Cambridge Part III Maths

Michaelmas 2020

Fluid Dynamics of Climate

based on a course given by
John Taylor & Peter Haynes

written up by
Charles Powell

Notes created using Josh Kirklin's packages & classes. Please send errors and suggestions to cwp29@cam.ac.uk.

Contents

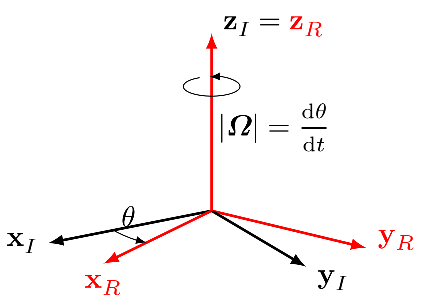
Lecture 1
12/10/20

1 Fluid motion in a rotating reference frame

In a non-rotating frame, the *Navier-Stokes* equations are

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p - \rho\nabla\phi + \rho\boldsymbol{F}$$

The body forces are assumed to be conservative with potential ϕ, e.g. $\phi = gz$ for gravitational force. $\boldsymbol{F}$ is the frictional force.
Consider a reference frame rotating about the z-axis with constant angular velocity $\boldsymbol{\Omega}$. Axes in the inertial frame are denoted with a subscript $_I$ and axes in the rotating frame are denoted with a subscript $_R$.



For a point with position vector $\boldsymbol{x}$ and velocity $\boldsymbol{u}_R = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R$ in the rotating reference frame

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{x}$$

or equivalently $\boldsymbol{u}_I = \boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}$. Hence the acceleration is

$$\begin{aligned}\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}\right]\right)_R + \boldsymbol{\Omega}\times\left(\boldsymbol{u}_R + \boldsymbol{\Omega}\times\boldsymbol{x}\right)_R \\ &= \left(\frac{\mathrm{d}\boldsymbol{u}_R}{\mathrm{d}t}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{u}_R + \boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{x}\right)\end{aligned}$$

The first term is the acceleration in the rotating frame, the second term is the *Coriolis acceleration* and the third term is the *centrifugal accelertion*. Note that we can write the centrifugal acceleration in the form of a conservative force

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) = \nabla\phi_c$$
$$\phi_c = -\frac{1}{2}\left|\boldsymbol{\Omega}\times\boldsymbol{x}\right|^2$$

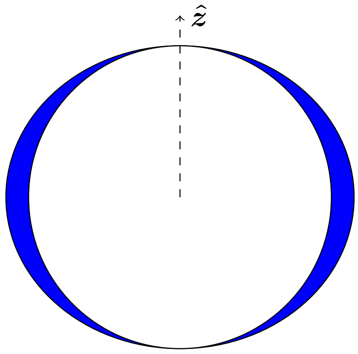


Figure 1: Geopotential ocean surface relative to a spherical Earth.

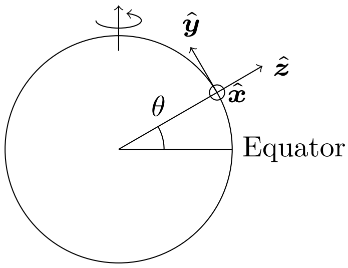


Figure 2: Local Cartesian coordinates

Hence the Navier-Stokes equations in a rotating reference frame are

$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)=-\nabla p-\rho\nabla\left(\phi+\phi_c\right)+\rho\boldsymbol{F} \tag{1}$$

We group the potential terms into a *geopotential* $\Phi\equiv\phi+\phi_c$. The surface of a stationary ocean or atmosphere has a constant *geopotential height* described by an oblate spheroid.
Imagine a spherical earth. At sea level, the polar radius is 21.4km smaller than the equatorial radius: see figure 1. In reality, the surface of the Earth is also very close to a geopotential surface. Hence *geopotential coordinates* are very useful for planetary scale motion.

1.1 Local Cartesian coordinates

For small motions, it is much more convenient to define *local Cartesian coordinates* (figure 2).
In this coordinate system $\boldsymbol{\Omega}=(0,\Omega\cos\theta,\Omega\sin\theta)$. Hence if $\boldsymbol{u}=(u,v,w)$ then

$$\begin{aligned}2\boldsymbol{\Omega}\times\boldsymbol{u}&=(2\Omega w\cos\theta-2\Omega v\sin\theta,2\Omega u\sin\theta,-2\Omega u\cos\theta)\\&=(-fv+f^*w,fu-f^*u)\end{aligned}$$

where $f\equiv2\Omega\sin\theta$ is the *Coriolis parameter* and $f^*\equiv2\Omega\cos\theta$.

Example. In Cambridge, $\theta=52.1^\circ N$ so

$$\begin{aligned}f&=2\Omega\sin\theta\\&=2\cdot\frac{2\pi}{3600\cdot24}\cdot0.79s^{-1}\\&\approx1.14\times10^{-4}s^{-1}\end{aligned}$$

At mid-latitudes, $f \sim 10^{-4}$ is a good approximation.

We can simplify the Coriolis acceleration expression; often $f^* w \ll fv$ and $f^* u \ll g$. Hence

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \approx (-fv, fu, 0) = f\hat{\boldsymbol{z}} \times \boldsymbol{u}$$

This is the *traditional approximation.* This is *not* always a good approximation, particularly at intermediate scales.

1.2 Scale analysis.

Define characteristic scales for length L, time T, and velocity U. Non-dimensional variables are denoted with a superscript star: $\boldsymbol{u}^* = \boldsymbol{u}/U$, etc.
Using these scalings with $\boldsymbol{F} = \nu \nabla^2 \boldsymbol{u}$ we have

$$\frac{U}{T}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \frac{U^2}{L}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + fU\hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho}\nabla\left(p + \rho\Phi\right) + \frac{\nu U}{L^2}\nabla_*^2 \boldsymbol{u}^*$$

Dividing through by fU leaves the Coriolis acceleration term ord(1) with other terms scaled relatively.

$$\frac{1}{fT}\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \mathrm{Ro}\boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* + \hat{\boldsymbol{z}} \times \boldsymbol{u}^* = -\frac{1}{\rho f U}\nabla\left(p + \rho\Phi\right) + \mathrm{E}\nabla_*^2 \boldsymbol{u}^*$$

where $\mathrm{Ro} \equiv \frac{U}{fL}$ is the *Rossby number* and $\mathrm{E} \equiv \frac{\nu}{fL^2}$ is the *Ekman number.*

Example. For an atmospheric storm, $U \sim 10ms^{-1}, L \sim 1000km, f \sim 10^{-4}s^{-1}$. Thus $\mathrm{Ro} \sim 0.1, \mathrm{E} \sim 10^{-13}$.

Lecture 2
14/10/2020

Further, if $T = L/U$, then $\mathrm{Ro} = U/fL = 1/fT$. For small Ro, E, on surfaces of constant Φ, $f\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla p$. This is *geostrophic balance.* In components, we have

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The equations of geostrophic balance can be arranged to give the horizontal velocity: $\boldsymbol{u}_H$

$$\boldsymbol{u}_H \equiv (u, v) = \frac{1}{\rho f}\hat{\boldsymbol{z}} \times \nabla p$$

Horizontal velocity is perpendicular to ∇p and hence parallel to isobars (lines of constant p), i.e. pressure acts like a streamfunction (see figure 3).
In the Northern Hemisphere, air moves clockwise around high p and anticlockwise around low p. A *cyclonic* rotation is in the same sense as $\boldsymbol{\Omega}$, *anticyclonic* in the opposite sense as $\boldsymbol{\Omega}$.

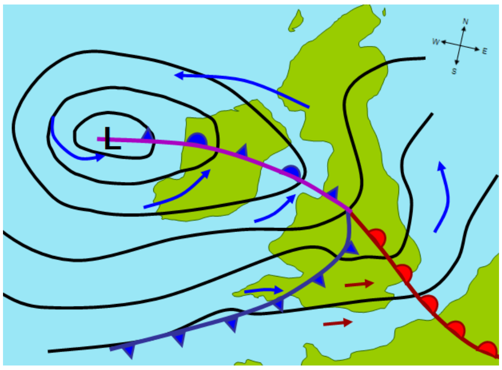


Figure 3: Lines of constant pressure p act as streamlines for the horizontal flow.

.

1.3 Taylor-Proudman Theorem

Consider an incompressible, ideal fluid in geostrophic balance (small Ro, E)

$$\nabla \cdot \boldsymbol{u} = 0$$
$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p \tag{2}$$

Taking the curl of (2) we have

$$\begin{aligned}
\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) &= \varepsilon_{ijk}\partial_j \varepsilon_{klm}\Omega_l u_m \\
&= \varepsilon_{kij}\varepsilon_{klm}\Omega_l \partial_j u_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\Omega_l \partial_j u_m \\
&= \Omega_i \partial_j u_j - \Omega_j \partial_j u_i
\end{aligned}$$

The first term is 0 by incompressibility. Thus

$$-\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0$$

For $\boldsymbol{\Omega} = (0, 0, \Omega)$, this implies $\frac{\partial w}{\partial z} = 0$. If $w = 0$ on some horizontal surface (e.g. ground) then $w = 0$ everywhere.
Also, $u_x + v_y = 0$, i.e. horizontal velocity is non-divergent in geostrophic balance. Fluid moves in 'columns' parallel to $\boldsymbol{\Omega}$, called *Taylor columns*.

