# 1 Gaussian transformations

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## 1.1 Hamiltonian

### 1.1.1 a)

$$H_G^\dagger = a_i^\dagger a_j^\dagger A_{ij}^* + a_j^\dagger a_i B_{ij}^* + a_i a_j C_{ij}^* + F_i^* a_i^\dagger + L_i^* a_i = H$$

Giving the conditions,

$$\begin{aligned} C_{ij} &= A_{ij}^* \\ B_{ij} &= B_{ji}^* \\ L_i &= F_i^* \end{aligned}$$

and so we can write $H_G$ as,

$$H_G = a_i a_j A_{ij} + a_i^\dagger a_j^\dagger A_{ij}^* + a_i^\dagger a_j B_{ij} + F_i a_i + F_i^* a_i^\dagger$$

### 1.1.2 b)

$$\begin{aligned} [H_G, a_k] &= \left[a_i a_j A_{ij} + a_i^\dagger a_j^\dagger A_{ij}^* + a_i^\dagger a_j B_{ij} + F_i a_i + F_i^* a_i^\dagger, a_k\right] \\ &= \left[a_i^\dagger a_j^\dagger A_{ij}^*, a_k\right] + \left[a_i^\dagger a_j B_{ij}, a_k\right] + F_i^* \left[a_i^\dagger, a_k\right] \\ &= A_{ij}^* \left[a_i^\dagger a_j^\dagger, a_k\right] (\delta_{i=k} + \delta_{i\neq k}) (\delta_{j=k} + \delta_{j\neq k}) - B_{kj} a_j - F_k^* \\ &= A_{kk}^* \left[a_k^\dagger a_k^\dagger, a_k\right] + A_{ik}^* a_i^\dagger \left[a_k^\dagger, a_k\right] \delta_{i\neq k} + A_{kj}^* a_j^\dagger \left[a_k^\dagger, a_k\right] \delta_{j\neq k} - B_{kj} a_j - F_k^* \\ &= -2A_{kk}^* a_k^\dagger - A_{ik}^* a_i^\dagger \delta_{i\neq k} - A_{kj}^* a_j^\dagger \delta_{j\neq k} - B_{kj} a_j - F_k^* \\ &= -\left(A_{jk}^* + A_{kj}^*\right) a_j^\dagger - B_{kj} a_j - F_k^* \\ &= -\left(A_{kj}^{T*} + A_{kj}^*\right) a_j^\dagger - B_{kj} a_j - F_k^* \end{aligned}$$

By taking the adjoint,

$$\left[H_G, a_k^\dagger\right] = \left(A_{kj}^T + A_{kj}\right) a_j + B_{kj}^* a_j^\dagger + F_k$$

and so in vector form we got,

$$\begin{aligned} [H_G, a] &= -Ba - \left(A^T + A\right)^* a^\dagger - F^* \\ \left[H_G, a^\dagger\right] &= \left(A^T + A\right) a + B^* a^\dagger + F \end{aligned}$$

### 1.1.3 c)

**I)** We do a few cases and extrapolate the pattern. Note that $[H_G, f] = 0$ for a constant vector $f$.

$$\left[(H_G)^1, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] = (-iG) \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \begin{pmatrix} -F^* \\ F \end{pmatrix}$$

$$\begin{aligned} \left[(H_G)^2, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] &= (-iG) \left[H_G, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] \\ &= (-iG)^2 \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + (-iG) \begin{pmatrix} -F^* \\ F \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \left[(H_G)^3, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] &= (-iG)^2 \left[H_G, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] \\ &= (-iG)^3 \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + (-iG)^2 \begin{pmatrix} -F^* \\ F \end{pmatrix}, \end{aligned}$$

so by extrapolation we find for $s \geq 1$,

$$\left[(H_G)^s, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] = (-iG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + (-iG)^{s-1} \begin{pmatrix} -F^* \\ F \end{pmatrix}$$

**II)** From the BCH lemma,

$$\begin{aligned} & e^{itH_G} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{-itH_G} \\ &= \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \sum_{s=1}^{\infty} \frac{1}{s!} (it)^s \left[(H_G)^s, \begin{pmatrix} a \\ a^\dagger \end{pmatrix}\right] \\ &= \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \sum_{s=1}^{\infty} \frac{1}{s!} (it)^s (-iG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \\ &\quad + \sum_{s=1}^{\infty} \frac{1}{s!} (it)^s (-iG)^{s-1} \begin{pmatrix} -F^* \\ F \end{pmatrix} \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} (tG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \\ &\quad + \sum_{s=0}^{\infty} \frac{1}{(s+1)!} (it)^{s+1} (-iG)^s \begin{pmatrix} -F^* \\ F \end{pmatrix} \end{aligned}$$

