Cambridge Part III Maths

Michaelmas 2020

Slow Viscous Flow

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Contents

1 Basic Fluid Mechanics

Lecture 1
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'Infinitesimal' fluid particles have well-defined density $\rho(\boldsymbol{x}, t)$, velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $p(\boldsymbol{x}, t)$ where $\boldsymbol{x}(t)$ is the position of the fluid particles.

Definition. The *Eulerian* or *material* derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$

is the rate of change following the fluid particle.

1.1 Mass Conservation

In general, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \iff \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = 0$$

For an incompressible fluid, $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \iff \nabla \cdot \boldsymbol{u} = 0$

1.2 The Stress Tensor

The *stress* $\boldsymbol{\tau}$ is the force per unit area acting across a surface. Force balance on an 'infinitesimal' fluid tetahedron shows that the stress $\boldsymbol{\tau}$ is linearly related to the surface normal $\boldsymbol{n}$:

$$\boldsymbol{\tau} = \sigma \cdot \boldsymbol{n}$$

where σ is the *stress tensor* and $\boldsymbol{\tau}$ is stress exerted by the outside fluid on the inside of a surface with outward normal $\boldsymbol{n}$. Angular momentum balance shows that σ is symmetric in most fluids.

1.3 Momentum equation

The *Cauchy momentum equation* states in general

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{F} + \nabla \cdot \sigma$$

1.4 Energy equation

In the case of an incompressible fluid, the rate of local inertial *viscous dissipation* is derived by contracting the Cauchy momentum equation with the fluid velocity and integrating over a volume. We have

$$\mathcal{D} = \int_V e_{ij}\sigma_{ij}\,\mathrm{d}V = \int_V e : \sigma\,\mathrm{d}V$$

where $e_{ij} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$ is the *rate of strain* tensor. Note $e_{ii} = 0$ by incompressibility and $e_{ij} = e_{ji}$.
The rate of working by external surface forces on the fluid is

$$\int_{\partial V} u_i \sigma_{ij} n_j \mathrm{d}S$$

1.5 Newtonian Fluids

Definition. Fluid deformation produces internal viscous stresses. If the relationship between fluid deformation $\frac{\partial u_i}{\partial x_j}$ and stress σ_{ij} is local, linear, instantaneous and isotropic, then the fluid is *Newtonian*.

If the fluid is also incompressible, then the stress tensor takes the form

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

where μ is the *dynamic viscosity* and $2\mu e_{ij}$ is the *deviatoric stress*. Note that there is no dependence on the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.

For an incompressible Newtonian fluid with uniform viscosity we have the *Navier-Stokes equations*

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{F} + \mu \nabla^2 \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

The rate of viscous dissipation is

$$\mathcal{D} = 2\mu \int e_{ij} e_{ij} \, \mathrm{d}V$$

Often body forces are conservative $\boldsymbol{F} = -\nabla\phi$ and we incorporate $\boldsymbol{F}$ into a *modified pressure* $p + \phi$.

1.6 Boundary conditions

Kinematic boundary conditions on a fluid-fluid interface are

- $[\boldsymbol{u} \cdot \boldsymbol{n}]_-^+ = 0$ by mass conservation
- $[\boldsymbol{u} \times \boldsymbol{n}]_-^+ = \boldsymbol{0}$ to avoid infinite stresses

Kinematic boundary conditions on a rigid boundary are

- No flux: $\boldsymbol{u} \cdot \boldsymbol{n} = 0$
- No slip: $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$

Dynamic boundary conditions in the absence of surface tension are

$$[\sigma \cdot \boldsymbol{n}]_-^+ = \boldsymbol{0}$$

Note that modified pressure should not be used here.
With surface tension included, the condition becomes

$$[\sigma \cdot \boldsymbol{n}]_-^+ = \gamma \kappa \boldsymbol{n} - \nabla_s \gamma$$

where $\kappa = \nabla_s \cdot \boldsymbol{n}$ is the *curvature* and γ is the *surface tension*.

1.7 Reynolds number

Suppose $U, L, L/U$ are representative velocity, length, and time scales of the flow. Then

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \sim \rho \frac{U^2}{L}$$
$$\mu \nabla^2 \boldsymbol{u} \sim \mu \frac{U}{L^2}$$

Definition. The *Reynolds number* is the ratio of these quantities and determines the important of inertial vs. viscous stresses.

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

If $\mathrm{Re} \ll 1$ then inertia is negligible and we have the *Stokes equations*

$$\mu \nabla^2 \boldsymbol{u} = \nabla p - \boldsymbol{F}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Stokes equations are useful in many regimes.

- Large μ, e.g. magma, glass, ice sheets
- Small L, e.g. microorganisms, microfluid devices
- Thin film flows, e.g. lubrication theory

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Example. Sperm cell – intrinsic length scales $L \sim 5\mu m, U \sim 100\mu m \cdot s^{-1}, \nu \sim 10^{-2} cm^2 \cdot s^{-1} = 10^6 \mu m^2 \cdot s^{-1}$. Therefore $Re \sim 5 \times 10^{-4}$ so can be described by the Stokes equations.

Example. Mantle convection – intrinsic length scales $L \sim 1000km = 10^8 cm, U \sim 2cm\text{year}^{-1} \sim 10^7 cm \cdot s^{-1}, \nu \sim 10^{21} cm^2 \cdot s^{-1}$. Thus $Re \sim 10^{-20}$.

There are some caveats which come with the use of intrinsic length scales.

- $\boldsymbol{u} \cdot \nabla$ and ∇^2 may not involve the same length scale L, e.g. in lubrication theory there is a short length scale for the depth of the flow, which is small compared to other length scales of the flow.
- L may vary in the flow e.g. in the far field of a moving body $Re \sim \frac{Ur}{\nu}$.
- T may not equal L/U if there is an external time scale, e.g. oscillating body with $T \sim \omega^{-1}$.

2 The Stokes Equations

$$
\begin{aligned}
&\nabla \cdot \sigma = \mu \nabla^2 \boldsymbol{u} - \nabla p = -\boldsymbol{F} \\
&\nabla \cdot \boldsymbol{u} = 0
\end{aligned}
$$

2.1 Simple Properties

2.1.1 Instantaneous

The Stokes equations involve no ∂_t term, so there is no inertia, no memory, and the flow only 'knows' about the current boundary conditions and applied forces, and responds immediately to changes. With moving boundaries (i.e. changing boundary conditions) the flow is *quasi-steady*.

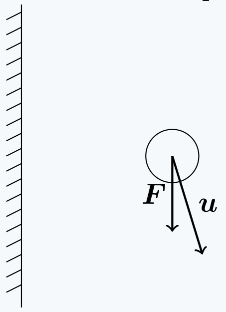
2.1.2 Linear

The Stokes equations are linear in $\boldsymbol{F}, p$, and $\boldsymbol{u}$. Therefore the fluid response is proportional to forcing and solutions for a given geometry can be superposed.

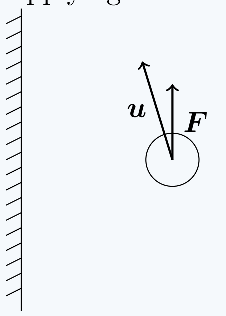
2.1.3 Reversible

If all the forces change sign, then $\boldsymbol{u}$ changes sign. Thus if we reverse all the forces and the history of their application, the flow returns to its original state. Reversibility can sometimes be used with a symmetry to rule out certain behaviours of the flow.

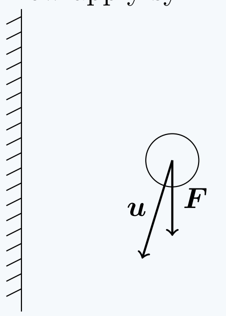
Example. Sedimenting sphere – consider a sphere sedimenting in a Stokes flow next to a rigid wall. Will the sphere migrate laterally?



Applying reversibility, change $\boldsymbol{F} \to -\boldsymbol{F}$ so $\boldsymbol{u} \to -\boldsymbol{u}$:



Now apply symmetry: reflect the geometry top to bottom.



Comparing with the original situation, we see there can be no lateral component of $\boldsymbol{u}$.

2.1.4 Forces balance

Since there is no inertia, the forces must balance. From the equations,

$$\nabla \cdot \sigma = -\boldsymbol{F} \implies \int_{\partial V} \sigma \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{F} \mathrm{d}V = \boldsymbol{0}$$

This is a consistency check on stress boundary conditions.
Similarly, in the absence of fluid sources,

$$\nabla \cdot \boldsymbol{u} = 0 \implies \int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$$

This is a consistency check on velocity boundary conditions.
Likewise, torques balance, giving another consistency check on stress boundary conditions.

2.1.5 Work balances dissipation

Intuitively, the flow has no kinetic energy (no inertia) so any work done on the fluid must be viscously dissipated instantaneously. We have

$$\begin{aligned}
\mathcal{D} &= 2\mu \int_V e_{ij}e_{ij}\,\mathrm{d}V \\
&= \int (\sigma_{ij} + p\delta_{ij})e_{ij}\,\mathrm{d}V \\
&= \int \sigma_{ij}\frac{\partial u_i}{\partial x_j} + pe_{ii}\,\mathrm{d}V \\
&= \int \frac{\partial}{\partial x_j}\sigma_{ij}u_i - u_i\frac{\partial \sigma_{ij}}{\partial x_j}\,\mathrm{d}V \\
&= \int_{\partial V} \boldsymbol{u}\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S + \int \boldsymbol{u}\cdot\boldsymbol{F}\mathrm{d}V
\end{aligned}$$

The first term is the work done by surface forces at the boundary, and the second term is the work done by body forces.

2.1.6 Three Theorems Based on Dissipation Integrals

Lemma 1. *If $\boldsymbol{u}^I$ is an incompressible flow and $\boldsymbol{u}^S$ is a Stokes flow with body force $\boldsymbol{F}^S$ then*

$$2\mu \int e^I : e^S\,\mathrm{d}V = \int_{\partial V} \boldsymbol{u}^I\cdot\sigma^S\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{u}^I\cdot\boldsymbol{F}^S\,\mathrm{d}V$$

Proof. Same as 'work balances dissipation'.

Theorem 1. *Uniqueness theorem. Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2$ are Stokes flows with the same boundary conditions and body forces, i.e. $\boldsymbol{F}_1 = \boldsymbol{F}_2$ in V and either $\boldsymbol{u}_1 = \boldsymbol{u}_2$ or $\sigma_1\cdot\boldsymbol{n} = \sigma_2\cdot\boldsymbol{n}$ on ∂V. Then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.*

Proof. Let $\boldsymbol{u}^* = \boldsymbol{u}_1 - \boldsymbol{u}_2$. From lemma 1,

$$2\mu \int_V e^* : e^*\,\mathrm{d}V = 0$$

Thus $e^* = 0$ in V. Hence we can deduce $\boldsymbol{u}^*$ consists entirely of rigid body motion: $\boldsymbol{u}^* = \boldsymbol{U} + \boldsymbol{\Omega}\times\boldsymbol{u}$. Using the boundary conditions, we have $\boldsymbol{U} = \boldsymbol{\Omega} = 0$ thus $\boldsymbol{u}_1 = \boldsymbol{u}_2$, i.e. Stokes flows are unique.

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Theorem 2. *Reciprocal theorem. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are Stokes flows in V then*

$$\int_{\partial V} \boldsymbol{u}_1\cdot\sigma_2\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{u}_1\cdot\boldsymbol{F}_2\,\mathrm{d}V = \int_{\partial V} \boldsymbol{u}_2\cdot\sigma_1\cdot\boldsymbol{n}\,\mathrm{d}S + \int_V \boldsymbol{u}_2\cdot\boldsymbol{F}_1\,\mathrm{d}V$$

That is, work done by forces of flow 1 against flow 2 = work done by forces of flow 2 against flow 1.

Proof. Apply the lemma twice.

Theorem 3. *Minimum Dissipation theorem. Among all the incompressible flows in V that satisfy given velocity boundary conditions, the dissipation is minimised by the Stokes flow $\boldsymbol{u}^S$ with $\boldsymbol{F}^s = \boldsymbol{0}$ satisfying the same velocity boundary conditions.*

Proof. We have

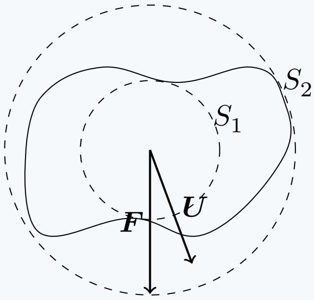
$$\begin{aligned}0 &\leq 2\mu \int (e - e^S) : (e - e^S)\, \mathrm{d}V \\ &\leq 2\mu \int e : e - e^S : e^S\, \mathrm{d}V + 4\mu \int e^S : (e^S - e)\, \mathrm{d}V\end{aligned}$$

Applying the lemma with $\boldsymbol{u}^I = \boldsymbol{u}^S - \boldsymbol{u}$, the last term is 0 since $\boldsymbol{u}^I = 0$ on ∂V and $\boldsymbol{F}^S = 0$ on V. Thus

$$0 \leq \mathcal{D} - \mathcal{D}^S$$

Example.

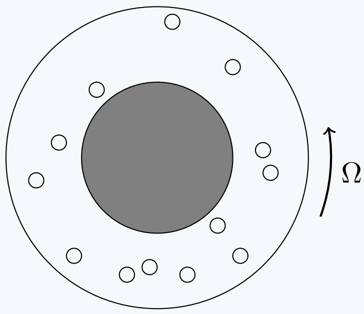
1. Consider an irregularly shaped body in a Stokes flow with inscribing circle S_1 with radius a_1 and circumscribing circle S_2 with radius a_2. Suppose the body experiences a force $\boldsymbol{F}$ and has uniform velocity $\boldsymbol{U}$.



Applying the theorem by taking $\boldsymbol{U}^S$ to be the Stokes flow past S_1 and $\boldsymbol{U}^I$ to be the Stokes flow past S_2 superposed with solid body motion in the gap between S_1 and S_2, we have

$$(6\pi\mu a_1 U)U \leq \boldsymbol{F} \cdot \boldsymbol{U} \leq (6\pi\mu a_2 U)U$$

2. Adding *rigid* particles to a Stokes flow with given *external* velocity boundary conditions increases dissipation and, if the particles are *force-free* and *torque-free*, the apparent viscosity also increases.



3. Inertia increases drag: consider $\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ as $\boldsymbol{F}$.

2.2 Representation by Potentials

Assume $\boldsymbol{F} = 0$, or that $\boldsymbol{F}$ is conservative and absorbed by the modified pressure. Consider the Stokes equations

$$\mu \nabla^2 \boldsymbol{u} = \nabla p \tag{1}$$
$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

From these equations we have

$$\begin{aligned}
\nabla \cdot (1) \& (2) &\implies \nabla^2 p = 0 \implies p \text{ is harmonic} \\
\nabla \times (1) &\implies \nabla^2 \boldsymbol{\omega} = 0 \implies \text{vorticity } \boldsymbol{\omega} = \nabla \times \boldsymbol{u} \text{ is harmonic} \\
\nabla^2 (1) &\implies \nabla^4 \boldsymbol{u} = \mathbf{0} \implies \boldsymbol{u} \text{ is } \textit{bi-harmonic}
\end{aligned}$$

In two dimensions, we can use a stream-function so that $\boldsymbol{u} = \nabla \times (0, 0, \psi)$. Then

$$\omega_z = -\nabla^2 \psi \implies \nabla^4 \psi = 0$$

Similarly, in axisymmetric spherical polars, $\boldsymbol{u} = \nabla \times (0, 0, \frac{\Psi}{r \sin\theta})$. Then $\omega_\phi = -\frac{E^2 \Psi}{r \sin\theta}$ and $E^4 \Psi = 0$ where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right)$$

Many exact solutions can be found in coordinate systems where the operators ∇^2, ∇^4, E^2, etc are separable.

2.2.1 Complex Variable Theory in 2D Flow

Writing $z = x + iy, \bar{z} = x - iy$ gives

$$\nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Thus $f(x, y)$ analytic implies $f = f(z)$ or equivalently $\frac{\partial f}{\partial \bar{z}} = 0$. Thus $\Re f$ and $\Im f$ are harmonic.
Similarly $\nabla^4 \psi = 0$ implies ψ can be written as $\psi = \Im(\bar{z}\phi + \chi)$ where $\phi(z), \chi(z)$ are analytic.
We can find clever exact solutions to difficult problems using this theory, but it is limited to 2D.

2.2.2 Papkovich-Neuber Solution

Let $p = \nabla^2 \pi$ where

$$\pi(\boldsymbol{x}) = -\frac{1}{4\pi} \int \frac{p(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}V$$

Then the Stokes equations can be written $\nabla^2 (\mu \boldsymbol{u} - \nabla \pi) = 0$. Thus

$$\mu \boldsymbol{u} = \nabla \pi - \boldsymbol{\Phi}$$

where $\nabla^2 \boldsymbol{\Phi} = 0$. Now $\nabla \cdot \boldsymbol{u} = 0$ implies $\nabla^2 \pi = \nabla \cdot \boldsymbol{\Phi}$. Then

$$\pi = \frac{1}{2} (\boldsymbol{x} \cdot \boldsymbol{\Phi} + \chi)$$

where $\nabla^2 \chi = 0$. Thus *any* Stokes flow with $\boldsymbol{F} = 0$ can be written in terms of a harmonic vector $\boldsymbol{\Phi}$ and a harmonic scalar χ. The Stokes equations are then

$$\begin{aligned}
2\mu \boldsymbol{u} &= \nabla (\boldsymbol{x} \cdot \boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi} \\
p &= \nabla \cdot \boldsymbol{\Phi}
\end{aligned}$$

which may also be re-written with the 2μ factor absorbed by p.
Note the following.

1. Any irrotational flow $2\mu\boldsymbol{u} = \nabla\chi$ is also a Stokes flow, though $p = 0$ and $\boldsymbol{\sigma} = \nabla\nabla\chi$ which is different from an inviscid irrotational flow.

2. It is sometimes possible to find a harmonic scalar ϕ with $\chi = \boldsymbol{x} \cdot \nabla\phi - 2\phi$. If so, χ can be eliminated by writing $\boldsymbol{\Phi}' = \boldsymbol{\Phi} + \nabla\phi$. For example, if χ has a spherical harmonic expansion we can eliminate all of the terms except the uniform strain $\chi/2\mu = \frac{1}{2}\boldsymbol{x} \cdot \mathsf{E} \cdot \boldsymbol{x} \iff \boldsymbol{u} = \mathsf{E} \cdot \boldsymbol{x}$, since $\chi = r^n Y_n^m(\theta, \phi) \iff \phi = \frac{r^n}{n-2} Y_n^m(\theta, \phi)$ which fails for $n = 2$.

3. Conversely, if $\boldsymbol{\Phi} = \nabla\phi$ then we can get the same $\boldsymbol{u}$ from $\chi = \boldsymbol{x} \cdot \nabla\phi - 2\phi$, which is easier to calculate.

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2.3 Solutions for points, spheres, and cylinders

A point or sphere has no intrinsic direction or orientation, thus solutions on these geometries should also have no intrinsic direction or orientation.

