Hw 3

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Problems 5.4

$\boxed{11.}$ $x_1' = x_1 - 2x_2$;

$x_2' = 2x_1 + x_2$;

$x_1(0) = 0$, $x_2(0) = 4$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x' = A \quad x$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 + 4$$

$$= \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm i4}{2} = 1 \pm i2$$

Find Eigenvector $\lambda = 1 + 2i$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$i \begin{bmatrix} -2i \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ -2i \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ i2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2i \end{bmatrix}$$

$$x = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = 1 - 2i$

$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-i \begin{bmatrix} 2i \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2i \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -2i \end{bmatrix} + \begin{bmatrix} -2 \\ 2i \end{bmatrix}$$

$$x = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Eigenvector $= \left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} ; \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$ → only need one eigenvalue and one eigenvectors.

General Solution:

$$x(t) = v e^{\lambda t}$$

$$= \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{t} (\cos 2t + i \sin 2t) = e^{t} \begin{bmatrix} -\sin 2t + i \cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix}$$

real imj

$x_1(t) = e^{t} [-\sin 2t \quad \cos 2t]^T$

$x_2(t) = e^{t} [\cos 2t \quad \sin 2t]^T$

Solution for $X(t) = C_1 x_1(t) + C_2 x_2(t)$

$$= C_1 e^{t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

$$x(t) = e^{t} \begin{bmatrix} -C_1 \sin 2t + C_2 \cos 2t \\ C_1 \cos 2t + C_2 \sin 2t \end{bmatrix}$$

$x_1(t) = e^{t}(-C_1 \sin 2t + C_2 \cos 2t)$

$x_2(t) = e^{t}(C_1 \cos 2t + C_2 \sin 2t)$

for:

$x_1(0) = 0$; $x_2(0) = 4$

$x_1(0) = C_2 = 0$

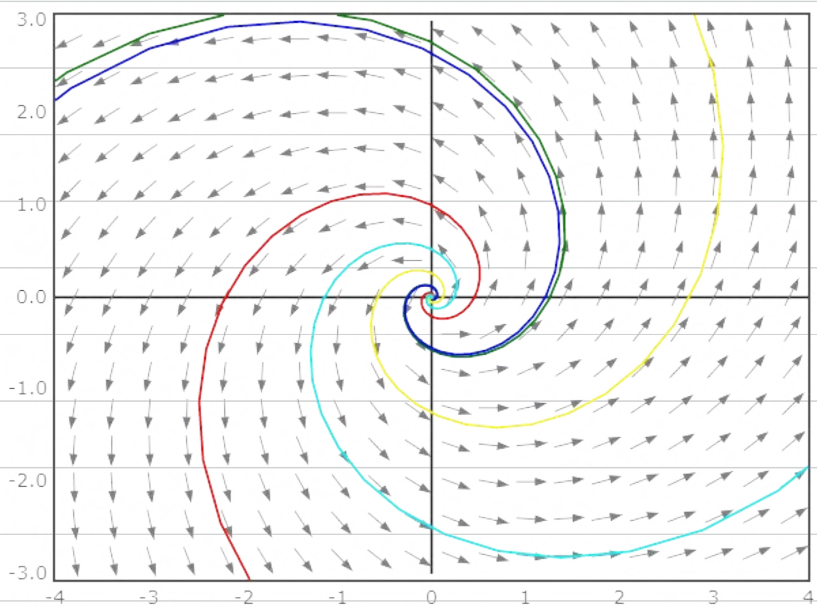
$x_2(0) = C_1 = 4$

Thus:

$x_1(t) = e^{t}(-4 \sin 2t)$

$x_2(t) = e^{t}(4 \cos 2t)$ //

Slope Field image:



$\boxed{26}$ Find Particular Solution

$x_1' = 3x_1 + x_3$ while $x_1(0)=0$; $x_2(0)=0$, and $x_3(0)=17$

$x_2' = 9x_1 - x_2 + 2x_3$

$x_3' = -9x_1 + 4x_2 - x_3$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x' = A \quad x$$

Eigenvalues:

