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Homework I

Problem 1. Vector spaces and Sub spaces

Proof whether the following sets form subspaces (the scalar multiplication and vector addition are defined as usual)

$\boxed{a}$ $\{(x_1, x_2, x_3)\} \in \mathbb{R}^3 : x_1 = 2x_3$

Proof :

1) Prove that this set is in the zero vector!

$x_1 = 2x_3$

So if $x_1 ; x_2 ; \& x_3 = 0$, then $x_1 = 2x_3 \rightarrow 0 = 2(0) = 0$.

$(x_1, x_2, x_3) \in S$

2) Addition proving.

$v = (v_1, v_2, v_3)$; $w = (w_1, w_2, w_3) \in \mathbb{R}^3$

$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3) \in S$ because

$v_1, v_2, v_3, w_1, w_2, w_3$ is real number. And for $v_1 = 2v_3$;

$w_1 = 2w_3$ is also a real number, then

$v + w = (2v_3 + 2w_3, v_2 + w_2, v_3 + w_3)$ is $\in S$

3) Scalar multiplication proving

If α is a nonzero real number, then $\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3) \in \mathbb{R}^3$

because $x_1, x_2, x_3 \in \mathbb{R}$. Also $x_1 = 2x_3$. That means

$\alpha x = (2\alpha x_3, \alpha x_2, \alpha x_3)$ and it is $\in \mathbb{R}^3$. Any

number of α will result in $\mathbb{R}^3$ and it is $\in S$.

This statisfy all the condition. Thus the set is a subspaces

b) $\{(x_1, x_2, x_3)\} \in \mathbb{R}^3$: $x_1 = 3x_2 + c$, $c \in \mathbb{R}$: any non zero real constent

Proof :

1) Proof that this set is in the zero vector !

So if $x_2 = 0$, then $x_1 = 3x_2 + c$ will be $x_1 = 3(0) + c$. C is a nonnegatif constan so $x_1 = c \rightarrow (c, 0, 0) \neq$ zero vecter

Now if $x_1 = 0$, then

$0 = 3x_2 + c$

$-3x_2 = c$

$\rightarrow x_2 = -\frac{c}{3}$. $\rightarrow \begin{bmatrix} 0 \\ -\frac{c}{3} \\ 0 \end{bmatrix} \neq$ zero vector

If $x_1, x_2 = 0$, then $0 = 3(0) + c$ while $c \neq 0$. $\rightarrow$ [

The set cannot be in the zero vector so its not statify all the condition. Thus, the set is $\notin S$

c) The set of linear combination of row A

Proof :

1) Prove that this linear combination of row A is in the zero vector !

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$ if $A_{11}, \ldots, A_{mn}$ is equal to zero, the matrix A is in the zero vector and it is in the subspaces.

2) Addition proving !

if we have V and W matrix . $V + W$ will equal to

$$V + W = \begin{bmatrix} (V_{11} + W_{11}) & \cdots & (V_{1n} + W_{1n}) \\ (V_{m1} + W_{m1}) & \cdots & (V_{mn} + W_{mn}) \end{bmatrix}$$. This will result the matrix is $\in \mathbb{R}^{mn}$.

The addition of the matrix is in the subspaces.

3) Scalar multiplication proving !

Let's say c is a non negatif constant, then, $c \cdot A$ will be

$$c \cdot A = \begin{bmatrix} cA_{11}, & \dots & cA_{1n} \\ \vdots & & \\ cA_{m1} & \dots & cA_{mn} \end{bmatrix}.$$

$c \cdot A_{11}, \dots, c \cdot A_{mn}$ is $\in \mathbb{R}$. Thus the matrix is in the subspaces.

4) Linear Combination of row A

$$V = \begin{bmatrix} V_{11}, & \dots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1}, & \dots & V_{mn} \end{bmatrix}$$

if c is a vector in $\mathbb{R}^{1 \times m}$. then

$$c \cdot V = [c_{11}, \dots c_{1m}] \begin{bmatrix} V_{11}, & \dots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1}, & \dots & V_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (V_{11}, \dots V_{1n}) c_{11} \\ \vdots \\ (V_{m1}, \dots V_{mn}) c_{1m} \end{bmatrix}$$

Now if $cv + dw$ it will be

$$\begin{bmatrix} (V_{11} \dots V_{1n}) c_{11} \\ \vdots \\ (V_{m1} \dots V_{mn}) c_{1m} \end{bmatrix} + \begin{bmatrix} (W_{11} \dots W_{1n}) d_{11} \\ (W_{m1} \dots W_{mn}) d_{1m} \end{bmatrix}$$

$$= \begin{bmatrix} (c_{11} V_{11} + d_{11} W_{11}) \dots (c_{11} V_{1n} + d_{11} W_{1n}) \\ \vdots \\ (c_{1m} V_{m1} + d_{1m} W_{m1}) \dots (c_{1m} V_{mn} + d_{1m} W_{mn}) \end{bmatrix}$$

$\in \mathbb{R}^{mn}$

The result linear combanition of $cv + dw$ is still in the subspace.

Thus it statisfy all the condition. The linear combination of rows A is $\in S$ //

d The set all solution to $Ax = b$

1) Prove that this set is in the zero vector !

$Ax = b$ can be repesent as the Linear combination of column A.

If $x = \begin{bmatrix} xp + xn \\ yp + yn \end{bmatrix}$, then if $xp = xn = yp = yn = 0$, then if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Ax = (0+0)\begin{bmatrix} a \\ c \end{bmatrix} + (0+0)\begin{bmatrix} b \\ d \end{bmatrix}$.

