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$\boxed{3.39}$ From a sack of fruit containing 3 oranges, 2 apples, and 3 bananas, a random sample of 4 pieces of fruit is selected. If X is the number of oranges and Y is the number of apples in the sample, Find

a) the joint probability distribution of X and Y;

b) $P_r[X, Y \in A]$ where A is the region that is given by $\{(x,y) \mid x+y \leq 2\}$

Answer:

a) Sample : $\binom{8}{4} = \frac{8!}{4!\,4!} = \frac{\cancel{8}^{2} \cdot 7 \cdot 6 \cdot 5}{\cancel{4} \cdot \cancel{6}} = 70$

Possible outcomes: $(3,0)$; $(2,0)$; $(1,0)$; $(3,1)$; $(2;1)$; $(1,1)$; $(0,1)$; $(2,2)$; $(1,2)$; $(0,2)$

The formula : $\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}$

The joint probability distribution of oranges and apples in X and Y :

$$f(x,y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}} & \text{for } 1 \leq x+y \leq 4 \\ 0 & \text{elsewhere.} \end{cases}$$

b) $F_{X,Y}(x+y \le 2)$

the possible outcome: $(2,0)$; $(1,0)$ $(1,1)$; $(0,1)$ $(0,2)$

- $f(2,0) = \dfrac{\binom{3}{2}\binom{2}{0}\binom{3}{2}}{70} = \dfrac{3 \cdot 1 \cdot 3}{70} = \dfrac{9}{70}$

- $f(1,0) = \dfrac{\binom{3}{1}\binom{2}{0}\binom{3}{3}}{70} = \dfrac{3}{70}$

- $f(1,1) = \dfrac{\binom{3}{1}\binom{2}{1}\binom{3}{2}}{70} = \dfrac{3 \cdot 2 \cdot 3}{70} = \dfrac{18}{70}$

- $f(0,1) = \dfrac{\binom{3}{0}\binom{2}{1}\binom{3}{3}}{70} = \dfrac{1 \cdot 2 \cdot 1}{70} = \dfrac{2}{70}$

- $f(0,2) = \dfrac{\binom{3}{0}\binom{2}{2}\binom{3}{2}}{70} = \dfrac{1 \cdot 1 \cdot 3}{70} = \dfrac{3}{70}$

$$f(2,0) + f(1,0) + f(1,1) + f(0,1) + f(0,2) = \frac{9+3+18+2+3}{70} = \frac{35}{70} = \frac{1}{2}$$

its make sense because the possible outcome for $F_{XY}(x+y \le 2)$ is 5 which is from 10 total outcome.

$\boxed{4.44}$ Find the covariance of the random variables X and Y of Exercise 3.39

Find all probality :

$$f(3,0) = \frac{\binom{3}{3}\binom{2}{0}\binom{3}{1}}{70} = \frac{3}{70}$$

$$f(2,0) = \frac{9}{7}$$

$$f(1,0) = \frac{3}{70}$$

$$f(3,1) = \frac{\binom{3}{3}\binom{2}{1}\binom{3}{0}}{70} = \frac{2}{70}$$

$$f(2,1) = \frac{\binom{3}{2}\binom{2}{1}\binom{3}{1}}{70} = \frac{3 \cdot 2 \cdot 3}{70} = \frac{18}{70}$$

$$f(1,1) = \frac{18}{70}$$

$$f(0,1) = \frac{2}{70}$$

$$f(2,2) = \frac{\binom{3}{2}\binom{2}{2}\binom{3}{0}}{70} = \frac{3}{70}$$

$$f(1,2) = \frac{\binom{3}{2}\binom{2}{2}\binom{3}{1}}{70} = \frac{9}{70}$$

$$f(0,2) = \frac{3}{70}$$

$\text{Cov}[X,Y] = E[XY] - E[X]E[Y]$

Find $E[XY] = \sum_x \sum_y xy\, f_{XY}(x,y)$

$= 3 \cdot \frac{2}{70} + 2 \cdot \frac{18}{70} + \frac{18}{70} + 2 \cdot 2 \cdot \frac{3}{70} + 1 \cdot 2 \cdot \frac{9}{70} = \frac{90}{70} = \frac{9}{7}$

