Cambridge Part III Maths

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Fluid Dynamics of Climate

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Contents

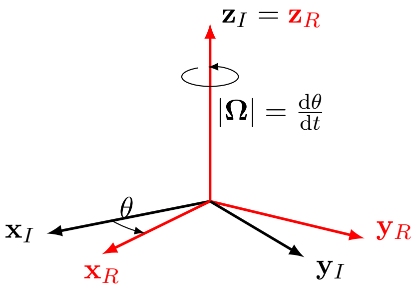
Lecture 1
12/10/20

1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p - \rho \nabla \phi + \rho \boldsymbol{F}$$

The body forces are assumed to be conservative with potential ϕ, e.g. $\phi = gz$ for gravitational force. $\boldsymbol{F}$ is the frictional force.
Consider a reference frame rotating about the z-axis with constant angular velocity $\boldsymbol{\Omega}$. Axes in the inertial frame are denoted with a subscript $_I$ and axes in the rotating frame are denoted with a subscript $_R$.



For a point with position vector $\boldsymbol{x}$ and velocity $\boldsymbol{u}_R = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R$ in the rotating reference frame

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega} \times \boldsymbol{x}$$

or equivalently $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}$. Hence the acceleration is

$$\begin{aligned} \left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}\right]\right)_R + \boldsymbol{\Omega} \times \left(\boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{x}\right)_R \\ &= \left(\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t}\right)_R + 2\boldsymbol{\Omega} \times \boldsymbol{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) \end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal accelertion.* Note that we can write the centrifugal acceleration in the form of a conservative force

$$\begin{aligned} \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) &= \nabla \phi_c \\ \phi_c &= -\frac{1}{2} \left|\boldsymbol{\Omega} \times \boldsymbol{x}\right|^2 \end{aligned}$$

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u}\right) = -\nabla p - \rho \nabla \left(\phi + \phi_c\right) + \rho \boldsymbol{F} \tag{1}$$

We group the potential terms into a *geopotential* $\Phi \equiv \phi + \phi_c$. The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.
Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

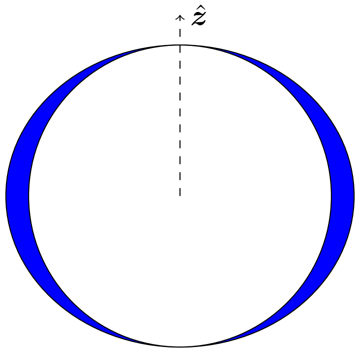


Figure 1: Geopotential ocean surface relative to a spherical Earth.

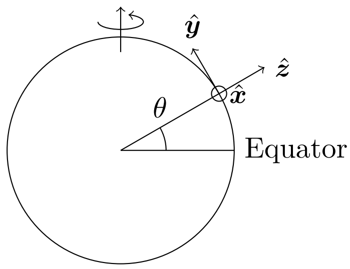


Figure 2: Local Cartesian coordinates

1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2). In this coordinate system $\boldsymbol{\Omega} = (0, \Omega\cos\theta, \Omega\sin\theta)$. Hence if $\boldsymbol{u} = (u, v, w)$ then

$$\begin{aligned} 2\boldsymbol{\Omega} \times \boldsymbol{u} &= (2\Omega w\cos\theta - 2\Omega v\sin\theta, 2\Omega u\sin\theta, -2\Omega u\cos\theta) \\ &= (-fv + f^*w, fu - f^*u) \end{aligned}$$

where $f \equiv 2\Omega\sin\theta$ is the *Coriolis parameter* and $f^* \equiv 2\Omega\cos\theta$.

Example. In Cambridge, $\theta = 52.1°N$ so

$$\begin{aligned} f &= 2\Omega\sin\theta \\ &= 2 \cdot \frac{2\pi}{3600 \cdot 24} \cdot 0.79s^{-1} \\ &\approx 1.14 \times 10^{-4}s^{-1} \end{aligned}$$

At mid-latitudes, $f \sim 10^{-4}$ is a good approximation.

We can simplify the Coriolis acceleration expression; often $f^*w \ll fv$ and $f^*u \ll g$. Hence

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \approx (-fv, fu, 0) = f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

This is the *traditional approximation.* This is *not* always a good approximation, particularly at intermediate scales.

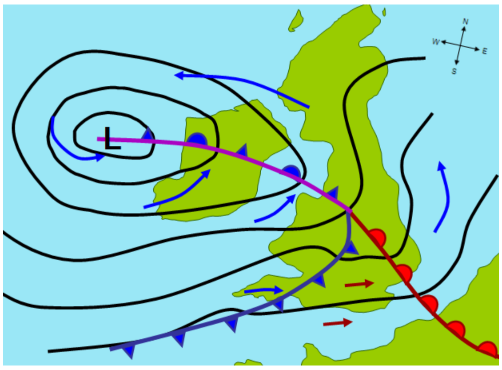


Figure 3: Lines of constant pressure p act as streamlines for the horizontal flow.
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1.2 Scale analysis.

Define characteristic scales for length L, time T, and velocity U. Non-dimensional variables are denoted with a superscript star: $\boldsymbol{u}^* = \boldsymbol{u}/U$, etc.
Using these scalings with $\boldsymbol{F} = \nu \nabla^2 \boldsymbol{u}$ we have

$$\frac{U}{T}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \frac{U^2}{L}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + fU\hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho}\nabla\left(p + \rho\Phi\right) + \frac{\nu U}{L^2}\nabla_*^2 \boldsymbol{u}^*$$

Dividing through by fU leaves the Coriolis acceleration term ord(1) with other terms scaled relatively.

$$\frac{1}{fT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \mathrm{Ro}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + \hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho f U}\nabla\left(p + \rho\Phi\right) + \mathrm{E}\nabla_*^2 \boldsymbol{u}^*$$

where $\mathrm{Ro} \equiv \frac{U}{fL}$ is the *Rossby number* and $\mathrm{E} \equiv \frac{\nu}{fL^2}$ is the *Ekman number.*

Example. For an atmospheric storm, $U \sim 10ms^{-1}, L \sim 1000km, f \sim 10^{-4}s^{-1}$. Thus $\mathrm{Ro} \sim 0.1, \mathrm{E} \sim 10^{-13}$.

