combined mass of the two bodies. Let $\boldsymbol{R}_1(t)$ and $\boldsymbol{R}_2(t)$ be the position vectors of the centres of mass of the two bodies, and $\boldsymbol{d} = \boldsymbol{R}_2 - \boldsymbol{R}_1$ their separation.

The gravitational potential due to body 2 (treated as a point mass or spherical mass) at position $\boldsymbol{R}_1 + \boldsymbol{x}$ within body 1 is

$$-\frac{GM_2}{|\boldsymbol{d} - \boldsymbol{x}|} = -GM_2 \left(|\boldsymbol{d}|^2 - 2\boldsymbol{d}\cdot\boldsymbol{x} + |\boldsymbol{x}|^2\right)^{-1/2}$$

$$
\begin{aligned}
&= -\frac{GM_2}{|\boldsymbol{d}|}\left(1-\frac{2\boldsymbol{d}\cdot\boldsymbol{x}}{|\boldsymbol{d}|^2}+\frac{|\boldsymbol{x}|^2}{|\boldsymbol{d}|^2}\right)^{-1/2} \\
&= -\frac{GM_2}{|\boldsymbol{d}|}\left[1-\frac{1}{2}\left(-\frac{2\boldsymbol{d}\cdot\boldsymbol{x}}{|\boldsymbol{d}|^2}+\frac{|\boldsymbol{x}|^2}{|\boldsymbol{d}|^2}\right)+\frac{3}{8}\left(-\frac{2\boldsymbol{d}\cdot\boldsymbol{x}}{|\boldsymbol{d}|^2}+\frac{|\boldsymbol{x}|^2}{|\boldsymbol{d}|^2}\right)^2+\cdots\right] \\
&= -\frac{GM_2}{|\boldsymbol{d}|}\left[1+\frac{\boldsymbol{d}\cdot\boldsymbol{x}}{|\boldsymbol{d}|^2}+\frac{3(\boldsymbol{d}\cdot\boldsymbol{x})^2-|\boldsymbol{d}|^2|\boldsymbol{x}|^2}{2|\boldsymbol{d}|^4}+O\left(\frac{|\boldsymbol{x}|^3}{|\boldsymbol{d}|^3}\right)\right]. \qquad (11.79)
\end{aligned}
$$

In this Taylor expansion, the term independent of $\boldsymbol{x}$ is a uniform potential that has no effect. The term linear in $\boldsymbol{x}$ gives rise to a uniform acceleration $GM_2\boldsymbol{d}/|\boldsymbol{d}|^3$, which causes the orbital motion of body 1. The remaining terms constitute the tidal potential Ψ; the quadratic terms written here are the tidal potential in the quadrupolar approximation.

For a circular orbit, the coordinate system can be chosen such that

$$\boldsymbol{d} = (a\cos\Omega_o t,\, a\sin\Omega_o t,\, 0). \qquad (11.80)$$

Introduce spherical polar coordinates within body 1 such that

$$\boldsymbol{x} = (r\sin\theta\cos\phi,\, r\sin\theta\sin\phi,\, r\cos\phi). \qquad (11.81)$$

Then

$$\boldsymbol{d}\cdot\boldsymbol{x} = ar\sin\theta\cos(\phi-\Omega_o t), \qquad (11.82)$$

$$
\begin{aligned}
\Psi &= \frac{GM_2r^2}{2a^3}\left[1-3\sin^2\theta\cos^2(\phi-\Omega_o t)\right] \\
&= \frac{GM_2r^2}{4a^3}[2-3\sin^2\theta-3\sin^2\theta\cos(2\phi-2\Omega_o t)]. \qquad (11.83)
\end{aligned}
$$

The first two terms are static; the remaining oscillatory part can be written as

$$\mathrm{Re}\left[-\frac{3GM_2r^2\sin^2\theta}{4a^3}\,\mathrm{e}^{2\mathrm{i}(\phi-\Omega_o t)}\right], \qquad (11.84)$$

which involves the spherical harmonic function $Y_2^2(\theta,\phi)\propto\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}$.

The tidal frequency in a non-rotating frame is $2\Omega_o$. In a frame rotating with the spin angular velocity Ω_s of body 1, the tidal frequency is $2(\Omega_o-\Omega_s)$, owing to an angular Doppler shift.

If the tidal frequency is sufficiently small, it might be assumed that body 1 responds hydrostatically to the tidal potential. Under this assumption of an equilibrium tide, body 1 is deformed into a spheroid with a tidal bulge that points instantaneously towards body 2, and no tidal torque is exerted.

We can allow for a more general linear response, including dissipation and wave-like disturbances (a dynamical tide) as follows. The most important aspect of the tidally deformed body is its exterior gravitational potential perturbation $\delta\Phi$, because it is only through gravity that the bodies communicate and exchange energy and angular momentum. We write the linear response as

$$\delta\Phi = \mathrm{Re}\left[-k\frac{3GM_2\sin^2\theta}{4a^3}\frac{R_1^5}{r^3}\,\mathrm{e}^{2\mathrm{i}(\phi-\Omega_o t)}\right], \qquad (11.85)$$

where R_1 is the radius of body 1 (or some appropriate measure of its radius if it is deformed by its rotation) and k is the potential Love number, a dimensionless complex number that describes the amplitude and phase of the tidal response. Note that $\delta\Phi$ involves the same frequency and the same spherical harmonic $Y_2^2(\theta,\phi)$, but combined with r^{-3} rather than r^2 to make it a valid solution of Laplace's equation in the exterior of body 1. The factor of R_1^5 is introduced so that k is dimensionless and measures the ratio of $\delta\Phi$ and Ψ at the surface of body 1.

The imaginary part of k determines the part of the tidal response that is out of phase with the tidal forcing, and which is associated with dissipation and irreversible evolution. The torque acting on the orbit of body 2 is

$$\begin{aligned} -T &= M_2 r\sin\theta\left(-\frac{1}{r\sin\theta}\frac{\partial\delta\Phi}{\partial\phi}\right)\bigg|_{r=a,\,\theta=\pi/2,\,\phi=\Omega_o t} \\ &= \mathrm{Re}\left(M_2 k\frac{3GM_2}{4a^3}\frac{R_1^5}{a^3}2\mathrm{i}\right) \\ &= -\mathrm{Im}(k)\frac{3GM_2^2R_1^5}{2a^6}. \end{aligned} \tag{11.86}$$

By Newton's third law, there is an equal and opposite torque, $+T$, acting on the spin of body 1.

The orbital angular momentum about the centre of mass is

$$L_o = \mu(GMa)^{1/2}, \tag{11.87}$$

where $\mu = M_1M_2/M$ is the reduced mass of the system. This result can be obtained by considering

$$M_1\left(\frac{M_2a}{M}\right)^2\Omega_o + M_2\left(\frac{M_1a}{M}\right)^2\Omega_o = \frac{M_1M_2}{M}(GMa)^{1/2} \tag{11.88}$$

(see figure 22). It evolves according to

$$\frac{\mathrm{d}L_o}{\mathrm{d}t} = -T, \tag{11.89}$$

which determines the rate of orbital migration:

$$\left.\begin{aligned} \frac{1}{2}\frac{M_1M_2}{M}(GMa)^{1/2}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} &= -\mathrm{Im}(k)\frac{3GM_2^2R_1^5}{2a^6} \\ -\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} &= 3\,\mathrm{Im}(k)\frac{M_2}{M_1}\left(\frac{R_1}{a}\right)^5\Omega_o. \end{aligned}\right\} \tag{11.90}$$

The orbital energy

$$E_o = -\mu\frac{GM}{2a} \tag{11.91}$$

evolves according to

$$\frac{\mathrm{d}E_o}{\mathrm{d}t} = \mu\frac{GM}{2a}\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = -\Omega_oT. \tag{11.92}$$

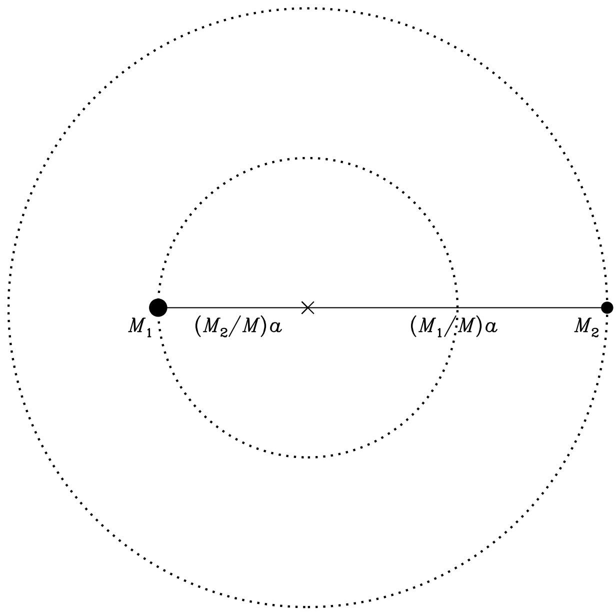

FIGURE 22. A binary star with two components in circular orbital motion about the centre of mass.

