# 1 Gaussian transformations

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**Prerequisites: Thermal states, Completeness relations** In this exercise we explore unitary transformations that are generated by Hamiltonians that are at most quadratic in the ladder operators.

Such unitaries are called gaussian transformations. We consider unitaries $U_G = e^{-itH_G}$ generated by Hamiltonians of the form,

$$H_G = a_i a_j A_{ij} + a_i^\dagger a_j B_{ij} + a_i^\dagger a_j^\dagger C_{ij} + F_i a_i + L_i a_i^\dagger$$

where a repeated index implies summation (Einstein convention). $a_i$ is the annihilation operator associated with mode $i$. We also introduce the vector notation,

$$a = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^T$$

and the element $i, j$ of matrix $A$ is written as, $(A)_{ij} = A_{ij}$.

## 1.1 Hamiltonian

### 1.1.1 a)

Show that hermiticity of $H_G$ implies

$$\begin{aligned} C_{ij} &= A_{ij}^* \\ B_{ij} &= B_{ji}^* \\ L_i &= F_i^* \end{aligned}$$

i.e. $B$ is an hermitian matrix.

### 1.1.2 b)

Verify the commutator

$$[H_G, a] = -\left(Ba + \left(A + A^T\right)^* a^\dagger + F^*\right).$$

Note that this is a vector relation. $*$ indicates elementwise complex conjugation (not the conjugate transpose).

### 1.1.3 c)

We let $D = A + A^T$. Then using the commutator from exercise b we have,

$$\left[H_G, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] = \begin{pmatrix} -B, & -D^* \\ D & B^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \begin{pmatrix} -F^* \\ F \end{pmatrix}.$$

We now seek to show that we have the transformation,

$$e^{itH_G} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{-itH_G} = e^{tG} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix},$$

where,

$$G = i \begin{pmatrix} -B, & -D^* \\ D & B^* \end{pmatrix} = \begin{pmatrix} -iB, & (iD)^* \\ iD & (-iB)^* \end{pmatrix}.$$

We have the Baker-Campbell-Hausdorff (BCH) lemma, see
https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula

$$e^X Y e^{-X} = \sum_{s=0}^{\infty} \frac{[(X)^s, Y]}{s!}$$

where

$$[(X)^s, Y] \equiv \underbrace{[X, \cdots [X, [X}_{s \text{ times}}, Y]] \cdots], \; \left[(X)^0, Y\right] \equiv Y.$$

**I)** Show that for $s \geq 1$,

$$\left[(H_G)^s, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] = (-iG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + (-iG)^{s-1} \begin{pmatrix} -F^* \\ F \end{pmatrix}$$

**II)** Now show that,

$$e^{itH_G} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{-itH_G} = \sum_{s=0}^{\infty} \frac{1}{s!} (tG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \sum_{s=0}^{\infty} \frac{1}{(s+1)!} (it)^{s+1} (-iG)^s \begin{pmatrix} -F^* \\ F \end{pmatrix}.$$

**III)** Prove the identity,

$$\frac{1}{s+1}(it)^{s+1} = i\int_0^t d\tau (i\tau)^s,$$

for $s \geq 0$.

**IV)** Verify that we have,

$$e^{itH_G}\begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{-itH_G} = e^{tG}\begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \int_0^t d\tau e^{\tau G}\begin{pmatrix} (iF)^* \\ iF \end{pmatrix}.$$

## 1.2 Bogoliubov matrices

Even without calculating the matrix exponential,

$$M_+ = e^{tG},$$

it is possible to make some general observations about the structure of $M_+$, which is called a Bogoliubov matrix. We explore this structure in this exercise. To make our notation more concise, we introduce,

$$C_a = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, C_b = \begin{pmatrix} b \\ b^\dagger \end{pmatrix}$$
$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, D_+ = \int_0^t d\tau e^{\tau G}\begin{pmatrix} (iF)^* \\ iF \end{pmatrix}$$

where $a, b$ are vectors of annihilation operators. We then have the transformation,

$$e^{itH_G}C_a e^{-itH_G} = M_+ C_a + D_+$$

### 1.2.1 a)

We have the ladder operator commutation relations in matrix form,

$$C_a \otimes C_a^T - C_a^T \otimes C_a = \Omega$$

Suppose that,

$$C_b = e^{itH_G}C_a e^{-itH_G}$$

Argue that $C_b$ must likewise satisfy

$$C_b \otimes C_b^T - C_b^T \otimes C_b = \Omega$$

### 1.2.2 b)

Argue that $M_+$ and $D_+$ must have the block matrix form,

$$M_+ = \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix}$$
$$D_+ = \begin{pmatrix} z \\ z^* \end{pmatrix}.$$