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Further, if $T = L/U$, then $\mathrm{Ro} = U/fL = 1/fT$. For small Ro, E, on surfaces of constant Φ, $f\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla p$. This is *geostrophic balance*. In components, we have

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The equations of geostrophic balance can be arranged to give the horizontal velocity: $\boldsymbol{u}_H$

$$\boldsymbol{u}_H \equiv (u, v) = \frac{1}{\rho f}\hat{\boldsymbol{z}} \times \nabla p$$

Horizontal velocity is perpendicular to ∇p and hence parallel to isobars (lines of constant p), i.e. pressure acts like a streamfunction (see figure 3).
In the Northern Hemisphere, air moves clockwise around high p and anticlockwise around low p. A *cyclonic* rotation is in the same sense as $\boldsymbol{\Omega}$, *anticyclonic* in the opposite sense as $\boldsymbol{\Omega}$.

1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small Ro, E)

$$\nabla \cdot \boldsymbol{u} = 0$$
$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \tag{2}$$

Taking the curl of (2) we have

$$\begin{aligned}\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\Omega_l u_m \\ &= \varepsilon_{kij}\varepsilon_{klm}\Omega_l \partial_j u_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\Omega_l \partial_j u_m \\ &= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i\end{aligned}$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0$$

For $\boldsymbol{\Omega} = (0, 0, \Omega)$, this implies $\frac{\partial w}{\partial z} = 0$. If $w = 0$ on some horizontal surface (e.g. ground) then $w = 0$ everywhere.
Also, $u_x + v_y = 0$, i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in 'columns' parallel to $\boldsymbol{\Omega}$, called *Taylor columns*.

2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density ρ_0 with angular velocity $\boldsymbol{\Omega} = (0, 0, f/2)$. Assume small ampltiude motions (i.e. $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$), i.e. neglect $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$. From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \tag{3}$$
$$v_t + fu = -\frac{p_y}{\rho_0} \tag{4}$$
$$w_t = -\frac{p_z}{\rho_0} \tag{5}$$
$$u_x + v_y + w_z = 0 \tag{6}$$

We will eliminate variables in favour of p.

$$\begin{aligned}\nabla \cdot ((3) - (5)) &\implies \nabla^2 p = \rho_0 f\,(v_x - u_y) \\ \partial_x(4) - \partial_y(3) \& (6) &\implies (v_x - u_y)_t = f w_z\end{aligned}$$

Combining these and using (5) we have

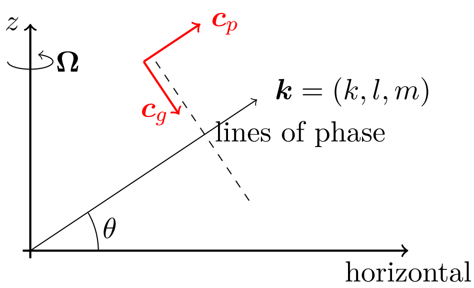
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for p. Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed $\boldsymbol{c}_p = w/\boldsymbol{k}$ and group velocity

$$\boldsymbol{c}_g = \frac{\partial w}{\partial \boldsymbol{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\boldsymbol{k}|^{3/2}}$$

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Note that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$. Also note $|\omega| \leq |f|$.

2.1 Inertial (free) oscillations

Assume $\nabla p = \mathbf{0}$. The x and y components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus $u = U \sin ft$ where f is the *inertial frequency*. Similarly, we have $v = U \cos ft$. For a particle with position (x_p, y_p) floating on an ocean surface $z = 0$ moving with the fluid velocity, we have

$$\frac{\mathrm{d}x_p}{\mathrm{d}t} = u \implies x_p = -\frac{U}{f} \cos ft + x_0$$
$$\frac{\mathrm{d}y_p}{\mathrm{d}t} = v \implies y_p = -\frac{U}{f} \sin ft + y_0$$

Thus the motion of fluid particles describes describes *inertial circles* with radius $\frac{2U}{f}$.

2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress $\boldsymbol{\tau}_w$. Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e. $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e. $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e. $w(z = 0) = 0$
4. No flow in deep ocean, i.e. $\lim_{z \to -\infty} \boldsymbol{u} = \mathbf{0}$
5. Constant density ρ
6. Traditional approximation

Continuity (incompressibility) says $u_x + v_y + w_z = 0$. Assumptions 2 and 3 then imply $w = 0$ everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$
$$fu = \nu v_{zz} \tag{8}$$

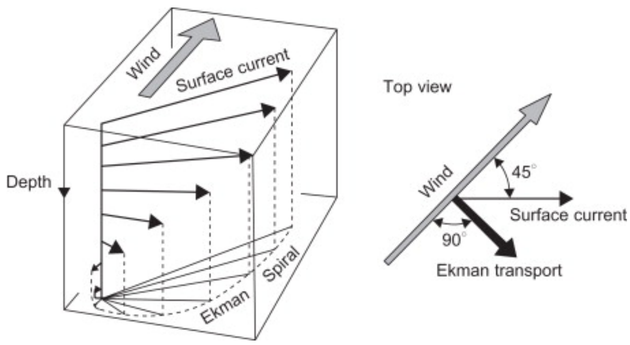

Figure 4: Ekman spiral.

Define the *complex velocity* $\mathcal{V} \equiv u + iv$. Then

$$\mathcal{V}_{zz} = \frac{if}{\nu}\mathcal{V} \tag{9}$$

Without loss of generality, assume $\boldsymbol{\tau}_w$ is aligned with the x-axis: $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$. Boundary conditions for (9) are

$$\mathcal{V}_z = \left(\frac{\tau_w}{\rho\nu}, 0\right) \quad \text{at } z = 0$$
$$\mathcal{V} = (0, 0) \quad \text{as } z \to -\infty$$

Thus $\mathcal{V} = Ae^{(1+i)z/\delta}$ where $\delta = \sqrt{\frac{2\nu}{f}}$, $A = \frac{\tau_w \delta(1-i)}{2\rho\nu}$. In terms of the velocity components, we have

$$u = \frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$
$$v = -\frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta} \sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where $\nu\boldsymbol{u}_z \approx 0$ at $z = -h$. Since $\nu\boldsymbol{u}_z(z = 0) = (\tau_w/\rho, 0)$, the *Ekman transport* $\boldsymbol{U}_T$ is

$$U_T \equiv \int_{-h}^{0} u \,\mathrm{d}z = 0$$
$$V_T \equiv \int_{-h}^{0} v \,\mathrm{d}z = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented 90° to the right of the applied wind shear stress (in the Northern Hemisphere).