The spin angular momentum $L_s = I_1 \Omega_s$ and spin energy $E_s = (I_1 \Omega_s^2)/2$, where I_1 is the moment of inertia of body 1, evolve according to

$$\frac{\mathrm{d}L_s}{\mathrm{d}t} = T, \quad \frac{\mathrm{d}E_s}{\mathrm{d}t} = \Omega_s T. \tag{11.93a,b}$$

The total energy therefore satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_o + E_s) = (\Omega_s - \Omega_o)T = -D, \tag{11.94}$$

where $D > 0$ is the rate of dissipation of energy. To ensure $D > 0$, the sign of $\mathrm{Im}(k)$ should be the same as the sign of the tidal frequency $2(\Omega_o - \Omega_s)$.

In a dissipative spin–orbit coupling, the tidal torque T tries to bring about an equalization of the spin and orbital angular velocities. Its action, mediated by gravity, is comparable to a frictional interaction between differential rotating components in a mechanical system.

In binary stars, and other cases in which the spin angular momentum is small compared to the orbital angular momentum, there is indeed a tendency towards synchronization of the spin with the orbital motion (as the Moon is synchronized with its orbit around the Earth). However, in systems of extreme mass ratio in which the spin of the large body contains most of the angular momentum, the tidal torque instead causes orbital migration away from the synchronous orbit at which $\Omega = \Omega_s$ (figure 23).

This situation applies to the moons of solar-system planets, most of which migrate outwards, and to extrasolar planets in close orbits around their host stars, where the migration is usually inward and may lead to the destruction of the planet.

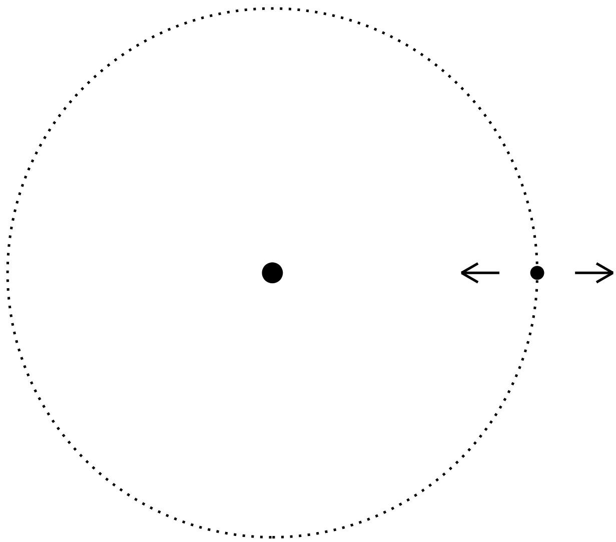

FIGURE 23. Orbital migration away from the synchronous orbit driven by tidal dissipation in a system of extreme mass ratio.

11.6. *Rotating fluid bodies*

Note: in this subsection (r, ϕ, z) are cylindrical polar coordinates.

11.6.1. *Equilibrium*

The equations of ideal gas dynamics in cylindrical polar coordinates are

$$\left.\begin{aligned}\frac{\mathrm{D}u_r}{\mathrm{D}t}-\frac{u_\phi^2}{r}&=-\frac{\partial\Phi}{\partial r}-\frac{1}{\rho}\frac{\partial p}{\partial r},\\ \frac{\mathrm{D}u_\phi}{\mathrm{D}t}+\frac{u_ru_\phi}{r}&=-\frac{1}{r}\frac{\partial\Phi}{\partial\phi}-\frac{1}{\rho r}\frac{\partial p}{\partial\phi},\\ \frac{\mathrm{D}u_z}{\mathrm{D}t}&=-\frac{\partial\Phi}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z},\\ \frac{\mathrm{D}\rho}{\mathrm{D}t}&=-\rho\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{\partial u_z}{\partial z}\right],\\ \frac{\mathrm{D}p}{\mathrm{D}t}&=-\gamma p\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{\partial u_z}{\partial z}\right],\end{aligned}\right\} \tag{11.95}$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\phi}{r}\frac{\partial}{\partial\phi}+u_z\frac{\partial}{\partial z}. \tag{11.96}$$

Consider a steady, axisymmetric basic state with density $\rho(r, z)$, pressure $p(r, z)$, gravitational potential $\Phi(r, z)$ and with differential rotation

$$\boldsymbol{u}=r\Omega(r, z)\,\boldsymbol{e}_\phi. \tag{11.97}$$

For equilibrium we require

$$-r\Omega^2\,\boldsymbol{e}_r=-\boldsymbol{\nabla}\Phi-\frac{1}{\rho}\boldsymbol{\nabla}p. \tag{11.98}$$

Take the curl to obtain

$$-r\frac{\partial \Omega^2}{\partial z}\boldsymbol{e}_\phi = \boldsymbol{\nabla} p \times \boldsymbol{\nabla}\left(\frac{1}{\rho}\right) = \boldsymbol{\nabla} T \times \boldsymbol{\nabla} s. \tag{11.99}$$

This is just the vorticity equation in a steady state. It is sometimes called the thermal wind equation. The equilibrium is called barotropic if $\boldsymbol{\nabla} p$ is parallel to $\boldsymbol{\nabla}\rho$, otherwise it is called baroclinic. In a barotropic state the angular velocity is independent of z: $\Omega = \Omega(r)$. This is a version of the Taylor–Proudman theorem[20] which states that under certain conditions the velocity in a rotating fluid is independent of height.

We can also write

$$\frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{g} = -\boldsymbol{\nabla}\Phi + r\Omega^2\boldsymbol{e}_r, \tag{11.100}$$

where $\boldsymbol{g}$ is the effective gravitational acceleration, including the centrifugal force associated with the (non-uniform) rotation.

In a barotropic state with $\Omega(r)$ we can write

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi_{cg}, \quad \Phi_{cg} = \Phi(r,z) + \Psi(r), \quad \Psi = -\int r\Omega^2\,\mathrm{d}r. \tag{11.101}$$

Also, since $p = p(\rho)$ in the equilibrium state, we can define the pseudoenthalpy $\tilde{h}(\rho)$ such that $\mathrm{d}\tilde{h} = \mathrm{d}p/\rho$. An example is a polytropic model for which

$$p = K\rho^{1+1/m}, \quad \tilde{h} = (m+1)K\rho^{1/m}. \tag{11.102a,b}$$

($\tilde{h}$ equals the true enthalpy only if the equilibrium is homentropic.) The equilibrium condition then reduces to

$$\boldsymbol{0} = -\boldsymbol{\nabla}\Phi_{cg} - \boldsymbol{\nabla}\tilde{h} \tag{11.103}$$

or

$$\Phi + \Psi + \tilde{h} = C = \text{const.} \tag{11.104}$$

An example of a rapidly and differentially rotating equilibrium is an accretion disc around a central mass M. For a non-self-gravitating disc $\Phi = -GM(r^2+z^2)^{-1/2}$. Assume the disc is barotropic and let the arbitrary additive constant in $\tilde{h}$ be defined (as in the polytropic example above) such that $\tilde{h} = 0$ at the surfaces $z = \pm H(r)$ of the disc where $\rho = p = 0$. Then

$$-GM(r^2+H^2)^{-1/2} + \Psi(r) = C, \tag{11.105}$$

from which

$$r\Omega^2 = -\frac{\mathrm{d}}{\mathrm{d}r}\left[GM(r^2+H^2)^{-1/2}\right]. \tag{11.106}$$

For example, if $H = \epsilon r$ with $\epsilon = \text{const.}$ being the aspect ratio of the disc, then

$$\Omega^2 = (1+\epsilon^2)^{-1/2}\frac{GM}{r^3}. \tag{11.107}$$

The thinner the disc is, the closer it is to Keplerian rotation. Once we have found the relation between $\Omega(r)$ and $H(r)$, equation (11.104) then determines the spatial distribution of $\tilde{h}$ (and therefore of ρ and p) within the disc.

[20]Joseph Proudman (1888–1975), British.

