Swimming Problem

Description

A person is swimming across several rivers.

The widths of rivers are: ***S1, S2, ..., Sn***.

Speeds of those rivers are different: ***v1, v2, ..., vn***. To simplify this problem, only consider the speed in vertical direction.

The person's swimming speed is a constant ***v***. The angle of the person's velocity to ho horizontal line is ***a1, a2, ..., an***.

The total time for swimming is ***T***. And the person must pass those rivers.

Task

Find out an equation to determine by choosing what angles (***a1, a2, ..., an***) the person can get maxmimum distance in vertical direction. That is to say, please maximize ***dh*** by determine ***a1, a2, ..., an***) under the total time T.

You are not required to give out concrete angle numbers, a ***cost function*** that can be derived from is enough.

Tips: A mathematical tool you may need is called ***Lagrangian Multiplier***, which means, when you provide a formula, say ***E***, which still need to satisfy some more conditions, say ***a>1***, for the convenience of calculating, we can write those 2 parts (formula ***E*** and condition ***a>1***) together as one new formula. Here the new formula will be:

$$\boldsymbol{E - \lambda(\alpha - 1)}$$

Solution

When the person swims cross the i^{th} river,

Horizontal speed: $vcos\alpha_i$

Time: $t_i = \frac{s_i}{v_{horizontal}} = \frac{s_i}{vcos\alpha_i}$

Vertical speed: $v_i + vsin\alpha_i$

Vertical distance: $d_i = t_i(v_i + vsin\alpha_i) = (v_i + vsin\alpha_i)\frac{s_i}{vcos\alpha_i} = \frac{v_i s_i}{vcos\alpha_i} + s_i tan\alpha_i$

Total time: $T = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \frac{s_i}{vcos\alpha_i}$

Total vertical distance: $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \frac{v_i s_i}{vcos\alpha_i} + s_i tan\alpha_i$

The question becomes: $\begin{cases} \max(\sum_{i=1}^{n} d_i) = \max(\sum_{i=1}^{n} \frac{v_i s_i}{vcos\alpha_i} + s_i tan\alpha_i) \\ T = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \frac{s_i}{vcos\alpha_i} \end{cases}$

Build Lagrangian Multiplier:

$$\boldsymbol{L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} (\frac{v_i s_i}{vcos\alpha_i} + s_i tan\alpha_i) - \lambda(\sum_{i=1}^{n} \frac{s_i}{vcos\alpha_i} - T)}$$

Let the first-order partial derivative be equal to 0:

$$\begin{cases} \frac{\partial L}{\partial \alpha_i} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} = \begin{cases} \frac{\partial L}{\partial \alpha_i} = 0 \\ \sum_{i=1}^{n} \frac{s_i}{vcos\alpha_i} = T \end{cases}$$

There are $n + 1$ unknown numbers: n for α_i and 1 for λ.

There are $n + 1$ equations: n for α_i and 1 for λ.

So the equations are resoluble.