Linear System Differential Equaton

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Problem 5.2

Show that the systems in Problem 23 through 25 are degenerate. In each problem determine – by attempting to solve the system – whether it has infinitely many solutions or no solutions.

$\boxed{23.}$

$$(D+2)x + (D+2)y = e^{-3t}$$
$$(D+3)x + (D+3)y = e^{-2t}$$

Matrix form:

$$\begin{bmatrix}(D+2) & (D+2)\\(D+3) & (D+3)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}e^{-3t}\\e^{-2t}\end{bmatrix}\longrightarrow\begin{bmatrix}D+2 & D+2\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}e^{-3t}\\e^{-2t}-e^{-3t}\frac{D+3}{D+2}\end{bmatrix}$$

$$R_2 = R_2 - \frac{D+3}{D+2}R_1$$

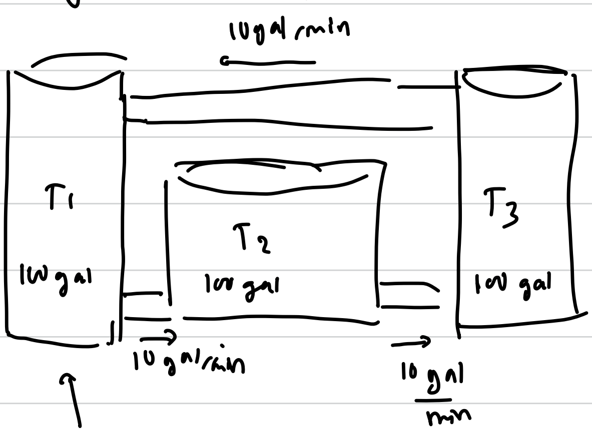
$$e^{-2t} - e^{-7t}$$

$$\begin{bmatrix}D+2 & D+2\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}e^{-3t}\\ \dfrac{-2e^{-2t}+2e^{-2t}-(-3e^{-7t}+3e^{-3t})}{D+2}\end{bmatrix}=\begin{bmatrix}e^{-3t}\\0\end{bmatrix}$$

→ its degenerate
→ and infinitely many solution.

The system is degenerate and it has infinitely many solutions.

$\boxed{34.}$



100 lb salt = x_1

Using konsentrations equations

$$C = \frac{x_1}{V} = \frac{100\text{ lb}}{100} = 1$$

$$10x_1' = -x_1 + x_3$$
$$10x_2' = x_1 - x_2$$
$$10x_3' = x_2 - x_3$$

$$10x_1' + x_1 - x_3 = 0 \rightarrow (10D+1)x_1 - x_3 = 0$$
$$10x_2' - x_1 + x_2 = 0 \rightarrow -x_1 + (10D+1)x_2 = 0$$
$$10x_3' - x_2 + x_3 = 0 \rightarrow -x_2 + (10D+1)x_3 = 0$$

$$\begin{bmatrix}(10D+1) & 0 & -1\\-1 & (10D+1) & 0\\0 & -1 & (10D+1)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

determinant

$$\begin{bmatrix} (10D+1) & 0 & -1 \\ -1 & (10D+1) & 0 \\ 0 & -1 & (10D+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \left| \begin{array}{ccc|cc} (10D+1) & 0 & -1 & (10D+1) & 0 \\ -1 & (10D+1) & 0 & -1 & (10D+1) \\ 0 & -1 & (10D+1) & 0 & -1 \end{array} \right| = 0$$

$$(10D+1)^3 - 1 - (0) = 1000D^3 + 300D^2 + 30D + 1 - 1 = 10D\left[100D^2 + 30D + 3\right] = 0$$

$$\Leftrightarrow 10r\left(100r^2 + 30r + 3\right) = 0$$

$$r_1 = 0$$

$$r_{2,3} = \frac{-30 \pm \sqrt{900 - 1200}}{2(100)} = \frac{-30 \pm \sqrt{-300}}{200} = \frac{-3 \pm i\sqrt{3}}{20}$$

$$x_1(t) = Ae^{0t} + e^{-\frac{3t}{20}}\left(B\cos\left(\frac{\sqrt{3}}{20}t\right) + C\sin\left(\frac{\sqrt{3}}{20}t\right)\right)$$

