1 Determining the q-quadrature wavefunction

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We let $a, a^\dagger$ be vectors of n ladder operators associated to n modes.

$$a = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^T.$$

q and x are vectors of quadrature operators and numbers respectively,

$$q = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^T$$
$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T,$$

where q_i is the q-quadrature operator for mode i. The relation between quadratures and ladder operators are,

$$q = \frac{k_c}{2}\left(a^\dagger + a\right)$$
$$p = \frac{k_c}{2} i \left(a^\dagger - a\right),$$

where $k_c$is a constant chosen according to convention.

Suppose we have a gaussian state, obtained by a gaussian unitary acting on vacuum,

$$|G\rangle = e^{-itH_G}|0\rangle.$$

Then we have the vector relation,

$$e^{-itH_G} a e^{itH_G}|G\rangle = 0,$$

where the right hand side is the zero vector.

We will now show that using the above condition, we can determine the q-quadrature wavefunction to be of the form,

$$\langle x|G\rangle = \psi_G(x) = \mathcal{N} \exp\left[-\frac{1}{k_c^2} x^T \Gamma_x x - \frac{2}{k_c} x^T u_z\right]$$

where $|x\rangle$ is a q quadrature eigenvector,

$$q|x\rangle = x|x\rangle,$$

note that this equation is a vector relation.

1.1 Bogoliubov transformations

Using relations from the exercise on gaussian transformations we have,

$$
\begin{aligned}
e^{-itH_G} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} e^{itH_G} &= M_- \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + D_- \\
&= \begin{pmatrix} V^H & -J^T \\ -J^H & V^T \end{pmatrix} \begin{pmatrix} a - z \\ a^\dagger - z^* \end{pmatrix},
\end{aligned}
$$

where $D_+^T = \begin{pmatrix} z^T & z^{*T} \end{pmatrix}$. There is n modes in total, giving V and J the dimension $n \times n$.

We find that we can rewrite $\langle x| e^{-itH_G} a e^{itH_G} |G\rangle = 0$ as,

$$
\begin{aligned}
&\langle x| e^{-itH_G} a e^{itH_G} |G\rangle \\
&= V^H \left(\langle x|a|G\rangle - z\langle x|G\rangle\right) - J^T \left(\langle x|a^\dagger|G\rangle - z^*\langle x|G\rangle\right) = 0.
\end{aligned}
$$

From the exercise set on representation theory we have that,

$$
\begin{aligned}
\langle x|a|G\rangle &= k_\mathrm{c}^{-1} \left(x + \frac{k_\mathrm{c}^2}{2} \partial_x \right) \psi_G(x) \\
\langle x|a^\dagger|G\rangle &= k_\mathrm{c}^{-1} \left(x - \frac{k_\mathrm{c}^2}{2} \partial_x \right) \psi_G(x),
\end{aligned}
$$

where ∂_x is the gradient,

$$
\partial_x = \begin{pmatrix} \partial_{x_1} & \partial_{x_2} & \cdots & \partial_{x_n} \end{pmatrix}^T.
$$

We find that we can rewrite as,

$$
\begin{aligned}
&\langle x| e^{-itH_G} a e^{itH_G} |G\rangle \\
&= \left(V^H - J^T\right) x\psi_G(x) + \frac{k_\mathrm{c}^2}{2} \left(V^H + J^T\right) \partial_x \psi_G(x) + k_\mathrm{c} \left(J^T z^* - V^H z\right) \psi_G(x) = 0.
\end{aligned}
$$

1.2 Invertibility of $V^H + J^T$

We will now verify that the matrix $V^H + J^T$ is invertible.