2 Departures from geostrophy

Consider an incompressible, rotating fluid with constant density ρ_0 with angular velocity $\boldsymbol{\Omega} = (0, 0, f/2)$. Assume small ampltiude motions (i.e. $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$), i.e. neglect $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nu \nabla^2 \boldsymbol{u}$. From (1),

$$u_t - fv = -\frac{p_x}{\rho_0} \tag{3}$$
$$v_t + fu = -\frac{p_y}{\rho_0} \tag{4}$$
$$w_t = -\frac{p_z}{\rho_0} \tag{5}$$
$$u_x + v_y + w_z = 0 \tag{6}$$

We will eliminate variables in favour of p.

$$\begin{aligned} \nabla \cdot ((3) - (5)) &\implies \nabla^2 p = \rho_0 f \left(v_x - u_y\right) \\ \partial_x (4) - \partial_y (3) \& (6) &\implies (v_x - u_y)_t = f w_z \end{aligned}$$

Combining these and using (5) we have

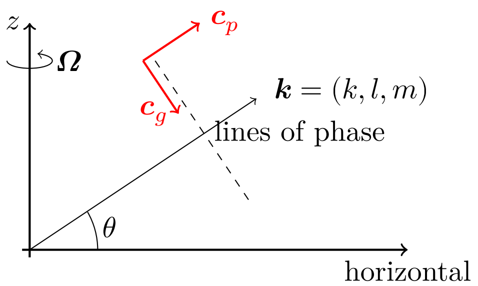
$$\nabla^2 p_{tt} + f^2 p_{zz} = 0$$

which is a wave equation for p. Seek plane wave solutions with ansatz

$$p = \hat{p} e^{i(kx + ly + mz - \omega t)}$$

and dispersion relation

$$\omega^2 = \frac{f^2 m^2}{k^2 + l^2 + m^2} = f^2 \sin^2 \theta$$



This is the dispersion relation for rotating internal waves. They have phase speed $\boldsymbol{c}_p = w / \boldsymbol{k}$ and group velocity

$$\boldsymbol{c}_g = \frac{\partial w}{\partial \boldsymbol{k}} = \pm f \frac{(-km, -lm, k^2 + l^2)}{|\boldsymbol{k}|^{3/2}}$$

Lecture 3 16/10/2020

Note that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$. Also note $|\omega| \leq |f|$.

2.1 Inertial (free) oscillations

Assume $\nabla p = \mathbf{0}$. The x and y components of geostrophic balance (3), (4) give

$$u_{tt} + f^2 u = 0$$

Thus $u = U \sin ft$ where f is the *inertial frequency*. Similarly, we have $v = U \cos ft$. For a particle with position (x_p, y_p) floating on an ocean surface $z = 0$ moving with the fluid velocity, we have

$$\begin{aligned} \frac{\mathrm{d}x_p}{\mathrm{d}t} = u &\implies x_p = -\frac{U}{f} \cos ft + x_0 \\ \frac{\mathrm{d}y_p}{\mathrm{d}t} = v &\implies y_p = -\frac{U}{f} \sin ft + y_0 \end{aligned}$$

Thus the motion of fluid particles describes describes *inertial circles* with radius $\frac{2U}{f}$.

2.2 Ekman layer

Look for a *steady* ocean response to a constant wind stress $\boldsymbol{\tau}_w$. Use local Cartesian coordinates and make the following assumptions:

1. Steady, i.e. $\partial_t \equiv 0$
2. Neglect horizontal variations, i.e. $\partial_x = \partial_y = 0$
3. Neglect surface waves, i.e. $w(z=0) = 0$
4. No flow in deep ocean, i.e. $\lim_{z\to-\infty} \boldsymbol{u} = \boldsymbol{0}$
5. Constant density ρ
6. Traditional approximation

Continuity (incompressibility) says $u_x + v_y + w_z = 0$. Assumptions 2 and 3 then imply $w = 0$ everywhere. The horizontal momentum equations are

$$-fv = \nu u_{zz} \tag{7}$$
$$fu = \nu v_{zz} \tag{8}$$

Define the *complex velocity* $\mathcal{V} \equiv u + iv$. Then

$$\mathcal{V}_{zz} = \frac{if}{\nu}\mathcal{V} \tag{9}$$

Without loss of generality, assume $\boldsymbol{\tau}_w$ is aligned with the x-axis: $\boldsymbol{\tau}_w = (\tau_w, 0) = (\rho\nu u_z, 0)$. Boundary conditions for (9) are

$$\mathcal{V}_z = \left(\frac{\tau_w}{\rho\nu}, 0\right) \quad \text{at } z = 0$$
$$\mathcal{V} = (0,0) \quad \text{as } z \to -\infty$$

Thus $\mathcal{V} = Ae^{(1+i)z/\delta}$ where $\delta = \sqrt{\frac{2\nu}{f}}$, $A = \frac{\tau_w\delta(1-i)}{2\rho\nu}$. In terms of the velocity components, we have

$$u = \frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta}\cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$
$$v = -\frac{\tau_w}{\rho\sqrt{\nu f}} e^{z/\delta}\sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)$$

A top view of the ocean shows an *Ekman spiral*: see figure 4.

2.3 Ekman transport

Integrate the horizontal momentum equations (7),(8) to the base of the Ekman layer where $\nu\boldsymbol{u}_z \approx 0$ at $z = -h$. Since $\nu\boldsymbol{u}_z(z=0) = (\tau_w/\rho, 0)$, the *Ekman transport* $\boldsymbol{U}_T$ is

$$U_T \equiv \int_{-h}^{0} u\,\mathrm{d}z = 0$$
$$V_T \equiv \int_{-h}^{0} v\,\mathrm{d}z = -\frac{\tau_w}{\rho f}$$

This is the net transport of fluid in the Ekman layer and is oriented 90° to the right of the applied wind shear stress (in the Northern Hemisphere).

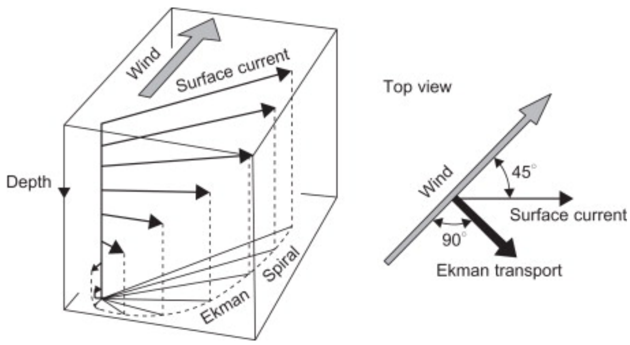


Figure 4: Ekman spiral.

2.4 Ekman pumping

Consider a wind stress $\tau_w(y)$ that varies over large scales. Then from incompressibility

$$\int_{-h}^{0} w_z \, \mathrm{d}z = -\int_{-h}^{0} u_x \, \mathrm{d}z - \int_{-h}^{0} v_y \, \mathrm{d}z$$

Thus for h constant,

$$-w(z=-h) = -\frac{\partial V_T}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\tau_w}{\rho f}\right)$$

In general we have

$$w(z=-h) = \hat{\boldsymbol{z}} \cdot \nabla \times \frac{\boldsymbol{\tau}_w}{\rho f}$$

Lecture 4
19/10/20

3 Rotating shallow water equations

Consider a thin layer of fluid with constant density ρ. Define characteristic scales

- length $L =$ horiz., $H =$ vert.
- velocity U
- time T
- pressure P

such that $\partial_x, \partial_y \sim \frac{1}{L}, \partial_z \sim \frac{1}{H}$. Define the *aspect ratio* $\delta \equiv H/L$. We will assume $\delta \ll 1$. From continuity (incompressibility) we have

$$\begin{aligned} \frac{\partial w}{\partial z} &= -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \implies \frac{w}{H} &= \mathcal{O}(U/L) \\ \implies w &= \mathcal{O}(\delta U) \end{aligned}$$

Using the traditional approximation and assuming the fluid is inviscid, the x-momentum equation

$$\begin{array}{lccccccc} & \dfrac{\partial u}{\partial t} & + u\dfrac{\partial u}{\partial x} & + v\dfrac{\partial u}{\partial y} & + w\dfrac{\partial u}{\partial z} & - fv & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \text{scaling:} & \dfrac{U}{T} & \dfrac{U^2}{L} & \dfrac{U^2}{L} & \dfrac{wU}{H} & fU & = & \dfrac{P}{\rho L} \end{array} \quad (10)$$

Thus if p_x appears at leading order then

$$P \sim \rho U \max(L/T, U, fL)$$

Similarly the z-momentum equation and its scalings are

$$\begin{array}{lcccccc} & \dfrac{\partial w}{\partial t} & +u\dfrac{\partial w}{\partial x} & +v\dfrac{\partial w}{\partial y} & +w\dfrac{\partial w}{\partial z} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} - g \\ \text{scaling:} & \dfrac{w}{T} & \dfrac{Uw}{L} & \dfrac{Uw}{L} & \dfrac{w^2}{H} & = & \dfrac{P}{\rho H} \end{array} \tag{11}$$

