Cambridge Part III Maths

Lent 2020

Fluid Dynamics of Environment

based on a course given by
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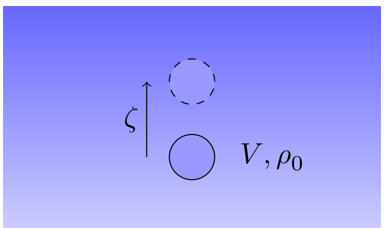
Contents

Lecture 1
22/01/21

1 Internal waves

1.1 Intuitive version



Consider a fluid parcel of volume V and density ρ_0 in a fluid with density profile $\hat{\rho}(z)$. Suppose the parcel is moved upwards by ζ. The parcel experiences a *buoyancy force* $B = gV\zeta\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$. Newton's second law gives

$$\ddot{\zeta} + \left(-\frac{g}{\rho_0}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}\right)\zeta = 0$$

The *buoyancy frequency* (or Brunt-Väisälä frequency) is defined as

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}$$

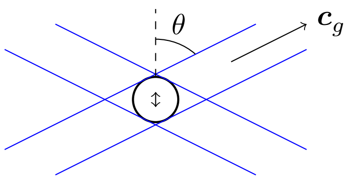
which has general solution

$$\zeta = A\cos Nt + B\sin Nt$$

If we instead consider a fluid slab inclined at angle θ with the vertical rather than a fluid parcel, the slab can fall in its plane much more easily than in the vertical. Hence in this situation we have

$$\ddot{\zeta} + N^2\cos^2\theta\zeta = 0$$

The dispersion relation is thus $\omega/N = \cos\theta$.
Now consider a sphere oscillating at frequency ω in the vertical in a stratified fluid with density $\rho(z)$. The fluid resonates in bands at angle θ satisfying the dispersion relation, provided $\omega < N$. Intuitively, the group velocity must be out of the beams as energy is radiated away.



At the leading edge of the rays, baroclinic vorticity is generated by the movement of fluid of different density to its surroundings. This provides the mechanism for the instability.

1.2 Rigorous derivation

Consider a fluid in which the mean pressure $p_0(z)$ and the mean density $\rho_0(z)$ are in hydrostatic balance when the fluid is at rest:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g$$

Assume that the vertical lengthscale for ρ_0 variation is L. Motion is governed by the Navier-Stokes equations (1) and (2) with $\nu = 0$, and mass conservation (3).

$$\nabla\cdot\boldsymbol{u} = 0 \tag{1}$$

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p - \rho g\hat{\boldsymbol{z}} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\rho = 0 \tag{3}$$

Following the Boussinesq approximation, assume small perturbations to the mean state: $\rho = \rho_0(z)+\tilde{\rho}$ and $p = p_0(z) + \tilde{p}$ where $\tilde{p} \ll p_0, \tilde{\rho} \ll \rho_0$. Under this approximation, the momentum equation (2) becomes

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}g\hat{\boldsymbol{z}}\\ &= -\frac{1}{\rho_0}\nabla(p + \rho_0 gz) - g'\hat{\boldsymbol{z}}\end{aligned}$$

where $g' = g(\rho - \rho_0)/\rho_0$ is the *reduced gravity.* We now linearise $\boldsymbol{u}$ about a state of rest, ignoring second order quantities in the velocity disturbance $\boldsymbol{u}'$. It is now further desirable to split the disturbance components into $\tilde{\rho} = \hat{\rho} + \rho', \tilde{p} = \hat{p} + p'$ where $\hat{\rho}, \hat{p}$ are in hydrostatic balance. We have

$$\begin{aligned}
\nabla \cdot \boldsymbol{u}' &= 0 \\
\frac{\partial \rho'}{\partial t} + w' \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z} = \frac{\partial \rho'}{\partial t} - w' \frac{\rho_0}{g} N^2 &= 0 \\
\frac{\partial \boldsymbol{u}'}{\partial t} &= -\frac{1}{\rho_0}\nabla(p_0 + \hat{p}) - \frac{g\hat{\rho}}{\rho_0}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}
\end{aligned}$$

Hydrostatic balance eliminates the first two RHS terms of the momentum equation; the hydrostatic pressure field is

$$p_0 + \hat{p} = -\int g\hat{\rho}\,\mathrm{d}z$$

Finally we have

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \frac{g\rho'}{\rho_0}\hat{\boldsymbol{z}}$$

Define buoyancy $b = -g\rho'/\rho_0$. The governing equations are now

$$\begin{aligned}
\frac{\partial b}{\partial t} &= -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}} \\
\frac{\partial \boldsymbol{u}'}{\partial t} &= -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}}
\end{aligned}$$

