Introduction to quantum mechanics

2.1

1 - 1 + 1 2 - 2 1 = 0 0

2.2

A0 = $A_{11}0 + A_{21}1 = 1 \Rightarrow A_{11} = 0,\ A_{21} = 1$

$A1 = A_{12}0 + A_{22}1 = 0 \Rightarrow A_{12} = 1,\ A_{22} = 0$

$A =$ 01 10

input: $\{0, 1\}$, output: $\{1, 0\}$

A0= $A_{11}1 + A_{21}0 = 1 \Rightarrow A_{11} = 1,\ A_{21} = 0$

$A1 = A_{12}1 + A_{22}0 = 0 \Rightarrow A_{12} = 0,\ A_{22} = 1$

$A =$ 10 01

2.3

From eq (2.12)

A v$_i = \sum_j A_{ji} w_j$

$Bw_j = \sum_k B_{kj} x_k$

Thus BA v$_i = B\left(\sum_j A_{ji} w_j\right)$

on $\mathcal{C}^n$ is $((y_1, \cdots, y_n), (z_1, \cdots, z_n)) = \sum_i y_i^* z_i$. Verify (1) of eq (2.13).

$((y_1, \cdots, y_n), \sum_i \lambda_i (z_{i1}, \cdots, z_{in})) = \sum_i y_i^* \left(\sum_j \lambda_j z_{ji} \right) = \sum_{i,j} y_i^* \lambda_j z_{ji} = \sum_j \lambda_j \left(\sum_i y_i^* z_{ji} \right) = \sum_j \lambda_j \left((y_1, \cdots, y_n), (z_{j1}, \cdots, z_{jn}) \right) = \sum_i \lambda_i \left((y_1, \cdots, y_n), (z_{i1}, \cdots, z_{in}) \right).$

Verify (2) of eq (2.13), $((y_1, \cdots, y_n), (z_1, \cdots, z_n))^* = \left(\sum_i y_i^* z_i \right)^* = \left(\sum_i y_i z_i^* \right) = \left(\sum_i z_i^* y_i \right) = ((z_1, \cdots, z_n), (y_1, \cdots, y_n)).$

Verify (3) of eq (2.13), $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = \sum_i y_i^* y_i = \sum_i |y_i|^2$

Since $|y_i|^2 \geq 0$ for all i. Thus $\sum_i |y_i|^2 = ((y_1, \cdots, y_n), (y_1, \cdots, y_n)) \geq 0$.

From now on, I will show the following statement, $((y_1, \cdots, y_n), (y_1, \cdots, y_n)) = 0 iff (y_1, \cdots, y_n) = 0$.

($\Leftarrow$) This is obvious.

($\Rightarrow$) Supbecause $v|A|v$ is real. Hence $\beta = v|C|v = 0$ for all v, i.e. $C = 0$.

Therefore $A = A^{\dagger}$.

Reference: MIT 8.05 Lecture note by Prof. Barton Zwiebach. https://ocw.mit.edu/courses/physics/8-05-quantum-physics-ii-fall-2013/lecture-notes/MIT$8_0 5F13_C hap_0 3.pdf$

Let T be a linear operator in a complex vector space V.

If $(u, Tv) = 0$ for all $u, v \in V$, then $T = 0$.

Suppose $u = Tv$. Then $(Tv, Tv) = 0$ for all v implies

i
$i0$
$=$
$02i$
$2i0$
$=$
$2iX$
$[Z, X] =$
10
$0-$
101
$10-$
01
1010
$0-$
1
$=$
$2i0-$
i
$i0$
$=$
$2iY$
2.41
$\{\sigma_1, \sigma_2\} =$
$\sigma_1\sigma_2+$
$\sigma_2\sigma_1$
$=$
01
$100-$
i
$i0+$
$0-$
i
$i001$
10
$=$
$i0$
$0-$
$i+$
$-i0$
$0i$
$=$
0
$\{\sigma_2, \sigma_3\} =$
$0-$
i
$i010$
$0-$
$1+$
10
$0-$
$10-$
i
$i0$
$=$
0
$\{\sigma_3, \sigma_1\} =$
10
$0-$
101
$10+$
01
1010
$0-$
1
$=$
0
$\sigma_0^2 =$
$I^2 =$
I
$\sigma_1^2 =$
01
$10^2 =$
I
$\sigma_2^2 =$
$0-$
i

the problem more abstractly. In order to prove, we use the following properties of $\vec{v}\cdot\vec{\sigma}$

$\vec{v}\cdot\vec{\sigma}$ is Hermitian

$(\vec{v}\cdot\vec{\sigma})^2 = I$ where $\vec{v}$ is a real unit vector.

We can easily check above conditions.

$$
\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^\dagger &= (v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)^\dagger \\
&= v_1\sigma_1^\dagger + v_2\sigma_2^\dagger + v_3\sigma_3^\dagger \\
&= v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \quad (Pauli matrices are Hermitian.) \\
&= \vec{v}\cdot\vec{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
(\vec{v}\cdot\vec{\sigma})^2 &= \sum_{j,k=1}^3 (v_j\sigma_j)(v_k\sigma_k) \\
&= \sum_{j,k=1}^3 v_j v_k \sigma_j\sigma_k \\
&= \sum_{j,k=1}^3 v_j v_k \left(\delta_{jk} I + i\sum_{l=1}^3 \epsilon_{jkl}\sigma_l\right) \quad (eqn(2.78)\ page 78) \\
&= \sum_{j,k=1}^3 v_j v_k \delta_{jk} I +
\end{aligned}
$$

$= \sum_i p_i^2 i$

$(\rho^2) = \left(\sum_i p_i^2 i\right) = \sum_i p_i^2 (i) = \sum_i p_i^2 i|i = \sum_i p_i^2 \le \sum_i p_i = 1$ $(p_i^2 \le p_i)$

Suppose $(\rho^2) = 1.$ Then $\sum_i p_i^2 = 1.$ Since $p_i^2 < p_i$ for $0 < p_i < 1,$ only single p_i should be 1 and otherwise have to vanish. Therefore $\rho = \psi_i.$ It is a pure state.

Conversely if ρ is pure, then $\rho = \psi.$ $(\rho^2) = (\psi\psi|\psi\psi) = (\psi) = \psi|\psi = 1.$

2.72

(1) Since density matrix is Hermitian, matrix represen-