Hence $\frac{\mathrm{D}w}{\mathrm{D}t} \sim \max(\frac{w}{T}, \frac{Uw}{L})$. Comparing with the pressure term, we have

$$\begin{aligned} \frac{\frac{\mathrm{D}w}{\mathrm{D}t}}{\frac{1}{\rho}\frac{\partial p}{\partial z}} &\sim \frac{\max(\frac{w}{T}, \frac{Uw}{L})}{\frac{U}{H}\max(\frac{L}{T}, \frac{U}{L}, f)} \\ &\sim \delta^2 \frac{\max(\frac{1}{T}, \frac{U}{L})}{\max(\frac{1}{T}, \frac{U}{L}, f)} \end{aligned}$$

Therefore to $\mathcal{O}(\delta^2)$ we have *hydrostatic balance*. To this order, (11) becomes

$$\frac{\partial p}{\partial z} - \rho g \implies p = \rho g(\eta - z)$$

assuming $p = 0$ at $z = \eta(x, y, t)$. Similarly, we have $\frac{1}{\rho}p_x = g\eta_x$ and $\frac{1}{\rho}p_y = g\eta_y$. Hence horizontal acceleration (i.e. the LHS of (10)) is independent of z. Motivated by this, we *assume* that horizontal velocity is also independent of z. For $Ro \ll 1$, this follows from the Tayor-Proudman theorem. Re-writing (10) with these results we have

$$u_t + uu_x + vu_y - fv = -g\eta_x \tag{12}$$

$$v_t + uv_x + vv_y + fu = -g\eta_y \tag{13}$$

since $u_z = v_z = 0$ by assumption. Integrating the continuity equation gives

$$w = -z(u_x + v_y) + A(x, y, t)$$

where A is to be determined by the boundary conditions. Requiring no normal flow at $z = -H_0 + h_b$ is imposed by $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$ where $\boldsymbol{n} = \nabla(z - h_b)$. Thus

$$-u\frac{\partial h_b}{\partial x} - v\frac{\partial h_b}{\partial y} + w = 0$$

Hence

$$A(x, y, t) = u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$$

The kinematic boundary condition at $z = \eta$ is $\frac{\mathrm{D}\eta}{\mathrm{D}t} = w$ which may be written as

$$\eta_t + u\eta_x + v\eta_y - w = 0$$

where $w = -\eta(u_x + v_y) + u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} + (-H_0 + h_b)(u_x + v_y)$. Combining these boundary conditions gives

$$\eta_t + [(H_0 - h_b + \eta)u]_x + [(H_0 - h_b + \eta)v]_y = 0 \tag{14}$$

If $H \equiv H_0 - h_b + \eta$ is the total depth of the fluid, then since $H_t = \eta_t$,

$$H_t + (uH)_x + (vH)_y = 0 \tag{15}$$

which is a statement of the conservation of volume (equivalently mass, since ρ is constant). Equations (12), (13), and (14) are the *rotating shallow water* (SW) equations.

3.1 Potential vorticity (PV)

Denote the vertical vorticity by $\zeta = v_x - u_y$. Consider $\partial_x(13) - \partial_y(12)$, which gives

$$\zeta_t + u\zeta_x + v\zeta_y + vf_y = -(\zeta + f)(u_x + v_y)$$

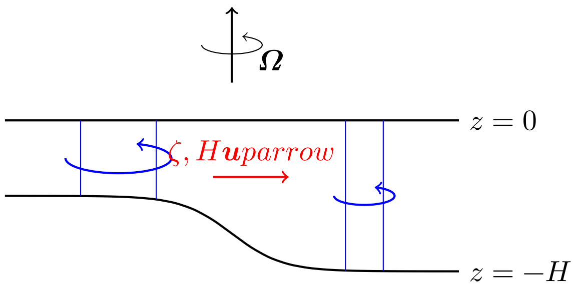
Now from conservation of volume (15),

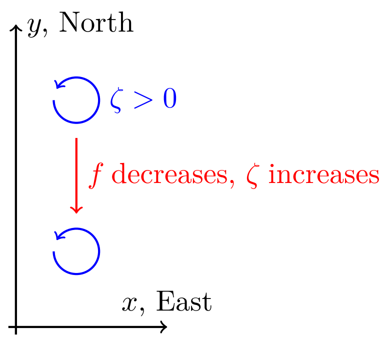
$$u_x + v_y = -\frac{1}{H}\frac{\mathrm{D}H}{\mathrm{D}t}$$

Combining these relates the material derivative of ζ and H by

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} + \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\zeta + f}{H}\frac{\mathrm{D}H}{\mathrm{D}t} \implies \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{16}$$

Let $q \equiv \frac{\zeta+f}{H}$, the *shallow water potential vorticity* (SWPV). SWPV is conserved following fluid motion. We call ζ the *relative vorticity* and f the *planetary vorticity*. ζ and f will change as a fluid moves to conserve SWPV (changing f) and angular momentum (changing depth).





Lecture 5
21/10/20

4 Small amplitude motions in rotating SW

Consider a stationary fluid with depth $H_s(x, y) = H_0 - h_b$. The fluid surface is then perturbed by $\eta(x, y, t)$ where $\eta \ll H_s$. The total depth is $H(x, y, t) = H_s + \eta$. For $|\boldsymbol{u}|^2 \ll |\boldsymbol{u}|$, linearise the shallow water equations:

$$u_t - fv = -g\eta_x \tag{17}$$

$$v_t + fu = -g\eta_y \tag{18}$$

$$\eta_t + (uH_s)_x + (vH_s)_y = 0$$

Assuming f is constant, we have from $\partial_x(17) + \partial_y(18)$ and $\partial_y(17) - \partial_x(18)$:

$$\partial_t\left[\left(\partial_t^2 + f^2\right)\eta - \nabla \cdot (gH_s\nabla\eta)\right] - fgJ(H_s, \eta) = 0 \tag{19}$$

where the Jacobian $J(a, b) = a_x b_y - a_y b_x$. For the velocity components we have

$$\left(\partial_t^2 + f^2\right)u = -g\left(\eta_{xt} + f\eta_y\right) \tag{20}$$

$$\left(\partial_t^2 + f^2\right)v = -g\left(\eta_{yt} + f\eta_x\right) \tag{21}$$

4.1 Steady flows

We now assume $\partial_t = 0$. From (20), (21),

$$u = -\frac{g}{f}\eta_y, \qquad v = \frac{g}{f}\eta_x$$

This is *shallow water geostrophic balance*: the surface displacement η acts as a streamfunction. Applying the steady assumption to (19) gives $J(H_s, \eta) = 0$ which implies $\eta = \eta(H_s(x, y))$. Hence linearised steady geostrophic flow in shallow water follows contours of constant depth. Steady PV conservation follows from (16) with $\partial_t = 0$ and assuming $\zeta \ll f$

$$\boldsymbol{u} \cdot \nabla \frac{f}{H_s} = 0$$

Thus when f varies, the flow follows contours of constant f/H_s.

4.2 Waves in an unbounded domain

Assume H_s is constant. From (19), we have

$$\left(\partial_t^2 + f^2\right)\eta - gH_s\nabla^2\eta = 0$$

Seek plane wave solutions to this wave equation with ansatz $\eta = \eta_0 \exp(i(kx + ly - \omega t))$. The dispersion relation is then

$$\omega^2 = f^2 + gH_s(k^2 + l^2) \tag{22}$$

If $f = 0$, i.e. no rotation, then the frequency is $\omega = \pm\sqrt{gH_s}|\boldsymbol{k}| = \omega_0$ and the phase speed is $|c_p| = \frac{|\omega|}{|\boldsymbol{k}|} = \sqrt{gH_s} = c_0$. For $f \neq 0$, we get *Poincaré* waves with

$$\omega^2 > \omega_0^2, \qquad |c_p| > c_0$$

i.e. rotation increases the frequency and phase speed. Define the *Rossby deformation scale* $R_D \equiv \frac{c_0}{f}$. From (22),

$$\frac{\omega^2}{f^2} = 1 + R_D^2|\boldsymbol{k}|^2$$

Without loss of generality, let $l = 0$, by reorienting x and y. If $\eta = \eta_0\cos(kx - \omega t)$ then (20), (21) imply the fluid velocity is

$$u = \frac{\omega_0\eta_0}{kH_s}\cos(kx - \omega t)$$
$$v = \frac{f\eta_0}{kH_s}$$

Thus the motion is an ellipse, also known as a *tidal ellipse*, which reduces to intertial circles if $\omega_0 = f$:

$$u^2 + \frac{\omega_0^2}{f^2}v^2 = \frac{\omega_0^2\eta_0^2}{k^2H_s^2}$$

Since $\omega > f$, the fluid moves anticylonically. The Rossby deformation scale R_D is the length scale for which rotation becomes important. Consider short and long waves:

- Short waves: $|\boldsymbol{k}|R_D \gg 1$. We have $\omega^2 \to gH_s|\boldsymbol{k}|^2$ i.e. non-rotating shallow water gravity waves.
- Long waves: $|\boldsymbol{k}|R_D \ll 1$. We have $\omega^2 \to f^2$ i.e. inertial waves where fluid moves in inertial circles. Gravity is not involved.