11.6.2. *Linear perturbations*

The basic state is independent of t and ϕ, allowing us to consider linear perturbations of the form

$$\mathrm{Re}[\delta u_r(r, z)\exp(-\mathrm{i}\omega t + \mathrm{i}m\phi)], \quad \text{etc.,} \tag{11.108}$$

where m is the azimuthal wavenumber (an integer). The linearized equations in the Cowling approximation are

$$\left.\begin{aligned}
-\mathrm{i}\hat{\omega}\delta u_r - 2\Omega\,\delta u_\phi &= -\frac{1}{\rho}\frac{\partial \delta p}{\partial r} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial r}, \\
-\mathrm{i}\hat{\omega}\delta u_\phi + \frac{1}{r}\delta\boldsymbol{u}\cdot\nabla(r^2\Omega) &= -\frac{\mathrm{i}m\,\delta p}{\rho r}, \\
-\mathrm{i}\hat{\omega}\delta u_z &= -\frac{1}{\rho}\frac{\partial \delta p}{\partial z} + \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial z}, \\
-\mathrm{i}\hat{\omega}\delta\rho + \delta\boldsymbol{u}\cdot\nabla\rho &= -\rho\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta u_r) + \frac{\mathrm{i}m\,\delta u_\phi}{r} + \frac{\partial \delta u_z}{\partial z}\right], \\
-\mathrm{i}\hat{\omega}\delta p + \delta\boldsymbol{u}\cdot\nabla p &= -\gamma p\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta u_r) + \frac{\mathrm{i}m\,\delta u_\phi}{r} + \frac{\partial \delta u_z}{\partial z}\right],
\end{aligned}\right\} \tag{11.109}$$

where

$$\hat{\omega} = \omega - m\Omega \tag{11.110}$$

is the intrinsic frequency, i.e. the angular frequency of the wave measured in a frame of reference that rotates with the local angular velocity of the fluid.

Eliminate δu_ϕ and $\delta\rho$ to obtain

$$\left.\begin{aligned}
(\hat{\omega}^2 - A)\,\delta u_r - B\,\delta u_z &= -\frac{\mathrm{i}\hat{\omega}}{\rho}\left(\frac{\partial \delta p}{\partial r} - \frac{\partial p}{\partial r}\frac{\delta p}{\gamma p}\right) + 2\Omega\frac{\mathrm{i}m\,\delta p}{\rho r}, \\
-C\,\delta u_r + (\hat{\omega}^2 - D)\,\delta u_z &= -\frac{\mathrm{i}\hat{\omega}}{\rho}\left(\frac{\partial \delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right),
\end{aligned}\right\} \tag{11.111}$$

where

$$\left.\begin{aligned}
A &= \frac{2\Omega}{r}\frac{\partial}{\partial r}(r^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial r}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right), \\
B &= \frac{2\Omega}{r}\frac{\partial}{\partial z}(r^2\Omega) - \frac{1}{\rho}\frac{\partial p}{\partial r}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}\right), \\
C &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right), \\
D &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}\right).
\end{aligned}\right\} \tag{11.112}$$

Note that A, B, C and D involve radial and vertical derivatives of the specific angular momentum $r^2\Omega$ and the specific entropy s. The thermal wind equation implies

$$B = C, \tag{11.113}$$

so the matrix

$$\boldsymbol{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \tag{11.114}$$

is symmetric.

11.6.3. *The Høiland criteria*

It can be useful to introduce the Lagrangian displacement $\boldsymbol{\xi}$ such that

$$\Delta \boldsymbol{u} = \delta \boldsymbol{u} + \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t}, \tag{11.115}$$

i.e.

$$\delta u_r = -\mathrm{i}\hat{\omega}\xi_r, \quad \delta u_\phi = -\mathrm{i}\hat{\omega}\xi_\phi - r\boldsymbol{\xi} \cdot \nabla \Omega, \quad \delta u_z = -\mathrm{i}\hat{\omega}\xi_z. \tag{11.116a–c}$$

Note that

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta u_r) + \frac{\mathrm{i}m\,\delta u_\phi}{r} + \frac{\partial \delta u_z}{\partial z} = -\mathrm{i}\hat{\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{\mathrm{i}m\xi_\phi}{r} + \frac{\partial \xi_z}{\partial z}\right]. \tag{11.117}$$

The linearized equations constitute an eigenvalue problem for ω but it is not self-adjoint except when $m = 0$. We specialize to the case $m = 0$ (axisymmetric perturbations). Then

$$\left.\begin{aligned} (\omega^2 - A)\xi_r - B\xi_z &= \frac{1}{\rho}\left(\frac{\partial \delta p}{\partial r} - \frac{\partial p}{\partial r}\frac{\delta p}{\gamma p}\right), \\ -B\xi_r + (\omega^2 - D)\xi_z &= \frac{1}{\rho}\left(\frac{\partial \delta p}{\partial z} - \frac{\partial p}{\partial z}\frac{\delta p}{\gamma p}\right), \end{aligned}\right\} \tag{11.118}$$

with

$$\delta p = -\gamma p \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla p. \tag{11.119}$$

Multiply the first of (11.118) by $\rho\xi_r^*$ and the second by $\rho\xi_z^*$ and integrate over the volume V of the fluid (using the boundary condition $\delta p = 0$) to obtain

$$\begin{aligned} \omega^2 \int_V \rho(|\xi_r|^2 + |\xi_z|^2)\,\mathrm{d}V &= \int_V \left[\rho Q(\boldsymbol{\xi}) + \boldsymbol{\xi}^* \cdot \nabla \delta p - \frac{\delta p}{\gamma p}\boldsymbol{\xi}^* \cdot \nabla p\right]\,\mathrm{d}V \\ &= \int_V \left[\rho Q(\boldsymbol{\xi}) - \frac{\delta p}{\gamma p}(\gamma p \nabla \cdot \boldsymbol{\xi}^* + \boldsymbol{\xi}^* \cdot \nabla p)\right]\,\mathrm{d}V \\ &= \int_V \left(\rho Q(\boldsymbol{\xi}) + \frac{|\delta p|^2}{\gamma p}\right)\,\mathrm{d}V, \end{aligned} \tag{11.120}$$

where

$$Q(\boldsymbol{\xi}) = A|\xi_r|^2 + B(\xi_r^*\xi_z + \xi_z^*\xi_r) + D|\xi_z|^2 = \begin{bmatrix}\xi_r^* & \xi_z^*\end{bmatrix}\begin{bmatrix}A & B \\ B & D\end{bmatrix}\begin{bmatrix}\xi_r \\ \xi_z\end{bmatrix} \tag{11.121}$$

is the (real) Hermitian form associated with the matrix $\boldsymbol{M}$.

Note that this integral involves only the meridional components of the displacement. If we had not made the Cowling approximation there would be the usual negative definite contribution to ω^2 from self-gravitation.

The above integral relation therefore shows that ω^2 is real, and a variational property ensures that instability to axisymmetric perturbations occurs if and only if the integral on the right-hand side can be made negative by a suitable trial displacement. If Q is positive definite then $\omega^2 > 0$ and we have stability. Now the characteristic equation of the matrix $\boldsymbol{M}$ is

$$\lambda^2 - (A + D)\lambda + AD - B^2 = 0. \tag{11.122}$$

The eigenvalues $\lambda_\pm$ are both positive if and only if

$$A + D > 0 \quad \text{and} \quad AD - B^2 > 0. \tag{11.123a,b}$$

If these conditions are satisfied throughout the fluid then $Q > 0$, which implies $\omega^2 > 0$, so the fluid is stable to axisymmetric perturbations (neglecting self-gravitation). These conditions are also necessary for stability. If one of the eigenvalues is negative in some region of the meridional plane, then a trial displacement can be found which is localized in that region, has $\delta p = 0$ and $Q < 0$, implying instability. (By choosing $\boldsymbol{\xi}$ in the correct direction and tuning $\nabla \cdot \boldsymbol{\xi}$ appropriately, it is possible to arrange for δp to vanish.)

Using $\ell = r^2\Omega$ (specific angular momentum) and $s = c_p(\gamma^{-1} \ln p - \ln \rho) + \text{const.}$ (specific entropy) for a perfect ideal gas, we have

$$\left.\begin{aligned} A &= \frac{1}{r^3}\frac{\partial \ell^2}{\partial r} - \frac{g_r}{c_p}\frac{\partial s}{\partial r}, \\ B &= \frac{1}{r^3}\frac{\partial \ell^2}{\partial z} - \frac{g_r}{c_p}\frac{\partial s}{\partial z} = -\frac{g_z}{c_p}\frac{\partial s}{\partial r}, \\ D &= -\frac{g_z}{c_p}\frac{\partial s}{\partial z}, \end{aligned}\right\} \tag{11.124}$$

so the two conditions become

$$\frac{1}{r^3}\frac{\partial \ell^2}{\partial r} - \frac{1}{c_p}\boldsymbol{g} \cdot \nabla s > 0 \tag{11.125}$$

and

$$-g_z\left(\frac{\partial \ell^2}{\partial r}\frac{\partial s}{\partial z} - \frac{\partial \ell^2}{\partial z}\frac{\partial s}{\partial r}\right) > 0. \tag{11.126}$$

These are the Høiland stability criteria[21].

(If the criteria are marginally satisfied a further investigation may be required.)

