1 Thermal states

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1.0.1 a)

$$
\begin{aligned}
\chi_{\mathrm{Th}}(\alpha,\alpha^*) &= \mathrm{Tr}\left\{\rho_{\mathrm{th}} D(\alpha)\right\} \\
&= \left(1-e^{-k}\right)\mathrm{Tr}\left\{D(z)e^{-ka^\dagger a}D(-z)D(\alpha)\right\} \\
&= \left(1-e^{-k}\right)\mathrm{Tr}\left\{e^{-ka^\dagger a}D(-z)D(\alpha)D(z)\right\} \\
&= \left(1-e^{-k}\right)e^{\alpha z^*-\alpha^* z}\mathrm{Tr}\left\{e^{-ka^\dagger a}D(\alpha)\right\} \\
&= \left(1-e^{-k}\right)e^{\alpha z^*-\alpha^* z}\frac{1}{\pi}\int_{\mathbb{C}} d^2\beta\langle\beta|e^{-ka^\dagger a}D(\alpha)|\beta\rangle \\
&= \left(1-e^{-k}\right)e^{\alpha z^*-\alpha^* z}\frac{1}{\pi}\int_{\mathbb{C}} d^2\beta\langle 0|D(-\beta)e^{-ka^\dagger a}D(\alpha)D(\beta)|0\rangle \\
&= \frac{\left(1-e^{-k}\right)}{\pi}e^{\alpha z^*-\alpha^* z}\int_{\mathbb{C}} d^2\beta\langle 0|e^{-ka^\dagger a}e^{ka^\dagger a}D(-\beta)e^{-ka^\dagger a}D(\alpha)D(\beta)|0\rangle
\end{aligned}
$$

where from Baker-Campbell-Hausdorff we got,

$$
\begin{aligned}
e^{ka^\dagger a}ae^{-ka^\dagger a} &= e^{-k}a \\
e^{ka^\dagger a}a^\dagger e^{-ka^\dagger a} &= e^{k}a^\dagger
\end{aligned}
$$

and so

$$
e^{ka^\dagger a}D(-\beta)e^{-ka^\dagger a} = e^{-\beta e^k a^\dagger+\beta^* e^{-k}a} = e^{-\frac{1}{2}|\beta|^2}e^{-\beta e^k a^\dagger}e^{\beta^* e^{-k}a}
$$

and the characteristic function becomes,

$$
\begin{aligned}
\chi(\alpha,\alpha^*) &= \frac{\left(1-e^{-k}\right)}{\pi}e^{\alpha z^*-\alpha^* z}\int_{\mathbb{C}} d^2\beta e^{-\frac{1}{2}|\beta|^2}\langle 0|e^{\beta^* e^{-k}a}D(\alpha)D(\beta)|0\rangle \\
&= \frac{\left(1-e^{-k}\right)}{\pi}e^{\alpha z^*-\alpha^* z}\int_{\mathbb{C}} d^2\beta e^{-\frac{1}{2}|\beta|^2}\langle 0|D(\alpha)D(\beta)e^{\beta^* e^{-k}(a+\alpha+\beta)}|0\rangle \\
&= \frac{\left(1-e^{-k}\right)}{\pi}e^{\alpha z^*-\alpha^* z}\int_{\mathbb{C}} d^2\beta e^{-\frac{1}{2}|\beta|^2}e^{\beta^* e^{-k}(\alpha+\beta)}\langle 0|D(\alpha)D(\beta)|0\rangle \\
&= \frac{\left(1-e^{-k}\right)}{\pi}e^{\alpha z^*-\alpha^* z}e^{-\frac{1}{2}|\alpha|^2}\int_{\mathbb{C}} d^2\beta \exp\left[-\left(1-e^{-k}\right)|\beta|^2-\alpha^*\beta+\beta^* e^{-k}\alpha\right] \\
&= \exp\left[-\frac{1}{2}\left(\frac{1+e^{-k}}{1-e^{-k}}\right)|\alpha|^2+\alpha z^*-\alpha^* z\right] \\
&= \exp\left[-\frac{1}{2}\begin{pmatrix}\alpha & \alpha^*\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\nu \\ \frac{1}{2}\nu & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix}-\begin{pmatrix}z & z^*\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \alpha^*\end{pmatrix}\right]
\end{aligned}
$$

