1 Thermal states

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Prerequisites: Completeness relations A displaced oscillator thermal state ρ_{th} is given by,

$$\rho_{\text{th}} = \left(1 - e^{-k}\right) D(z) e^{-ka^\dagger a} D(-z),$$

where k is a real positive number, given as the ratio of the oscillator energy over the thermal energy. z is a complex number referred to as the displacement of the state.

1.0.1 a)

We now show that the characteristic function of the thermal state is,

$$\chi_{\text{Th}}(\alpha, \alpha^*) = \exp\left[-\frac{1}{2}\begin{pmatrix} \alpha & \alpha^* \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} - \begin{pmatrix} z & z^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}\right],$$

where

$$\nu = \frac{1 + e^{-k}}{1 - e^{-k}}.$$

I) Show that we have the identity,

$$\begin{aligned} \chi_{\text{Th}}(\alpha, \alpha^*) &= \text{Tr}\left\{\rho_{\text{th}} D(\alpha)\right\} \\ &= \left(1 - e^{-k}\right) e^{\alpha z^* - \alpha^* z} \text{Tr}\left\{e^{-ka^\dagger a} D(\alpha)\right\} \end{aligned}$$

II) Perform the trace in the coherent basis and show that,

$$\chi_{\text{Th}}(\alpha, \alpha^*) = \frac{\left(1 - e^{-k}\right)}{\pi} e^{\alpha z^* - \alpha^* z} \int_{\mathbb{C}} d^2\beta \langle 0| e^{-ka^\dagger a} e^{ka^\dagger a} D(-\beta) e^{-ka^\dagger a} D(\alpha) D(\beta) |0\rangle$$

III) Verify that,

$$\langle 0| e^{-ka^\dagger a} = \langle 0|,$$

and

$$
\begin{aligned}
e^{ka^\dagger a} D(-\beta) e^{-ka^\dagger a} &= e^{-\beta e^k a^\dagger + \beta^* e^{-k} a} \\
&= e^{-\frac{1}{2}|\beta|^2} e^{-\beta e^k a^\dagger} e^{\beta^* e^{-k} a},
\end{aligned}
$$

Hint: Use the Baker-Campbell-Hausdorff lemma.
https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula

IV) Show that we can rearrange to obtain,

$$
\chi_{\mathrm{Th}}(\alpha, \alpha^*) = \frac{(1 - e^{-k})}{\pi} e^{\alpha z^* - \alpha^* z} \int_{\mathbb{C}} d^2\beta e^{-\frac{1}{2}|\beta|^2} e^{\beta^* e^{-k}(\alpha + \beta)} \langle 0 | D(\alpha) D(\beta) | 0 \rangle
$$

V) Once we have evaluated $\langle 0 | D(\alpha) D(\beta) | 0 \rangle$ we can perform the integration over β. Perform the integral and show that,

$$
\chi_{\mathrm{Th}}(\alpha, \alpha^*) = \exp\left[-\frac{1}{2} \begin{pmatrix} \alpha & \alpha^* \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}\nu \\ \frac{1}{2}\nu & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} - \begin{pmatrix} z & z^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \right]
$$

Verify that the limit of $k \to \infty$ yields the characteristic function of the displaced vacuum state.

Hint: Use the formula for gaussian integrals,
https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory

1.0.2 b)

We calculate the displacement and covariance matrix of a thermal state. We define the notation,

$$
\begin{aligned}
C_\alpha &= \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}, C_a = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \\
\bar{C} &= \begin{pmatrix} z \\ z^* \end{pmatrix}, \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}
$$

We define the covariance matrix,

$$
\Sigma = \frac{1}{2} \mathrm{Tr}\left\{ \rho \left(C_a \otimes C_a^T + C_a^T \otimes C_a \right) \right\} - \mathrm{Tr}\left\{ \rho C_a^T \right\} \otimes \mathrm{Tr}\left\{ \rho C_a \right\}
$$

$$
= \frac{1}{2} \mathrm{Tr}\left\{ \rho \left[\begin{pmatrix} aa, & aa^\dagger \\ a^\dagger a, & a^\dagger a^\dagger \end{pmatrix} + \begin{pmatrix} aa, & a^\dagger a \\ aa^\dagger, & a^\dagger a^\dagger \end{pmatrix} \right] \right\} - \begin{pmatrix} \langle a \rangle \langle a \rangle, & \langle a^\dagger \rangle \langle a \rangle \\ \langle a \rangle \langle a^\dagger \rangle, & \langle a^\dagger \rangle \langle a^\dagger \rangle \end{pmatrix}
$$

where $\langle a \rangle = \mathrm{Tr}\{\rho a\}$ and the action of taking the expectation is elementwise over an operator matrix. The $\otimes$ is a Kronecker product, see
https://en.wikipedia.org/wiki/Kronecker_product .

