

Swimming Problem

Description

A person is swimming across several rivers.

The widths of rivers are: S_1, S_2, \dots, S_n .

Speeds of those rivers are different: v_1, v_2, \dots, v_n . To simplify this problem, only consider the speed in vertical direction.

The person's swimming speed is a constant v . The angle of the person's velocity to horizontal line is $\alpha_1, \alpha_2, \dots, \alpha_n$.

The total time for swimming is T . And the person must pass those rivers.

Task

Find out an equation to determine by choosing what angles ($\alpha_1, \alpha_2, \dots, \alpha_n$) the person can get maximum distance in vertical direction. That is to say, please maximize dh by determine $\alpha_1, \alpha_2, \dots, \alpha_n$ under the total time T .

You are not required to give out concrete angle numbers, a *cost function* that can be derived from is enough.

Tips: A mathematical tool you may need is called *Lagrangian Multiplier*, which means, when you provide a formula, say E , which still need to satisfy some more conditions, say $a > 1$, for the convenience of calculating, we can write those 2 parts (formula E and condition $a > 1$) together as one new formula. Here the new formula will be:

$$E - \lambda(\alpha - 1)$$

Solution

When the person swims cross the i^{th} river,

Horizontal speed: $v \cos \alpha_i$

$$\text{Time: } t_i = \frac{s_i}{v_{horizontal}} = \frac{s_i}{v \cos \alpha_i}$$

Vertical speed: $v_i + v \sin \alpha_i$

$$\text{Vertical distance: } d_i = t_i(v_i + v \sin \alpha_i) = (v_i + v \sin \alpha_i) \frac{s_i}{v \cos \alpha_i} = \frac{v_i s_i}{v \cos \alpha_i} + s_i \tan \alpha_i$$

$$\text{Total time: } T = \sum_{i=1}^n t_i = \sum_{i=1}^n \frac{s_i}{v \cos \alpha_i}$$

$$\text{Total vertical distance: } \sum_{i=1}^n d_i = \sum_{i=1}^n \frac{v_i s_i}{v \cos \alpha_i} + s_i \tan \alpha_i$$

$$\text{The question becomes: } \begin{cases} \max(\sum_{i=1}^n d_i) = \max(\sum_{i=1}^n \frac{v_i s_i}{v \cos \alpha_i} + s_i \tan \alpha_i) \\ T = \sum_{i=1}^n t_i = \sum_{i=1}^n \frac{s_i}{v \cos \alpha_i} \end{cases}$$

Build Lagrangian Multiplier:

$$L(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{i=1}^n \frac{v_i s_i}{v \cos \alpha_i} + s_i \tan \alpha_i - \lambda \left(\sum_{i=1}^n \frac{s_i}{v \cos \alpha_i} - T \right)$$

Let the first-order partial derivative be equal to 0:

$$\begin{cases} \frac{\partial L}{\partial \alpha_i} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} = \begin{cases} \frac{\partial L}{\partial \alpha_i} = 0 \\ \sum_{i=1}^n \frac{s_i}{v \cos \alpha_i} = T \end{cases}$$

There are $n + 1$ unknown numbers: n for α_i and 1 for λ .

There are $n + 1$ equations: n for α_i and 1 for λ .

So the equations are resolvable.