Hint: Use the fact that $G^* = XGX$ where,

$$X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, X^2 = I.$$

Examine $XM_+X$ and $XD_+$ using the definition of $M_+$ and $D_+$.

### 1.2.3 c)

In order that $M_+$ preserves the commutation relation, there must exist certain conditions on the matrices $V, J$.

Verify the following relation,

$$M_+ \Omega M_+^T = \Omega$$

and verify that the following conditions on $V, J$ follow from this relation,

$$VJ^T - JV^T = 0$$
$$VV^H - JJ^H = I$$

where $H$ indicates the conjugate transpose.

### 1.2.4 d)

Using the relations from exercise c, we argue that $V$ is always invertible.

**I)** Argue that $VV^H = I + JJ^H$ implies that $VV^H$ is positive definite.

**II)** Note that $VV^H$ is hermitian. We let $v_k, \lambda_k$ be the eigenvectors and eigenvalues of $VV^H$, such that,

$$VV^H v_k = \lambda_k v_k.$$

Argue that $V$ can always be expanded as,

$$V = \sum_{k,m} \alpha_{km} v_k \otimes v_m^H,$$

and verify that the expansion coefficients $\alpha_{km}$ satisfy,

$$\sum_m \alpha_{km} \alpha_{sm}^* = \lambda_k \delta_{ks}.$$

Hint: Use the spectral decomposition of $VV^H$.

**III)** Verify that $V$ has the inverse,

$$V^{-1} = \sum_{k,m} \frac{\alpha_{km}^*}{\lambda_k} v_k^H \otimes v_m,$$

and that this matrix always exists.

### 1.2.5 e)

Using the relations from c), namely the relation,

$$M_+ \Omega M_+^T = \Omega$$

verify that we have the inverse transformation,

$$M_+^{-1} = \Omega M_+^T \Omega^T = \begin{pmatrix} V^H & -J^T \\ -J^H & V^T \end{pmatrix}$$

where $H$ indicates the conjugate transpose.

Argue that $M_+^{-1}$ also must preserve the commutator, and state further conditions on $V, J$ from this fact. I.e. argue that,

$$\begin{aligned} -V^H J + J^T V^* &= 0 \\ V^H V - J^T J^* &= I \end{aligned}$$

### 1.2.6 f)

Show that the determinant of $M_+$, i.e.$|M_+|$, is always $+1$.

Hint: Use the formulae

$$\begin{aligned} M_+ &= \exp[tG] \\ \left|e^{tG}\right| &= e^{\mathrm{Tr}\{tG\}} \end{aligned}$$

#### 1.2.7 g)

Find the displacement $D_-$ and matrix $M_-$ associated with the inverse transformation, such that,

$$e^{-itH_G}C_a e^{itH_G} = M_- C_a + D_-.$$

Show that

$$\begin{aligned} M_- &= M_+^{-1} \\ D_- &= -M_+^{-1}D_+ \end{aligned}$$

### 1.3 Transformation of the state

Using Glauber's formula, and the fact that gaussian transformations are linear transformations on the ladder operators, we derive general statements about how a quantum state $\rho$ transforms under a gaussian transformation. We write a gaussian transformation as,

$$U_G = e^{-itH_G}.$$

We introduce some additional notation, we write,

$$C_\alpha = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}$$

Then the $n$-mode displacement operator $D(\alpha)$ can be written,

$$D(\alpha) = D\left(C_\alpha\right) = \exp\left[C_\alpha^T \Omega C_a\right]$$

and Glauber's formula can be written,

$$\begin{aligned} \rho = \frac{1}{\pi^n}\int_{\mathbb{C}^n} d^{2n}\alpha \chi_\rho\left(C_\alpha\right)\exp\left[-C_\alpha^T\Omega C_a\right] \\ \chi_\rho(C_\alpha) = \mathrm{Tr}\left\{\rho\exp\left[C_\alpha^T\Omega C_a\right]\right\}. \end{aligned}$$

