3.2 Integration by Parts

Integrals of the form $\int f(t)g(t)\,dt$ can be integrated by parts and **may** so yield asymptotic expansions; one automatically obtains the remainder.

Example 1. See §2.1 for $\mathrm{erf}(z)$.

Example 2. Consider the *exponential integral*

$$E_1(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\int_0^\infty \frac{e^{-t}\,dt}{x+t}\ .$$

Then integrating by parts

$$\begin{aligned}
E_1(x) &= \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \\
&= \frac{e^{-x}}{x}\left(1-\frac{1}{x}+\frac{2!}{x^2}+\ldots+\frac{m!}{(-x)^m}\right)+R_m(x)\ ,
\end{aligned}$$

where

$$R_m(x) = (-)^{m+1}(m+1)!\int_x^\infty \frac{e^{-t}}{t^{m+2}}\,dt\,.$$

Hence

$$|R_m(x)| \leqslant \frac{(m+1)!e^{-x}}{x^{m+2}}\,,$$

and as in §3.1, the remainder is asymptotically smaller than the retained terms on truncation with $m\sim x$.

Example 3. The sine and cosine integrals.

$$\begin{aligned}
-\mathrm{Ci}(x) - i\,\mathrm{si}(x) &= -\mathrm{Ci}(x)+i\left(\frac{\pi}{2}-\mathrm{Si}(x)\right) \equiv \int_x^\infty \frac{e^{it}\,dt}{t} \\
&= -\frac{e^{ix}}{ix}\left(1+\frac{1}{ix}+\frac{2!}{(ix)^2}+\ldots+\frac{m!}{(ix)^m}\right)+R_m(x)\ ,
\end{aligned}$$

where

$$R_m(x) = i(m+1)!\int_x^\infty \frac{e^{it}\,dt}{(it)^{m+2}}\ .$$

If we proceed as before

$$|R_m| \leqslant (m+1)!\int_x^\infty \frac{dt}{t^{m+2}} = \frac{m!}{x^{m+1}} = \mathcal{O}(\text{last term})\ ,$$

so this does not demonstrate asymptoticness. We seek an improved error estimate by integrating by parts:

$$R_m = \left[\frac{(m+1)!\,e^{it}}{(it)^{m+2}}\right]_x^\infty + i(m+2)!\int_x^\infty \frac{e^{it}\,dt}{(it)^{m+3}}\ ,$$

hence

$$|R_m| \leqslant \frac{(m+1)!}{x^{m+2}}+\frac{(m+1)!}{x^{m+2}} = \mathcal{O}\left(\frac{1}{x^{m+2}}\right)\ .$$

3.3 Integrals with Algebraic Parameter Dependence

Example 1. Consider the integral

$$I(\varepsilon) = \int_0^1 \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}\,dx = 2\left(\sqrt{1+\varepsilon}-\sqrt{\varepsilon}\right).$$

The leading-order ($\varepsilon \to 0$) estimate is just

$$I(0) = \underbrace{\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx}_{\substack{\text{global contribution from}\\ \text{all of integration range}}} = 2\ .$$

In order to obtain an improved estimate one cannot expand $(1+\varepsilon/x)^{-1/2}$ throughout the range as

$$(1+\varepsilon/x)^{-1/2} = 1 - \varepsilon/2x + \dots\,,$$

since for $0 \leqslant x \ll \varepsilon$ the expansion is not convergent.[1] Further, we note that when $x = \text{ord}(\varepsilon)$, the integrand is $\text{ord}\left(\varepsilon^{-1/2}\right) \Rightarrow$ contribution to the integral for this range of x will be $\text{ord}\left(\varepsilon^{-1/2}\cdot\varepsilon\right)$, i.e. $\text{ord}\left(\varepsilon^{1/2}\right)$.

To account for this correction, one could subtract the leading-order estimate exactly; then

$$I = 2 + \underbrace{\int_0^1 \left[\frac{1}{(x+\varepsilon)^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}}\right] dx}_{\substack{x=\text{ord}(\varepsilon),\ \text{integrand } =\text{ord}\left(\varepsilon^{-1/2}\right),\ \text{contribution to } \int = \text{ord}\left(\varepsilon^{1/2}\right)\\ x=\text{ord}(1),\ \text{integrand } =\text{ord}\left(\varepsilon\right),\ \text{contribution to } \int = \text{ord}\left(\varepsilon\right)}}$$

The major contribution is from near $x=0$, so put $x = \varepsilon\xi$ ($\xi = \text{ord}(1)$), then

$$\begin{aligned} I &= 2 + \varepsilon^{\frac{1}{2}} \int_0^{\frac{1}{\varepsilon}\approx\infty} \left[\frac{1}{(1+\xi)^{\frac{1}{2}}} - \frac{1}{\xi^{\frac{1}{2}}}\right] d\xi \\ &\approx 2 - 2\varepsilon^{\frac{1}{2}} \end{aligned}$$

Further corrections can be obtained by now subtracting out this contribution, but this method is tedious
02/03 and difficult! There must be a better way.

