Homework 1 (by Hand)

1. Find SVD and pseudoinverse of the B and C matrices;

$$B = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 2 & 10 & 2 \\ 5 & 11 & 5 & 11 \end{bmatrix}$$

a) $B = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$ First Find $B^T B$!

$$B^T B = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

Find eigenvalue $B^T B$

$$\begin{bmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{bmatrix} = 0 \longrightarrow \begin{array}{l} (5-\lambda)(5-\lambda) - 25 = 0 \\ 25 - 10\lambda + \lambda^2 - 25 = 0 \\ \lambda(\lambda - 10) = 0 \end{array} \longrightarrow \begin{array}{l} \lambda_1 = 10 \\ \lambda_2 = 0 \end{array}$$

Eigenvector :

$\lambda_1 = 10$

$$\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} V_1 = 0 \longrightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow V_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 0$

$$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} V_2 = 0 \longrightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad V_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Sigma = \sqrt{\Lambda} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

Find U! $\longrightarrow$ $B = U \Sigma V^T$

$$U = B V \Sigma^{-1}$$

$$U = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & 0 \\ \frac{-2\sqrt{5}}{5} & 0 \end{bmatrix}$$

See, u_2 is zero vector, it cannot be zero, so find the $N(u_1 \cdot u_2)$

$$\begin{bmatrix} \frac{\sqrt{5}}{5} & -2\frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \longrightarrow \left.\begin{matrix} x_1 = 2 \\ x_2 = 1 \end{matrix}\right\} \quad u_2 = \begin{bmatrix} \frac{2}{5}\sqrt{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix}$$

Hence

$$B = U \Sigma V^T = \underset{2\times1}{\begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2}{5}\sqrt{5} \\ \frac{-2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}} \underset{1}{\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}} \underset{1\times2}{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}} //$$

Now, find

$$B^+ = V \Sigma^+ U^T$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \Sigma^+ \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{2}{5}\sqrt{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$$

$$B^+ = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{2}{5}\sqrt{5} & \frac{\sqrt{5}}{5} \end{bmatrix} //$$

1×2

$$C = \begin{bmatrix} 10 & 2 & 10 & 2 \\ 5 & 11 & 5 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

Find $CC^T = \begin{bmatrix} 10 & 2 & 10 & 2 \\ 5 & 11 & 5 & 11 \end{bmatrix} \begin{bmatrix} 10 & 5 \\ 2 & 11 \\ 10 & 5 \\ 2 & 11 \end{bmatrix}$ $\boxed{}$

$$CC^T = \begin{bmatrix} 100+4+100+4 & 50+22+50+22 \\ 50+22+50+22 & 25+121+25+121 \end{bmatrix} = \begin{bmatrix} 208 & 144 \\ 144 & 292 \end{bmatrix}$$

Find eigenvalue :

$$\begin{bmatrix} 208-\lambda & 144 \\ 144 & 292-\lambda \end{bmatrix} = \lambda^2 - 500\lambda + 40000$$
$$= (\lambda - 100)(\lambda - 400) \rightarrow \lambda_1 = 400$$
$$\lambda_2 = 100$$

Eigenvector:

$\lambda_1 = 400$

$$\begin{bmatrix} -192 & 144 \\ 144 & -108 \end{bmatrix} x_1 = 0 \rightarrow x_1 = \begin{bmatrix} \frac{12}{16} \\ 1 \end{bmatrix} \rightarrow u_1 = \frac{1}{\sqrt{\frac{400}{256}}} \begin{bmatrix} \frac{12}{16} \\ 1 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

$\lambda_2 = 100$

$$\begin{bmatrix} 108 & 144 \\ 144 & 192 \end{bmatrix} x_2 = 0 \rightarrow x_2 = \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix} \rightarrow u_2 = \frac{3}{5} \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$$

$$U = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \quad ; \quad \Sigma = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix}$$