$|A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 0 & 1 & 3-\lambda & 0 \\ 9 & -1-\lambda & 2 & 9 & -1-\lambda \\ -9 & 4 & -1-\lambda & -9 & 4 \end{bmatrix} = (3-\lambda)(-1-\lambda)(-1-\lambda) + 36 - (-9)(-1-\lambda)(1) - 8(3-\lambda)$$

$$= -\lambda^3 + \lambda^2 + 4\lambda + 6 = 0$$

$$\lambda_1 = 3 \ ; \ \lambda_{2,3} = -1 \pm i$$

Eigenvectors :

$\lambda = 3$

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} v_1 = 0 \ ; \ v_1 = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

$\lambda = -1 + i$

$$\begin{bmatrix} 4-j & 0 & 1 \\ 9 & -j & 2 \\ -9 & 4 & -j \end{bmatrix} v_2 = 0 \ ; \ v_2 = \begin{bmatrix} 1 \\ 2-j \\ -4+j \end{bmatrix}$$

$$\bar{v} \quad e^{\lambda t}$$

$$= \begin{bmatrix} 1 \\ 2-i \\ -4+i \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} e^{-t}(\cos t + i \sin t) \\ e^{-t}(2\cos t + \sin t + i(2\sin t - \cos t)) \\ e^{-t}(-4\cos t - \sin t + i(-4\sin t + \cos t)) \end{bmatrix}$$

real imj

$$x(t) = \begin{bmatrix} 4c_1 e^{3t} \\ 9c_1 e^{3t} \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 \cos t\, e^{-t} \\ c_2(2\cos t + \sin t)e^{-t} \\ c_2(-4\cos t - \sin t)e^{-t} \end{bmatrix} + \begin{bmatrix} c_3 \sin t\, e^{-t} \\ c_3(2\sin t - \cos t)e^{-t} \\ c_3(-4\sin t + \cos t)e^{-t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 4c_1 e^{3t} + e^{-t}(c_2 \cos t + c_3 \sin t) \\ 9c_1 e^{3t} + e^{-t}((2c_3 + c_2)\sin t + (2c_2 - c_3)\cos t) \\ 0 + e^{-t}((-4c_3 - c_2)\sin t + (-4c_2 + c_3)\cos t) \end{bmatrix}$$

For: $x_1(0) = 0$, $x_2(0) = 0$; and $x_3(0) = 0$

$x_1(0) = 4c_1 + c_2 = 0$

$x_2(0) = 9c_1 + 2c_2 - c_3 = 0 = 9c_1 + 2(-4c_1) - c_3 = 0 = c_1 = c_3$

$x_3(0) = -4c_2 + c_3 = 17 \;|\; 4 = -16c_2 + 4c_3 = 68$

$4c_3 + c_2 = 0 \;|\; 1 \quad 4c_3 + c_2 = 0$ —

$-17c_2 = 68$

$\boxed{c_2 = -4}$

$4c_1 + -4 = 0$

$\boxed{c_1 = 1 = c_3}$

Particular Solutions :

$$x_1(t) = 4e^{3t} + e^{-t}(-4\cos t + \sin t)$$
$$x_2(t) = 9e^{3t} + e^{-t}(-9\cos t - 2\sin t)$$
$$x_3(t) = e^{-t}(17\cos t)$$

//

[28] $V_1 = 25$ gal , $V_2 = 40$ gal

$$x_1' = -k_1 x_1$$
$$x_2' = k_1 x_1 - k_2 x_2$$
$$k_i = \frac{r}{V_i} \quad ; \quad r = 10 \ \frac{\text{gal}}{\text{min}}$$

For that

$$x_1' = -\frac{r}{25} x_1 = -\frac{2}{5} x_1$$
$$x_2' = \frac{2}{5} x_1 - \frac{1}{4} x_2$$

$$x' = \begin{bmatrix} -\frac{2}{5} & 0 \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find Eigenvalues.

$$\begin{bmatrix} -\frac{2}{5} - \lambda & 0 \\ \frac{2}{5} & -\frac{1}{4} - \lambda \end{bmatrix} = 0 \implies \left(-\frac{2}{5} - \lambda\right)\left(-\frac{1}{4} - \lambda\right) = 0$$

$$\lambda_1 = -\frac{2}{5}$$
$$\lambda_2 = -\frac{1}{4}$$

Eigenvectors

$\lambda = -\frac{2}{5}$

$$\begin{bmatrix} 0 & 0 \\ \frac{2}{5} & \frac{3}{20} \end{bmatrix} V_1 = 0 \rightarrow V_1 = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