Alternative 1: Solve a differential equation. Let

$$J(x) = \int_0^x \frac{1}{(q+\varepsilon)^{\frac{1}{2}}}\,dq\ .$$

Then we need to find $J(1)$. This can be done by solving the differential equation

$$\frac{dJ}{dx} = \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}$$

subject to the initial condition $J(0) = 0$. We will discover how to do this in §5.

Alternative 2: Divide & Conquer. In this method we split the range of integration. Split $[0,1]$ at $x = \delta$ where $\varepsilon \ll \delta \ll 1$, and then use Taylor series when we can use Taylor series:

$$\begin{aligned} I &= \int_0^\delta \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} + \int_\delta^1 \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} \\ &= \varepsilon^{\frac{1}{2}} \int_0^{\delta/\varepsilon} \frac{d\xi}{(1+\xi)^{\frac{1}{2}}} + \int_\delta^1 \frac{1}{x^{\frac{1}{2}}}\left(1 - \frac{\varepsilon}{2x} + \frac{3\varepsilon^2}{8x^2} + \dots\right) dx \\ &= 2\varepsilon^{\frac{1}{2}}\left(\left(\frac{\delta}{\varepsilon}+1\right)^{\frac{1}{2}} - 1\right) + 2 - 2\delta^{\frac{1}{2}} + \varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}} + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}},\,\varepsilon^2\right) \\ &= 2\delta^{\frac{1}{2}} + \frac{\varepsilon}{\delta^{\frac{1}{2}}} - 2\varepsilon^{\frac{1}{2}} + 2 - 2\delta^{\frac{1}{2}} + \varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}} + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}},\,\varepsilon^2\right) \\ &= 2 - 2\varepsilon^{\frac{1}{2}} + \varepsilon + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}},\,\varepsilon^2\right)\ . \end{aligned}$$

[1] And in this case there is no exponentially small multiplier.

Remarks.

- The error term is definitely small if $\varepsilon^{\frac{2}{3}} \ll \delta \ll 1$.
- Since δ is arbitrary, all terms containing a δ must cancel.
- To organise the algebra it is sometimes helpful to tie δ to ε, e.g.

$$\delta = K\varepsilon^{\frac{3}{4}} \ ,$$

  and then the answer must be independent of K.

Example 2. Suppose that we wish to estimate the integral

$$I(m,\varepsilon) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(1-m^2\cos^2\theta\right)^2\sin^2\theta+\varepsilon^2}\,d\theta \qquad 0 < m < \infty \ ,$$

for $0 < \varepsilon \ll 1$. It turns out that there are three cases to consider: $0 < m < 1$; $|m-1| \ll 1$; $m > 1$.

(a) $\quad 0 < m < 1$

θ	integrand	contribution to $\int$
ord(1)	ord(1)	ord(1)
ord(ε)	ord(1)	ord(ε)

$\uparrow \left(1-m^2\cos^2\theta\right)^2\sin^2\theta \sim \varepsilon^2$

We will find the solution correct to $\mathcal{O}\left(\varepsilon^2\right)$; to this end let $0 < \varepsilon \ll \delta \ll 1$. Then

$$\begin{aligned}
I &= \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{\sin^2(\varepsilon u)}{\left(1-m^2\cos^2(\varepsilon u)\right)^2\sin^2(\varepsilon u)+\varepsilon^2}\,du + \int_\delta^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(1-m^2\cos^2\theta\right)^2\sin^2\theta+\varepsilon^2}\,d\theta \\
&= \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{u^2\,du}{\left(1-m^2\right)^2u^2+1} + \int_\delta^{\frac{\pi}{2}} \frac{1}{\left(1-m^2\cos^2\theta\right)^2}\,d\theta + \mathcal{O}\left(\varepsilon^2\right) \\
&= \varepsilon\left[\frac{(1-m^2)u-\tan^{-1}\left((1-m^2)u\right)}{(1-m^2)^3}\right]_0^{\frac{\delta}{\epsilon}} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \int_0^\delta \frac{d\theta}{\left(1-m^2\cos^2\theta\right)^2} + \mathcal{O}\left(\varepsilon^2\right) \\
&\qquad\qquad \text{via a } \tan\theta = t = (1-m^2)^{\frac{1}{2}}\tan\psi \text{ substitution} \\
&= \frac{\delta}{(1-m^2)^2} - \frac{\varepsilon\pi}{2(1-m^2)^3} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \frac{\delta}{(1-m^2)^2} + \mathcal{O}\left(\varepsilon^2,\delta^2,\frac{\varepsilon^2}{\delta}\right) \\
&\qquad\qquad \text{since } \arctan\left(\tfrac{1}{\Delta}\right) \sim \tfrac{\pi}{2} - \Delta
\end{aligned}$$

$$I = \underbrace{\frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}}}_{\text{global}} - \underbrace{\frac{\varepsilon\pi}{2(1-m^2)^3}}_{\text{local}} + \dots \tag{3.1}$$