Consider first the non-rotating case $\ell = 0$. The first criterion reduces to the Schwarzschild criterion for convective stability,

$$-\frac{1}{c_p}\boldsymbol{g} \cdot \nabla s \equiv N^2 > 0. \tag{11.127}$$

In the homentropic case $s = \text{const.}$ (which is a barotropic model) they reduce to the Rayleigh criterion[22] for centrifugal (inertial) stability,

$$\frac{\mathrm{d}\ell^2}{\mathrm{d}r} > 0, \tag{11.128}$$

which states that the specific angular momentum should increase with r for stability.

[21] Einar Høiland (1907–1974), Norwegian.

[22] John William Strutt, Lord Rayleigh (1842–1919), British.

The second Høiland criterion is equivalent to

$$(\boldsymbol{e}_r \times (-\boldsymbol{g})) \boldsymbol{\cdot} (\nabla \ell^2 \times \nabla s) > 0. \tag{11.129}$$

In other words the vectors $\boldsymbol{e}_r \times (-\boldsymbol{g})$ and $\nabla \ell^2 \times \nabla s$ should be parallel (rather than antiparallel). In a rotating star in which the specific entropy increases outwards, for stability we require that the specific angular momentum should increase with r on each surface of constant entropy.

Related example: A.26.

Appendix A. Examples

A.1. *Validity of a fluid approach*

The Coulomb cross-section for 'collisions' (i.e. large-angle scatterings) between electrons and protons is $\sigma \approx 1 \times 10^{-4}(T/\mathrm{K})^{-2}\ \mathrm{cm}^2$. Why does it depend on the inverse square of the temperature?

Using the numbers quoted in § 2.9.3 (or elsewhere), estimate the order of magnitude of the mean free path and the collision frequency in (i) the centre of the Sun, (ii) the solar corona, (iii) a molecular cloud and (iv) the hot phase of the interstellar medium. Is a fluid approach likely to be valid in these systems?

A.2. *Vorticity equation*

Show that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ of an ideal fluid without a magnetic field satisfies the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nabla p \times \nabla v, \tag{A 1}$$

where $v = 1/\rho$ is the specific volume. Explain why the last term, which acts as a source of vorticity, can also be written as $\nabla T \times \nabla s$. Under what conditions does this 'baroclinic' source term vanish, and in what sense(s) can the vorticity then be said to be 'conserved'?

Show that the (Rossby–Ertel) potential vorticity $(1/\rho)\,\boldsymbol{\omega} \boldsymbol{\cdot} \nabla s$ is conserved, as a material invariant, even when the baroclinic term is present.

A.3. *Homogeneous expansion or contraction*

(This question explores a very simple fluid flow in which compressibility and self-gravity are important.)

A homogeneous perfect gas of density $\rho = \rho_0(t)$ occupies the region $|\boldsymbol{x}| < R(t)$, surrounded by a vacuum. The pressure is $p = p_0(t)(1 - |\boldsymbol{x}|^2/R^2)$ and the velocity field is $\boldsymbol{u} = A(t)\boldsymbol{x}$, where $A = \dot{R}/R$.

Using either Cartesian or spherical polar coordinates, show that the equations of Newtonian gas dynamics and the boundary conditions are satisfied provided that

$$\rho_0 \propto R^{-3}, \quad p_0 \propto R^{-3\gamma}, \quad \ddot{R} = -\frac{4\pi G \rho_0 R}{3} + \frac{2p_0}{\rho_0 R}. \tag{A 2a–c}$$

Deduce the related energy equation

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi G \rho_0 R^2}{3} + \frac{2p_0}{3(\gamma - 1)\rho_0} = \text{const.}, \tag{A 3}$$

and interpret the three contributions. Discuss the dynamics qualitatively in the two cases $\gamma > 4/3$ and $1 < \gamma < 4/3$.[23]

A.4. *Dynamics of ellipsoidal bodies*

(This question uses Cartesian tensor notation and the summation convention.)

A fluid body occupies a time-dependent ellipsoidal volume centred on the origin. Let $f(\boldsymbol{x}, t) = 1 - S_{ij}x_ix_j$, where $S_{ij}(t)$ is a symmetric tensor with positive eigenvalues, such that the body occupies the region $0 < f \leqslant 1$ with a free surface at $f = 0$. The velocity field is $u_i = A_{ij}x_j$, where $A_{ij}(t)$ is a tensor that is not symmetric in general. Assume that the gravitational potential inside the body has the form $\Phi = B_{ij}x_ix_j +$ const., where $B_{ij}(t)$ is a symmetric tensor.

Show that the equations of Newtonian gas dynamics and the boundary conditions are satisfied if the density and pressure are of the form

$$\rho = \rho_0(t)\hat{\rho}(f), \quad p = \rho_0(t)T(t)\hat{p}(f), \tag{A 4a,b}$$

where the dimensionless functions $\hat{p}(f)$ and $\hat{\rho}(f)$ are related by $\hat{p}'(f) = \hat{\rho}(f)$ with the normalization $\hat{\rho}(1) = 1$ and the boundary condition $\hat{p}(0) = 0$, provided that the coefficients evolve according to

$$\left.\begin{aligned} \dot{S}_{ij} + S_{ik}A_{kj} + S_{jk}A_{ki} &= 0,\\ \dot{A}_{ij} + A_{ik}A_{kj} &= -2B_{ij} + 2TS_{ij},\\ \dot{\rho}_0 &= -\rho_0 A_{ii},\\ \dot{T} &= -(\gamma-1)TA_{ii}. \end{aligned}\right\} \tag{A 5}$$

Examples of the spatial structure are the homogeneous body: $\hat{\rho} = 1$, $\hat{p} = f$, and the polytrope of index n: $\hat{\rho} = f^n$, $\hat{p} = f^{n+1}/(n+1)$. Show that Poisson's equation cannot be satisfied if the body is inhomogeneous.[24]

Show how the results of the previous question are recovered in the case of a homogeneous, spherically symmetric body.

A.5. *Resistive MHD*

Ohm's Law for a medium of electrical conductivity σ is $\boldsymbol{J} = \sigma\boldsymbol{E}$, where $\boldsymbol{E}$ is the electric field measured in the rest frame of the conductor. Show that, in the presence of a finite and uniform conductivity, the ideal induction equation is modified to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}, \tag{A 6}$$

where $\eta = 1/(\mu_0\sigma)$ is the magnetic diffusivity, proportional to the resistivity of the fluid. Hence argue that the effects of finite conductivity are small if the magnetic Reynolds number $Rm = LU/\eta$ is large, where L and U are characteristic scales of length and velocity for the fluid flow.[25]

[23]This flow is similar in form to the cosmological 'Hubble flow' and can be seen as a homogeneous expansion or contraction centred on any point, if a Galilean transformation is made. In the limit $R \to \infty$ (for $\gamma > 4/3$), or if the pressure is negligible, the equations derived here correspond to the Friedmann equations for a 'dust' universe (i.e. negligible relativistic pressure $p \ll \rho c^2$) with a scale factor $a \propto R$, $\ddot{a}/a = -4\pi G\rho_0/3$ and $(\dot{a}^2 + \text{const.})/a^2 = 8\pi G\rho_0/3$. See Bondi (1960) for a discussion of Newtonian cosmology.

[24]It can be shown that the self-gravity of a homogeneous ellipsoid generates an interior gravitational potential of the assumed form. The behaviour of self-gravitating, homogeneous, incompressible ellipsoids was investigated by many great mathematicians, including Maclaurin, Jacobi, Dirichlet, Dedekind, Riemann and Poincaré, illustrating the equilibrium and stability of rotating and tidally deformed astrophysical bodies (Chandrasekhar 1969).

[25]The magnetic diffusivity in a fully ionized plasma is of the order of $10^{13}(T/\mathrm{K})^{-3/2}$ cm^2 s^{-1}. Simple estimates imply that $Rm \gg 1$ for observable solar phenomena.

A.6. *Flux freezing*

Consider a magnetic field that is defined in terms of two Euler potentials α and β by

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta. \tag{A 7}$$

(This is sometimes called a Clebsch representation.) Show that a vector potential of the form $\boldsymbol{A} = \alpha\nabla\beta + \nabla\gamma$ generates this magnetic field via $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and that the magnetic field lines are the intersections of the families of surfaces $\alpha = \text{const.}$ and $\beta = \text{const.}$ Show also that

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \nabla\left(\frac{\mathrm{D}\alpha}{\mathrm{D}t}\right) \times \nabla\beta + \nabla\alpha \times \nabla\left(\frac{\mathrm{D}\beta}{\mathrm{D}t}\right). \tag{A 8}$$

Deduce that the ideal induction equation is satisfied if the families of surfaces $\alpha = \text{const.}$ and $\beta = \text{const.}$ are material surfaces, in which case the magnetic field lines can also be identified with material curves.