2.3.1 Spherical harmonic functions

Let $r = |\boldsymbol{x}|$. Recall $\nabla^2(\frac{1}{r}) = 0$ for $r \neq 0$. All other spherical harmonic functions ϕ with $\phi \to 0$ as $r \to \infty$ are obtained from

$$\frac{1}{r}, \quad \nabla\frac{1}{r}, \quad \nabla\nabla\frac{1}{r}, \quad \text{etc.}$$

The harmonic functions which are bounded as $r \to 0$ are obtained from

$$r \cdot \frac{1}{r} = 1, \quad r^3\nabla\frac{1}{r} = -\boldsymbol{x}, \quad r^5\nabla\nabla\frac{1}{r}, \quad \dots, \quad r^{2n+1}\nabla^n\frac{1}{r}$$

Compare with separable solutions, for example the $2n + 1$ solutions in spherical polars given by

$$\begin{pmatrix} r^n \\ r^{-n-1} \end{pmatrix} P_n^m(\theta) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

where P_n^m are *associated Legendre functions* and $0 \leq m \leq n$.
Recall the following results.

$$\begin{aligned} \nabla\boldsymbol{x} &= \mathsf{I} \\ \nabla r &= \frac{\boldsymbol{x}}{r} \\ \nabla f(r) &= f'(r)\nabla r = f'(r)\frac{\boldsymbol{x}}{r} \end{aligned}$$

Hence we have

$$\begin{aligned} \nabla\frac{1}{r} &= -\frac{\boldsymbol{x}}{r^3} \\ \nabla\nabla\frac{1}{r} &= -\frac{\mathsf{I}}{r^3} + \frac{3\boldsymbol{x} \cdot \boldsymbol{x}}{r^5} \\ \nabla_i\nabla_j\nabla_k\frac{1}{r} &= \nabla_i\left(-\frac{\delta_{jk}}{r^3} + \frac{3x_j \cdot x_k}{r^5}\right) \\ &= \frac{3(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})}{r^5} - \frac{15x_ix_jx_k}{r^7} \end{aligned}$$

Note: these depend only on $\boldsymbol{x}$ and r and thus have no preferred direction, as hoped. We can use these functions to form Papkovich-Neuber potentials $\boldsymbol{\Phi}$ and χ by multiplying the harmonic functions above by constant scalars, vectors or tensors and taking an appropriate number of dot products, e.g. the following are all harmonic vectors

$$\boldsymbol{A}\frac{1}{r}, \quad \mathsf{B}\cdot\nabla\frac{1}{r}, \quad C\nabla\frac{1}{r}, \quad (\boldsymbol{D}\cdot\nabla)\nabla\frac{1}{r}, \quad (\mathsf{E}:\nabla\nabla)\nabla\frac{1}{r}, \quad \boldsymbol{\Omega}\times\nabla\frac{1}{r}$$

It is useful to distinguish between *true* and *pseudo* tensors. True / pseudo tensors keep / change sign upon reflection, e.g.

$$T'_{ijk} = \pm R_{il}R_{jm}R_{kn}T_{lmn}$$

Examples of true vectors are velocity $\boldsymbol{u}$; force $\boldsymbol{F}$; position $\boldsymbol{x}$; del ∇; identity I. Examples of pseudo vectors are angular velocity $\boldsymbol{\Omega}$; torque $\boldsymbol{G}$; $\boldsymbol{u}\times\boldsymbol{x}$; vorticity $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$. Products obey the obvious parity rules, e.g. helicity $\boldsymbol{u}\cdot\boldsymbol{\Omega}$ is a pseudo scalar.

2.3.2 Solution due to a point force

The Papkovich-Neuber solution due to a point force is a Green's function for the Stokes equations. This problem is also known as a 'Stokeslet'. Consider the problem

$$\begin{aligned}
\nabla\cdot\sigma &= \mu\nabla^2\boldsymbol{u} - \nabla p = -\boldsymbol{F}\delta(\boldsymbol{x}) \\
\nabla\cdot\boldsymbol{u} &= 0
\end{aligned}$$

with $\boldsymbol{u}\to 0$ at infinity. The answer must be linear in $\boldsymbol{F}$, but otherwise has no orientation. The only choice is $\boldsymbol{\Phi}=\alpha\frac{\boldsymbol{F}}{r}$. We could have tried $\boldsymbol{F}\times\nabla\frac{1}{r}$, but this is a pseudo vector whilst $\boldsymbol{\Phi}$ and χ need to be true since $\boldsymbol{u}$ is true. Similar arguments rule out other harmonic functions.

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We have

$$\begin{aligned}
2\mu\boldsymbol{u} &= \alpha\left(\nabla\left(\frac{\boldsymbol{F}\cdot\boldsymbol{x}}{r}\right) - 2\frac{\boldsymbol{F}}{r}\right) \\
&= \alpha\left(\frac{\boldsymbol{F}\cdot\mathsf{I}}{r} - \frac{(\boldsymbol{F}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3} - 2\frac{\boldsymbol{F}}{r}\right) \\
&= -\alpha\left(\frac{\boldsymbol{F}}{r} + \frac{(\boldsymbol{F}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3}\right)
\end{aligned}$$

Thus the stress tensor is

$$\sigma = \mu\left(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T\right) - (\nabla\cdot\boldsymbol{\Phi})\,\mathsf{I} = 3\alpha(\boldsymbol{F}\cdot\boldsymbol{x})\frac{\boldsymbol{x}\boldsymbol{x}}{r^5}$$

On any sphere $r=R$, $\boldsymbol{n}=\frac{\boldsymbol{x}}{R}$, and

$$\begin{aligned}
2\mu\boldsymbol{u}\cdot\boldsymbol{n} &= -\frac{2\alpha}{R}\boldsymbol{F}\cdot\boldsymbol{n} \\
\sigma\cdot\boldsymbol{n} &= 3\alpha\frac{(\boldsymbol{F}\cdot\boldsymbol{n})\boldsymbol{n}}{R^2}
\end{aligned}$$

To determine the constant α we can consider the surface volume flux and the surface stress. The surface volume flux is

$$\int_{r=R}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = -\frac{\alpha\boldsymbol{F}}{\mu R}\cdot\int_{r=R}\boldsymbol{n}\,\mathrm{d}S = 0$$

which does not provide any information on α. The surface forces should equal $-\boldsymbol{F}$. We have

$$-\boldsymbol{F} = \int_{r=R}\sigma\cdot\boldsymbol{n}\,\mathrm{d}S = 3\alpha\boldsymbol{F}\cdot\int_{r=R}\boldsymbol{n}\boldsymbol{n}\,\frac{\mathrm{d}S}{R^2} = 3\alpha\boldsymbol{F}\cdot\frac{4\pi}{3}\mathsf{I} = 4\pi\alpha\boldsymbol{F}$$

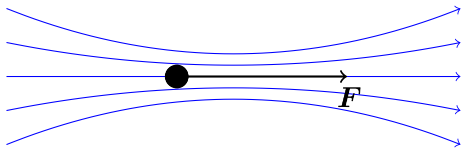


Figure 1: Stokeslet solution for a point force.

Hence we choose $\alpha = -1/4\pi$. Thus the final solution is

$$\boldsymbol{u} = \boldsymbol{F} \cdot \mathsf{J}(\boldsymbol{x}), \qquad \sigma = \boldsymbol{F} \cdot \mathsf{K}(\boldsymbol{x}), \qquad p = \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{4\pi r^3}$$

where J is the *Oseen tensor*:

$$\mathsf{J} = \frac{1}{8\pi\mu}\left(\frac{\mathsf{I}}{r} + \frac{\boldsymbol{x}\boldsymbol{x}}{r^3}\right), \qquad \mathsf{K} = -\frac{3}{4\pi}\frac{\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}}{r^5}$$

Finally, from incompressibility $\nabla \cdot \boldsymbol{u} = 0$ and the Stokes equations $\nabla \cdot \sigma = -\boldsymbol{F}\delta(\boldsymbol{x})$, we deduce

$$\nabla \cdot \mathsf{J} = 0, \qquad \nabla \cdot \mathsf{K} = -\mathsf{I}\delta(\boldsymbol{x})$$

Note that the velocity scales as $u \propto \frac{1}{r}$: see figure 1. This is slowly decaying compared to many forces e.g. gravity which scales as $\frac{1}{r^2}$. Thus particle interactions in Stokes flow can occur on much larger scales than (for example) charge interactions.

2.3.3 Source flow

Consider a source of strength Q with $\nabla \cdot \boldsymbol{u} = Q\delta(\boldsymbol{x})$. This is referred to as a *point volume source.* One can show the solution is

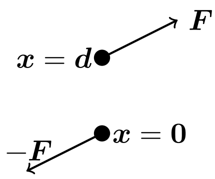
$$\boldsymbol{u} = \frac{Q\boldsymbol{x}}{4\pi r^3}$$

which is obtained using Papkovich-Neuber potentials

$$\chi = \alpha\frac{Q}{r}, \qquad \boldsymbol{\Phi} = \beta Q \nabla \frac{1}{r}$$

2.3.4 Force dipole, stresslet, rotlet

Further solutions for dipoles, quadrupoles, etc. can be found by taking gradients of the Stokeslet and source solutions. For example, consider a dipole.



The Stokeslet solution is

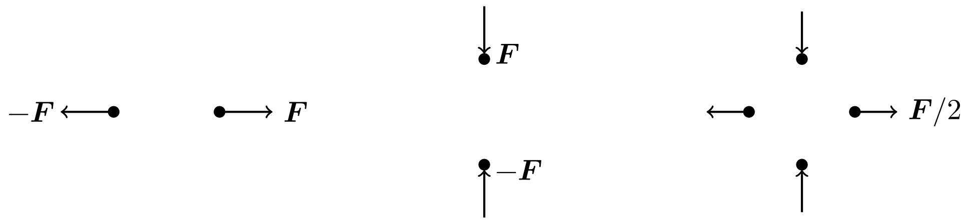
$$\boldsymbol{u} = \boldsymbol{F} \cdot \mathsf{J}(\boldsymbol{x} - \boldsymbol{d}) - \boldsymbol{F} \cdot \mathsf{J}(\boldsymbol{x}) = \boldsymbol{F} \cdot (-\boldsymbol{d} \cdot \nabla)\mathsf{J}(\boldsymbol{x}) + \text{h.o.t.}$$

Take the limit $\boldsymbol{d} \to \boldsymbol{0}$ with $\boldsymbol{F}\boldsymbol{d}$ fixed and split $-F_i d_j$ into

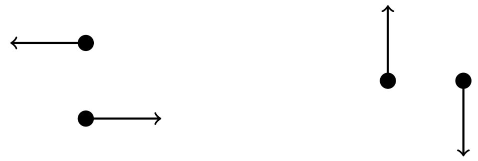
1. An isotropic part $-\frac{1}{3}F_k d_k \delta_{ij}$
2. A symmetric traceless part
$$s_{ij} = -\frac{1}{2}(F_i d_j + F_j d_i) + \frac{1}{3}F_k d_k \delta_{ij}$$
3. An antisymmetric part $-\frac{1}{2}\varepsilon_{ijk}G_k$ where $\boldsymbol{G} = \boldsymbol{d} \times \boldsymbol{F}$

The flow contribution from each of these components may then be calculated.

1. The isotropic component gives no flow since $\nabla \cdot J = 0$
2. This component is a *stresslet* representing the following components of motion



3. This component is a *rotlet* due to a point torque $\boldsymbol{G}$.



Both the stresslet and rotlet decay as $\frac{1}{r^2}$.

2.3.5 Rigid sphere with velocity U

Consider a rigid sphere of radius a moving uniformly with velocity $\boldsymbol{U}$ in a Stokes flow. We have $\nabla \cdot \boldsymbol{u} = 0$ and $\mu\nabla^2\boldsymbol{u} = \nabla p$ in $r > a$. We require $\boldsymbol{u} \to 0$ as $r \to \infty$ and $\boldsymbol{u} = \boldsymbol{U}$ on the sphere's surface $r = a$. The sphere is isotropic, so we need harmonic functions of $\boldsymbol{x}, \boldsymbol{U}$ which are linear in $\boldsymbol{U}$; decay at ∞; and are true tensors. We choose

$$\frac{1}{2\mu}\boldsymbol{\Phi} = \alpha\boldsymbol{U}\frac{1}{r}, \qquad \frac{1}{2\mu}\chi = \beta\boldsymbol{u} \cdot \nabla\frac{1}{r}$$

This gives a solution which is a superposition of a Stokeslet and a source dipole

$$\boldsymbol{u} = -\alpha\left(\frac{\boldsymbol{U}}{r} + \frac{(\boldsymbol{U}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3}\right) + \beta\left(-\frac{\boldsymbol{U}}{r^3} + 3\frac{(\boldsymbol{U}\cdot\boldsymbol{x})\boldsymbol{x}}{r^5}\right)$$

Enforcing the boundary condition $\boldsymbol{u} = \boldsymbol{U}$ on $r = a$ requires

$$-\frac{\alpha}{a} - \frac{\beta}{a^3} = 1, \qquad -\frac{\alpha}{a} + 3\frac{\beta}{a^3} = 0$$
$$\implies \alpha = -\frac{3a}{4}, \quad \beta = -\frac{a^3}{4}$$

Thus the final solution for a sphere in a Stokes flow is

$$\boldsymbol{u} = \frac{3}{4}\boldsymbol{U}\left(\frac{a}{r} + \frac{a^3}{3r^3}\right) + \frac{3}{4}\frac{(\boldsymbol{U}\cdot\boldsymbol{x})\boldsymbol{x}}{r^2}\left(\frac{a}{r} - \frac{a^3}{r^3}\right)$$

The corresponding pressure and vorticity, both of which are harmonic and due to the Stokeslet, are

$$p = \frac{3}{2}\mu a \frac{\boldsymbol{U}\cdot\boldsymbol{x}}{r^3}, \qquad \boldsymbol{\omega} = -\frac{1}{\mu}\nabla\times\boldsymbol{\Phi} = \frac{3a}{2}\frac{\boldsymbol{U}\times\boldsymbol{x}}{r^3}$$

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2.3.6 Force and stress on a translating sphere

The flow at large distances is dominated by the $\frac{1}{r}$ Stokeslet term with strength $\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}$. Since the force across $r = `\infty'$ must balance the force across $r = a$, we can see without further work that the force by the fluid on the sphere i.e. drag $= -6\pi\mu a\boldsymbol{U}$. In more detail:

$$\begin{aligned}
\sigma &= -pI + \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \\
&= -\frac{3}{2}\mu a\frac{\boldsymbol{U}\cdot\boldsymbol{x}}{r} + \frac{3}{4}\mu\left[\left(-\frac{a}{r^3} - \frac{a^3}{r^5}\right)(\boldsymbol{x}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{x})\right. \\
&\quad \left. + (\boldsymbol{U}\boldsymbol{x} + \boldsymbol{x}\boldsymbol{U} + 2\boldsymbol{U}\cdot\boldsymbol{x}I)\left(\frac{a}{r^3} - \frac{a^3}{r^5}\right) + (\boldsymbol{U}\cdot\boldsymbol{x})\left(-\frac{3a}{r^5} + \frac{5a^3}{r^7}\right)2\boldsymbol{x}\boldsymbol{x}\right] \\
\implies \sigma\cdot\boldsymbol{n}\,|_{r=a} &= -\frac{3}{2}\frac{\mu}{a}(\boldsymbol{U}\cdot\boldsymbol{n})\boldsymbol{n} + \frac{3\mu}{4a}\left[-2(\boldsymbol{n}(\boldsymbol{U}\cdot\boldsymbol{n}) + \boldsymbol{U}) + (\boldsymbol{U}\cdot\boldsymbol{n})(-2)2\boldsymbol{n}\right] \\
&= -\frac{3\mu}{2a}\boldsymbol{U}
\end{aligned}$$

Thus the drag on the sphere is

$$\boldsymbol{F} = \int_{r=a} \sigma\cdot\boldsymbol{n}\,\mathrm{d}S = -\frac{3\mu}{2a}\boldsymbol{U}\cdot 4\pi a^2 = -6\pi\mu a\boldsymbol{U}$$

as expected.

2.3.7 Gravitational settling

The force balance of weight, buoyancy, and drag gives the settling velocity of a rigid sphere in a Stokes flow.

$$\frac{4}{3}\pi a^3\rho_s\boldsymbol{g} - \frac{4}{3}\pi a^3\rho\boldsymbol{g} - 6\pi\mu a\boldsymbol{U} = 0$$
$$\implies \boldsymbol{U} = \frac{2a^2}{9\mu}(\rho_s - \rho)\boldsymbol{g} \propto a^2$$

2.3.8 2D potentials

The harmonic functions in two-dimensions which are bounded as $r \to \infty$ are

$$\ln r, \quad \nabla\ln r = \frac{\boldsymbol{x}}{r^2}, \quad \nabla\nabla\ln r = \frac{I}{r^2} - \frac{2\boldsymbol{x}\boldsymbol{x}}{r^4}, \quad \dots$$

Similar to the 3D spherical case, the harmonic functions in two-dimensions which are bounded as $r \to 0$ are

$$1, \quad r^2\nabla\ln r = \boldsymbol{x}, \quad ,\dots, \quad r^{2n}\nabla^n\ln r$$

The 2D Stokeslet solution follows from the Papkovich-Neuber potentials $\boldsymbol{\Phi} = \frac{\boldsymbol{F}}{2\pi}\ln r$ which gives

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{F}\cdot \mathsf{J}^{2D} \\
p &= \frac{\boldsymbol{F}\cdot\boldsymbol{x}}{2\pi r^2} \\
\sigma &= \boldsymbol{F}\cdot \mathsf{K}^{2D} \\
\mathsf{J}^{2D} &= \frac{1}{4\pi\mu}\left(-\ln r \mathsf{I} + \frac{\boldsymbol{x}\boldsymbol{x}}{r^2}\right) \\
\mathsf{K}^{2D} &= -\frac{1}{\pi}\frac{\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}}{r^4}
\end{aligned}$$

The 2D Stokeslet solution corresponds to a line force $\boldsymbol{F}$ *per unit length*. Note that $\boldsymbol{u} \not\to 0$ as $r\to\infty$ for a line force, though $\boldsymbol{u}\to 0$ for line dipoles, line quadrupoles, etc.

2.4 Motion of rigid particles

2.4.1 The resistance matrix

A rigid particle moving with velocity $\boldsymbol{U} + \boldsymbol{\Omega}\times\boldsymbol{x}$ through fluid otherwise at rest exerts a force $\boldsymbol{F}$ and a couple $\boldsymbol{G}$ on the fluid. By linearity,

$$\begin{pmatrix}\boldsymbol{F}\\ \boldsymbol{G}\end{pmatrix} = \begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{U}\\ \boldsymbol{\Omega}\end{pmatrix}$$

where the tensors $\mathsf{A}-\mathsf{D}$ depend on the size, shape, and orientation of the body. We refer to this matrix as the *resistance matrix*. We have

$$\int_{\text{body}} \boldsymbol{u}\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S = \int (\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{x})\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S = \boldsymbol{U}\cdot\boldsymbol{F} + \boldsymbol{\Omega}\cdot\boldsymbol{G}$$

Similarly, from the reciprocal theorem (with $\boldsymbol{F}=0$), for all $\boldsymbol{U}_1, \boldsymbol{\Omega}_1, \boldsymbol{U}_2, \boldsymbol{\Omega}_2$,

$$(\boldsymbol{U}_1, \boldsymbol{\Omega}_1)\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{U}_2\\ \boldsymbol{\Omega}_2\end{pmatrix} = (\boldsymbol{U}_2, \boldsymbol{\Omega}_2)\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}\begin{pmatrix}\boldsymbol{U}_1\\ \boldsymbol{\Omega}_1\end{pmatrix} = (\boldsymbol{U}_1, \boldsymbol{\Omega}_1)\begin{pmatrix}\mathsf{A}^T & \mathsf{C}^T\\ \mathsf{B}^T & \mathsf{D}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{U}_2\\ \boldsymbol{\Omega}_2\end{pmatrix}$$

Hence $\mathsf{A}=\mathsf{A}^T$, $\mathsf{D}=\mathsf{D}^T$ so A and D are diagonalisable, and $\mathsf{B}=\mathsf{C}^T$, i.e. the force from pure rotation is equal to the couple from pure translation. Since the viscous dissipation is positive, the resistance matrix is positive definite, so invertible.

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Example.

1. A body with 3 independent planes of reflectional symmetry has $\mathsf{B}=\mathsf{C}=0$.
2. A cube falls with the same speed (and no rotation) in all orientations since $\mathsf{A}\propto\mathsf{I}$.
3. A and D are known for ellipsoids, and therefore for rods and discs which are limits of ellipsoids.