To eliminate pressure, we take the curl of the momentum equation to get

$$\frac{\partial \boldsymbol{\zeta}'}{\partial t} = -\hat{\boldsymbol{z}} \times \nabla b$$

where $\boldsymbol{\zeta}' = \nabla \times \boldsymbol{u}'$ is the disturbance vorticity. Using the buoyancy equation we have

$$\left[\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2\right] w' = 0$$

where $\nabla_H = (\partial_x, \partial_y)$ is the horizontal gradient operator. This equation admits plane wave solutions

$$w'(\boldsymbol{x}, t) = \Re\left[\hat{w}(t) e^{i(k_x x + k_y y - \omega t)}\right]$$

where $\hat{w}$ satisfies

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)\hat{w} = 0$$

which has general solution

$$\begin{aligned}
\hat{w} &= \Re\left[Ae^{-inz} + Be^{inz}\right] \\
n^2 &= (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)
\end{aligned}$$

If $\omega > N$, n is imaginary and, defining $\gamma = \sqrt{1 - N^2/\omega^2}$, we have

$$w' = (Ae^{-\gamma k z} + Be^{\gamma k z})e^{i(k_x x + k_y y - \omega t)}$$

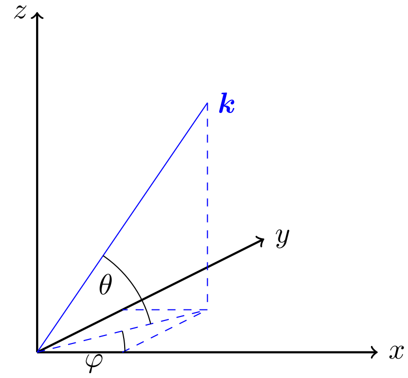
When $N = 0$, we get potential flow. If $0 < N < \omega$ then we have rescaled potential flow, with scaling γ. If $\omega < N$ then n is real and solutions are oscillatory with

$$n^2 = kz^2 = (k_x^2 + k_y^2)(\frac{N^2}{\omega^2} - 1)$$

The wavenumber vector is $\boldsymbol{k} = (k_x, k_y, k_z) = (k, l, m)$. Hence

$$\frac{\omega^2}{N^2} = \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} = 1 - \frac{k_z^2}{|\boldsymbol{k}|^2} = \cos^2\theta$$

where θ is the angle between $\boldsymbol{k}$ and the horizontal plane. Note that N is assumed constant.



We will use θ as the angle between the horizontal plane and $\boldsymbol{k}$, hence $\omega/N = \cos\theta$. Some authors use the polar inclination angle between $\hat{\boldsymbol{z}}$ and $\boldsymbol{k}$, in which case $\omega/N = \sin\theta$. The azimuthal angle between the x axis and the projection of $\boldsymbol{k}$ onto the horizontal plane is denoted by φ. We will also assume $\omega \geq 0$ going forward. The wavevector takes the form

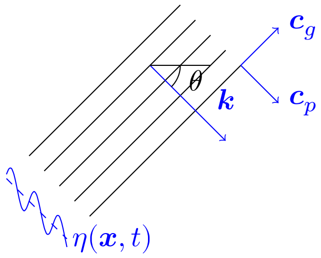
$$\boldsymbol{k} = |\boldsymbol{k}| \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}$$

The *phase* of the wave is defined to be $\phi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$. Note the following:

$$e^{i\phi} = e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$
$$\Re\left[e^{i\phi}\right] = \Re\left[e^{-i\phi}\right]$$
$$\Re\left[\tilde{\eta}e^{i\phi}\right] = \Re\left[\tilde{\eta}^* e^{-i\phi}\right]$$

We will focus on 2D waves. By suitable choice of coordinate system, 3D waves can be reduced to 2D. In the (x,z) plane, from $\nabla\cdot\boldsymbol{u} = 0$ and assuming $w(\boldsymbol{x}, t) = \tilde{w}e^{i\phi}$ with $\tilde{w} \in \mathbb{C}$ we have

$$u = \int -\frac{\partial w}{\partial z}\,\mathrm{d}x = -\frac{m}{k}\tilde{w}e^{i\phi} = -\tan\theta\tilde{w}e^{i\phi}$$