$$V_i = \frac{1}{\sigma_i} C^T u_i$$

$$V_1 = \frac{1}{20}\begin{bmatrix} 10 & 5 \\ 2 & 11 \\ 10 & 5 \\ 2 & 11 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 30+20 \\ 6+44 \\ 30+20 \\ 6+44 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 50 \\ 50 \\ 50 \\ 50 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_2 = \frac{1}{10}\begin{bmatrix} 10 & 5 \\ 2 & 11 \\ 10 & 5 \\ 2 & 11 \end{bmatrix} \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} -40+15 \\ -8+33 \\ -40+15 \\ -8+33 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} -25 \\ 25 \\ -25 \\ 25 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$ → becaus C is $\mathbb{R}^{2\times 4}$; V needs to be 4×4. Find nullspace of C to make the multi-plication into zero

$$N(C) \Rightarrow \begin{bmatrix} 10 & 2 & 10 & 2 \\ 5 & 11 & 5 & 11 \end{bmatrix} v = 0 \longrightarrow \begin{bmatrix} 10 & 2 & 10 & 2 \\ 0 & 9 & 0 & 9 \end{bmatrix} :$$

$r_2 = r_2 - \frac{1}{2}r_1$

$r_1 = r_1 - \frac{2}{9}r_2$

$$\longrightarrow \begin{bmatrix} 10 & 0 & 10 & 0 \\ 0 & 9 & 0 & 9 \end{bmatrix} v = 0 \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} v = 0$$

$r_1 = \frac{r_1}{10}$

$r_2 = \frac{r_2}{9}$

x_1	x_2	x_3	x_4	
-1	0	1	0	$= s_1$
0	-1	0	1	$= s_2$

$$s_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad \frac{s_1}{|s_1|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$s_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}; \quad \frac{s_2}{|s_2|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Hence: $C = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$

Now, $C^{+} = V \Sigma^{+} U^{T}$

$$= \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}}_{4\times 4} \underbrace{\begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{10} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{4\times 2} \frac{1}{5} \underbrace{\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}}_{2\times 2}$$

$\boxed{2}$ i) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+9 & 2+8+18 \\ 2+8+18 & 4+16+36 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$$

Find Eigenvalue

$$\begin{vmatrix} 14-\lambda & 28 \\ 28 & 56-\lambda \end{vmatrix} = (14-\lambda)(56-\lambda) - 28^2$$
$$= \lambda^2 - 70\lambda \longrightarrow \lambda(\lambda-70) = 0$$
$$\lambda_1 = 70$$
$$\lambda_2 = 0$$

Eigenvector:

$\lambda_1 = 70$

$$\begin{vmatrix} 14-70 & 28 \\ 28 & 56-70 \end{vmatrix} x_1 = \begin{vmatrix} -56 & 28 \\ 28 & -14 \end{vmatrix} x_1 = 0$$

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \qquad v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ii) $\lambda_2 = 0$

$$\begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix} x = 0$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; \quad v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 3\times 2$$

Find U :>

$$U = \frac{1}{\sigma_i} A \, V_i$$

$$u_1 = \frac{1}{\sqrt{70}} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5\sqrt{14}} \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$N(A^T) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} x = 0$$

$R_2 = R_2 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} x = 0$$

x_1	x_2	x_3	
-2	1	0	$\Rightarrow s_1$
-3	0	1	$\Rightarrow s_2$

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} ; \quad s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \longrightarrow u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} ; \quad u_3 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{\sqrt{14}}{14} & -\frac{2\sqrt{5}}{5} & \frac{-3}{\sqrt{10}} \\ \frac{\sqrt{14}}{7} & \frac{\sqrt{5}}{5} & 0 \\ \frac{3\sqrt{14}}{14} & 0 & \frac{\sqrt{10}}{10} \end{bmatrix}$$

Hence:

$$A = \underbrace{\begin{bmatrix} \frac{\sqrt{14}}{14} & -\frac{2\sqrt{5}}{5} & \frac{-3}{\sqrt{10}} \\ \frac{\sqrt{14}}{7} & \frac{\sqrt{5}}{5} & 0 \\ \frac{3\sqrt{14}}{14} & 0 & \frac{\sqrt{10}}{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{70} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_{V^T} \; //$$

• Find A^+

$$A^+ = V \Sigma^+ U^T$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{70} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{14}}{14} & \frac{\sqrt{14}}{7} & \frac{3}{14}\sqrt{14} \\ -\frac{2}{5}\sqrt{5} & \frac{\sqrt{5}}{5} & 0 \\ \frac{-3}{\sqrt{10}} & 0 & \frac{\sqrt{10}}{10} \end{bmatrix}$$