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Tugas 2nd

1. Tentukan titik kritis dan jenisnya dari fungsi f dengan

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Jawab :

$f_x = 6xy - 6x$ | $f_y = 3x^2 + 3y^2 - 6y$
$f_{xx} = 6y - 6$ | $f_{yy} = 6y - 6$
| $f_{xy} = 6x$

* $f_x = 6xy - 6x = 0$ $\quad x = 0$
$x(6y - 6) = 0$ $\quad y = 1$

* $f_y = 3x^2 + 3y^2 - 6y = 0$
• $f_y = 0 + 3y^2 - 6y = 0$ $\quad x = 0$
$3y(y-2) = 0$
$y = 0 \quad \vee \quad y = 2$

• $f_y = 3x^2 + 3(1) - 6(1) = 0$ $\quad y = 1$
$3x^2 - 3 = 0$
$x^2 = 1$
$x = 1 \quad \vee \quad x = -1$

* $f_x = 1(6y - 6) = 0$ $\quad x = 1$
$y = 1$

$f_x = -1(6y - 6) = 0$ $\quad x = -1$
$y = 1$

Titik Kritis :

$(0,1), (0,0), (0,2), (1,1), (-1,1)$

$$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$$

• $D(0,1) = 0 \cdot 0 - 0^2 = 0$
$f_{xx}(0,1) = 0$
} bisa local min. local max, atau saddle point.

• $D(0,0) = -6 \cdot (-6) - (0)^2 = 36 > 0$
$f_{xx}(0,0) = -6 < 0$ lokal maximum

• $D(0,2) = 6 \cdot 6 - (0)^2 = 36 > 0$
$f_{xx}(0,2) = 6 > 0$ lokal minimum

• $D(1,1) = 0 \cdot 0 - (6)^2 = -36 < 0$

$f_{xx}(1,1) = 0$ saddle point

• $D(-1,1) = 0 \cdot 0 - (-6)^2 = -36 < 0$
$f_{xx}(-1,1) = 0$ saddle point

Sehingga :
(0,1) tidak diketahui
(0,0) lokal maksimum
(0,2) lokal minimum
(1,1) saddle point
(-1,1) saddle point

2. Tentukan titik pada bidang $4x - 2y + z = 1$ yang memiliki jarak terdekat dengan $(-2,-1,5)$.

$f(x,y,z) = (x+2)^2 + (y+1)^2 + (z-5)^2$
$g(x,y,z) = 4x - 2y + z - 1 = 0$

Mencari turunan parsial $f(x,y,z)$ & $g(x,y,z)$

$\nabla f = (2(x+2), 2(y+1), 2(z-5))$
$\nabla g = (4, -2, 1)$

Persamaan Lagrange

$\nabla f = \lambda \nabla g$

$(2(x+2), 2(y+1), 2(z-5)) = \lambda(4,-2,1)$

$\therefore \hat{i} \Rightarrow 2x+4 = 4\lambda$

$2x = 4\lambda - 4 \rightarrow x = 2\lambda - 2$

$\hat{j} \Rightarrow 2y+2 = -2\lambda$

$2y = -2\lambda - 2 \rightarrow y = -\lambda - 1$

$\hat{k} \Rightarrow 2z - 10 = \lambda$

$2z = \lambda + 10 \rightarrow z = \frac{\lambda}{2} + 5$

Mencari nilai

$g(x,y,z) = 4x - 2y + z - 1 = 0$

Substitusikan x, y, dan z

$4(2\lambda-2) - 2(-\lambda-1) + \frac{\lambda}{2} + 5 - 1 = 0$

$8\lambda - 8 + 2\lambda + 2 + \frac{\lambda}{2} + 5 - 1 = 0$

$8\lambda + 2\lambda + \frac{\lambda}{2} - 2 = 0$

$16\lambda + 4\lambda + \lambda - 4 = 0$

$21\lambda = 4$

$\lambda = \frac{4}{21}$

Substitusi ke persamaan λ !

