1 Completeness relations

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1.1 Coherent states

1.1.1 a)

We have,

$$\begin{aligned} P|n\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha|\alpha\rangle\langle\alpha|n\rangle \\ &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha|\alpha\rangle e^{-\frac{1}{2}|\alpha|^2}\frac{1}{\sqrt{n!}}\alpha^{*n} \\ &= \frac{1}{\pi}\sum_{m=0}\frac{1}{\sqrt{m!n!}}|m\rangle\int_{\mathbb{C}} d^2\alpha e^{-|\alpha|^2}\alpha^m\alpha^{*n} \end{aligned}$$

We introduce polar coordinates $\alpha = re^{i\theta}$, then we get,

$$\begin{aligned} \int_{\mathbb{C}} d^2\alpha e^{-|\alpha|^2}\alpha^m\alpha^{*n} &= \int_0^\infty dr\int_0^{2\pi} d\theta e^{-r^2}e^{i\theta(m-n)}r^{m+n+1} \\ = 2\pi\delta_{m,n}\int_0^\infty dre^{-r^2}r^{2n+1} &= 2\pi\delta_{m,n}\int_0^\infty dre^{-r^2}r^{2n+1} = \pi\delta_{m,n}n! \end{aligned}$$

and so we get,

$$P|n\rangle = \frac{1}{\pi}\sum_{m=0}\frac{1}{\sqrt{m!n!}}|m\rangle\pi\delta_{m,n}n! = |n\rangle$$

1.1.2 b)

We define an arbitrary state

$$|\psi\rangle = \sum_{n=0}\psi_n|n\rangle$$

Applying P we get,

$$P|\psi\rangle = \sum_{n=0}\psi_n P|n\rangle = \sum_{n=0}\psi_n|n\rangle = |\psi\rangle$$

1.1.3 c)

We perform the trace in the Fock basis and rewrite using $P = I$,

$$
\begin{aligned}
\mathrm{Tr}\left\{\rho\right\} &= \sum_{n=0} \langle n|\rho|n\rangle \\
&= \sum_{n=0} \langle n| \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha |\alpha\rangle\langle\alpha|\rho \frac{1}{\pi} \int_{\mathbb{C}} d^2\beta |\beta\rangle\langle\beta|n\rangle \\
&= \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\alpha \int_{\mathbb{C}} d^2\beta \langle\alpha|\rho|\beta\rangle \sum_{n=0} \langle\beta|n\rangle\langle n|\alpha\rangle \\
&= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \langle\alpha|\rho \frac{1}{\pi} \int_{\mathbb{C}} d^2\beta |\beta\rangle\langle\beta|\alpha\rangle \\
&= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \langle\alpha|\rho|\alpha\rangle
\end{aligned}
$$

1.2 Displacement operators

1.2.1 a)

I) We use the Baker-Campbell-Hausdorff formula,

$$
D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}\left[\alpha a^\dagger, -\alpha^* a\right]} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}
$$

II) We use the Baker-Campbell-Hausdorff formula

$$
\begin{aligned}
D(\alpha)D(\beta) &= e^{\alpha a^\dagger - \alpha^* a} e^{\beta a^\dagger - \beta^* a} \\
&= \exp\left[\left(\alpha + \beta\right) a^\dagger - \left(\alpha^* + \beta^*\right) a + \frac{1}{2}\left[\alpha a^\dagger - \alpha^* a, \beta a^\dagger - \beta^* a\right]\right] \\
&= \exp\left[\left(\alpha + \beta\right) a^\dagger - \left(\alpha^* + \beta^*\right) a + \frac{1}{2}\left(\alpha\beta^* - \alpha^*\beta\right)\right] \\
&= e^{\frac{1}{2}\left(\alpha\beta^* - \alpha^*\beta\right)} D(\alpha + \beta)
\end{aligned}
$$

III) We use that $D(\alpha)$ is unitary,

$$
\begin{aligned}
D(\alpha)D(\beta) &= D(\alpha)D(\beta)D^\dagger(\alpha)D(\alpha) \\
&= e^{\beta\left(a^\dagger - \alpha^*\right) - \beta^*(a - \alpha)} D(\alpha) = e^{\beta^*\alpha - \beta\alpha^*} D(\beta)D(\alpha)
\end{aligned}
$$