2.4 Ekman pumping

Consider a wind stress $\tau_w(y)$ that varies over large scales. Then from incompressibility

$$\int_{-h}^{0} w_z \,\mathrm{d}z = -\int_{-h}^{0} u_x \,\mathrm{d}z - \int_{-h}^{0} v_y \,\mathrm{d}z$$

Thus for h constant,

$$-w(z=-h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\tau_w}{\rho f}\right)$$

In general we have

$$w(z=-h) = \hat{\boldsymbol{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$

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3 Rotating shallow water equations

Consider a thin layer of fluid with constant density ρ. Define characteristic scales

- length $L =$ horiz., $H =$ vert.
- velocity U
- time T
- pressure P

such that $\partial_x, \partial_y \sim \frac{1}{L}, \partial_z \sim \frac{1}{H}$. Define the *aspect ratio* $\delta \equiv H/L$. We will assume $\delta \ll 1$. From continuity (incompressibility) we have

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \implies \frac{w}{H} &= \mathcal{O}(U/L) \\ \implies w &= \mathcal{O}(\delta U) \end{aligned}$$

Using the traditional approximation and assuming the fluid is inviscid, the x-momentum equation

$$\begin{array}{lccccccc} & \dfrac{\partial u}{\partial t} & + u\dfrac{\partial u}{\partial x} & + v\dfrac{\partial u}{\partial y} & + w\dfrac{\partial u}{\partial z} & - fv & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \text{scaling:} & \dfrac{U}{T} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{wU}{H} & fU & = & \dfrac{P}{\rho L} \end{array} \tag{10}$$

Thus if p_x appears at leading order then

$$P \sim \rho U \max(L/T, U, fL)$$

Similarly the z-momentum equation and its scalings are

$$\begin{array}{lcccccc} & \dfrac{\partial w}{\partial t} & + u\dfrac{\partial w}{\partial x} & + v\dfrac{\partial w}{\partial y} & + w\dfrac{\partial w}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} - g \\ \text{scaling:} & \dfrac{w}{T} & \dfrac{Uw}{L} & \dfrac{Uw}{L} & \dfrac{w^2}{H} & = & \dfrac{P}{\rho H} \end{array} \tag{11}$$

Hence $\frac{\mathrm{D}w}{\mathrm{D}t} \sim \max(\frac{w}{T}, \frac{Uw}{L})$. Comparing with the pressure term, we have

$$\begin{aligned} \frac{\frac{\mathrm{D}w}{\mathrm{D}t}}{\frac{1}{\rho}\frac{\partial p}{\partial z}} &\sim \frac{\max(\frac{w}{T}, \frac{Uw}{L})}{\frac{U}{H}\max(\frac{L}{T}, \frac{U}{L}, f)} \\ &\sim \delta^2 \frac{\max(\frac{1}{T}, \frac{U}{L})}{\max(\frac{1}{T}, \frac{U}{L}, f)} \end{aligned}$$

Therefore to $\mathcal{O}(\delta^2)$ we have *hydrostatic balance*. To this order, (11) becomes

$$\frac{\partial p}{\partial z} - \rho g \implies p = \rho g(\eta - z)$$

assuming $p = 0$ at $z = \eta(x, y, t)$. Similarly, we have $\frac{1}{\rho} p_x = g\eta_x$ and $\frac{1}{\rho} p_y = g\eta_y$. Hence horizontal acceleration (i.e. the LHS of (10)) is independent of z. Motivated by this, we *assume* that horizontal velocity is also independent of z. For $Ro \ll 1$, this follows from the Tayor-Proudman theorem. Re-writing (10) with these results we have

$$u_t + uu_x + vu_y - fv = -g\eta_x \tag{12}$$

$$v_t + uv_x + vv_y + fu = -g\eta_y \tag{13}$$

since $u_z = v_z = 0$ by assumption. Integrating the continuity equation gives

$$w = -z(u_x + v_y) + A(x, y, t)$$

where A is to be determined by the boundary conditions. Requiring no normal flow at $z = -H_0 + h_b$ is imposed by $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ where $\boldsymbol{n} = \nabla(z - h_b)$. Thus

$$-u\frac{\partial h_b}{\partial x} - v\frac{\partial h_b}{\partial y} + w = 0$$

Hence

$$A(x, y, t) = u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$$

The kinematic boundary condition at $z = \eta$ is $\frac{\mathrm{D}\eta}{\mathrm{D}t} = w$ which may be written as

$$\eta_t + u\eta_x + v\eta_y - w = 0$$

where $w = -\eta(u_x + v_y) + u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$. Combining these boundary conditions gives

$$\eta_t + [(H_0 - h_b + \eta)u]_x + [(H_0 - h_b + \eta)v]_y = 0 \tag{14}$$

If $H \equiv H_0 - h_b + \eta$ is the total depth of the fluid, then since $H_t = \eta_t$,

$$H_t + (uH)_x + (vH)_y = 0 \tag{15}$$

which is a statement of the conservation of volume (equivalently mass, since ρ is constant). Equations (12), (13), and (14) are the *rotating shallow water* (SW) equations.