Denoting the wave displacement by $\eta(\boldsymbol{x},t) = \tilde{\eta}e^{i\phi}$ we have the following:

$$\begin{aligned}\frac{\partial \eta}{\partial t} &= -i\omega\tilde{\eta}e^{i\phi}\\ u(\boldsymbol{x},t) &= \tilde{u}e^{i\phi} = -i\omega\sin\theta\tilde{\eta}e^{i\phi}\\ w(\boldsymbol{x},t) &= \tilde{w}e^{i\phi} = i\omega\cos\theta\tilde{\eta}e^{i\phi}\end{aligned}$$

Using the buoyancy equation $\partial b/\partial t = -wN^2$ we also find

$$i\omega\tilde{b} = -\tilde{w}N^2 \implies b = -\tilde{\eta}\frac{\omega^2}{\cos\theta}e^{i\phi} = -\tilde{\eta}\omega N e^{i\phi}$$

Similarly the pressure field follows from the momentum equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \implies \tilde{p} = i\frac{\omega^2}{k}\tilde{\eta}\sin\theta$$

1.3 Wave velocities

1.3.1 Phase velocity

For the phase $\phi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ we can write

$$\begin{aligned}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x_i} &= 0\\ \implies k_i\frac{\partial \phi}{\partial t} + \omega\frac{\partial \phi}{\partial x_i} &= 0\\ \implies \frac{\partial \phi}{\partial t} + \frac{\omega}{|\boldsymbol{k}|^2}k_i\frac{\partial \phi}{\partial x_i}&\end{aligned}$$

Defining the phase velocity $\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k}$ as the velocity at which the phase is advected, we have

$$\frac{\partial \phi}{\partial t} + (\boldsymbol{c}_p\cdot\nabla)\phi = 0$$

For all waves, this is the speed at which wave crests move. For deep water waves with dispersion relation $\omega = \sqrt{gk}$ we find $c_p = \sqrt{g/k}$.

1.3.2 Group velocity

By symmetry of partial differentiation, we may write

$$\begin{aligned}\frac{\partial^2 \phi}{\partial t\partial x_i} - \frac{\partial^2 \phi}{\partial t\partial x_i} &= 0\\ \implies \frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} &= 0\end{aligned} \tag{4}$$

If $\omega = \omega(\boldsymbol{k})$ then from chain rule

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j}\frac{\partial k_j}{\partial x_i}$$

We also know from definition of phase

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right) = \frac{\partial k_i}{\partial x_j}$$

Hence (4) may be written as

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = 0$$

Thus we define the group velocity as the velocity at which the wavevector is advected, i.e. $\boldsymbol{c}_g = \frac{\partial \omega}{\partial k_i}$ and

$$\left(\frac{\partial}{\partial t} + \boldsymbol{c}_g \cdot \nabla\right)\boldsymbol{k} = 0$$

For deep water waves, we thus have $c_g = \sqrt{g/k}/2 = c_p/2$.

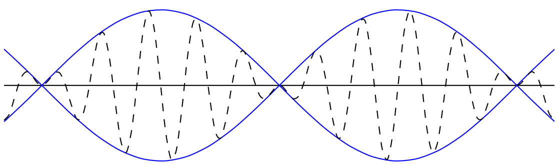
1.3.3 Superposition

Consider two waves with slightly different wavenumber and frequency superposed:

$$\begin{aligned}\eta &= \cos\left((k+\delta k)x - (\omega+\delta\omega)t\right) + \cos\left((k-\delta k)x - (\omega-\delta\omega)t\right)\\ &= 2\cos\left(\delta k x - \delta\omega t\right)\cos\left(kx - \omega t\right)\end{aligned}$$

This is referred to as *modulation* with $|\delta k| \ll |k|, |\delta\omega| \ll |\omega|$. Equivalently, in the limit as $|\delta k| \to 0, |\delta\omega| \to 0$, we have

$$\eta = 2\cos\left((x - \frac{\partial\omega}{\partial k}t)\delta k\right)\cos\left(kx - \omega t\right)$$



1.3.4 Internal gravity wave velocities

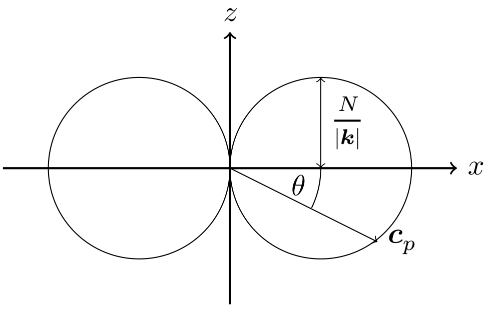
For internal gravity waves we have dispersion relation