IV) We proceed via normal ordering of the operators using the disentangling identity from I),

$$\begin{aligned}
\langle\alpha|D(\gamma)|\beta\rangle &= \langle 0|D(-\alpha)D(\gamma)D(\beta)|0\rangle \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}\langle 0|e^{\alpha^* a}e^{\beta\left(a^\dagger-\gamma^*\right)}D(\gamma)|0\rangle \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\gamma|^2\right)}e^{-\beta\gamma^*}\langle 0|e^{\alpha^* a}e^{(\beta+\gamma)a^\dagger}|0\rangle \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\gamma|^2\right)}e^{\alpha^*\beta}e^{\gamma\alpha^*-\gamma^*\beta}
\end{aligned}$$

V)

$$\begin{aligned}
\langle\alpha|\beta\rangle &= \langle 0|D(-\alpha)D(\beta)|0\rangle \\
&= e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}\langle 0|D(\beta-\alpha)|0\rangle \\
&= e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}\langle 0|\beta-\alpha\rangle \\
&= e^{\frac{1}{2}(-\alpha\beta^*+\alpha^*\beta)}e^{-\frac{1}{2}(\beta-\alpha)(\beta^*-\alpha^*)} \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}e^{\alpha^*\beta}
\end{aligned}$$

1.2.2 b)

We perform the trace in the coherent state basis,

$$\begin{aligned}
\mathrm{Tr}\left\{D(\alpha)\right\} &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\beta\langle\beta|D(\alpha)|\beta\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\beta\langle\beta|e^{\alpha a^\dagger-\alpha^* a}|\beta\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\beta\langle\beta|e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}e^{-\alpha^* a}|\beta\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2}\frac{1}{\pi}\int_{\mathbb{C}} d^2\beta e^{\alpha\beta^*-\alpha^*\beta} \\
&= e^{-\frac{1}{2}|\alpha|^2}\frac{1}{\pi}\int_{-\infty}^{\infty} d\beta_R\int_{-\infty}^{\infty} d\beta_I e^{2i\alpha_I\beta_R-i2\alpha_R\beta_I}
\end{aligned}$$

where $\beta_R = \mathrm{Re}\left\{\beta\right\}$ and $\beta_I = \mathrm{Im}\left\{\beta\right\}$, then,

$$\begin{aligned}
&= \pi e^{-\frac{1}{2}|\alpha|^2}\int_{-\infty}^{\infty}\frac{d\beta_R}{\pi}e^{2i\alpha_I\beta_R}\int_{-\infty}^{\infty}\frac{d\beta_I}{\pi}e^{-i2\alpha_R\beta_I} \\
&= \pi e^{-\frac{1}{2}|\alpha|^2}\int_{-\infty}^{\infty} d\gamma_R e^{2\pi i\alpha_I\gamma_R}\int_{-\infty}^{\infty} d\gamma_I e^{-i2\pi\alpha_R\gamma_I} \\
&= \pi e^{-\frac{1}{2}|\alpha|^2}\delta\left(\alpha_I\right)\delta\left(\alpha_R\right) = \pi\delta^{(2)}\left(\alpha\right)
\end{aligned}$$

1.2.3 c)

We multiply Glauber's formula by $D(\beta)$ and take the trace,

$$\begin{aligned}
\mathrm{Tr}\left\{\rho D(\beta)\right\} &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\chi(\alpha,\alpha^*)\mathrm{Tr}\left\{D^\dagger(\alpha)D(\beta)\right\} \\
&= \frac{1}{\pi}\int d^2\alpha\chi(\alpha,\alpha^*)e^{\frac{1}{2}(\alpha^*\beta-\alpha\beta^*)}\mathrm{Tr}\left\{D(\beta-\alpha)\right\} \\
&= \frac{1}{\pi}\int d^2\alpha\chi(\alpha,\alpha^*)e^{\frac{1}{2}(\alpha^*\beta-\alpha\beta^*)}\pi\delta^{(2)}\left(\beta-\alpha\right) \\
&= \chi(\beta,\beta^*)
\end{aligned}$$

1.2.4 d)