**III)** For $s \geq 0$,

$$\int_0^t d\tau (i\tau)^s = \left[(-i)\frac{1}{s+1}(i\tau)^{s+1}\right]_0^t$$
$$= (-i)\frac{1}{s+1}(it)^{s+1}$$

**IV)** We rewrite the sums,

$$\sum_{s=0}^{\infty} \frac{1}{s!} (tG)^s \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = e^{tG} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

and

$$\sum_{s=0}^{\infty} \frac{1}{(s+1)!} (it)^{s+1} (-iG)^s \begin{pmatrix} -F^* \\ F \end{pmatrix}$$
$$= \sum_{s=0}^{\infty} \frac{1}{s!} i \int_0^t d\tau (i\tau)^s (-iG)^s \begin{pmatrix} -F^* \\ F \end{pmatrix}$$
$$= i \int_0^t d\tau \sum_{s=0}^{\infty} \frac{1}{s!} (\tau G)^s \begin{pmatrix} -F^* \\ F \end{pmatrix}$$
$$= i \int_0^t d\tau e^{\tau G} \begin{pmatrix} -F^* \\ F \end{pmatrix}$$

Combining these results we obtain,

$$e^{itH_G} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{-itH_G} =$$
$$e^{tG} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + i \int_0^t d\tau e^{\tau G} \begin{pmatrix} -F^* \\ F \end{pmatrix}$$

as was claimed.

## 1.2 Bogoliubov matrices

### 1.2.1 a)

We have,

$$C_b \otimes C_b^T - C_b^T \otimes C_b$$
$$= e^{itH_G} C_a e^{-itH_G} \otimes e^{itH_G} C_a^T e^{-itH_G} - e^{itH_G} C_a^T e^{-itH_G} \otimes e^{itH_G} C_a e^{-itH_G}$$
$$= e^{itH_G} \left(C_a \otimes C_a^T - C_a^T \otimes C_a\right) e^{-itH_G}$$
$$= e^{itH_G} \Omega e^{-itH_G} = \Omega$$

where multiplication with $e^{itH_G}$ is elementwise over the array.

### 1.2.2 b)

We write $M_+$ and $D_+$ in block form,

$$M_+ = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, D_+ = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

The symmetry of $M_+$ follows from the symmetri of $G$, we define,

$$X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Then,

$$G = (XGX)^*.$$

Note that $X^2 = I$. It then follows that,

$$(XM_+X)^* = \left(Xe^{tG}X\right)^* = e^{t(XGX)^*} = e^{tG} = M_+.$$

As a result,

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M_{22}^* & M_{21}^* \\ M_{12}^* & M_{11}^* \end{pmatrix},$$

and so we can write $M_+$ in the form,

$$M_+ = \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix}.$$

We have,

$$D_+ = \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix},$$

and so,

$$\begin{aligned}
(XD_+)^* &= \int_0^t d\tau \left(Xe^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix}\right)^* \\
&= \int_0^t d\tau \left(Xe^{\tau G}X\right)^* X \begin{pmatrix} iF \\ (iF)^* \end{pmatrix} \\
&= \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix} = D_+.
\end{aligned}$$

It follows that $D_+$ has the symmetry,

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_2^* \\ d_1^* \end{pmatrix},$$

and so we can write $D_+$ in the form,

$$D_+ = \begin{pmatrix} z \\ z^* \end{pmatrix}.$$

### 1.2.3 c)

Let

$$C_b = M_+ C_a + D_+.$$

Then we require that the transformation preserves the commutation relations,

$$C_b \otimes C_b^T - C_b^T \otimes C_b = \Omega.$$

We can rewrite the commutator as,

$$\begin{aligned}
(M_+ C_a + D_+) \otimes \left(C_a^T M_+^T + D_+^T\right) - \left(C_a^T M_+^T + D_+^T\right) \otimes (M_+ C_a + D_+) \\
= M_+ C_a \otimes C_a^T M_+^T - C_a^T M_+^T \otimes M_+ C_a \\
+ M_+ C_a \otimes D_+^T - D_+^T \otimes M_+ C_a \\
+ D_+ \otimes C_a^T M_+^T - C_a^T M_+^T \otimes D_+ \\
+ D_+ \otimes D_+^T - D_+^T \otimes D_+
\end{aligned}$$