The resistance matrix and its inverse the *mobility matrix* are sometimes extended to describe a rigid particle placed in a background linear flow $\boldsymbol{u}^\infty = \boldsymbol{U}^\infty + \boldsymbol{\Omega}^\infty\times\boldsymbol{x} + \mathsf{E}^\infty\cdot\boldsymbol{x}$, as

$$\begin{pmatrix}\boldsymbol{F}\\ \boldsymbol{G}\\ \mathsf{S}\end{pmatrix} = \begin{pmatrix} & 12\times 12 & \end{pmatrix}\begin{pmatrix}\boldsymbol{U}-\boldsymbol{U}^\infty\\ \boldsymbol{\Omega}-\boldsymbol{\Omega}^\infty\\ \mathsf{E}^\infty\end{pmatrix}$$

where S is the stresslet exerted by the particle.
If $\mathsf{E}^\infty = 0$, a force-free, couple-free particle just translates and rotates with the flow. If $\mathsf{E}^\infty \neq 0$ then the particle generates a stresslet. In general, the extra dissipation

$$\boldsymbol{F} \cdot (\boldsymbol{U} - \boldsymbol{U}^\infty) + \boldsymbol{G} \cdot (\boldsymbol{\Omega} - \boldsymbol{\Omega}^\infty) + \mathsf{S} : \mathsf{E}^\infty$$

is positive by the minimum dissipation theorem, thus the extended matrix is positive definite, so invertible.
This is useful because it gives the leading order effects for a small particle of size a in a flow of larger lengthscale L:

$$\boldsymbol{u}^\infty(\boldsymbol{x}) = \boldsymbol{u}^\infty(0) + \boldsymbol{x} \cdot \nabla \boldsymbol{u}^\infty(0) + \mathcal{O}(a^2/L^2)$$

2.5 Faxén relations

Consider the behaviour of a rigid particle placed in an arbitrary unbounded Stokes flow $\boldsymbol{u}^\infty(\boldsymbol{x})$. Can we find the particle motion $\boldsymbol{u}^P = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{x}$? Let $\boldsymbol{u}'$ be the *perturbation flow* $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{u}^\infty$, which has boundary conditions

$$\begin{aligned} \boldsymbol{u}' &\to 0 \quad \text{as } |\boldsymbol{x}| \to \infty \\ \boldsymbol{u}' &= \boldsymbol{u}^P - \boldsymbol{u}^\infty(\boldsymbol{x}) \quad \text{on particle} \end{aligned}$$

and $\hat{\boldsymbol{u}}$ a 'test' flow due to particle translation with arbitrary velocity $\hat{\boldsymbol{V}}$. By linearity,

$$\hat{\sigma} = \hat{\Sigma}(\boldsymbol{x}) \cdot \hat{\boldsymbol{V}}$$

for some third rank tensor $\hat{\Sigma}$. The reciprocal theorem for $\boldsymbol{u}'$ and $\hat{\boldsymbol{u}}$ gives

$$\hat{\boldsymbol{V}} \cdot \int_{S_P} \sigma' \cdot \boldsymbol{n} \, \mathrm{d}S = \hat{\boldsymbol{V}} \cdot \int (\boldsymbol{u}^P - \boldsymbol{u}^\infty(\boldsymbol{x})) \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$$

Now $\hat{\boldsymbol{V}}$ is arbitrary and contribution from $\int \boldsymbol{u}^P \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$ is given by

$$\begin{pmatrix} \boldsymbol{U} & \boldsymbol{\Sigma} \end{pmatrix} \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B}^T & \mathsf{D} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{V}} \\ \boldsymbol{0} \end{pmatrix}$$

Hence we have *Faxén's first formula*

$$\boldsymbol{F} = \mathsf{A} \cdot \boldsymbol{U} + \mathsf{B} \cdot \boldsymbol{\Omega} - \int \boldsymbol{u}^\infty(\boldsymbol{x}) \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$$

Similar Faxén relations can be obtained for the couple $\boldsymbol{G}$ and stresslet S exerted, by using other test flows.

Example. For a sphere, we have $\hat{\Sigma} \cdot \boldsymbol{n} = \frac{3\mu}{2a} \mathsf{I}$ on $r = a$, where $\boldsymbol{n}$ is *into* the particle. We also have

$$\mathsf{A} = 6\pi\mu a \mathsf{I}, \qquad \mathsf{B} = 0$$

which gives a translation velocity

$$\boldsymbol{U} = \frac{\boldsymbol{F}}{6\pi\mu a} + \frac{1}{4\pi a^2} \int \boldsymbol{u}^\infty(\boldsymbol{x}) \, \mathrm{d}S$$

where the last term is an average of $\boldsymbol{u}^\infty$ over the surface. Moreover, we can Taylor expand this far field velocity

$$\boldsymbol{u}^\infty(\boldsymbol{x}) = \boldsymbol{u}^\infty(\boldsymbol{0}) + \boldsymbol{x} \cdot \nabla \boldsymbol{u}^\infty(\boldsymbol{0}) + \frac{1}{2}\boldsymbol{x}\boldsymbol{x} : \nabla\nabla \boldsymbol{u}^\infty(\boldsymbol{0}) + ...$$

Therefore we have

$$\frac{1}{4\pi a^2}\int_{r=a} \boldsymbol{u}^\infty(\boldsymbol{x})\,\mathrm{d}S = \boldsymbol{u}^\infty(\boldsymbol{0}) + 0 + \frac{1}{8\pi a^2}\int_{r=a} \boldsymbol{x}\boldsymbol{x}\,\mathrm{d}S : \nabla\nabla \boldsymbol{u}^\infty(\boldsymbol{0}) + ...$$

Odd terms, for example $\int_{r=a} \boldsymbol{x}\boldsymbol{x}\boldsymbol{x}\,\mathrm{d}S = 0$ by symmetry. Even terms are isotropic and give $\nabla^{2n}\boldsymbol{u}^\infty(\boldsymbol{0})$, but by Stokes equations $\nabla^4\boldsymbol{u}^\infty = \boldsymbol{0}$. Hence

$$\boldsymbol{U} = \frac{\boldsymbol{F}}{6\pi\mu a} + \boldsymbol{u}^\infty(\boldsymbol{0}) + \frac{a^2}{6}\nabla^2\boldsymbol{u}^\infty(\boldsymbol{0})$$

Application to two sedimenting spheres. Consider two rigid spheres of radius a sedimenting under a given force $\boldsymbol{F}$. We wish to calculate the flal speed $\boldsymbol{U}$ to $\mathcal{O}(a^3/R^3)$. One sphere moves in the far-field flow of the other sphere, which itself depends on the force, couple, and stresslet exerted at leading order. $\boldsymbol{F}$ and $\boldsymbol{G} = 0$ are the same for each sphere, hence this far field is the same as for an

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isolated sphere up to $\mathcal{O}(a^4/R^4)$. Faxén gives an $\mathcal{O}(a^3/R^3)$ correction to $\boldsymbol{U}$ from the $\frac{a^2}{6}\nabla^2$ term.

2.6 Integral representations of Stokes flow

2.6.1 Basic integral identity

Consider a Stokes flow $\boldsymbol{u}$ and a Stokeslet flow $\boldsymbol{u}^S$ due to a point force $\boldsymbol{F}$ at $\boldsymbol{y}$. We have $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{u}^S = 0$ and

$$\nabla \cdot \sigma = -\boldsymbol{f}, \qquad \nabla \cdot \sigma^S = -\boldsymbol{F}\delta(\boldsymbol{x}-\boldsymbol{y})$$

Apply the reciprocal theorem with $\boldsymbol{u}_1 = \boldsymbol{u}, \boldsymbol{u}_2 = \boldsymbol{u}^S$. We have

$$\int_V \boldsymbol{u}\cdot\boldsymbol{F}\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}V + \int_{\partial V} \boldsymbol{u}\cdot(\boldsymbol{F}\cdot K(\boldsymbol{x}-\boldsymbol{y}))\cdot\boldsymbol{n}\,\mathrm{d}S = \int_v (\boldsymbol{F}\cdot J(\boldsymbol{x}-\boldsymbol{y}))\cdot\boldsymbol{f}\,\mathrm{d}V + \int_{\partial V} (\boldsymbol{F}\cdot J(\boldsymbol{x}-\boldsymbol{y}))\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S$$

Note by definition J is even and K is odd, so $J(\boldsymbol{x}-\boldsymbol{y}) = J(\boldsymbol{y}-\boldsymbol{x})$ and $K(\boldsymbol{x}-\boldsymbol{y}) = -K(\boldsymbol{y}-\boldsymbol{x})$. Also, $\boldsymbol{F}$ is arbitrary so may be factored out by the quotient theorem. We also have from the sampling property of the delta function

$$\int_V \phi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}V = \begin{cases} \phi(\boldsymbol{y}) & \boldsymbol{y} \in V \\ 0 & \boldsymbol{y} \notin V \\ \frac{1}{2}\phi(\boldsymbol{y}) & \boldsymbol{y} \in \partial V \end{cases}$$

Note we require $\boldsymbol{y} \in \partial V$ to be a smooth point of ∂V for this to hold. Hence

$$\int_V J(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}V + \int_{\partial V} J(\boldsymbol{y}-\boldsymbol{x})\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\partial V} \boldsymbol{u}(\boldsymbol{x})\cdot K(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S = \begin{cases} \boldsymbol{u}(\boldsymbol{y}) & \boldsymbol{y} \in V \\ 0 & \boldsymbol{y} \notin V \\ \frac{1}{2}\boldsymbol{u}(\boldsymbol{y}) & \boldsymbol{y} \notin \partial V \end{cases}$$

1. The above follows the sign conventions: $\boldsymbol{n}$ is out of v, and we have $\boldsymbol{y}-\boldsymbol{x}$ as the argument.
2. $\boldsymbol{f}$ is often not included, i.e. there is no body force or $\boldsymbol{f}$ is absorbed into modified pressure
3. The jump in the RHS comes from the K integral which depends on $\frac{1}{r}$ which jumps when $\boldsymbol{y}$ crosses ∂V.
4. Usually we only know $\sigma\cdot\boldsymbol{n}$ or $\boldsymbol{u}$ on ∂V. In general, we solve as an integral equation for whichever is not specified with $\boldsymbol{y}\in\partial V$ and then substitute to find $\boldsymbol{u}$ elsewhere.

2.6.2 Far-field approximations/multipole expansion for a moving body

For a body of size a and for $|\boldsymbol{y}|\gg a$ we have by Taylor expanding

$$\begin{aligned}\mathsf{J}(\boldsymbol{y}-\boldsymbol{x})&\approx\mathsf{J}(\boldsymbol{y})-\boldsymbol{x}\cdot\nabla\mathsf{J}(\boldsymbol{y})+\mathcal{O}(\frac{a^2}{y^3})\\ \mathsf{K}(\boldsymbol{y}-\boldsymbol{x})&\approx\mathsf{K}(\boldsymbol{y})+\mathcal{O}(\frac{a^2}{y^3})\end{aligned}$$

Assuming $\boldsymbol{f}=0$, substitute this into the basic integral representation to obtain after some messy algebra

$$\boldsymbol{u}(\boldsymbol{y})\approx\boldsymbol{F}\cdot\mathsf{J}(\boldsymbol{y})+\frac{\boldsymbol{G}\times\boldsymbol{y}}{8\pi\mu|\boldsymbol{y}|^3}+\frac{Q\boldsymbol{y}}{4\pi|\boldsymbol{y}|^3}-\frac{3(\boldsymbol{y}\cdot\mathsf{S}\cdot\boldsymbol{y})\boldsymbol{y}}{8\pi\mu|\boldsymbol{y}|^5}+\mathcal{O}(\frac{a^2}{y^3})$$

where the first term is $\mathcal{O}(\frac{1}{y})$, the three following terms are $\mathcal{O}(\frac{a}{y^2})$ and (using $\boldsymbol{n}$ out of body)

$$\begin{aligned}\boldsymbol{F}&=-\int\sigma\cdot\boldsymbol{n}\,\mathrm{d}S=\text{force exerted by body}\\ \boldsymbol{G}&=-\int\boldsymbol{x}\times\sigma\cdot\boldsymbol{n}\,\mathrm{d}S=\text{couple exerted by body}\\ Q&=\int\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S=\text{source strength (vanishes for rigid body)}\\ \mathsf{S}&=\int\left[\frac{1}{2}\left(\boldsymbol{x}(\sigma\cdot\boldsymbol{n})-2\mu\boldsymbol{u}\boldsymbol{n}\right)+\frac{1}{2}\left(\boldsymbol{x}(\sigma\cdot\boldsymbol{n})-2\mu\boldsymbol{u}\boldsymbol{n}\right)^T-\frac{1}{3}\mathsf{I}\,\mathrm{tr}\left(\boldsymbol{x}(\sigma\cdot\boldsymbol{n})-2\mu\boldsymbol{u}\boldsymbol{n}\right)\right]\mathrm{d}S=\text{stresslet exerted}\end{aligned}$$

Note the terms involving $\boldsymbol{u}$ in the integrand for S vanish for a rigid body.

1. The far-field of a moving body is related to the force exerted by the body, not its velocity.
2. The far-field of a force-free, couple-free, incompressible body is a stresslet.

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2.6.3 Representation for droplets

Consider a droplet occupying a volume V_1 with viscosity $\lambda\mu$ in a surrounding fluid volume V_2 with viscosity μ. Let $\boldsymbol{n}$ be *out* of the droplet. Use viscosity μ in the Stokeslet tensor J. We assume for simplicity that there is no body force, $\boldsymbol{f}=0$, and the far field is at rest, $\boldsymbol{u}\to 0$ as $r\to\infty$.

Applying the basic integral identity to each of V_1 and V_2 we have

$$\int_{\partial V} \frac{1}{\lambda} J \cdot \sigma_{\mathbf{1}} \cdot \boldsymbol{n} + \boldsymbol{u}_1 \cdot K \cdot \boldsymbol{n} \, \mathrm{d}S = \begin{cases} \boldsymbol{u}_1 & \boldsymbol{y} \in V_1 \\ 0 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}\boldsymbol{u}_1 & \boldsymbol{y} \in \partial V \end{cases} \tag{3}$$

$$-\int_{\partial V} \frac{1}{\lambda} J \cdot \sigma_{\mathbf{2}} \cdot \boldsymbol{n} + \boldsymbol{u}_2 \cdot K \cdot \boldsymbol{n} \, \mathrm{d}S = \begin{cases} \boldsymbol{u}_2 & 0 \in V_1 \\ \boldsymbol{u}_2 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}\boldsymbol{u}_2 & \boldsymbol{y} \in \partial V \end{cases} \tag{4}$$

Note the $-$ sign in (4) is because $\boldsymbol{n}$ points into V_2. Now $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$ (say) on ∂V. So $\lambda(4) + (3)$ implies

$$\int_{\partial V} J(\boldsymbol{y}-\boldsymbol{x}) \cdot (\sigma_{\mathbf{1}} \cdot \boldsymbol{n} - \sigma_{\mathbf{2}} \cdot \boldsymbol{n}) \, \mathrm{d}S_x + (\lambda - 1) \int_{\partial V} \boldsymbol{u} \cdot K(\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n} \, \mathrm{d}S_x = \begin{cases} \lambda \boldsymbol{u}_1 & \boldsymbol{y} \in V_1 \\ \boldsymbol{u}_2 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}(\lambda+1)\boldsymbol{u} & \boldsymbol{y} \in \partial V \end{cases}$$

This integral equation is the basis for powerful numerical methods to determine droplet shapes and evolution.

Example. Relaxation of a drop under constant surface tension. The flow is driven by $\sigma_{\mathbf{2}} \cdot \boldsymbol{n} - \sigma_{\mathbf{1}} \cdot \boldsymbol{n} = \gamma(\nabla \cdot \boldsymbol{n})\boldsymbol{n}$. We consider $\boldsymbol{y} \in \partial V$ to find the interfacial motion:

1. $\lambda = 1$: we have

$$\boldsymbol{u}(\boldsymbol{y}) = -\gamma \int_{\partial V} (\nabla \cdot \boldsymbol{n})\boldsymbol{n} \cdot J(\boldsymbol{y}-\boldsymbol{x}) \, \mathrm{d}S_x$$

hence the interfacial velocity is determined directly by the current shape of the droplet. It requires only an integral of the forcing over the boundary (a 'membrane' of Stokeslets) to determine $\boldsymbol{u}$.

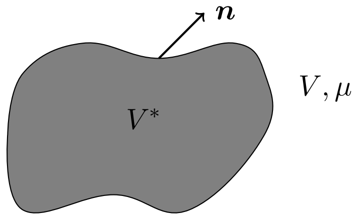
2. $\lambda \neq 1$: to solve the integral equation

$$\frac{\lambda+1}{2}\boldsymbol{u}(\boldsymbol{y}) = -\gamma \int_{\partial V} (\nabla \cdot \boldsymbol{n})\boldsymbol{n} \cdot J \, \mathrm{d}S_x + (\lambda - 1) \int_{\partial V} \boldsymbol{u}(\boldsymbol{x}) \cdot K \cdot \boldsymbol{n} \, \mathrm{d}S_x$$

for $\boldsymbol{u}$, we discretize the integrals and invert the resulting matrix equation. The integral equation is singular (so the matrix equation is singular) if $\lambda = 0$ or $\lambda = \infty$. The null eigenmodes are the 6 rigid body motions ($\lambda = \infty$) and bubble enlargement ($\lambda = 0$); the corresponding solubility conditions are the consistency conditions on boundary conditions.

2.6.4 Representation by Stokeslets alone

Consider the flow outside a moving body, e.g. swimming micro-organisms. Though we are interested in the flow $\boldsymbol{u}$ in V, we imagine a flow $\boldsymbol{u}^*$ in V^* with $\mu^* = \mu$ and $\boldsymbol{u}^* = \boldsymbol{u}$ on ∂V. Such a flow exists if $\int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$.



The previous result for drops gives

$$\boldsymbol{u}(\boldsymbol{y}) = \int_{\partial V} \mathsf{J}(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{f}_S(\boldsymbol{x}) \, \mathrm{d}S$$

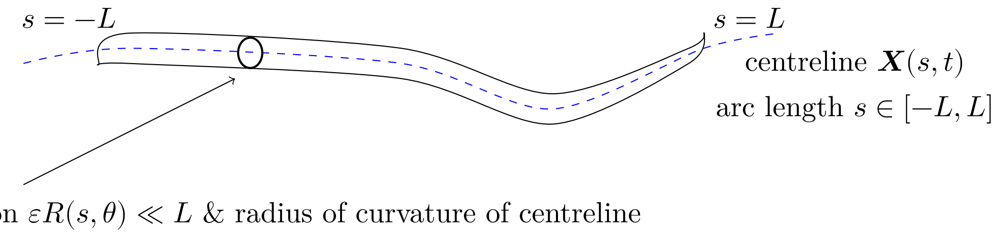
where $\boldsymbol{f}_S = (\sigma^* - \sigma) \cdot \boldsymbol{n}$ is the Stokeslet density. Note:

1. σ^* may not be the real stress in V^* unless the body really is fluid with $\mu = \mu^*$.
2. If the body is rigid then $\boldsymbol{u}^*$ is rigid-body motion and $\sigma^* = \mathbf{0}$. In this case $\boldsymbol{f}_S$ is the stress $-\sigma \cdot \boldsymbol{n}$ exerted by the body on the fluid. The external flow depends *only* on $\sigma \cdot \boldsymbol{n}$ and *not* on $\boldsymbol{u}$.

3 Approximations and applications

3.1 Slender-body theory

Consider the case of a solid, perhaps flexible, body e.g. flagellum, or fibre in a flow. Our aim is to calculate resistance to prescribed motion $\boldsymbol{V}(s) = \dot{\boldsymbol{X}}$ in background flow $\boldsymbol{u}^\infty(\boldsymbol{x})$.