A.7. *Equilibrium of a solar prominence*

A simple model for a prominence or filament in the solar atmosphere involves a two-dimensional magnetostatic equilibrium in the (x, z) plane with uniform gravity $\boldsymbol{g} = -g\,\boldsymbol{e}_z$. The gas is isothermal with isothermal sound speed c_s. The density and magnetic field depend only on x and the field lines become straight as $|x| \to \infty$.

Show that the solution is of the form

$$B_z = B_0 \tanh(kx), \tag{A 9}$$

where k is a constant to be determined. Sketch the field lines and find the density distribution.

A.8. *Equilibrium of a magnetic star*

A star contains an axisymmetric and purely toroidal magnetic field $\boldsymbol{B} = B(r, z)\,\boldsymbol{e}_\phi$, where (r, ϕ, z) are cylindrical polar coordinates. Show that the equation of magnetostatic equilibrium can be written in the form

$$\boldsymbol{0} = -\rho\nabla\Phi - \nabla p - \frac{B}{\mu_0 r}\nabla(rB). \tag{A 10}$$

Assuming that the equilibrium is barotropic such that ∇p is everywhere parallel to $\nabla\rho$, show that the magnetic field must be of the form

$$B = \frac{1}{r} f(r^2\rho), \tag{A 11}$$

where f is an arbitrary function. Sketch the topology of the contour lines of $r^2\rho$ in a star and show that a magnetic field of this form is confined to the interior.

A.9. *Force-free magnetic fields*

(i) Show that an axisymmetric force-free magnetic field satisfies

$$B_\phi = \frac{f(\psi)}{r}, \tag{A 12}$$

where ψ is the poloidal magnetic flux function, r is the cylindrical radius and f is an arbitrary function. Show also that ψ satisfies the equation

$$r^2\nabla\cdot(r^{-2}\nabla\psi) + f\frac{\mathrm{d}f}{\mathrm{d}\psi} = 0. \tag{A 13}$$

(ii) Let V be a fixed volume bounded by a surface S. Show that the rate of change of the magnetic energy in V is

$$\frac{1}{\mu_0}\int_S [(\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{B} - B^2\boldsymbol{u}]\cdot\mathrm{d}\boldsymbol{S} - \int_V \boldsymbol{u}\cdot\boldsymbol{F}_m\,\mathrm{d}V, \tag{A 14}$$

where $\boldsymbol{F}_m$ is the Lorentz force per unit volume. If V is an axisymmetric volume containing a magnetic field that remains axisymmetric and force free, and if the velocity on S consists of a differential rotation $\boldsymbol{u} = r\Omega(r, z)\,\boldsymbol{e}_\phi$, deduce that the instantaneous rate of change of the magnetic energy in V is

$$\frac{2\pi}{\mu_0}\int f(\psi)\Delta\Omega(\psi)\,\mathrm{d}\psi, \tag{A 15}$$

where $\Delta\Omega(\psi)$ is the difference in angular velocity of the two end points on S of the field line labelled by ψ, and the range of integration is such as to cover S once.

A.10. *Helicity*

The magnetic helicity in a volume V is

$$H_m = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V. \tag{A 16}$$

A thin, untwisted magnetic flux tube is a thin tubular structure consisting of the neighbourhood of a smooth curve C, such that the magnetic field is confined within the tube and is parallel to C.

(i) Consider a simple example of a single, closed, untwisted magnetic flux tube such that

$$\boldsymbol{B} = B(r, z)\,\boldsymbol{e}_\phi, \tag{A 17}$$

where (r, ϕ, z) are cylindrical polar coordinates and $B(r, z)$ is a positive function localized near $(r = a, z = 0)$. The tube is contained entirely within V. Show that the magnetic helicity of this field is uniquely defined and equal to zero.

(ii) Use the fact that H_m is conserved in ideal MHD to argue that the magnetic helicity of any single, closed, untwisted and unknotted flux tube contained within V is also zero.

(iii) Consider a situation in which V contains two such flux tubes T_1 and T_2. Let F_1 and F_2 be the magnetic fluxes associated with T_1 and T_2. By writing $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, etc., and assuming that the tubes are thin, show that

$$H_m = \pm 2F_1F_2 \tag{A 18}$$

if the tubes are simply interlinked, while $H_m = 0$ if they are unlinked.

A.11. *Variational principles*

The magnetic energy in a volume V bounded by a surface S is

$$E_m = \int_V \frac{B^2}{2\mu_0}\,\mathrm{d}V. \tag{A 19}$$

(i) Making use of the representation $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ of the magnetic field in terms of a magnetic vector potential, show that the magnetic field that minimizes E_m, subject to the tangential components of $\boldsymbol{A}$ being specified on S, is a potential field. Argue that this constraint corresponds to specifying the normal component of $\boldsymbol{B}$ on S.

(ii) Making use of the representation $\boldsymbol{B} = \nabla\alpha \times \nabla\beta$ of the magnetic field in terms of Euler potentials, show that the magnetic field that minimizes E_m, subject to α and β being specified on S, is a force-free field. Argue that this constraint corresponds to specifying the normal component of $\boldsymbol{B}$ on S and also the way in which points on S are connected by magnetic field lines.

A.12. *Friedrichs diagrams*

The dispersion relations $\omega(\boldsymbol{k})$ for Alfvén and magnetoacoustic waves in a uniform medium are given by

$$v_p^2 = v_a^2 \cos^2\theta, \tag{A 20}$$

$$v_p^4 - (v_s^2 + v_a^2)v_p^2 + v_s^2 v_a^2 \cos^2\theta = 0, \tag{A 21}$$

where $v_p = \omega/k$ is the phase velocity and θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}$. Use the form of $v_p(\theta)$ for each mode to calculate the group velocities $\boldsymbol{v}_g = \partial\omega/\partial\boldsymbol{k}$, determining their components parallel and perpendicular to $\boldsymbol{B}$.

Sketch the phase and group diagrams by tracking $\boldsymbol{v}_p = v_p\hat{\boldsymbol{k}}$ and $\boldsymbol{v}_g$, respectively, over the full range of θ. Treat the cases $v_s > v_a$ and $v_s < v_a$ separately. By analysing the limit $\theta \to \pi/2$, show that the group diagram for the slow wave has a cusp at speed $v_s v_a (v_s^2 + v_a^2)^{-1/2}$.

A.13. *Shock relations*

The Rankine–Hugoniot relations in the rest frame of a non-magnetic shock are

$$[\rho u_x]_1^2 = 0, \tag{A 22}$$

$$[\rho u_x^2 + p]_1^2 = 0, \tag{A 23}$$

$$\left[\rho u_x \left(\tfrac{1}{2}u_x^2 + h\right)\right]_1^2 = 0, \tag{A 24}$$

where $u_x > 0$ and $[Q]_1^2 = Q_2 - Q_1$ is the difference between the downstream and upstream values of any quantity Q. Show that the velocity, density and pressure ratios

$$U = \frac{u_2}{u_1}, \quad D = \frac{\rho_2}{\rho_1}, \quad P = \frac{p_2}{p_1} \tag{A 25a–c}$$

across a shock in a perfect gas are given by

$$D = \frac{1}{U} = \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2}, \quad P = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{(\gamma+1)}, \tag{A 26a,b}$$

where $\mathcal{M} = u_x/v_s$ is the Mach number, and also that

$$\mathcal{M}_2^2 = \frac{(\gamma - 1)\mathcal{M}_1^2 + 2}{2\gamma\mathcal{M}_1^2 - (\gamma - 1)}. \tag{A 27}$$

Show that the entropy change in passing through the shock is given by

$$\frac{[s]_1^2}{c_v} = \ln P - \gamma \ln \left[\frac{(\gamma + 1)P + (\gamma - 1)}{(\gamma - 1)P + (\gamma + 1)}\right] \tag{A 28}$$

and deduce that only compression shocks ($D > 1$, $P > 1$) are physically realizable.

A.14. *Oblique shocks*

For a hydrodynamic shock, let u_{X2} and u_{Y2} be the downstream velocity components parallel and perpendicular, respectively, to the upstream velocity vector $\boldsymbol{u}_1$. In the limit of a strong shock, $\mathcal{M}_1 \gg 1$, derive the relation

$$u_{Y2}^2 = (|\boldsymbol{u}_1| - u_{X2}) \left[u_{X2} - \left(\frac{\gamma - 1}{\gamma + 1}\right) |\boldsymbol{u}_1|\right]. \tag{A 29}$$

Sketch this relation in the (u_{X2}, u_{Y2}) plane. Hence show that the maximum angle through which the velocity vector can be deflected on passing through a stationary strong shock is $\arcsin(1/\gamma)$.

A.15. *The Riemann problem*

A perfect gas flows in one dimension in the absence of boundaries, gravity and magnetic fields.