Find $E[X] = \sum_x \sum_y x\, f_{XY}(x,y)$

$= \frac{9}{70} + \frac{18}{70} + \frac{3}{70} + \frac{6}{70} + \frac{36}{70} + \frac{18}{70} + \frac{6}{70} + \frac{9}{70} = \frac{105}{70} = \frac{3}{2}$

Find $E[Y] = \sum_x \sum_y y\, f_{XY}(x,y)$

$= \frac{2}{70} + \frac{18}{70} + \frac{18}{70} + \frac{6}{70} + \frac{18}{70} + \frac{6}{70} + \frac{2}{70} = \frac{70}{70} = 1$

$\text{Cov}[X,Y] = E[X,Y] - E[X]E[Y]$

$= \frac{9}{7} - \frac{3}{2} \cdot 1 = \frac{18-21}{14} = -\frac{3}{14}$ // weakly anticolerated.

$\boxed{4.51}$ For the random variables X and Y in Excercise 3.39, determine the coleration coefficeient between X and Y.

$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\text{Cov}[XY]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}}$

Find $E[X^2] = \sum_x \sum_y x^2 f_{XY}(x,y)$

$= \frac{27}{70} + \frac{36}{70} + \frac{3}{70} + \frac{18}{70} + \frac{72}{70} + \frac{18}{70} + \frac{12}{70} + \frac{9}{70} = \frac{195}{70}$

$$\begin{array}{c} 195 \\ \wedge \\ 3 \quad 65 \\ \wedge \\ 5 \quad 13 \end{array}$$

Find $E[Y^2] = \sum_x \sum_y y^2 f_{XY}(x,y)$

$= \frac{2}{70} + \frac{18}{70} + \frac{18}{70} + \frac{2}{70} + \frac{12}{70} + \frac{36}{70} + \frac{12}{70} = \frac{100}{70}$

Find $Var[X] = E[X^2] - E[X]^2$

$= \frac{195}{70} - \left(\frac{3}{2}\right)^2 = \frac{195}{70} - \frac{9}{4} = \frac{15}{28}$

Find $Var[Y] = E[Y^2] - E[Y]^2$

$= \frac{100}{70} - 1 = \frac{30}{70} = \frac{3}{7}$

$\rho_{XY} = \frac{Cov[XY]}{\sqrt{Var[X]\,Var[Y]}} = \frac{-3/14}{\sqrt{\frac{15}{28} \cdot \frac{3}{7}}} = \frac{-3}{14} \frac{\sqrt{28} \cdot \sqrt{7}}{\sqrt{45}} = \frac{-\cancel{3}}{14} \frac{\sqrt{28} \cdot \sqrt{7}}{\cancel{3}\sqrt{5}}$

$= \frac{-1}{14} \frac{\sqrt{7 \cdot 4 \cdot 7}}{\sqrt{5}} = \frac{-1}{\cancel{2}} \frac{\cancel{\sqrt{4}}}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$ // $\rightarrow$ weakly collerated //

$\boxed{3.47}$ Temperature $= Y$

Reaction time $= X$

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find

a) $P(0 \le X \le \frac{1}{2} \text{ and } \frac{1}{4} \le Y \le \frac{1}{2})$

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^{\frac{1}{2}} 4xy \, dy \, dx = \int_0^{\frac{1}{2}} 2xy^2 \, dx \Big]_{\frac{1}{4}}^{\frac{1}{2}} = \int_0^{\frac{1}{2}} 2x\frac{1}{4} \, dx - \int_0^{\frac{1}{2}} 2x\frac{1}{16} \, dx$$

$$= \int_0^{\frac{1}{2}} 2x\frac{3}{16} \, dx$$

$$= x^2\frac{3}{16}\Big]_0^{\frac{1}{2}} = \frac{3}{64} \,//$$

b) $P(X < Y) = P(0 < x < 1;\ x < y < 1)$

$$= \int_0^1\int_x^1 4xy \, dy \, dx = \int_0^1 2xy^2 \, dx \Big]_x^1 = \int_0^1 2x \, dx - \int_0^1 2x^3 \, dx$$

$$= x^2\Big]_0^1 - \frac{1}{2}x^4\Big]_0^1 = 1 - \frac{1}{2} = \frac{1}{2} \,//$$