For vectors $A, B$ with commuting vector elements we have $A \otimes B^T - B^T \otimes A = 0$, and so we get,

$$\begin{aligned}
C_b \otimes C_b^T - C_b^T \otimes C_b \\
= M_+ C_a \otimes C_a^T M_+^T - C_a^T M_+^T \otimes M_+ C_a \\
= M_+ C_a \otimes C_a^T M_+^T - \left(M_+ C_a \otimes C_a^T M_+^T\right)^T \\
= M_+ C_a \otimes C_a^T M_+^T - M_+ \left(C_a \otimes C_a^T\right)^T M_+^T \\
= M_+ \left(C_a \otimes C_a^T - C_a^T \otimes C_a\right) M_+^T \\
= M_+ \Omega M_+^T = \Omega
\end{aligned}$$

Giving the condition,

$$M_+ \Omega M_+^T = \Omega.$$

Performing the matrix product in block form, we get the conditions on $V, J$,

$$\begin{aligned} &\begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} V^T & J^H \\ J^T & V^H \end{pmatrix} \\ &= \begin{pmatrix} VJ^T - JV^T, & VV^H - JJ^H \\ J^*J^T - V^*V^T, & J^*V^H - V^*J^H \end{pmatrix}. \end{aligned}$$

Giving the conditions,

$$\begin{aligned} VJ^T - JV^T &= 0 \\ VV^H - JJ^H &= I, \end{aligned}$$

where $H$ indicates the conjugate transpose.

### 1.2.4 d)

**I)** We have,

$$VV^H = I + JJ^H$$

$JJ^H$ is clearly positive semi-definite, i.e.

$$u^H JJ^H u = \left(J^H u\right)^H \left(J^H u\right) \geq 0$$

for all vectors $u$. It follows that $VV^H$ is then necessarily positive definite,

$$u^H VV^H u = u^H u + u^H JJ^H u > 0.$$

**II)** Since $VV^H$ is hermitian, it has a spectral decomposition,

$$VV^H = \sum_k \lambda_k v_k \otimes v_k^H$$

where $v_k$ is a normalized vector and $\lambda_k$ a real number, that are solutions of the hermitian eigenproblem,

$$VV^H v_k = \lambda_k v_k.$$

Since $VV^H$ is positive definite then $\lambda_k$ are all positive (non-zero). Note that the eigenvectors of a $n$-dimensional hermitian matrix form an orthonormal basis over the $n$-dimensional space,

$$\sum_k v_k \otimes v_k^H = I_n.$$

Then we can always expand $V$ in the basis $v_k$,

$$V = \sum_{k,m} \alpha_{km} v_k \otimes v_m^H.$$

This leads to a condition on $\alpha_{km}$,

$$\begin{aligned}
VV^H &= \left( \sum_{k,m} \alpha_{km} v_k \otimes v_m^H \right) \left( \sum_{s,l} \alpha_{sl} v_s \otimes v_l^H \right)^H \\
&= \sum_{k,m} \sum_{s,l} \alpha_{km} \alpha_{sl}^* v_k \otimes v_m^H . v_s^H \otimes v_l \\
&= \sum_{k,m} \sum_{s,l} \alpha_{km} \alpha_{sl}^* v_k \otimes v_s^H \left( v_m^H . v_l \right) \\
&= \sum_{k,s} \left( \sum_m \alpha_{km} \alpha_{sm}^* \right) v_k \otimes v_s^H \\
&= \sum_k \lambda_k v_k \otimes v_k^H .
\end{aligned}$$

We find that the expansion coefficients of $V$ must satisfy,

$$\sum_m \alpha_{km} \alpha_{sm}^* = \lambda_k \delta_{ks}.$$

**III)** Division by $\lambda_k$ is always possible because all $\lambda_k$ are positive (non-zero). Then we can then construct the matrix,