3.1 Potential vorticity (PV)

Denote the vertical vorticity by $\zeta = v_x - u_y$. Consider $\partial_x(13) - \partial_y(12)$, which gives

$$\zeta_t + u\zeta_x + v\zeta_y + vf_y = -(\zeta + f)(u_x + v_y)$$

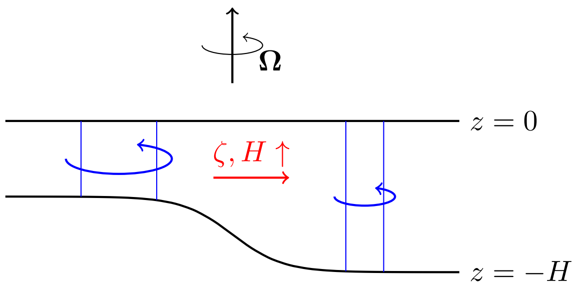
Now from conservation of volume (15),

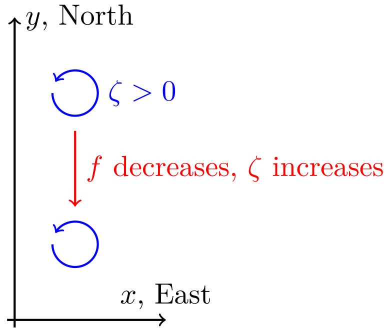
$$u_x + v_y = -\frac{1}{H}\frac{\mathrm{D}H}{\mathrm{D}t}$$

Combining these relates the material derivative of ζ and H by

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\zeta + f}{H}\frac{\mathrm{D}H}{\mathrm{D}t} \implies \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{16}$$

Let $q \equiv \frac{\zeta+f}{H}$, the *shallow water potential vorticity* (SWPV). SWPV is conserved following fluid motion. We call ζ the *relative vorticity* and f the *planetary vorticity*. ζ and f will change as a fluid moves to conserve SWPV (changing f) and angular momentum (changing depth).





Lecture 5
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4 Small amplitude motions in rotating SW

Consider a stationary fluid with depth $H_s(x, y) = H_0 - h_b$. The fluid surface is then perturbed by $\eta(x, y, t)$ where $\eta \ll H_s$. The total depth is $H(x, y, t) = H_s + \eta$. For $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$, linearise the shallow water equations:

$$u_t - fv = -g\eta_x \tag{17}$$

$$v_t + fu = -g\eta_y \tag{18}$$

$$\eta_t + (uH_s)_x + (vH_s)_y = 0$$

Assuming f is constant, we have from $\partial_x(17) + \partial_y(18)$ and $\partial_y(17) - \partial_x(18)$:

$$\partial_t \left[\left(\partial_t^2 + f^2 \right) \eta - \nabla \cdot (gH_s \nabla \eta) \right] - fgJ(H_s, \eta) = 0 \tag{19}$$

where the Jacobian $J(a, b) = a_x b_y - a_y b_x$. For the velocity components we have

$$\left(\partial_t^2 + f^2 \right) u = -g \left(\eta_{xt} + f\eta_y \right) \tag{20}$$

$$\left(\partial_t^2 + f^2 \right) v = -g \left(\eta_{yt} + f\eta_x \right) \tag{21}$$

4.1 Steady flows

We now assume $\partial_t = 0$. From (20), (21),

$$u = -\frac{g}{f}\eta_y, \qquad v = \frac{g}{f}\eta_x$$

This is *shallow water geostrophic balance*: the surface displacement η acts as a streamfunction. Applying the steady assumption to (19) gives $J(H_s, \eta) = 0$ which implies $\eta = \eta(H_s(x, y))$. Hence linearised steady geostrophic flow in shallow water follows contours of constant depth. Steady PV conservation follows from (16) with $\partial_t = 0$ and assuming $\zeta \ll f$

$$\boldsymbol{u} \cdot \nabla \frac{f}{H_s} = 0$$

Thus when f varies, the flow follows contours of constant f/H_s.

4.2 Waves in an unbounded domain

Assume H_s is constant. From (19), we have

$$\left(\partial_t^2 + f^2\right)\eta - gH_s\nabla^2\eta = 0$$

Seek plane wave solutions to this wave equation with ansatz $\eta = \eta_0\exp(i(kx + ly - \omega t))$. The dispersion relation is then

$$\omega^2 = f^2 + gH_s(k^2 + l^2) \tag{22}$$

If $f = 0$, i.e. no rotation, then the frequency is $\omega = \pm\sqrt{gH_s}|\boldsymbol{k}| = \omega_0$ and the phase speed is $|c_p| = \frac{|\omega|}{|\boldsymbol{k}|} = \sqrt{gH_s} = c_0$. For $f \neq 0$, we get *Poincaré* waves with

$$\omega^2 > \omega_0^2, \qquad |c_p| > c_0$$

i.e. rotation increases the frequency and phase speed. Define the *Rossby deformation scale* $R_D \equiv \frac{c_0}{f}$. From (22),

$$\frac{\omega^2}{f^2} = 1 + R_D^2|\boldsymbol{k}|^2$$

Without loss of generality, let $l = 0$, by reorienting x and y. If $\eta = \eta_0\cos(kx - \omega t)$ then (20), (21) imply the fluid velocity is

$$\begin{aligned} u &= \frac{\omega_0\eta_0}{kH_s}\cos(kx - \omega t) \\ v &= \frac{f\eta_0}{kH_s} \end{aligned}$$

Thus the motion is an ellipse, also known as a *tidal ellipse*, which reduces to intertial circles if $\omega_0 = f$:

$$u^2 + \frac{\omega_0^2}{f^2}v^2 = \frac{\omega_0^2\eta_0^2}{k^2H_s^2}$$

Since $\omega > f$, the fluid moves anticylonically. The Rossby deformation scale R_D is the length scale for which rotation becomes important. Consider short and long waves:

- Short waves: $|\boldsymbol{k}|R_D \gg 1$. We have $\omega^2 \to gH_s|\boldsymbol{k}|^2$ i.e. non-rotating shallow water gravity waves.
- Long waves: $|\boldsymbol{k}|R_D \ll 1$. We have $\omega^2 \to f^2$ i.e. inertial waves where fluid moves in inertial circles. Gravity is not involved.