Lecture 6
23/10/20

5 Geostrophic adjustment

Consider the response of rotating shallow water to an initial state *not* in geostrophic balance. Here, we consider $\eta(x, y,) = \eta_0 \mathrm{sgn}(x)$, $\boldsymbol{u}(x, y, 0) = \mathbf{0}$, so the initial PV is 0.
Assume f is constant, the perturbation is small $\eta_0 \ll H$, the PV is small $\zeta \ll f$, and the bottom is flat $H_s = H_0$. Linearise the shallow water PV:

$$q = \frac{f + \zeta}{H_0 + \eta} = \frac{f}{H_0}\left(1 + \frac{\zeta}{f} + \dots\right)\left(1 - \frac{\eta}{H_0} + \dots\right) \approx \frac{f}{H_0}\left(1 + \frac{\zeta}{f} - \frac{\eta}{H_0}\right)$$

Since PV is conserved, we have

$$\frac{\zeta}{f} - \frac{\eta}{H_0} = -\frac{\eta_0}{H_0}\mathrm{sgn}(x) \qquad \forall t \tag{23}$$

By symmetry, $\partial_y \equiv 0$ so the PV is $\zeta = v_x$. The linearised shallow water equations in this case

$$\begin{aligned} u_t - fv &= -g\eta_x \\ v_t + fu &= 0 \\ \eta_t + H_0 u_x &= 0 \end{aligned}$$

Using these equations we have

$$\zeta = v_x = \frac{u_{xt} + g\eta_{xx}}{f} = -\frac{1}{fH_0}\eta_{tt} + \frac{g}{f}\eta_{xx}$$

Now conservation of potential vorticity (23) gives

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = f^2\eta_0\mathrm{sgn}(x)$$

where $c^2 \equiv gH_0$. This is a *Klein-Gordon equation* where the $f^2\eta$ term adds elasticity to the waves.

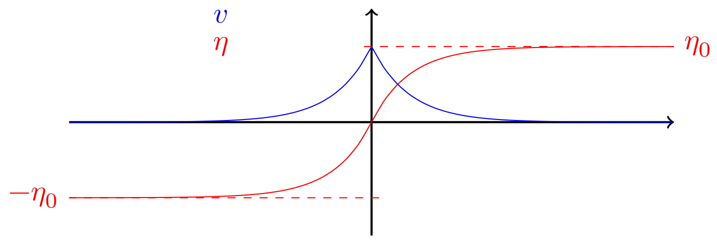
5.1 Steady solutions

Consider steady solutions. Owing to the step forcing, our BCs are to match η_x and η at $x = 0$. We find

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R_d} & x > 0 \\ -1 + e^{x/R_d} & x < 0 \end{cases} \tag{24}$$

where $R_d \equiv \sqrt{gH_0}/f$ is the *deformation radius*. From the equations of geotrophic balance we have the velocity components

$$u = 0, \qquad v = \frac{g\eta_0}{fR_d}e^{-|x|/R_d}$$



5.2 Transients

The steady solution (24) solves the geostrophic adjustment equation, but it does not match the initial conditions. We add this particular solution to a solution to the homogeneous equation

$$\eta_{tt} - c^2\eta_{xx} + f^2\eta = 0$$

with initial condition

$$\eta = \eta_0 \operatorname{sgn}(x) - \eta_{\text{steady}} = \eta_0 e^{-|x|/R_d} \operatorname{sgn}(x)$$

We seek solutions of plane wave form

$$\eta = \hat{\eta} e^{i(kx-\omega t)}$$

with $\omega^2 = f^2 + c^2k^2$. These are Poincaré waves.

5.3 Energetics

The change in potential energy per unit length in the y direction is

$$\begin{aligned}
PE_{\text{initial}} - PE_{\text{final}} &= \int_{-\infty}^{\infty}\int_0^{\eta_i} \rho_0 gz \,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty} \rho_0 gz\,\mathrm{d}z\,\mathrm{d}x \\
&= 2\rho_0 g\left[\int_0^\infty \frac{\eta_i^2}{2}\,\mathrm{d}x - \int_0^\infty \frac{\eta_f^2}{2}\,\mathrm{d}x\right] \\
&= \rho_0 g\eta_0^2 \int_0^\infty \left[1 - (1 - e^{-x/R_d})^2\right]\,\mathrm{d}x \\
&= \frac{3}{2}\rho_0 g\eta_0^2 R_d
\end{aligned}$$

The change in kinetic energy per unit length in the y direction is

$$\begin{aligned}
KE_{\text{initial}} - KE_{\text{final}} &= \int_{-\infty}^{\infty}\int_{-H}^{\eta_i} \frac{1}{2}\rho_0 v_i^2\,\mathrm{d}z\,\mathrm{d}x - \int_{-\infty}^{\infty}\int_{-H}^{\eta_f} \frac{1}{2}\rho_0 v_f^2\,\mathrm{d}z\,\mathrm{d}x \\
&\approx 0 - \frac{1}{2}\rho_0 \int_{-\infty}^{\infty} H_s v_f^2\,\mathrm{d}x \\
&= -\rho_0 H_s \int_0^\infty \frac{g^2\eta_0^2}{f^2R_d^2} e^{-2x/R_d}\,\mathrm{d}x \\
&= -\rho_0 \frac{R_d^2 g\eta_0^2}{R_d^2} \cdot -\frac{R_d}{2} \cdot \left[e^{-2x/R_d}\right]_0^\infty \\
&= -\rho_0 g\eta_0^2 \frac{R_d}{2}
\end{aligned}$$

Only $\frac{1}{3}$ of the potential energy released is converted into kinetic energy of the geostrophic flow. The remainder is radiated away by Poincaré waves.

Lecture 7 26/10/20

6 Quasi-geostrophic equations

Large scale motions in the ocean and atmosphere are associated with small Rossby number $Ro \equiv \frac{U}{fL} \ll 1$. In this limit, the rotating shallow water equations are approximated by the SW quasi-geostrophic (SW QG) equation. Start from the SW PV equation:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = 0 \tag{25}$$

Assumption 1: $Ro \ll 1$ Assuming a small Rossby number implies the flow is close to geostrophic balance with

$$f\hat{\boldsymbol{k}} \times \boldsymbol{u} \approx -g\nabla\eta$$

where $\hat{\boldsymbol{k}}$ is the vertical unit vector. Define the *geostrophic streamfunction* $\psi \equiv \frac{g\eta}{f}$. In terms of this streamfunction we have

$$\begin{aligned}\boldsymbol{u} &\approx -\nabla \times (\psi\hat{\boldsymbol{k}})\\ \zeta &= (\nabla \times \boldsymbol{u})\hat{\boldsymbol{k}} \approx \nabla^2\psi\end{aligned}$$

Assumption 2: small changes in f Recall the Coriolis parameter $f = 2\Omega\sin\theta$ where θ is latitude. Expand in a Taylor series about $\theta = \theta_0$ to get

$$f = f_0 + y\frac{\mathrm{d}f}{\mathrm{d}y}|_{\theta_0} + \cdots \approx f_0 + \beta y$$

where y is in the direction of local North, $f_0 = 2\Omega\sin\theta_0$ and β is defined as

$$\beta = \frac{1}{R}\frac{\mathrm{d}f}{\mathrm{d}\theta}|_{\theta_0} = \frac{2\Omega}{R}\cos\theta_0$$

with R the radius of Earth. For characteristic length scale L, assume $\frac{\beta L}{f_0} \ll 1$. This is the *$\beta$-plane approximation.*

Assumption 3: small changes in fluid height. This is consistent with small Rossby number: from geostrophic balance, we know $\eta \sim \frac{fUL}{g}$ and $\frac{\eta}{H_0} \sim \frac{fUL}{gH_0} = \frac{U}{fL}\frac{L^2}{R_D^2}$. Therefore $\eta/H_0 \ll 1$ if $Ro \ll \frac{R_D^2}{L^2}$. For $L \sim R_D$, $Ro \ll 1$ implies $\eta/H_0 \ll 1$. Further, we assume $h_b/H_0 \ll 1$.

Quasi-geostrophic equations. With these assumptions, SWPV becomes

$$\begin{aligned}\frac{\zeta + f}{H_0 - h_b + \eta} &\approx \frac{f_0}{H_0}\frac{1 + \frac{\beta y}{f_0} + \frac{\zeta}{f_0}}{1 - \frac{h_b}{H_0} + \frac{\eta}{H_0}}\\ &\approx \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\nabla^2\psi}{f_0} + \frac{h_b}{H_0} - \frac{f_0\psi}{gH_0}\right)\\ &= \frac{f_0}{H_0}P_g\end{aligned}$$

where P_g is the *quasi-geostrophic potential vorticity* and $\zeta = \nabla^2\psi, \eta = \frac{f_0\psi}{g}$. Hence from SWPV conservation (25),

$$\frac{\partial P_g}{\partial t} + \boldsymbol{u}\cdot\nabla P_g \approx 0$$

Using $\boldsymbol{u} \approx -\nabla \times (\psi\hat{\boldsymbol{k}})$, $\boldsymbol{u} = -\psi_y, v = \psi_x$ so

$$\frac{\partial P_g}{\partial t} + J(\psi, P_g) \approx 0 \tag{26}$$

This is the *shallow water Quasi-geostrophic* (SWQG) equation, which is one equaiton for one unknown ψ, as opposed to SWPV with 2 unknowns ζ, η.