1.0.2 b)

I) We have,

$$
\begin{aligned}
\mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a\right\} &= \mathrm{Tr}\left\{\left(1-e^{-k}\right) D(z) e^{-k a^\dagger a} D(-z) C_a\right\} \\
&= \left(1-e^{-k}\right) \mathrm{Tr}\left\{e^{-k a^\dagger a} D(-z) C_a D(z)\right\} \\
&= \left(1-e^{-k}\right) \mathrm{Tr}\left\{e^{-k a^\dagger a}\left(C_a+\bar{C}\right)\right\} \\
&= \bar{C}+\left(1-e^{-k}\right) \sum_{n=0} e^{-k n}\langle n|C_a|n\rangle=\bar{C}
\end{aligned}
$$

II)

$$
\begin{aligned}
\Sigma_{\mathrm{Th}} &= \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a^T \otimes C_a\right\}+\frac{1}{2} \Omega-\mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_{\mathrm{Th}} C_a\right\} \\
&= \left(1-e^{-k}\right) \mathrm{Tr}\left\{e^{-k a^\dagger a} D(-z)\left(\begin{array}{cc} a a, & a^\dagger a \\ a a^\dagger, & a^\dagger a^\dagger \end{array}\right) D(z)\right\}+\frac{1}{2} \Omega-\bar{C}^T \otimes \bar{C} \\
&= \left(1-e^{-k}\right) \mathrm{Tr}\left\{e^{-k a^\dagger a}\left(\begin{array}{cc} (a+z)^2, & \left(a^\dagger+z^*\right)(a+z) \\ (a+z)\left(a^\dagger+z^*\right), & \left(a^\dagger+z^*\right)^2 \end{array}\right)\right\}+\frac{1}{2} \Omega-\bar{C}^T \otimes \bar{C} \\
&= \left(1-e^{-k}\right) \sum_{n=0}\langle n| e^{-k a^\dagger a}\left(\begin{array}{cc} (a+z)^2, & \left(a^\dagger+z^*\right)(a+z) \\ (a+z)\left(a^\dagger+z^*\right), & \left(a^\dagger+z^*\right)^2 \end{array}\right)|n\rangle+\frac{1}{2} \Omega-\bar{C}^T \otimes \bar{C} \\
&= \left(1-e^{-k}\right) \sum_{n=0} e^{-k n}\left(\begin{array}{cc} z^2, & n+|z|^2 \\ n+1+|z|^2, & z^{*2} \end{array}\right)+\frac{1}{2} \Omega-\left(\begin{array}{cc} z z, & z^* z \\ z z^*, & z^* z^* \end{array}\right) \\
&= \left(\begin{array}{cc} 0, & \frac{1}{2} \frac{1+e^{-k}}{1-e^{-k}} \\ \frac{1}{2} \frac{1+e^{-k}}{1-e^{-k}}, & 0 \end{array}\right)
\end{aligned}
$$

Where we used,

$$
\left(1-e^{-k}\right) \sum_{n=0} e^{-k n} n=\frac{e^{-k}}{1-e^{-k}}
$$

1.0.3 c)

We write Σ in block matrix form,

$$
\Sigma=\left(\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array}\right),
$$

and we define,

$$\mathrm{Tr}\left\{\rho C_a\right\} = \begin{pmatrix} z \\ z^* \end{pmatrix}$$

where z is a vector of complex numbers of length n.
We have,

$$\Sigma = \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a \otimes C_a^T + C_a^T \otimes C_a\right)\right\} - \mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\},$$

$$= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[\begin{pmatrix} a \otimes a^T, & a \otimes a^{\dagger T} \\ a^\dagger \otimes a^T, & a^\dagger \otimes a^{\dagger T} \end{pmatrix} + \begin{pmatrix} a^T \otimes a, & a^{\dagger T} \otimes a \\ a^T \otimes a^\dagger, & a^{\dagger T} \otimes a^\dagger \end{pmatrix}\right]\right\} - \begin{pmatrix} z^T \otimes z, & z^{*T} \otimes z \\ z^T \otimes z^*, & z^{*T} \otimes z^* \end{pmatrix}$$