Initial value of $x_1(0) = 100$ lb ; $x_2(0) = 0$; $x_3(0) = 0$

- $x_1(0) = A + B = 100$ lb

$$B = 100 - A$$

- $10x_1' + x_1 - x_3 = 0$

$$x_1' = -\frac{3}{20}(100-A)e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}}{20}t\right) - (100-A)e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\sin\left(\frac{\sqrt{3}}{20}t\right) - \frac{3}{20}Ce^{-\frac{3t}{20}}\sin\left(\frac{\sqrt{3}}{20}t\right) + Ce^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\cos\left(\frac{\sqrt{3}}{20}t\right)$$

$$x_3 = 10x_1' + x_1$$

$$x_3(0) = 0 = 10\left(-\frac{3}{20}(100-A) + \frac{\sqrt{3}}{20}C\right) + 100$$

$$= \frac{-300+3A}{2} + \frac{C\sqrt{3}}{2} + 100 = 0$$

$$= \frac{-100 + 3A + C\sqrt{3}}{2} = 0 \longrightarrow C\sqrt{3} = 100 - 3A$$

$$C = \frac{100\sqrt{3} - \sqrt{3}A}{3}$$

$$x_3(t) = 10\left(-\frac{3}{20}(100-A)e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}}{20}t\right) - (100-A)e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\sin\left(\frac{\sqrt{3}}{20}t\right) - \frac{3}{20}\left(\frac{100\sqrt{3}}{3}-A\sqrt{3}\right)e^{-\frac{3t}{20}}\sin\left(\frac{\sqrt{3}}{20}t\right) + \left(\frac{100\sqrt{3}}{3}-\sqrt{3}A\right)e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\cos\left(\frac{\sqrt{3}}{20}t\right)\right) + 100$$

$$x_3(t) = \left(\frac{-300+3A}{2}\right)e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}}{20}t\right) - (100-A)\frac{\sqrt{3}}{2}e^{-\frac{3}{20}t}\sin\left(\frac{\sqrt{3}}{20}t\right) - \left(50\sqrt{3} - \frac{3}{2}A\sqrt{3}\right)e^{-\frac{3}{20}t}\sin\left(\frac{\sqrt{3}}{20}t\right) + \left(50 - \frac{3A}{2}\right)e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}}{20}t\right) + 100$$

$$x_3' = \left(\frac{-300+3A}{2}\right)\left[-\frac{3}{20}e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}}{20}t\right) - e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\sin\left(\frac{\sqrt{3}}{20}t\right)\right] - (100-A)\frac{\sqrt{3}}{2}\left[-\frac{3}{20}e^{-\frac{3}{20}t}\sin\left(\frac{\sqrt{3}}{20}t\right) + \frac{\sqrt{3}}{20}e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}t}{2}\right)\right]$$
$$- \left(50\sqrt{3} - \frac{3}{2}A\sqrt{3}\right)\left[-\frac{3}{20}e^{-\frac{3}{20}t}\sin\left(\frac{\sqrt{3}}{20}t\right) + e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\cos\left(\frac{\sqrt{3}}{20}t\right)\right] + \left(50-\frac{3A}{2}\right)\left[-\frac{3}{20}e^{-\frac{3}{20}t}\cos\left(\frac{\sqrt{3}t}{20}\right) - e^{-\frac{3}{20}t}\frac{\sqrt{3}}{20}\sin\left(\frac{\sqrt{3}t}{20}\right)\right]$$

$$x_2 = 10x_3' + x_3$$

$$x_2(0) = 10\left[\left(\frac{-300+3A}{2}\right)\left(-\frac{3}{20}\right) - (100-A)\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{20}\right) - \left(50\sqrt{3} - \frac{3A}{2}\sqrt{3}\right)\frac{\sqrt{3}}{20} + \left(50-\frac{3A}{2}\right)\frac{-3}{20}\right]$$

$$= \left[\left(\frac{-300+3A}{2}\right)\left(-\frac{3}{2}\right) - (100-A)\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{100}{2}\sqrt{3} - \frac{3A}{2}\sqrt{3}\right)\frac{\sqrt{3}}{2} + \left(\frac{100}{2}-\frac{3A}{2}\right)\frac{-3}{2}\right]$$

$$= \left[\frac{900 - 9A - 300 + 3A - 300 + 9A - 300 + 9A}{4}\right]$$

$$x_2 = \left[\frac{-\cancel{9A} + 3A + \cancel{9A} + 9A}{4}\right] = 0$$

$$\frac{12A}{4} = 0$$

Sorry, I cannot continue the calculation.