Since the body is slender, we can approximate the surface distribution of Stokeslets by a distribution along the centreline, i.e.

$$\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{u}^\infty(\boldsymbol{y}) + \int_{-L}^{L} \mathsf{J}(\boldsymbol{y} - \boldsymbol{X}(s)) \cdot \boldsymbol{f}(s) \, \mathrm{d}s$$

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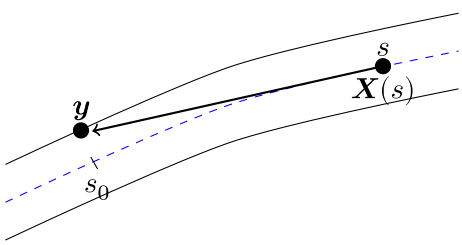
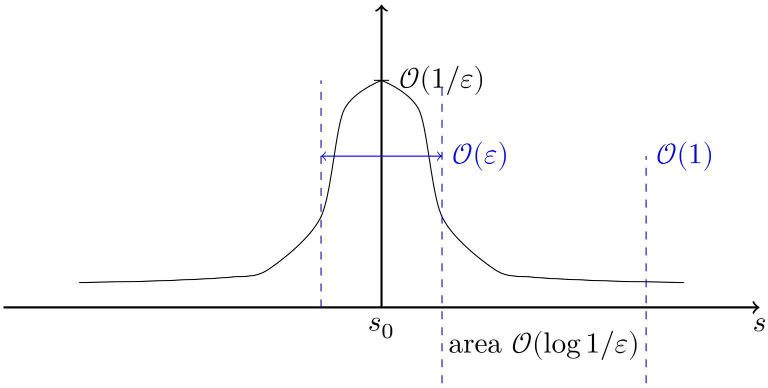
Note the integral is the first term in a far-field expansion for a ring of Stokeslets. Consider a point $\boldsymbol{y} = \boldsymbol{X}(s_0) + \varepsilon \boldsymbol{R}(s_0, \theta)$ on the surface of the body.



The integrand is proportional to $\frac{1}{|\boldsymbol{y}-\boldsymbol{X}(s)|}$ and behaves like so:



We see the integral is dominated by the contribution from $\varepsilon R \ll |s - s_0| \ll L$, where

$$\begin{aligned} \boldsymbol{f}(s) &\approx \boldsymbol{f}(s_0) \\ \boldsymbol{y} - \boldsymbol{X}(s) &\approx \boldsymbol{X}'(s_0)(s_0 - s) \end{aligned}$$

Hence we have

$$\boldsymbol{V}(s_0) = \boldsymbol{u}^\infty(\boldsymbol{X}(s_0)) + \frac{1}{8\pi\mu}\left(I + \boldsymbol{X}'(s_0)\boldsymbol{X}'(s_0)\right) \cdot \boldsymbol{f}(s_0) \left[\int_L^{s_0 - \mathcal{O}(\varepsilon)} + \int_{s_0+\mathcal{O}(\varepsilon)}^L \frac{\mathrm{d}s}{|s - s_0|}\right]$$

The integral $[\cdot] \sim 2\log\frac{1}{\varepsilon} + \mathcal{O}(1)$ and since $|\boldsymbol{X}'| = 1$,

$$[I + \boldsymbol{X}'\boldsymbol{X}']^{-1} = I - \frac{1}{2}\boldsymbol{X}'\boldsymbol{X}'$$

Thus we have the *leading-order slender-body approximation*

$$\boldsymbol{f}(s_0) \approx \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\left(I - \frac{1}{2}\boldsymbol{X}'\boldsymbol{X}'\right) \cdot \left(\boldsymbol{V}(s_0) - \boldsymbol{u}^\infty(\boldsymbol{X}(s_0))\right)$$

Note the following:

1. At this order, there is no dependence on the detailed cross-section, and resistance is local.
2. $\mathcal{O}\left((\log\frac{1}{3})^{-2}\right)$ corrections are not much smaller
3. There exists an ad hoc generalisation, called *resistive force theory*, of the form

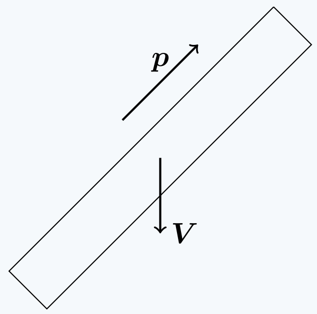
   $$\boldsymbol{f} = \left[k_\perp I + (k_\parallel - k_\perp)\boldsymbol{X}'\boldsymbol{X}'\right] \cdot [\boldsymbol{u} - \boldsymbol{u}^\infty]$$

   where

   $$k_\perp = \frac{4\pi\mu}{\log\frac{1}{\varepsilon} + c_1}, \qquad k_\parallel = \frac{2\pi\mu}{\log\frac{1}{\varepsilon} + c_2}$$

   This generalisation is still local.
4. A real improvement of accuracy, to $\mathcal{O}(\varepsilon^2\log\frac{1}{\varepsilon})$, comes from numerical solution of the 1D integral equation, which includes non-local effects.

Example. Translation of a rigid straight rod with velocity $\boldsymbol{V}(s)$, centre-line tangent vector $\boldsymbol{X}'(s) = \boldsymbol{p}$ both constant, and no background flow $\boldsymbol{u}^\infty = \boldsymbol{0}$.



If the rod falls perpendicular to its velocity (broadside) then $\boldsymbol{p} \cdot \boldsymbol{V} = 0$ so

$$\boldsymbol{f} = \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\boldsymbol{V}, \qquad \boldsymbol{F} = \frac{8\pi\mu L}{\log\frac{1}{\varepsilon}}\boldsymbol{V}$$

If the rod falls parallel to its velocity (lengthwise), we have $(\boldsymbol{p} \cdot \boldsymbol{V})\boldsymbol{p} = \boldsymbol{V}$ so

$$\boldsymbol{f} = \frac{2\pi\mu}{\log\frac{1}{\varepsilon}}\boldsymbol{V}, \qquad \boldsymbol{F} = \frac{4\pi\mu L}{\log\frac{1}{\varepsilon}}\boldsymbol{V}$$

1. The rod falls lengthwise only twice as fast as broadside.
2. The drag $\boldsymbol{F}$ scales with L, not εR, apart from $\log\frac{1}{\varepsilon}$, c.f. $\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}$ for a sphere.
3. Finite length more important than inertia if $\varepsilon = \frac{a}{L} \gg \mathrm{Re}$.

Example. Swimming flagellum.

Suppose the centreline is

$$\boldsymbol{X}(s) = (-Ut + s, a\cos(ks - \omega t))$$

i.e. a linearised wave with $ak \ll 1$. Then

$$\begin{aligned}\boldsymbol{V}(s) &= (-U, a\omega\sin(ks - \omega t))\\ \boldsymbol{X}'(s) &= (1, -ak\sin(ks - \omega t))\end{aligned}$$

Write $S = \sin(ks - \omega t)$. Slender-body theory gives

$$\begin{aligned}\boldsymbol{f} &= \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\begin{pmatrix}1 - \frac{1}{2} & \frac{1}{2}akS\\ \frac{1}{2}akS & 1 - \frac{1}{2}(akS)^2\end{pmatrix}\begin{pmatrix}-U\\ a\omega S\end{pmatrix}\\ &= \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\begin{pmatrix}-\frac{1}{2}U + \frac{1}{2}a^2k\omega S^2\\ -\frac{1}{2}akUS + a\omega S\end{pmatrix}\end{aligned}$$

Time averaging $\boldsymbol{f}$ gives

$$\langle \boldsymbol{f} \rangle = \frac{2\pi\mu}{\log\frac{1}{\varepsilon}}\left(-U + \frac{1}{2}a^2 k\omega\right)$$

There is no net force on the body, therefore $\langle \boldsymbol{f} \rangle = 0$ which implies $U = \frac{1}{2}a^2 k\omega$.

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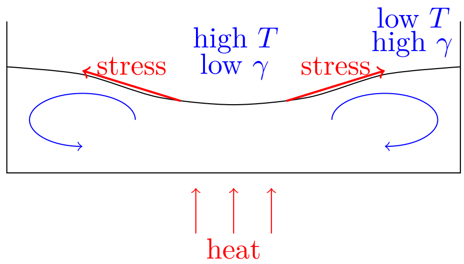
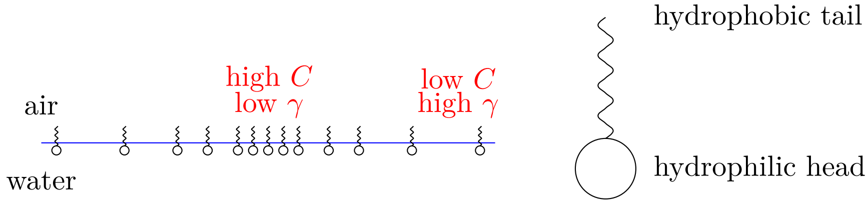
3.2 Marangoni Flows

The surface tension γ between two immiscible fluids depends on the fluids properties;

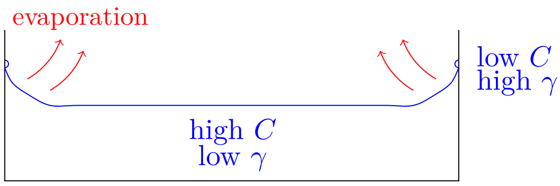
- temperature - γ decreases as T increases. For water, $\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}T} \sim -\frac{1}{50K}$. For example, *Marangoni convection*. Heat applied to a fluid with surface tension generates circulation:



- concentration of surfactants (surface active agents) - usually γ decreases as the concentration C increases. For example, detergent molecules spread over a surface. Detergent molecules have hydrophobic tails and hydrophilic heads.



Another example is alcohol in water; 'wine tears' creep up the side to form a rim on the meniscus due to high surface tension due to evaporating alcohol decreasing concentration.



3.2.1 Boundary conditions

Surface tension can be represented by a surface stress (in $N \cdot m^{-1}$, derived from a surface energy in $J \cdot m^{-2}$) acting isotropically across lines in the surface with stress tensor

$$\sigma^s = \gamma(\mathsf{I} - \boldsymbol{nn})$$

The force balance on an arbitrary small area of the interface gives

$$\nabla_s \cdot \sigma^s + [\sigma \cdot \boldsymbol{n}]_-^+ = 0$$

where $\boldsymbol{n}$ points out of the fluid and $\nabla_S = (\mathsf{I} - \boldsymbol{nn}) \cdot \nabla$ is the gradient operator in the surface. Hence the jump in fluid stress is

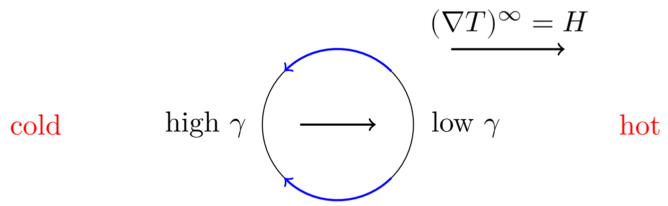
$$\begin{aligned}
[\sigma \cdot \boldsymbol{n}]_-^+ &= -\nabla_s \cdot \gamma (\mathsf{I} - \boldsymbol{nn}) \\
&= -(\mathsf{I} - \boldsymbol{nn}) \cdot \nabla_s \gamma + \gamma (\nabla_s \cdot \boldsymbol{n}) \boldsymbol{n} + \cancel{\gamma(\boldsymbol{n} \cdot \nabla_S)}\boldsymbol{n} \\
&= -\nabla_s \gamma + \gamma \kappa \boldsymbol{n}
\end{aligned}$$

The first term is the surface gradient of γ and the second is tension $\times$ curvature, where curvature is denoted by $\kappa \equiv \nabla_s \cdot \boldsymbol{n}$.
Equivalently, in perpendicular and parallel components we have

$$\begin{aligned}
[\boldsymbol{n} \cdot \sigma \cdot \boldsymbol{n}]_-^+ &= \gamma \kappa \\
[\boldsymbol{n} \times \sigma \cdot \boldsymbol{n}]_-^+ &= -\boldsymbol{n} \times \nabla_s \gamma
\end{aligned}$$

3.2.2 Thermophoresis

An immiscible drop in a temperature gradient migrates from cold regions to hot.



Consider a scaling argument. We have $\nabla T \sim Ha \implies \nabla \gamma \sim Ha\gamma'$ where $\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}T} < 0$. The driving force F from $\nabla\gamma$ acting across the equator length $\sim a$ implies $F \sim Ha^2\gamma'$. Now $\sigma \sim \frac{\mu U}{a}$ acts over an area $\sim a^2$ to give viscous resistive force $\sim \mu U a$ (c.f. drag on a sphere). Hence

$$U \sim \frac{\gamma' H a}{\mu}$$

Typical values are $H \sim 1^\circ C/cm, \gamma' \sim 1\text{dyn}/cm/^\circ C, a \sim 10^{-2}cm, \rho \sim 1g \cdot cm^{-3}, \nu \sim 10^{-2}cm$. Therefore $U \sim 1cm \cdot s^{-1}$.

Thermal problem. Consider the problem

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$

with boundary conditions $\nabla T \to \boldsymbol{H}$ as $|\boldsymbol{x}| \to \infty$ and

$$[T]_-^+ = [k\boldsymbol{n} \cdot \nabla T]_-^+ = 0 \qquad \text{on } |\boldsymbol{x}| = a$$

where k is the conductivity ($\boldsymbol{q} = -k\nabla T$) and $\kappa = \frac{k}{\rho C_p}$ is the thermal diffusivity. Assume the *Peclet* number satisfies

$$\mathrm{Pe} = \frac{Ua}{\kappa} = \frac{\text{advection}}{\text{diffusion}} \ll 1$$

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For Pe $\ll 1$, the problem reduces to solving $\nabla^2 T = 0$. Then, for example, with an insulating drop ($k_1 \ll k_2$)

$$T = T_0 + \boldsymbol{H}\cdot\boldsymbol{x}\left(1+\frac{a^3}{2r^3}\right)$$

and on $r=a$, $T = T_0 + \frac{3}{2}\boldsymbol{H}\cdot\boldsymbol{x}$. Hence

$$\gamma(T) = \gamma_0 + \gamma'(T-T_0) = \gamma_0 + \frac{3}{2}\gamma'\boldsymbol{H}\cdot\boldsymbol{x}$$

Fluid problem. The flow is driven by an interfacial force density

$$\begin{aligned}\boldsymbol{f}_s &= \sigma_1\cdot\boldsymbol{n} - \sigma_2\cdot\boldsymbol{n}\\ &= \nabla_s\gamma - \gamma\boldsymbol{n}\nabla_s\cdot\boldsymbol{n}\\ &= (\mathsf{I}-\boldsymbol{nn})\cdot\frac{3}{2}\gamma'\boldsymbol{H} - \frac{2}{a}\left(\gamma_0+\frac{3}{2}\gamma'\boldsymbol{H}\cdot\boldsymbol{x}\right)\boldsymbol{n}\\ &= -\frac{2}{a}\gamma_0\boldsymbol{n} + \frac{3}{2}\gamma'\boldsymbol{H}(\mathsf{I}-3\boldsymbol{nn})\end{aligned}$$

where the second term drives the motion. For the simple case of equal viscosities, $\lambda = 1$,

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{y}) &= \int_{r=a}\mathsf{J}(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{f}_s(\boldsymbol{x})\cdot\mathrm{d}S\\ &= \frac{3\gamma'}{16\pi\mu}\boldsymbol{H}\cdot\int(\mathsf{I}-3\boldsymbol{nn})\cdot\left(\frac{\mathsf{I}}{R}+\frac{\boldsymbol{RR}}{R^3}\right)\mathrm{d}S \quad \text{where } \boldsymbol{R}=\boldsymbol{y}-\boldsymbol{x}\\ &= \frac{3\gamma' a}{16\pi\mu}\boldsymbol{H}\cdot\mathsf{G}(\boldsymbol{y}/a)\end{aligned}$$

where, by symmetry of the integral and dimensions,

$$\mathsf{G} = \alpha(r/a)\mathsf{I} + \beta(r/a)\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$$

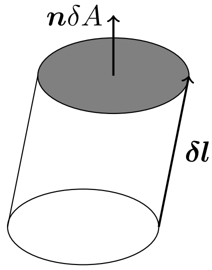
and $\hat{\boldsymbol{y}} = \boldsymbol{y}/a$. To determine the functions α, β one can evaluate $G_{ii} = 3\alpha+\beta$ and $\hat{y}_i G_{ij}\hat{y}_j = \alpha+\beta$. Now $\boldsymbol{u}\cdot\hat{\boldsymbol{y}} = \frac{3\gamma' a}{16\pi\mu}(\alpha+\beta)\boldsymbol{H}\cdot\hat{\boldsymbol{y}}$ which shows the spherical drop remains spherical and translates with velocity

$$\boldsymbol{U} = \frac{3\gamma' a}{16\pi\mu}\left(\alpha(1)+\beta(1)\right) = -\frac{1}{5}\frac{a}{\mu}\gamma\boldsymbol{H}$$

3.2.3 Surfactant effects

The concentration C in a small material area of interface δ changes due to

1. surface diffusion with flux $-D_s\nabla_s C$, where $D_s \sim 10^{-9}m^2s^{-1}$ typically
2. absorbtion from the bulk fluid, if soluble, with rate $-k(C-C_0)$ from chemical equilibrium and (linearised) kinetics $k \sim 10^3 s^{-1}$
3. change of area $\Delta(\delta A\, C) = 0 \implies \Delta C = -\frac{C}{\delta A}\Delta\delta A$. To get material $\frac{\mathrm{d}}{\mathrm{d}t}\delta A$, consider a material volume



Now $\delta V = \boldsymbol{\delta l} \cdot \boldsymbol{n}\delta A$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta V = \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{\delta l}\right] \cdot \boldsymbol{n}\delta A + \boldsymbol{\delta l} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{n}\delta A\right]$$

We have $\frac{\mathrm{d}}{\mathrm{d}t}\delta V = \delta V \nabla \cdot \boldsymbol{u}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\delta l} = (\boldsymbol{\delta l} \cdot \nabla)\boldsymbol{u}$ which, since $\boldsymbol{\delta l}$ is arbitrary, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{n}\delta A\right] = (I\nabla \cdot \boldsymbol{u} - \nabla\boldsymbol{u}) \cdot \boldsymbol{n}\delta A$$

Taking the dot product with $\boldsymbol{n}$ and noting $\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0$ we have

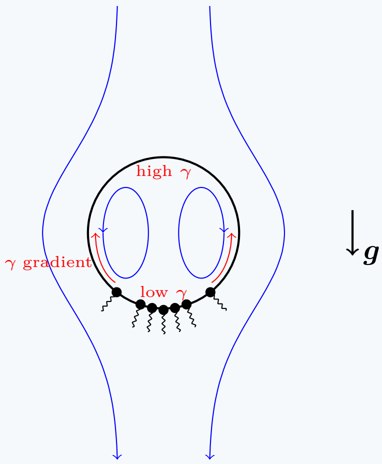
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\delta A &= (\nabla \cdot \boldsymbol{u} - \boldsymbol{n}\boldsymbol{n} : \nabla\boldsymbol{u})\, \delta A \\
&= \nabla_s \cdot \boldsymbol{u}\delta A \\
&= \delta A \nabla_s \cdot \left[\boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}\right] \\
&= \delta A \left[\nabla_s \cdot \boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\nabla_s \cdot \boldsymbol{n} + \cancel{(\boldsymbol{n} \cdot \nabla_s)(\boldsymbol{u} \cdot \boldsymbol{n})}\right]
\end{aligned}$$

Hence we have the *transport equation*

$$\frac{\mathrm{D}C}{\mathrm{D}t} = -C\left[\nabla_s \cdot \boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}\right] + D_s \nabla_s^2 C - k(C - C_0)$$

Example.