I) We note that,

$$\begin{aligned}
\Sigma^T &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a \otimes C_a^T + C_a^T \otimes C_a\right)^T\right\} - \left(\mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\}\right)^T \\
&= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(\left(C_a \otimes C_a^T\right)^T + \left(C_a^T \otimes C_a\right)^T\right)\right\} - \mathrm{Tr}\left\{\rho C_a\right\} \otimes \mathrm{Tr}\left\{\rho C_a^T\right\} \\
&= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a^T \otimes C_a + C_a \otimes C_a^T\right)\right\} - \mathrm{Tr}\left\{\rho C_a\right\} \otimes \mathrm{Tr}\left\{\rho C_a^T\right\}
\end{aligned}$$

and since $\mathrm{Tr}\left\{\rho C_a\right\}$ is a vector of numbers (not operators) we have,

$$\mathrm{Tr}\left\{\rho C_a\right\} \otimes \mathrm{Tr}\left\{\rho C_a^T\right\} = \mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\}.$$

It then follows that,

$$\Sigma = \Sigma^T$$

from which we can extract,

$$\begin{aligned}
\Sigma_{11} &= \Sigma_{11}^T \\
\Sigma_{22} &= \Sigma_{22}^T \\
\Sigma_{12} &= \Sigma_{21}^T.
\end{aligned}$$

Let $(\Sigma_{12})_{ij}$ be element i,j of the matrix Σ_{12}. We note that,

$$(\Sigma_{12})_{ij} = \frac{1}{2}\mathrm{Tr}\left\{\rho\left[a_i a_j^\dagger + \ a_j^\dagger a_i\ \right]\right\} - z_j^* z_i$$

and so,

$$\left(\Sigma_{12}^{T}\right)_{ij} = \left(\Sigma_{12}\right)_{ji} = \frac{1}{2}\mathrm{Tr}\left\{\rho\left[a_j a_i^\dagger + \ a_i^\dagger a_j \ \right]\right\} - z_i^* z_j$$

and also,

$$\begin{aligned}\left(\Sigma_{12}^{*}\right)_{ij} &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[\left(a_i a_j^\dagger\right)^\dagger + \left(\ a_j^\dagger a_i \ \right)^\dagger\right]\right\} - z_j z_i^* \\ &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[a_j a_i^\dagger + a_i^\dagger a_j\right]\right\} - z_i^* z_j\end{aligned}$$

and we recognize that,

$$\left(\Sigma_{12}^{T}\right)_{ij} = \left(\Sigma_{12}^{*}\right)_{ij} .$$

Likewise,

$$\begin{aligned}\left(\Sigma_{11}^{*}\right)_{ij} &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[(a_i a_j)^\dagger + (a_j a_i)^\dagger\right]\right\} - z_j^* z_i^* \\ &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger\right]\right\} - z_j^* z_i^* \\ &= \left(\Sigma_{22}\right)_{ij} ,\end{aligned}$$

finally,

$$\begin{aligned}\left(\Sigma_{22}^{*}\right)_{ij} &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[\left(a_i^\dagger a_j^\dagger\right)^\dagger + \left(a_j^\dagger a_i^\dagger\right)^\dagger\right]\right\} - z_j z_i \\ &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left[a_i a_j + a_j a_i\right]\right\} - z_j z_i \\ &= \left(\Sigma_{11}\right)_{ij}\end{aligned}$$

and so we get,

$$\begin{aligned}\Sigma_{12}^{T} &= \Sigma_{12}^{*} \\ \Sigma_{11}^{*} &= \Sigma_{22} \\ \Sigma_{22}^{*} &= \Sigma_{11}\end{aligned}$$

