ALIN HW # 1

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Problem 1

Prove the following properties of column operation:

(A) Column Exchanges :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}$$

A way to prove the problem is using the double transpose method !

1) $$\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix}^T \right)^T = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ Let's rearrange until we get the result that we after

$$\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix}^T \right)^T = \left(\begin{vmatrix} a & c \\ b & d \end{vmatrix} \right)^T \xrightarrow{\text{change row}} - \left(\begin{vmatrix} b & d \\ a & c \end{vmatrix} \right)^T$$

$$\xrightarrow{\text{do the remaining transpose}} - \left(\begin{vmatrix} b & d \\ a & c \end{vmatrix} \right)^T = - \begin{vmatrix} b & a \\ d & c \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ //

it is proved that column exchange is the same result as row exchange.

(B) Linearity :

•) $\begin{vmatrix} a+a' & b \\ c+c' & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b \\ c' & d \end{vmatrix}$

Prove:

$\begin{vmatrix} a+a' & b \\ c+c' & d \end{vmatrix} = ad + a'd - cb - c'b$

$= ad - cb + a'd - c'b = (ad - cb) + (a'd - c'b)$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b \\ c' & d \end{vmatrix}$ //

•) $\begin{vmatrix} ta & b \\ tc & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Prove

$\begin{vmatrix} ta & b \\ tc & d \end{vmatrix} = tad - tcb = t(ad - cb)$

$= t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ //

Both of the problems are proved to be valid. //

(c) Subtracting a multiple of one column from another column:

$$\begin{vmatrix} a & b - \alpha a \\ c & d - \alpha c \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

First we know that $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$ because the matrix does not have basis at $\mathbb{R}^{m \times m}$ dimension so it has no determinant.

Prove:

$$\begin{vmatrix} a & b - \alpha a \\ c & d - \alpha c \end{vmatrix} = (ad - \alpha ac) - (cb - \alpha ca)$$

$$= ad - \alpha ac - cb + \alpha ca$$

$$= ad - cb + \alpha ca - \alpha ac$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \alpha a & \alpha c \\ a & c \end{vmatrix}$$

With property number 3 of determinant, we can simplify as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \alpha \begin{vmatrix} a & c \\ a & c \end{vmatrix}$$

As we know, $\begin{vmatrix} a & c \\ a & c \end{vmatrix}$ has same row properties, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \alpha(0) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ It is proved that the equation is valid.

Problem # 2 : Calculating Determinant of Matrices

$$\begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 & 0 \\ 0 & 1 & & \vdots & & 0 & 0 \\ \vdots & & \ddots & \vdots & & & \\ 0 & 0 & \cdots & x_i & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \cdots & x_{n-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & x_n & & 0 & 1 \end{vmatrix} = x_i$$

By adopting the properties of both row and column operation prove it! Using Lower and Upper triangular method!

To find Upper triangular matrices, we can do

$$\boxed{R_n = R_n - \frac{x_n}{x_i} R_i}$$

R = row vector

Do this until R_{n-1} except R_i and we will get :

$$U = \begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 & 0 \\ 0 & \ddots & & \vdots & & \vdots & \vdots \\ \vdots & & \ddots & x_i & & \vdots & 0 \\ 0 & & & \vdots & \ddots & 0 & \vdots \\ \vdots & & & \vdots & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \cdots & 0 & 1 \end{vmatrix}$$

and the lower triangle is

$$L \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ & 1 & \vdots & & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & & x_{i+1} & \ddots & 1 \\ 0 & & \vdots & & \\ 0 & 0 \cdots & x_n & \cdots & 1 \end{vmatrix}$$

We know that the determinant of the triangular matrices is the multiple of its diagonal value so

$$|U| = \begin{vmatrix} 1 & & x_1 & 0 & 0 & 0 \\ 0 & 1 & x_i & 1 & 1 & \\ 0 & 0 & 0 & & & 1 \end{vmatrix} = x_i$$

$$|L| = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & & 1 & 0 & 0 & 0 \\ 0 & & & 1 & 1 & 0 \\ 0 & & x_n & & & \end{vmatrix} = 1$$

So the answer is

$$|L|\,|U| = (1)(x_i) = x_i$$ //

Problem # 3 : Calculating Determinant of Block Diagonal Matrices

Consider the following Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad ; \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{21} \\ 0 & 0 & b_{12} & b_{22} \end{bmatrix}$$

Show the relation between $|A|$, $|B|$ and $|C|$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}b_{21}$$

$$|C| = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{vmatrix}$$

$$= + a_{11} \underbrace{\begin{vmatrix} a_{22} & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{vmatrix}}_{x} - a_{12} \underbrace{\begin{vmatrix} a_{21} & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{vmatrix}}_{y} + (0) - (0)$$

$$x = + a_{22} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} - (0) + (0) \quad ; \quad y = + a_{21} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} - (0) + (0)$$

$$x = a_{22}(b_{11}b_{22} - b_{12}b_{21}) \qquad y = a_{21}(b_{11}b_{22} - b_{12}b_{21})$$

Subtitute the x and y!

$$|C| = a_{11}(x) - a_{12}(y) = a_{11}(a_{22}(b_{11}b_{22} - b_{12}b_{21})) - a_{12}(a_{21}(b_{11}b_{22} - b_{12}b_{21}))$$

$$= (b_{11}b_{22} - b_{12}b_{21})(a_{11}a_{22} - a_{12}a_{21})$$

$$|C| = |B|\,|A| \;//$$

So, the relationship between $|A|$, $|B|$, and $|C|$ is that $|C| = |B|\,|A|$ //