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21/481767/TK/53170

ALIN Homework 2

Diagonalize these Hermitian matrices to reach $S = Q \Lambda Q^H$

a) $S = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}$

Find eigenvalues:

$$\begin{vmatrix} -\lambda & 1-i \\ i+1 & 1-\lambda \end{vmatrix} = -\lambda + \lambda^2 - 2 \iff \lambda^2 - \lambda - 2 \iff \lambda_{12} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{1 \pm 3}{2}$$

$$\lambda_{12} = 2 ; -1$$

Find Eigenvectors:

$\lambda = 2$

$$\begin{bmatrix} -2 & 1-i \\ i+1 & -1 \end{bmatrix} x_1 = 0 \rightarrow \begin{bmatrix} -2 & 1-i \\ 0 & 0 \end{bmatrix} x_1 = 0$$

$R_2' = R_2 + R_1\left(\frac{i+1}{2}\right)$

x_1	x_2
0	0
$\frac{1-i}{2}$	1

$$x_1 = \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}, \quad Q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

$-2x_1 + 1 - i = 0$

$x_1 = \frac{-i+1}{2}$

$\lambda = -1$

$$\begin{bmatrix} 1 & 1-i \\ i+1 & 2 \end{bmatrix} x_2 = 0 \longrightarrow \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix} x_2 = 0$$

$R_2' = R_2 - R_1(i+1)$

x_1	x_2
0	0
$-1+i$	1

$\iff x_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$

$$Q_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

Hence $SQ = Q\Lambda$

$$\begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1-i}{6}\sqrt{6} & \frac{-1+i}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{6}\sqrt{6} & \frac{-1+i}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S = Q\Lambda Q^H$$

$$= \begin{bmatrix} \frac{1-i}{6}\sqrt{6} & \frac{-1+i}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & \frac{1}{3}\sqrt{3} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1+i}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \\ \frac{-1-i\sqrt{3}}{3} & \frac{1}{3}\sqrt{6} \end{bmatrix} =$$

$$\therefore S = \begin{bmatrix} 2 & 1+i \\ i-1 & 3 \end{bmatrix}$$

Find eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1+i \\ i-1 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 - (1+i)(i-1)$$

$$= \lambda^2 - 5\lambda + 6 - (i-1-1-i) = \lambda^2 - 5\lambda + 8 = 0 \rightarrow \lambda_{1,2} = \frac{5 \pm \sqrt{25-32}}{2} = \frac{5 \pm i\sqrt{7}}{2}$$

Find Eigenvectors:

$$\lambda_1 = \frac{5+i\sqrt{7}}{2}$$

$$\begin{bmatrix} \frac{-1-i\sqrt{7}}{2} & 1+i \\ i-1 & \frac{1-i\sqrt{7}}{2} \end{bmatrix} x_1 = 0 \longrightarrow \begin{bmatrix} \frac{-1-i\sqrt{7}}{2} & 1+i \\ 0 & 0 \end{bmatrix} x_1 = 0$$

$$R_2' = R_2 + R_1 \frac{(1-i)2}{-1-i\sqrt{7}}$$

x_1	x_2
0	0
$\frac{2+2\sqrt{7}+i(2-2\sqrt{7})}{8}$	1

$$\frac{-1-i\sqrt{7}}{2}x_1 + 1+i = 0$$

$$\frac{-1-i\sqrt{7}}{2}x_1 = -1-i$$

$$x_1 = \frac{-2-i2}{-1-i\sqrt{7}} = \frac{2+i2}{1+i\sqrt{7}} \cdot \frac{1-i\sqrt{7}}{1-i\sqrt{7}} = \frac{2-2i\sqrt{7}+2i+2\sqrt{7}}{8}$$

$$x_1 = \begin{bmatrix} 2+2\sqrt{7}+i(2-2\sqrt{7}) \\ 8 \end{bmatrix} \quad ; \quad Q_1 = \frac{1}{8\sqrt{2}} \begin{bmatrix} 2+2\sqrt{7}+i(2-2\sqrt{7}) \\ 8 \end{bmatrix}$$

$$\left| 2+2\sqrt{7} + i(2-2\sqrt{7}) \right|^2 = a^2+b^2 = 4+\cancel{4\sqrt{7}}+28+4-\cancel{4\sqrt{7}}+28 = 64$$

this is the calculations.

$$\lambda_2 = \frac{5-i\sqrt{7}}{2}$$

$$\begin{bmatrix} \frac{-1+i\sqrt{7}}{2} & 1+i \\ i-1 & \frac{1+i\sqrt{7}}{2} \end{bmatrix} x_2 = 0 \longrightarrow \begin{bmatrix} \frac{-1+i\sqrt{7}}{2} & 1+i \\ 0 & 0 \end{bmatrix} x_2 = 0$$

$$R_2' = R_2 + R_1 \frac{(2-i2)}{(-1+i\sqrt{7})}$$

x_1	x_2
0	0
$\frac{2-2\sqrt{7}+i(2+2\sqrt{7})}{8}$	1

$$\frac{-1+i\sqrt{7}}{2} x_1 + 1+i = 0$$

$$\frac{-1+i\sqrt{7}}{2} x_1 = -1-i$$

$$x_1 = \frac{-2-i2}{-1+i\sqrt{7}} = \frac{2+i2}{1-i\sqrt{7}} \cdot \frac{1+i\sqrt{7}}{1+i\sqrt{7}} = \frac{2+2i\sqrt{7}+2i-2\sqrt{7}}{8}$$

$$x_2 = \begin{bmatrix} 2-2\sqrt{7}+i(2+2\sqrt{7}) \\ 8 \end{bmatrix} \quad ; \quad Q_2 = \frac{1}{8\sqrt{2}} \begin{bmatrix} 2-2\sqrt{7}+i(2+2\sqrt{7}) \\ 8 \end{bmatrix}$$

$$S = Q \Lambda Q^H$$

$$= \frac{1}{8\sqrt{2}} \begin{bmatrix} 2+2\sqrt{7}+i(2-2\sqrt{7}) & 2-2\sqrt{7}+i(2+2\sqrt{7}) \\ 8 & 8 \end{bmatrix} \begin{bmatrix} \frac{5+i\sqrt{7}}{2} & 0 \\ 0 & \frac{5-i\sqrt{7}}{2} \end{bmatrix} \frac{1}{8\sqrt{2}} \begin{bmatrix} 2+2\sqrt{7}-i(2-2\sqrt{7}) & 8 \\ 2-2\sqrt{7}-i(2+2\sqrt{7}) & 8 \end{bmatrix}$$

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