Note that this is a non-uniform approximation as $m \to 1$. There is a loss of ordering of the series solution when

$$\frac{1}{(1-m^2)^{\frac{3}{2}}} \sim \frac{\varepsilon}{(1-m^2)^3}$$

i.e. when

$$(1-m^2) \sim \varepsilon^{\frac{2}{3}} \quad \text{and} \quad I \sim \frac{1}{\varepsilon}\,.$$

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(b) This suggests that when $|m-1| \ll 1$, we should introduce a scaled parameter: viz.

$$m = 1 - \varepsilon^{\frac{2}{3}}\lambda \, . \tag{3.2}$$

First let us examine the local contribution from near $\theta = 0$ (since on the basis of the estimates above it will be leading order). Put $\theta = \varepsilon^{\beta} u$, then

$$\left(1 - m^2\cos^2\theta\right)^2 \sin^2\theta + \varepsilon^2 = \left(\varepsilon^{2\beta}u^2 + 2\varepsilon^{\frac{2}{3}}\lambda\right)^2 \varepsilon^{2\beta}u^2 + \varepsilon^2 + \dots$$

All leading order terms balance if $\beta = \frac{1}{3}$; this is referred to as a *distinguished scaling*. As a first guess, let us assume that this is the scaling in θ to consider. Then

$$\begin{array}{llll} \theta = \operatorname{ord}(\varepsilon^{\frac{1}{3}}); & \text{integrand } = \operatorname{ord}\left(\varepsilon^{\frac{2}{3}}/\varepsilon^2\right); & \text{contribution to } \int = \operatorname{ord}\left(1/\varepsilon\right) \\ \theta = \operatorname{ord}(1)\ ; & \text{integrand } = \operatorname{ord}\left(1\right) \quad ; & \text{contribution to } \int = \operatorname{ord}\left(1\right) \end{array}$$

The 'local' contribution dominates. Hence introduce $\varepsilon^{\frac{1}{3}} \ll \delta \ll 1$, and split the integral:

$$\begin{aligned} I &= \int_0^{\delta} \dots \, d\theta + \int_{\delta}^{\frac{\pi}{2}} \dots \, d\theta \\ &\sim \frac{1}{\varepsilon}\int_0^{\delta\varepsilon^{-\frac{1}{3}}} \frac{u^2\, du}{\left(u^2+2\lambda\right)^2 u^2 + 1} \sim \frac{1}{\varepsilon} f(\lambda) \end{aligned}$$

where

$$f(\lambda) = \int_0^{\infty} \frac{u^2\, du}{\left(u^2+2\lambda\right)^2 u^2 + 1} \, .$$

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Hence for a given λ (or equivalently m), we have a leading order asymptotic estimate. However, we should check that as $\lambda \to \infty$, we obtain the same estimate as in (a). In particular, when $\lambda \gg 1$

$$\begin{array}{lll} u = \operatorname{ord}(1)\ , \text{ integrand } = \operatorname{ord}\left(1/\lambda^2\right), & \text{contribution to } \int = \operatorname{ord}\left(1/\lambda^2\right) \\ u = \operatorname{ord}(\lambda^{\frac{1}{2}}), \text{ integrand } = \operatorname{ord}\left(1/\lambda^2\right), & \text{contribution to } \int = \operatorname{ord}\left(1/\lambda^{\frac{3}{2}}\right) \end{array}$$

This suggests that the largest contribution will come from where $v = \lambda^{-\frac{1}{2}}u = \operatorname{ord}(1)$. Hence estimate f in this range:

$$f(\lambda) \sim \frac{1}{\lambda^{\frac{3}{2}}}\int_0^{\infty} \frac{dv}{\left(2+v^2\right)^2} = \frac{\pi}{4\left(2\lambda\right)^{\frac{3}{2}}} \, ,$$

and

$$I \sim \frac{\pi}{4\varepsilon\left(2\lambda\right)^{\frac{3}{2}}} \sim \underbrace{\frac{\pi}{4\left(1-m^2\right)^{\frac{3}{2}}}}_{\downarrow \text{agrees with (3.1) for } m \approx 1} \tag{3.3}$$