$\boxed{3.47}$ Kerosene $= Y$

Sold $= X$

No refill $\Rightarrow X \le Y$

$$f(x,y) = \begin{cases} 2, & 0 < x \le y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Determine if X and Y are independent

if X and Y is independent then, $f_{XY}(x,y) = g_X(x)\, h_Y(y)$

Find $g_X(x) = \int_x^1 2 \, dy = 2 - 2x$

Find $h_Y(y) = \int_0^y 2 \, dx = 2y$

$$g_X(x) \cdot h_Y(y) = 2y(2-2x)$$
$$= 4y - 4xy \neq 2$$

b) Find $P(1/4 < X < 1/2 \mid Y = 3/4)$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} f_{X|Y}(x|y) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = h_Y(y) = 2y$$

Thus

$$\Leftrightarrow \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2}{2y}\,dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{y}\,dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4}{3}\,dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$ //

$\boxed{3.49}$ X = malfunction

Y = technicians being called.

Joint Probability Distribution:

$f(x,y)$	x = 1	2	3
1	0,05	0,05	0,1
3	0,05	0,1	0,35
5	0,00	0,2	0,1

a) Evaluate the marginal distribution of X

$$f_X(x) = \sum_y f_{XY}(x,y) = \text{Total of columns}$$

$Pr[X=1] = 0,1$

$Pr[X=2] = 0,35$

$Pr[X=3] = 0,55$

X	1	2	3
$f_X(x)$	0,1	0,35	0,55

b) Evaluate the marginal distribution of Y

$$f_Y(y) = \sum_x f_{XY}(x,y) = \text{Total of rows}$$

$Pr[Y=1] = 0,2$ $Pr[Y=5] = 0,3$

(Continue in the next pages) *

$Pr[Y=2] = 0,5$

Marginal distribution of Y

y	1	3	5
$f_Y(y)$	0,2	0,5	0,3

c) Find $Pr(Y=3 \mid X=2)$

$$Pr[Y=3 \mid X=2] = \frac{Pr[Y=3, X=2]}{Pr[X=2]} = \frac{0,1}{0,35} = \frac{10}{35} = \frac{2}{7}$$

$\boxed{4.45}$ Find the covariances of random variables X and Y of 3.49

Find $E[XY] = \sum_x \sum_y xy \, f_{XY}(x,y)$

$= 0,05 + 0,1 + 0,3 + 0,15 + 0,6 + 3,15 + 0,00 + 2 + 1,5$

$= 7,85$

Find $E[X] = \sum_x x \, f_X(x)$

$= 0,1 + 0,7 + 1,65 = 2,45$

Find $E[Y] = \sum_y y \, f_Y(y)$

$= 0,2 + 1,5 + 1,5 = 3,2$

$Cov[XY] = E[X,Y] - E[X]E[Y] = 7,85 - (2,45)(3,2) = 0,1$

$\boxed{4.52}$ $f_{XY}(x,y) = \begin{cases} 2 & 0 < x \leq y < 1 \\ 0 & \text{othe wise} \end{cases}$

Determine the correlation coefficient between X and Y

$$\rho_{XY} = \frac{Cov[XY]}{\sqrt{Var[X]Var[Y]}} =$$

Find $COV[XY] = E[XY] - E[X]E[Y]$

∘) Find $E[X] = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2-2x)\,dx = \int_0^1 2x - 2x^2\,dx = x^2 - \frac{2}{3}x^3 \Big]_0^1$
$= \frac{1}{3}$

∘) Find $E[Y] = \int_0^1 y f_Y(y)\,dy = \int_0^1 y 2y\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}\Big]_0^1 = \frac{2}{3}$,