#### 1.3.1 a)

Suppose we have a characteristic function $\chi_i(C_\alpha)$ such that,

$$\chi_i(C_\alpha) = \mathrm{Tr}\left\{\rho_i\exp\left[C_\alpha^T\Omega C_a\right]\right\}$$

Show that the transformed state $\rho_f$,

$$\rho_f = e^{-itH_G}\rho_i e^{itH_G},$$

has the characteristic functions,

$$\begin{aligned}\chi_f(C_\alpha) &= \mathrm{Tr}\left\{\rho_f \exp\left[C_\alpha^T \Omega C_a\right]\right\} \\ &= \chi_i\left(M_+^{-1} C_\alpha\right)\exp\left[C_\alpha^T \Omega D_+\right].\end{aligned}$$

It follows that the action of a gaussian transformation is to deform phase space by application of the invertible map $M_+^{-1}$.

Hint: The followings relation will be helpful,

$$\Omega M_+ = M^{-T}\Omega$$

### 1.3.2 b)

We define the displacement,

$$\bar{C}_i = \mathrm{Tr}\left\{\rho_i C_a\right\},$$

and covariance matrix,

$$\begin{aligned}\Sigma_i &= \frac{1}{2}\mathrm{Tr}\left\{\rho_i\left(C_a \otimes C_a^T + C_a^T \otimes C_a\right)\right\} - \mathrm{Tr}\left\{\rho_i C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_i C_a\right\} \\ &= \mathrm{Tr}\left\{\rho_i C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \mathrm{Tr}\left\{\rho_i C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_i C_a\right\}\end{aligned}$$

Given that,

$$\rho_f = e^{-itH_G}\rho_i e^{itH_G}$$

Show that,

$$\bar{C}_f = \mathrm{Tr}\left\{\rho_f C_a\right\} = M_+\bar{C}_i + D_+$$

$$\Sigma_f = \mathrm{Tr}\left\{\rho_f C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \mathrm{Tr}\left\{\rho_f C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_f C_a\right\} = M_+\Sigma_i M_+^T$$

### 1.3.3 c)

We have the characteristic function of an $n$-mode thermal state,

$$\chi_{\mathrm{Th}}(C_\alpha) = \exp\left[\frac{1}{2} C_\alpha^T \Omega^T \Sigma_{\mathrm{Th}} \Omega C_\alpha - \bar{C}^T \Omega C_\alpha\right]$$

where $\Sigma_{\mathrm{Th}}$ is the covariance matrix of $\rho_{\mathrm{Th}}$ and $\bar{C}$ is the displacement of $\rho_{\mathrm{Th}}$. Show that applying a gaussian transformation to a thermal state,

$$\rho_f = e^{-itH_G} \rho_{\mathrm{Th}} e^{itH_G},$$

results in the characteristic function,

$$\chi_f(C_\alpha) = \exp\left[\frac{1}{2} C_\alpha^T \Omega^T M_+ \Sigma_{\mathrm{Th}} M_+^T \Omega C_\alpha - \left(M_+ \bar{C} + D_+\right)^T \Omega C_\alpha\right]$$

Hint:
Use the relations,

$$M_+^{-1} = \Omega M_+^T \Omega^T$$
$$\Omega\Omega = -I, \ \Omega^T = -\Omega$$

### 1.3.4 d)

The class of states that can be obtained by applying a gaussian transformation to a thermal state are called gaussian states.

Argue based on the result from c), that the characteristic function of an arbitrary gaussian state can be written as,

$$\chi_G(C_\alpha) = \exp\left[\frac{1}{2} C_\alpha^T \Omega^T \Sigma \Omega C_\alpha - \bar{C}^T \Omega C_\alpha\right]$$

where $\bar{C}, \Sigma$ are the displacement and covariance matrix of the gaussian state.