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21/481767/TK/53170

Tugas 7.

5.1

The random variables $Y_1, \dots Y_4$ have the joint PDF

$$f_{Y_1 \dots} f_{Y_4}(y_1, \dots y_4) = \begin{cases} 4 & 0 \le y_1 \le y_2 \le 1,\ 0 \le y_3 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let C donate the event that $\max_i Y_i \le 1/2$. Find $Pr[C]$

$Pr[C] = Pr[Y_i \le 1/2]$ means every upper bound random variables is restricted equally or bellow $1/2$.

$0 \le y_1 \le y_2$; $0 \le y_2 \le \frac{1}{2}$; $0 \le y_3 \le y_4$; $0 \le y_4 \le \frac{1}{2}$

$$Pr[Y_i \le 1/2] = \int_0^{1/2}\int_0^{y_2}\int_0^{1/2}\int_0^{y_4} f_{Y_1 \dots Y_4}(y_1 \dots y_4)\, dy_2\, dy_1\, dy_4\, dy_3$$

$$= \int_0^{1/2}\int_0^{y_4}\int_0^{1/2}\int_0^{y_2} 4 \quad dy_1\, dy_2\, dy_3\, dy_4$$

$$= 1/4_{\,//}$$

5.3 $\mathbf{X} = [X_1, X_2, X_3]'$ has PDF

$$f_X(\mathbf{X}) = \begin{cases} 6 & 0 \le x_1 \le x_2 \le x_3 \le 1 \\ 0 & \end{cases}$$

find Marginal PDF :

- $f_{X_1 X_2}[x_1 x_2] = \int_{x_2}^{1} 6 \ dx_3 = 6(1-x_2)_{//}$
- $f_{X_1 X_3}[x_1 x_3] = \int_{x_1}^{x_3} 6 \ dx_2 = 6(x_3 - x_1)$
- $f_{X_2 X_3}[x_2 x_3] = \int_{0}^{x_2} 6 \ dx_1 = 6x_2{}_{//}$

• $f_{X_1}(x_1) = \int_{x_1}^{1} f_{X_1 X_2}(x_1, x_2)\, dx_2 = \int_{x_1}^{1} 6(1-x_2)\, dx_2 = 6\left[x_2 - \frac{1}{2}x_2^2\right]_{x_1}^{1}$

$= 6\left(\frac{1}{2} - x_1 + \frac{1}{2}x_1^2\right) = 3(1 - 2x_1 + x_1^2) = 3(x_1 - 1)^2$ //

• $f_{X_2}(x_2) = \int_0^{x_2} 6(1-x_2)\, dx_1 = 6\left[x_1 - x_2 x_1\right]_0^{x_2} = 6(x_2 - x_2^2) = 6x_2(1-x_2)$

• $f_{X_3}(x_3) = \int_0^{x_3} f_{X_2 X_3}(x_2, x_3)\, dx_2 = \int_0^{x_3} 6x_2\, dx_2 = 6\left(\frac{1}{2}x_2^2\right]_0^{x_3} = 3x_3^2$,

$\boxed{5.6.1}$ Random Variables X_1 and X_2 have zero expected value and $Var[X_1] = 4$ and $Var[X_2] = 9$. Their covariances is $Cov[X_1, X_2] = 3$

a) Find the covariance matrix of $\mathbf{X} = [X_1, X_2]^T$

b) X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$Y_1 = X_1 - 2X_2$

$Y_2 = 3X_1 + 4X_2$

Find the covariances matrix of $Y = [Y_1\ Y_2]^T$

a) $Cov[\mathbf{X}] = R\,\mathbf{X}\mathbf{X}^T - E(\mathbf{X})E(\mathbf{X})^T = \begin{bmatrix} Var[\mathbf{X}] & Cov[\mathbf{X}] \\ Cov[\mathbf{X}] & Var[\mathbf{X} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$ //

b) $C_{\mathbf{Y}} = A\, Cov[\mathbf{X}]\, A^T$

with $A \longrightarrow \mathbf{Y} = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}}_{A} \mathbf{X}$

So $C_{\mathbf{Y}} = A \, Cov[\mathbf{X}] \, A^T$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -15 \\ 24 & 45 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$

$\boxed{5.6.4}$ 4- dimensional random vector:

$$f_{\mathbf{X}}(x) = \begin{cases} 1 & 0 \leq x_i \leq 1, \quad i = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

• Find $E[X] = [E[X_1] \;\; E[X_2] \;\; E[X_3] \;\; E[X_4]]^T$

$$E[x_i] = \int_0^1 x_i f_{X_i}(x_i) \, dx_i = \int_0^1 x_i \, dx_i = \frac{1}{2} x_i^2 \Big]_0^1 = \frac{1}{2}$$

$$E[\mathbf{X}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} //$$

• Find $R_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^T] =:$

$$\mathbf{X}\mathbf{X}^T = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & x_1x_4 \\ x_2x_1 & x_2^2 & x_2x_3 & x_2x_4 \\ x_3x_1 & x_3x_2 & x_3^2 & x_3x_4 \\ x_4x_1 & x_4x_2 & x_4x_3 & x_4^2 \end{bmatrix} \rightarrow R_{\mathbf{X}} = \begin{bmatrix} E[x_1^2] & E[x_1x_2] & E[x_1x_3] & E[x_1x_4] \\ E[x_2x_1] & E[x_2^2] & E[x_2x_3] & E[x_2x_4] \\ E[x_3x_1] & E[x_3x_2] & E[x_3^2] & E[x_3x_4] \\ E[x_4x_1] & E[x_4x_2] & E[x_4x_3] & E[x_4^2] \end{bmatrix}$$

$$E[X_i^2] = \int_0^1 x_i^2 \, f_{X_i}(x_i) \, dx_i = \frac{1}{3}$$

$$E[X_i X_j] = \int_0^1 \int_0^1 x_i x_j \, f_{X_i X_j}(x_i x_j) \, dx_i \, dx_j$$

$$= \int_0^1 \frac{1}{2} x_i^2 x_j \, dx_j \Big]_0^1 = \int_0^1 \frac{1}{2} x_j \, dx_j = \frac{1}{4} x_j^2 \Big]_0^1 = \frac{1}{4}$$

$$R_X = \begin{bmatrix} 1/3 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/3 \end{bmatrix}_{/\!/}$$

•) Find $C_X = R_X - E[X]\,E[X]^T$

$$= \begin{bmatrix} 1/3 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/3 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/3 \end{bmatrix} - \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/12 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & 1/12 \end{bmatrix}_{/\!/}$$