Matdis B - Homework 04: Induction

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4 Let $P(n)$ be the statement that $1^3 + 2^3 + \ldots + n^3 = (n(n+1)/2)^2$ for positive integer n.

$\forall n \in \mathbb{Z}^+$

a) What is the statement $P(1)$

$P(1): 1^3 = 1 = (1(1+1)/2)^2$

b) Show that $P(1)$ is true, completing the basis step of the proof of $P(n)$ for all positive integers n

Basis Steps : $P(1): 1 = (1(1+1)/2)^2 = \left(\frac{2}{2}\right)^2 = \frac{4}{4} = 1$

c) What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?

Inductive Steps:
$P(k) \Rightarrow P(k+1)$

Inductive Hypothesis (IH)
$P(k): 1^3 + 2^3 + \ldots + k^3 = (k(k+1)/2)^2$

d.) We need to proof if $P(k)$ is true, than $P(k+1)$ should be true to all the statement.

e) IH : $P(k): 1^3 + 2^3 + \ldots + k^3 = (k(k+1)/2)^2$
Show $P(k+1)$
$P(k+1): 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = ((k+1)(k+1+1)/2)^2$

$(k(k+1)/2)^2 + (k+1)^3 = ((k+1)(k+2)/2)^2$

$\left(k\frac{(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

$\cancel{(k+1)^2}\left(\left(\frac{k}{2}\right)^2 + k+1\right) = \frac{\cancel{(k+1)^2}(k+2)^2}{4}$

$\frac{k^2 + 4k + 4}{\cancel{4}} = \frac{(k+2)^2}{\cancel{4}}$

$(k+2)^2 = (k+2)^2$ ■ tomb stone

f) Two explain this formula is true, we can assume that all the positive integer numbers are represented as blocks of domino. When block number 1 is true, than block will falls and hits the second block and continue the dominoes effect until it reaches K^{th} block. If the block hits k^{th} block, then it should also hit $(k+1)^{th}$ block. These will indicate that all the formula of hitting the first domino block will also work when we hit the k^{th} or $(k+1)^{th}$ block to start the dominoes effect.

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a) Find a formula for

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}$

* $n = 1$; $P(1) : \frac{1}{2^1} = \frac{1}{2}$

* $n = 2$; $P(2) : \frac{1}{2^1} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{2+1}{4} = \frac{3}{4}$

* $n = 3$; $P(3) : \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}$

* $n = 4$; $P(4) : \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^4} = \frac{8+4+2+1}{16} = \frac{15}{16}$

So from the experiment, the formula is

$$P(n) : \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

b) Prove that for every positive integer n is true!

* Basic Step

$$P(1) : \frac{1}{2} = \frac{2^1 - 1}{2^1} = \frac{1}{2} \quad \checkmark$$

* Inductive Steps

Inductive Hypothesis:

$$P(k) : \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^k} = \frac{2^k - 1}{2^k}$$

Show that $P(k+1)$ is true!

$$P(k+1) : \underbrace{\frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^k}} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

$$P(k) + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

$$\frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

$$\frac{2^k - 1}{2^k} + \frac{1}{2 \cdot 2^k} = \frac{2 \cdot 2^k - 1}{2 \cdot 2^k}$$

$$\frac{1}{\cancel{2^k}}\left(2^k - 1 + \frac{1}{2}\right) = \frac{2 \cdot 2^k - 1}{2 \cdot \cancel{2^k}}$$

$$\frac{2 \cdot 2^k - 2 + 1}{\cancel{2}} = \frac{2 \cdot 2^k - 1}{\cancel{2}}$$

$$2^{k+1} - 1 = 2^{k+1} - 1 \quad \blacksquare$$

tomb stone

So, $P(k) \Rightarrow P(k+1)$ is also true for $n \in \mathbb{Z}^+$

$\boxed{27}$ Prove that every positif integer n is true!

27 Prove that every positif integer n is true!

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1)$$

※ Basic Steps :

$P(1): 1 > 2(\sqrt{2}-1)$ ✓ true!

※ Inductive Steps

∴ Inductive Hypothesis

$$P(k): 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{k}} > 2(\sqrt{k+1}-1)$$

Shows that $P(k+1)$ is also true!

$$P(k+1): 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2}-1)$$

$$P(k): \quad P(k) > 2(\sqrt{k+1}-1)$$

$$P(k)+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1}-1)+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2}-1)$$

We want to prove that the lef hand side is greater than the rigth hand side!

$$P(k)+\frac{1}{\sqrt{k+1}} > \frac{2(\sqrt{k+1}-1)(\sqrt{k+1})+1}{\sqrt{k+1}} > 2(\sqrt{k+2}-1)$$

$$\frac{2(k+1-\sqrt{k+1})+1}{\sqrt{k+1}} > 2(\sqrt{k+2}-1)$$

* multiply both side by "$\frac{1}{\sqrt{k+1}}$"

$$2k+2-2\sqrt{k+1}+1 > 2(\sqrt{k+2}-1)(\sqrt{k+1})$$

$$2k-2\sqrt{k+1}+3 > 2(\sqrt{k+2}\sqrt{k+1}-\sqrt{k+1})$$

$$2k-2\sqrt{k+1}+3 > 2\sqrt{k+2}\sqrt{k+1}-2\sqrt{k+1}$$

$$2k+3 > 2\sqrt{k+2}\sqrt{k+1}$$

* square both side

$$(2k+3)^2 > (2\sqrt{k^2+3k+2})^2$$

* we know that the left side is greater than the right side

$$4k^2+12k+9 > 4k^2+12k+8 \quad ✓$$

$$P(k+1): \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2}-1)$$

This prove us that if $P(k)$ then $P(k+1)$ is true for $\forall n \in \mathbb{Z}^+$ ▨ tomb stone