1. *Chemophoresis* – migration in a background concentration gradient ∇C^∞
2. *Rigidifcation of interfaces* – for example flow past a bubble.



Flow produces a concentration gradient ∇C, which produces a surface tension gradient $\nabla\gamma$, which produces an opposing flow. In practice, a small contamination has a big effect.

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A steady flow in the frame of the bubble has $\frac{\partial C}{\partial t} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. Consider a linearised calculation with $C = C_0 + C'(x)$ where $C' \ll C_0$. Then the transport equation reduces to

$$D_s \nabla^2 C' - kC' = C_0 \nabla_s \cdot \boldsymbol{u}_s$$

where the non-linear terms $C' \nabla_s \cdot \boldsymbol{u}_s$ and $\boldsymbol{u}_s \cdot \nabla_s C'$ have been neglected. Note k vanishes if the fluid is insoluble. This is a linear problem in the rise velocity $\boldsymbol{U}$ and from spherical symmetry we deduce

$$\boldsymbol{u}_s = A\,(\mathsf{I} - \boldsymbol{nn}) \cdot \boldsymbol{U}$$

for some constant A to be determined. To calculate $\nabla_s \cdot \boldsymbol{u}_s$, note

$$\begin{aligned}
\nabla_s \boldsymbol{n} &= (\mathsf{I} - \boldsymbol{nn}) \cdot \nabla \frac{\boldsymbol{x}}{a} = \frac{1}{a}(\mathsf{I} - \boldsymbol{nn}) \\
\nabla_s \cdot \boldsymbol{n} &= \frac{2}{a} \\
\implies \nabla_s \cdot \boldsymbol{u}_s &= -A\left[(\nabla_s \cdot \boldsymbol{n})\boldsymbol{n} + \cancel{(\boldsymbol{n} \cdot \nabla_s)}\boldsymbol{n}\right] \cdot \boldsymbol{U} = -\frac{2}{a} A \boldsymbol{U} \cdot \boldsymbol{n}
\end{aligned}$$

using the curvature $\kappa = \frac{2}{a}$ for a sphere of radius a. Guessing $C' \propto \boldsymbol{U} \cdot \boldsymbol{n}$, and since

$$\nabla_s^2(\boldsymbol{U} \cdot \boldsymbol{n}) = \nabla_s \cdot \left[\frac{1}{a}(\mathsf{I} - \boldsymbol{nn}) \cdot \boldsymbol{U}\right] = -\frac{2}{a^2} \boldsymbol{U} \cdot \boldsymbol{n}$$

we find

$$C' = \frac{2C_0 A \frac{\boldsymbol{U} \cdot \boldsymbol{n}}{a}}{k + 2\frac{D_s}{a^2}}$$

Therefore the linearisation $C' \ll C_0$ is valid if $\frac{UA}{D_s} \ll 1$ (fast diffusion) or $\frac{U}{ka} \ll 1$ (fast adsorption). The problem is completed by using a Papkovich-Neuber representation inside and outside the droplet with boundary conditions

$$\begin{aligned}
\boldsymbol{u} &\to -\boldsymbol{U} \text{ as } r \to \infty \\
\boldsymbol{u} &\text{ finite at } r = 0 \\
\boldsymbol{u} &= \boldsymbol{u}_s \text{ at } r = a^+_- \\
[\boldsymbol{n} \times \sigma \cdot \boldsymbol{n}]^+_- &= -\boldsymbol{n} \times \nabla C' \frac{\mathrm{d}\gamma}{\mathrm{d}C} \text{ at } r = a
\end{aligned}$$

These are 5 conditions for 4 potentials and A. We find the drag on the bubble is

$$\boldsymbol{F} = -4\pi\mu a \boldsymbol{U} \left(\frac{\frac{3}{2}(\lambda^* + \lambda) + 1}{\lambda^* + \lambda + 1}\right)$$

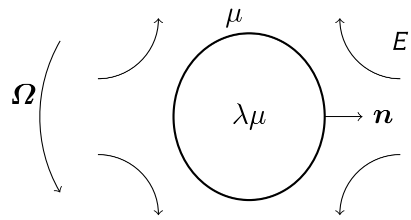
where

$$\lambda^* = -\frac{\frac{1}{3} a C_0 \frac{\mathrm{d}\gamma}{\mathrm{d}C}}{(a^2 k + 2D_s)\mu}$$

Note as $\lambda^* \to \infty$ we get a 'rigid' sphere.

3.3 Deformation of droplets.

To leading order, a small force-free droplet moves with the flow $\boldsymbol{u}^\infty$ and is deformed by the local gradient $(\boldsymbol{x}\cdot\nabla)\boldsymbol{u}^\infty$ against the restoring action of surface tension. We assume the background flow is composed of a straining component with symmetric traceless E and a rotational component with angular velocity $\boldsymbol{\Omega}$. Hence $\boldsymbol{u} \to \boldsymbol{\Omega}\times\boldsymbol{x} + \mathsf{E}\cdot\boldsymbol{x}$ as $|\boldsymbol{x}|\to\infty$.



The importance of viscous stress vs. surface tension is characterised by the *capillary number* Ca where

$$\frac{\text{viscous stresses}}{\text{surface tension}} \sim \frac{\mu E}{\gamma/a} = \mathrm{Ca} = \frac{\mu U}{\gamma}$$

3.3.1 Small deformations

Here we consider $\mathrm{Ca}\ll 1$ and a droplet with small deformation in a pure strain flow $\boldsymbol{u}^\infty = \mathsf{E}\cdot\boldsymbol{x}$. Deformation is due to symmetric traceless E. Hence for small (linearised) deformations we expect (to first order in D)

$$r = a\left[1 + \frac{\boldsymbol{x}\cdot\mathsf{D}\cdot\boldsymbol{x}}{r^2}\right]$$

where D is symmetric, traceless (from volume conservation), and $\mathcal{O}(\mathrm{Ca})$. The surface normal is then

$$\begin{aligned}\boldsymbol{n} &= \nabla\left(r - a\left[1+\frac{\boldsymbol{x}\cdot\mathsf{D}\cdot\boldsymbol{x}}{r^2}\right]\right)\\ &= \frac{\boldsymbol{x}}{r} - 2a\left(\frac{\mathsf{D}\cdot\boldsymbol{x}}{r^2} - \frac{(\boldsymbol{x}\cdot\mathsf{D}\cdot\boldsymbol{x})\boldsymbol{x}}{r^4}\right) + \mathcal{O}(\mathsf{D}^2)\end{aligned}$$

Note $|\boldsymbol{n}| = 1$ to $\mathcal{O}(\mathsf{D}^2)$ as required. Also, $|\boldsymbol{n}|=1$ implies $\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla)\boldsymbol{n} = 0$. Hence

$$\begin{aligned}\nabla_s\cdot\boldsymbol{n} &= \nabla\cdot\boldsymbol{n}\\ &= \frac{2}{r} + 6a\left.\frac{\boldsymbol{x}\cdot\mathsf{D}\cdot\boldsymbol{x}}{r^4}\right|_s\\ &= \frac{2}{a} + 4\frac{\boldsymbol{x}\cdot\mathsf{D}\cdot\boldsymbol{x}}{a^3} + \mathcal{O}(\mathsf{D}^2)\end{aligned}$$

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To solve the problem, use a Papkovich-Neuber solution with

$$\begin{aligned}&\chi = \frac{1}{2}\boldsymbol{x}\cdot\mathsf{E}\cdot\boldsymbol{x} + \frac{a^5}{3}\mathsf{Q}:\nabla\nabla\frac{1}{r}, && \boldsymbol{\Phi} = \frac{a^2}{3}\mathsf{P}\cdot\nabla\frac{1}{r} && r>a\\ &\chi = \frac{1}{2}\boldsymbol{x}\cdot\mathsf{e}\cdot\boldsymbol{x}, && \boldsymbol{\Phi} = \frac{r^7}{6a^2}\mathsf{p}:\nabla\nabla\nabla\frac{1}{r} && r<a\end{aligned}$$

where

- the second rank tensors $\mathsf{P},\mathsf{Q},\mathsf{p},\mathsf{e}$ are linearly dependent on D,E, and hence are all symmetric and traceless;

- the use of χ is forced by strain; the contribution of Q to χ is equivalent to $\boldsymbol{\Phi} = \frac{a^5}{r^5}\mathsf{Q} : \nabla\nabla\nabla\frac{1}{r}$;
- the coefficients are used to get the dimensions correction and to reduce numerical factors.

The fluid velocity is then

$$\boldsymbol{u} = \begin{cases} \mathsf{e}\cdot\boldsymbol{x} + \left(\frac{5r^2}{a^2}\mathsf{p}\cdot\boldsymbol{x} - \frac{2}{a^2}(\boldsymbol{x}\cdot\mathsf{p}\cdot\boldsymbol{x})\boldsymbol{x}\right) & \text{inside} \\ \mathsf{E}\cdot\boldsymbol{x} + \frac{a^3}{r^5}(\boldsymbol{x}\cdot\mathsf{P}\cdot\boldsymbol{x})\boldsymbol{x} + \left(\frac{2a^5}{r^5}\mathsf{Q}\cdot\boldsymbol{x} - \frac{5a^5}{r^7}(\boldsymbol{x}\cdot\mathsf{Q}\cdot\boldsymbol{x})\boldsymbol{x}\right) & \text{outside} \end{cases}$$

and similar expressions for $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ on $r = a$ for linearised BCs. Our boundary conditions are continuity of normal velocity; continuity of tangential velocity; continuity of tangential stress; and jump in normal stress equal to $\frac{2\gamma}{\mu a}\mathsf{D}$. The kinematic boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = \dot{r} = a\boldsymbol{n}\cdot\dot{\mathsf{D}}(t)\cdot\boldsymbol{n}$ gives

$$\frac{\mathrm{d}\mathsf{D}}{\mathrm{d}t} = \frac{5}{2\lambda+3}\mathsf{E} - \frac{\gamma}{\mu a}\frac{40(\lambda+1)}{(2\lambda+3)(19\lambda+16)}\mathsf{D}$$

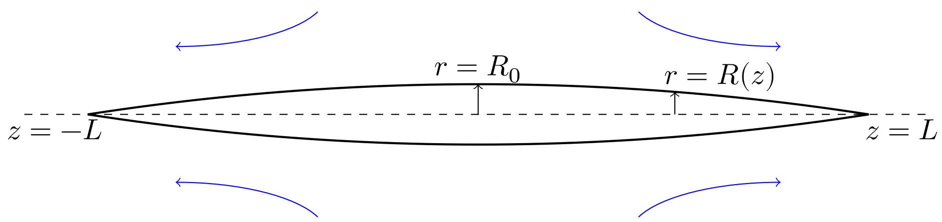
which tends monotonically to a linearised state

$$\mathsf{D} = \frac{19\lambda+16}{8(\lambda+1)}\frac{\mu a}{\gamma}\mathsf{E}$$

3.3.2 Larger deformations

Experiments and numerics show that the deformation increases monotonically in the capillary number Ca up to a critical value $\mathrm{Ca}_{\mathrm{crit}}$ where the drop breaks. This critical value of Ca depends on the flow type and viscosity ratio λ. Taylor (1934) showed that large λ drops do not break in a simple shear. Since a simple shear flow is composed of rotation and strain, we find the drop rotates nearly rigidly with angular velocity $\boldsymbol{\Omega}$ and sees an oscillating strain tensor E, with steady deformation $\sim \frac{\mathrm{Ca}}{\lambda}$. In pure strain, we find $\mathrm{Ca}_{\mathrm{crit}} \sim \lambda^{-1/6}$ for small λ and $\mathrm{Ca}_{\mathrm{crit}}$ tends to ~ 0.1 as $\lambda \to \infty$. Large λ drops just take longer to extend and break. Small λ drops (bubbles) are hard to break.

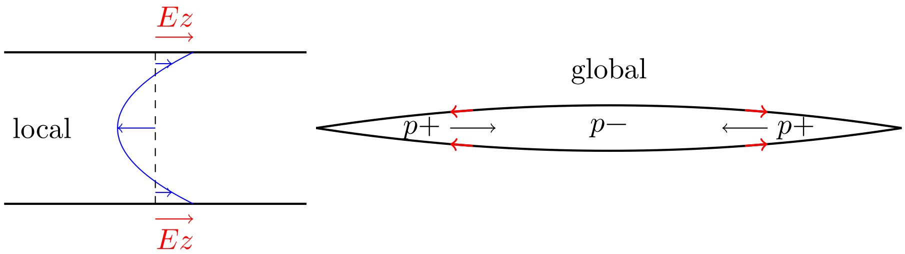
Large deformations at $\lambda \ll 1$. In experiment, we see drops extending axisymmetrically and forming pointed ends. We assume the drop is defined by $r = R(z)$ with $-L \leq z \leq L$ and $L \gg R_0$, the maximum width of the droplet.



Since $\lambda \ll 1$, we neglect the shear stress exerted by the internal flow, so that $u_z^{\mathrm{ext}} = Ez \sim EL$. We start by neglecting pressure gradients in the bubble: moving along the bubble with a material slice of the external fluid, we see a circular hole of radius $R(z)$ collapsing with velocity $Ez\frac{\mathrm{d}R}{\mathrm{d}z}$ with strain rate $\sim \frac{\text{velocity}}{r}$, due to a stress difference $\frac{\gamma}{R}$ between inside and out. Hence

$$\mu E \sim \frac{\gamma}{R_0} \implies \frac{a}{R_0} \sim \mathrm{Ca}$$

The volume constraint gives $R_0^2 L \sim a^3$ so that $\frac{L}{R_0} \sim \mathrm{Ca}^3$. The flow inside the drop is (nearly) Poiseuille, but with wall velocity Ez and no net flux.



To balance the pressure gradient we require

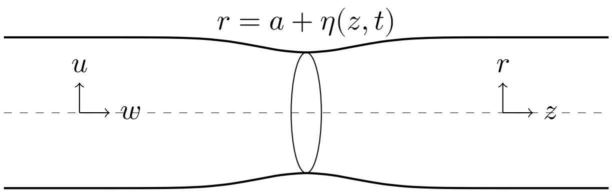
$$\frac{\Delta p}{L} \sim \frac{\lambda \mu E L}{R_0^2}$$

This is possible provided $\Delta p \le \frac{\gamma}{R_0}$, i.e. $\lambda \ll \frac{\gamma}{R_0 \mu E} \frac{R_0^2}{L^2}$. Note the factor $\gamma / R_0 \mu E \sim 1$ so

$$\lambda \ll \frac{R_0^2}{L^2} \sim \mathrm{Ca}^{-6}$$

3.4 Rayleigh-Plateau Instability

Consider small axisymmetric perturbations to the shape of a cylinder of one fluid surrounded by another.



The cylinder has radius $r = a + \eta(z,t)$ with $\eta \ll a$ and $\eta_z \ll 1$. The surface normal is

$$\boldsymbol{n} = (1, 0, -\eta_z) + \mathcal{O}(\eta^2)$$

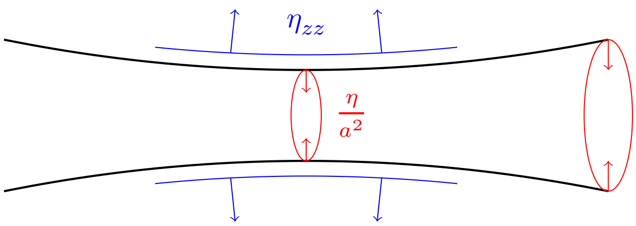
Hence

$$\nabla \cdot \boldsymbol{n} = \frac{1}{r} - \eta_{zz} \approx \frac{1}{a} - \frac{\eta}{a^2} - \eta_{zz} + \mathcal{O}(\eta^2)$$

The η/a^2 term accounts of azimuthal curvature and the η_{zz} term accounts for axial curvature. **Lecture 15** 11/11/20
Consider disturbances of the form $\eta = \hat{\eta} e^{st} \cos kz$. Then

$$\nabla \cdot \boldsymbol{n} = \frac{1}{a} - \frac{1 - k^2 a^2}{a^2} \eta$$

The disturbance is stabilised by axial curvature which acts to reduce the disturbance, and is destabilised by azimuthal curvature which acts to constrict the cylinder further.



The disturbance is unstable to long wavelengths with $k^2a^2 < 1$. The growth rate s depends on the dynamics. The full case including inertia, internal and external pressure, and internal and external viscosity is given by Tomotika (1935). We will sketch the simple case of no inertia and no external viscosity. The linearised boundary conditions are

$$
\begin{aligned}
\text{kinematic} \qquad \eta_t &= u\big|_{r=a} \qquad \text{neglecting } w\eta_z \\
\implies \eta &= \frac{u}{s} \text{ on } r = a \\
\text{dynamic} \quad [\sigma \cdot \boldsymbol{n}]_-^+ &= \gamma(\nabla \cdot \boldsymbol{n})\boldsymbol{n} \\
\sigma_{\text{out}} &= 0
\end{aligned}
$$

Hence the stress components are, on $r = a$,

$$
\begin{aligned}
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} &= 0 && \text{tangential} \\
\cancel{-p_0} - p + 2\mu\frac{\partial u}{\partial r} &= \gamma\left(\frac{1 - k^2a^2}{a^2}\right)\frac{u}{s} - \cancel{\frac{\gamma}{a}} && \text{normal}
\end{aligned}
$$

where in the normal component the balanced terms $-p_0 = -\frac{\gamma}{a}$ are excluded since they play no role in the dynamics. To find the Stokes flow inside the cylinder, we need the axisymmetric harmonic functions which are proportional to $\cos kz$, derived by separation of variables. We have

$$
\begin{aligned}
\chi &= AI_0(kr)\cos kz \\
\boldsymbol{\Phi} &= (BI_1(kr)\cos kz, 0, 0)
\end{aligned}
$$

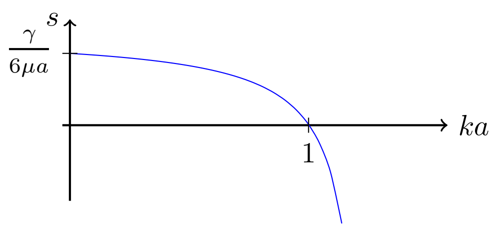
where I_0 and I_1 are modified Bessel functions which are well behaved at the origin. Note that the $\hat{\boldsymbol{z}}$ component of $\boldsymbol{\Phi}$ vanishes to avoid $w \propto z$ as we only want oscillatory motion in the z direction. Note also that $I_0'(x) = I_1(x)$ and $xI_0 = (xI_x(x))'$. We find

$$
\begin{aligned}
u &= (AkI_1 + BkrI_0 - 2BI_1)\cos kz \\
w &= (-AkI_0 - BkrI_1)\sin kz \\
p &= 2\mu BkI_0 \cos kz
\end{aligned}
$$

Applying the boundary conditions yields the growth rate as

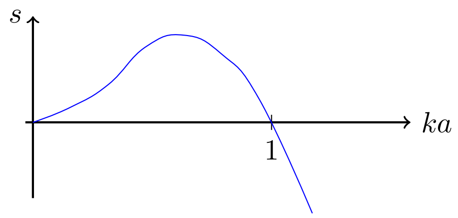
$$
s = \frac{\gamma}{2\mu a}(1 - k^2a^2)\frac{I_1(ka)^2}{k^2a^2I_0(ka)^2 - (1 + k^2a^2)I_1(ka)^2}
$$

as derived by Lord Rayleigh (1892).



1. The most unstable disturbance has infinite wavelength since this minimises the internal deformation and we have ignored external drag. A cylindrical bubble in viscous fluid is also most unstable at infinite wavelengths, since it minimises external $\frac{\partial u}{\partial z}$ and ignores internal Poiseuille flow.