Lecture 6
23/10/20

5 Geostrophic adjustment

Consider the response of rotating shallow water to an initial state *not* in geostrophic balance. Here, we consider $\eta(x, y,) = \eta_0\mathrm{sgn}(x)$, $\boldsymbol{u}(x, y, 0) = \boldsymbol{0}$, so the initial PV is 0.
Assume f is constant, the perturbation is small $\eta_0 \ll H$, the PV is small $\zeta \ll f$, and the bottom is flat $H_s = H_0$. Linearise the shallow water PV:

$$q = \frac{f + \zeta}{H_0 + \eta} = \frac{f}{H_0}\left(1 + \frac{\zeta}{f} + \dots\right)\left(1 - \frac{\eta}{H_0} + \dots\right) \approx \frac{f}{H_0}\left(1 + \frac{\zeta}{f} - \frac{\eta}{H_0}\right)$$

Since PV is conserved, we have

$$\frac{\zeta}{f} - \frac{\eta}{H_0} = -\frac{\eta_0}{H_0}\mathrm{sgn}(x) \qquad \forall t \tag{23}$$

By symmetry, $\partial_y \equiv 0$ so the PV is $\zeta = v_x$. The linearised shallow water equations in this case

$$\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= 0 \\ \eta_t + H_0 u_x &= 0 \end{aligned}$$

Using these equations we have

$$\zeta = v_x = \frac{u_{xt} + g\eta_{xx}}{f} = -\frac{1}{fH_0}\eta_{tt} + \frac{g}{f}\eta_{xx}$$

Now conservation of potential vorticity (23) gives

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = f^2\eta_0\mathrm{sgn}(x)$$

where $c^2 \equiv gH_0$. This is a *Klein-Gordon equation* where the $f^2\eta$ term adds elasticity to the waves.

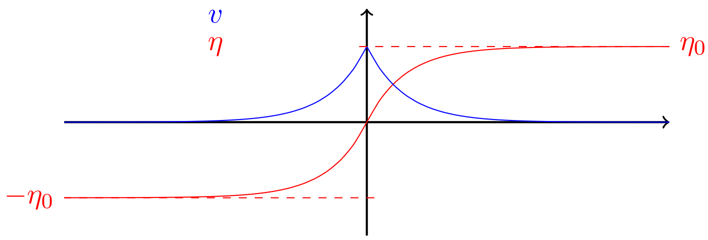
5.1 Steady solutions

Consider steady solutions. Owing to the step forcing, our BCs are to match η_x and η at $x = 0$. We find

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R_d} & x > 0 \\ -1 + e^{x/R_d} & x < 0 \end{cases} \tag{24}$$

where $R_d \equiv \sqrt{gH_0}/f$ is the *deformation radius*. From the equations of geotrophic balance we have the velocity components

$$u = 0, \qquad v = \frac{g\eta_0}{fR_d}e^{-|x|/R_d}$$



5.2 Transients

The steady solution (24) solves the geostrophic adjustment equation, but it does not match the initial conditions. We add this particular solution to a solution to the homogeneous equation

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = 0$$

with initial condition

$$\eta = \eta_0\mathrm{sgn}(x) - \eta_{\text{steady}} = \eta_0 e^{-|x|/R_d}\mathrm{sgn}(x)$$

We seek solutions of plane wave form

$$\eta = \hat{\eta}e^{i(kx - \omega t)}$$

with $\omega^2 = f^2 + c^2k^2$. These are Poincaré waves.

5.3 Energetics

The change in potential energy per unit length in the y direction is

$$\begin{aligned} PE_{\text{initial}} - PE_{\text{final}} &= \int_{-\infty}^{\infty} \int_0^{\eta_i} \rho_0 g z \, \mathrm{d}z \, \mathrm{d}x - \int_{-\infty}^{\infty} \rho_0 g z \, \mathrm{d}z \, \mathrm{d}x \\ &= 2\rho_0 g \left[\int_0^{\infty} \frac{\eta_i^2}{2} \, \mathrm{d}x - \int_0^{\infty} \frac{\eta_f^2}{2} \, \mathrm{d}x \right] \\ &= \rho_0 g \eta_0^2 \int_0^{\infty} \left[1 - (1 - e^{-x/R_d})^2 \right] \mathrm{d}x \\ &= \frac{3}{2} \rho_0 g \eta_0^2 R_d \end{aligned}$$

The change in kinetic energy per unit length in the y direction is

$$\begin{aligned} KE_{\text{initial}} - KE_{\text{final}} &= \int_{-\infty}^{\infty} \int_{-H}^{\eta_i} \frac{1}{2} \rho_0 v_i^2 \, \mathrm{d}z \, \mathrm{d}x - \int_{-\infty}^{\infty} \int_{-H}^{\eta_f} \frac{1}{2} \rho_0 v_f^2 \, \mathrm{d}z \, \mathrm{d}x \\ &\approx 0 - \frac{1}{2} \rho_0 \int_{-\infty}^{\infty} H_s v_f^2 \, \mathrm{d}x \\ &= -\rho_0 H_s \int_0^{\infty} \frac{g^2 \eta_0^2}{f^2 R_d^2} e^{-2x/R_d} \, \mathrm{d}x \\ &= -\rho_0 \frac{R_d^2 g \eta_0^2}{R_d^2} \cdot -\frac{R_d}{2} \cdot \left[e^{-2x/R_d} \right]_0^{\infty} \\ &= -\rho_0 g \eta_0^2 \frac{R_d}{2} \end{aligned}$$

Only $\frac{1}{3}$ of the potential energy released is converted into kinetic energy of the geostrophic flow.
Lecture 7 26/10/20 The remainder is radiated away by Poincaré waves.