6.1 Waves in QG

Assume a flat bottom $h_b = 0$. Linearise (26) about a state of rest (i.e. neglect terms $\mathcal{O}(\psi^2)$). Then

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{f_0^2}{gH_0}\psi\right) + \frac{\partial\psi}{\partial x}\beta = 0$$

Seek plane wave solutions of the form

$$\psi = \psi_0 e^{i(kx+ly-\omega t)}$$

with dispersion relation

$$\omega = \frac{-k\beta}{k^2 + l^2 + R_D^{-2}}, \qquad R_D \equiv \frac{\sqrt{gH_0}}{f_0}$$

This is the *Rossby wave dispersion relation.* Note $\omega = 0$ (i.e. no waves) if $\beta = 0$. Also, if $h_b = 0$ and $\beta = 0$ there are no wave solutions unlike rotating SW. Thus the QG system 'filters' out Poincaré waves. Note that $\beta = \frac{2\Omega}{R}\cos\theta \geq 0$, hence $c_p = \frac{\omega}{k} \leq 0$. Rossby wave speed is always directed to the *west*.

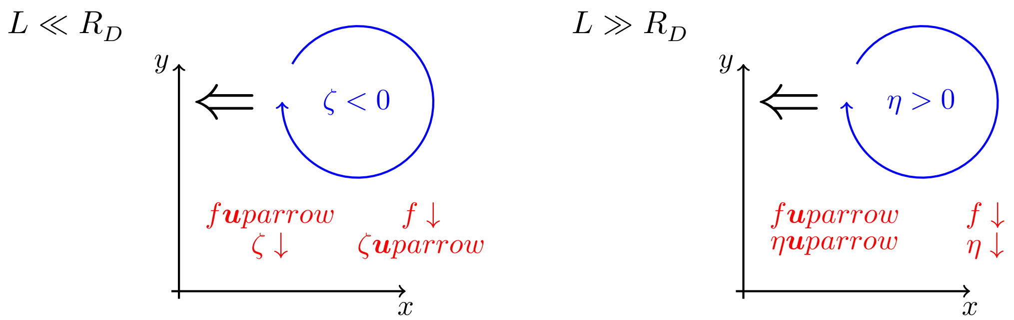
Consider the size of the dynamic terms in P_g, specifically the ratio of relative vorticity to surface height

$$\frac{\nabla^2\psi}{-\frac{f_0^2\psi}{gH_0}} \sim \frac{R_D^2}{L^2}$$

Hence relativity vorticity dominates at scales small compared to R_D whilst surface height dominates at scales large compared to R_D.

6.2 Physical interpretation of Rossby waves

Consider $L \ll R_d$ ($L \gg R_D$) and a small perturbation in the dominant term for the scale, ζ (η). For $L \ll R_D$, the planetary vorticity increases (thus ζ decreases) on the westward side, whilst the planetary vorticity decreases (thus ζ increases) on the eastward side. Hence the perturbation propagates westwards. For $L \gg R_D$, the planetary vorticity increases (η increases) on the westward size and decreases (η decreases) on the eastward side as before. Thus the perturbation propagates to the west also. These are Rossby waves.



7 Large scale ocean circulation

7.1 Sverdrup flow

Seek steady solutions for rotating shallow water driven by a wind stress $\boldsymbol{\tau}_w$. We have

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} \tag{27}$$

$$H_t + \nabla \cdot (\boldsymbol{u}H) = 0 \tag{28}$$

Consider $\nabla \times (27) \cdot \hat{\boldsymbol{k}}$ and (28) which implies modified PV conservation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\zeta + f}{H}\right) = \frac{1}{H}\nabla \times \left(\frac{\boldsymbol{\tau}_w}{\rho H}\right) \cdot \hat{\boldsymbol{k}} \tag{29}$$

Thus we see frictional forcing modifies PV conservation. Assuming H is constant, $\zeta \ll f$ ($Ro \ll 1$), and using the β-plane approximation $f = f_0 + \beta y$, (29) becomes

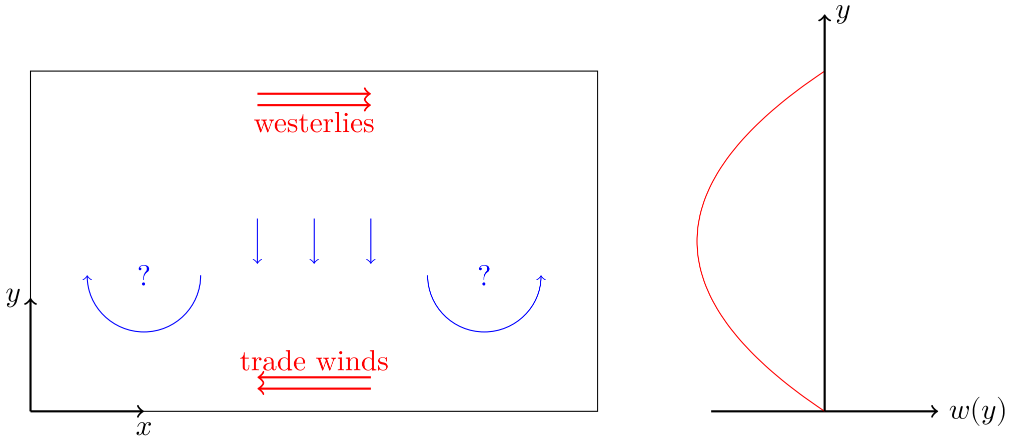
$$\beta v = \frac{1}{\rho H}(\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}} \tag{30}$$

This is called *Sverdrup balance*. Physically, the North/South advection of planetary vorticity $\boldsymbol{u} \cdot \nabla f$ balances the vorticity input by wind.

7.2 Western boundary currents

Consider steady circulation in a rectangular basin, driven by a wind stress curl

$$w(y) = \frac{(\nabla \times \boldsymbol{\tau}_w) \cdot \hat{\boldsymbol{k}}}{\rho H}$$



From (30), $w < 0 \implies v < 0$. Recall $\boldsymbol{u} = -\nabla \times \psi\hat{\boldsymbol{k}}$. Boundary conditions are no normal flow at the boundaries, i.e. ψ is constant. Sverdrup balance (30) $\beta\psi_x = w(y)$ gives

$$\psi = \frac{xw(y)}{\beta} + G(y)$$

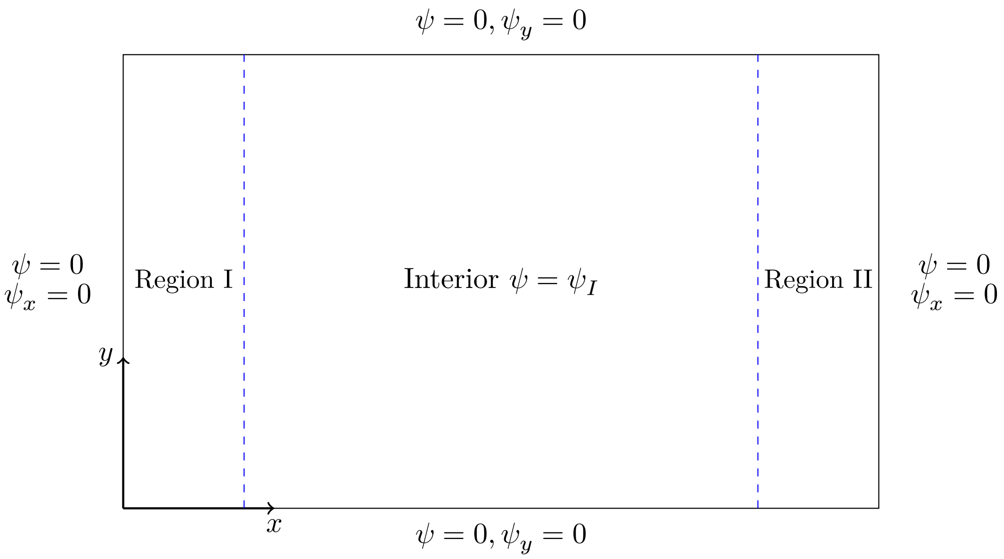
for some arbitary function $G(y)$. This presents a problem: we cannot meet the boundary conditions at both $x = 0$ and $x = L$. Hence we need extra terms and boundary layers. Following Musk, we include horizontal friction in (27):

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -g\nabla\eta + \frac{\boldsymbol{\tau}_w}{\rho H} + \nu\nabla^2\boldsymbol{u} \tag{31}$$

Note here we are using the horizontal gradient $\nabla \equiv (\partial_x, \partial_y)$. Consider $\nabla \times (31) \cdot \hat{\boldsymbol{k}}$ with $\zeta \ll f$. Then

$$\beta\psi_x = w(y) + \nu\nabla^4\psi \tag{32}$$

The PDE is now fourth order, so we need four boundary conditions.