$Ax = \begin{bmatrix} 0 + 0 \\ 0 + 0 \end{bmatrix}$ is a zero vector and $\in S$

2) Addition proving !

if $Vx = b_1$ and $Wy = b_2$ then $Vx + Wy = b_1 + b_2$.

$x = \begin{bmatrix} x_{p_1} + x_{n_1} \\ x_{p_2} + x_{n_2} \end{bmatrix}$ and $y = \begin{bmatrix} y_{p_1} + y_{n_1} \\ y_{p_2} + y_{n_2} \end{bmatrix}$, $v = [V_1, V_2]$ and $w = [W_1, W_2]$, then $Vx + Wy$

is equal to

$Vx + Wy = [(x_{p_1} + x_{n_1})V_1 + (x_{p_2} + x_{n_2})V_2 + [(y_{p_1} + y_{n_1})W_1 + (y_{p_2} + y_{n_2})W_2] = [x_{p_1}V_1 + x_{n_1}V_1 + x_{p_2}V_2 + x_{n_2}V_2 + y_{p_1}W_1 + y_{n_1}W_1 + y_{p_2}W_2 + y_{n_2}W_2]$

The $x_1, x_2, y_1, y_2, V_1, V_2, W_1, W_2$ is $\in$ Real number so the $Vx + Wx \in S$

3) Scalar muliplication proving !

If $pAx = pb$ and p is nonnegatif real number, then the expersion can be proof as :

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $x = \begin{bmatrix} x_p + x_n \\ y_p + x_n \end{bmatrix}$, $pAx = px\begin{bmatrix} a \\ c \end{bmatrix} + py\begin{bmatrix} b \\ d \end{bmatrix} = pb$

$(px_p + px_n)\begin{bmatrix} a \\ c \end{bmatrix} + (py_p + py_n)\begin{bmatrix} b \\ d \end{bmatrix}$ is still part of the sub spaces.

4) This also mean that the linear combination of $pAx + qBy$ is also in the subspaces with the expession :

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $x = \begin{bmatrix} x_{p_1} + x_{n_1} \\ x_{p_2} + x_{n_2} \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, $y = \begin{bmatrix} y_{p_1} + y_{n_1} \\ y_{p_2} + y_{n_2} \end{bmatrix}$

$pAx + qBy = px_{p_1}\begin{bmatrix} a \\ c \end{bmatrix} + px_{n_1}\begin{bmatrix} a \\ c \end{bmatrix} + px_{p_2}\begin{bmatrix} b \\ d \end{bmatrix} + px_{n_2}\begin{bmatrix} b \\ d \end{bmatrix} + qy_{p_1}\begin{bmatrix} e \\ g \end{bmatrix} + qy_{n_1}\begin{bmatrix} e \\ g \end{bmatrix} + qy_{p_2}\begin{bmatrix} f \\ h \end{bmatrix}$
$qy_{n_2}\begin{bmatrix} f \\ h \end{bmatrix} \in S$ because $x_{p_1} + x_{n_1}$, $x_{p_2} + x_{n_2}$, $y_{p_1} + y_{n_1}$, $y_{p_2} + y_{n_2}$, p, and q is real numbers.

the linear combination is resulting in real number. Thus the solution of x_1, x_2, y_1, y_2 is also a real number and it is $\in S$

Thus it statisfy all the condition, that means solution $Ax = b$ is in the Subspaces.

$\boxed{e}$ The set of differentialble functions

1) Porve that the set of differentiable function is in the zero vector !

Let's say $v = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$. A differentiable function is a function that has derivative in every point in its domain. But not all function have solutions for zero. For example

$f(x) = x^2 + 1 = 0$ there is no solution for that equation. Hences the set of differentiable
$x^2 = -1$ function is not in the subspaces $\notin S$, and it fails to statisfy the conditions !

Problem 2 : Nullspace and The Complete Solution to $Ax = b$

Consider the set of linear equations $Ax = b$ with

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

a) Find the null space of A ($N(A)$)

$Ax = b$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 \end{array}\right]$$

row 2 = row 2 − row 1

row 3 = row 3 − 2 (row 1) $\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & -3 & -2 & -2 & -3 \end{array}\right]$

row 3 = row 3 − 3 (row 2) $\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right]$

row 1 = row 1 + 2 (row 2) $\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right]$

row 1 = row 1 + $\frac{1}{2}$ (row 3) $\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right]$

its a full row rank with $r = 3$

free variable

To find Null spaces, find the solution for $Ax = 0$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{aligned} x_1 + \tfrac{1}{2} x_4 &= 0 \\ -x_2 - x_4 &= 0 \\ -2 x_3 + x_4 &= 0 \end{aligned}$$

free	pivot			
x_4	x_1	x_2	x_3	
1	$-1/2$	-1	$1/2$	$\to s_1$
0	0	0	0	$\to s_2$

$$N(A) = c_1 s_1 + c_2 s_2$$

$$= c_1 \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix}$$ this is the nullspaces for all $c_1 \in \mathbb{R}$

b) Find the particular solution $Ax = b$

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$x_1 + 1/2\, x_4 = 1$$
$$-x_2 - x_4 = -1$$
$$-2x_3 + x_4 = 0$$

$$x_1 = 1 - 1/2\, x_4$$
$$x_2 = 1 - x_4$$
$$x_3 = \tfrac{1}{2} x_4$$

x_4	x_1 x_2 x_3	
1	1/2 0 1/2	→ linear combination N(A)
0	1 1 0	→ x_p

$$x_p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ //

c) The complete solution using a) and b)

$$Ax = b$$
$$A(x_p + x_n) = b \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$ //