2. If $0 < \lambda < \infty$ then the maximum s is at finite k.



3. Consider the scaling requierd to be able to neglect inertia. Suppose $ak = \mathcal{O}(1)$, then we have from the boundary conditions

$$\frac{\mu u}{a^2} \sim \frac{p}{a} \sim \frac{\gamma}{a^2}, \qquad u \sim as$$

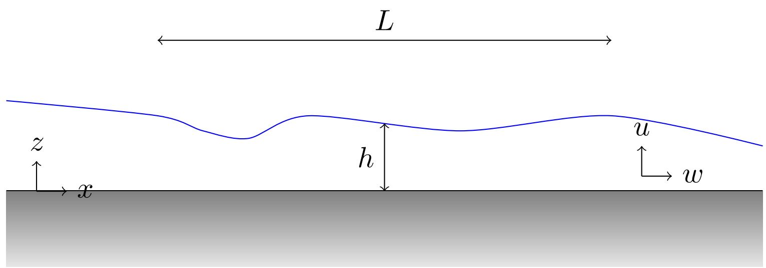
Thus we find the growth rate scales as $s \sim \frac{\gamma}{\mu a}$ as we expected from the solution. If inertia is much smaller than viscous terms then we have

$$\rho us \ll \frac{\mu u}{a^2} \implies \frac{\mu^2}{\gamma \rho a} \gg 1$$

4. There are interesting self-similar dynamics in the non-linear regime as the point of breakup is reached.

3.5 Long Thin Flows I: Lubrication Theory

Here we consider thin flow with (at least one) effectively no-slip boundary. Denote the fluid depth as h and the horizontal lengthscale L. We assume $h \ll L$.



Assume the velocity gradients satisfy $\partial_x \sim \frac{1}{L} \ll \frac{1}{h} \sim \partial_z$. Consider the scalings for the governing equations. From incompressibility we have

$$\nabla \cdot \boldsymbol{u} = 0 \implies \frac{u}{L} \sim \frac{w}{h} \implies w \ll u$$

From Navier-Stokes we have

$$\begin{array}{lccccccc}
\text{NS (steady)} & \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & = & -\nabla p & + & \mu\nabla^2\boldsymbol{u} & & \\
\parallel \text{ to boundary} & \rho\dfrac{u^2}{L} & & \dfrac{p}{L} & & \mu\left(\dfrac{u}{L^2}\right. & + & \left.\dfrac{u}{h^2}\right) \\
\implies & \dfrac{uh}{\nu}\dfrac{h}{L} & & \dfrac{p}{\left(\frac{\mu u L}{h^2}\right)} & & \dfrac{h^2}{L^2} & & 1
\end{array}$$

Hence we can neglect inertia provided the *modified Reynolds number* $\frac{uh}{\nu}\frac{h}{L} \ll 1$. Note we can also approximate ∇^2 by $\frac{\partial^2}{\partial z^2}$, and we expect $p \sim \frac{\mu u L}{h^2}$ (cf. pipe flow). The component perpendicular to the boundary scales as

$$
\begin{array}{llcccc}
\text{NS (steady)} & \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & = & -\nabla p & + & \mu\nabla^2\boldsymbol{u} \\
\perp \text{ to boundary} & \dfrac{\rho u w}{L} & & \dfrac{p}{h} & & \mu\left(\dfrac{w}{L^2} + \dfrac{w}{h^2}\right) \\
\implies & \left(\dfrac{uh}{\nu}\right)\dfrac{h^2}{L^2} & & 1 & & \dfrac{h^4}{L^4} + \dfrac{h^2}{L^2}
\end{array}
$$

so at leading order we have $\frac{\partial p}{\partial z} = 0$, i.e. $p = p(x, y, t)$ only. The flow in the thin layer is quasi-parallel to the boundary. Let $\boldsymbol{u} \equiv (u, v)$ and $\nabla \equiv (\partial_x, \partial_y)$ denote the parallel components of velocity and gradient. Our governing equation is now **Lecture 16** 13/11/20

$$\mu\frac{\partial^2\boldsymbol{u}}{\partial z^2} = \nabla p$$

Define the *depth-integrated parallel volume flux*

$$\boldsymbol{q} = \int_0^h \boldsymbol{u}\,\mathrm{d}z$$

Depth-integrated conservation of mass can then be written as

$$\frac{\partial h}{\partial t} + \nabla\cdot\boldsymbol{q} = 0$$

Note on a curve surface ∇ should be replaced by ∇_s. These three equations can be solved with various boundary conditions at $z = 0, h$, or equivalently $z = h_1, h_2$.

3.5.1 Squeeze films

Here we have prescribed velocity boundary conditions

$$\boldsymbol{u} = \begin{cases} \boldsymbol{U}_1 & z = 0 \\ \boldsymbol{U}_2 & z = h \end{cases}$$

Subject to these boundary conditions we find

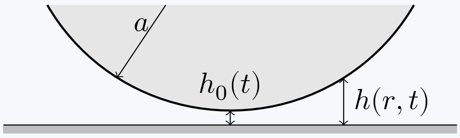
$$
\begin{aligned}
\boldsymbol{u} &= -\frac{\nabla p}{2\mu}z(h-z) + \boldsymbol{U}_1 + (\boldsymbol{U}_2 - \boldsymbol{U}_1)\frac{z}{h} \\
\boldsymbol{q} &= -\frac{h^3}{12\mu}\nabla p + \frac{h}{2}(\boldsymbol{U}_1 + \boldsymbol{U}_2)
\end{aligned}
$$

Applying mass conservation yields *Reynold's (lubrication) equation*

$$\frac{\partial h}{\partial t} = \frac{1}{12\mu}\nabla\cdot(h^3\nabla p) - \frac{1}{2}\nabla\cdot h(\boldsymbol{U}_1 + \boldsymbol{U}_2)$$

Usually given the velocities, we first calculate the pressure field p and then the force as an integral of pressure over the surface.

Example. Rigid sphere settling towards a plane wall.



The gap thickness is given by

$$h(r,t) = h_0(t) + \frac{r^2}{2a} + \mathcal{O}(\frac{r^4}{a^3})$$

with $\partial_t h = \dot{h}_0 < 0$. By mass conservation (i.e. either integrate the mass conservation equation or consider flux across $r =$constant) we have

$$-\pi r^2 \dot{h}_0 \hat{\boldsymbol{r}} = 2\pi r \boldsymbol{q}(r)$$

and for these rigid-rigid boundary conditions we solve to find

$$\boldsymbol{u} = -\frac{1}{2\mu}\frac{\partial p}{\partial r} y(h-y)\hat{\boldsymbol{r}}$$
$$\boldsymbol{q} = -\frac{h^3}{12\mu}\frac{\partial p}{\partial r}\hat{\boldsymbol{r}}$$

We can use the expression for $\boldsymbol{q}$ and mass conservation to find $\frac{\partial p}{\partial r}$ and integrate to get

$$p(r,t) = \int^r \frac{6\mu\dot{h}_0 r}{h^3}\,\mathrm{d}r = -\frac{6\mu\dot{h}_0}{h_0^3}\int_r^\infty \frac{r}{(1+r^2/2ah_0)^3}\,\mathrm{d}r = p_\infty - \frac{3\mu\dot{h}_0 a}{h_0^2(1+r^2/2ah_0)^2}$$

Hence the force on the sedimenting sphere is

$$F = 2\pi\int_0^\infty (p-p_\infty) r\,\mathrm{d}r = -\frac{6\pi\mu a^2\dot{h}_0}{h_0}$$

Example. Hele-Shaw flow. A simple case of the above is $\boldsymbol{U}_i = 0$ with h uniform and constant. Then

$$\nabla^2 p = 0$$
$$\boldsymbol{q} = -\frac{h^3}{12\mu}\nabla p$$

Boundary conditions on p or $\boldsymbol{q}\cdot\boldsymbol{n}$ are needed. This set-up is often used to visualise flow past an object.

3.5.2 Free-surface flows

Consider the boundary conditions

$$\begin{aligned} \boldsymbol{u} &= 0 \quad \text{on } z = 0 \\ \mu\frac{\partial \boldsymbol{u}}{\partial z} &= 0 \quad \text{on } z = h \end{aligned}$$

Subject to these conditions, the solution is

$$\begin{aligned} \boldsymbol{u} &= -\frac{\nabla p}{2\mu} z(2h - z) \\ \boldsymbol{q} &= -\frac{h^3}{3\mu}\nabla p \\ \frac{\partial h}{\partial t} &= \frac{1}{3\mu}\nabla \cdot (h^3 \nabla p) \end{aligned}$$

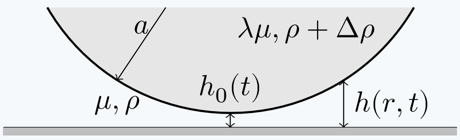
Usually, p is specified, for example from gravity and surfacae tension $p = \rho g(h - z) - \gamma\nabla^2 h$, from which $\frac{\partial h}{\partial t}$ is calculated.

3.5.3 Marangoni flows

Consider replacing the top rigid or free surface boundary condition with a surface tension condition

$$\mu\frac{\partial \boldsymbol{u}}{\partial z} = \nabla\gamma$$

Example. Viscous drop approaching a plane wall.



Assume to start with that

1. surface tension is present and keeps the drop spherical;
2. $\lambda \gg 1$ so the drop is effectively rigid.

Consider the following scalings.

$$\begin{aligned} L &\sim \sqrt{ah_0} && \text{from geometry} \\ \mu &\sim \frac{L}{h_0}\dot{h}_0 && \text{from mass conservation} \\ p &\sim \frac{\mu u L}{h_0^2} \sim \frac{\mu a}{h_0^2}\dot{h}_0 && \text{from pipe flow} \\ F &\sim pL^2 \sim \frac{\mu a^2}{h_0}\dot{h}_0 && (\sim \Delta\rho g a^3 \text{ if balancing weight}) \end{aligned}$$

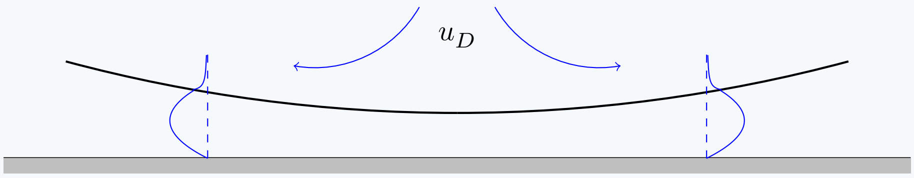
From these we can check our assumptions.

1. High pressure in the middle of the gap deforms the drop by Δh over lengthscale L. Then $p \sim \gamma \Delta\kappa$ and $\Delta\kappa \sim \frac{\Delta h}{L^2}$. Hence

$$\frac{\Delta h}{h_0} \sim \frac{p}{\gamma/a} \sim \frac{\mu \dot{h}_0}{\gamma}\frac{a^2}{h_0^2}$$

Thus deformation is negligible if the capillary number $\mathrm{Ca} \equiv \frac{\mu \dot{h}_0}{\gamma} \ll \frac{h_0^2}{a^2}$, or the *Bond number* $\mathrm{Bo} \equiv \frac{\Delta\rho g a^2}{\gamma} \ll \frac{h_0}{a}$. Note the Bond number indicates the importance of weight over capillary pressure. This requirement is eventually invalid as h_0 decreases.

2. The shear stress $\sim \frac{\mu u}{h_0}$ in the gap drives flow u_D in the drop over a lengthscale L.



Stress balance at interface

$$\frac{\mu u}{h_0} \sim \frac{\lambda \mu u_D}{L}$$
$$\implies \frac{u_D}{u} \sim \sqrt{\frac{a}{h_0}}\frac{1}{\lambda}$$

Rigid boundary condition $u(h) = 0$ is valid provided $\lambda \gg \sqrt{\frac{a}{h_0}}$, which is eventually invalid.

This problem can be extended in a number of ways.

1. If $\lambda \ll \sqrt{\frac{a}{h_0}}$, e.g. bubble, can neglect shear stress exerted by the drop and apply $\mu\frac{\partial u}{\partial z} = 0$ at $z = h$. Hence $\dot{h}_0/F$ increases by a factor of $(\frac{1}{3})/(\frac{1}{12}) = 4$.

2. If $\lambda \sim \sqrt{\frac{a}{h_0}}$ then we need to couple $u(h)$ and $\mu\frac{\partial u}{\partial z}$ at $z = h$ by integral representation for the flow in the drop.

3. If $\mathrm{Bo} \gg \frac{h_0}{a}$, we get a dimple in the middle of the gap where the pressure is greatest.

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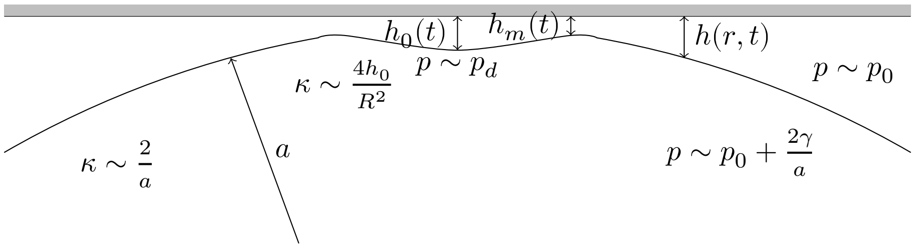
3.5.4 Dimple drainage between bubble and wall

Here we will sketch the solution for dimple drainage of a bubble close to a wall. A full treatment can be found in Jones & Wilson, JFM 1978, and Yiantso & Davis, JFM 1990. For simplicity we neglect the viscosity of the bubble, $\lambda = 0$. Further we assume $\frac{h_0}{a} \ll \mathrm{Bo} = \frac{\Delta\rho g a^2}{\gamma} \ll 1$, so that the majority of the fluid in the dimple has drained. Finally, we assume and must check that $h_m \ll h_0 \ll R \ll a$ as $t \to \infty$, where R is the horizontal extent of the dimple.



The key idea is the small gap h_m controls a slow leakage flux from the dimple, which is thus roughly stagnant with uniform pressure

$$p_d = p_0 + \gamma \left(\frac{2}{a} - \nabla^2 h \right)$$

We now sketch the solution. From vertical force balance we have

$$(p_d - p_0)\pi R \approx \frac{4}{3}\pi a^3 \Delta\rho g$$

Hence we find in $r < R$

$$R \sim a\mathrm{Bo}^{1/2}$$
$$\nabla^2 h \ll \frac{2}{a}$$

Dimple: in $r < R$, since p_d is uniform, we have

$$\nabla^2 h = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r} \right) \approx \text{const.}$$

Solving for h we find

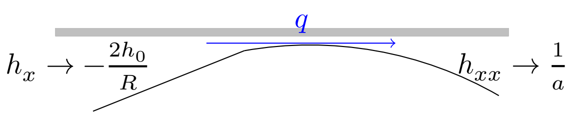
$$h \approx h_0 \left(1 - \frac{r^2}{R^2} \right)$$

and the trapped volume $V \sim h_0 R^2$.

Small gap: free surface lubrication theory gives leakage flux

$$q = \frac{h^3}{3\mu}(\gamma h_{xx})_x$$

where $x = r - R$ is a local coordinate near h_m. Clearly $\dot{V} \approx -2\pi R q$ from which we can deduce an equation for $\dot{h}_0$.



We now look for a similarity solution. The scalings are

$$\frac{h}{x} \sim \frac{h_0}{r}, \qquad \frac{h^4}{x^3} \sim q \sim \frac{h_0}{t}, \qquad \frac{h}{x^2} \sim 1$$

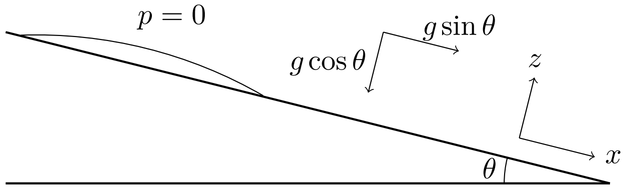
Hence we have $h, h_m \propto t^{-1/2}$, $h_0 \sim t^{-1/4}$, $x \sim t^{-1/4}$, $q \sim t^{-5/4}$, with $\xi = \frac{x}{Bt^{-1/4}}$, $h = At^{-1/2}H(\xi)$ and suitable constants A, B, chosen so that the unknown flux ~ 1 rather than $H'' \to 1$ as $\xi \to \infty$. From this we deduce an equation for H:

$$H^3 H''' = 1$$

with boundary conditions $H' \to -1$ as $\xi \to -\infty$, which is infact 2 conditions that the coefficient of ξ^2 goes to 0 and the coefficient of ξ goes to -1. Hence we find a unique solution to within translation, with $H_{\text{min}} = 1.2571$ and $H \sim \frac{1}{2}C\xi^2$ as $\xi \to \infty$ with $C = 1.2098$. The unscaled matching condition $h \sim \dfrac{1}{2a}x^2$ fixes the remaining constants.

3.5.5 Gravity current on an inclined plane

Consider a viscous fluid placed on a plane inclined at angle θ to the horizontal. We use x as the coordinate parallel and down the plane, y parallel and across the plane, and z perpendicular to the plane. We will neglect surface tension.



In our original derivation of the lubrication equations, we neglected body forces. Hence we must modify the equations to include gravitational force. The modified z-balance is

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho g \cos\theta \\ \implies p &= \rho g(h(x,y,t) - z)\cos\theta \end{aligned}$$

The modified x-balance is similarly

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} - \rho g \sin\theta$$

We solve for $\boldsymbol{u}$ to get $\boldsymbol{q}$, then using mass conservation we find the evolution equation

$$\frac{\partial h}{\partial t} + \frac{g\sin\theta}{3\nu}\frac{\partial h^3}{\partial x} = \frac{g\cos\theta}{3\nu}\nabla \cdot (h^3 \nabla h)$$

We can find various similarity solutions to this equation. See Lister, JFM 1992.

3.5.6 Similarity solutions

Note we often find similarity solutions of the form $At^\alpha f(\xi \equiv \frac{x}{Bt^\beta})$. Then

$$\begin{aligned} \frac{\partial}{\partial x}\left[At^\alpha f(\frac{x}{Bt^\beta})\right] &= \frac{At^\alpha}{Bt^\beta} f'(\xi) \\ \int \mathrm{d}x &= Bt^\beta \int \mathrm{d}\xi \\ \frac{\partial}{\partial t}\left[At^\alpha f(\frac{x}{Bt^\beta})\right] &= \frac{At^\alpha}{t}(\alpha f - \beta \xi f'(\xi)) \end{aligned}$$

Hence operations work in the same way, but with extra factors, in the same way the scalings are derived. For this reason, the factors often cancel in the governing equations.

Example. Thermal diffusion of a heat pulse in 3D. The governing equation is

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r}$$

with volume constraint

$$4\pi \int T r^2 \, \mathrm{d}r = Q$$

Consider the scalings of these two equations.

$$\frac{T}{t} \sim \frac{\kappa T}{r^2}, \qquad T r^3 \sim Q$$

Hence $r \sim \sqrt{\kappa t}$, $T \sim \frac{Q}{(\kappa t)^{3/2}}$. Motivated by these scalings, try similarity solution $T = \frac{Q}{(\kappa t)^{3/2}} f(\xi \equiv \frac{r}{\sqrt{\kappa t}})$. Then

$$-\frac{3}{2} f - \frac{1}{2} \xi f' = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^2 \frac{\partial f}{\partial \xi}$$

and volume constraint

$$4\pi \int_0^\infty f \xi^2 \, \mathrm{d}\xi = 1$$

Hence we find $f = Ae^{-\xi^2/4}$ with $A = \frac{1}{(2\pi)^{3/2}}$ by the integral constraint.