$x = 2\left(\frac{4}{21}\right) - 2 = \frac{8}{21} - 2 = -\frac{34}{21}$

$y = -\frac{4}{21} - 1 = \frac{-25}{21}$

$z = \frac{4}{42} + 5 = \frac{4+210}{42} = \frac{214}{42}$

$= \frac{107}{21}$

titik terdekatnya adalah

$\left(-\frac{34}{21}, \frac{-25}{21}, \frac{107}{21}\right)$

Kenapa titik tersebut merupakan titik terdekat karena berdasarkan persamaan Lagrange, jarak terdekat akan didapatkan ketika ∇f sama dengan nilai ∇g. Namun, besarnya perbedaan antara arah dan nilai tidak diketahui sehingga diperlukannya konstanta λ untuk menyamakan nilai ∇f dengan nilai ∇g.

3. Tentukan nilai a jika bidang $\alpha : x+y+z=1$ merupakan bidang singgung dari $z = x^2 + ay^2$

Persamaan bidang α : $x+y+z-1=0$

$\bar{N} = \nabla f(x,y,z)$; $f(x,y,z) = x^2 + ay^2 - z$

$\nabla f(x,y,z) = (2x, 2ay, -1) = \bar{N}$

$\therefore$ Persamaan bidang α : $\bar{N} \cdot \overline{P_0P}$ dengan

$\bar{N} \perp \overline{P_0P}$ dan $\overline{P_0P} = (x-x_0, y-y_0, z-z_0)$

$\bar{N} \cdot \overline{P_0P} = 0 = \bar{N}\big|_{x_0,y_0,z_0} \cdot (x-x_0, y-y_0, z-z_0)$

$= (2x_0, 2ay_0, -1) \cdot (x-x_0, y-y_0, z-z_0)$

$= 2x_0x - 2x_0^2 + 2ay_0y - 2ay_0^2 + -z + z_0 = 0$

$2x_0x + 2ay_0y - z - 2x_0^2 - 2ay_0^2 + z_0 = 0$

dikali -1

$-2x_0x - 2ay_0y + z + 2x_0^2 + 2ay_0^2 - z_0 = 0$

pada bidang α, nilai $d = 1$

Sehingga $2x_0^2 + 2ay_0^2 - z_0 = 1$

Sehingga $\alpha = \bar{N} \cdot \overline{P_0P}$

$-2x_0x - 2ay_0y + z + 2x_0^2 - 2ay_0^2 - z_0 = x + y + z - 1$

Samakan bentuknya :

- $-2x_0x = x$, $x_0 = -\frac{1}{2}$
- $-2ay_0y = y$, $y_0 = -\frac{1}{2a}$

Kita tahu nilai $d = 1$

$d = 2x_0^2 + 2ay_0^2 - z_0 = -1$

$= 2\left(-\frac{1}{2}\right)^2 + 2a\left(-\frac{1}{2a}\right)^2 - z_0 = -1$

$= \frac{1}{2} + \frac{1}{2a} - z_0 = -1$

$z_0 = \frac{1}{2} + \frac{1}{2a} + 1$

titik $P_0 = \left(-\frac{1}{2}, -\frac{1}{2a}, \frac{1}{2} + \frac{1}{2a} + 1\right)$

Karena P_0 berada pada $z = x^2 + ay^2$, artinya nilai x_0, y_0, z_0 memenuhi persamaan tersebut sehingga

$z = x^2 + ay^2 \longrightarrow \left(-\frac{1}{2}, -\frac{1}{2a}, \frac{1}{2} + \frac{1}{2a} + 1\right)$

$\frac{1}{2} + \frac{1}{2a} + 1 = \frac{1}{4} + a\frac{1}{4a^2}$

$\frac{3}{2} + \frac{1}{2a} = \frac{1}{4} + \frac{1}{4a}$

$\frac{3}{2} - \frac{1}{4} = \frac{1}{4a} - \frac{1}{2a}$

$\frac{6-1}{4} = \frac{1-2}{4a}$

$\frac{5}{4} = \frac{-1}{4 \cdot a}$

$a = -\frac{1}{5}$

Nilai dari a adalah $-\frac{1}{5}$.