II)

$$\begin{aligned}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^{T} & \Sigma_{11}^{*} \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^{*} & \Sigma_{11}^{*} \end{pmatrix}\end{aligned}$$

defining $\Sigma_D = \Sigma_{11}$ and $\Sigma_A = \Sigma_{12}$ we get,

$$\Sigma = \begin{pmatrix} \Sigma_D & \Sigma_A \\ \Sigma_A^* & \Sigma_D^* \end{pmatrix}$$

III) By matrix multiplication,

$$\begin{aligned} X\Sigma X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \Sigma_D & \Sigma_A \\ \Sigma_A^* & \Sigma_D^* \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \Sigma_A, & \Sigma_D \\ \Sigma_D^*, & \Sigma_A^* \end{pmatrix} \\ = \begin{pmatrix} \Sigma_D^*, & \Sigma_A^* \\ \Sigma_A, & \Sigma_D \end{pmatrix} = \Sigma^* \end{aligned}$$

1.0.4 d)

We have the matrix,

$$\mathrm{Tr}\left\{\rho_{\mathrm{Th}} a \otimes a^T\right\}$$

Then for $i \neq j$,

$$\begin{aligned} \left(\mathrm{Tr}\left\{\rho_{\mathrm{Th}} a \otimes a^T\right\}\right)_{ij} = \mathrm{Tr}\left\{\rho_{\mathrm{Th}} a_i a_j\right\} \\ = \mathrm{Tr}\left\{\rho_{\mathrm{th}}^{(i)} a_i\right\} \mathrm{Tr}\left\{\rho_{\mathrm{th}}^{(j)} a_j\right\} = z_i z_j. \end{aligned}$$

Whereas for $i = j$,

$$\begin{aligned} \left(\mathrm{Tr}\left\{\rho_{\mathrm{Th}} a \otimes a^T\right\}\right)_{ii} = \mathrm{Tr}\left\{\rho_{\mathrm{th}}^{(i)} a_i a_i\right\} \\ = \left(1 - e^{-k_i}\right) \mathrm{Tr}\left\{D(z_i) e^{-k_i a^\dagger a} D(-z_i) a_i a_i\right\} \\ = z_i^2 \left(1 - e^{-k_i}\right) \sum_{n=0} e^{-k_i n} = z_i^2. \end{aligned}$$

Likewise $\langle a \rangle_i = \mathrm{Tr}\left\{\rho_{\mathrm{th}}^{(i)} a_i\right\} = z_i$, and so,

$$\left(\langle a \rangle^T \otimes \langle a \rangle\right)_{ij} = \langle a \rangle_j \langle a \rangle_i = z_j z_i.$$

It follows that the upper left block Σ_D of the covariance matrix is zero,

$$\Sigma_D = \frac{1}{2} \mathrm{Tr}\left\{\rho_{\mathrm{Th}} \left[a \otimes a^T + a^T \otimes a\right]\right\} - \langle a \rangle^T \otimes \langle a \rangle = 0$$

For the upper right block Σ_A,

$$\begin{aligned}\left(\mathrm{Tr}\left\{\rho_{\mathrm{Th}} a \otimes a^{\dagger T}\right\}\right)_{ij} &= \mathrm{Tr}\left\{\rho_{\mathrm{Th}} a_i a_j^\dagger\right\} \\ &= \mathrm{Tr}\left\{\rho_{\mathrm{Th}} a_j^\dagger a_i\right\} + \delta_{i=j} \\ &= z_j^* z_i \delta_{i\neq j} + \left(\frac{e^{-k_i}}{1-e^{-k_i}} + |z_i|^2 + 1\right)\delta_{i=j} \\ &= z_j^* z_i + \left(\frac{e^{-k_i}}{1-e^{-k_i}} + 1\right)\delta_{i=j}\end{aligned}$$

and

$$\begin{aligned}\left(\mathrm{Tr}\left\{\rho_{\mathrm{Th}} a^{\dagger T} \otimes a\right\}\right)_{ij} &= \mathrm{Tr}\left\{\rho_{\mathrm{Th}} a_j^\dagger a_i\right\} \\ &= z_j^* z_i + \frac{e^{-k_i}}{1-e^{-k_i}}\delta_{i=j}\end{aligned}$$