Problem 5.3

4. $x = \begin{bmatrix} 2t \\ e^{-t} \end{bmatrix}$ and $y = \begin{bmatrix} t^2 \\ \sin t \\ \cos t \end{bmatrix}$ $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 5 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 3 \\ -7 & 0 \\ 3 & -2 \end{bmatrix}$

Find Ay and Bx

$$Ay = \begin{bmatrix} 2 & 0 & -1 \\ \cdot & -4 & 5 \end{bmatrix} \begin{pmatrix} t^2 \\ \sin t \\ \cos t \end{pmatrix} = \begin{bmatrix} 2t^2 - \cos t \\ 3t^2 - 4\sin t + 5\cos t \end{bmatrix}_{//}$$

$$By = \begin{bmatrix} 1 & 3 \\ -7 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2t \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 2t + 3e^{-t} \\ -14t \\ 6t - 2e^{-t} \end{bmatrix}_{//}$$

The product of Ay and By is not defined because the numbers of colom Ay is not the same as the row of By. So it is not defined.

11. $x' = -3y$, $y' = 3x$

$x' = 0x - 3y + 0$

$y' = 3x + 0 + 0$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x' = P(t) \quad x \quad f(t)$$

$$\mathbf{x}' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \mathbf{x}$$

12. $x' = 3x - 2y$; $y' = 2x + y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x' = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} x$$

13. $x' = 2x + 4y + 3e^t$; $y' = 5x - y - t^2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

$$x' = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} x + \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$$

Problem 5.3

$\boxed{24}$ $x' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} x$; $x_1 = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_2 = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$x_1 = \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix}$ $x_2 = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$

$x_1' = \begin{bmatrix} 3e^{3t} \\ -3e^{3t} \end{bmatrix}$; $x_2' = \begin{bmatrix} 2e^{2t} \\ -4e^{2t} \end{bmatrix}$

$\rightarrow x_1' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ -3e^{3t} \end{bmatrix} = x_1'$

$x_2' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -4e^{2t} \end{bmatrix} = x_2'$

x_1 and x_2 is the given solution for the system.

* Wronkian Solutions :

$$\begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = -2e^{5t} + e^{5t} = -e^{5t} \neq 0 \Rightarrow \text{its linearly independent.}$$

x_1 x_2

* General solutions

$$x(t) = a\begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} + b\begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} ae^{3t} + be^{2t} \\ -ae^{3t} - 2be^{2t} \end{bmatrix},$$

$\boxed{25}$ $x' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} x$; $x_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$; $x_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

$x_1' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$; $x_2' = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$

$$x_1' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 12e^{2t} - 6e^{2t} \\ 18e^{2t} - 14e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = x_1'$$

$$x_2' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} 4e^{-5t} - 9e^{-5t} \\ 6e^{-5t} - 21e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} = x_2'$$

it's the solution for the system.

* Wronskian Solution: $\begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 9e^{-3t} - 2e^{-3t} = 7e^{-3t} \neq 0 \rightarrow$ linearly independent.

* General Solution: $x(t) = \begin{bmatrix} 3ae^{2t} + be^{-5t} \\ 2ae^{2t} + 3be^{-5t} \end{bmatrix}$

Problem 5.4.

$\boxed{6}$ $x_1' = 9x_1 + 5x_2$; $x_2' = -6x_1 - 2x_2$; $x_1(0) = 1$; $x_2(0) = 0$

$x' = \begin{bmatrix} 9 & 5 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\rightsquigarrow$ find eigenvalues and eigenvectors.

"In the next page!"

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = 0 \qquad (9-\lambda)(-2-\lambda) + 30 = 0$$

$$-18 - 9\lambda + 2\lambda + \lambda^2 + 30 = 0$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0 \qquad \boxed{\lambda_1 = 3 \; ; \; \lambda_2 = 4}$$

Eigenvectors:

$\lambda_1 = 3$

$$\begin{vmatrix} 6 & 5 \\ -6 & -5 \end{vmatrix} x = 0 \quad \rightsquigarrow \quad v_1 = \begin{bmatrix} -\frac{5}{6} \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$\lambda_2 = 4$

$$\begin{vmatrix} 5 & 5 \\ -6 & -6 \end{vmatrix} x = 0 \quad \rightsquigarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

General Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n$

$$x(t) = c_1 e^{3t} \begin{bmatrix} -5 \\ 6 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \cdot -5 + c_2 e^{4t} \\ c_1 e^{3t} \, 6 + c_2 e^{4t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -5c_1 + c_2 \\ 6c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} -5 & 1 & 1 \\ 0 & \frac{1}{5} & \frac{6}{5} \end{array}\right]$$