4 Tentukan nilai maksimum dari fungsi $f(x,y,z) = x^2 + y^2 + z^2$ dengan syarat $z = x+y$ dan $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$

Nilai maksimum fungsi-fungsi tersebut dapat dicari dengan Lagrange multiplier

$f(x,y,z) = x^2 + y^2 + z^2$

$g_1(x,y,z) = x + y - z = 0$

$g_2(x,y,z) = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0$

Persamaan Lagrange

$\nabla f(x,y,z) = \lambda g_1(x,y,z) + \mu g_2(x,y,z)$

$(2x, 2y, 2z) = \lambda(1, 1, -1) + \mu\left(\frac{1}{2}x, \frac{2}{5}y, \frac{2}{25}z\right)$

- $2x = \lambda + \frac{1}{2}x\mu$

  $\lambda = 2x - \frac{1}{2}x\mu$

  $\lambda = x\left(2 - \frac{1}{2}\mu\right)$

  $x = \frac{\lambda}{2 - \frac{1}{2}\mu} = \frac{2\lambda}{4-\mu}$

- $2y = \lambda + \frac{2}{5}y\mu$

  $\lambda = 2y - \frac{2}{5}y\mu$

  $\lambda = y\left(2 - \frac{2}{5}\mu\right)$

  $y = \frac{\lambda}{2 - \frac{2}{5}\mu} = \frac{5\lambda}{10-2\mu}$

- $2z = -\lambda + \frac{2}{25}z\mu$

  $\lambda = \frac{2}{25}z\mu - 2z$

  $\lambda = z\left(\frac{2}{25}\mu - 2\right)$

  $z = \frac{\lambda}{\frac{2}{25}\mu - 2}$

  $z = \frac{25\lambda}{2\mu - 50}$

Substitusikan x, y, z ke persamaan $g_1(x,y,z)$

$g_1(x,y,z) = x + y - z = 0$

$= \frac{2\lambda}{4-\mu} + \frac{5\lambda}{10-2\mu} - \frac{25\lambda}{2\mu - 50} = 0$

$= \frac{2\lambda(10-2\mu)(2\mu-50) + 5\lambda(4-\mu)(2\mu-50) - 25\lambda(4-\mu)(10-2\mu)}{(4-\mu)(10-2\mu)(2\mu-50)}$

$= 240\lambda\mu - 1000\lambda - 8\lambda\mu^2 + 290\lambda\mu - 1000\lambda - 10\lambda\mu^2 - 1000\lambda + 450\lambda\mu - 50\lambda\mu^2 = 0$

$= 980\lambda\mu - 3000\lambda - 68\lambda\mu^2 = 0$

$4\lambda(245\mu - 750 - 17\mu^2) = 0$

$-\lambda(17\mu^2 - 245\mu + 750) = 0$

$\lambda(17\mu^2 - 75\mu - 170\mu + 750) = 0$

$\lambda(\mu(17\mu - 75) - 10(17\mu + 75)) = 0$

$\lambda(17\mu - 75)(\mu - 10) = 0$

$\lambda = 0$

$\mu = \frac{75}{17} \quad \vee \quad \mu = 10$

Substitusikan ke persamaan x, y, z !
untuk mencari λ

•)

$M = 75/17$

$\rightarrow \quad x\left(2 - \frac{1}{2}\left(\frac{75}{17}\right)\right) = \lambda$