We define,

$$
\Theta = \left(V^H + J^T\right)\left(V - J^*\right).
$$

Using exercise e) from the exercise on gaussian transformations, we have,

$$V^H V - J^T J^* = I$$

and so we find,

$$\begin{aligned}\Theta &= V^H V - V^H J^* + J^T V - J^T J^* \\ &= I + J^T V - V^H J^*.\end{aligned}$$

We notice that,

$$K = i\left(J^T V - V^H J^*\right),$$

is hermitian,

$$K^H = -i\left(V^H J^* - J^T V\right) = K.$$

Then we can rewrite Θ as,

$$\Theta = I - iK.$$

We examine the product,

$$\begin{aligned}\Theta^H \Theta &= (I + iK)(I - iK) \\ &= I + KK.\end{aligned}$$

We now argue that $\Theta^H \Theta$ is positive definite.

Since K is hermitian and finite dimensional, we can associate a complete orthonormal basis to K. By the spectral theorem, we can write K as,

$$K = \sum_i \lambda_i v_i \otimes v_i^H,$$

where v_i and λ_i are eigenvectors and eigenvalues,

$$K v_i = \lambda_i v_i,$$

and λ_i is real.

Then for an arbitrary vector u, we have,

$$
\begin{aligned}
u^H \Theta^H \Theta u &= u^H \left(I + KK\right) u \\
&= u^H \left(I + \sum_i \lambda_i^2 v_i \otimes v_i^H\right) u \\
&= u^H u + \sum_i \lambda_i^2 \left|v_i^H u\right|^2 .
\end{aligned}
$$

Since λ_i^2 is necessarily non-negative, then the above inner product is always positive, for non-zero vectors u. It follows that $\Theta^H \Theta$ is positive definite and hermitian. Then we know that the determinant is non-zero,

$$
\left|\Theta^H \Theta\right| = \left|\Theta^*\right| \left|\Theta\right| \neq 0,
$$

but this implies $|\Theta| \neq 0$, or upon inserting the definition of Θ,

$$
\left|V^H + J^T\right| \left|V - J^*\right| \neq 0,
$$

and this implies that,

$$
\left|V^H + J^T\right| \neq 0,
$$

and so $V^H + J^T$ is invertible.

1.3 Showing that Γ_x is symmetric

Examining the differential equation defining ψ_G,

$$
\left(V^H - J^T\right) x \psi_G(x) + \frac{k_c^2}{2} \left(V^H + J^T\right) \partial_x \psi_G(x) + k_c \left(J^T z^* - V^H z\right) \psi_G(x) = 0.
$$

We multiply by $\left(V^H + J^T\right)^{-1}$ and rearrange,

$$
\partial_x \psi_G(x) = -\frac{2}{k_c^2} \left(V^H + J^T\right)^{-1} \left(V^H - J^T\right) x \psi_G(x) - \frac{2}{k_c} \left(V^H + J^T\right)^{-1} \left(J^T z^* - V^H z\right) \psi_G(x).
$$

The solution to this equation will be a multivariate complex gaussian provided that $\left(V^H + J^T\right)^{-1} \left(V^H - J^T\right)$ is symmetric.

We verify that,

$$
\Gamma_x = \left(V^H + J^T\right)^{-1} \left(V^H - J^T\right),
$$

is indeed symmetric. We know from d) in the exercise set on gaussian transformations, that V is invertible,

$$\begin{aligned}\Gamma_x &= \left(V^H + J^T\right)^{-1}\left(V^H - J^T\right)\\ &= \left(V^H\left(I + V^{-H}J^T\right)\right)^{-1}\left(V^H\left(I - V^{-H}J^T\right)\right)\\ &= \left(I + V^{-H}J^T\right)^{-1}\left(I - V^{-H}J^T\right).\end{aligned}$$

We note the commutator,

$$\left[\left(I + V^{-H}J^T\right), \left(I - V^{-H}J^T\right)\right] = 0,$$

and so it follows that,

$$\left[\left(I + V^{-H}J^T\right)^{-1}, \left(I - V^{-H}J^T\right)\right] = 0.$$