We expand the displacement operator,

$$\begin{aligned}
&\int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{|\alpha\rangle\langle\beta|D(\gamma)\right\}D^\dagger(\gamma) \\
&= \int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\langle\beta|D(\gamma)|\alpha\rangle D^\dagger(\gamma) \\
&= \int_{\mathbb{C}}\frac{d^2\lambda}{\pi}\int_{\mathbb{C}}\frac{d^2\eta}{\pi}\left[\int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\langle\beta|D(\gamma)|\alpha\rangle\langle\lambda|D^\dagger(\gamma)|\eta\rangle\right]|\lambda\rangle\langle\eta| \\
&= \int_{\mathbb{C}}\frac{d^2\lambda}{\pi}\int_{\mathbb{C}}\frac{d^2\eta}{\pi}I_1(\lambda,\eta)|\lambda\rangle\langle\eta|
\end{aligned}$$

We have,

$$\begin{aligned}
I_1(\lambda,\eta) &= \int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\langle\beta|D(\gamma)|\alpha\rangle\langle\lambda|D^\dagger(\gamma)|\eta\rangle \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\lambda|^2+|\eta|^2\right)}e^{\lambda^*\eta+\beta^*\alpha}\int_{\mathbb{C}}\frac{d^2\gamma}{\pi}e^{-|\gamma|^2}e^{\gamma(\beta^*-\lambda^*)+\gamma^*(\eta-\alpha)} \\
&= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2+|\lambda|^2+|\eta|^2\right)}e^{\lambda^*\eta+\beta^*\alpha}I_2(\lambda,\eta)
\end{aligned}$$

where

$$\begin{aligned}
I_2(\lambda,\eta) &= \int_{\mathbb{C}}\frac{d^2\gamma}{\pi}e^{-|\gamma|^2}e^{\gamma(\beta^*-\lambda^*)+\gamma^*(\eta-\alpha)} \\
&= \frac{1}{\pi}\int_{\mathbb{R}}d\gamma_R e^{-\gamma_R^2}e^{\gamma_R(\beta^*-\lambda^*+\eta-\alpha)}\int_{\mathbb{R}}d\gamma_I e^{-\gamma_I^2}e^{\gamma_I i(\beta^*-\lambda^*+\alpha-\eta)} \\
&= e^{\frac{1}{4}\left[(\beta^*-\lambda^*+\eta-\alpha)^2-(\beta^*-\lambda^*+\alpha-\eta)^2\right]} \\
&= e^{(\beta^*-\lambda^*)(\eta-\alpha)}
\end{aligned}$$

and so

$$
\begin{aligned}
I_1(\lambda,\eta) = \left(e^{-\frac{1}{2}\left(|\beta|^2+|\eta|^2\right)}e^{\beta^*\eta}\right)\left(e^{-\frac{1}{2}\left(|\alpha|^2+|\lambda|^2\right)}e^{\lambda^*\alpha}\right) \\
= \langle\beta|\eta\rangle\langle\lambda|\alpha\rangle
\end{aligned}
$$

and so we find,

$$
\begin{aligned}
&\int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{|\alpha\rangle\langle\beta|D(\gamma)\right\}D^\dagger(\gamma) \\
&= \int_{\mathbb{C}} \frac{d^2\lambda}{\pi}\int_{\mathbb{C}} \frac{d^2\eta}{\pi}\langle\beta|\eta\rangle\langle\lambda|\alpha\rangle|\lambda\rangle\langle\eta| \\
&= \int_{\mathbb{C}} \frac{d^2\lambda}{\pi}|\lambda\rangle\langle\lambda|\alpha\rangle\int_{\mathbb{C}} \frac{d^2\eta}{\pi}\langle\beta|\eta\rangle\langle\eta| \\
&= |\alpha\rangle\langle\beta|
\end{aligned}
$$

as was claimed.