$$\frac{\omega^2}{N^2} = \frac{k^2+l^2}{k^2+l^2+m^2} = \cos^2\theta$$

Hence the phase velocity is

$$\begin{aligned}\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k} &= N\frac{(k^2+l^2)^{1/2}}{(k^2+l^2+m^2)^{3/2}}\boldsymbol{k}\\ &= \frac{N\cos\theta}{|\boldsymbol{k}|^2}\begin{pmatrix}|\boldsymbol{k}|\cos\varphi\cos\theta\\ |\boldsymbol{k}|\sin\varphi\cos\theta\\ |\boldsymbol{k}|\sin\theta\end{pmatrix}\\ &= \frac{N|\cos\theta|}{|k|}\begin{pmatrix}\cos\varphi\cos\theta\\ \sin\varphi\cos\theta\\ \sin\theta\end{pmatrix}\end{aligned}$$

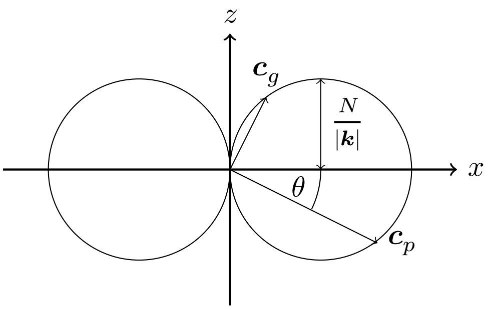
The prefactor is the magnitude of the phase velocity. The locus of possible phase velocities for given N and $\boldsymbol{k}$ is two circles:



The group velocity is

$$\boldsymbol{c}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{1}{2\omega}\frac{\partial \omega^2}{\partial \boldsymbol{k}} = \frac{\omega}{|\boldsymbol{k}|^2}\left(\frac{N^2}{\omega^2}K\boldsymbol{k} - m\hat{\boldsymbol{z}}) - \boldsymbol{k}\right) = \frac{N}{|\boldsymbol{k}|}|\sin\theta| \begin{pmatrix}\cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ -\cos\theta\end{pmatrix}$$

Hence the magnitude of the group velocity is $N|\sin\theta|/|\boldsymbol{k}|$. The group velocity is perpendicular to the phase velocity:



Note that

$$\boldsymbol{c}_g + \boldsymbol{c}_p = \frac{N}{|\boldsymbol{k}|}\left[|\cos\theta|\begin{pmatrix}\cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta\end{pmatrix} + |\sin\theta|\begin{pmatrix}\cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ -\cos\theta\end{pmatrix}\right] = \frac{N}{|\boldsymbol{k}|}\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix}$$

Hence $\left|\boldsymbol{c}_p + \boldsymbol{c}_g\right| = N/|\boldsymbol{k}|$ and $c_{pz} = -c_{gz}$. Note also that $\boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$.

1.4 Equipartition of energy

We can form an energy equation from the momentum equation dotted with $\boldsymbol{u}$:

$$\boldsymbol{u}\cdot\left(\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p + \rho g\hat{\boldsymbol{z}}\right) = 0$$

Using the Boussinesq approximation and linearising, we get

$$\boldsymbol{u}\cdot\left(\rho_0\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p' + \rho' g\hat{\boldsymbol{z}}\right) = 0$$
$$\frac{1}{2}\rho_0\frac{\partial}{\partial t}|\boldsymbol{u}|^2 + \boldsymbol{u}\cdot p' + w\rho' g = 0$$

The first term is the rate of change of kinetic energy, the second is the work against pressure gradients, and the last is the rate of change of potential energy.

Using linearised conservation of mass we have

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z} = \frac{\partial \rho'}{\partial t} - w\frac{\rho_0}{g}N^2 = 0 \implies w = \frac{g}{\rho_0 N^2}\frac{\partial \rho'}{\partial t}$$

Hence also using $\nabla \cdot \boldsymbol{u} = 0$ we have the energy equation

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \frac{1}{2}\frac{g^2}{\rho_0 N^2}\rho'^2\right] + \nabla \cdot (p'\boldsymbol{u}) = 0$$

The term proportional to ρ'^2 is *potential energy.* To see this, consider a parcel of fluid raised vertically by some amount ζ. The buoyant force on the parcel is

$$F = Vg\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}(z - z_0)$$

Hence the potential energy gained is

$$\mathrm{PE} = \int_{z_0}^{z_0+\zeta} g\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}(z - z_0)\,\mathrm{d}z = \frac{1}{2}\rho_0 N^2\zeta^2$$