We might also be interested in the other limit, i.e. $\lambda \to -\infty$. This estimate is a little more tricky, since $\left(u^2+2\lambda\right)$ can now have a zero (when $|\lambda| \gg 1$, this term normally dominates the denominator). First we test for a significant contribution from near this zero by introducing a scaled coordinate:

$$u = \left(-2\lambda\right)^{\frac{1}{2}} + \left(-\lambda\right)^{\gamma} w \, .$$

Then

$$\begin{aligned} 1 + u^2\left(u^2+2\lambda\right)^2 &\sim 1 + \left(-2\lambda\right)\left(2\left(-2\lambda\right)^{\frac{1}{2}}\left(-\lambda\right)^{\gamma}w + \dots\right)^2 \\ &\sim 1 + 16\lambda^2(-\lambda)^{2\gamma}w^2 + \dots \, . \end{aligned}$$

There is a distinguished scaling for the choice $\gamma = -1$, in which case the contribution to the integral from near the zero can be estimated as follows:

$$u = \left(-2\lambda\right)^{\frac{1}{2}} + \operatorname{ord}\left(1/\left|\lambda\right|\right); \ \text{integrand} = \operatorname{ord}\left(\left|\lambda\right|/1\right); \ \text{contribution to } \int = \operatorname{ord}(1) \, .$$

This is a much larger contribution than we found for $\lambda \gg 1$.

In order to estimate the contribution set

$$u = (-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)} \tag{3.4}$$

then

$$f(\lambda) = \int_{-2^{1/2}(-\lambda)^{3/2}\approx-\infty}^{\infty} \frac{(-2\lambda+\dots)\,dw}{(-\lambda)\,[1+16w^2\dots]} \sim \frac{\pi}{2}\,.$$

Hence as $\lambda \to -\infty$, the value of the integral tends to a large constant, viz.

$$I \sim \frac{\pi}{2\varepsilon}\,. \tag{3.5}$$

(c) Finally consider the case when $m > 1$.

The limit $\lambda \to -\infty$ (i.e. $0 < (m-1) \ll 1$) suggests that the main contribution will be local, and will come from the region close to the point where

$$m^2\cos^2\theta = 1\,.$$

Define

$$\theta_m = \cos^{-1}\left(\frac{1}{m}\right) \qquad\qquad \left(0 < \theta_m < \frac{\pi}{2}\right)\,.$$

In order to deduce the coordinate scaling that is appropriate close to θ_m, we note from (3.2) and (3.4) that the 'inner' scaling for $0 < m-1 \ll 1$ can be written in the form

$$\theta = \varepsilon^{\frac{1}{3}}u = \varepsilon^{\frac{1}{3}}\left((-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)}\right) = (2(m-1))^{\frac{1}{2}} + \frac{\varepsilon w}{(m-1)} \sim \theta_m + \frac{2\varepsilon w}{\theta_m^2}\,.$$

This suggests that for $(m-1) = \mathcal{O}(1)$ we should try the scaling

$$\theta = \theta_m + \varepsilon t\,,$$

in which case

$$\left(1 - m^2\cos^2\theta\right)^2\sin^2\theta + \varepsilon^2 \sim 4\varepsilon^2 m^2\sin^4\theta_m t^2 + \dots + \varepsilon^2$$

and

$$\begin{aligned} I &\sim \int_{-\frac{1}{\varepsilon}\theta_m\approx-\infty}^{\frac{1}{\varepsilon}\left(\frac{\pi}{2}-\theta_m\right)\approx+\infty} \frac{\varepsilon\sin^2(\theta_m+\varepsilon t)\;dt}{\varepsilon^2\left(4m^2t^2\sin^4\theta_m + 1\right)+\dots} \\ &\sim \frac{1}{\varepsilon}\cdot\frac{\pi}{2m} \end{aligned} \tag{3.6}$$

05/01 We note that (3.6) agrees with (3.5) in the limit $m \to 1$.
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3.4 Logarithms

Consider

$$\int_0^a f(x,\varepsilon)\,dx \quad \text{with} \quad f(x,\varepsilon) = \begin{cases} \operatorname{ord}(\varepsilon^{-\alpha}) & x = \operatorname{ord}(\varepsilon) \\ x^{-\alpha} & \varepsilon \ll x \ll 1 \\ \operatorname{ord}(1) & x = \operatorname{ord}(1). \end{cases}$$

e.g.

$$f = \frac{1}{(x+\varepsilon)^\alpha}\frac{1}{1+x}\,.$$

There are three possibilities for the leading-order contribution depending on the value of α: