Slow Viscous Flow

based on a course given by   written up by

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Contents

1 Basic Fluid Mechanics

Lecture 1
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'Infinitesimal' fluid particles have well-defined density $\rho(\boldsymbol{x}, t)$, velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $p(\boldsymbol{x}, t)$ where $\boldsymbol{x}(t)$ is the position of the fluid particles.

> **Definition.** The *Eulerian* or *material* derivative
>
> $$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$$
>
> is the rate of change following the fluid particle.

1.1 Mass Conservation

In general, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \iff \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = 0$$

For an incompressible fluid, $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \iff \nabla \cdot \boldsymbol{u} = 0$

1.2 The Stress Tensor

The *stress* $\boldsymbol{\tau}$ is the force per unit area acting across a surface. Force balance on an 'infinitesimal' fluid tetrahedron shows that the stress $\boldsymbol{\tau}$ is linearly related to the surface normal $\boldsymbol{n}$:

$$\boldsymbol{\tau} = \sigma \cdot \boldsymbol{n}$$

where σ is the *stress tensor* and $\boldsymbol{\tau}$ is stress exerted by the outside fluid on the inside of a surface with outward normal $\boldsymbol{n}$. Angular momentum balance shows that σ is symmetric in most fluids.

1.3 Momentum equation

The *Cauchy momentum equation* states in general

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{F} + \nabla \cdot \sigma$$

1.4 Energy equation

In the case of an incompressible fluid, the rate of local inertial *viscous dissipation* is derived by contracting the Cauchy momentum equation with the fluid velocity and integrating over a volume. We have

$$\mathcal{D} = \int_V e_{ij}\sigma_{ij} \,\mathrm{d}V = \int_V e : \sigma \,\mathrm{d}V$$

where $e_{ij} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the *rate of strain* tensor. Note $e_{ii} = 0$ by incompressibility and $e_{ij} = e_{ji}$.

The rate of working by external surface forces on the fluid is

$$\int_{\partial V} u_i \sigma_{ij} n_j \mathrm{d}S$$

1.5 Newtonian Fluids

Definition. Fluid deformation produces internal viscous stresses. If the relationship between fluid deformation $\frac{\partial u_i}{\partial x_j}$ and stress σ_{ij} is local, linear, instantaneous and isotropic, then the fluid is *Newtonian.*

If the fluid is also incompressible, then the stress tensor takes the form

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

where μ is the *dynamic viscosity* and $2\mu e_{ij}$ is the *deviatoric stress.* Note that there is no dependence on the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.

For an incompressible Newtonian fluid with uniform viscosity we have the *Navier-Stokes equations*

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{F} + \mu\nabla^2\boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

The rate of viscous dissipation is

$$\mathcal{D} = 2\mu \int e_{ij}e_{ij} \,\mathrm{d}V$$

Often body forces are conservative $\boldsymbol{F} = -\nabla\phi$ and we incorporate $\boldsymbol{F}$ into a *modified pressure $p + \phi$.*

1.6 Boundary conditions

Kinematic boundary conditions on a fluid-fluid interface are

- $[\boldsymbol{u} \cdot \boldsymbol{n}]_-^+ = 0$ by mass conservation

- $[\boldsymbol{u} \times \boldsymbol{n}]_-^+ = \boldsymbol{0}$ to avoid infinite stresses

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Kinematic boundary conditions on a rigid boundary are

- No flux: $\boldsymbol{u} \cdot \boldsymbol{n} = 0$

- No slip: $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$

Dynamic boundary conditions in the absence of surface tension are

$$[\sigma \cdot \boldsymbol{n}]_-^+ = \boldsymbol{0}$$

Note that modified pressure should not be used here.
With surface tension included, the condition becomes

$$[\sigma \cdot \boldsymbol{n}]_-^+ = \gamma \kappa \boldsymbol{n} - \nabla_s \gamma$$

where $\kappa = \nabla_s \cdot \boldsymbol{n}$ is the *curvature* and γ is the *surface tension*.

1.7 Reynolds number

Suppose $U, L, L/U$ are representative velocity, length, and time scales of the flow. Then

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \sim \rho \frac{U^2}{L}$$
$$\mu \nabla^2 \boldsymbol{u} \sim \mu \frac{U}{L^2}$$

> **Definition.** The *Reynolds number* is the ratio of these quantities and determines the important of inertial vs. viscous stresses.
> $$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$

If $\mathrm{Re} \ll 1$ then inertia is negligible and we have the *Stokes equations*

$$\mu \nabla^2 \boldsymbol{u} = \nabla p - \boldsymbol{F}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Stokes equations are useful in many regimes.

- Large μ, e.g. magma, glass, ice sheets

- Small L, e.g. microorganisms, microfluid devices

- Thin film flows, e.g. lubrication theory

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> **Example.** Sperm cell – intrinsic length scales $L \sim 5\mu m, U \sim 100\mu m \cdot s^{-1}, \nu \sim 10^{-2} cm^2 \cdot s^{-1} = 10^6 \mu m^2 \cdot s^{-1}$. Therefore $Re \sim 5 \times 10^{-4}$ so can be described by the Stokes equations.

> **Example.** Mantle convection – intrinsic length scales $L \sim 1000km = 10^8 cm, U \sim 2cm \text{year}^{-1} \sim 10^7 cm \cdot s^{-1}, \nu \sim 10^{21} cm^2 \cdot s^{-1}$. Thus $Re \sim 10^{-20}$.

There are some caveats which come with the use of intrinsic length scales.

- $\boldsymbol{u} \cdot \nabla$ and ∇^2 may not involve the same length scale L, e.g. in lubrication theory there is a short length scale for the depth of the flow, which is small compared to other length scales of the flow.

- L may vary in the flow e.g. in the far field of a moving body $Re \sim \frac{Ur}{\nu}$.

- T may not equal L/U if there is an external time scale, e.g. oscillating body with $T \sim \omega^{-1}$.

2 The Stokes Equations

$$\nabla \cdot \sigma = \mu \nabla^2 \boldsymbol{u} - \nabla p = -\boldsymbol{F}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

2.1 Simple Properties

2.1.1 Instantaneous

The Stokes equations involve no ∂_t term, so there is no inertia, no memory, and the flow only 'knows' about the current boundary conditions and applied forces, and responds immediately to changes. With moving boundaries (i.e. changing boundary conditions) the flow is *quasi-steady.*

2.1.2 Linear

The Stokes equations are linear in $\boldsymbol{F}, p$, and $\boldsymbol{u}$. Therefore the fluid response is proportional to forcing and solutions for a given geometry can be superposed.

2.1.3 Reversible

If all the forces change sign, then $\boldsymbol{u}$ changes sign. Thus if we reverse all the forces and the history of their application, the flow returns to its original state. Reversibility can sometimes be used with a symmetry to rule out certain behaviours of the flow.

Example. Sedimenting sphere – consider a sphere sedimenting in a Stokes flow next to a rigid wall. Will the sphere migrate laterally?
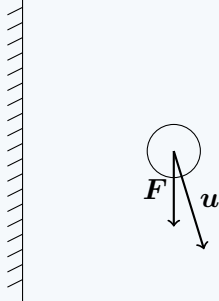
Applying reversibility, change $\boldsymbol{F} \to -\boldsymbol{F}$ so $\boldsymbol{u} \to -\boldsymbol{u}$:
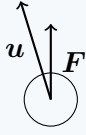
Now apply symmetry: reflect the geometry top to bottom.

Comparing with the original situation, we see there can be no lateral component of $\boldsymbol{u}$.

2.1.4 Forces balance

Since there is no inertia, the forces must balance. From the equations,

$$\nabla \cdot \sigma = -\boldsymbol{F} \implies \int_{\partial V} \sigma \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{F} \mathrm{d}V = \boldsymbol{0}$$

This is a consistency check on stress boundary conditions.
Similarly, in the absence of fluid sources,

$$\nabla \cdot \boldsymbol{u} = 0 \implies \int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = 0$$

This is a consistency check on velocity boundary conditions.
Likewise, torques balance, giving another consistency check on stress boundary conditions.

2.1.5 Work balances dissipation

Intuitively, the flow has no kinetic energy (no inertia) so any work done on the fluid must be viscously dissipated instantaneously. We have

$$\mathcal{D} = 2\mu \int_V e_{ij} e_{ij} \, \mathrm{d}V$$

$$= \int (\sigma_{ij} + p\delta_{ij}) e_{ij} \, \mathrm{d}V$$

$$= \int \sigma_{ij} \frac{\partial u_i}{\partial x_j} + p e_{ii} \, \mathrm{d}V$$

$$= \int \frac{\partial}{\partial x_j} \sigma_{ij} u_i - u_i \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V$$

$$= \int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S + \int \boldsymbol{u} \cdot \boldsymbol{F} \mathrm{d}V$$

The first term is the work done by surface forces at the boundary, and the second term is the work done by body forces.

2.1.6 Three Theorems Based on Dissipation Integrals

Lemma 1. *If $\boldsymbol{u}^I$ is an incompressible flow and $\boldsymbol{u}^S$ is a Stokes flow with body force $\boldsymbol{F}^S$ then*

$$2\mu \int e^I : e^S \, \mathrm{d}V = \int_{\partial V} \boldsymbol{u}^I \cdot \sigma^S \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{u}^I \cdot \boldsymbol{F}^S \, \mathrm{d}V$$

Proof. Same as 'work balances dissipation'.

Theorem 1. *Uniqueness theorem. Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2$ are Stokes flows with the same boundary conditions and body forces, i.e. $\boldsymbol{F}_1 = \boldsymbol{F}_2$ in V and either $\boldsymbol{u}_1 = \boldsymbol{u}_2$ or $\sigma_1 \cdot \boldsymbol{n} = \sigma_2 \cdot \boldsymbol{n}$ on ∂V. Then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.*

Proof. Let $\boldsymbol{u}^* = \boldsymbol{u}_1 - \boldsymbol{u}_2$. From lemma 1,

$$2\mu \int_V e^* : e^* \, \mathrm{d}V = 0$$

Thus $e^* = 0$ in *V*. Hence we can deduce $\boldsymbol{u}^*$ consists entirely of rigid body motion: $\boldsymbol{u}^* = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{u}$. Using the boundary conditions, we have $\boldsymbol{U} = \boldsymbol{\Omega} = 0$ thus $\boldsymbol{u}_1 = \boldsymbol{u}_2$, i.e. Stokes flows are unique.

Theorem 2. *Reciprocal theorem. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are Stokes flows in V then*

$$\int_{\partial V} \boldsymbol{u}_1 \cdot \sigma_2 \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{u}_1 \cdot \boldsymbol{F}_2 \, \mathrm{d}V = \int_{\partial V} \boldsymbol{u}_2 \cdot \sigma_1 \cdot \boldsymbol{n} \, \mathrm{d}S + \int_V \boldsymbol{u}_2 \cdot \boldsymbol{F}_1 \, \mathrm{d}V$$

That is, work done by forces of flow 1 against flow 2 = work done by forces of flow 2 against flow 1.

Proof. Apply the lemma twice.

Theorem 3. *Minimum Dissipation theorem. Among all the incompressible flows in V that satisfy given velocity boundary conditions, the dissipation is minimised by the Stokes flow $\boldsymbol{u}^S$ with $\boldsymbol{F}^s = \boldsymbol{0}$ satisfying the same velocity boundary conditions.*

Proof. We have

$$0 \leq 2\mu \int (e - e^S) : (e - e^S) \,\mathrm{d}V$$

$$\leq 2\mu \int e : e - e^S : e^S \,\mathrm{d}V + 4\mu \int e^S : (e^S - e) \,\mathrm{d}V$$

Applying the lemma with $\boldsymbol{u}^I = \boldsymbol{u}^S - \boldsymbol{u}$, the last term is 0 since $\boldsymbol{u}^I = 0$ on ∂V and $\boldsymbol{F}^S = 0$ on V. Thus

$$0 \leq \mathcal{D} - \mathcal{D}^S$$

Example.

1. Consider an irregularly shaped body in a Stokes flow with inscribing circle S_1 with radius a_1 and circumscribing circle S_2 with radius a_2. Suppose the body experiences a force $\boldsymbol{F}$ and has uniform velocity $\boldsymbol{U}$.
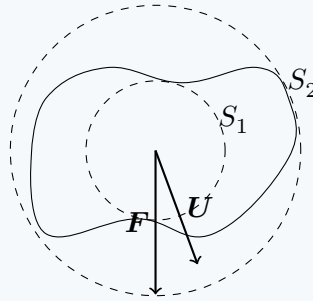
   Applying the theorem by taking $\boldsymbol{U}^S$ to be the Stokes flow past S_1 and $\boldsymbol{U}^I$ to be the Stokes flow past S_2 superposed with solid body motion in the gap between S_1 and S_2, we have

$$(6\pi\mu a_1 U)U \leq \boldsymbol{F} \cdot \boldsymbol{U} \leq (6\pi\mu a_2 U)U$$

2. Adding *rigid* particles to a Stokes flow with given *external* velocity boundary conditions increases dissipation and, if the particles are *force-free* and *torque-free*, the apparent viscosity also increases.
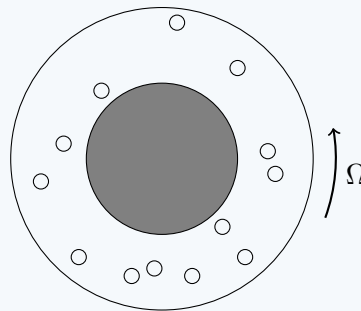
3. Inertia increases drag: consider $\rho\dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ as $\boldsymbol{F}$.

2.2 Representation by Potentials

Assume $\boldsymbol{F} = 0$, or that $\boldsymbol{F}$ is conservative and absorbed by the modified pressure. Consider the Stokes equations

$$\mu \nabla^2 \boldsymbol{u} = \nabla p \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

From these equations we have

$$\nabla \cdot (1) \& (2) \implies \nabla^2 p = 0 \implies p \text{ is harmonic}$$
$$\nabla \times (1) \implies \nabla^2 \boldsymbol{\omega} = 0 \implies \text{vorticity } \boldsymbol{\omega} = \nabla \times \boldsymbol{u} \text{ is harmonic}$$
$$\nabla^2 (1) \implies \nabla^4 \boldsymbol{u} = \boldsymbol{0} \implies \boldsymbol{u} \text{ is } \textit{bi-harmonic}$$

In two dimensions, we can use a stream-function so that $\boldsymbol{u} = \nabla \times (0, 0, \psi)$. Then

$$\omega_z = -\nabla^2 \psi \implies \nabla^4 \psi = 0$$

Similarly, in axisymmetric spherical polars, $\boldsymbol{u} = \nabla \times (0, 0, \frac{\Psi}{r \sin \theta})$. Then $\omega_\phi = -\frac{E^2 \Psi}{r \sin \theta}$ and $E^4 \Psi = 0$ where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right)$$

Many exact solutions can be found in coordinate systems where the operators ∇^2, ∇^4, E^2, etc are separable.

2.2.1 Complex Variable Theory in 2D Flow

Writing $z = x + iy, \bar{z} = x - iy$ gives

$$\nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Thus $f(x, y)$ analytic implies $f = f(z)$ or equivalently $\frac{\partial f}{\partial \bar{z}} = 0$. Thus $\Re f$ and $\Im f$ are harmonic. Similarly $\nabla^4 \psi = 0$ implies ψ can be written as $\psi = \Im(\bar{z}\phi + \chi)$ where $\phi(z), \chi(z)$ are analytic. We can find clever exact solutions to difficult problems using this theory, but it is limited to 2D.

2.2.2 Papkovich-Neuber Solution

Let $p = \nabla^2 \pi$ where

$$\pi(\boldsymbol{x}) = -\frac{1}{4\pi} \int \frac{p(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}V$$

Then the Stokes equations can be written $\nabla^2 (\mu \boldsymbol{u} - \nabla \pi) = 0$. Thus

$$\mu \boldsymbol{u} = \nabla \pi - \boldsymbol{\Phi}$$

where $\nabla^2 \boldsymbol{\Phi} = 0$. Now $\nabla \cdot \boldsymbol{u} = 0$ implies $\nabla^2 \pi = \nabla \cdot \boldsymbol{\Phi}$. Then

$$\pi = \frac{1}{2} \left(\boldsymbol{x} \cdot \boldsymbol{\Phi} + \chi \right)$$

where $\nabla^2 \chi = 0$. Thus *any* Stokes flow with $\boldsymbol{F} = 0$ can be written in terms of a harmonic vector $\boldsymbol{\Phi}$ and a harmonic scalar χ. The Stokes equations are then

$$2\mu \boldsymbol{u} = \nabla \left(\boldsymbol{x} \cdot \boldsymbol{\Phi} + \chi \right) - 2\boldsymbol{\Phi}$$
$$p = \nabla \cdot \boldsymbol{\Phi}$$

which may also be re-written with the 2μ factor absorbed by p.

Note the following.

1. Any irrotational flow $2\mu\boldsymbol{u} = \nabla\chi$ is also a Stokes flow, though $p = 0$ and $\boldsymbol{\sigma} = \nabla\nabla\chi$ which is different from an inviscid irrotational flow.

2. It is sometimes possible to find a harmonic scalar ϕ with $\chi = \boldsymbol{x}\cdot\nabla\phi - 2\phi$. If so, χ can be eliminated by writing $\boldsymbol{\Phi}' = \boldsymbol{\Phi} + \nabla\phi$. For example, if χ has a spherical harmonic expansion we can eliminate all of the terms except the uniform strain $\chi/2\mu = \frac{1}{2}\boldsymbol{x}\cdot E\cdot\boldsymbol{x} \iff \boldsymbol{u} = E\cdot\boldsymbol{x}$, since $\chi = r^n Y_n^m(\theta,\phi) \iff \phi = \dfrac{r^n}{n-2}Y_n^m(\theta,\phi)$ which fails for $n = 2$.

3. Conversely, if $\boldsymbol{\Phi} = \nabla\phi$ then we can get the same $\boldsymbol{u}$ from $\chi = \boldsymbol{x}\cdot\nabla\phi - 2\phi$, which is easier to calculate.

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2.3 Solutions for points, spheres, and cylinders

A point or sphere has no intrinsic direction or orientation, thus solutions on these geometries should also have no intrinsic direction or orientation.

2.3.1 Spherical harmonic functions

Let $r = |\boldsymbol{x}|$. Recall $\nabla^2(\frac{1}{r}) = 0$ for $r \neq 0$. All other spherical harmonic functions ϕ with $\phi \to 0$ as $r \to \infty$ are obtained from

$$\frac{1}{r}, \quad \nabla\frac{1}{r}, \quad \nabla\nabla\frac{1}{r}, \quad \text{etc.}$$

The harmonic functions which are bounded as $r \to 0$ are obtained from

$$r\cdot\frac{1}{r} = 1, \quad r^3\nabla\frac{1}{r} = -\boldsymbol{x}, \quad r^5\nabla\nabla\frac{1}{r}, \quad \dots, \quad r^{2n+1}\nabla^n\frac{1}{r}$$

Compare with separable solutions, for example the $2n + 1$ solutions in spherical polars given by

$$\begin{pmatrix} r^n \\ r^{-n-1} \end{pmatrix} P_n^m(\theta) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

where P_n^m are *associated Legendre functions* and $0 \leq m \leq n$.

Recall the following results.

$$\nabla\boldsymbol{x} = I$$
$$\nabla r = \frac{\boldsymbol{x}}{r}$$
$$\nabla f(r) = f'(r)\nabla r = f'(r)\frac{\boldsymbol{x}}{r}$$

Hence we have

$$\nabla\frac{1}{r} = -\frac{\boldsymbol{x}}{r^3}$$
$$\nabla\nabla\frac{1}{r} = -\frac{I}{r^3} + \frac{3\boldsymbol{x}\cdot\boldsymbol{x}}{r^5}$$
$$\nabla_i\nabla_j\nabla_k\frac{1}{r} = \nabla_i\left(-\frac{\delta_{jk}}{r^3} + \frac{3x_j\cdot x_k}{r^5}\right)$$
$$= \frac{3(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})}{r^5} - \frac{15x_i x_j x_k}{r^7}$$

Note: these depend only on $\boldsymbol{x}$ and r and thus have no preferred direction, as hoped. We can use these functions to form Papkovich-Neuber potentials $\boldsymbol{\Phi}$ and χ by multiplying the harmonic functions above by constant scalars, vectors or tensors and taking an appropriate number of dot products, e.g. the following are all harmonic vectors

$$\boldsymbol{A}\frac{1}{r}, \quad \boldsymbol{B}\cdot\nabla\frac{1}{r}, \quad C\nabla\frac{1}{r}, \quad (\boldsymbol{D}\cdot\nabla)\nabla\frac{1}{r}, \quad (\boldsymbol{E}:\nabla\nabla)\nabla\frac{1}{r}, \quad \boldsymbol{\Omega}\times\nabla\frac{1}{r}$$

It is useful to distinguish between *true* and *pseudo* tensors. True / pseudo tensors keep / change sign upon reflection, e.g.

$$T'_{ijk} = \pm R_{il}R_{jm}R_{kn}T_{lmn}$$

Examples of true vectors are velocity $\boldsymbol{u}$; force $\boldsymbol{F}$; position $\boldsymbol{x}$; del ∇; identity $\boldsymbol{l}$. Examples of pseudo vectors are angular velocity $\boldsymbol{\Omega}$; torque $\boldsymbol{G}$; $\boldsymbol{u}\times\boldsymbol{x}$; vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$. Products obey the obvious parity rules, e.g. helicity $\boldsymbol{u}\cdot\boldsymbol{\Omega}$ is a pseudo scalar.

2.3.2 Solution due to a point force

The Papkovich-Neuber solution due to a point force is a Green's function for the Stokes equations. This problem is also known as a 'Stokeslet'. Consider the problem

$$\nabla\cdot\sigma = \mu\nabla^2\boldsymbol{u} - \nabla p = -\boldsymbol{F}\delta(\boldsymbol{x})$$
$$\nabla\cdot\boldsymbol{u} = 0$$

with $\boldsymbol{u}\to 0$ at infinity. The answer must be linear in $\boldsymbol{F}$, but otherwise has no orientation. The only choice is $\boldsymbol{\Phi} = \alpha\dfrac{\boldsymbol{F}}{r}$. We could have tried $\boldsymbol{F}\times\nabla\dfrac{1}{r}$, but this is a pseudo vector whilst $\boldsymbol{\Phi}$ and χ need to be true since $\boldsymbol{u}$ is true. Similar arguments rule out other harmonic functions.

We have

$$2\mu\boldsymbol{u} = \alpha\left(\nabla\left(\frac{\boldsymbol{F}\cdot\boldsymbol{x}}{r}\right) - 2\frac{\boldsymbol{F}}{r}\right)$$
$$= \alpha\left(\frac{\boldsymbol{F}\cdot\boldsymbol{l}}{r} - \frac{(\boldsymbol{F}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3} - 2\frac{\boldsymbol{F}}{r}\right)$$
$$= -\alpha\left(\frac{\boldsymbol{F}}{r} + \frac{(\boldsymbol{F}\cdot\boldsymbol{x})\boldsymbol{x}}{r^3}\right)$$

Thus the stress tensor is

$$\sigma = \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) - (\nabla\cdot\boldsymbol{\Phi})\,\boldsymbol{l} = 3\alpha(\boldsymbol{F}\cdot\boldsymbol{x})\frac{\boldsymbol{x}\boldsymbol{x}}{r^5}$$

On any sphere $r = R$, $\boldsymbol{n} = \dfrac{\boldsymbol{x}}{R}$, and

$$2\mu\boldsymbol{u}\cdot\boldsymbol{n} = -\frac{2\alpha}{R}\boldsymbol{F}\cdot\boldsymbol{n}$$
$$\sigma\cdot\boldsymbol{n} = 3\alpha\frac{(\boldsymbol{F}\cdot\boldsymbol{n})\boldsymbol{n}}{R^2}$$

To determine the constant α we can consider the surface volume flux and the surface stress. The surface volume flux is

$$\int_{r=R}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = -\frac{\alpha\boldsymbol{F}}{\mu R}\cdot\int_{r=R}\boldsymbol{n}\,\mathrm{d}S = 0$$

which does not provide any information on α. The surface forces should equal $-\boldsymbol{F}$. We have

$$-\boldsymbol{F} = \int_{r=R}\sigma\cdot\boldsymbol{n}\,\mathrm{d}S = 3\alpha\boldsymbol{F}\cdot\int_{r=R}\boldsymbol{n}\boldsymbol{n}\frac{\mathrm{d}S}{R^2} = 3\alpha\boldsymbol{F}\cdot\frac{4\pi}{3}\boldsymbol{l} = 4\pi\alpha\boldsymbol{F}$$

Figure 1: Stokeslet solution for a point force.

Hence we choose $\alpha = -1/4\pi$. Thus the final solution is

$$\boldsymbol{u} = \boldsymbol{F} \cdot J(\boldsymbol{x}), \qquad \sigma = \boldsymbol{F} \cdot K(\boldsymbol{x}), \qquad p = \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{4\pi r^3}$$

where J is the *Oseen tensor*:

$$J = \frac{1}{8\pi\mu} \left(\frac{I}{r} + \frac{\boldsymbol{xx}}{r^3} \right), \qquad K = -\frac{3}{4\pi} \frac{\boldsymbol{xxx}}{r^5}$$

Finally, from incompressibility $\nabla \cdot \boldsymbol{u} = 0$ and the Stokes equations $\nabla \cdot \sigma = -\boldsymbol{F}\delta(\boldsymbol{x})$, we deduce

$$\nabla \cdot J = 0, \qquad \nabla \cdot K = -I\delta(\boldsymbol{x})$$

Note that the velocity scales as $u \propto \frac{1}{r}$: see figure 1. This is slowly decaying compared to many forces e.g. gravity which scales as $\frac{1}{r^2}$. Thus particle interactions in Stokes flow can occur on much larger scales than (for example) charge interactions.

2.3.3 Source flow

Consider a source of strength Q with $\nabla \cdot \boldsymbol{u} = Q\delta(\boldsymbol{x})$. This is referred to as a *point volume source.* One can show the solution is

$$\boldsymbol{u} = \frac{Q\boldsymbol{x}}{4\pi r^3}$$

which is obtained using Papkovich-Neuber potentials

$$\chi = \alpha \frac{Q}{r}, \qquad \boldsymbol{\Phi} = \beta Q \nabla \frac{1}{r}$$

2.3.4 Force dipole, stresslet, rotlet

Further solutions for dipoles, quadrupoles, etc. can be found by taking gradients of the Stokeslet and source solutions. For example, consider a dipole.

The Stokeslet solution is

$$\boldsymbol{u} = \boldsymbol{F} \cdot J(\boldsymbol{x} - \boldsymbol{d}) - \boldsymbol{F} \cdot J(\boldsymbol{x}) = \boldsymbol{F} \cdot (-\boldsymbol{d} \cdot \nabla)J(\boldsymbol{x}) + \text{h.o.t.}$$

Take the limit $\boldsymbol{d} \to \boldsymbol{0}$ with $\boldsymbol{Fd}$ fixed and split $-F_i d_j$ into

1. An isotropic part $-\frac{1}{3}F_k d_k \delta_{ij}$

2. A symmetric traceless part

$$s_{ij} = -\frac{1}{2}(F_i d_j + F_j d_i) + \frac{1}{3}F_k d_k \delta_{ij}$$

3. An antisymmetric part $-\frac{1}{2}\varepsilon_{ijk}G_k$ where $\boldsymbol{G} = \boldsymbol{d} \times \boldsymbol{F}$

The flow contribution from each of these components may then be calculated.

1. The isotropic component gives no flow since $\nabla \cdot J = 0$

2. This component is a *stresslet* representing the following components of motion

3. This component is a *rotlet* due to a point torque $\boldsymbol{G}$.

Both the stresslet and rotlet decay as $\frac{1}{r^2}$.

2.3.5 Rigid sphere with velocity U

Consider a rigid sphere of radius a moving uniformly with velocity $\boldsymbol{U}$ in a Stokes flow. We have $\nabla \cdot \boldsymbol{u} = 0$ and $\mu \nabla^2 \boldsymbol{u} = \nabla p$ in $r > a$. We require $\boldsymbol{u} \to 0$ as $r \to \infty$ and $\boldsymbol{u} = \boldsymbol{U}$ on the sphere's surface $r = a$. The sphere is isotropic, so we need harmonic functions of $\boldsymbol{x}, \boldsymbol{U}$ which are linear in $\boldsymbol{U}$; decay at ∞; and are true tensors. We choose

$$\frac{1}{2\mu}\boldsymbol{\Phi} = \alpha \boldsymbol{U}\frac{1}{r}, \qquad \frac{1}{2\mu}\chi = \beta \boldsymbol{u} \cdot \nabla \frac{1}{r}$$

This gives a solution which is a superposition of a Stokeslet and a source dipole

$$\boldsymbol{u} = -\alpha \left(\frac{\boldsymbol{U}}{r} + \frac{(\boldsymbol{U} \cdot \boldsymbol{x})\boldsymbol{x}}{r^3} \right) + \beta \left(-\frac{\boldsymbol{U}}{r^3} + 3\frac{(\boldsymbol{U} \cdot \boldsymbol{x})\boldsymbol{x}}{r^5} \right)$$

Enforcing the boundary condition $\boldsymbol{u} = \boldsymbol{U}$ on $r = a$ requires

$$-\frac{\alpha}{a} - \frac{\beta}{a^3} = 1, \qquad -\frac{\alpha}{a} + 3\frac{\beta}{a^3} = 0$$

$$\implies \alpha = -\frac{3a}{4}, \quad \beta = -\frac{a^3}{4}$$

Thus the final solution for a sphere in a Stokes flow is

$$\boldsymbol{u} = \frac{3}{4}\boldsymbol{U}\left(\frac{a}{r} + \frac{a^3}{3r^3} \right) + \frac{3}{4}\frac{(\boldsymbol{U} \cdot \boldsymbol{x})\boldsymbol{x}}{r^2}\left(\frac{a}{r} - \frac{a^3}{r^3} \right)$$

The corresponding pressure and vorticity, both of which are harmonic and due to the Stokeslet, are

$$p = \frac{3}{2}\mu a \frac{\boldsymbol{U} \cdot \boldsymbol{x}}{r^3}, \qquad \boldsymbol{\omega} = -\frac{1}{\mu}\nabla \times \boldsymbol{\Phi} = \frac{3a}{2}\frac{\boldsymbol{U} \times \boldsymbol{x}}{r^3}$$

2.3.6 Force and stress on a translating sphere

The flow at large distances is dominated by the $\frac{1}{r}$ Stokeslet term with strength $\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}$. Since the force across $r =\text{'} \infty'$ must balance the force across $r = a$, we can see without further work that the force by the fluid on the sphere i.e. drag $= -6\pi\mu a\boldsymbol{U}$. In more detail:

$$\begin{aligned}
\sigma &= -pl + \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \\
&= -\frac{3}{2}\mu a\frac{\boldsymbol{U} \cdot \boldsymbol{x}}{r} + \frac{3}{4}\mu\left[\left(-\frac{a}{r^3} - \frac{a^3}{r^5}\right)(\boldsymbol{xU} + \boldsymbol{Ux})\right. \\
&\quad\left. + (\boldsymbol{Ux} + \boldsymbol{xU} + 2\boldsymbol{U}\cdot\boldsymbol{x}l)\left(\frac{a}{r^3} - \frac{a^3}{r^5}\right) + (\boldsymbol{U}\cdot\boldsymbol{x})\left(-\frac{3a}{r^5} + \frac{5a^3}{r^7}\right)2\boldsymbol{xx}\right] \\
\implies \sigma\cdot\boldsymbol{n}\mid_{r=a} &= -\frac{3}{2}\frac{\mu}{a}(\boldsymbol{U}\cdot\boldsymbol{n})\boldsymbol{n} + \frac{3\mu}{4a}\left[-2(\boldsymbol{n}(\boldsymbol{U}\cdot\boldsymbol{n}) + \boldsymbol{U}) + (\boldsymbol{U}\cdot\boldsymbol{n})(-2)2\boldsymbol{n}\right] \\
&= -\frac{3\mu}{2a}\boldsymbol{U}
\end{aligned}$$

Thus the drag on the sphere is

$$\boldsymbol{F} = \int_{r=a}\sigma\cdot\boldsymbol{n}\,\mathrm{d}S = -\frac{3\mu}{2a}\boldsymbol{U}\cdot 4\pi a^2 = -6\pi\mu a\boldsymbol{U}$$

as expected.

2.3.7 Gravitational settling

The force balance of weight, buoyancy, and drag gives the settling velocity of a rigid sphere in a Stokes flow.

$$\frac{4}{3}\pi a^3\rho_s\boldsymbol{g} - \frac{4}{3}\pi a^3\rho\boldsymbol{g} - 6\pi\mu a\boldsymbol{U} = 0$$

$$\implies \boldsymbol{U} = \frac{2a^2}{9\mu}(\rho_s - \rho)\boldsymbol{g} \propto a^2$$

2.3.8 2D potentials

The harmonic functions in two-dimensions which are bounded as $r \to \infty$ are

$$\ln r, \quad \nabla\ln r = \frac{\boldsymbol{x}}{r^2}, \quad \nabla\nabla\ln r = \frac{l}{r^2} - \frac{2\boldsymbol{xx}}{r^4}, \quad \dots$$

Similar to the 3D spherical case, the harmonic functions in two-dimensions which are bounded as $r \to 0$ are

$$1, \quad r^2\nabla\ln r = \boldsymbol{x}, \quad , \dots, \quad r^{2n}\nabla^n\ln r$$

The 2D Stokeslet solution follows from the Papkovich-Neuber potentials $\boldsymbol{\Phi} = \dfrac{\boldsymbol{F}}{2\pi} \ln r$ which gives

$$\boldsymbol{u} = \boldsymbol{F} \cdot J^{2D}$$

$$p = \frac{\boldsymbol{F} \cdot \boldsymbol{x}}{2\pi r^2}$$

$$\sigma = \boldsymbol{F} \cdot K^{2D}$$

$$J^{2D} = \frac{1}{4\pi\mu} \left(-\ln rI + \frac{\boldsymbol{x}\boldsymbol{x}}{r^2} \right)$$

$$K^{2D} = -\frac{1}{\pi} \frac{\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}}{r^4}$$

The 2D Stokeslet solution corresponds to a line force $\boldsymbol{F}$ *per unit length*. Note that $\boldsymbol{u} \not\to 0$ as $r \to \infty$ for a line force, though $\boldsymbol{u} \to 0$ for line dipoles, line quadrupoles, etc.

2.4 Motion of rigid particles

2.4.1 The resistance matrix

A rigid particle moving with velocity $\boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{x}$ through fluid otherwise at rest exerts a force $\boldsymbol{F}$ and a couple $\boldsymbol{G}$ on the fluid. By linearity,

$$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\Omega} \end{pmatrix}$$

where the tensors $A - D$ depend on the size, shape, and orientation of the body. We refer to this matrix as the *resistance matrix*. We have

$$\int_{\text{body}} \boldsymbol{u} \cdot \sigma \cdot \boldsymbol{n} \, \mathrm{d}S = \int (\boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \sigma \cdot \boldsymbol{n} \, \mathrm{d}S = \boldsymbol{U} \cdot \boldsymbol{F} + \boldsymbol{\Omega} \cdot \boldsymbol{G}$$

Similarly, from the reciprocal theorem (with $\boldsymbol{F} = 0$), for all $\boldsymbol{U}_1, \boldsymbol{\Omega}_1, \boldsymbol{U}_2, \boldsymbol{\Omega}_2$,

$$(\boldsymbol{U}_1, \boldsymbol{\Omega}_1) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_2 \\ \boldsymbol{\Omega}_2 \end{pmatrix} = (\boldsymbol{U}_2, \boldsymbol{\Omega}_2) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_1 \\ \boldsymbol{\Omega}_1 \end{pmatrix} = (\boldsymbol{U}_1, \boldsymbol{\Omega}_1) \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_2 \\ \boldsymbol{\Omega}_2 \end{pmatrix}$$

Hence $A = A^T$, $D = D^T$ so A and D are diagonalisable, and $B = C^T$, i.e. the force from pure rotation is equal to the couple from pure translation. Since the viscous dissipation is positive, the resistance matrix is positive definite, so invertible.

> **Example.**
>
> 1. A body with 3 independent planes of reflectional symmetry has $B = C = 0$.
>
> 2. A cube falls with the same speed (and no rotation) in all orientations since $A \propto I$.
>
> 3. A and D are known for ellipsoids, and therefore for rods and discs which are limits of ellipsoids.

The resistance matrix and its inverse the *mobility matrix* are sometimes extended to describe a rigid particle placed in a background linear flow $\boldsymbol{u}^\infty = \boldsymbol{U}^\infty + \boldsymbol{\Omega}^\infty \times \boldsymbol{x} + E^\infty \cdot \boldsymbol{x}$, as

$$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \\ S \end{pmatrix} = \begin{pmatrix} & & \\ & 12 \times 12 & \\ & & \end{pmatrix} \begin{pmatrix} \boldsymbol{U} - \boldsymbol{U}^\infty \\ \boldsymbol{\Omega} - \boldsymbol{\Omega}^\infty \\ E^\infty \end{pmatrix}$$

where S is the stresslet exerted by the particle.

If $E^\infty = 0$, a force-free, couple-free particle just translates and rotates with the flow. If $E^\infty \neq 0$ then the particle generates a stresslet. In general, the extra dissipation

$$\boldsymbol{F} \cdot (\boldsymbol{U} - \boldsymbol{U}^\infty) + \boldsymbol{G} \cdot (\boldsymbol{\Omega} - \boldsymbol{\Omega}^\infty) + S : E^\infty$$

is positive by the minimum dissipation theorem, thus the extended matrix is positive definite, so invertible.

This is useful because it gives the leading order effects for a small particle of size a in a flow of larger lengthscale L:

$$\boldsymbol{u}^\infty(\boldsymbol{x}) = \boldsymbol{u}^\infty(0) + \boldsymbol{x} \cdot \nabla \boldsymbol{u}^\infty(0) + \mathcal{O}(a^2/L^2)$$

2.5 Faxén relations

Consider the behaviour of a rigid particle placed in an arbitrary unbounded Stokes flow $\boldsymbol{u}^\infty(\boldsymbol{x})$. Can we find the particle motion $\boldsymbol{u}^P = \boldsymbol{U} + \boldsymbol{\Omega} \times \boldsymbol{x}$? Let $\boldsymbol{u}'$ be the *perturbation flow* $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{u}^\infty$, which has boundary conditions

$$\boldsymbol{u}' \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty$$
$$\boldsymbol{u}' = \boldsymbol{u}^P - \boldsymbol{u}^\infty(\boldsymbol{x}) \quad \text{on particle}$$

and $\hat{\boldsymbol{u}}$ a 'test' flow due to particle translation with arbitrary velocity $\hat{\boldsymbol{V}}$. By linearity,

$$\hat{\sigma} = \hat{\Sigma}(\boldsymbol{x}) \cdot \hat{\boldsymbol{V}}$$

for some third rank tensor $\hat{\Sigma}$. The reciprocal theorem for $\boldsymbol{u}'$ and $\hat{\boldsymbol{u}}$ gives

$$\hat{\boldsymbol{V}} \cdot \int_{S_P} \sigma' \cdot \boldsymbol{n} \, \mathrm{d}S = \hat{\boldsymbol{V}} \cdot \int (\boldsymbol{u}^P - \boldsymbol{u}^\infty(\boldsymbol{x})) \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$$

Now $\hat{\boldsymbol{V}}$ is arbitrary and contribution from $\int \boldsymbol{u}^P \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$ is given by

$$\begin{pmatrix} \boldsymbol{U} & \boldsymbol{\Sigma} \end{pmatrix} \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{V}} \\ \boldsymbol{0} \end{pmatrix}$$

Hence we have *Faxén's first formula*

$$\boldsymbol{F} = A \cdot \boldsymbol{U} + B \cdot \boldsymbol{\Omega} - \int \boldsymbol{u}^\infty(\boldsymbol{x}) \cdot \hat{\Sigma} \cdot \boldsymbol{n} \, \mathrm{d}S$$

Similar Faxén relations can be obtained for the couple $\boldsymbol{G}$ and stresslet S exerted, by using other test flows.

Example. For a sphere, we have $\hat{\Sigma} \cdot \boldsymbol{n} = \frac{3\mu}{2a} I$ on $r = a$, where $\boldsymbol{n}$ is *into* the particle. We also have

$$A = 6\pi\mu a I, \qquad B = 0$$

which gives a translation velocity

$$\boldsymbol{U} = \frac{\boldsymbol{F}}{6\pi\mu a} + \frac{1}{4\pi a^2} \int \boldsymbol{u}^\infty(\boldsymbol{x}) \, \mathrm{d}S$$

> where the last term is an average of $\boldsymbol{u}^\infty$ over the surface. Moreover, we can Taylor expand this far field velocity
>
> $$\boldsymbol{u}^\infty(\boldsymbol{x}) = \boldsymbol{u}^\infty(\boldsymbol{0}) + \boldsymbol{x}\cdot\nabla\boldsymbol{u}^\infty(\boldsymbol{0}) + \frac{1}{2}\boldsymbol{x}\boldsymbol{x} : \nabla\nabla\boldsymbol{u}^\infty(\boldsymbol{0}) + \ldots$$
>
> Therefore we have
>
> $$\frac{1}{4\pi a^2}\int_{r=a}\boldsymbol{u}^\infty(\boldsymbol{x})\,\mathrm{d}S = \boldsymbol{u}^\infty(\boldsymbol{0}) + 0 + \frac{1}{8\pi a^2}\int_{r=a}\boldsymbol{x}\boldsymbol{x}\,\mathrm{d}S : \nabla\nabla\boldsymbol{u}^\infty(\boldsymbol{0}) + \ldots$$
>
> Odd terms, for example $\int_{r=a}\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}\,\mathrm{d}S = 0$ by symmetry. Even terms are isotropic and give $\nabla^{2n}\boldsymbol{u}^\infty(\boldsymbol{0})$, but by Stokes equations $\nabla^4\boldsymbol{u}^\infty = \boldsymbol{0}$. Hence
>
> $$\boldsymbol{U} = \frac{\boldsymbol{F}}{6\pi\mu a} + \boldsymbol{u}^\infty(\boldsymbol{0}) + \frac{a^2}{6}\nabla^2\boldsymbol{u}^\infty(\boldsymbol{0})$$

Application to two sedimenting spheres. Consider two rigid spheres of radius a sedimenting under a given force $\boldsymbol{F}$. We wish to calculate the flal speed $\boldsymbol{U}$ to $\mathcal{O}(a^3/R^3)$. One sphere moves in the far-field flow of the other sphere, which itself depends on the force, couple, and stresslet exerted at leading order. $\boldsymbol{F}$ and $\boldsymbol{G} = 0$ are the same for each sphere, hence this far field is the same as for an isolated sphere up to $\mathcal{O}(a^4/R^4)$. Faxén gives an $\mathcal{O}(a^3/R^3)$ correction to $\boldsymbol{U}$ from the $\frac{a^2}{6}\nabla^2$ term.

2.6 Integral representations of Stokes flow

2.6.1 Basic integral identity

Consider a Stokes flow $\boldsymbol{u}$ and a Stokeslet flow $\boldsymbol{u}^S$ due to a point force $\boldsymbol{F}$ at $\boldsymbol{y}$. We have $\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{u}^S = 0$ and

$$\nabla\cdot\sigma = -\boldsymbol{f}, \qquad \nabla\cdot\sigma^S = -\boldsymbol{F}\delta(\boldsymbol{x}-\boldsymbol{y})$$

Apply the reciprocal theorem with $\boldsymbol{u}_1 = \boldsymbol{u}, \boldsymbol{u}_2 = \boldsymbol{u}^S$. We have

$$\int_V \boldsymbol{u}\cdot\boldsymbol{F}\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}V + \int_{\partial V}\boldsymbol{u}\cdot(\boldsymbol{F}\cdot K(\boldsymbol{x}-\boldsymbol{y}))\cdot\boldsymbol{n}\,\mathrm{d}S = \int_v (\boldsymbol{F}\cdot J(\boldsymbol{x}-\boldsymbol{y}))\cdot\boldsymbol{f}\,\mathrm{d}V + \int_{\partial V}(\boldsymbol{F}\cdot J(\boldsymbol{x}-\boldsymbol{y}))\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S$$

Note by definition J is even and K is odd, so $J(\boldsymbol{x}-\boldsymbol{y}) = J(\boldsymbol{y}-\boldsymbol{x})$ and $K(\boldsymbol{x}-\boldsymbol{y}) = -K(\boldsymbol{y}-\boldsymbol{x})$. Also, $\boldsymbol{F}$ is arbitrary so may be factored out by the quotient theorem. We also have from the sampling property of the delta function

$$\int_V \phi(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}V = \begin{cases}\phi(\boldsymbol{y}) & \boldsymbol{y}\in V \\ 0 & \boldsymbol{y}\notin V \\ \frac{1}{2}\phi(\boldsymbol{y}) & \boldsymbol{y}\in\partial V\end{cases}$$

Note we require $\boldsymbol{y}\in\partial V$ to be a smooth point of ∂V for this to hold. Hence

$$\int_V J(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}V + \int_{\partial V}J(\boldsymbol{y}-\boldsymbol{x})\cdot\sigma\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\partial V}\boldsymbol{u}(\boldsymbol{x})\cdot K(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}\,\mathrm{d}S = \begin{cases}\boldsymbol{u}(\boldsymbol{y}) & \boldsymbol{y}\in V \\ 0 & \boldsymbol{y}\notin V \\ \frac{1}{2}\boldsymbol{u}(\boldsymbol{y}) & \boldsymbol{y}\notin\partial V\end{cases}$$

1. The above follows the sign conventions: $\boldsymbol{n}$ is out of v, and we have $\boldsymbol{y} - \boldsymbol{x}$ as the argument.

2. $\boldsymbol{f}$ is often not included, i.e. there is no body force or $\boldsymbol{f}$ is absorbed into modified pressure

3. The jump in the RHS comes from the K integral which depends on $\frac{1}{r}$ which jumps when $\boldsymbol{y}$ crosses ∂V.

4. Usually we only know $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ or $\boldsymbol{u}$ on ∂V. In general, we solve as an integral equation for whichever is not specified with $\boldsymbol{y} \in \partial V$ and then substitute to find $\boldsymbol{u}$ elsewhere.

2.6.2 Far-field approximations/multipole expansion for a moving body

For a body of size a and for $|\boldsymbol{y}| \gg a$ we have by Taylor expanding

$$J(\boldsymbol{y} - \boldsymbol{x}) \approx J(\boldsymbol{y}) - \boldsymbol{x} \cdot \nabla J(\boldsymbol{y}) + \mathcal{O}(\frac{a^2}{y^3})$$

$$K(\boldsymbol{y} - \boldsymbol{x}) \approx K(\boldsymbol{y}) + \mathcal{O}(\frac{a^2}{y^3})$$

Assuming $\boldsymbol{f} = 0$, substitute this into the basic integral representation to obtain after some messy algebra

$$\boldsymbol{u}(\boldsymbol{y}) \approx \boldsymbol{F} \cdot J(\boldsymbol{y}) + \frac{\boldsymbol{G} \times \boldsymbol{y}}{8\pi\mu|\boldsymbol{y}|^3} + \frac{Q\boldsymbol{y}}{4\pi|\boldsymbol{y}|^3} - \frac{3(\boldsymbol{y} \cdot S \cdot \boldsymbol{y})\boldsymbol{y}}{8\pi\mu|\boldsymbol{y}|^5} + \mathcal{O}(\frac{a^2}{y^3})$$

where the first term is $\mathcal{O}(\frac{1}{y})$, the three following terms are $\mathcal{O}(\frac{a}{y^2})$ and (using $\boldsymbol{n}$ out of body)

$$\boldsymbol{F} = -\int \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = \text{force exerted by body}$$

$$\boldsymbol{G} = -\int \boldsymbol{x} \times \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}S = \text{couple exerted by body}$$

$$Q = \int \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = \text{source strength (vanishes for rigid body)}$$

$$S = \int \left[\frac{1}{2}\left(\boldsymbol{x}(\boldsymbol{\sigma} \cdot \boldsymbol{n}) - 2\mu\boldsymbol{u}\boldsymbol{n}\right) + \frac{1}{2}\left(\boldsymbol{x}(\boldsymbol{\sigma} \cdot \boldsymbol{n}) - 2\mu\boldsymbol{u}\boldsymbol{n}\right)^T - \frac{1}{3}I\mathrm{tr}\left(\boldsymbol{x}(\boldsymbol{\sigma} \cdot \boldsymbol{n}) - 2\mu\boldsymbol{u}\boldsymbol{n}\right) \right] \mathrm{d}S = \text{stresslet exerted}$$

Note the terms involving $\boldsymbol{u}$ in the integrand for S vanish for a rigid body.

1. The far-field of a moving body is related to the force exerted by the body, not its velocity.

2. The far-field of a force-free, couple-free, incompressible body is a stresslet.

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2.6.3 Representation for droplets

Consider a droplet occupying a volume V_1 with viscosity $\lambda\mu$ in a surrounding fluid volume V_2 with viscosity μ. Let $\boldsymbol{n}$ be *out* of the droplet. Use viscosity μ in the Stokeslet tensor J. We assume for simplicity that there is no body force, $\boldsymbol{f} = 0$, and the far field is at rest, $\boldsymbol{u} \to 0$ as $r \to \infty$.

Applying the basic integral identity to each of V_1 and V_2 we have

$$\int_{\partial V} \frac{1}{\lambda} J \cdot \boldsymbol{\sigma_1} \cdot \boldsymbol{n} + \boldsymbol{u}_1 \cdot K \cdot \boldsymbol{n} \, dS = \begin{cases} \boldsymbol{u}_1 & \boldsymbol{y} \in V_1 \\ 0 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}\boldsymbol{u}_1 & \boldsymbol{y} \in \partial V \end{cases} \tag{3}$$

$$-\int_{\partial V} \frac{1}{\lambda} J \cdot \boldsymbol{\sigma_2} \cdot \boldsymbol{n} + \boldsymbol{u}_2 \cdot K \cdot \boldsymbol{n} \, dS = \begin{cases} \boldsymbol{u}_2 & 0 \in V_1 \\ \boldsymbol{u}_2 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}\boldsymbol{u}_2 & \boldsymbol{y} \in \partial V \end{cases} \tag{4}$$

Note the $-$ sign in (4) is because $\boldsymbol{n}$ points into V_2. Now $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$ (say) on ∂V. So $\lambda(4) + (3)$ implies

$$\int_{\partial V} J(\boldsymbol{y} - \boldsymbol{x}) \cdot (\boldsymbol{\sigma_1} \cdot \boldsymbol{n} - \boldsymbol{\sigma_2} \cdot \boldsymbol{n}) \, dS_x + (\lambda - 1) \int_{\partial V} \boldsymbol{u} \cdot K(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{n} \, dS_x = \begin{cases} \lambda \boldsymbol{u}_1 & \boldsymbol{y} \in V_1 \\ \boldsymbol{u}_2 & \boldsymbol{y} \in V_2 \\ \frac{1}{2}(\lambda + 1)\boldsymbol{u} & \boldsymbol{y} \in \partial V \end{cases}$$

This integral equation is the basis for powerful numerical methods to determine droplet shapes and evolution.

Example. Relaxation of a drop under constant surface tension. The flow is driven by $\boldsymbol{\sigma_2} \cdot \boldsymbol{n} - \boldsymbol{\sigma_1} \cdot \boldsymbol{n} = \gamma(\nabla \cdot \boldsymbol{n})\boldsymbol{n}$. We consider $\boldsymbol{y} \in \partial V$ to find the interfacial motion:

1. $\lambda = 1$: we have

$$\boldsymbol{u}(\boldsymbol{y}) = -\gamma \int_{\partial V} (\nabla \cdot \boldsymbol{n})\boldsymbol{n} \cdot J(\boldsymbol{y} - \boldsymbol{x}) \, dS_x$$

   hence the interfacial velocity is determined directly by the current shape of the droplet. It requires only an integral of the forcing over the boundary (a 'membrane' of Stokeslets) to determine $\boldsymbol{u}$.

2. $\lambda \neq 1$: to solve the integral equation

$$\frac{\lambda + 1}{2}\boldsymbol{u}(\boldsymbol{y}) = -\gamma \int_{\partial V} (\nabla \cdot \boldsymbol{n})\boldsymbol{n} \cdot J \, dS_x + (\lambda - 1) \int_{\partial V} \boldsymbol{u}(\boldsymbol{x}) \cdot K \cdot \boldsymbol{n} \, dS_x$$

   for $\boldsymbol{u}$, we discretize the integrals and invert the resulting matrix equation. The integral equation is singular (so the matrix equation is singular) if $\lambda = 0$ or $\lambda = \infty$. The null eigenmodes are the 6 rigid body motions ($\lambda = \infty$) and bubble enlargement ($\lambda = 0$); the corresponding solubility conditions are the consistency conditions on boundary conditions.

2.6.4 Representation by Stokeslets alone

Consider the flow outside a moving body, e.g. swimming micro-organisms. Though we are interested in the flow $\boldsymbol{u}$ in V, we imagine a flow $\boldsymbol{u}^*$ in V^* with $\mu^* = \mu$ and $\boldsymbol{u}^* = \boldsymbol{u}$ on ∂V. Such a flow exists if $\int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, dS = 0$.

The previous result for drops gives

$$\boldsymbol{u}(\boldsymbol{y}) = \int_{\partial V} \mathsf{J}(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{f}_S(\boldsymbol{x}) \, \mathrm{d}S$$

where $\boldsymbol{f}_S = (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}) \cdot \boldsymbol{n}$ is the Stokeslet density. Note:

1. $\boldsymbol{\sigma}^*$ may not be the real stress in V^* unless the body really is fluid with $\mu = \mu^*$.

2. If the body is rigid then $\boldsymbol{u}^*$ is rigid-body motion and $\boldsymbol{\sigma}^* = \boldsymbol{0}$. In this case $\boldsymbol{f}_S$ is the stress $-\boldsymbol{\sigma} \cdot \boldsymbol{n}$ exerted by the body on the fluid. The external flow depends *only* on $\boldsymbol{\sigma} \cdot \boldsymbol{n}$ and *not* on $\boldsymbol{u}$.

3 Approximations and applications

3.1 Slender-body theory

Consider the case of a solid, perhaps flexible, body e.g. flagellum, or fibre in a flow. Our aim is to calculate resistance to prescribed motion $\boldsymbol{V}(s) = \dot{\boldsymbol{X}}$ in background flow $\boldsymbol{u}^\infty(\boldsymbol{x})$.

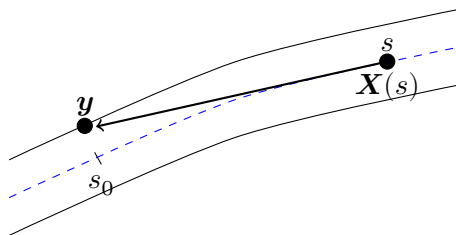Since the body is slender, we can approximate the surface distribution of Stokeslets by a distribution along the centreline, i.e.
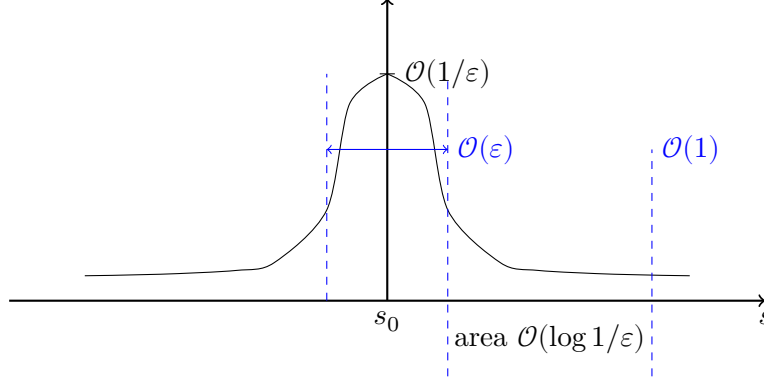
$$\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{u}^\infty(\boldsymbol{y}) + \int_{-L}^{L} \mathsf{J}(\boldsymbol{y} - \boldsymbol{X}(s)) \cdot \boldsymbol{f}(s) \, \mathrm{d}s$$

Note the integral is the first term in a far-field expansion for a ring of Stokeslets. Consider a point $\boldsymbol{y} = \boldsymbol{X}(s_0) + \varepsilon \boldsymbol{R}(s_0, \theta)$ on the surface of the body.

The integrand is proportional to $\dfrac{1}{|\boldsymbol{y}-\boldsymbol{X}(s)|}$ and behaves like so:

We see the integral is dominated by the contribution from $\varepsilon R \ll |s - s_0| \ll L$, where

$$\boldsymbol{f}(s) \approx \boldsymbol{f}(s_0)$$
$$\boldsymbol{y} - \boldsymbol{X}(s) \approx \boldsymbol{X}'(s_0)(s_0 - s)$$

Hence we have

$$\boldsymbol{V}(s_0) = \boldsymbol{u}^\infty(\boldsymbol{X}(s_0)) + \frac{1}{8\pi\mu}\left(I + \boldsymbol{X}'(s_0)\boldsymbol{X}'(s_0)\right) \cdot \boldsymbol{f}(s_0)\left[\int_L^{s_0 - \mathcal{O}(\varepsilon)} + \int_{s_0 + \mathcal{O}(\varepsilon)}^L \frac{\mathrm{d}s}{|s - s_0|}\right]$$

The integral $[\cdot] \sim 2\log\frac{1}{\varepsilon} + \mathcal{O}(1)$ and since $|\boldsymbol{X}'| = 1$,

$$[I + \boldsymbol{X}'\boldsymbol{X}']^{-1} = I - \frac{1}{2}\boldsymbol{X}'\boldsymbol{X}'$$

Thus we have the *leading-order slender-body approximation*

$$\boldsymbol{f}(s_0) \approx \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\left(I - \frac{1}{2}\boldsymbol{X}'\boldsymbol{X}'\right) \cdot (\boldsymbol{V}(s_0) - \boldsymbol{u}^\infty(\boldsymbol{X}(s_0)))$$

Note the following:

1. At this order, there is no dependence on the detailed cross-section, and resistance is local.

2. $\mathcal{O}\left((\log\frac{1}{3})^{-2}\right)$ corrections are not much smaller

3. There exists an ad hoc generalisation, called *resistive force theory*, of the form

   $$\boldsymbol{f} = \left[k_\perp I + (k_\parallel - k_\perp)\boldsymbol{X}'\boldsymbol{X}'\right] \cdot [\boldsymbol{u} - \boldsymbol{u}^\infty]$$
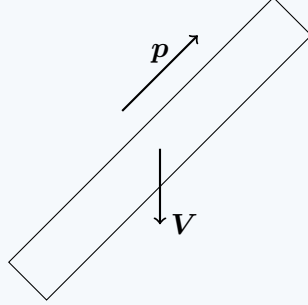
   where

   $$k_\perp = \frac{4\pi\mu}{\log\frac{1}{\varepsilon} + c_1}, \qquad k_\parallel = \frac{2\pi\mu}{\log\frac{1}{\varepsilon} + c_2}$$

   This generalisation is still local.

4. A real improvement of accuracy, to $\mathcal{O}(\varepsilon^2 \log\frac{1}{\varepsilon})$, comes from numerical solution of the 1D integral equation, which includes non-local effects.

Example. Translation of a rigid straight rod with velocity $\boldsymbol{V}(s)$, centre-line tangent vector $\boldsymbol{X}'(s) = \boldsymbol{p}$ both constant, and no background flow $\boldsymbol{u}^\infty = \boldsymbol{0}$.