6 Quasi-geostrophic equations

Large scale motions in the ocean and atmosphere are associated with small Rossby number $Ro \equiv \frac{U}{fL} \ll 1$. In this limit, the rotating shallow water equations are approximated by the SW quasi-geostrophic (SW QG) equation. Start from the SW PV equation:

$$\frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\zeta + f}{H} \right) = 0 \tag{25}$$

Assumption 1: $Ro \ll 1$ Assuming a small Rossby number implies the flow is close to geostrophic balance with

$$f \hat{\boldsymbol{k}} \times \boldsymbol{u} \approx -g \nabla \eta$$

where $\hat{\boldsymbol{k}}$ is the vertical unit vector. Define the *geostrophic streamfunction* $\psi \equiv \frac{g\eta}{f}$. In terms of this streamfunction we have

$$\begin{aligned} \boldsymbol{u} &\approx -\nabla \times (\psi \hat{\boldsymbol{k}}) \\ \zeta &= (\nabla \times \boldsymbol{u}) \hat{\boldsymbol{k}} \approx \nabla^2 \psi \end{aligned}$$

Assumption 2: small changes in f Recall the Coriolis parameter $f = 2\Omega \sin\theta$ where θ is latitude. Expand in a Taylor series about $\theta = \theta_0$ to get

$$f = f_0 + y\frac{\mathrm{d}f}{\mathrm{d}y}|_{\theta_0} + \cdots \approx f_0 + \beta y$$

where y is in the direction of local North, $f_0 = 2\Omega\sin\theta_0$ and β is defined as

$$\beta = \frac{1}{R}\frac{\mathrm{d}f}{\mathrm{d}\theta}|_{\theta_0} = \frac{2\Omega}{R}\cos\theta_0$$

with R the radius of Earth. For characteristic length scale L, assume $\frac{\beta L}{f_0} \ll 1$. This is the *$\beta$-plane approximation.*

Assumption 3: small changes in fluid height. This is consistent with small Rossby number: from geostrophic balance, we know $\eta \sim \frac{fUL}{g}$ and $\frac{\eta}{H_0} \sim \frac{fUL}{gH_0} = \frac{U}{fL}\frac{L^2}{R_D^2}$. Therefore $\eta/H_0 \ll 1$ if $Ro \ll \frac{R_D^2}{L^2}$. For $L \sim R_D$, $Ro \ll 1$ implies $\eta/H_0 \ll 1$. Further, we assume $h_b/H_0 \ll 1$.

Quasi-geostrophic equations. With these assumptions, SWPV becomes

$$\begin{aligned}\frac{\zeta + f}{H_0 - h_b + \eta} &\approx \frac{f_0}{H_0}\frac{1 + \frac{\beta y}{f_0} + \frac{\zeta}{f_0}}{1 - \frac{h_b}{H_0} + \frac{\eta}{H_0}} \\ &\approx \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\nabla^2\psi}{f_0} + \frac{h_b}{H_0} - \frac{f_0\psi}{gH_0}\right) \\ &= \frac{f_0}{H_0}P_g\end{aligned}$$

where P_g is the *quasi-geostrophic potential vorticity* and $\zeta = \nabla^2\psi, \eta = \frac{f_0\psi}{g}$. Hence from SWPV conservation (25),

$$\frac{\partial P_g}{\partial t} + \boldsymbol{u}\cdot\nabla P_g \approx 0$$

Using $\boldsymbol{u} \approx -\nabla\times(\psi\hat{\boldsymbol{k}})$, $\boldsymbol{u} = -\psi_y, v = \psi_x$ so

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) \approx 0 \tag{26}$$

This is the *shallow water Quasi-geostrophic* (SWQG) equation, which is one equaiton for one unknown ψ, as opposed to SWPV with 2 unknowns ζ, η.

6.1 Waves in QG

Assume a flat bottom $h_b = 0$. Linearise (26) about a state of rest (i.e. neglect terms $\mathcal{O}(\psi^2)$). Then

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{f_0^2}{gH_0}\psi\right) + \frac{\partial\psi}{\partial x}\beta = 0$$

Seek plane wave solutions of the form

$$\psi = \psi_0 e^{i(kx+ly-\omega t)}$$

with dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2 + R_D{-2}}, \qquad R_D \equiv \frac{\sqrt{gH_0}}{f_0}$$

This is the *Rossby wave dispersion relation*. Note $\omega = 0$ (i.e. no waves) if $\beta = 0$. Also, if $h_b = 0$ and $\beta = 0$ there are no wave solutions unlike rotating SW. Thus the QG system 'filters' out Poincaré waves. Note that $\beta = \frac{2\Omega}{R}\cos\theta \geq 0$, hence $c_p = \frac{\omega}{k} \leq 0$. Rossby wave speed is always directed to the *west*.

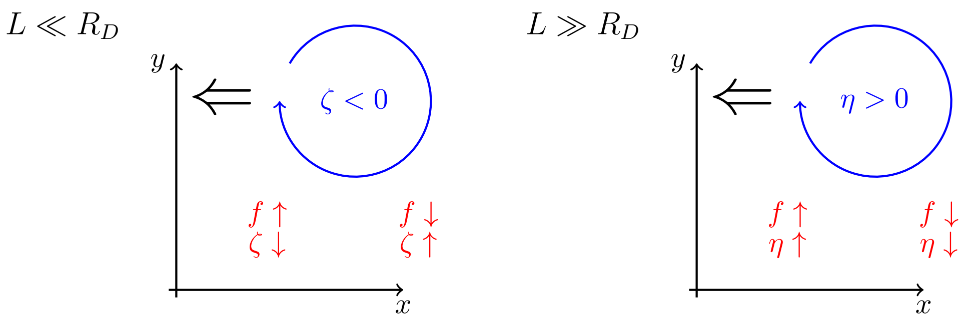
Consider the size of the dynamic terms in P_g, specifically the ratio of relative vorticity to surface height

$$\frac{\nabla^2\psi}{-\frac{f_0^2\psi}{gH_0}} \sim \frac{R_D^2}{L^2}$$

Hence relativity vorticity dominates at scales small compared to R_D whilst surface height dominates at scales large compared to R_D.

6.2 Physical interpretation of Rossby waves

Consider $L \ll R_d$ ($L \gg R_D$) and a small perturbation in the dominant term for the scale, ζ (η). For $L \ll R_D$, the planetary vorticity increases (thus ζ decreases) on the westward side, whilst the planetary vorticity decreases (thus ζ increases) on the eastward side. Hence the perturbation propagates westwards. For $L \gg R_D$, the planetary vorticity increases (η increases) on the westward size and decreases (η decreases) on the eastward side as before. Thus the perturbation propagates to the west also. These are Rossby waves.