Now from hydrostatic balance

$$\rho' = -\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}z}\zeta = \frac{\rho_0}{g}N^2\zeta$$

Hence the potential energy can be written in the following equivalent ways

$$\mathrm{PE} = \frac{1}{2}\rho_0 N^2\zeta^2 = \frac{1}{2}\frac{g^2}{\rho_0 N^2}\rho'^2 = \frac{1}{2}\rho_0\frac{b^2}{N^2}$$

So we have the energy equation

$$\frac{\partial}{\partial t}\left[\mathrm{KE} + \mathrm{PE}\right] + \nabla \cdot (p'\boldsymbol{u}) = 0$$

Integrating over a volume V we have

$$\int_V \frac{\partial}{\partial t}\left[\mathrm{KE} + \mathrm{PE}\right]\mathrm{d}V + \int_S p'\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = 0$$

Define the flux of energy, or work against pressure, as $\boldsymbol{F}_E = p'\boldsymbol{u}$. Recall that for a wave $\eta = \tilde{\eta}e^{i\phi}$, ,the dynamical variables u, w, b, p' can be expressed for $\tilde{\eta} \in \mathbb{R}$ as

$$\begin{aligned} u &= \tilde{\eta}\omega\sin\theta\sin\phi \\ w &= -\tilde{\eta}\omega\cos\theta\sin\phi \\ b &= \tilde{\eta}\frac{\omega^2}{\cos^2\theta} \\ p' &= \tilde{\eta}\rho_0\frac{\omega^2}{|\boldsymbol{k}|}\tan\theta\sin\phi \end{aligned}$$

Therefore we can write the KE and PE as

$$\begin{aligned} \mathrm{KE} &= \frac{1}{2}\rho_0(u^2 + w^2) = \frac{1}{2}\rho_0\omega^2\tilde{\eta}^2\sin^2\phi \\ \mathrm{PE} &= \frac{1}{2}\rho_0\omega^2\tilde{\eta}^2\cos^2\phi \\ \implies \overline{\mathrm{KE}} &= \overline{\mathrm{PE}} = \frac{1}{4}\rho_0\omega^2\tilde{\eta}^2 \end{aligned}$$

where $\bar{\cdot}$ denotes an average of a wavelength or period. Equipartition of energy is the statement $\overline{E} = \overline{\text{KE}} + \overline{\text{PE}} = 2\overline{\text{KE}} = 2\overline{\text{PE}}$. We may write the energy flux as

$$\boldsymbol{F}_E = p'\overline{u} = \rho_0 \omega^2 \tilde{\eta}^2 \sin^2\phi \frac{N}{|\boldsymbol{k}|} \sin\theta \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

The average energy flux is then

$$\overline{\boldsymbol{F}_E} = \frac{1}{2}\rho_0 \omega^2 \tilde{\eta}^2 \boldsymbol{c}_g = \overline{E}\boldsymbol{c}_g$$

which is the average energy multipled by the group velocity, which justifies the term 'energy flux', and shows energy is transported with the group velocity.

1.5 Amplitude decay along a beam

Consider a point source which generates a beam of internal waves, in a 2D incompressible fluid. We assume the buoyancy frequency N is constant, and set it equal to 1 for convenience. Also assume $\kappa = 0$, i.e. no diffusion. The governing equations are

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0 \\ \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial \rho}{\partial x} &= \nu \nabla^2 u \\ \frac{\partial w}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} - b &= \nu \nabla^2 w \\ \frac{\partial b}{\partial t} + N^2 w &= 0 \end{aligned}$$

where $b = -g\frac{\rho - \rho_0}{\rho_0}$. In 2D, we may introduce a streamfunction $\boldsymbol{\psi} = (0, \psi, 0)$ so that

$$\boldsymbol{u} = \nabla \times \boldsymbol{\psi} e^{-i\omega t} = (-\frac{\partial \psi}{\partial z}, 0, \frac{\partial \psi}{\partial x})e^{-i\omega t}$$

It is convenient to transform to a co-ordinate system (ξ, ζ) where ξ is in the direction of $\boldsymbol{k}$ and ζ is in the direction of $\boldsymbol{c}_g$, i.e.

$$\begin{aligned} \xi &= x\cos\theta - z\sin\theta \\ \zeta &= x\sin\theta + z\cos\theta \end{aligned}$$

In this co-ordinate system, the three governing equations can be reduced to two:

$$\begin{aligned} \frac{\partial b}{\partial t} + \frac{\partial \psi}{\partial \xi}\cos\theta + \frac{\partial \psi}{\partial \zeta}\sin\theta &= 0 \\ -i\omega\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2}\right) - \frac{\partial b}{\partial \zeta}\sin\theta - \frac{\partial b}{\partial \xi}\cos\theta - \nu\nabla^4\psi &= 0 \end{aligned}$$