Note that the commutator on $\left[a, a^\dagger\right] = 1$ implies,

$$C_a \otimes C_a^T - C_a^T \otimes C_a = \Omega$$

and so

$$\Sigma = \mathrm{Tr}\left\{\rho C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\}$$

I) Show that we have the displacement,

$$\mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a\right\} = \bar{C}$$

II) Show that the covariance matrix of the single mode thermal state is,

$$\Sigma_{\mathrm{Th}} = \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a\right\} = \frac{1}{2}\begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix}$$

1.0.3 c)

We generalize the results of a) and b) to the n-mode case. We update our notation to,

$$\begin{aligned} \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}^T, a = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^T, \\ z = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \end{pmatrix}^T, \Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\ \Omega\Omega = \Omega^T\Omega^T = -I, \Omega^T = -\Omega. \end{aligned}$$

Note that we allow the dimension of the identity matrix I to vary with context.

We have the n-mode covariance matrix,

$$\Sigma = \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a \otimes C_a^T + C_a^T \otimes C_a\right)\right\} - \mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\},$$

which we write in block form as,

$$\begin{aligned} \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \frac{1}{2}\mathrm{Tr}\left\{\rho\left[\begin{pmatrix} a \otimes a^T, & a \otimes a^{\dagger T} \\ a^\dagger \otimes a^T, & a^\dagger \otimes a^{\dagger T} \end{pmatrix} + \begin{pmatrix} a^T \otimes a, & a^{\dagger T} \otimes a \\ a^T \otimes a^\dagger, & a^{\dagger T} \otimes a^\dagger \end{pmatrix}\right]\right\} \\ - \begin{pmatrix} \langle a\rangle^T \otimes \langle a\rangle, & \langle a\rangle^{*T} \otimes \langle a\rangle \\ \langle a\rangle^T \otimes \langle a\rangle^*, & \langle a\rangle^{*T} \otimes \langle a\rangle^* \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \operatorname{Tr}\{\rho a\} &= \langle a \rangle \\ \operatorname{Tr}\left\{\rho a^\dagger\right\} &= \langle a^\dagger \rangle = \langle a \rangle^* \end{aligned}$$

We now state some properties of these blocks.

I) Argue that we have the relations,

$$\begin{aligned} \Sigma_{11} &= \Sigma_{11}^T \\ \Sigma_{22} &= \Sigma_{22}^T \\ \Sigma_{12} &= \Sigma_{21}^T \\ \Sigma_{12}^T &= \Sigma_{12}^* \\ \Sigma_{11}^* &= \Sigma_{22} \\ \Sigma_{22}^* &= \Sigma_{11}. \end{aligned}$$

II) Argue that the identities from cI) jointly imply that Σ can be written in block matrix form,

$$\Sigma = \begin{pmatrix} \Sigma_D & \Sigma_A \\ \Sigma_A^* & \Sigma_D^* \end{pmatrix}$$

each block of dimension n.

III) Verify that we can conjugate as,

$$X \Sigma X = \Sigma^*$$

where

$$X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

1.0.4 d)

Verify that the covariance matrix Σ_{Th} of an n-mode thermal state can be written as,

$$
\Sigma_{\mathrm{Th}} = \frac{1}{2}\begin{pmatrix} 0 & \nu_{\mathrm{th}} \\ \nu_{\mathrm{th}} & 0 \end{pmatrix},
$$

$$
\nu_{\mathrm{th}} = \begin{pmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \nu_n \end{pmatrix},
$$

and $\nu_i = \frac{1+e^{-k_i}}{1-e^{-k_i}}$.

1.0.5 e)

Argue that the characteristic function of an n-mode thermal state can be written as,

$$
\chi_{\mathrm{Th}}(C_\alpha) = \exp\left[-\frac{1}{2}C_\alpha^T \Sigma_{\mathrm{Th}} C_\alpha - \bar{C}^T \Omega C_\alpha\right]
$$

where $\bar{C}$ is the n-mode displacement $\bar{C} = \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a\right\}$, and Σ_{Th} is the thermal state covariance matrix.

1.0.6 f)

For reasons that will later become apparent, it is advantageous to symmetrize $\chi_{\mathrm{Th}}(C_\alpha)$ a bit further. Show that $\chi_{\mathrm{Th}}(C_\alpha)$ can be written as,

$$
\chi_{\mathrm{Th}}(C_\alpha) = \exp\left[\frac{1}{2}\left(\Omega C_\alpha\right)^T \Sigma_{\mathrm{Th}} \Omega C_\alpha - \bar{C}^T \Omega C_\alpha\right]
$$