$\lambda = -\frac{1}{4}$

$$\begin{bmatrix} -\frac{3}{20} & 0 \\ \frac{2}{5} & 0 \end{bmatrix} V_2 = 0 \rightarrow V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(t) = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$
$$= C_1 \begin{bmatrix} -3 \\ 8 \end{bmatrix} e^{-\frac{2}{5}t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\frac{1}{4}t} = \begin{bmatrix} -3C_1 e^{-\frac{2}{5}t} + 0 \\ 8C_1 e^{-\frac{2}{5}t} + C_2 e^{-\frac{1}{4}t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} -3c_1 e^{-\frac{2}{5}t} + 0 \\ 8c_1 e^{-\frac{2}{5}t} + c_2 e^{-\frac{1}{4}t} \end{bmatrix}$$

initial condition:

$x_1(0) = 15$; $x_2(0) = 0$

For that:

$x_1(0) = -3c_1 = 15$

$c_1 = -5$

$x_2(0) = 8c_1 + c_2 = 0$

$= -40 + c_2 = 0$

$c_2 = 40$

So, the particular solutions is:

$x_1(t) = 15\, e^{-\frac{2}{5}t}$

$x_2(t) = -40\, e^{-\frac{2}{5}t} + 40\, e^{-\frac{1}{4}t}$

Find the maximum amount of salt ever in tank 2.

The slope for maximum x_2 is $x_2' = 0$

$x_2' = 16 e^{-\frac{2}{5}t} - 10 e^{-\frac{1}{4}t} = 0$

$16\, e^{-\frac{2}{5}t} = 10\, e^{-\frac{1}{4}t}$

$\frac{8}{5}\, e^{-\frac{2}{5}t} = e^{-\frac{1}{4}t}$

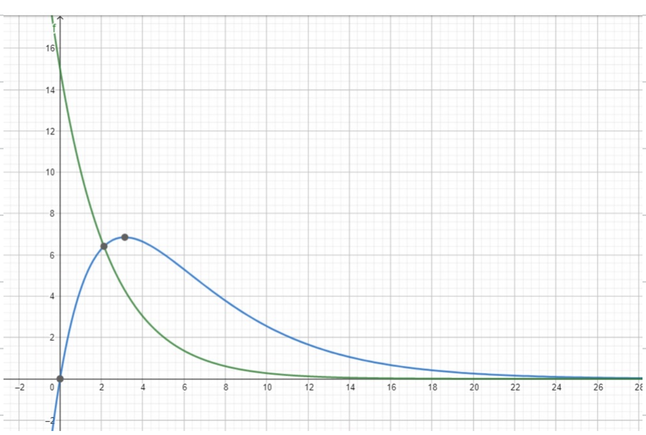
$\frac{8}{5} = e^{-\frac{1}{4}+\frac{2}{5}t} = e^{\frac{3}{20}t}$

$\ln \frac{8}{5} = \frac{3}{20}t$

$\frac{20}{3} \ln \frac{8}{5} = t$ // subtitute into $x_2(t)$

$x_2\left(\frac{20}{3}\ln\frac{8}{5}\right) = -40\, e^{-\frac{2}{5}\left(\frac{20}{3}\ln\frac{8}{5}\right)} + 40\, e^{-\frac{1}{4}\left(\frac{20}{3}\ln\frac{8}{5}\right)}$

$= -40\, e^{-\frac{8}{3}\ln\frac{8}{5}} + 40\, e^{-\frac{5}{3}\ln\frac{8}{5}}$ // $\approx 6,85317$ //

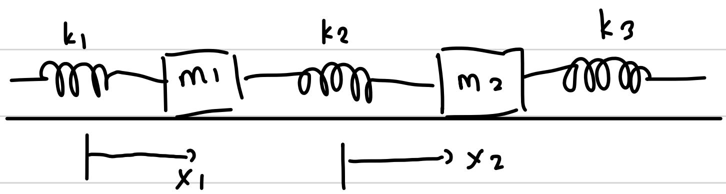


The graph of $x_1(t)$ = green

$x_2(t)$ = blue

Problems 5.5

$\boxed{1}$



no walls

$$K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}$$

with : $m_1 = m_2 = 1$; $k_1 = 0$, $k_2 = 2$, $k_3 = 0$ (no walls)

$Mx'' = Kx$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} x'' = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x$$

$$x'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x$$

Find the eigenvalues.