Assuming the viscosity is small, we expand b and ψ in the small parameter $\varepsilon = \nu/2$:

$$\begin{aligned} b &= (b_0 + \varepsilon b_1 + \dots)e^{-i\omega t} \\ \psi &= (\psi_0 + \varepsilon \psi_1 + \dots)e^{-i\omega t} \end{aligned}$$

Further we define a scaled coordinate $\chi = \varepsilon\zeta/\sin\theta$. At order ε^0 we have:

$$\begin{aligned} \frac{\partial \psi_0}{\partial \xi} &= ib_0 \\ \frac{\partial^2 \psi_0}{\partial \xi^2} &= i\frac{\partial b_0}{\partial \xi} \end{aligned}$$

At order ε^1:

$$\omega\frac{\partial\psi_1}{\partial\xi} - i\omega b_1 = -\frac{\partial\psi_0}{\partial\chi}$$
$$i\omega\frac{\partial^2\psi_1}{\partial\xi^2} + \omega\frac{\partial b_1}{\partial\xi} = i\frac{\partial^2\psi_0}{\partial\xi\partial\chi} - 2\frac{\partial^4\psi_0}{\partial\xi^4}$$

From these we form a governing equation for ψ_0:

$$\frac{\partial^4\psi_0}{\partial\xi^4} = i\frac{\partial^2\psi_0}{\partial\xi\partial\chi} \implies \frac{\partial^3\psi_0}{\partial\xi^3} = i\frac{\partial\psi_0}{\partial\chi} + f(\chi)$$

For a point source, as $|\xi| \to \infty$, we except ψ will tend to a constant. Hence $f(\chi) \equiv 0$. The governing equation is separable: suppose $\psi_0 = F(\xi)G(\chi)$. Then

$$\frac{F'''}{F} = i\frac{G'}{G} = -ik^3$$
$$\implies G(\chi) = e^{-k^3\chi}, \quad F(\xi) = e^{ik\xi}$$

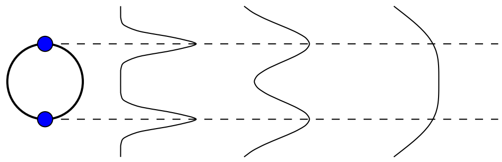
The streamfunction is therefore $\psi = Ae^{-k^3\chi}e^i\phi$. If N is constant but *not* equal to 1, we have

$$\chi = \frac{\nu}{2N\sin\theta}\zeta$$
$$\implies \psi = A\exp\left(\frac{-k^3\nu\zeta}{2N\sin\theta}\right)e^{i\phi}$$

If A depends on k then

$$\psi = \int_{-\infty}^{\infty} A(k)e^{-i\omega t}\exp\left(ik\xi - \frac{\nu k^3}{2N\sin\theta}\zeta\right)\,\mathrm{d}k$$

In the case of an oscillating cylinder, the beams are well described by point sources at the tangent points on the cylinder, and from above the amplitude is initially bimodal but smooths out (decays) to become unimodal:



We can form a Reynolds number to characterise these waves. Recall $|\boldsymbol{c}_g| = N\sin\theta/|\boldsymbol{k}|$. Hence

$$-\frac{\nu k^3}{2N\sin\theta}\zeta = -\frac{\nu k}{2}\frac{k}{N\sin\theta}k\zeta = -\pi\frac{\nu}{\lambda|\boldsymbol{c}_g|}k\zeta = -\pi\mathrm{Re}^{-1}k\zeta$$

where $\lambda = 2\pi/k$ is the wavelength. Thus we have

$$\mathrm{Re} = \frac{\lambda|\boldsymbol{c}_g|}{\nu}$$

in analogy with the usual definition of Reynolds number, UL/ν.