All $E[X]$ and $E[Y]$ is known from question no. 3 Problem 3.47

∘) Find $E[XY] = \int_0^1\int_0^y xy\,2\,dx\,dy = \int_0^1 x^2y\Big]_0^y\,dy = \int_0^1 y^3\,dy = \frac{1}{4}y^4\Big]_0^1$
$= \frac{1}{4}$

$Cov[XY] = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \frac{9}{36} - \frac{8}{36} = \frac{1}{36}$

∘) Find $Var[X] = E[X^2] - E[X]^2$

Find $E[X^2] = \int_0^1 x^2(2-2x)\,dx = \int_0^1 2x^2 - 2x^3\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^4\Big]_0^1 = \frac{2}{3} - \frac{1}{2}$
$= \frac{1}{6}$

$Var[X] = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18}$,

∘) Find $Var[Y] = E[Y^2] - E[Y]^2$

Find $E[Y^2] = \int_0^1 y^2 2y\,dy = \int_0^1 2y^3\,dy = \frac{1}{2}y^4\Big]_0^1 = \frac{1}{2}$

$Var[Y] = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$

$$\rho XY = \frac{Cov[XY]}{\sqrt{Var[X]Var[Y]}} = \frac{1/36}{1/18} = \frac{1}{2}$$ //

6. Buktikan secara matematis bahwa koefisien korelasi selalu berkisar antara -1 dan 1

$$\rho XY = \frac{Cov[XY]}{\sqrt{Var[X]Var[Y]}}$$

Donate $Z = x - ay$

Now : $Var[Z] = Var[x-ay] = E[(x-ay)^2] - E[x-ay]^2$

$$= E[x^2] + a^2 E[y^2] - 2a E[xy]$$
$$- E[x]^2 - a^2 E[Y]^2 + 2a E[X] E[Y]$$

We can write as $Var[X]$ and $Var[Y]$:

$$Var[Z] = Var[X] + a^2 Var[Y] - 2a Cov[XY]$$

This $Var[Z] \geq 0$

So : $Cov[XY] \leq \frac{Var[X] + a^2 Var[Y]}{2a}$

for maximum value : $\frac{Var[X] + a^2 Var[Y]}{2a}$ has derivative equall to 0

$$\frac{d}{da} \frac{Var[X] + a^2 \ Var[Y]}{2a} = 0$$

$$-\frac{Var[X]}{a^2} + Var[Y] = 0$$

$$a^2 = \frac{Var[X]}{Var[Y]} \longrightarrow a = \pm\sqrt{\frac{Var[X]}{Var[Y]}}$$

for $a = +\sqrt{\frac{Var[X]}{Var[Y]}}$

So: $Cov[XY] \leq \dfrac{Var[X] + \dfrac{Var[X]}{\cancel{Var[Y]}}\cancel{Var[Y]}}{2\sqrt{\dfrac{Var[X]}{Var[Y]}}} = \sqrt{Var[X]Var[Y]}$

$\dfrac{Cov[XY]}{\sqrt{Var[X]Var[Y]}} \leq 1$ devided by $\sqrt{Var[X]Var[Y]}$

for $a = -\sqrt{\frac{Var[X]}{Var[Y]}}$

Subtitute into: $Var[X] + a^2 Var[Y] - 2a\,Cov[XY] \geq 0$

$= Var[X] + \dfrac{Var[X]}{\cancel{Var[Y]}}\cancel{Var[Y]} - 2\left(-\dfrac{\sqrt{Var[X]}}{\sqrt{Var[Y]}}\right)Cov[XY] \geq 0$

$= 2Var[X] + 2\sqrt{\dfrac{Var[X]}{Var[Y]}}\;Cov[XY] \geq 0$

$= Cov[XY] \geq -\dfrac{\cancel{2}Var[X]}{\cancel{2}\sqrt{\dfrac{Var[X]}{Var[Y]}}}$

$= Cov[XY] \geq -\sqrt{Var[X]Var[Y]}$ //

$\dfrac{Cov[XY]}{\sqrt{Var[X]Var[Y]}} \geq -1$

proved //