(i) Determine all possible smooth local solutions of the equations of one-dimensional gas dynamics that depend only on the variable $\xi = x/t$ for $t > 0$. Show that one such solution is a rarefaction wave in which $\mathrm{d}u/\mathrm{d}\xi = 2/(\gamma + 1)$. How do the adiabatic sound speed and specific entropy vary with ξ?

(ii) At $t = 0$ the gas is initialized with uniform density ρ_L, pressure p_L and velocity u_L in the region $x < 0$ and with uniform density ρ_R, pressure p_R and velocity u_R in the region $x > 0$. Explain why the subsequent flow is of the similarity form described in part (i). What constraints must be satisfied by the initial values if the subsequent evolution is to involve only two uniform states connected by a rarefaction wave? Give a non-trivial example of such a solution.

(iii) Explain why, for more general choices of the initial values, the solution cannot have the simple form described in part (ii), even if $u_R > u_L$. What other features will appear in the solution? (Detailed calculations are not required.)

A.16. *Nonlinear waves in incompressible MHD*

Show that the equations of ideal MHD in the case of an incompressible fluid of uniform density ρ can be written in the symmetrical form

$$\frac{\partial z_\pm}{\partial t} + z_\mp \boldsymbol{\cdot} \nabla z_\pm = -\nabla \psi, \tag{A 30}$$

$$\nabla \boldsymbol{\cdot} z_\pm = 0, \tag{A 31}$$

where

$$z_{\pm} = \boldsymbol{u} \pm \boldsymbol{v}_a \tag{A 32}$$

are the Elsässer variables, $\boldsymbol{v}_a = (\mu_0 \rho)^{-1/2} \boldsymbol{B}$ is the vector Alfvén velocity and $\psi = \Phi + (\Pi/\rho)$ is a modified pressure.

Consider a static basic state in which the magnetic field is uniform and $\psi = \text{const.}$ Write down the exact equations governing perturbations $(z'_{\pm}, \psi')$ (i.e. without performing a linearization). Hence show that there are special solutions in which disturbances of arbitrary amplitude propagate along the magnetic field lines in one direction or other without change of form. How do these relate to the MHD wave modes of a compressible fluid? Why does the general argument for wave steepening not apply to these nonlinear simple waves?

A.17. *Spherical blast waves*

A supernova explosion of energy E occurs at time $t = 0$ in an unmagnetized perfect gas of adiabatic exponent γ. The surrounding medium is initially cold and has non-uniform density $Cr^{-\beta}$, where C and β are constants (with $0 < \beta < 3$) and r is the distance from the supernova.

(i) Explain why a self-similar spherical blast wave may be expected to occur, and deduce that the radius $R(t)$ of the shock front increases as a certain power of t.

(ii) Write down the self-similar form of the velocity, density and pressure for $0 < r < R(t)$ in terms of three undetermined dimensionless functions of $\xi = r/R(t)$. Obtain a system of dimensionless ordinary differential equations governing these functions, and formulate the boundary conditions on the dimensionless functions at the strong shock front $\xi = 1$.

(iii) Show that special solutions exist in which the radial velocity and the density are proportional to r for $r < R(t)$, if

$$\beta = \frac{7-\gamma}{\gamma+1}. \tag{A 33}$$

For the case $\gamma = 5/3$ express the velocity, density and pressure for this special solution in terms of the original dimensional variables.

A.18. *Accretion on to a black hole*

Write down the equations of steady, spherical accretion of a perfect gas in an arbitrary gravitational potential $\Phi(r)$.

Accretion on to a black hole can be approximated within a Newtonian theory by using the Paczyński–Wiita potential

$$\Phi = -\frac{GM}{r - r_h}, \tag{A 34}$$

where $r_h = 2GM/c^2$ is the radius of the event horizon and c is the speed of light.

Show that the sonic radius r_s is related to r_h and the nominal accretion radius $r_a = GM/2v_{s0}^2$ (where v_{s0} is the sound speed at infinity) by

$$2r_s^2 - \left[(5-3\gamma)r_a + 4r_h\right] r_s + 2r_h^2 - 4(\gamma-1)r_a r_h = 0. \tag{A 35}$$

Argue that the accretion flow passes through a unique sonic point for any value of $\gamma > 1$. Assuming that $v_{s0} \ll c$, find approximations for r_s in the cases (i) $\gamma < 5/3$, (ii) $\gamma = 5/3$ and (iii) $\gamma > 5/3$.

A.19. *Spherical flow in a power-law potential*

For steady, spherically symmetric, adiabatic flow in a gravitational potential $\Phi = -Ar^{-\beta}$, where A and β are positive constants, show that a necessary condition for either (i) an inflow that starts from rest at $r = \infty$ or (ii) an outflow that reaches $r = \infty$ to pass through a sonic point is

$$\gamma < f(\beta), \tag{A 36}$$

where $\gamma > 1$ is the adiabatic exponent and $f(\beta)$ is a function to be determined.

Assuming that this condition is satisfied, calculate the accretion rate of a transonic accretion flow in terms of A, β, γ and the density and sound speed at $r = \infty$. Evaluate your expression in each of the limits $\gamma \to 1$ and $\gamma \to f(\beta)$. (You may find it helpful to define $\delta = \gamma - 1$.)

A.20. *Rotating outflows*

The wind from a rotating star can be modelled as a steady, axisymmetric, adiabatic flow in which the magnetic field is neglected. Let $\psi(r, z)$ be the mass flux function, such that

$$\rho \boldsymbol{u}_p = \nabla\psi \times \nabla\phi, \tag{A 37}$$

where (r, ϕ, z) are cylindrical polar coordinates and $\boldsymbol{u}_p$ is the poloidal part of the velocity. Show that the specific entropy, the specific angular momentum and the Bernoulli function are constant along streamlines, giving rise to three functions $s(\psi)$, $\ell(\psi)$ and $\varepsilon(\psi)$. Use the remaining dynamical equation to show that ψ satisfies the partial differential equation

$$\frac{1}{\rho}\nabla\boldsymbol{\cdot}\left(\frac{1}{\rho r^2}\nabla\psi\right) = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\psi} - T\frac{\mathrm{d}s}{\mathrm{d}\psi} - \frac{\ell}{r^2}\frac{\mathrm{d}\ell}{\mathrm{d}\psi}. \tag{A 38}$$

A.21. *Critical points of magnetized outflows*

The integrals of the equations of ideal MHD for a steady axisymmetric outflow are

$$\boldsymbol{u} = \frac{k\boldsymbol{B}}{\rho} + r\omega\boldsymbol{e}_\phi, \tag{A 39}$$

$$u_\phi - \frac{B_\phi}{\mu_0 k} = \frac{\ell}{r}, \tag{A 40}$$

$$s = s(\psi), \tag{A 41}$$

$$\tfrac{1}{2}|\boldsymbol{u} - r\omega\boldsymbol{e}_\phi|^2 + \Phi - \tfrac{1}{2}r^2\omega^2 + h = \tilde{\varepsilon}, \tag{A 42}$$

where $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\tilde{\varepsilon}(\psi)$ are surface functions. Assume that the magnetic flux function $\psi(r, z)$ is known from a solution of the Grad–Shafranov equation, and let the cylindrical radius r be used as a parameter along each magnetic field line. Then the poloidal magnetic field $\boldsymbol{B}_p = \nabla\psi \times \nabla\phi$ is a known function of r on each field line. Assume further that the surface functions $k(\psi)$, $\omega(\psi)$, $\ell(\psi)$, $s(\psi)$ and $\tilde{\varepsilon}(\psi)$ are known.

Show that (A 39)–(A 41) can then be used, in principle, and together with the equation of state, to determine the velocity $\boldsymbol{u}$ and the specific enthalpy h as functions of ρ and r on each field line. Deduce that (A 42) has the form

$$f(\rho, r) = \tilde{\varepsilon} = \text{const.} \tag{A 43}$$

on each field line.

Show that

$$-\rho\frac{\partial f}{\partial \rho} = \frac{u_p^4 - (v_s^2 + v_a^2)u_p^2 + v_s^2 v_{ap}^2}{u_p^2 - v_{ap}^2}, \tag{A 44}$$

where v_s is the adiabatic sound speed, v_a is the (total) Alfvén speed and the subscript 'p' denotes the poloidal (meridional) component. Deduce that the flow has critical points where u_p equals the phase speed of axisymmetric fast or slow magnetoacoustic waves. What condition must be satisfied by $\partial f/\partial r$ for the flow to pass through these critical points?

A.22. *Radial oscillations of a star*

Show that purely radial (i.e. spherically symmetric) oscillations of a spherical star satisfy the Sturm–Liouville equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\gamma p}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r)\right] - \frac{4}{r}\frac{\mathrm{d}p}{\mathrm{d}r}\xi_r + \rho\omega^2\xi_r = 0. \tag{A 45}$$

How should ξ_r behave near the centre of the star and near the surface $r = R$ at which $p = 0$?