1.6 Mass diffusivity

In some situations, the fluid mass may diffuse, in which case mass conservation is modified to

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \kappa\nabla^2\rho$$

If $\rho = \rho(S, T, \phi)$ then

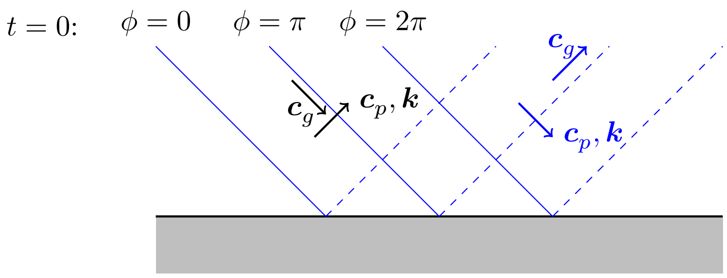
$$\frac{\mathrm{D}S}{\mathrm{D}t} = \kappa_S\nabla^2 S, \qquad \frac{\mathrm{D}T}{\mathrm{D}t} = \kappa_T\nabla^2 T$$

The ratio of viscosity to thermal diffusivity is characterised by the *Prandtl number* $\mathrm{Pr} = \nu/\kappa_T$. For heat diffusion in air, $\mathrm{Pr} \sim 0.7$ and in water, $\mathrm{Pr} \sim 7$. A more general *Schmidt number* $\mathrm{Sc} = \nu/\kappa_S$ characterises viscosity versus diffusion of some concentration, e.g. salt. Salt in water has $\mathrm{Sc} \sim 700$.

1.7 Reflections of waves

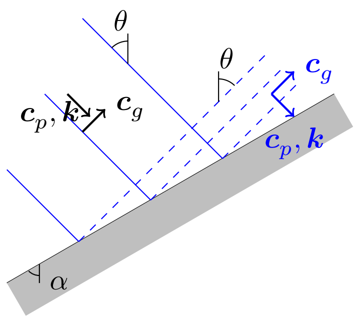
1.7.1 Properties of beams

For simplicity, assume N^2 constant. The usual laws of optics imply the incident angle is equal to the reflection angle. Thus $\omega/N = \cos\theta$ is conserved, assuming no Doppler shift, i.e. the boundary is stationary.



In the case of a horizontal reflection boundary, the vertical component of $\boldsymbol{u}$ is reversed, whilst the horizontal component is preserved. Energy conservation implies the magnitude of the flux of energy $\overline{\boldsymbol{F}} = \overline{E}\boldsymbol{c}_g$ is conserved, i.e. $|\overline{\boldsymbol{F}}_i| = |\overline{\boldsymbol{F}}_r|$. We also have wavelength conserved, $\lambda_i = \lambda_r$, wavevector magnitude conserved, $|\boldsymbol{k}_i| = |\boldsymbol{k}_r|$, and group velocity magnitude conserved, $|\boldsymbol{c}_{gi}| = |\boldsymbol{c}_{gr}|$. Consequently, the wave energy $\overline{E}_i = \overline{E}_r$ is conserved.

In the case of a vertical reflection boundary, the energy flux remains conserved, but instead the vertical component of $\boldsymbol{u}$ is reversed and the vertical component is preserved.



Finally, with a boundary at some angle α to the vertical, since θ is determined by the dispersion relation it is still conserved after reflection. Waves also preserve frequency, since it is set by the source, so $\theta_i = \theta_r$ and $\omega_i = \omega_r$. The wave displacement satisfies

$$|\tilde{\eta}_r| = \gamma|\tilde{\eta}_i|$$

where the magnitude is scaled by

$$\gamma = \left|\frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}\right| > 1$$

1.7.2 Energy density upon reflection

For reflection from a boundary at general angle α, we have scalings

$$\begin{aligned}
|\boldsymbol{k}_r| &= \gamma|\boldsymbol{k}_i| \iff \lambda_r = \frac{1}{\gamma}\lambda_i \\
|\tilde{w}_r| &= \gamma|\tilde{w}_i| \\
|\tilde{\eta}_r| &= \gamma|\tilde{\eta}_i| \\
|\boldsymbol{c}_{gr}| &= \frac{1}{\gamma}|\boldsymbol{c}_{gi}| \\
\left|\tilde{\boldsymbol{F}}_r\right| &= \left|\tilde{\boldsymbol{F}}_i\right|
\end{aligned}$$

where $\tilde{\boldsymbol{F}}$ is the energy flux per unit wavelength. The energy density per unit wavelength is

$$\tilde{E} = \int_0^\lambda \mathrm{PE} + \mathrm{KE}\,\mathrm{d}\xi$$

with $\tilde{\boldsymbol{F}} = \tilde{E}\boldsymbol{c}_g$ and hence $\tilde{E}_r = \gamma\tilde{E}_i$, since

$$\tilde{E} \sim \lambda(\mathrm{PE}+\mathrm{KE}) \sim \lambda(|\tilde{\eta}|^2 + |\tilde{w}|^2)$$

and λ scales by $1/\gamma$, whilst the terms in the bracket scale as γ^2.
The energy density per unit *length* is instead

$$\hat{E} = \frac{1}{\lambda}\int_0^\lambda \mathrm{PE} + \mathrm{KE}\,\mathrm{d}\xi = \frac{1}{\lambda}\tilde{E}$$

