Cambridge Part III Maths

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Astrophysical Fluid Dynamics

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Contents

Lecture 1
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1 Introduction

1.1 Areas of application

Astrophysical fluid dynamics (AFD) is relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, the interstellar medium, the intergalactic medium, and cosmology itself. A fluid description is not applicable in regions that are solidified, such as the rocky or icy cores of giant planets and the crusts of neutron stars, and also in very genuous regions where the medium is not sufficiently collisional.

1.2 Theoretical varieties

Various flavours of AFD are in use. The basic models we will consider are:

Hydrodynamics (HD) / Newtonian gas dynamics: This model is non-relativistic, compressible, ideal (inviscid and adiabatic), self-gravitating, and usually assumes a perfect gas.

Magnetohydrodynamics (MHD): This model is the same as above, with the addition of a magnetic field. We will often use ideal MHD, which assumes a perfectly conducting fluid.

1.3 Characteristic features

The elements of theory often important in AFD are compressibility, gravitation, and thermal physics. Sometimes, magnetic fields, radiation forces, and relativity are important. Rarely important aspects are viscosity, surface tension, and solid boundaries.

1.4 Useful data

Some useful data for the course, in CGS (centimetres, grams, seconds) units:

Newton's constant	$G = 6.674 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$
Boltzmann's constant	$k = 1.381 \times 10^{-16}\,\mathrm{erg}\,\mathrm{K}^{-1}$
Stefan's constant	$\sigma = 5.670 \times 10^{-5}\,\mathrm{erg}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{K}^{-4}$
Speed of light	$c = 2.998 \times 10^{1-}\,\mathrm{cm}\,\mathrm{s}^{-1}$
Hydrogen mass	$m_H = 1.674 \times 10^{-24}\,\mathrm{g}$
Solar mass	$M_s = 1.988 \times 10^{33}\,\mathrm{g}$
Solar radius	$R_s = 6.957 \times 10^{10}\,\mathrm{cm}$
Solar luminosity	$L_s = 3.828 \times 10^{33}\,\mathrm{ergs}^{-1}$
Parsec	$pc = 3.086 \times 10^{18}\,\mathrm{cm}$
Astronomial unit (AU)	$au = 1.496 \times 10^{13}\,\mathrm{cm}$
Joule erg conversion	$1\mathrm{J} = 10^7\mathrm{erg}$

2 Ideal gas dynamics

2.1 Fluid variables

A fluid is characterised by a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The *specific volume* (volume per unit mass) is $v = 1/\rho$.
We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionisation and recombination.

2.2 Eulerian and Lagrangian viewpoints

In the *Eulerian* viewpoint we consider how fluid properties vary in time at a point which is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian time derivative is simply ∂_t.
In the *Lagrangian* viewpoint we consider how fluid properties vary in time at a point which moves with the fluid at velocity $\boldsymbol{u}(\boldsymbol{x}, t)$. The Lagrangian time derivative (or material derivative) is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

2.3 Material points and structures

A material point is an idealised fluid element, a point that moves with the bulk velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ of the fluid. Note that the true particles of which the fluid is composed also have random thermal motion.
Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are deformed by it. An infinitesimal material line element $\delta\boldsymbol{x}$ evolves according to

$$\frac{\mathrm{D}\delta\boldsymbol{x}}{\mathrm{D}t} = \delta\boldsymbol{u} = \delta\boldsymbol{x}\cdot\nabla\boldsymbol{u}$$

It changes its length and/or orientation in the presence of a velocity gradient.
Infinitesimal material surface or volume elements can be defined from two or three material line elements according to the vector product and the triple scalar product.

$$\begin{aligned}\delta\boldsymbol{S} &= \delta\boldsymbol{x}^{(1)}\times\delta\boldsymbol{x}^{(2)}\\ \delta V &= \delta\boldsymbol{x}^{(1)}\cdot\delta\boldsymbol{x}^{(2)}\times\delta\boldsymbol{x}^{(3)}\end{aligned}$$

They evolve according to

$$\begin{aligned}\frac{\mathrm{D}\delta\boldsymbol{S}}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta\boldsymbol{S} - \nabla\boldsymbol{u}\cdot\delta\boldsymbol{S}\\ \frac{\mathrm{D}\delta V}{\mathrm{D}t} &= (\nabla\cdot\boldsymbol{u})\delta V\end{aligned}$$

The second result is easier to understand: the volume element increases when the flow is divergent.