$$V^{-1} = \sum_{k,m} \frac{\alpha_{km}^*}{\lambda_k} v_k^H \otimes v_m,$$

we verify that this is indeed the inverse of $V$,

$$
\begin{aligned}
&VV^{-1}\\
&= \left(\sum_{k,m} \alpha_{km} v_k \otimes v_m^H\right)\left(\sum_{s,l} \frac{\alpha_{sl}^*}{\lambda_s} v_s^H \otimes v_l\right)\\
&= \sum_{k,m}\sum_{s,l} \alpha_{km} \frac{\alpha_{sl}^*}{\lambda_s} v_k \otimes v_m^H . v_s^H \otimes v_l\\
&= \sum_{k,m}\sum_{s,l} \alpha_{km} \frac{\alpha_{sl}^*}{\lambda_s} v_k \otimes v_s^H \left(v_m^H . v_l\right)\\
&= \sum_{k,s} \frac{1}{\lambda_s}\left(\sum_m \alpha_{km}\alpha_{sm}^*\right) v_k \otimes v_s^H\\
&= \sum_{k,s} \frac{1}{\lambda_s} \lambda_k \delta_{ks} v_k \otimes v_s^H\\
&= \sum_k v_k \otimes v_k^H = I_n
\end{aligned}
$$

It follows that $V$ is always invertible.

### 1.2.5 e)

We perform the matrix product,

$$
M_+ M_+^{-1} = \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix}\begin{pmatrix} V^H & -J^T \\ -J^H & V^T \end{pmatrix}
$$
$$
= \begin{pmatrix} VV^H - JJ^H, & -VJ^T + JV^T \\ J^*V^H - V^*J^H, & -J^*J^T + V^*V^T \end{pmatrix} = \begin{pmatrix} I, & 0 \\ 0, & I \end{pmatrix},
$$

We may argue that $M_+^{-1}$ preserves the commutator by rewriting the condition,

$$
M_+ \Omega M_+^T = \Omega,
$$

via multiplication with $M_+^{-1}$ from the left and $M_+^{-T}$ from the right,

$$
\Omega = M_+^{-1} \Omega M_+^{-T}.
$$

Since the inverse transformation also preserve the commutator, we have the same conditions as before but with $V \to V^H$ and $J \to -J^T$,

$$
\begin{aligned}
-V^H J + J^T V^* &= 0\\
V^H V - J^T J^* &= I
\end{aligned}
$$

### 1.2.6 f)

We have,

$$\begin{aligned} |M_+| &= |\exp[tG]| \\ &= \exp[t\mathrm{Tr}\{G\}] \\ &= \exp[it(\mathrm{Tr}\{B^*\} - \mathrm{Tr}\{B\})]. \end{aligned}$$

Since $B$ is hermitian it must have real values on the diagonal, and so,

$$\begin{aligned} \mathrm{Tr}\{B^*\} - \mathrm{Tr}\{B\} &= 0 \\ |M_+| &= 1 \end{aligned}$$

### 1.2.7 g)

We have,

$$e^{itH_G} C_a e^{-itH_G} = e^{tG} C_a + \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix}$$

Multiplying with $e^{-itH_G}$ from the left and $e^{itH_G}$ from the right,

$$\begin{aligned} C_a &= e^{-itH_G} e^{tG} C_a e^{itH_G} + e^{-itH_G} \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix} e^{itH_G} \\ &= e^{tG} e^{-itH_G} C_a e^{itH_G} + \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix} \end{aligned}$$

Rearranging,

$$e^{-itH_G} C_a e^{itH_G} = e^{-tG} C_a - e^{-tG} \int_0^t d\tau e^{\tau G} \begin{pmatrix} (iF)^* \\ iF \end{pmatrix}$$

We identify

$$\begin{aligned} M_- &= e^{-tG} = M_+^{-1} \\ D_- &= -M_+^{-1} D_+ \end{aligned}$$

## 1.3 Transformation of the state

### 1.3.1 a)

We have the characteristic function,

$$
\begin{aligned}
\chi_f(C_\alpha) &= \mathrm{Tr}\left\{\rho_f \exp\left[C_\alpha^T \Omega C_a\right]\right\} \\
&= \mathrm{Tr}\left\{\rho_i e^{itH_G} \exp\left[C_\alpha^T \Omega C_a\right] e^{-itH_G}\right\} \\
&= \mathrm{Tr}\left\{\rho_i \exp\left[C_\alpha^T \Omega \left(M_+ C_a + D_+\right)\right]\right\} \\
&= \mathrm{Tr}\left\{\rho_i \exp\left[C_\alpha^T \Omega M_+ C_a\right]\right\} \exp\left[C_\alpha^T \Omega D_+\right] \\
&= \mathrm{Tr}\left\{\rho_i \exp\left[\left(M^{-1} C_\alpha\right)^T \Omega C_a\right]\right\} \exp\left[C_\alpha^T \Omega D_+\right] \\
&= \chi_i(M^{-1} C_\alpha) \exp\left[C_\alpha^T \Omega D_+\right]
\end{aligned}
$$