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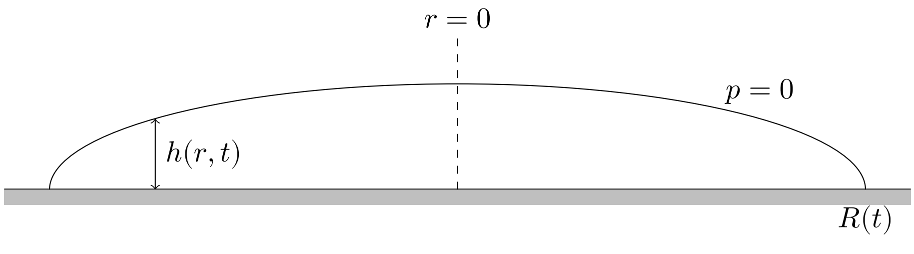
3.5.7 Axisymmetric viscous gravity current

Consider a fixed volume V of viscous fluid spreading axisymmetrically on a horizontal plane under air (i.e. a free-surface condition). The equations of motion and global conservation of mass are

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu} \frac{1}{r} \frac{\partial}{\partial r} \left[r h^3 \frac{\partial h}{\partial r} \right]$$

$$V = 2\pi \int_0^{R(t)} h r \, \mathrm{d}r$$



Scaling gives $h/t \sim gh^4/\nu R^2$ and $hR^2 \sim V$, from which we deduce

$$h \sim \left(\frac{\nu V}{gt}\right)^{1/4}$$

$$r \sim \left(\frac{gV^3 t}{\nu}\right)^{1/8}$$

Hence try similarity solution

$$h(r,t) = \left(\frac{\nu V}{gt}\right)^{1/4} H(\eta), \qquad \eta \equiv \frac{r}{\left(\frac{gV^3 t}{\nu}\right)^{1/8}}$$

Substituting into the equation of motion and mass conservation, as expected the constants cancel and we find

$$2\pi \int_0^\eta H\eta \,\mathrm{d}\eta = 1$$
$$\frac{1}{3\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta H^3 \frac{\mathrm{d}H}{\mathrm{d}\eta}\right] = -\frac{1}{4}H - \frac{1}{8}\eta H'$$

subject to $H(\eta_N) = 0$ where the 'nose' of the current is at $\eta_N = R/(gV^3 t/\nu)^{1/8}$. The solution is

$$H(\eta) = \left(\frac{9}{16}(\eta_N^2 - \eta^2)\right)^{1/3}$$

where $\eta_N = (1024/243\pi^3)^{1/8}$.

3.6 Long Thin Flows II: Extensional flows

We now consider thin flows with effectively stress-free boundaries.

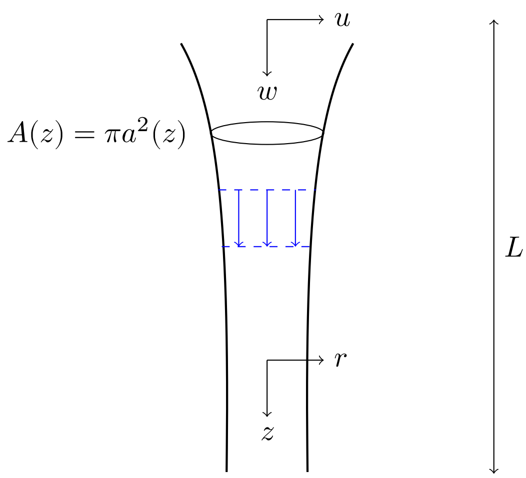
3.6.1 Axisymmetric case

Assume that $\partial_z, \partial_r \sim \frac{1}{L} \ll \frac{1}{a}$. Assuming no tangential stress, we have $\frac{\partial w}{\partial r} = 0$ hence $w = w(z,t)$ is plug flow. Mass conservation on a slice gives

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(Aw) = 0$$
$$\implies \frac{\mathrm{D}A}{\mathrm{D}t} = -A\frac{\partial w}{\partial z}$$

i.e. there is thinning of the flow by stretching in the z direction. Local mass conservation $\nabla \cdot \boldsymbol{u} = 0$ gives

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$
$$\implies u = -\frac{1}{2}r\frac{\partial w}{\partial z} \tag{5}$$



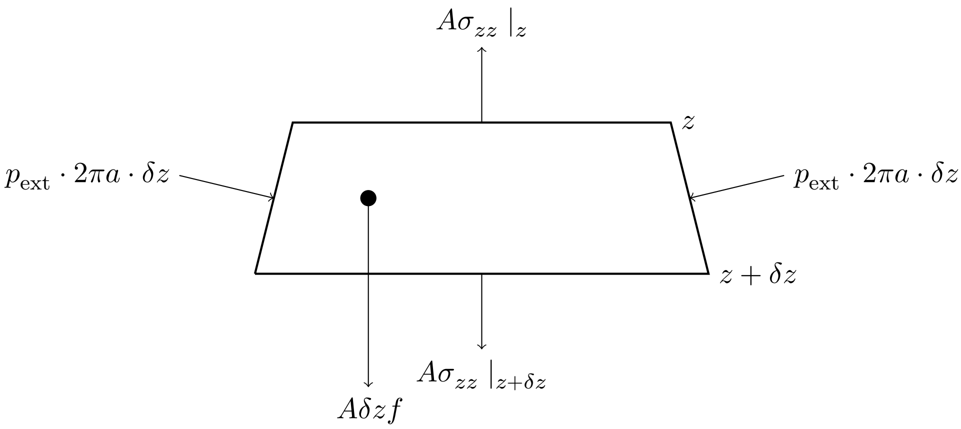
This makes physical sense: there are 3 directions of straining which must sum to 0. Horizontally, there is strain in 2 directions each equal to u_r, and the vertical strain is w_z. The radial stress balance is

$$-p + 2\mu\frac{\partial u}{\partial r} = -p_{\text{ext}}$$
$$\implies -p = -p_{\text{ext}} + \mu\frac{\partial w}{\partial z}$$

The axial stress is

$$\begin{aligned}\sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z}\\ &= -p_{\text{ext}} + 3\mu\frac{\partial w}{\partial z}\end{aligned}$$

Axial force balance on a slice includes the differential stress across the slice, the external pressure acting normal to the sides, and the axial body force.



$$\frac{\partial}{\partial z}(A\sigma_{zz}) + p_{\text{ext}}\frac{\partial A}{\partial z} + Af = 0$$

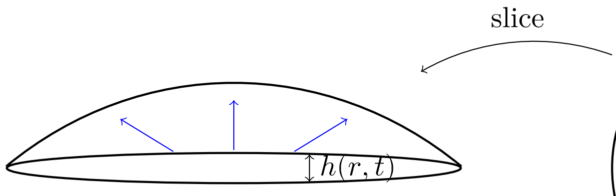
Hence using above results

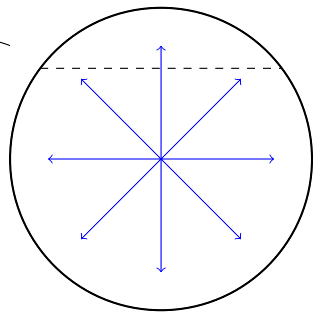
$$\frac{3\mu}{A}\frac{\partial}{\partial z}(A\frac{\partial w}{\partial z}) - \frac{\partial p_{\text{ext}}}{\partial z} + f = 0 \tag{6}$$

Equations (6) and (5) can be used to solve the problem.

Variations.

1. Add surface tension: add $\frac{\gamma}{a}$ to p_{int} and add $\gamma\,[2\pi a(z+\delta z) - 2\pi a(z)]$ to the axial force balance. Equivalently (and more simply) add $\frac{\gamma}{a}$ to p_{ext} in (6).
2. Unidirectional extension of a sheet simply has $u_x = -w_z$ and A is replaced with h, and an extensional viscosity 4μ instead of 3μ. General extension of a sheet has coupling terms between u and w. For example;
3. Radial extension of a sheet.





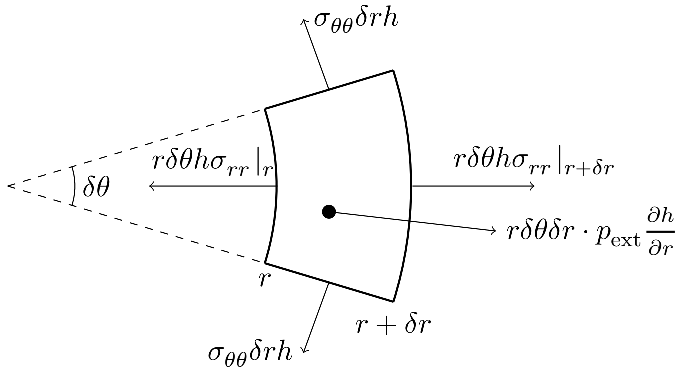
The velocity has only radial and vertical components, $\boldsymbol{u} = (u, 0, w)$. We have $u = u(r,t)$ and from incompressibility

$$\frac{\partial w}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(ru)$$

Following a similar derivation as before we also find

$$
\begin{aligned}
-p + 2\mu\frac{\partial w}{\partial z} &= -p_{\text{ext}} \\
e_{rr} &= \frac{\partial u}{\partial r} \\
e_{\theta\theta} &= \frac{u}{r} \\
e_{\theta\theta} + e_{rr} + e_{zz} &= 0
\end{aligned}
$$

From these results we can find σ_{rr} and Consider a force balance on a 'pineapple slice' of small angle $\delta\theta$ and radial length δr.



$$\implies \frac{\partial}{\partial r}(rh\sigma_{rr}) - h\sigma_{\theta\theta} + rp_{\text{ext}}\frac{\partial h}{\partial r} = 0$$

For further details, see SVF exam 2019 question 2.

4. Can include inertia $\rho\frac{\mathrm{D}w}{\mathrm{D}t}$ on RHS of (6) when $\frac{wL}{\nu} \sim 1$.
5. Adding a lateral force gives a rapid response by bending (low resistance compared to stretching).

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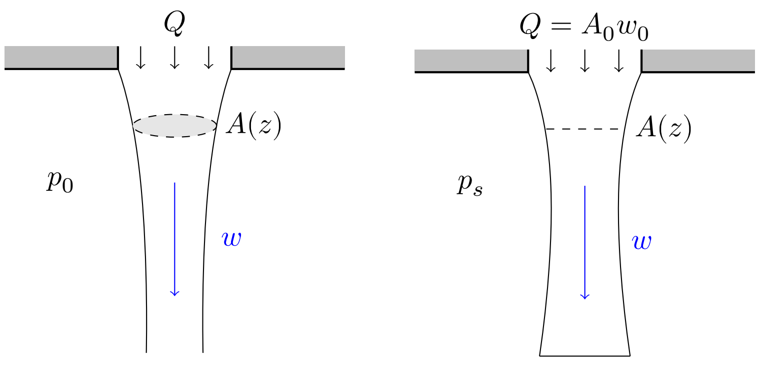
3.6.2 Viscous thread falling under gravity

Consider a thread of viscous fluid injected through a gap with volume flux Q. The cross-sectional area of the thread is $A(z)$ with velocity $\boldsymbol{u} = w\hat{\boldsymbol{z}}$ only. We will consider both the steady behaviour and the transient case when the thread has length $L(t)$ and a stress free bottom. The internal pressure is assumed hydrostatic.



Steady Transient

The governing equations in both cases are

$$3\mu\frac{\partial}{\partial z}\left[A\frac{\partial w}{\partial z}\right] = -\rho g A$$
$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}[Aw] = 0$$
$$\text{or} \quad \frac{\mathrm{D}A}{\mathrm{D}t} = -A\frac{\partial w}{\partial z}$$

Steady case. From mass conservation, we have $Aw = Q$. Hence

$$\frac{w}{Q}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{Q}{w}\frac{\mathrm{d}w}{\mathrm{d}z}\right] = -\frac{\rho g}{3\mu}$$

This can be solved exactly in the general case. The solution with $A\frac{\mathrm{d}w}{\mathrm{d}z} \to 0$ as $z \to \infty$ is

$$w = \frac{g}{6\nu}(z - z_0)^2$$
$$A = \frac{6Q\nu}{g}\frac{1}{(z - z_0)^2}$$

where the constant z_0 is an origin chosen so that $A = A_0$ at $z = 0$. We now check the validity of our assumptions.

- Slenderness is valid if $\frac{a}{L} \ll 1$. Hence $A^{1/2}/(z - z_0) \ll 1$

$$\implies z - z_0 \gg \left(\frac{6Q\nu}{g}\right)^{1/4} \sim a_0$$

- Inertia is negligible if $\frac{wL}{\nu} \sim \frac{g(z-z_0)^3}{6\nu^2} \ll 1$

$$\implies z - z_0 \ll \left(\frac{6\nu^2}{g}\right)^{1/3}$$

For syrup with $\nu \sim 300\,cm^2 s^{-1}$, this distance is approximately $9\,cm$.

Note that for large z, the dominant balance is

$$\rho w\frac{\partial w}{\partial z} \sim \rho g \implies w \sim \sqrt{2gz}$$

hence the parts of the thread far from $z = 0$ are in free fall.

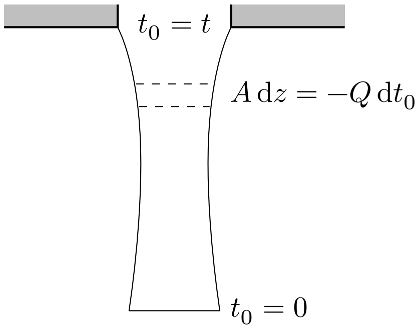
Transient evolution. Integrating the governing equation for w we have

$$3\mu A\frac{\partial w}{\partial z} = \rho g\int_z^{L(t)} A\,\mathrm{d}z$$

This says that viscous stress balances the weight of the fluid. Re-arranging this equation allows use of the governing equation for A:

$$A\frac{\partial w}{\partial z} = \frac{g}{3\nu}\int_z^{L(t)} A\,\mathrm{d}z = -\frac{\mathrm{D}A}{\mathrm{D}t}$$

Hence a material slice thins due to the weight below it. To proceed we label material slices by their 'release time' t_0. Hence we replace $\frac{\mathrm{D}}{\mathrm{D}t}$ with $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}$ and note $\int_z^L A\,\mathrm{d}z = Qt_0$.



Hence

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{gQt_0}{3\nu}$$
$$\implies A = A_0 - \frac{gQt_0}{3\nu}(t - t_0)$$

Hence $A \to 0$ as $t \to t_0 + \frac{3\nu A_0}{gQt_0}$ which is minimised at

$$t^* = 2\left(\frac{3\nu A_0}{gQ}\right)^{1/2}$$

and $t_0 = t^*/2$. Hence half the mass breaks off and the evolution continues with a new weight. We know $A(t; t_0)$, and want to find the shape of the flow, i.e. $A(z, t)$. Now

$$A\,\mathrm{d}z = -Q\,\mathrm{d}t_0$$
$$\implies z(t; t_0) = \int_{t_0}^{t} \frac{Q\,\mathrm{d}t_0'}{A(t; t_0')}$$

Let $T = t/(\frac{gQ}{3\nu A_0})^{1/2}$ and $Z = z/(\frac{gQ^3}{3\nu A_0^3})^{1/2}$. Then

$$A = 1 - T_0(T - T_0)$$
$$Z = \frac{2}{\sqrt{4 - T^2}}\left[\arctan\left(\frac{T}{\sqrt{4 - T^2}}\right) - \arctan\left(\frac{2T_0 - T}{\sqrt{4 - T^2}}\right)\right]$$

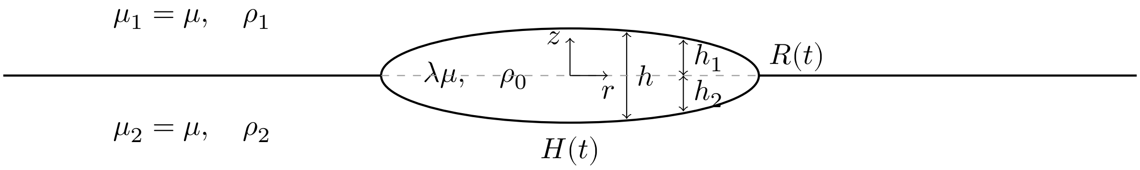
As $T \to 2$ we get break-up, with the slice $T_0 = 1$ breaking.

3.7 Long Thin Flows III: Two-fluid case

When are boundary conditions effectively rigid or stress free?

3.7.1 Gravity current along interface between two fluids.

For simplicity, we assume that the two semi-infinite fluids have the same viscosity μ, and the fluid pool between the two interfaces has viscosity $\lambda\mu$.



Lecture 20 23/11/20

At large times, we expect $R \gg H$. Hence the vertical balance is hydrostatic. We can deduce the form of pressure in two ways, either by traversing through the upper fluid or the lower fluid. We have

$$\begin{aligned} p &= p_0 - \rho_1 g h_1 + \rho_0 g(h_1 - z) \quad \text{upper path} \\ &= p_0 + \rho_2 g h_2 - \rho_0 g(h_2 + z) \quad \text{lower path} \end{aligned}$$

Given these two forms must coincide, we deduce $(\rho_0 - \rho_1)h_1 = (\rho_2 - \rho_0)h_2$, i.e. the gravity current floats iceberg-like due to differences in density. We also find from the above

$$\frac{\partial p}{\partial r} = \Delta\rho g \frac{\partial h}{\partial r}$$
$$\text{where } \Delta\rho = \frac{(\rho_2 - \rho_0)(\rho_0 - \rho_1)}{\rho_2 - \rho_1}$$

drives the radial spreading. The rate of spreading is determined by the viscous resistance. Consider the scalings for $\frac{R}{H} \gg 1$:

$$\text{Kinematics} \implies U \sim \frac{R}{t}$$
$$\text{Volume conservation} \implies R^2 H \sim a^3$$

where a is the radius of the sphere when the current is initially undeformed, or equivalently $a = V^{1/3}$ where V is the volume of the initial current. The radial dynamics give

$$\text{Total spreading force } \sim \frac{\partial p}{\partial r} \times \text{volume} \sim \Delta\rho g a^3 \frac{H}{R}$$
$$\text{Viscous resistance} \sim \mu_s \frac{U}{L_s} A_s$$

where μ_s is the viscosity, L_s is the lengthscale of the dominant source of viscous stress, and A_s is the area on which it acts. Hence the radial balance is

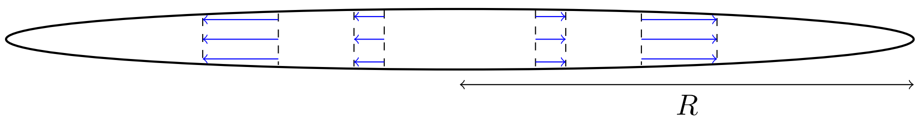
$$\Delta\rho g a^3 \frac{H}{R} \sim \mu_s \frac{R}{tL_s} A_s$$

Define a dimensionless time $\tau \equiv \frac{\Delta\rho g a t}{\mu}$. Then

$$\frac{R^2 A_s}{a^2 H L_s} \frac{\mu_s}{\mu} \sim \tau$$
$$R^2 H \sim a^3$$

which is 2 equations for $R(\tau), H(\tau)$ once A_s, L_s, μ_s determined. There are four possibilities for (μ_s, L_s, A_s) depending on λ and R/H, in which the dominant behaviour is determined by the relative important of extreme viscosity contrasts and extreme aspect ratio.

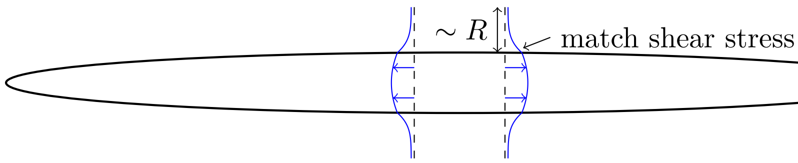
1. $\lambda \gg R/H$: internal strain dominates. The external fluid appears inviscid. The very viscous internal fluid stretches in extensional flow (cf. radial version of extensional flow in section 3.6). The internal flow behaves like plug flow.