Show that the associated variational principle can be written in the equivalent forms

$$\begin{aligned}\omega^2\int_0^R \rho|\xi_r|^2\, r^2\,\mathrm{d}r &= \int_0^R\left[\frac{\gamma p}{r^2}\left|\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r)\right|^2 + 4r\frac{\mathrm{d}p}{\mathrm{d}r}|\xi_r|^2\right]\mathrm{d}r\\ &= \int_0^R\left[\gamma p r^4\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\xi_r}{r}\right)\right|^2 + (4-3\gamma)r\frac{\mathrm{d}p}{\mathrm{d}r}|\xi_r|^2\right]\mathrm{d}r,\end{aligned} \tag{A 46}$$

where γ is assumed to be independent of r. Deduce that the star is unstable to purely radial perturbations if and only if $\gamma < 4/3$. Why does it not follow from the first form of the variational principle that the star is unstable for all values of γ?

Can you reach the same conclusion using only the virial theorem?

A.23. *Waves in an isothermal atmosphere*

Show that linear waves of frequency ω and horizontal wavenumber k_h in a plane-parallel isothermal atmosphere satisfy the equation

$$\frac{\mathrm{d}^2\xi_z}{\mathrm{d}z^2} - \frac{1}{H}\frac{\mathrm{d}\xi_z}{\mathrm{d}z} + \frac{(\gamma-1)}{\gamma^2H^2}\xi_z + (\omega^2 - N^2)\left(\frac{1}{v_s^2} - \frac{k_h^2}{\omega^2}\right)\xi_z = 0, \tag{A 47}$$

where H is the isothermal scale height, N is the Brunt–Väisälä frequency and v_s is the adiabatic sound speed.

Consider solutions of the vertically wave-like form

$$\xi_z \propto \mathrm{e}^{z/2H}\exp(\mathrm{i}k_z z), \tag{A 48}$$

where k_z is real, so that the wave energy density (proportional to $\rho|\boldsymbol{\xi}|^2$) is independent of z. Obtain the dispersion relation connecting ω and $\boldsymbol{k}$. Assuming that $N^2>0$, show that propagating waves exist in the limits of high and low frequencies, for which

$$\omega^2 \approx v_s^2 k^2 \text{ (acoustic waves)} \quad \text{and} \quad \omega^2 \approx \frac{N^2 k_h^2}{k^2} \text{ (gravity waves)} \tag{A 49a,b}$$

respectively. Show that the minimum frequency at which acoustic waves propagate is $v_s/2H$.

Explain why the linear approximation must break down above some height in the atmosphere.

A.24. *Gravitational instability of a slab*

An isothermal ideal gas of sound speed c_s forms a self-gravitating slab in hydrostatic equilibrium with density $\rho(z)$, where (x, y, z) are Cartesian coordinates.

(i) Verify that

$$\rho \propto \operatorname{sech}^2\left(\frac{z}{H}\right), \tag{A 50}$$

and relate the scale height H to the surface density

$$\Sigma = \int_{-\infty}^{\infty} \rho \, \mathrm{d}z. \tag{A 51}$$

(ii) Assuming that the perturbations are also isothermal, derive the linearized equations governing displacements of the form

$$\mathrm{Re}\left[\boldsymbol{\xi}(z)\, \mathrm{e}^{\mathrm{i}(kx-\omega t)}\right], \tag{A 52}$$

where k is a real wavenumber. Show that ω^2 is real for disturbances satisfying appropriate conditions as $|z| \to \infty$.

(iii) For a marginally stable mode with $\omega^2 = 0$, derive the associated Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(1-\tau^2)\frac{\mathrm{d}\,\delta\Phi}{\mathrm{d}\tau}\right] + \left(2 - \frac{\nu^2}{1-\tau^2}\right)\delta\Phi = 0, \tag{A 53}$$

where $\tau = \tanh(z/H)$, $\nu = kH$ and $\delta\Phi$ is the Eulerian perturbation of the gravitational potential. Verify that two solutions of this equation are

$$\left(\frac{1+\tau}{1-\tau}\right)^{\nu/2}(\nu-\tau) \quad \text{and} \quad \left(\frac{1-\tau}{1+\tau}\right)^{\nu/2}(\nu+\tau). \tag{A 54a,b}$$

Deduce that the marginally stable mode has $|k| = 1/H$ and $\delta\Phi \propto \operatorname{sech}(z/H)$. Would you expect the unstable modes to have wavelengths greater or less than $2\pi H$?

A.25. *Magnetic buoyancy instabilities*

A perfect gas forms a static atmosphere in a uniform gravitational field $-g\boldsymbol{e}_z$, where (x, y, z) are Cartesian coordinates. A horizontal magnetic field $B(z)\boldsymbol{e}_y$ is also present.

Derive the linearized equations governing small displacements of the form

$$\mathrm{Re}[\boldsymbol{\xi}(z)\,\exp(-\mathrm{i}\omega t + \mathrm{i}k_x x + \mathrm{i}k_y y)], \tag{A 55}$$

where k_x and k_y are real horizontal wavenumbers, and show that

$$\omega^2 \int_a^b \rho|\boldsymbol{\xi}|^2 \, \mathrm{d}z = [\xi_z^* \, \delta\Pi]_a^b$$
$$+ \int_a^b \left(\frac{|\delta\Pi|^2}{\gamma p + \dfrac{B^2}{\mu_0}} - \frac{\left|\rho g \xi_z + \dfrac{B^2}{\mu_0} \mathrm{i} k_y \xi_y\right|^2}{\gamma p + \dfrac{B^2}{\mu_0}} + \frac{B^2}{\mu_0} k_y^2 |\boldsymbol{\xi}|^2 - g\frac{\mathrm{d}\rho}{\mathrm{d}z}|\xi_z|^2 \right) \mathrm{d}z, \qquad \text{(A 56)}$$

where $z = a$ and $z = b$ are the lower and upper boundaries of the atmosphere, and $\delta\Pi$ is the Eulerian perturbation of total pressure. (Self-gravitation may be neglected.)

You may assume that the atmosphere is unstable if and only if the integral on the right-hand side can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, which are such that $[\xi_z^* \, \delta\Pi]_a^b = 0$. You may also assume that the horizontal wavenumbers are unconstrained. Explain why the integral can be minimized with respect to ξ_x by letting $\xi_x \to 0$ and $k_x \to \infty$ in such a way that $\delta\Pi = 0$.

Hence show that the atmosphere is unstable to disturbances with $k_y = 0$ if and only if

$$-\frac{\mathrm{d} \ln \rho}{\mathrm{d}z} < \frac{\rho g}{\gamma p + \dfrac{B^2}{\mu_0}} \qquad \text{(A 57)}$$

at some point.

Assuming that this condition is not satisfied anywhere, show further that the atmosphere is unstable to disturbances with $k_y \neq 0$ if and only if

$$-\frac{\mathrm{d} \ln \rho}{\mathrm{d}z} < \frac{\rho g}{\gamma p} \qquad \text{(A 58)}$$

at some point.

How does these stability criteria compare with the hydrodynamic stability criterion $N^2 < 0$?

A.26. *Waves in a rotating fluid*

Write down the equations of ideal gas dynamics in cylindrical polar coordinates (r, ϕ, z), assuming axisymmetry. Consider a steady, axisymmetric basic state in uniform rotation, with density $\rho(r, z)$, pressure $p(r, z)$ and velocity $\boldsymbol{u} = r\Omega \boldsymbol{e}_\phi$. Determine the linearized equations governing axisymmetric perturbations of the form

$$\mathrm{Re}[\delta\rho(r, z) \, \mathrm{e}^{-\mathrm{i}\omega t}], \qquad \text{(A 59)}$$

etc. If the basic state is homentropic and self-gravity may be neglected, show that the linearized equations reduce to

$$-\mathrm{i}\omega\delta u_r - 2\Omega\delta u_\phi = -\frac{\partial W}{\partial r}, \qquad \text{(A 60)}$$

$$-\mathrm{i}\omega\delta u_\phi + 2\Omega\delta u_r = 0, \qquad \text{(A 61)}$$

$$-\mathrm{i}\omega\delta u_z = -\frac{\partial W}{\partial z}, \qquad \text{(A 62)}$$

$$-\mathrm{i}\omega W+\frac{v_s^2}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\rho\delta u_r)+\frac{\partial}{\partial z}(\rho\delta u_z)\right]=0, \tag{A 63}$$

where $W=\delta p/\rho$.

Eliminate $\delta\boldsymbol{u}$ to obtain a second-order partial differential equation for W. Is the equation of elliptic or hyperbolic type? What are the relevant solutions of this equation if the fluid has uniform density and fills a cylindrical container $\{r<a,\ 0<z<H\}$ with rigid boundaries?