### 1.3.2 b)

For the mean we have,

$$
\begin{aligned}
\bar{C}_f &= \mathrm{Tr}\left\{\rho_f C_a\right\} \\
&= \mathrm{Tr}\left\{\rho_i e^{itH_G} C_a e^{-itH_G}\right\} \\
&= M_+ \mathrm{Tr}\left\{\rho_i C_a\right\} + \mathrm{Tr}\left\{\rho_i\right\} D_+ \\
&= M_+ \bar{C}_i + D_+
\end{aligned}
$$

For the covariance matrix we have,

$$
\begin{aligned}
\Sigma_f &= \mathrm{Tr}\left\{\rho_f C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \mathrm{Tr}\left\{\rho_f C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho_f C_a\right\} \\
&= \mathrm{Tr}\left\{\rho_i \left(C_a^T M_+^T + D_+^T\right) \otimes \left(M_+ C_a + D_+\right)\right\} + \frac{1}{2}\Omega \\
&\quad - \left(\bar{C}_i^T M_+^T + D_+^T\right) \otimes \left(M_+ \bar{C}_i + D_+\right) \\
&= \mathrm{Tr}\left\{\rho_i \left(C_a^T M_+^T \otimes M_+ C_a\right)\right\} - \bar{C}_i^T M_+^T \otimes M_+ \bar{C}_i + \frac{1}{2}\Omega \\
&= \mathrm{Tr}\left\{\rho_i \left(M_+ C_a \otimes C_a^T M_+^T\right)^T\right\} - \left(M_+ \bar{C}_i \otimes \bar{C}_i^T M_+^T\right)^T + \frac{1}{2}\Omega \\
&= M_+ \left[\mathrm{Tr}\left\{\rho_i C_a^T \otimes C_a\right\} + \frac{1}{2}\Omega - \bar{C}_i^T \otimes \bar{C}_i\right] M_+^T \\
&= M_+ \Sigma_i M_+^T
\end{aligned}
$$

### 1.3.3 c)

Using the result from exercise a),

$$
\begin{aligned}
\chi_f(C_\alpha) &= \chi_{\mathrm{Th}}\left(M_+^{-1}C_\alpha\right)\exp\left[C_\alpha^T\Omega D_+\right] \\
= \exp\left[\frac{1}{2}C_\alpha^T M_+^{-T}\Omega^T\Sigma_{\mathrm{Th}}\Omega M_+^{-1}C_\alpha\right] &\exp\left[-\bar{C}^T\Omega M_+^{-1}C_\alpha - D_+^T\Omega C_\alpha\right] \\
= \exp\left[\frac{1}{2}C_\alpha^T\Omega^T M_+\Sigma_{\mathrm{Th}}M_+^T\Omega C_\alpha\right] &\exp\left[-\left(M_+\bar{C}+D_+\right)^T\Omega C_\alpha\right]
\end{aligned}
$$

### 1.3.4 d)

From c) we have that the density matrix of a general gaussian state $\rho_f$, obtained as,

$$\rho_f = e^{-itH_G}\rho_{\mathrm{Th}}e^{itH_G},$$

can be written,

$$\chi_f(K_\alpha) = \exp\left[\frac{1}{2}C_\alpha^T\Omega^T M_+\Sigma_{\mathrm{Th}}M_+^T\Omega C_\alpha - \left(M_+\bar{C}+D_+\right)^T\Omega C_\alpha\right].$$

Since $\Sigma_{\mathrm{Th}}$ is the covariance matrix of $\rho_{\mathrm{Th}}$ and $\bar{C}$ is the displacement of $\rho_{\mathrm{Th}}$, then by comparison with exercise b), we recognize,

$$
\begin{aligned}
\bar{C}_f &= M_+\bar{C} + D_+ \\
\Sigma_f &= M_+\Sigma_{\mathrm{Th}}M_+^T
\end{aligned}
$$

as the displacement and covariance of $\rho_f$. The characteristic function of the arbitrary gaussian state $\rho_f$ can then be written as,

$$\chi_f(C_\alpha) = \exp\left[\frac{1}{2}C_\alpha^T\Omega^T\Sigma_f\Omega C_\alpha - \bar{C}_f^T\Omega C_\alpha\right].$$