1.2.5 e)

Since the coherent states form a complete basis, we may expand ρ as,

$$
\begin{aligned}
\rho &= \int_{\mathbb{C}} \frac{d^2\alpha}{\pi}\int_{\mathbb{C}} \frac{d^2\beta}{\pi}\langle\alpha|\rho|\beta\rangle|\alpha\rangle\langle\beta| \\
&= \int_{\mathbb{C}} \frac{d^2\alpha}{\pi}\int_{\mathbb{C}} \frac{d^2\beta}{\pi}\langle\alpha|\rho|\beta\rangle\int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{|\alpha\rangle\langle\beta|D(\gamma)\right\}D^\dagger(\gamma) \\
&= \int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{\int_{\mathbb{C}} \frac{d^2\alpha}{\pi}\int_{\mathbb{C}} \frac{d^2\beta}{\pi}|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|D(\gamma)\right\}D^\dagger(\gamma) \\
&= \int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{\rho D(\gamma)\right\}D^\dagger(\gamma)
\end{aligned}
$$

1.2.6 f)

$$
\begin{aligned}
\mathrm{Tr}\left\{AB\right\} &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha\int_{\mathbb{C}} d^2\beta\chi_A(\alpha,\alpha^*)\chi_B(\beta,\beta^*)\mathrm{Tr}\left\{D^\dagger(\alpha)D^\dagger(\beta)\right\} \\
&= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha\int_{\mathbb{C}} d^2\beta\chi_A(\alpha,\alpha^*)\chi_B(\beta,\beta^*)e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\mathrm{Tr}\left\{D(-\alpha-\beta)\right\} \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\int_{\mathbb{C}} d^2\beta\chi_A(\alpha,\alpha^*)\chi_B(\beta,\beta^*)e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\delta^{(2)}(\alpha+\beta) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\chi_A(\alpha,\alpha^*)\chi_B(-\alpha,-\alpha^*)
\end{aligned}
$$

1.2.7 g)

Expressing $|\alpha\rangle\langle\beta|$ using Glauber's formula,

$$
\begin{aligned}
|\alpha\rangle\langle\beta| &= \bigotimes_{k=1}^{n} |\alpha_k\rangle\langle\beta_k| \\
&= \bigotimes_{k=1}^{n} \int_{\mathbb{C}} \frac{d^2\gamma_k}{\pi} \mathrm{Tr}\left\{D(\gamma_k)|\alpha_k\rangle\langle\beta_k|\right\} D^\dagger(\gamma_k) \\
&= \left\{\prod_{k=1}^{n} \int_{\mathbb{C}} \frac{d^2\gamma_k}{\pi}\right\} \left\{\prod_{k=1}^{n} \mathrm{Tr}\left\{D(\gamma_k)|\alpha_k\rangle\langle\beta_k|\right\}\right\} \left\{\bigotimes_{k=1}^{n} D^\dagger(\gamma_k)\right\}
\end{aligned}
$$

where the brackets $\{\cdot\}$ indicate a product, then,

$$
|\alpha\rangle\langle\beta| = \int_{\mathbb{C}^n} \frac{d^{2n}\gamma}{\pi^n} \mathrm{Tr}\left\{D(\gamma)|\alpha\rangle\langle\beta|\right\} D^\dagger(\gamma)
$$

Since the coherent states form a complete basis, we have the n-mode operator expansion,

$$
\begin{aligned}
\rho &= \int_{\mathbb{C}^n} d^{2n}\alpha \int_{\mathbb{C}^n} d^{2n}\beta \langle\alpha|\rho|\beta\rangle |\alpha\rangle\langle\beta| \\
&= \int_{\mathbb{C}^n} d^{2n}\alpha \int_{\mathbb{C}^n} d^{2n}\beta \langle\alpha|\rho|\beta\rangle \int_{\mathbb{C}^n} \frac{d^{2n}\gamma}{\pi^n} \mathrm{Tr}\left\{D(\gamma)|\alpha\rangle\langle\beta|\right\} D^\dagger(\gamma) \\
&= \int_{\mathbb{C}^n} \frac{d^{2n}\gamma}{\pi^n} \mathrm{Tr}\left\{D(\gamma) \int_{\mathbb{C}^n} d^{2n}\alpha \int_{\mathbb{C}^n} d^{2n}\beta \langle\alpha|\rho|\beta\rangle |\alpha\rangle\langle\beta|\right\} D^\dagger(\gamma) \\
&= \int_{\mathbb{C}^n} \frac{d^{2n}\gamma}{\pi^n} \mathrm{Tr}\left\{D(\gamma)\rho\right\} D^\dagger(\gamma)
\end{aligned}
$$