If the rod falls perpendicular to its velocity (broadside) then $\boldsymbol{p} \cdot \boldsymbol{V} = 0$ so

$$\boldsymbol{f} = \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\boldsymbol{V}, \qquad \boldsymbol{F} = \frac{8\pi\mu L}{\log\frac{1}{\varepsilon}}\boldsymbol{V}$$

If the rod falls parallel to its velocity (lengthwise), we have $(\boldsymbol{p} \cdot \boldsymbol{V})\boldsymbol{p} = \boldsymbol{V}$ so

$$\boldsymbol{f} = \frac{2\pi\mu}{\log\frac{1}{\varepsilon}}\boldsymbol{V}, \qquad \boldsymbol{F} = \frac{4\pi\mu L}{\log\frac{1}{\varepsilon}}\boldsymbol{V}$$

1. The rod falls lengthwise only twice as fast as broadside.

2. The drag $\boldsymbol{F}$ scales with L, not εR, apart from $\log\frac{1}{\varepsilon}$, c.f. $\boldsymbol{F} = 6\pi\mu a\boldsymbol{U}$ for a sphere.

3. Finite length more important than inertia if $\varepsilon = \frac{a}{L} \gg \mathrm{Re}$.

Example. Swimming flagellum.

Suppose the centreline is
$$\boldsymbol{X}(s) = (-Ut + s, a\cos(ks - \omega t))$$
i.e. a linearised wave with $ak \ll 1$. Then

$$\boldsymbol{V}(s) = (-U, a\omega\sin(ks - \omega t))$$
$$\boldsymbol{X}'(s) = (1, -ak\sin(ks - \omega t))$$

Write $S = \sin(ks - \omega t)$. Slender-body theory gives

$$\boldsymbol{f} = \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\begin{pmatrix} 1 - \frac{1}{2} & \frac{1}{2}akS \\ \frac{1}{2}akS & 1 - \frac{1}{2}(akS)^2 \end{pmatrix}\begin{pmatrix} -U \\ a\omega S \end{pmatrix}$$

$$= \frac{4\pi\mu}{\log\frac{1}{\varepsilon}}\begin{pmatrix} -\frac{1}{2}U + \frac{1}{2}a^2k\omega S^2 \\ -\frac{1}{2}akUS + a\omega S \end{pmatrix}$$

> Time averaging $\boldsymbol{f}$ gives
>
> $$\langle \boldsymbol{f} \rangle = \frac{2\pi\mu}{\log\frac{1}{\varepsilon}} \left(-U + \tfrac{1}{2}a^2 k\omega \right)$$
>
> There is no net force on the body, therefore $\langle \boldsymbol{f} \rangle = 0$ which implies $U = \tfrac{1}{2}a^2 k\omega$.

Lecture 11 2/11/20

3.2 Marangoni Flows

The surface tension γ between two immiscible fluids depends on the fluids properties;

- temperature - γ decreases as T increases. For water, $\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}T} \sim -\frac{1}{50K}$. For example, *Marangoni convection*. Heat applied to a fluid with surface tension generates circulation:

- concentration of surfactants (surface active agents) - usually γ decreases as the concentration C increases. For example, detergent molecules spread over a surface. Detergent molecules have hydrophobic tails and hydrophilic heads.
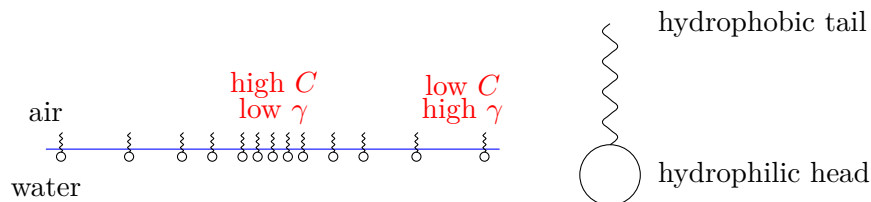
Another example is alcohol in water; 'wine tears' creep up the side to form a rim on the meniscus due to high surface tension due to evaporating alcohol decreasing concentration.

3.2.1 Boundary conditions

Surface tension can be represented by a surface stress (in $N \cdot m^{-1}$, derived from a surface energy in $J \cdot m^{-2}$) acting isotropically across lines in the surface with stress tensor

$$\sigma^s = \gamma\left(I - \boldsymbol{nn}\right)$$

The force balance on an arbitrary small area of the interface gives

$$\nabla_s \cdot \sigma^s + \left[\sigma \cdot \boldsymbol{n}\right]_-^+ = 0$$

where $\boldsymbol{n}$ points out of the fluid and $\nabla_S = (I - \boldsymbol{nn}) \cdot \nabla$ is the gradient operator in the surface. Hence the jump in fluid stress is

$$[\boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = -\nabla_s \cdot \gamma (I - \boldsymbol{nn})$$
$$= -(I - \boldsymbol{nn}) \cdot \nabla_s \gamma + \gamma (\nabla_s \cdot \boldsymbol{n}) \boldsymbol{n} + \gamma \cancel{(\boldsymbol{n} \cdot \nabla_S)} \boldsymbol{n}$$
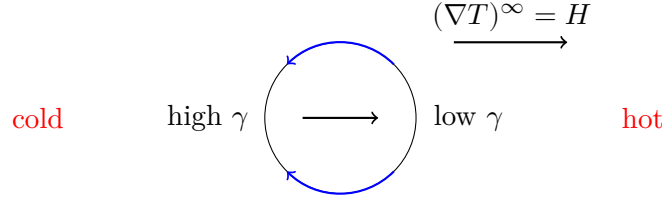$$= -\nabla_s \gamma + \gamma \kappa \boldsymbol{n}$$

The first term is the surface gradient of γ and the second is tension $\times$ curvature, where curvature is denoted by $\kappa \equiv \nabla_s \cdot \boldsymbol{n}$.

Equivalently, in perpendicular and parallel components we have

$$[\boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = \gamma \kappa$$
$$[\boldsymbol{n} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = -\boldsymbol{n} \times \nabla_s \gamma$$

3.2.2 Thermophoresis

An immiscible drop in a temperature gradient migrates from cold regions to hot.

Consider a scaling argument. We have $\nabla T \sim Ha \implies \nabla \gamma \sim Ha\gamma'$ where $\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}T} < 0$. The driving force F from $\nabla \gamma$ acting across the equator length $\sim a$ implies $F \sim Ha^2 \gamma'$. Now $\sigma \sim \frac{\mu U}{a}$ acts over an area $\sim a^2$ to give viscous resistive force $\sim \mu Ua$ (c.f. drag on a sphere). Hence

$$U \sim \frac{\gamma' Ha}{\mu}$$

Typical values are $H \sim 1°C/cm, \gamma' \sim 1\mathrm{dyn}/cm/°C, a \sim 10^{-2} cm, \rho \sim 1 g \cdot cm^{-3}, \nu \sim 10^{-2} cm$. Therefore $U \sim 1 cm \cdot s^{-1}$.

Thermal problem.   Consider the problem

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T$$

with boundary conditions $\nabla T \to \boldsymbol{H}$ as $|\boldsymbol{x}| \to \infty$ and

$$[T]_-^+ = [k\boldsymbol{n} \cdot \nabla T]_-^+ = 0 \qquad \text{on } |\boldsymbol{x}| = a$$

where k is the conductivity ($\boldsymbol{q} = -k\nabla T$) and $\kappa = \frac{k}{\rho C_p}$ is the thermal diffusivity. Assume the *Peclet* number satisfies

$$\mathrm{Pe} = \frac{Ua}{\kappa} = \frac{\text{advection}}{\text{diffusion}} \ll 1$$

For Pe $\ll 1$, the problem reduces to solving $\nabla^2 T = 0$. Then, for example, with an insulating drop ($k_1 \ll k_2$)

$$T = T_0 + \boldsymbol{H} \cdot \boldsymbol{x} \left(1 + \frac{a^3}{2r^3}\right)$$

and on $r = a$, $T = T_0 + \frac{3}{2}\boldsymbol{H} \cdot \boldsymbol{x}$. Hence

$$\gamma(T) = \gamma_0 + \gamma'(T - T_0) = \gamma_0 + \frac{3}{2}\gamma'\boldsymbol{H} \cdot \boldsymbol{x}$$

Fluid problem. The flow is driven by an interfacial force density

$$
\begin{aligned}
\boldsymbol{f}_s &= \sigma_1 \cdot \boldsymbol{n} - \sigma_2 \cdot \boldsymbol{n} \\
&= \nabla_s\gamma - \gamma\boldsymbol{n}\nabla_s \cdot \boldsymbol{n} \\
&= (I - \boldsymbol{n}\boldsymbol{n}) \cdot \frac{3}{2}\gamma'\boldsymbol{H} - \frac{2}{a}\left(\gamma_0 + \frac{3}{2}\gamma'\boldsymbol{H} \cdot \boldsymbol{x}\right)\boldsymbol{n} \\
&= -\frac{2}{a}\gamma_0\boldsymbol{n} + \frac{3}{2}\gamma'\boldsymbol{H}(I - 3\boldsymbol{n}\boldsymbol{n})
\end{aligned}
$$

where the second term drives the motion. For the simple case of equal viscosities, $\lambda = 1$,

$$
\begin{aligned}
\boldsymbol{u}(\boldsymbol{y}) &= \int_{r=a} \mathsf{J}(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{f}_s(\boldsymbol{x}) \cdot \mathrm{d}S \\
&= \frac{3\gamma'}{16\pi\mu}\boldsymbol{H} \cdot \int (I - 3\boldsymbol{n}\boldsymbol{n}) \cdot \left(\frac{I}{R} + \frac{\boldsymbol{R}\boldsymbol{R}}{R^3}\right)\mathrm{d}S \quad \text{where } \boldsymbol{R} = \boldsymbol{y} - \boldsymbol{x} \\
&= \frac{3\gamma'a}{16\pi\mu}\boldsymbol{H} \cdot \mathsf{G}(\boldsymbol{y}/a)
\end{aligned}
$$

where, by symmetry of the integral and dimensions,

$$\mathsf{G} = \alpha(r/a)I + \beta(r/a)\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$$

and $\hat{\boldsymbol{y}} = \boldsymbol{y}/a$. To determine the functions α, β one can evaluate $G_{ii} = 3\alph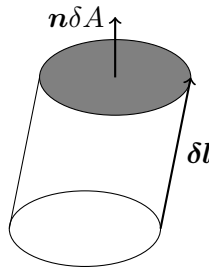a + \beta$ and $\hat{y}_i G_{ij}\hat{y}_j = \alpha + \beta$. Now $\boldsymbol{u} \cdot \hat{\boldsymbol{y}} = \frac{3\gamma'a}{16\pi\mu}(\alpha + \beta)\boldsymbol{H} \cdot \hat{\boldsymbol{y}}$ which shows the spherical drop remains spherical and translates with velocity

$$\boldsymbol{U} = \frac{3\gamma'a}{16\pi\mu}\left(\alpha(1) + \beta(1)\right) = -\frac{1}{5}\frac{a}{\mu}\gamma\boldsymbol{H}$$