and so,

$$\begin{aligned}(\Sigma_A)_{ij} &= \frac{1}{2}\left(\mathrm{Tr}\left\{\rho\left[a \otimes a^{\dagger T} + a^{\dagger T} \otimes a\right]\right\}\right)_{ij} - \left(\langle a\rangle^{*T} \otimes \langle a\rangle\right)_{ij} \\ &= z_j^* z_i + \frac{e^{-k_i}}{1-e^{-k_i}}\delta_{i=j} + \frac{1}{2}\delta_{i=j} - z_j^* z_i \\ &= \left(\frac{e^{-k_i}}{1-e^{-k_i}} + \frac{1}{2}\right)\delta_{i=j} = \frac{1}{2}\frac{1+e^{-k_i}}{1-e^{-k_i}}\delta_{i=j}\end{aligned}$$

and so

$$\Sigma_A = \frac{1}{2}\nu_{\mathrm{th}}$$

and so we find that the covariance matrix of a thermal state is,

$$\Sigma_{\mathrm{Th}} = \begin{pmatrix} 0 & \frac{1}{2}\nu_{\mathrm{th}} \\ \frac{1}{2}\nu_{\mathrm{th}} & 0 \end{pmatrix}$$

1.0.5 e)

The n-mode thermal state is a product of thermal states,

$$\rho_{\mathrm{th}} = \bigotimes_{k=1}^{n} \rho_{\mathrm{th}}^{(k)}$$

The characteristic function is then the product of the characteristic functions for each mode,

$$\chi_{\text{Th}}(C_\alpha) = \text{Tr}\left\{\rho_{\text{th}} D(C_\alpha)\right\} = \text{Tr}\left\{\bigotimes_{k=1}^{n} \rho_{\text{th}}^{(k)} \bigotimes_{k=1}^{n} D(\alpha_k)\right\} = \prod_{k=1}^{n} \text{Tr}\left\{\rho_{\text{th}}^{(k)} D(\alpha_k)\right\} = \prod_{k=1}^{n} \chi_{\text{Th}}^{(k)}(\alpha_k)$$

Taking the product,

$$\begin{aligned}
\chi_{\text{Th}}(C_\alpha) &= \prod_{k=1}^{n} \chi_{\text{Th}}^{(k)}(\alpha_k) \\
&= \exp\left[-\frac{1}{2}\sum_{k=1}^{n} v_k|\alpha_k|^2 - \sum_{k=1}^{n}(z_n\alpha_n^* - z_n^*\alpha_n)\right] \\
&= \exp\left[-\frac{1}{2}\alpha^T \nu_{\text{th}}\alpha^* - \left(z^T\alpha^* - z^{*T}\alpha\right)\right] \\
&= \exp\left[-\frac{1}{2}\left(\frac{1}{2}\alpha^T\nu_{\text{th}}\alpha^* + \frac{1}{2}\alpha^{*T}\nu_{\text{th}}\alpha\right) - \begin{pmatrix} z^T & z^{*T} \end{pmatrix}\begin{pmatrix} \alpha^* \\ -\alpha \end{pmatrix}\right] \\
&= \exp\left[-\frac{1}{2}\begin{pmatrix} \alpha^T & \alpha^{*T} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2}\nu_{\text{th}} \\ \frac{1}{2}\nu_{\text{th}} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} - \begin{pmatrix} z^T & z^{*T} \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}\right]
\end{aligned}$$

1.0.6 f)

We note that,

$$\Omega^T \Sigma_{\text{Th}} \Omega = -\Sigma_{\text{Th}}$$

and so we get,

$$\begin{aligned}
\chi_{\text{Th}}(C_\alpha) &= \exp\left[-\frac{1}{2}C_\alpha^T \Sigma_{\text{Th}} C_\alpha - \bar{C}^T \Omega C_\alpha\right] \\
&= \exp\left[\frac{1}{2}C_\alpha^T \Omega^T \Sigma_{\text{Th}} \Omega C_\alpha - \bar{C}^T \Omega C_\alpha\right]
\end{aligned}$$