7 Large scale ocean circulation

7.1 Sverdrup flow

Seek steady solutions for rotating shallow water driven by a wind stress $\boldsymbol{\tau}_w$. We have

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} \tag{27}$$

$$H_t + \nabla \cdot (\boldsymbol{u}H) = 0 \tag{28}$$

Consider $\nabla \times (27) \cdot \hat{\boldsymbol{k}}$ and (28) which implies modified PV conservation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = \frac{1}{H}\nabla \times \left(\frac{\boldsymbol{\tau}_w}{\rho H}\right) \cdot \hat{\boldsymbol{k}} \tag{29}$$

Thus we see frictional forcing modifies PV conservation. Assuming H is constant, $\zeta \ll f$ ($Ro \ll 1$), and using the β-plane approximation $f = f_0 + \beta y$, (29) becomes

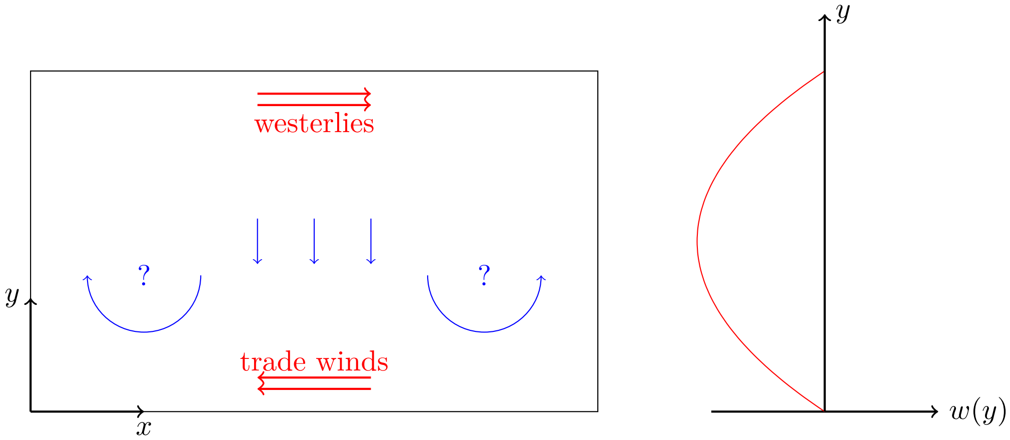
$$\beta v = \frac{1}{\rho H} (\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}} \tag{30}$$

This is called *Sverdrup balance*. Physically, the North/South advection of planetary vorticity $\boldsymbol{u} \cdot \nabla f$ balances the vorticity input by wind.

7.2 Western boundary currents

Consider steady circulation in a rectangular basin, driven by a wind stress curl

$$w(y) = \frac{(\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}}}{\rho H}$$



From (30), $w < 0 \implies v < 0$. Recall $\boldsymbol{u} = -\nabla \times \psi \hat{\boldsymbol{k}}$. Boundary conditions are no normal flow at the boundaries, i.e. ψ is constant. Sverdrup balance (30) $\beta \psi_x = w(y)$ gives

$$\psi = \frac{x w(y)}{\beta} + G(y)$$

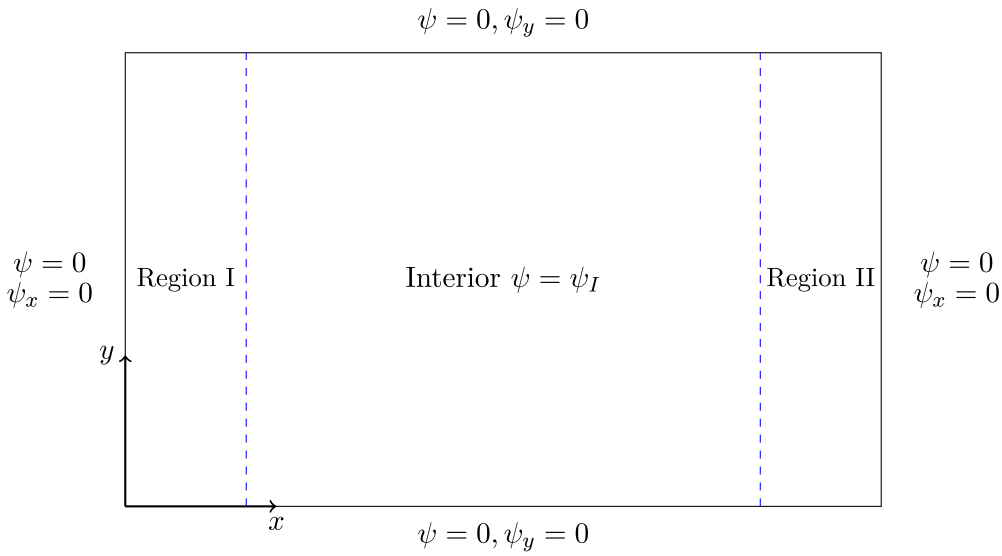
for some arbitary function $G(y)$. This presents a problem: we cannot meet the boundary conditions at both $x = 0$ and $x = L$. Hence we need extra terms and boundary layers. Following Musk, we include horizontal friction in (27):

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} + \nu \nabla^2 \boldsymbol{u} \tag{31}$$

Note here we are using the horizontal gradient $\nabla \equiv (\partial_x, \partial_y)$. Consider $\nabla \times (31) \cdot \hat{\boldsymbol{k}}$ with $\zeta \ll f$. Then

$$\beta \psi_x = w(y) + \nu \nabla^4 \psi \tag{32}$$

The PDE is now fourth order, so we need four boundary conditions.



In region I we have $\psi \approx \psi_I + \psi^{(1)}$ and in region II we have $\psi \approx \psi_I + \psi^{(2)}$. The full solution is $\psi = \psi_I + \psi^{(1)} + \psi^{(2)}$ with interior flow $\psi_I = x\frac{w(y)}{\beta} + G(y)$.