3.3 Surfactant effects

The concentration C in a small material area of interface δ changes due to

1. surface diffusion with flux $-D_s\nabla_s C$, where $D_s \sim 10^{-9}m^2s^{-1}$ typically

2. absorbtion from the bulk fluid, if soluble, with rate $-k(C - C_0)$ from chemical equilibrium and (linearised) kinetics $k \sim 10^3 s^{-1}$

3. change of area $\Delta(\delta A\, C) = 0 \implies \Delta C = -\frac{C}{\delta A}\Delta\delta A$. To get material $\frac{\mathrm{d}}{\mathrm{d}t}\delta A$, consider a material volume

Now $\delta V = \boldsymbol{\delta l} \cdot \boldsymbol{n} \delta A$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta V = \frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{\delta l} \right] \cdot \boldsymbol{n} \delta A + \boldsymbol{\delta l} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{n} \delta A \right]$$

We have $\frac{\mathrm{d}}{\mathrm{d}t} \delta V = \delta V \nabla \cdot \boldsymbol{u}$ and $\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\delta l} = (\boldsymbol{\delta l} \cdot \nabla)\boldsymbol{u}$ which, since $\boldsymbol{\delta l}$ is arbitrary, gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\boldsymbol{n} \delta A \right] = (I \nabla \cdot \boldsymbol{u} - \nabla \boldsymbol{u}) \cdot \boldsymbol{n} \delta A$$

Taking the dot product with $\boldsymbol{n}$ and noting $\boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \delta A &= \left(\nabla \cdot \boldsymbol{u} - \boldsymbol{n}\boldsymbol{n} : \nabla \boldsymbol{u} \right) \delta A \\
&= \nabla_s \cdot \boldsymbol{u} \delta A \\
&= \delta A \nabla_s \cdot \left[\boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \right] \\
&= \delta A \left[\nabla_s \cdot \boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\nabla_s \cdot \boldsymbol{n} + \underline{(\boldsymbol{n} \cdot \cancel{\nabla_s})\cancel{(\boldsymbol{u} \cdot \boldsymbol{n})}} \right]
\end{aligned}$$

Hence we have the *transport equation*

$$\frac{\mathrm{D}C}{\mathrm{D}t} = -C \left[\nabla_s \cdot \boldsymbol{u}_s + (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} \right] + D_s \nabla_s^2 C - k(C - C_0)$$

Example.

1. *Chemophoresis* – migration in a background concentration gradient ∇C^∞

2. *Rigidifcation of interfaces* – for example flow past a bubble.
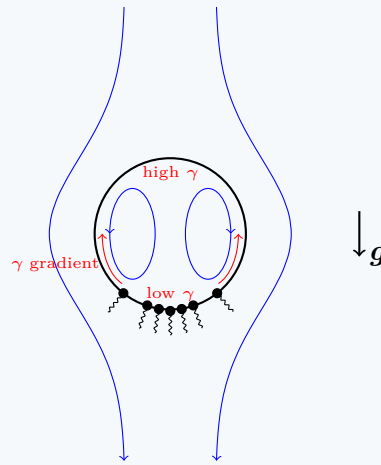
Flow produces a concentration gradient ∇C, which produces a surface tension gradient $\nabla \gamma$, which produces an opposing flow. In practice, a small contamination has a big effect.

A steady flow in the frame of the bubble has $\frac{\partial C}{\partial t} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. Consider a linearised calculation with $C = C_0 + C'(x)$ where $C' \ll C_0$. Then the transport equation reduces to

$$D_s \nabla^2 C' - kC' = C_0 \nabla_s \cdot \boldsymbol{u}_s$$

where the non-linear terms $C' \nabla_s \cdot \boldsymbol{u}_s$ and $\boldsymbol{u}_s \cdot \nabla_s C'$ have been neglected. Note k vanishes if the fluid is insoluble. This is a linear problem in the rise velocity $\boldsymbol{U}$ and from spherical symmetry we deduce

$$\boldsymbol{u}_s = A \left(I - \boldsymbol{n}\boldsymbol{n} \right) \cdot \boldsymbol{U}$$

for some constant A to be determined. To calculate $\nabla_s \cdot \boldsymbol{u}_s$, note

$$\nabla_s \boldsymbol{n} = (I - \boldsymbol{nn}) \cdot \nabla \frac{\boldsymbol{x}}{a} = \frac{1}{a}(I - \boldsymbol{nn})$$

$$\nabla_s \cdot \boldsymbol{n} = \frac{2}{a}$$

$$\implies \nabla_s \cdot \boldsymbol{u}_s = -A\left[(\nabla_s \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{n} \cdot \nabla_s)\boldsymbol{n}\right] \cdot \boldsymbol{U} = -\frac{2}{a}A\boldsymbol{U} \cdot \boldsymbol{n}$$

using the curvature $\kappa = \frac{2}{a}$ for a sphere of radius a. Guessing $C' \propto \boldsymbol{U} \cdot \boldsymbol{n}$, and since

$$\nabla_s^2(\boldsymbol{U} \cdot \boldsymbol{n}) = \nabla_s \cdot \left[\frac{1}{a}(I - \boldsymbol{nn}) \cdot \boldsymbol{U}\right] = -\frac{2}{a^2}\boldsymbol{U} \cdot \boldsymbol{n}$$

we find

$$C' = \frac{2C_0 A\frac{\boldsymbol{U} \cdot \boldsymbol{n}}{a}}{k + 2\frac{D_s}{a^2}}$$

Therefore the linearisation $C' \ll C_0$ is valid if $\frac{UA}{D_s} \ll 1$ (fast diffusion) or $\frac{U}{ka} \ll 1$ (fast adsorption). The problem is completed by using a Papkovich-Neuber representation inside and outside the droplet with boundary conditions

$$\boldsymbol{u} \to -\boldsymbol{U} \text{ as } r \to \infty$$

$$\boldsymbol{u} \text{ finite at } r = 0$$

$$\boldsymbol{u} = \boldsymbol{u}_s \text{ at } r = a_-^+$$

$$[\boldsymbol{n} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}]_-^+ = -\boldsymbol{n} \times \nabla C' \frac{\mathrm{d}\gamma}{\mathrm{d}C} \text{ at } r = a$$

These are 5 conditions for 4 potentials and A. We find the drag on the bubble is

$$\boldsymbol{F} = -4\pi\mu a \boldsymbol{U} \left(\frac{\frac{3}{2}(\lambda^* + \lambda) + 1}{\lambda^* + \lambda + 1}\right)$$

where

$$\lambda^* = -\frac{\frac{1}{3}aC_0\frac{\mathrm{d}\gamma}{\mathrm{d}C}}{(a^2 k + 2D_s)\mu}$$

Note as $\lambda^* \to \infty$ we get a 'rigid' sphere.

3.4 Deformation of droplets.

To leading order, a small force-free droplet moves with the flow $\boldsymbol{u}^\infty$ and is deformed by the local gradient $(\boldsymbol{x} \cdot \nabla)\boldsymbol{u}^\infty$ against the restoring action of surface tension. We assume the background flow is composed of a straining component with symmetric traceless E and a rotational component with angular velocity $\boldsymbol{\Omega}$. Hence $\boldsymbol{u} \to \boldsymbol{\Omega} \times \boldsymbol{x} + E \cdot \boldsymbol{x}$ as $|\boldsymbol{x}| \to \infty$.
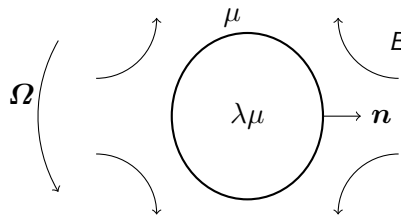
The importance of viscous stress vs. surface tension is characterised by the *capillary number* Ca where

$$\frac{\text{viscous stresses}}{\text{surface tension}} \sim \frac{\mu E}{\gamma/a} = \text{Ca} = \frac{\mu U}{\gamma}$$

3.4.1 Small deformations

Here we consider Ca $\ll$ 1 and a droplet with small deformation in a pure strain flow $\boldsymbol{u}^\infty = E \cdot \boldsymbol{x}$. Deformation is due to symmetric traceless E. Hence for small (linearised) deformations we expect

$$r = a \left[1 + \frac{\boldsymbol{x} \cdot D \cdot \boldsymbol{x}}{r^2} \right]$$

where D is symmetric, traceless (from volume conservation), and $\mathcal{O}(\text{Ca})$. The surface normal is then

$$\boldsymbol{n} = \nabla \left(r - a \left[1 + \frac{\boldsymbol{x} \cdot D \cdot \boldsymbol{x}}{r^2} \right] \right)$$
$$= \frac{\boldsymbol{x}}{r} - 2a \left(\frac{D \cdot \boldsymbol{x}}{r^2} - \frac{(\boldsymbol{x} \cdot D \cdot \boldsymbol{x})\boldsymbol{x}}{r^4} \right) + \mathcal{O}(D^2)$$

Note $|\boldsymbol{n}| = 1$ to $\mathcal{O}(D^2)$ as required. Also, $|\boldsymbol{n}| = 1$ implies $\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla)\boldsymbol{n} = 0$. Hence

$$\nabla_s \cdot \boldsymbol{n} = \nabla \cdot \boldsymbol{n}$$
$$= \frac{2}{r} + 6a \left. \frac{\boldsymbol{x} \cdot D \cdot \boldsymbol{x}}{r^4} \right|_s$$
$$= \frac{2}{a} + 4\frac{\boldsymbol{x} \cdot D \cdot \boldsymbol{x}}{a^3} + \mathcal{O}(D^2)$$