2.4 Equation of mass conservation

The equation of mass conservation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

has typical form of conservation law: rate of change of a density and divergence of a flux. Here, ρ is mass density and $\rho\boldsymbol{u}$ is mass flux density. An alternative form of the same equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u}$$

If $\delta m = \rho\delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0$$

2.5 Equation of motion

The equation of motion

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho\nabla\Phi - \nabla p$$

derives from Newton's second law per unit volume with gravitational and pressure forces. The gravitational potential is $\Phi(\boldsymbol{x}, t)$ and $\boldsymbol{g} = -\nabla\Phi$ is the gravitational field.
The force due to pressure acting on a volume V with bounding surface S is

$$-\int_S p\,\mathrm{d}\boldsymbol{S} = \int_V (-\nabla p)\,\mathrm{d}V$$

Viscous forces are neglected in ideal gas dynamics.

2.6 Poisson's equation

The gravitational potential is related to the mass density by Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho$$

where G is Newton's constant. The solution

$$\Phi = \Phi_{\rm int} + \Phi_{\rm ext} = -G\int_V \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}' - G\int_{\hat{V}} \frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,\mathrm{d}^3\boldsymbol{x}'$$

generally involves contributions form both the fluid region V under consideration and the exterior region $\hat{V}$. A *non-self-gravitating* fluid is one of negligible mass for which $\Phi_{\rm int}$ can be neglected. More generally, the *Cowling approximation* consists of treating Φ as being specified in advance, so that Poisson's equation is not coupled to the other equations.

2.7 Thermal energy equation and equation of state

In the absence of non-adiabatic heating (e.g. by viscous dissipation or nuclear reactions) and cooling (e.g. by radiation or conduction),

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0$$

where s is the *specific entropy* (entropy per unit mass). Fluid element undergo reversible thermodynamic changes and preserve their entropy (adiabatic flow). This condition is violated in shocks (see section 6).
The thermal variables (T, s) can be related to the dynamical variables (P, ρ) via an *equation of state* and standard thermodynamic identities. The most important case is that of an *ideal gas* with *blackbody radiation*

$$p = p_g + p_r = \frac{k\rho T}{\mu m_H} + \frac{4\sigma T^4}{3c}$$

where k is Boltzmann's constant, m_H is mass of a hydrogen atom, σ is Stefan's constant, c is the speed of light, and μ is the mean molecular weight, defined as the average mass of the particles in units of m_H, equal to

- 2.0 for molecular hydrogen
- 1.0 for atomic hydrogen
- 0.5 for fully ionised hydrogen
- about 0.6 for ionised matter of typical cosmic abundances.

The component p_g is the *gas pressure* and p_r is the *radiation pressure*. Radiation pressure is usually negligible except in the centres of high mass stars and in the immediate environments of neutron stars and black holes. The pressure of an ideal gas is often written in the form $\mathcal{R}\rho T/\mu$ where $\mathcal{R} = k/m_H$ is a version of the universal gas constant.
We define the *first adiabatic exponent*

$$\Gamma_1 = \left(\frac{\partial \log p}{\partial \log \rho}\right)_S$$

which is related to the ratio of specific heat capacities

$$\gamma = \frac{c_p}{c_v} = \frac{T\left(\frac{\partial s}{\partial T}\right)_p}{T\left(\frac{\partial s}{\partial T}\right)_V}$$

by $\Gamma_1 = \chi_\rho \gamma$ where

$$\chi_\rho = \left(\frac{\partial \log p}{\partial \log \rho}\right)_T$$

can be found from the equation of state. We can then rewrite the thermal energy equation as

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\Gamma_1 p \nabla \cdot \boldsymbol{u}$$

For an ideal gas with negligible radiation pressure, $\chi_\rho = 1$ and so $\Gamma_1 = \gamma$. Adopting this very common assumption, we write

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \boldsymbol{u}$$