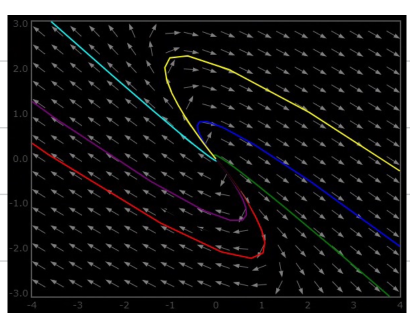
$$\frac{1}{5} c_2 = \frac{6}{5} \longrightarrow c_2 = 6$$

$$-5c_1 + 6 = 1$$

$$c_1 = 1$$

$$x_1(t) = -5e^{3t} + 6e^{4t}$$

$$x_2(t) = 6e^{3t} - 6e^{4t}$$



Field

$\boxed{7}$ $x_1' = -3x_1 + 4x_2$; $x_2' = 6x_1 - 5x_2$

$$x' = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix} x \rightsquigarrow \text{Finding Eigenvalues.}$$

$$\begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} = 0 = (-3-\lambda)(-5-\lambda) - 24 = \lambda^2 + 8\lambda - 9 = 0$$

$$= 15 + 3\lambda + 5\lambda + \lambda^2 - 24 \qquad (\lambda+9)(\lambda-1) = 0 \qquad \lambda_1 = -9 \; ; \; \lambda_2 = 1$$

Eigenvectors : $\lambda_1 = -9$

$$\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix} v_1 = 0 \longrightarrow v_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

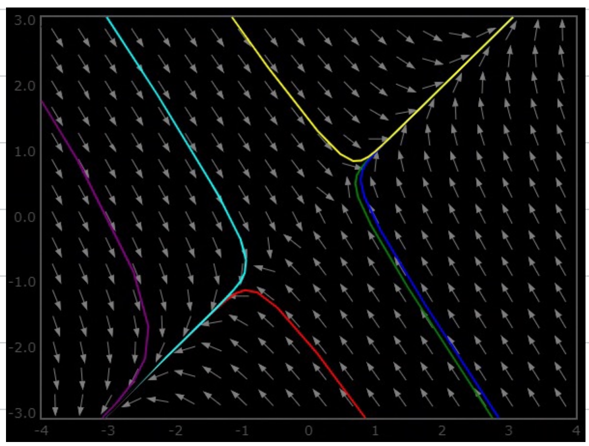
$\lambda_2 = 1$

$$\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} v_2 = 0 \longrightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = c_1 e^{-9t} \begin{bmatrix} -2 \\ 3 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = -2c_1 e^{-9t} + c_2 e^{t}$$

$$x_2(t) = 3c_1 e^{-9t} + c_2 e^{t}$$



Directional Field.

$\boxed{26}$

$$\begin{aligned} x_1' &= 3x_1 + x_3 \\ x_2' &= 9x_1 - x_2 + 2x_3 \\ x_3' &= -9x_1 + 4x_2 - x_3 \end{aligned} \rightsquigarrow x' = \begin{bmatrix} 3 & 0 & 1 \\ 9 & -1 & 2 \\ -9 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

↳ Find eigen values

$$\left|\begin{array}{ccc|cc} 3-\lambda & 0 & 1 & 3-\lambda & 0 \\ 9 & -1-\lambda & 2 & 9 & -1-\lambda \\ -9 & 4 & -1-\lambda & -9 & 4 \end{array}\right| = (3-\lambda)(-1-\lambda)^2 + 36 - (-1-\lambda)(-9) - 8(3-\lambda)$$

$$= -\lambda^3 + \lambda^2 + 4\lambda + 6 = 0$$

using calculator : $\lambda_1 = 3$; $\lambda_{2,3} = -1 \pm i$

Eigenvectors: $\lambda_1 = 3$

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} v_1 = 0 \quad ; \quad v_1 = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

$\lambda_2 = -1 + j$

$$\begin{bmatrix} 4-j & 0 & 1 \\ 9 & -j & 2 \\ -9 & 4 & -j \end{bmatrix} v_2 = 0 \quad ; \quad v_2 = \begin{bmatrix} 1 \\ 2-j \\ -4+j \end{bmatrix}$$

$$x_1(t) = 4c_1 e^{3t} + e^{-t}(c_2 \cos t + c_3 \sin t)$$
$$x_2(t) = 9c_1 e^{3t} + e^{-t}((2c_2 - c_3)\cos t + (c_2 + 2c_3)\sin t)$$
$$x_3(t) = e^{-t}((-4c_2 + c_3)\cos t + (-c_2 - 4c_3)\sin t)$$