$x = -\frac{34}{7}\lambda$

$M = 10$

$x\left(2 - \frac{1}{2}(10)\right) = \lambda$

$x(-3) = \lambda$

$x = -\frac{1}{3}\lambda$

•) $M = 75/17$

$y\left(2 - \frac{2}{5}\left(\frac{75}{17}\right)\right) = \lambda$

$y\left(\frac{4}{17}\right) = \lambda$

$y = \frac{17\lambda}{4}$

$M = 10$

$y\left(2 - \frac{2}{5}(10)\right) = \lambda$

$y(-2)$

$y = -\frac{1}{2}\lambda$

•) $M = 75/17$

$z\left(\frac{2}{25}\left(\frac{75}{17}\right) - 2\right) = \lambda$

$z\left(\frac{6 - 34}{17}\right) = \lambda$

$z\left(\frac{-28}{17}\right) = \lambda$

$z = \frac{-17}{28}\lambda$

•) $M = 10$

$z = -\frac{5}{6}\lambda$

dengan $\lambda = 0$ kita dapat titik pertama yakni $(0,0,0)$

Sekarang masukkan $x\ y\ z$ ke persamaan

$g_2(x,y,z) = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0$

Untuk $M = 10$

$$\frac{\left(-\frac{1}{3}\lambda\right)^2}{4} + \frac{\left(-\frac{1}{2}\lambda\right)^2}{5} + \frac{\left(\frac{-5}{6}\lambda\right)^2}{25} - 1 = 0$$

dengan kalkulator

$\lambda = -\frac{6\sqrt{95}}{19} \quad ; \quad \lambda = \frac{6\sqrt{95}}{19}$

untuk $M = \frac{75}{17}$

dengan kalkulator

$\lambda = -\frac{70\sqrt{646}}{5491} \quad ; \quad \lambda = \frac{70\sqrt{646}}{5491}$

nilai-nilai λ dan M yang kita punya adalah

$M = \frac{75}{17} \quad ; \quad 10$

$\lambda = \pm\frac{6\sqrt{95}}{19} \quad ; \quad \pm\frac{70\sqrt{646}}{5491} \quad ; \quad 0$

Titik-titiknya adalah

$x \rightarrow \quad 0, \pm\frac{20\sqrt{646}}{323}, \pm\frac{2\sqrt{95}}{19}$

$y \rightarrow \quad 0, \pm\frac{35\sqrt{646}}{646}, \pm\frac{3\sqrt{95}}{19}$

$z \rightarrow \quad 0, \pm\frac{5\sqrt{646}}{646}, \pm\frac{5\sqrt{95}}{19}$

titik-titiknya adalah : $(0,0,0)$,

$\left(\pm\frac{20\sqrt{646}}{323}, \pm\frac{35\sqrt{646}}{646}, \pm\frac{5\sqrt{646}}{646}\right)$,

$\left(\pm\frac{2\sqrt{95}}{19}, \pm\frac{3\sqrt{95}}{19}, \pm\frac{5\sqrt{95}}{19}\right)$

Kita tahu bahwa fungsi $f(x,y,z) = x^2 + y^2 + z^2$; sehingga titik-titik yang dimasukkan akan tetap bernilai positif meskipun titik yang dimasukkan bernilai negatif karena akan dikuadratkan.

$f(x,y,z) = x^2+y^2+z^2$

•) $f(0,0,0) = 0$

•) $f\left(\pm\frac{20\sqrt{646}}{323}, \pm\frac{35\sqrt{646}}{646}, \pm\frac{5\sqrt{646}}{646}\right)$

$= \left(\pm\frac{20\sqrt{646}}{323}\right)^2 + \left(\pm\frac{35\sqrt{646}}{646}\right)^2 + \left(\pm\frac{5\sqrt{646}}{646}\right)^2$

$= \frac{75}{17}$

•) $f\left(\pm\frac{2\sqrt{95}}{19}, \pm\frac{3\sqrt{95}}{19}, \pm\frac{5\sqrt{95}}{19}\right)$

$= \left(\pm\frac{2\sqrt{95}}{19}\right)^2 + \left(\pm\frac{3\sqrt{95}}{19}\right)^2 + \left(\pm\frac{5\sqrt{95}}{19}\right)^2$

$= 10$ // (maksimum)

Nilai maksimal dari fungsi $f(x,y,z)$ adalah 10 // dengan titik

$\left(\pm\frac{2\sqrt{95}}{19}, \pm\frac{3\sqrt{95}}{19}, \pm\frac{5\sqrt{95}}{19}\right)$