In region I we have $\psi \approx \psi_I + \psi^{(1)}$ and in region II we have $\psi \approx \psi_I + \psi^{(2)}$. The full solution is
Lecture 8 30/10/20 $\psi = \psi_I + \psi^{(1)} + \psi^{(2)}$ with interior flow $\psi_I = x\frac{w(y)}{\beta} + G(y)$.

Region I. Let $\varepsilon = \nu$ with $\varepsilon \ll 1$. Define a rescaled coordinate $\tilde{x} \equiv \frac{x}{\varepsilon^a}$ with $\partial_x = \varepsilon^{-a}\partial_{\tilde{x}}$. Note: if $a > 0$ then $\partial_x \gg \partial_y$. This is the *method of undetermined coefficients*. From the PDE (32) for ψ we have

$$\cancel{\beta\psi^I_x} + \beta\varepsilon^{-a}\tilde{\psi}^{(1)}_{\tilde{x}} = \varepsilon^{1-4a}\tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \cancel{w}$$

Matching exponents, we have $a = \frac{1}{3}$. Hence

$$\beta\tilde{\psi}^{(1)}_{\tilde{x}} = \tilde{\psi}^{(1)}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$$

Seek solutions of the form $\tilde{\psi} = \tilde{\psi}_0 e^{r\tilde{x}}$. Then $r^4 - \beta r = 0$ so $r = 0, \beta^{1/3}, -\frac{1}{2}\beta^{1/3} \pm i\frac{\sqrt{3}}{2}\beta^{1/3}$. The general solution is therefore

$$\tilde{\psi}^{(1)} = A(y) + B(y)e^{\beta^{1/3}\tilde{x}} + C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}} + D(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}e^{-i\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x}}$$

In order for the interior and boundary layer flows to match asymptotically, we apply the *matching condition* $\lim_{\tilde{x}\to\infty}\tilde{\psi}^{(1)} = 0$. Thus $A(y) = B(y) = 0$. For convenience we re-define C and D to get

$$\tilde{\psi}^{(1)} = C(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} + D(y)\right)$$

We now apply the boundary conditions. $\psi = 0$ at $x = 0$ gives $\tilde{\psi}^{(1)} = -\psi^I\mid_{x=0}$. Hence

$$C(y)\cos D(y) = -G(y)$$

$\psi_x = 0$ at $x = 0$ gives $\psi_x^{(1)} = -\psi_x^I\mid_{x=0}$. Hence

$$\varepsilon^{-1/3}\tilde{\psi}_{\tilde{x}}^{(1)} = -\frac{w(y)}{\beta}$$

$$\varepsilon^{-1/3}(-\frac{1}{2}\beta^{1/3})C(y)\cos D(y) - \varepsilon^{-1/3}\frac{\sqrt{3}}{2}\beta^{1/3}C(y)\sin D(y) = -\frac{w(y)}{\beta}$$

Since $\varepsilon \ll 1$ and can be taken arbitrarily small, we require

$$-\frac{1}{2}\cos D(y) = \frac{\sqrt{3}}{2}\sin D(y)$$
$$\implies \tan D(y) = -\frac{1}{\sqrt{3}}$$
$$\implies D(y) = -\frac{\pi}{6}$$

Combining the boundary conditions we also have $C(y) = -\frac{2}{\sqrt{3}}G(y)$. Finally we have

$$\tilde{\psi}^{(1)} = -\frac{2}{\sqrt{3}}G(y)e^{-\beta^{1/3}\frac{\tilde{x}}{2}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\tilde{x} - \frac{\pi}{6}\right)$$

Region II. Here, we define a rescaled coordinate $\tilde{x} = \frac{x-L}{\varepsilon^{1/3}}$. The same PDE is satisfied in region II, so the general solution is the same. Here, the matching condition is $\lim_{\tilde{x}\to-\infty}\tilde{\psi}^{(2)} = 0$ which gives $A(y) = C(y) = D(y) = 0$, so

$$\tilde{\psi}^{(2)} = B(y)e^{\beta^{1/3}\tilde{x}}$$

We now apply the boundary conditions. $\psi_x = 0$ at $x = L$ gives

$$\varepsilon^{-1/3}\tilde{\psi}^{(2)} = -\psi_x^I \quad \text{at} \quad x = L$$
$$\varepsilon^{-1/3}\beta^{1/3}B(y) = -\frac{w(y)}{\beta}$$
$$\implies B(y) = -\frac{\varepsilon^{1/3}w(y)}{\beta^{4/3}}$$

To enforce $\psi = 0$ at $x = L$, note $\lim_{\varepsilon\to 0}B(y) = 0$, so $\tilde{\psi}^{(2)}\mid_{x=L}\to 0$ as $\varepsilon \to 0$ so we instead require $\psi^I\mid_{x=L}= 0$.

$$\implies G(y) = -\frac{w(y)L}{\beta}$$

Hence we have

$$\tilde{\psi}^{(2)} = -\varepsilon^{1/3}w(y)\beta^{-4/3}e^{\beta^{1/3}\tilde{x}}$$

Full solution. The full solution $\psi = \psi^I + \psi^{(1)} + \psi^{(2)}$ is

$$\begin{aligned}
\psi &= \frac{x-L}{\beta}w(y) && \text{interior}\\
&+ \frac{2w(y)L}{\sqrt{3}\beta}e^{-\beta^{1/3}\frac{x}{2\nu^{1/3}}}\cos\left(\frac{\sqrt{3}}{2}\beta^{1/3}\nu^{-1/3}x - \frac{\pi}{6}\right) && \text{western boundary correction}\\
&- \nu^{1/3}\beta^{-4/3}w(y)e^{\beta^{1/3}\frac{x-L}{\nu^{1/3}}} && \text{eastern boundary correction}
\end{aligned}$$

Note that the Eastern boundary correction is $\mathcal{O}(\nu^{1/3})$ whilst the Western boundary correction is $\mathcal{O}(1)$.

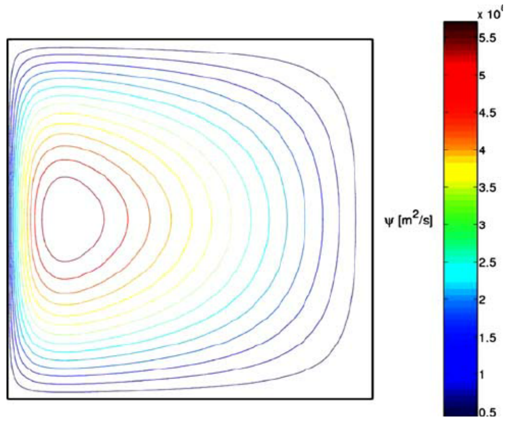
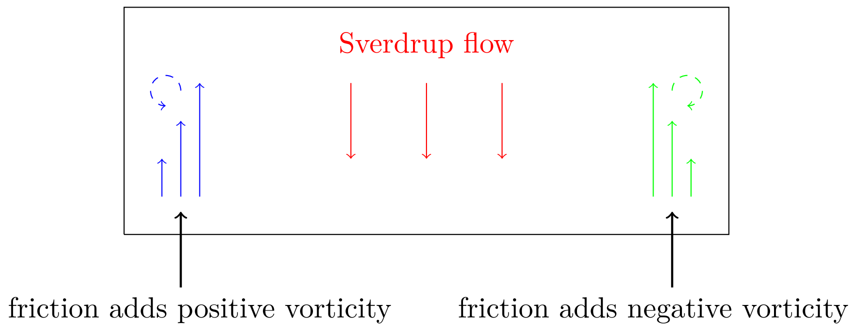


Figure 5: Streamlines of ψ demonstrating western boundary currents.

Physical explanation. The cause of western boundary currents can be physically explained by vorticity. The wind stress curl $w < 0$ inputs negative vorticity in the interior flow. The flow in the western boundary layer inputs positive vorticity to compensate.