$x_1(0) = 4c_1 + c_2 = 0$

$x_2(0) = 9c_1 + 2c_2 - c_3 = 0 \;=\; 9c_1 + 2(-4c_1) - c_3 = 0 \;=\; c_1 = c_3$

$x_3(0) = -4c_2 + c_3 = 17 \;\big|\; 4 \;=\; -16c_2 + 4c_3 = 68$

$ 4c_3 + c_2 = 0 \;\big|\; 1 \qquad\quad 4c_3 + c_2 = 0 \;-$

$-17c_2 = 68$

$\boxed{c_2 = -4}$

$4c_1 + -4 = 0$

$\boxed{c_1 = 1 = c_3}$

$$x_1(t) = 4e^{3t} + e^{-t}(-4\cos t + \sin t)$$
$$x_2(t) = 9e^{3t} + e^{-t}(-9\cos t - 2\sin t)$$
$$x_3(t) = e^{-t}(17\cos t)$$

$\boxed{41}$.

$$x' = \begin{bmatrix} 4 & 1 & 1 & 7 \\ 1 & 4 & 10 & 1 \\ 1 & 10 & 4 & 1 \\ 7 & 1 & 1 & 4 \end{bmatrix} x$$

$\lambda_1 = -3$ $\quad$ $x_1(0) = 3$

$\lambda_2 = -6$ $\quad$ $x_2(0) = x_3(0) = 1$

$\lambda_3 = 10$ $\quad$ $x_4(0) = 3$

$\lambda_4 = 15$

Eigenvector: $\lambda_1 = -3$ $\begin{bmatrix} 7 & 1 & 1 & 7 \\ 1 & 7 & 10 & 1 \\ 1 & 10 & 7 & 1 \\ 7 & 1 & 1 & 7 \end{bmatrix} V_1 = 0 \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_2 = -6$

$$\begin{bmatrix} 10 & 1 & 1 & 7 \\ 1 & 10 & 10 & 1 \\ 1 & 10 & 10 & 1 \\ 7 & 1 & 1 & 10 \end{bmatrix} V_2 = 0 \rightarrow v_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 10$

$$\begin{bmatrix} -6 & 1 & 1 & 7 \\ 1 & -6 & 10 & 1 \\ 1 & 10 & -6 & 1 \\ 7 & 1 & 1 & -6 \end{bmatrix} V_3 = 0 \longrightarrow V_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

$\lambda_4 = 15$

$$\begin{bmatrix} -11 & 1 & 1 & 7 \\ 1 & -11 & 10 & 1 \\ 1 & 10 & -11 & 1 \\ 7 & 1 & 1 & -11 \end{bmatrix} V_3 = 0 \longrightarrow V_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$x_1(t) = -c_1 e^{-3t} + 2c_3 e^{10t} + c_4 e^{15t}$

$x_2(t) = -c_2 e^{-6t} - c_3 e^{10t} + 2c_4 e^{15t}$

$x_3(t) = c_2 e^{-6t} - c_3 e^{10t} + 2c_4 e^{15t}$

$x_4(t) = c_1 e^{-3t} + 2c_3 e^{10t} + c_4 e^{15t}$

$x_1(0) = -c_1 + 2c_3 + c_4 = 3$

$x_2(0) = -c_2 - c_3 + 2c_4 = 1$

$x_3(0) = c_2 - c_3 + 2c_4 = 1$

$x_4(0) = c_1 + 2c_3 + c_4 = 3$

$x_1(0) = -C_1 + 2C_3 + C_4 = 3$ $\qquad C_1 = 0 ; \; C_2 = 0 ; \; C_3 = 1 ; \; C_4 = 1$

$x_2(0) = -C_2 - C_3 + 2C_4 = 1$

$x_3(0) = C_2 - C_3 + 2C_4 = 1$

$x_4(0) = C_1 + 2C_3 + C_4 = 3$

Particular solutions.

$x_1(t) = 2e^{10t} + e^{15t}$

$x_2(t) = -e^{10t} + 2e^{15t}$

$x_3(t) = -e^{10t} + 2e^{15t}$

$x_4(t) = 2e^{10t} + e^{15t}$