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Region I. Let $\varepsilon = \nu$ with $\varepsilon \ll 1$. Define a rescaled coordinate $\tilde{x} \equiv \frac{x}{\varepsilon^a}$ with $\partial_x = \varepsilon^{-a}\partial_{\tilde{x}}$. Note: if $a > 0$ then $\partial_x \gg \partial_y$. This is the *method of undetermined coefficients*. From the PDE (32) for ψ we have

$$\cancel{\beta\psi^I_x} + \beta\varepsilon^{-a}\tilde{\psi}^{(1)}_{\tilde{x}} = \varepsilon^{1-4a}\tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \cancel{w}$$

Matching exponents, we have $a = \frac{1}{3}$. Hence

$$\beta\tilde{\psi}^{(1)}_{\tilde{x}} = \tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$$

Seek solutions of the form $\tilde{\psi} = \tilde{\psi}_0 e^{r\tilde{x}}$. Then $r^4 - \beta r = 0$ so $r = 0, \beta^{1/3}, -\frac{1}{2}\beta^{1/3} \pm i\frac{\sqrt{3}}{2}\beta^{1/3}$. The general solution is therefore

$$\tilde{\psi}^{(1)} = A(y) + B(y)e^{\beta^{1/3}\tilde{x}} + C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}} + D(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{-i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}}$$

In order for the interior and boundary layer flows to match asymptotically, we apply the *matching condition* $\lim_{\tilde{x}\to\infty}\tilde{\psi}^{(1)} = 0$. Thus $A(y) = B(y) = 0$. For convenience we re-define C and D to get

$$\tilde{\psi}^{(1)} = C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} + D(y)\right)$$

We now apply the boundary conditions. $\psi = 0$ at $x = 0$ gives $\tilde{\psi}^{(1)} = -\psi^I \mid_{x=0}$. Hence

$$C(y)\cos D(y) = -G(y)$$

$\psi_x = 0$ at $x = 0$ gives $\psi^{(1)}_x = -\psi^I_x \mid_{x=0}$. Hence

$$\varepsilon^{-1/3}\tilde{\psi}^{(1)}_{\tilde{x}} = -\frac{w(y)}{\beta}$$

$$\varepsilon^{-1/3}(-\frac{1}{2}\beta^{1/3})C(y)\cos D(y) - \varepsilon^{-1/3}\frac{\sqrt{3}}{2}\beta^{1/3}C(y)\sin D(y) = -\frac{w(y)}{\beta}$$

Since $\varepsilon \ll 1$ and can be taken arbitrarily small, we require

$$\begin{aligned} -\frac{1}{2}\cos D(y) &= \frac{\sqrt{3}}{2}\sin D(y) \\ \implies \tan D(y) &= -\frac{1}{\sqrt{3}} \\ \implies D(y) &= -\frac{\pi}{6} \end{aligned}$$

Combining the boundary conditions we also have $C(y) = -\frac{2}{\sqrt{3}}G(y)$. Finally we have

$$\tilde{\psi}^{(1)} = -\frac{2}{\sqrt{3}}G(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} - \frac{\pi}{6}\right)$$

Region II. Here, we define a rescaled coordinate $\tilde{x} = \frac{x-L}{\varepsilon^{1/3}}$. The same PDE is satisfied in region II, so the general solution is the same. Here, the matching condition is $\lim_{\tilde{x}\to-\infty}\tilde{\psi}^{(2)} = 0$ which gives $A(y) = C(y) = D(y) = 0$, so

$$\tilde{\psi}^{(2)} = B(y)e^{\beta^{1/3}\tilde{x}}$$

We now apply the boundary conditions. $\psi_x = 0$ at $x = L$ gives

$$\begin{aligned} \varepsilon^{-1/3}\tilde{\psi}^{(2)} &= -\psi^I_x \quad \text{at} \quad x = L \\ \varepsilon^{-1/3}\beta^{1/3}B(y) &= -\frac{w(y)}{\beta} \\ \implies B(y) &= -\frac{\varepsilon^{1/3}w(y)}{\beta^{4/3}} \end{aligned}$$

To enforce $\psi = 0$ at $x = L$, note $\lim_{\varepsilon\to 0} B(y) = 0$, so $\tilde{\psi}^{(2)} \mid_{x=L} \to 0$ as $\varepsilon \to 0$ so we instead require $\psi^I \mid_{x=L} = 0$.

$$\implies G(y) = -\frac{w(y)L}{\beta}$$

Hence we have

$$\tilde{\psi}^{(2)} = -\varepsilon^{1/3}w(y)\beta^{-4/3}e^{\beta^{1/3}\tilde{x}}$$

Full solution. The full solution $\psi = \psi^I + \psi^{(1)} + \psi^{(2)}$ is

$$\begin{aligned} \psi = {} & \frac{x-L}{\beta}w(y) && \text{interior} \\ & + \frac{2w(y)L}{\sqrt{3}\beta}e^{-\beta^{1/3}\frac{x}{2\nu^{1/3}}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\nu^{-1/3}x - \frac{\pi}{6}\right) && \text{western boundary correction} \\ & - \nu^{1/3}\beta^{-4/3}w(y)e^{\beta^{1/3}\frac{x-L}{\nu^{1/3}}} && \text{eastern boundary correction} \end{aligned} \tag{33}$$

Note that the Eastern boundary correction is $\mathcal{O}(\nu^{1/3})$ whilst the Western boundary correction is $\mathcal{O}(1)$.

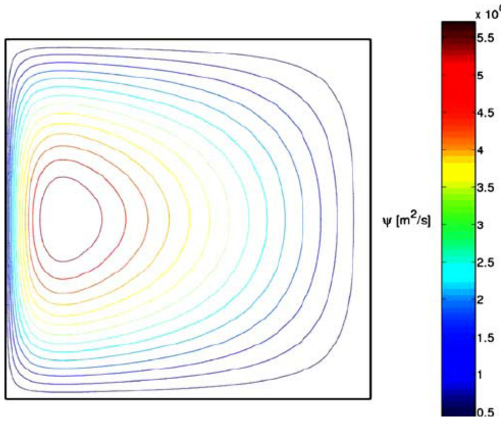

Figure 5: Streamlines of ψ defined by (33) demonstrating western boundary currents.

Physical explanation. The cause of western boundary currents can be physically explained by vorticity. The wind stress curl $w < 0$ inputs negative vorticity in the interior flow. The flow in the western boundary layer inputs positive vorticity to compensate.