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 \quad - 4 = \lambda^2 + 4\lambda = 0$$

$$\lambda_1 = 0 \quad ; \quad \lambda_2 = -4$$

Find Eigenvector :

$\lambda = 0$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} v_1 = 0 \iff v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -4$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} V_2 = 0 \iff V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{Eigenvetor} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$\lambda = \alpha^2 = -\omega^2$

$\alpha = \pm i\omega$

* $\lambda = 0 \rightarrow \alpha = 0$
* $\lambda = -4 = -\omega^2$

$\omega^2 = 4$

$\alpha^2 = -(-4) = 4$

$\alpha = 2$

Natural Frequency :

$\alpha = 0$

$x_1(t) = (a_1 + b_1 t)\, v_1$

$= (a_1 + b_1 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\alpha = 2$

$x(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$x(t) = \left(\frac{a_2}{\sqrt{a_2 + b^2}} \cos 2t + \frac{b_2}{\sqrt{a_2 + b^2}} \sin 2t\right)\left(\sqrt{a^2 + b^2}\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= C_2\left(\cos \alpha_2 \cos 2t + \sin \alpha_2 \sin 2t\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$x(t) = C_2 \cos(2t - \alpha_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -C_2 \cos(2t - \alpha_2) \\ C_2 \cos(2t - \alpha_2) \end{bmatrix}$

$x_1(t) = -C_2 \cos(2t - \alpha_2)$

$x_2(t) = C_2 \cos(2t - \alpha_2)$

Explanations:

- for $\alpha = 0$ the two mass have the same direction with the amplitudes
- for $\alpha = 2$ the two mass move in different ways ; but the value is the same for x_1 and x_2.

$\boxed{3}$ $m_1 = 1$, $m_2 = 2$; $k_1 = 1$, $k_2 = k_3 = 2$ $\quad K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}$

$M x'' = KX$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x'' = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} x$$

$$x'' = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} x = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} x$$

Find Eigenvalue:

$$\begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda) - 2$$

$$= 6 + 5\lambda + \lambda^2 - 2 = \lambda^2 + 5\lambda + 4 = 0$$

$$= (\lambda+1)(\lambda+4) = 0$$

$$\lambda_1 = -1 \quad \& \quad \lambda_2 = -4$$

Eigenvectors:

$\lambda_1 = -1$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} v_1 = 0 \; ; \; V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -4$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} V_2 = 0 \; ; \; V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\lambda = \alpha^2 = -\omega^2$

- $\lambda_1 = -1$

  $\alpha^2 = -(-1)$

  $\alpha = 1$

- $\lambda_2 = -4$

  $\alpha^2 = -(-4)$

  $\alpha = 2$

Natural Frequency:

- $\alpha = 1$

$$x(t) = (a_1 \cos t + b_2 \sin t)\begin{bmatrix}1\\1\end{bmatrix}$$

$$= c_1 \cos(t - \alpha_1)\begin{bmatrix}1\\1\end{bmatrix}$$

For $\lambda = -1$; $\alpha = 1$ the two masses move in the same direction as the amplitudes.

- $\alpha = 2$

$$x(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= C_2(\cos(2t - \alpha_2))\begin{bmatrix}-2\\1\end{bmatrix}$$

$$x(t) = \begin{bmatrix}-2C_2 \cos(2t - \alpha_2)\\ C_2 \cos(2t - \alpha_2)\end{bmatrix} \left.\begin{matrix}\\ \\ \end{matrix}\right\} \begin{matrix} x_1(t) = -2C_2 \cos(2t - \alpha_2) \\ x_2(t) = C_2 \cos(2t - \alpha_2)\end{matrix}$$

For $\lambda = -4$; $\alpha = 2$ the direction of two masses is moving in opposite ways and x_1 is twice the amplitude of x_2.