Lecture 9
2/11/20

8 Stratification

8.1 Boussinesq approximation

We will consider stably stratified flow under the *Boussinesq approximation*: we assume the density ρ may be split into two parts ρ_0 and ρ' with ρ_0 constant and $\rho'/\rho_0 \ll 1$. The pressure may then also be split into two parts, $p_0(z)$, such that $-p_0'(z) = \rho_0 g$, and the remainder p'. This says $p_0(z)$ is the hydrostatic pressure and p' is the excess. The vertical component of the momentum equations may then be written

$$\begin{aligned}
\frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho_0 + \rho'}\frac{\partial p_0}{\partial z} - \frac{1}{\rho_0 + \rho'}\frac{\partial p'}{\partial z} - g \\
&= -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} + \frac{\partial p_0}{\partial z}\left[\frac{1}{\rho_0} - \frac{1}{\rho_0 + \rho'}\right] - \frac{\partial p'}{\partial z}\frac{1}{\rho_0 + \rho'} - g \\
&\approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0} g
\end{aligned}$$

where terms including ρ' are discarded unless multiplying g. The *Boussinesq equations* are therefore

$$\begin{aligned}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} &= -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} &= 0 \\ \frac{\mathrm{D}\rho'}{\mathrm{D}t} &= 0\end{aligned}$$

These equations make clear the role of buoyancy: a light fluid parcel experiences an upward force and a heavy fluid parcel experiences a downward force. It is further useful to write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$ where $\rho_s(z)$ is a *background* or *reference* density and $\tilde{\rho}$ is the *disturbance density* which is zero for fluid at rest.
The stability of the reference density state is determined by the *buoyancy frequency* or *Brunt-Väisäla frequency* N defined by

$$N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_s}{\mathrm{d}z}$$

8.2 Atmosphere & ocean stratification

In the ocean, the buoyancy frequency N is typically $10^{-2}s^{-1}$ in the upper ocean where stratification is strong, and $5 \times 10^{-4}s^{-1}$ in the deep ocean, where stratification is weak.
In the atmosphere, calculating N needs to take account of compressibility, because the density ρ is not conserved by a fluid parcel in reversible, dissipationless motion. The quantity that is instead conserved is the *potential temperature*

$$\theta = T\left(\frac{p}{p_0}\right)^{-2/7}$$

where T is temperature. The corresponding buoyancy frequency is

$$N^2 = -\frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

In this course we will use the Boussinesq approximation for the atmosphere and ocean, despite issues with compressibility.

8.3 Internal gravity waves

We linearise about a resting state with density structure represented by the buoyancy frequency N. For simplicity, background rotation is ignored for the time being. We define the *buoyancy* $\sigma = -\rho' g/\rho_0$ for convenience.

$$\begin{aligned}\tilde{\boldsymbol{u}}_t &= -\frac{1}{\rho_0}\nabla\tilde{p} + \tilde{\sigma}\hat{\boldsymbol{z}} \\ \nabla \cdot \tilde{\boldsymbol{u}} &= 0 \\ \tilde{\sigma} + N^2\tilde{w} &= 0\end{aligned}$$

where the notation $\tilde{\boldsymbol{u}}$ denotes the disturbance quantities away from a state of rest. These can be combined into a single equation for $\tilde{w}$:

$$\nabla^2\tilde{w}_{tt} + N^2(\tilde{w}_{xx} + \tilde{w}_{yy}) = 0$$

Assuming N^2 is constant, seek plane wave solutions $\tilde{w} = \hat{w}e^{i(kx+ly+mz-\omega t)}$. This gives a dispersion relation

$$\omega^2 = N^2 \frac{k^2 + l^2}{k^2 + l^2 + m^2}$$

Note that if $N^2 > 0$ we get oscillatory motion, and if $N^2 < 0$ we get exponentially growing disturbances. We also have $0 \leq |\omega| \leq N$ with the lower limit achieved in the limit $k^2 + l^2 \ll m^2$. Define $\theta = \tan^{-1}(m(k^2 + l^2)^{-1/2})$, the angle a surface of constant phase makes with the vertical. Then $\omega = \pm N \cos\theta$. Owing to incompressibility $\nabla \cdot \boldsymbol{u} = 0$, the velocity vector is perpendicular to $\boldsymbol{k}$. Thus θ is also the angle that fluid parcel trajectories make with the vertical.
The fact $|\omega| \leq N$ implies only disturbances with a sufficiently low frequency can propagate as waves. Localised forcing with frequency greater than N will remain localised rather than propagating.
Further, note the group velocity is

$$\boldsymbol{c}_g \equiv \frac{\partial \omega}{\partial \boldsymbol{k}} = \pm \frac{N}{(k^2 + l^2)^{1/2}(k^2 + l^2 + m^2)^{3/2}} (km^2, lm^2, -m(k^2 + l^2))$$

Lecture 10 4/11/20 which gives $\boldsymbol{c}_g \cdot \boldsymbol{k} = 0$, i.e. the group velocity lies in surfaces of constant phase.

9 3D quasi-geostrophic equations

9.1 Basic facts about rotation & stratification

1. Assuming the buoyancy frequency N is constant and $\boldsymbol{f}$ is vertical, the dispersion relation for small amplitude waves is

   $$\omega^2 = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2}$$

   where $\boldsymbol{k} = (k, l, m)$. Thus the relative strength of stratification vs. rotation is N/L vs. f/D, where L is the horizonal lengthscale and D is the vertical lengthscale.

2. Typically, $N \gg f$. In the deep ocean, $N \sim 10^{-3} s^{-1}$ and in the upper ocean and atmosphere $N \sim 10^{-2} s^{-1}$, whilst $f \sim 10^{-4} s^{-1}$.

3. Rotation is important only if $L \gg D$, which implies vertical velocities are much smaller than horizontal velocities. Hence the *hydrostatic approximation* is valid.

4. Given $L \gg D$, the Coriolis force may be neglected in the vertical momentum equation, and in the horizontal momentum equation only the part of the Coriolis force associateds with the horizontal velocity is important. This can be seen as follows: let $\boldsymbol{f} = \boldsymbol{f}_h + \boldsymbol{f}_v$ and $\boldsymbol{u} = \boldsymbol{u}_h + \boldsymbol{u}_v$, where $_h$ and $_v$ denote horizontal and vertical components respectively. Then,

   $$\boldsymbol{f} \times \boldsymbol{f} = \boldsymbol{f}_h \times \boldsymbol{u}_h + \boldsymbol{f}_v \times \boldsymbol{u}_h + \boldsymbol{f}_h \times \boldsymbol{u}_v + \cancel{\boldsymbol{f}_v \times \boldsymbol{u}_h} \approx \boldsymbol{f}_h \times \boldsymbol{u}_h + \boldsymbol{f}_v \times \boldsymbol{u}_h$$

   where we have assumed $|\boldsymbol{u}_v| \ll |\boldsymbol{u}_h|$. At low latitude, this assumption fails since $\boldsymbol{f}_v \ll \boldsymbol{f}_h$. Following the traditional approximation, only the $\boldsymbol{f}_v \times \boldsymbol{u}_h$ contribution is retained in the horizontal momentum equation. This is equivalent to replacing $\boldsymbol{f}$ with its vertical component only.

5. Given the above assumptions, as well as assuming the fluid layer is thin compared to the radius of the Earth, we get the *primitive equations*.

Further, we invoke the β-plane approximation $f = f_0 + \beta y$. The full $3D$ Boussinesq primite equations on a β-plane are

$$\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} - (f_0 + \beta y)v &= -\frac{1}{\rho_0}p'_x \\
\frac{\mathrm{D}v}{\mathrm{D}t} + (f_0 + \beta y)u &= -\frac{1}{\rho_0}p'_y \\
p'_z &= -\rho' g \\
\frac{\mathrm{D}\rho'}{\mathrm{D}t} &= \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}$$

This set of equations is formed of three *prognostic* equations which can be used to evolve the five dependent variables, and two instantaneous constraints. There are strong similarities to the shallow water equations on a β-plane: at small Rossby number, the shallow water equations have two fast modes (e.g. Poincaré waves & Kelvin waves for the shallow-water equations, internal gravity waves for the primitive equations) and one slow mode of waves which are close to geostrophic balance.

9.2 Thermal wind equation

When the Rossby number is small, we expect the flow to be close to geostrophic balance, so that

$$\begin{aligned}
-fv &= -\frac{1}{\rho_0}p'_x \\
fu &= -\frac{1}{\rho_0}p'_y
\end{aligned}$$

Differentiating with respect to z and using the hydrostatic relation, we have the *thermal wind equations*

$$\begin{aligned}
fv_z &= -\frac{g}{\rho_0}\rho'_x \\
fu_z &= \frac{g}{\rho_0}\rho'_y
\end{aligned}$$

Here, the density perturbation ρ' can be viewed analogously to temperature.

9.3 Potential vorticity

In the shallow-water equations, the shallow-water potential vorticity was conserved. Under the Boussinesq primite equations, instead the *Rossby-Ertal potential vorticity* P is conserved materially, where

$$P = \frac{1}{\rho_0}(\boldsymbol{f} + \boldsymbol{\zeta}) \cdot \nabla\rho'$$

In terms of velocities this is equivalent to

$$P = \frac{1}{\rho_0}\left[(f_v + v_x - u_y)\rho'_z + u_z\rho'_y - v_z\rho'_x\right]$$

Note that forcing and dissipation terms are not yet included. These will give rise to features in the P field which can affect or drive the evolution of the flow.

