Homework 2

Qornain Aji
217481767 / TK / 53170

Prove that

$$x = A e^{-\frac{bt}{2m}} \cos(\omega t + \phi_0) \quad \text{with} \quad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{is the same as}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

Answer : Find $x'(t)$ and $x''(t)$.

• $x'(t) = \frac{d}{dt}\left(A e^{-\frac{bt}{2m}} \cos(\omega t + \phi_0)\right)$

$= A \left(\frac{d}{dt} e^{-\frac{b}{2m}t} \cos(\omega t + \phi_0)\right)$

$= A \left(\left(-\frac{b}{2m} e^{-\frac{b}{2m}t}\right)\left(\cos(\omega t + \phi_0)\right) + \left(e^{-\frac{b}{2m}t}\right)\left(-\omega \sin(\omega t + \phi_0)\right)\right)$

$= -A e^{-\frac{b}{2m}t}\left(\frac{b}{2m}\cos(\omega t + \phi_0) + \omega \sin(\omega t + \phi_0)\right)$

• $x''(t) = \frac{d^2}{dt^2} -A e^{-\frac{b}{2m}t}\left(\frac{b}{2m}\cos(\omega t + \phi_0) + \omega \sin(\omega t + \phi_0)\right)$

$= -A \frac{d^2}{dt^2} e^{-\frac{b}{2m}t}\left(\frac{b}{2m}\cos(\omega t + \phi_0) + \omega \sin(\omega t + \phi_0)\right)$

$= -A \left(\left(-\frac{b}{2m} e^{-\frac{b}{2m}t}\right)\left(\frac{b}{2m}\cos(\omega t + \phi_0) + \omega \sin(\omega t + \phi_0)\right) + \left(e^{-\frac{b}{2m}t}\right)\right.$
$\left.\left(-\frac{b}{2m}\omega \sin(\omega t + \phi_0) + \omega^2 \cos(\omega t + \phi_0)\right)\right)$

$= -A \left(-\frac{b^2}{4m^2} e^{-\frac{b}{2m}t} \cos(\omega t + \phi_0) - \frac{b}{2m} e^{-\frac{b}{2m}t} \omega \sin(\omega t + \phi_0) - \frac{b}{2m} e^{-\frac{b}{2m}t} \omega \sin(\omega t + \phi_0)\right.$
$\left. + e^{-\frac{b}{2m}t} \omega^2 \cos(\omega t + \phi_0)\right)$

$= A e^{-\frac{b}{2m}t}\left(\frac{b^2}{4m^2}\cos(\omega t + \phi_0) + \frac{b}{2m}\omega \sin(\omega t + \phi_0) + \frac{b}{2m}\omega \sin(\omega t + \phi_0)\right.$
$\left. - \omega^2 \cos(\omega t + \phi_0)\right)$

Now, Subtitute the equation into $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

$$\dot{x} = -Ae^{-\frac{b}{2m}t}\left(\frac{b}{2m}\cos(\omega t + \phi_0) + \omega\sin(\omega t + \phi_0)\right),$$

$$\ddot{x} = Ae^{-\frac{b}{2m}t}\left(\frac{b^2}{4m^2}\cos(\omega t + \phi_0) + \frac{b}{m}\omega\sin(\omega t + \phi_0) - \omega^2\cos(\omega t + \phi_0)\right)$$

Now, let say that : $\cos(\omega t + \phi_0) = u$
$\sin(\omega t + \phi_0) = v$

So, the final subtitution form will be

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$= Ae^{-\frac{b}{2m}t}\left(\frac{b^2}{4m^2}u + \frac{b}{m}\omega v - \omega^2 u\right) + \frac{b}{m}\left(-Ae^{-\frac{b}{2m}t}\left(\frac{b}{2m}u + \omega v\right)\right) + \frac{k}{m}Ae^{-\frac{b}{2m}t}u = 0$$

$$= Ae^{-\frac{b}{2m}t}\frac{b^2}{4m^2}u + \cancel{Ae^{-\frac{bt}{2m}}\frac{b}{m}\omega v} - Ae^{-\frac{b}{2m}t}\omega^2 u - Ae^{-\frac{bt}{2m}}\frac{b^2}{2m^2}u - \cancel{Ae^{-\frac{b}{2m}t}\frac{b}{m}\omega v} + \frac{k}{m}Ae^{-\frac{bt}{2m}}u = 0$$

$$= -Ae^{-\frac{bt}{2m}}\frac{b^2}{4m^2}u - Ae^{-\frac{b}{2m}t}\omega^2 u + \frac{k}{m}Ae^{-\frac{bt}{2m}}u = 0$$

$$= Ae^{-\frac{bt}{2m}}u\left(-\frac{b^2}{4m^2} - \omega^2 + \frac{k}{m}\right) = 0$$

$$-\frac{b^2}{4m^2} - \left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)^2 + \frac{k}{m} = 0$$

$$\cancel{\frac{-b^2}{4m^2}} - \frac{k}{m} + \cancel{\frac{b^2}{4m^2}} + \frac{k}{m} = 0$$

$$0 = 0 \quad \text{proves.}$$