We also note that $V^{-H}J^T$ is symmetric, as follows from the result of exercise e) in the exercise set on gaussian transformations,

$$J^T V^* = V^H J$$

or upon rearranging,

$$V^{-H}J^T = JV^{-*}.$$

Taking the transpose,

$$\left(V^{-H}J^T\right)^T = JV^{-*} = V^{-H}J^T.$$

As a consequence, we find that the matrix Γ_x is symmetric,

$$\begin{aligned}\Gamma_x^T &= \left[\left(I + V^{-H}J^T\right)^{-1}\left(I - V^{-H}J^T\right)\right]^T\\ &= \left(I - V^{-H}J^T\right)^T\left(I + V^{-H}J^T\right)^{-T}\\ &= \left(I - V^{-H}J^T\right)\left(I + V^{-H}J^T\right)^{-1}\\ &= \left(I + V^{-H}J^T\right)^{-1}\left(I - V^{-H}J^T\right) = \Gamma_x.\end{aligned}$$

1.4 Solving the differential equation

The wavefunction satisfies the differential equation,

$$\partial_x \psi_G(x) = -\frac{2}{k_c^2}\Gamma_x x \psi_G(x) - \frac{2}{k_c} u_z \psi_G(x).$$

where we've introduce the notation,

$$u_z = \left(V^H + J^T\right)^{-1}\left(J^T z^* - V^H z\right).$$

The above differential equation has the solution,

$$\psi_G(x) = \mathcal{N} \exp\left[-\frac{1}{k_c^2} x^T \Gamma_x x - \frac{2}{k_c} x^T u_z\right].$$

where $\mathcal{N}$is a normalization constant. We verify this by calculating the gradient of ψ_G. We write the exponent as a sum using the Einstein summation convention,

$$\psi_G(x) = \mathcal{N} \exp\left[-\frac{1}{k_c^2} x_i \left(\Gamma_x\right)_{ij} x_j - \frac{2}{k_c} x_i \left(u_z\right)_i\right],$$

and we let $\partial_i = \frac{\partial}{\partial x_i}$, then,

$$\begin{aligned}
\partial_x \psi_G(x) &= \begin{pmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \partial_n \end{pmatrix} \mathcal{N} \exp\left[-\frac{1}{k_c^2} x_i \left(\Gamma_x\right)_{ij} x_j - \frac{2}{k_c} x_i \left(u_z\right)_i\right] \\
&= \begin{pmatrix} -\frac{1}{k_c^2}\left(\Gamma_x\right)_{1j} x_j - \frac{1}{k_c^2} x_i \left(\Gamma_x\right)_{i1} - \frac{2}{k_c}\left(u_z\right)_1 \\ -\frac{1}{k_c^2}\left(\Gamma_x\right)_{2j} x_j - \frac{1}{k_c^2} x_i \left(\Gamma_x\right)_{i2} - \frac{2}{k_c}\left(u_z\right)_2 \\ \cdots \\ -\frac{1}{k_c^2}\left(\Gamma_x\right)_{nj} x_j - \frac{1}{k_c^2} x_i \left(\Gamma_x\right)_{in} - \frac{2}{k_c}\left(u_z\right)_n \end{pmatrix} \psi_G(x) \\
&= \left(-\frac{1}{k_c^2}\Gamma_x x - \frac{1}{k_c^2}\Gamma_x^T x - \frac{2}{k_c} u_z\right)\psi_G(x) \\
&= \left(-\frac{2}{k_c^2}\Gamma_x x - \frac{2}{k_c} u_z\right)\psi_G(x),
\end{aligned}$$

since $\Gamma_x^T = \Gamma_x$.
So we see that,

$$\psi_G(x) = \mathcal{N} \exp\left[-\frac{1}{k_c^2} x^T \Gamma_x x - \frac{2}{k_c} x^T u_z\right],$$

does indeed solve the differential equation,

$$\partial_x \psi_G(x) = -\frac{2}{k_\mathrm{c}^2}\Gamma_x x \psi_G(x) - \frac{2}{k_\mathrm{c}} u_z \psi_G(x).$$