Hence $\tilde{\boldsymbol{F}} = \lambda\hat{E}\boldsymbol{c}_g$. The flux of energy per unit wavelength is conserved, so

$$\lambda_r\hat{E}_r|\boldsymbol{c}_{gr}| = \frac{1}{\gamma}\lambda_i\hat{E}_r\frac{1}{\gamma}|\boldsymbol{g}_{gi}| = \lambda_i\hat{E}_i|\boldsymbol{c}_{gi}|$$

Hence $\hat{E}_r = \gamma^2\hat{E}_i$. Denote the lengthscale for wave crests as L_i for the incident wave and L_r for the reflected wave. The total energy is then

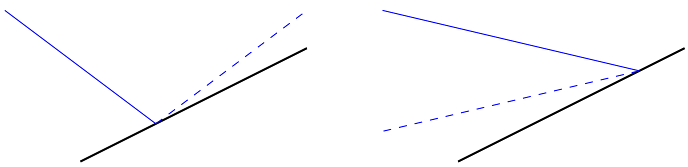
$$\begin{aligned}
\mathrm{TE}_r &= \int_{-\infty}^{\infty} \mathrm{PE}_r + \mathrm{KE}_r\,\mathrm{d}\xi \\
&= \int_{-L_r/2}^{L_r/2} \mathrm{PE}_r + \mathrm{KE}_r\,\mathrm{d}\xi \\
&= \gamma^2\int_{-L_i/2\gamma}^{L_i/2\gamma} \mathrm{PE}_i + \mathrm{KE}_i\,\mathrm{d}\xi \\
&= \gamma TE_i
\end{aligned}$$

Spectral energy denity $S(k)$ obeys

$$S_r(k) = \gamma S_i(k/\gamma)$$

1.7.3 Critical reflections

Subcritical reflections are where the vertical component of the group velocity reverses, whilst supercritical reflections are those where the horizontal component of the group velocity reverses.



At critical slope $\theta = \alpha$, $\gamma \to \infty$. If $\lambda \to \infty$ then $|\boldsymbol{k}_r| \to \infty$, $|\tilde{\eta}_r| \to \infty$ and $\left|\boldsymbol{c}_{gr}\right| \to 0$. As usual, $\omega_r = \omega_i$. Linear incident waves become non-linear after reflection close to criticality. The $e^{-\frac{\nu k^3 \zeta}{2N \sin\theta}}$ decay of an internal wave beam, combined with amplification of the wavenumber upon reflection and the no-slip boundary creates very rapid dissipation of energy, and hence very strong non-linearities.

1.7.4 Ray tracing

In 2D, internal waves satisfy the wave equation

$$\left(\nabla^2 \partial_t^2 + N^2 \nabla_H^2\right) \psi = 0$$

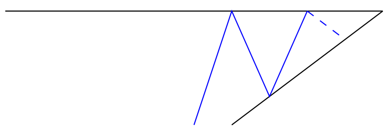
where $\psi = \tilde{\psi}(x, z) e^{-i\omega t}$ is the streamfunction. We have

$$(N^2 - \omega^2) \frac{\partial^2 \tilde{\psi}}{\partial x^2} - \omega^2 \frac{\partial^2 \tilde{\psi}}{\partial z^2} = 0$$

Define $\Lambda^2 = \omega^2 / (N^2 - \omega)^2$. Then we have the *Poincaré wave equation*

$$\left(\frac{\partial^2}{\partial x^2} - \Lambda^2 \frac{\partial^2}{\partial z^2}\right) \tilde{\psi} = 0$$

If the domain is bounded, $\tilde{\psi} = 0$ on the boundary but this is an ill-posed problem. Instead, we use ray tracing which effectively follows the energy around the domain. Consider a top-boundary with a sloping boundary below it:



The energy is trapped into the corner by repeated focusing reflections. For large $\mathrm{Re} = \left|\boldsymbol{c}_g\right| |\boldsymbol{k}| \nu$ then non-linearities will dominate and possible breaking giving mixing.

1.7.5 Reflections from rough topography

An incident monochromatic internal wave reflecting from a rough topography will in general become a beam with a spectrum of wavenumbers, i.e. $\boldsymbol{k}_r$ is a spectrum. Consider for aexample a sine wave topography of *small amplitude*. We may then linearise the topography.