9.4 3D quasi-geostrophic equations

Primitive equations. Following the same procedure as with the shallow-water equations, we aim to find a prognostic equations for the slow (close to geostrophic balance) motion from the Boussinesq primitive equations on a β-plane.
We write $\rho'(x, y, z, t) = \rho_s(z) + \tilde{\rho}(x, y, z, t)$ where $\rho_s(z)$ is in hydrostatic balance, and write $p'(x, y, z, t) = p_s(z) + \tilde{p}(x, y, z, t)$ where each term is in hydrostatic balance with the corresponding density term, i.e.

$$\frac{\mathrm{d}p_s}{\mathrm{d}z} = -\rho_s g$$
$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho} g$$

The density equation $\frac{\mathrm{D}\rho'}{\mathrm{D}t} = 0$ thus becomes

$$\frac{\mathrm{D}\tilde{\rho}}{\mathrm{D}t} + w\frac{\mathrm{d}\rho_s}{\mathrm{d}z} = 0$$

The velocity vield is divided into a part which is in geostrophic balance with the pressure field (assuming constant f_0) and a remainder, the *ageostrophic velocity*:

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a \qquad \text{where } f_0 \boldsymbol{k} \times \boldsymbol{u}_g = -\frac{1}{\rho_0}\nabla_h \tilde{p}$$

Note that the vertical component of the geostrophic velocity $\boldsymbol{u}_g$ is zero, and $\nabla \cdot \boldsymbol{u}_g = 0$. We also assume that the y-lengthscale L_y is sufficiently small that $\beta L_y \ll f_0$. Then if $Ro \ll 1$, it follows that $|\boldsymbol{u}_a| \ll |\boldsymbol{u}_g|$. Thus we are domain limited in latitude.

Lecture 11 6/11/20

The primitive equations may now be written

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a)\cdot\nabla\right](u_g + u_a) - f_0(v_g + v_a) - \beta y(v_g + v_a) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} \tag{33}$$
$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a)\cdot\nabla\right](v_g + v_a) + f_0(u_g + u_a) + \beta y(u_g + u_a) = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial y} \tag{34}$$
$$-\frac{\partial \tilde{p}}{\partial z} - \tilde{\rho} g = 0$$
$$\left[\frac{\partial}{\partial t} + (\boldsymbol{u}_g + \boldsymbol{u}_a)\cdot\nabla\right]\tilde{\rho} + w_a\frac{\mathrm{d}\rho_s}{\mathrm{d}z} = 0$$
$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} = 0$$

Approximation validity. Given that $Ro \ll 1$ we may approximate $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$ by $\frac{\mathrm{D}_g \boldsymbol{u}_g}{\mathrm{D}t}$ where $\frac{\mathrm{D}_g}{\mathrm{D}t} = \partial_t + \boldsymbol{u}_g \cdot \nabla$, and also $\beta y \boldsymbol{u}$ can be approximated by $\beta y \boldsymbol{u}_g$.
Note that $f_0(u_a, v_a)$ and $\beta y(u_g, v_g)$ are of similar size. Hence the requirement $\beta L_y/f_0 \ll 1$ is better expressed by $\beta L_y/f_0 \sim Ro$. Also, $w_a \frac{\mathrm{d}\rho_s}{\mathrm{d}z}$ is retained, but $w_a \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}z}$ is not, which requires $|\mathrm{d}\tilde{\rho}/\mathrm{d}z| \ll |\mathrm{d}\rho_s/\mathrm{d}z|$. Denoting the horizontal scale as L and the vertical scale as D, the thermal wind equation gives $g\tilde{\rho}/L\rho_0 \sim f_0 U/D$ where U is the typical horizontal velocity scale. Hence we have

$$\frac{\tilde{\rho}_z}{\rho_{s,z}} \sim \frac{f_0 U L \rho_0}{g D^2 \rho_{s,z}} = \frac{U}{f_0 L}\left(\frac{L f_0}{N D}\right)^2 = Ro Bu$$

where $Bu \equiv (L f_0/DN)^2$ is the *Burger number*. For our approximation to be valid, we require $RoBu \ll 1$. If $Bu \sim 1$, then this is implied by $Ro \ll 1$.

Quasi-geostrophic potential vorticity. To reduce the primitive equations to a single prognostic equation, we eliminate $\boldsymbol{u}_a$ by taking the curl of horizontal momentum, i.e. $\partial_x(34) - \partial_y(33)$. The non-divergence of geostrophic velocity then gives a vorticity equation

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] + \beta v_g + f_0\left[\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right] = 0$$

Finally, we eliminate u_a, v_a and w_a using the remaining primitive equations to give

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] + \beta v_g + f_0\frac{\partial}{\partial z}\left[\frac{\mathrm{D}_g\tilde{\rho}}{\mathrm{D}t} / \frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right] = 0$$

We now define a streamfunction $\psi = \frac{\tilde{p}}{\rho_0 f_0}$ analogous to the geostrophic streamfunction. Then $u = -\psi_y, v = \psi_x$ from hydrostatic balance $\tilde{\rho} = -\rho_0 f_0 \psi_z / g$. Our prognostic equation, called the *quasi-geostrophic potential vorticity equation* is then

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\left[\psi_{xx} + \psi_{yy} + \left(\frac{f_0^2\psi_z}{N^2}\right)_z\right] + \beta\psi_x = 0$$

or equivalently $\frac{\mathrm{D}_g q}{\mathrm{D}t} = 0$ where q is the quasi-geostrophic potential vorticity

$$q = \psi_{xx} + \psi_{yy} + \left(\frac{f_0^2\psi_z}{N^2}\right)_z + \beta y$$

and $N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_s}{\mathrm{d}z}$. In terms of the QGPV, the (geostrophic) material derivative is

$$\frac{\mathrm{D}_g}{\mathrm{D}t} = \frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial y} + \psi_x\frac{\partial}{\partial y}$$

Under the quasi-geostrophic approximation, q is conserved following the horizontal geostrophic flow. The QGPV equation is an approximation to the statemenet of material conservation of Rossby-Ertel potential vorticity following the flow along ρ' surfaces in a Boussinesq flow, or θ surfaces in a compressible flow.
If q is known, then ψ can be calculated via the *potential vorticity inversion operator*

$$\psi = \left[\partial_x^2 + \partial_y^2 + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\right)\right]^{-1}(q - \beta y)$$

Application of this operator requires boundary conditions on ψ or its derivatives.

Boundary conditions.

- At rigid side boundaries, we require the normal component of $\boldsymbol{u}$ is zero (no flux condition), which requires ψ constant along the boundary.
- At rigid top or bottom boundaries, we require the kinematic boundary condition to be satisfied, i.e. $\frac{\mathrm{D}z}{\mathrm{D}t} = w = \frac{\mathrm{D}h}{\mathrm{D}t}$ on the boundary $z = z_b + h$ where z_b is constant and h is the topographic perturbation. We may express w_a in terms of other variables via the density equation to get
$$w \sim w_a = -\frac{\mathrm{D}_g\tilde{\rho}}{\mathrm{D}t}\left(\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right)^{-1} = \frac{\mathrm{D}h}{\mathrm{D}t} \approx \frac{\mathrm{D}_g h}{\mathrm{D}t}$$

Hence $h \sim \tilde{\rho}\left(\frac{\mathrm{d}\rho_s}{\mathrm{d}z}\right)^{-1} \sim D\left(\frac{U}{f_0 L}\right)\left(\frac{L^2 f_0^2}{N^2 D^2}\right) = RoBuD$, so we require $h \ll D$. The boundary condition may then be linearised, so that it can be applied at $z = z_b$. Writing $\tilde{\rho}$ in terms of ψ, we have

$$\frac{\mathrm{D}_g}{\mathrm{D}t}\psi_z = -\frac{N^2}{f_0}\frac{\mathrm{D}_g}{\mathrm{D}t}h$$

at $z = z_b$. This is a prognostic equation for ψ_z at the bottom surface which relates (physically) to material rate of change of density or temperature.

Physical interpretation of QGPV. The density or temperature at horizontal boundaries have similar importance to the QGPV in the interior of the flow. The physical interpretations of different contributions to the quasi-geostrophic potential vorticity q are

$$q \;=\; \underbrace{\psi_{xx} + \psi_{yy}}_{\text{relative vorticity}} \;+\; \underbrace{\left(\frac{f_0^2}{N^2}\psi_z\right)_z}_{\text{stretching}} \;+\; \underbrace{\beta y}_{\text{planetary vorticity}}$$

The stretching measures vertical gradients in density perturbations, i.e. the amount by which nearby density surfaces move apart or together. The ratio of the relativity vorticity term to the stretching term is $1/Bu$. If $Bu \ll 1$ then relative vorticity dominates, whilst if $Bu \gg 1$ then stretching dominates. If $Bu \sim 1$ the terms are comparable and the ratio of horizontal to vertical scales $L/D \sim N/f_0$ is implied by this condition as is called *Prandtl's ratio of scales.*

The 3D quasi-geostrophic equations have a structural similarity to the equations of 2D vortex dynamics, in that there is a non-local dependence of ψ on q. In 2D vortex dynamics, the non-locality is purely horizontal, whilst in the 3D quasi-geostrophic equations the non-locality is also in the vertical and the PV field in a localised region at a given level influences the ψ field outside that region on the same level *and* at other levels.

If N is constant in height, then the PV operator is isotropic in scales coordinates $x, y, Nz/f_0$. The evolution equations are not isotropic since the flow only has horizontal components. Therefore we expect solutions of the QG equations to tend towards isotropy in the coordinates above, but the isotropy is likely not exact.