$\boxed{18}$ $m_1 = 8$ (tons) $= 500$

$m_2 = 16$ (tons) $: 1000$

$k = 3000$

$\frac{\pi}{3}$ seconds

$$x'' = \begin{bmatrix} -c_1 & c_1 \\ c_2 & -c_2 \end{bmatrix} x$$

$c_i = k / m_i$

$x_1'(t) = -\frac{1}{3} v_0$

$x_2'(t) = \frac{2}{3} v_0$

$$x'' = \begin{bmatrix} -6 & 6 \\ 3 & -3 \end{bmatrix} x$$

Eigenvalue :

$$\begin{vmatrix} -6-\lambda & 6 \\ 3 & -3-\lambda \end{vmatrix} = (6+\lambda)(3+\lambda) - 18$$

$$= 18 + 9\lambda + \lambda^2 + (-18) = \lambda^2 + 9\lambda = 0$$

$$\lambda_1 = 0 \;;\; \lambda_2 = -9$$

Eigen vectors: $\lambda = 0$

$$\begin{bmatrix} -6 & 6 \\ 3 & -3 \end{bmatrix} v_1 = 0 \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -9$

$$\begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} v_2 = 0 \longrightarrow v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Find α

$\lambda = \alpha^2 = -\omega^2$

* $\lambda = 0$

$\alpha = 0$

** $\lambda = -9$

$\alpha^2 = -(-9)$

$\alpha = 3$

* $\alpha = 0$

$$x(t) = a_1 + b_1 t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = x_2(t) = a_1 + b_1 t$$

* $\alpha = 3$

$$x(t) = (a_2 \cos 3t + b_2 \sin 3t) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2a_2 \cos 3t - 2b_2 \sin 3t \\ a_2 \cos 3t + b_2 \sin 3t \end{bmatrix}$$

General Solutions.

$x_1 = a_1 + b_1 t - 2a_2 \cos 3t - 2b_2 \sin 3t$ | $x_1' = b_1 + 6a_2 \sin 3t - 6b_2 \cos 3t$

$x_2 = a_1 + b_1 t + a_2 \cos 3t + b_2 \sin 3t$ | $x_2' = b_1 - 3a_2 \sin 3t + 3b_2 \cos 3t$

From problem 16, we know that the car 1 had velocity V_0 and car 2 velocity of 0

$x_1(0) = 0$; $x_1' = V_0$

$x_2(0) = 0$ $x_2' = 0$

Thus:

$x_1(0) = a_1 - 2a_2 = 0 \longrightarrow a_1 = 2a_2$

$x_2(0) = a_1 + a_2 = 0$

$2a_2 + a_2 = 0$

$a_2 = 0$

$a_1 = 0$

$x_1'(0) = b_1 - 6b_2 = V_0$

$x_2'(0) = b_1 + 3b_2 = 0$ $-$

$-9b_2 = V_0$

$\boxed{b_2 = -\frac{V_0}{9}}$

$b_1 + \frac{2}{3}V_0 = V_0 \longrightarrow \boxed{b_1 = \frac{1}{3}V_0}$

Particular Solutions :

$x_1(t) = \frac{1}{3}V_0 t + \frac{2V_0}{9}\sin 3t$

$x_2(t) = \frac{1}{3}V_0 t - \frac{V_0}{9}\sin 3t$

The colissions happen :

$x_2 - x_1 < 0$ until the collissions is done ;

$x_2 - x_1 = 0$

$-\frac{1}{3}V_0 \sin 3t = 0$

$t = \frac{\pi}{3}$ //

To prove that cars 1 and 2 is moving away ; we can use x_1' and x_2' at $t = \frac{\pi}{3}$

$x_1' = \frac{1}{3}V_0 + \frac{2}{3}V_0 \cos 3t$

$x_2' = \frac{1}{3}V_0 - \frac{1}{3}V_0 \cos 3t$

$t = \pi/3$

$\left.\begin{array}{l} x_1'(\frac{\pi}{3}) = \frac{1}{3}V_0 - \frac{2}{3}V_0 = -\frac{1}{3}V_0 \\ x_2'(\frac{\pi}{3}) = \frac{1}{3}V_0 + \frac{1}{3}V_0 = \frac{2}{3}V_0 \end{array}\right\}$ moving in opposite directions.

Proved //