Appendix B. Electromagnetic units

These lecture notes use rationalized units for electromagnetism, such that Maxwell's equations take the form

$$\frac{\partial\boldsymbol{B}}{\partial t}=-\nabla\times\boldsymbol{E},\quad \nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{B}=\mu_0\left(\boldsymbol{J}+\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right),\quad \nabla\cdot\boldsymbol{E}=\frac{\rho_e}{\epsilon_0}. \tag{B 1a–d}$$

These involve the vacuum permeability and permittivity μ_0 and ϵ_0, related to the speed of light c by $c=(\mu_0\epsilon_0)^{-1/2}$, but do not involve factors of 4π or c.

In astrophysics it is common to use Gaussian units for electromagnetism, such that Maxwell's equations take the form

$$\frac{\partial\boldsymbol{B}}{\partial t}=-c\nabla\times\boldsymbol{E},\quad \nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{B}=\frac{1}{c}\left(4\pi\boldsymbol{J}+\frac{\partial\boldsymbol{E}}{\partial t}\right),\quad \nabla\cdot\boldsymbol{E}=4\pi\rho_e. \tag{B 2a–d}$$

In the limit relevant for Newtonian MHD, the $\partial\boldsymbol{E}/\partial t$ term is neglected. Different factors then appear in several related equations. The magnetic energy density in Gaussian units is

$$\frac{B^2}{8\pi}\quad\text{rather than}\quad\frac{B^2}{2\mu_0}, \tag{B 3}$$

the electromagnetic energy flux density (Poynting vector) is

$$\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}\quad\text{rather than}\quad\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}, \tag{B 4}$$

the Maxwell stress is

$$\frac{1}{4\pi}\left(\boldsymbol{BB}-\frac{B^2}{2}\boldsymbol{I}\right)\quad\text{rather than}\quad\frac{1}{\mu_0}\left(\boldsymbol{BB}-\frac{B^2}{2}\boldsymbol{I}\right), \tag{B 5}$$

and the Lorentz force is

$$\frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}=\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\quad\text{rather than}\quad\boldsymbol{J}\times\boldsymbol{B}=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}. \tag{B 6}$$

The perfectly conducting fluid approximation of ideal MHD corresponds to

$$\boldsymbol{E}=-\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\quad\text{rather than}\quad\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}. \tag{B 7}$$

The fields $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{J}$ can be converted from rationalized to Gaussian units by replacing

$$\left.\begin{aligned} \boldsymbol{E} &\mapsto \left(\frac{1}{4\pi\epsilon_0}\right)^{1/2}\boldsymbol{E} = c\left(\frac{\mu_0}{4\pi}\right)^{1/2}\boldsymbol{E}, \\ \boldsymbol{B} &\mapsto \left(\frac{\mu_0}{4\pi}\right)^{1/2}\boldsymbol{B}, \\ \boldsymbol{J} &\mapsto (4\pi\epsilon_0)^{1/2}\boldsymbol{J} = \frac{1}{c}\left(\frac{4\pi}{\mu_0}\right)^{1/2}\boldsymbol{J}. \end{aligned}\right\} \tag{B 8}$$

For historical reasons, rationalized electromagnetic units are associated with the MKS (metre–kilogram–second) system of mechanical units, while Gaussian electromagnetic units are associated with the CGS (centimetre–gram–second) system. The most common system of rationalized units is SI units, in which μ_0 has the exact value $4\pi \times 10^{-7}$ (in units of N A^{-2} or H m^{-1}). In principle, rationalized units can be used within CGS, in which case μ_0 has the value 4π.

Appendix C. Summary of notation

A: poloidal Alfvén number
A_i: matrix describing hyperbolic structure
$\boldsymbol{A}$: magnetic vector potential
$\boldsymbol{a}$: particle acceleration; initial position vector
B: Bernoulli constant
$\boldsymbol{B}$: magnetic field
$\boldsymbol{B}_p$: poloidal magnetic field
C_{ij}: cofactor of deformation tensor
c: speed of light; velocity dispersion
c_p: specific heat capacity at constant pressure
c_s: isothermal sound speed
c_v: specific heat capacity at constant volume
$\mathrm{D}/\mathrm{D}t$: Lagrangian time-derivative
$\boldsymbol{E}$: electric field
e: specific internal energy
$\boldsymbol{e}$: basis (unit) vector
F: determinant of deformation tensor
F_{ij}: deformation tensor
$\boldsymbol{F}$: flux density of conserved quantity
$\boldsymbol{F}_m$: Lorentz force per unit volume
$\mathcal{F}$: force operator
f: distribution function
f_M: Maxwellian distribution function
G: Newton's constant
G_{ij}: inverse of deformation tensor
g: gravitational acceleration
$\boldsymbol{g}$: gravitational field

H: Heaviside step function; scale height
H_c: cross helicity
H_k: kinetic helicity
H_m: magnetic helicity
h: specific enthalpy
I: trace of inertia tensor; moment of inertia
I_{ij}: inertia tensor
$\boldsymbol{I}$: unit tensor
J: Jacobian determinant
J_0, J_1: Bessel functions
J_{ij}: Jacobian matrix
$\boldsymbol{J}$: electric current density
K: polytropic constant; kinetic energy
K_{ij}: kinetic energy tensor
k: Boltzmann's constant; wavenumber; mass loading; potential Love number
$\boldsymbol{k}$: wavevector
L: characteristic length scale; Lagrangian
$\mathcal{L}$: Lagrangian density
ℓ: angular momentum invariant; specific angular momentum
M: magnetic energy; mass
$\boldsymbol{M}$: Maxwell stress tensor
$\mathcal{M}$: Mach number
m: particle mass
m_H: mass of hydrogen atom
N: buoyancy frequency
n: number of degrees of freedom; number density
$\boldsymbol{n}$: unit normal vector
p: pressure
p_g: gas pressure
p_m: magnetic pressure
p_r: radiation pressure
q: electric charge; density of conserved quantity
R: shock radius
$R_\pm$: Riemann invariants
r: cylindrical radius; spherical radius
r_0: footpoint radius
r_a: Alfvén radius
r_s: sonic radius
S: bounding surface; action
s: specific entropy
T: temperature; characteristic time scale; torque
T_m: magnetic tension
$\boldsymbol{T}$: stress tensor

$\mathcal{T}$: trace of integrated stress tensor
$\mathcal{T}_{ij}$: integrated stress tensor
t : time
U : internal energy
$\boldsymbol{U}$: state vector
u_{sh} : shock speed
$\boldsymbol{u}$: velocity field
V : volume (occupied by fluid)
$\hat{V}$: exterior volume
V_{ij} : second-rank potential energy tensor
$V_{ijk\ell}$: fourth-rank potential energy tensor
v : specific volume; wave speed
v_s : adiabatic sound speed
$\boldsymbol{v}$: particle velocity; relative velocity of frames
$\boldsymbol{v}_a$: Alfvén velocity
$\boldsymbol{v}_g$: group velocity
$\boldsymbol{v}_p$: phase velocity
W : gravitational energy; potential energy functional
x : Cartesian coordinate
$\boldsymbol{x}$: position vector
Y : spherical harmonic
y : Cartesian coordinate
z : Cartesian coordinate
β : plasma beta
Γ_1 : first adiabatic exponent
γ : ratio of specific heats; adiabatic exponent
Δ : Lagrangian perturbation; divergence of displacement
δ : Eulerian perturbation; Dirac delta function
δm : material mass element
$\delta \boldsymbol{S}$: material surface element
$\delta \boldsymbol{u}$: velocity difference; velocity perturbation
δV : material volume element
$\delta \boldsymbol{x}$: material line element
$\delta \Phi$: material flux element
ϵ_{ijk} : Levi–Civita tensor
ε : energy invariant
$\boldsymbol{\eta}$: secondary displacement
θ : polar angle; angle between wavevector and magnetic field
λ : mean free path; force-free field scalar; scaling parameter
μ : mean molecular weight; scaling parameter
μ_0 : vacuum permeability
ξ : similarity variable
$\boldsymbol{\xi}$: (Lagrangian) displacement

Π : total pressure
ρ : mass density
ρ_e : charge density
σ : Stefan's constant; collisional cross-section
τ : relaxation time
Φ : gravitational potential
Φ_e : electrostatic potential
Φ_{ext} : external gravitational potential
Φ_{int} : internal (self-) gravitational potential
ϕ : azimuthal angle
φ : phase
χ : scalar field in gauge transformation
χ_ρ : inverse isothermal compressibility
Ψ : secondary gravitational potential
ψ : magnetic flux function
Ω : angular velocity
ω : wave frequency
$\boldsymbol{\omega}$: vorticity

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