The relevant viscosity is the internal viscosity, hence $\mu_s = \lambda\mu$. The relevant lengthscale in terms of the velocity gradient (which determines stresses) is the radius of the current, hence $L_s \sim R$. Finally, the stresses act over a 'band' of height H around the current at each radius, so $A_s \sim HR$. Then

$$\frac{R}{a} \sim \left(\frac{\tau}{\lambda}\right)^{1/2}$$

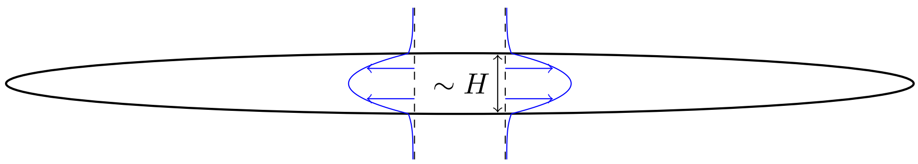
2. $\frac{H}{R} \ll \lambda \ll \frac{R}{H}$: external shear dominates. Internal fluid is still approximately plug flow, but now resisted by shear stress in the external fluid.



The external fluid sees the current as a flat disc of radial Stokeslets (cf. integral representations in section 2.6). Here, the relevant viscosity is the external viscosity so $\mu_s \sim \mu$. The external flow (which exerts the stress) extends over a lengthscale R so $L_s \sim R$ (see diagram) and the stress acts on the top surface of the current so $A_s \sim R^2$. Now

$$\frac{R}{a} \sim \tau^{1/5}$$

3. $\frac{H}{R \ln R/H} \ll \lambda \ll \frac{H}{R}$: internal sehar dominates. Poiseuille component of internal flow now much larger than the velocity at which external fluid is dragged along (cf. lubrication theory in section 3.5). Effectively no-slip boundaries.



In this case the internal viscosity is relevant to the stress, so $\mu_s \approx \lambda\mu$, and the flow extends over a lengthscale $L_s \sim H$ inside the drop. Finally, the shear stress is acting over the surface of the drop so $A_s \sim R^2$ as in the previous case. Hence

$$\frac{R}{a} \sim \left(\frac{\tau}{\lambda}\right)^{1/8}$$

Note this is the same scaling as the rigid surface case in section 3.5.7, as the balances are the same in both problems.

4. $\lambda \ll \frac{H}{R \ln R/H}$: 'push' at nose dominates, i.e. bubble regime. Very low viscosity internal fluid moves along the current with negligible pressure drop, but must push the external fluid out of the way at the nose. The current has a rounded nose and away from the nose, h is approximately constant.

The external fluid sees an expanding ring force (cf. slender-body theory in section 3.1). Hence the relevant viscosity is internal, so $\mu_s \approx \mu$. The stress is concentrated on a region of lengthscale H at the nose of the current, and we must include a modifying factor due to slender body theory so that $L_s \sim H \ln R/H$. Since the stress acts at the nose, the relevant area scale is $A_s \sim HR$, the thickness of the edge multiplied by the perimeter. Hence

$$\frac{R}{a \ln R/H} \sim \tau^{1/5}$$

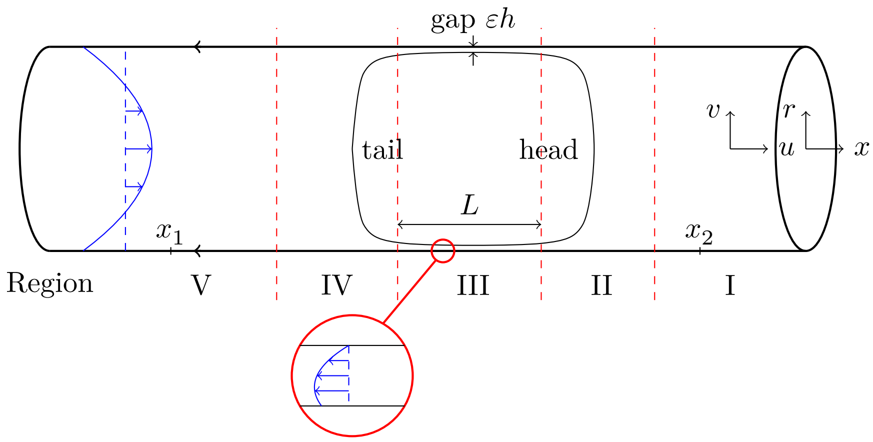
Note that in all four regimes, analytic similarity solutions exist. Also, for all λ, the eventual regime as $\tau \to \infty$ is case 2 (Lister & Kerr, JFM, 1989).

3.8 Pigs & Slugs

Lecture 21 27/11/20

3.8.1 Rigid particles in a tube

Consider a rigid tube of radius a containing fluid of viscosity μ and a tight-fitting, axisymmetric, force-free particle. Apply a pressure gradient $G^* = -\frac{\mathrm{d}p}{\mathrm{d}x}$. What is the steady speed U^* of the particle?



Far from the particle, in regions I and V, we have Poiseuille flow. In the frame of the *tube*:

$$u = \frac{G^*}{4\mu}(a^2 - r^2)$$
$$Q^* = \frac{\pi G^* a^4}{8\mu}$$
$$\bar{u} = \frac{G^* a^2}{8\mu}$$

We expect $u < \bar{u}$ in the thin gap and so $U^* > \bar{u}$ but only by $\mathcal{O}(\varepsilon)$, since the average must be $\bar{u}$ across the whole tube. It is convenient to work in the frame of the particle for regions II – IV. Fix

U^* and find G^* rather than vice versa. Scale lengths by a, speeds by U^*, and pressures by $\mu U^*/u$. We must solve

$$\nabla^2 \boldsymbol{u} = \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{7}$$

$$\boldsymbol{u} = -\hat{\boldsymbol{x}} \text{ on } r = 1 \tag{8}$$

$$\boldsymbol{u} = 0 \text{ on particle} \tag{9}$$

$$\int_{\text{particle}} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = 0 \tag{10}$$

We wish to find

$$G = \frac{G^* a^2}{\mu U^*} = \lim_{x \to \pm\infty} -\frac{\mathrm{d}p}{\mathrm{d}x}$$

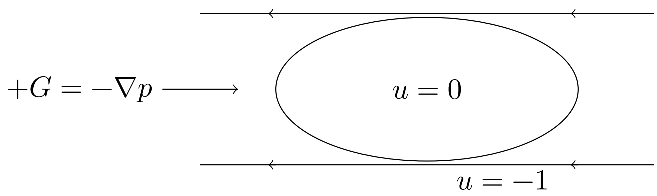
for the case $\varepsilon \ll 1$. For a long fluid control volume

$$V = \{x_1 \le x \le x_2; r \le 1; \text{outside particle}\}$$

where x_1 is in region V and x_2 in region I (see diagram), (10) and $\int_{\partial V} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$ gives

$$\pi \left[p(x_1) - p(x_2) \right] + 2\pi \int_{x_1}^{x_2} (u_r + v_x) \mid_{r=1} \, \mathrm{d}x = 0 \tag{11}$$

Key idea: G and hence the flux through the gap must be such as to satisfy this balance.



If $G = 0$ or $G \gg 1$, the particle is *not* force-free.

Regions I & V: From Poiseuille flow we non-dimensionalise to get

$$u = -1 + \frac{G}{4}(1 - r^2)$$

$$Q = \int u \, \mathrm{d}A = \pi\left(\frac{G}{8} - 1\right)$$

Note $\frac{\mathrm{d}Q}{\mathrm{d}x} = 0$ from (7), (8), (9) and from Q we may deduce G and hence G^* and U^*.

Regions II & IV: The region is of size $\mathcal{O}(1)$ and the full Stokes equations apply. Pressure variations are $\mathcal{O}(1)$ – details unimportant.

Region III: Assume region is of size $L = \mathcal{O}(1)$ (not true for a sphere – see ES4). Lubrication scalings give $u \sim 1$, $p \sim \frac{uL}{(\varepsilon h)^2} = \mathcal{O}(\varepsilon^{-2})$, $Q \sim u\varepsilon h = \mathcal{O}(\varepsilon)$. Set $y = \frac{1-r}{\varepsilon}$ with $0 \le y \le h(x)$. We expand u, p, Q as

$$u = u_0 + \varepsilon u_1 + \ldots$$

$$p = \varepsilon^{-2} p_{-2} + \varepsilon^{-1} p_{-1} + \ldots$$

$$Q = \varepsilon Q_1 + \ldots$$

Leading order solution follows from lubrication theory:

$$\frac{\partial^2 u_0}{\partial y^2} = \frac{\partial p_{-2}}{\partial x} \qquad \begin{cases} u(0) = -1 \\ u(h) = 0 \end{cases}$$

Hence

$$\begin{aligned} u_0 &= \frac{y}{h} - 1 - \frac{1}{2}\frac{\mathrm{d}p_{-2}}{\mathrm{d}x} y(h-y) \\ \frac{Q_1}{2\pi} &= \int_0^h u_0 \,\mathrm{d}y = -\frac{h}{2} - \frac{h^3}{12}\frac{\mathrm{d}p_{-2}}{\mathrm{d}x} \\ \implies p_{-2}(x_1) - p_{-2}(x_2) &= 6\left(\frac{Q_1}{\pi}\int_L h^{-3}\,\mathrm{d}x + \int_L h^{-2}\,\mathrm{d}x\right) \end{aligned}$$

Force balance (11) at $\mathcal{O}(\varepsilon^{-2})$ gives

$$\pi(p_{-2}(x_1) - p_{-2}(x_2)) = 0$$

since $u_r = \mathcal{O}(\varepsilon^{-1})$. Then we solve to get

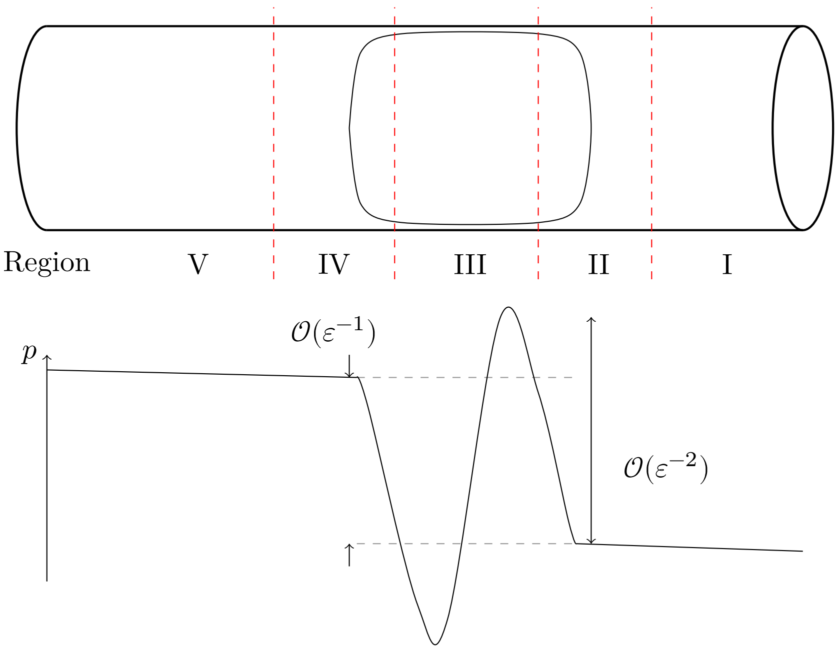
$$\begin{aligned} Q_1 &= -\frac{\pi \int h^{-2}\,\mathrm{d}x}{\int h^{-3}\,\mathrm{d}x} \\ G &= 8\left(1 + \frac{\varepsilon Q_1}{\pi} + \mathcal{O}(\varepsilon^2)\right) = \frac{G^* a^2}{\mu U^*} \end{aligned}$$

Force balance (11) at $\mathcal{O}(\varepsilon^{-1})$ gives

$$\pi(p_{-1}(x_1) - p_{-1}(x_2)) - 2\pi \int_L u_{0y}\,|_{y=0}\;\mathrm{d}x = 0$$

$$\implies \nabla p \sim \varepsilon^{-1}(p_{-1}(\text{tail}) - p_{-1}(\text{head})) = 2\varepsilon^{-1}\int \left(\frac{4}{h} + \frac{3Q_1}{\pi h^2}\right)\mathrm{d}x$$

Hence we find the pressure slowly decreases through regions V and IV, varies through region III and decreases slowly again through regions II and I.



3.8.2 Long bubble in a tube

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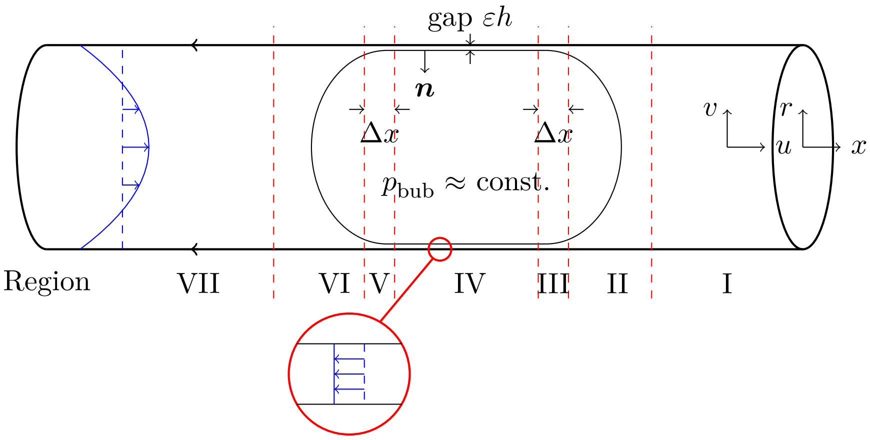
Here, we replace the rigid particle by a long bubble with capillary number $\mathrm{Ca} = \frac{\mu U^*}{\gamma} \ll 1$. See Bretherton, JFM 1960. In this problem, to find out the speed U^* of the bubble we also need the steady shape of the bubble, e.g. to determine the gap εh.



We take the same length, speed, and pressure scales as before. There is Poiseuille flow as before in regions I and VII. We have governing equations and boundary conditions

$$
\begin{aligned}
\nabla^2 \boldsymbol{u} &= \nabla p \\
\nabla \cdot \boldsymbol{u} &= 0 \\
\boldsymbol{u} &= -\hat{\boldsymbol{x}} \ \text{ on } \ r = 1 \\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 \ \text{ on particle} \\
-\sigma \cdot \boldsymbol{n} &= \frac{\kappa}{\mathrm{Ca}} \boldsymbol{n} \ \text{ on particle}
\end{aligned}
$$

Integrating the final (dynamic) boundary condition, we see there is no net force on the bubble. If the interface is at $r = R(x)$ then

$$
\begin{aligned}
\boldsymbol{n} &= (-1, R_x)(1 + R_x^2)^{-1/2} \ \text{ into bubble} \\
\kappa &= -\frac{1}{R(1 + R_x^2)^{1/2}} + \frac{R_{xx}}{(1 + R_x^2)^{3/2}}
\end{aligned}
$$

If $R = 1 - \varepsilon h$, then

$$
\kappa = -(1 + \varepsilon h + \varepsilon h_{xx}) + \mathcal{O}(\varepsilon^2)
$$

Key idea: strong surface tension ($\mathrm{Ca} \ll 1$) pulls the bubble into a cylinder in region IV where $\kappa = -1$ with spherical caps in II and IV where $\kappa = -2$. Hence $\nabla p = \mathcal{O}(1/\mathrm{Ca})$ across narrow regions III and V.

From lubrication scalings, we have

$$
\begin{aligned}
\frac{\Delta p}{\Delta x} &\sim \frac{u}{(\varepsilon h)^2} \\
\varepsilon h_{xx} &\sim 1
\end{aligned}
$$

Hence $\varepsilon \sim \mathrm{Ca}^{2/3}$, $\Delta x \sim \mathrm{Ca}^{1/3}$, $p \sim \mathrm{Ca}^{-1}$. We therefore define $\varepsilon \equiv \mathrm{Ca}^{2/3}$ and

$$
P \equiv \frac{p - p_{\text{bub}}}{\mathrm{Ca}^{-1}}
$$

Regions I & VII: Uniform Poiseuille flow as before so

$$Q = \pi(\frac{G}{8} - 1)$$

Regions II & VI: Full Stokes equations apply, with $\mathcal{O}(1)$ deviatoric stresses, as before. Hence $\nabla P = \mathcal{O}(\mathrm{Ca})$, i.e. P is uniform at leading order. Also $P = \kappa$ on bubble, so we have spherical caps since κ is uniform. Thus $\kappa_0 = -2, P_0 = -2$.

Region IV: Since $\kappa_0 = -1, P_0 = -1$ i.e. uniform. Lubrication theory gives leading order equation

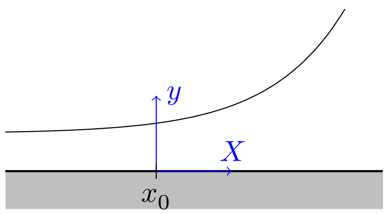
$$\frac{\partial^2 u_0}{\partial y^2} = 0$$

where $y = (1-r)/\varepsilon$ as before. The boundary condition $u_0 = -1$ on $y = 0$ gives $u_0 = -1$, i.e. plug
Lecture 23 2/12/20 flow as shown in the diagram. Hence $Q \sim \mathrm{Ca}^{2/3} Q_1$, and $\frac{Q_1}{2\pi} = -h_0$, constant.

Region III: Define local co-ordinates

$$y = \frac{1-r}{\mathrm{Ca}^{2/3}}, \qquad X = \frac{x - x_0}{\mathrm{Ca}^{1/3}}$$

Since we know the curvature in this region and in the neighbouring region II, to match we require $\varepsilon h_{xx} \to 1$ as $X \to \infty$. Similarly, matching with region IV we have $h \to h_0$ as $X \to -\infty$.



At leading order, lubrication theory gives

$$\begin{aligned} P_0 &= -(1 + h_{XX}) \\ u_0 &= -1 - \frac{1}{2} P_{0X}\, y(2h - y) \\ \implies q = \frac{Q_1}{2\pi} &= -h - \frac{h^3}{3} P_{0X} = \text{const.} \end{aligned}$$

Hence we have the governing equation

$$\frac{1}{3} h^3 h_{XXX} = h - h_0$$

with boundary conditions

$$\begin{aligned} h &\to h_0 \ \text{ as } \ X \to -\infty \\ h_{XX} &\to 1 \ \text{ as } \ X \to \infty \end{aligned}$$

To solve this ODE it is beneficial to remove the presence of h_0 by rescaling $h(X) = h_0 H(\xi)$ with $X = \frac{h_0}{3^{1/3}}\xi$. Then

$$\begin{aligned} H^3 H''' &= H - 1 \\ H \to 1 &\text{ as } \xi \to -\infty \\ H'' \to \frac{h_0}{3^{2/3}} &\text{ as } \xi \to \infty \end{aligned}$$

We can utilise the boundary conditions by first linearising about $H = 1$:

$$\delta H''' = \delta H \implies \delta H \propto e^{m\xi}$$

where $m = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Thus $H \to 1$ as $\xi \to -\infty$ suppresses both complex solutions for m. Hence the solution with $H \sim 1 + e^{\xi - \xi_0}$ as $\xi \to -\infty$ is unique, to within choice of origin ξ_0, and has $H \sim \frac{1}{2}C\xi^2$ as $\xi \to \infty$ with $C = 0.643$. Then we find $h_0 = 3^{2/3}C = 1.337$ and the gap $\varepsilon h \sim 1.337\mathrm{Ca}^{2/3}$ in region III.

Region V: We once again find $H^3 H''' = H - 1$ but now the boundary conditions require solution

$$H \sim 1 + e^{(\xi_1 - \xi)/2} \cos\left[\frac{\sqrt{3}}{2}(\xi - \xi_1 + \phi)\right]$$

as $\xi \to \infty$, and h_0 is already known. The phase ϕ is determined by the condition $H'' \sim C = 0.643$ as $\xi \to -\infty$ for matching region VI.

Pressure variation. As in the rigid particle problem, we can find the pressure drop across the bubble using a total force argument. We find the pressure drop across the bubble is $\mathcal{O}(\mathrm{Ca}^{-1/3})$ and is independent of the length of the bubble.