1 Quadratures and Wigner functions

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Prerequisites: Some representation theory, Gaussian transformations, Thermal states, Completeness relations We introduce quadratures,

$$q = \frac{k_\mathrm{c}}{2}\left(a^\dagger + a\right)$$
$$p = \frac{k_\mathrm{c}}{2} i\left(a^\dagger - a\right)$$

where k_c is a real constant, chosen according to convention. Common conventions in the litterature are, $k_\mathrm{c} = \left\{1, \sqrt{2}, 2\right\}$.

We have q-quadrature eigenstates,

$$q|x\rangle = x|x\rangle,$$

where $q, p, a, a^\dagger$ are vectors of operators and x is a vector of numbers,

$$q = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^T,$$
$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T.$$

Defining,

$$R_Q = \begin{pmatrix} q \\ p \end{pmatrix}, C_a = \begin{pmatrix} a \\ a^\dagger \end{pmatrix},$$

we have the vector relation,

$$R_Q = T_k C_a,$$

where

$$T_k = k_\mathrm{c} T$$
$$T = \begin{pmatrix} \frac{1}{2}I, & \frac{1}{2}I \\ -\frac{1}{2}iI, & \frac{1}{2}iI \end{pmatrix}, T^{-1} = 2T^H$$
$$T^T \Omega T = T\Omega T^T = \frac{i}{2}\Omega$$

We have the commutator,

$$[q_j, p_s] = i\frac{k_\mathrm{c}^2}{2}\delta_{js}.$$

1.1 Gaussian transformations

1.1.1 a)

We now show that a gaussian transformation e^{-itH_G} results in the quadrature transformation,

$$e^{itH_G}R_Q e^{-itH_G} = S_+ R_Q + \mu_+.$$

I) Show that S_+ and μ_+ are given by,

$$\begin{aligned} S_+ &= TM_+T^{-1} \\ \mu_+ &= T_k D_+. \end{aligned}$$

Verify that S_+ and μ_+ have only real elements.

II) Show that S_+ is a symplectic matrix, i.e. show that,

$$S_+ \Omega S_+^T = \Omega$$

Hint: For this part the following identities help,

$$\begin{aligned} T^T \Omega T = T \Omega T^T &= \frac{i}{2}\Omega \\ M_+ \Omega M_+^T &= \Omega \end{aligned}$$

1.1.2 b)

Show that the inverse transformation,

$$e^{-itH_G}R_Q e^{itH_G} = S_- R_Q + \mu_-,$$

satisfies,

$$\begin{aligned} S_- &= S_+^{-1} \\ \mu_- &= -S_+^{-1}\mu_+ \end{aligned}$$

1.1.3 c)

Verify that the determinant of S_+ is always 1.

Hint: Use that $|M_+| = 1$ as we proved in the exercise on gaussian transformations.

1.2 Glauber's formula and the quadrature characteristic function

1.2.1 a)

We have the n-mode displacement operator,

$$D(\alpha) = \exp\left[C_\alpha^T \Omega C_a\right],$$

with $C_\alpha^T = \begin{pmatrix} \alpha^T & \alpha^{*T} \end{pmatrix}$ and α is a vector of n complex numbers $\alpha = \alpha_R + i\alpha_I$.

Show that in terms of quadratures, we have the displacement operator,

$$\begin{aligned} D(\alpha) &= \exp\left[C_\alpha^T \Omega C_a\right] \\ &= \exp\left[iR_Q^T \Omega R_\Lambda\right], \end{aligned}$$

where

$$R_\Lambda = 2k_{\mathrm{c}}^{-1} \begin{pmatrix} \alpha_R \\ \alpha_I \end{pmatrix}.$$

We let $D(R_\Lambda) = \exp\left[iR_Q^T \Omega R_\Lambda\right]$.

1.2.2 b)

Given Glauber's formula for an n-mode operator ρ,

$$\rho = \frac{1}{\pi^n} \int_{\mathbb{C}^n} d^{2n}\alpha \chi_\rho\left(C_\alpha\right) \exp\left[-C_\alpha^T \Omega C_a\right].$$

Show that in terms of quadratures, Glauber's formula becomes,

$$\rho = \left(\frac{k_{\mathrm{c}}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n} R_\Lambda \chi_\rho^{(Q)}\left(R_\Lambda\right) \exp\left[-iR_Q^T \Omega R_\Lambda\right],$$

where $\chi_\rho^{(Q)}$ is the quadrature characteristic function for the operator ρ,

$$\chi_\rho^{(Q)}\left(R_\Lambda\right) = \chi_\rho\left(k_{\mathrm{c}} T^H R_\Lambda\right).$$

1.2.3 c)

Show that the characteristic function $\chi_\rho^{(Q)}\left(R_\Lambda\right)$ can be expressed as the following expectation value,

$$\chi_\rho^{(Q)}\left(R_\Lambda\right) = \mathrm{Tr}\left\{\rho \exp\left[iR_Q^T \Omega R_\Lambda\right]\right\}.$$

1.2.4 d)

Show that,

$$\mathrm{Tr}\left\{\exp\left[iR_Q^T\Omega R_\Lambda\right]\right\} = \left(\frac{4\pi}{k_\mathrm{c}^2}\right)^n \delta^{(n)}\left(\Lambda_p\right)\delta^{(n)}\left(\Lambda_q\right)$$

1.2.5 e)

We have the displacement and covariance matrix,

$$\bar{R} = \mathrm{Tr}\left\{\rho R_Q\right\}$$
$$Q = \frac{1}{2}\mathrm{Tr}\left\{\rho\left(R_Q\otimes R_Q^T + R_Q^T\otimes R_Q\right)\right\} - \mathrm{Tr}\left\{\rho R_Q^T\right\}\otimes\mathrm{Tr}\left\{\rho R_Q\right\}$$

Relate these to $\bar{C}$ and Σ,

$$\bar{C} = \mathrm{Tr}\left\{\rho C_a\right\},$$

$$\Sigma = \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a\otimes C_a^T + C_a^T\otimes C_a\right)\right\} - \mathrm{Tr}\left\{\rho C_a^T\right\}\otimes\mathrm{Tr}\left\{\rho C_a\right\}.$$

Show that

$$\bar{R} = T_k\bar{C}$$
$$Q = T_k\Sigma T_k^T$$

1.2.6 f)

Show that the covariance matrix of a gaussian state can be written as,

$$Q = \frac{k_\mathrm{c}^2}{4}S_+\begin{pmatrix}\nu_\mathrm{th} & 0\\ 0 & \nu_\mathrm{th}\end{pmatrix}S_+^T,$$

and verify that Q is symmetric, real, and positive definite. Finally, show that the determinant of Q is,

$$|Q| = \left(\frac{k_\mathrm{c}^2}{4}\right)^{2n}\left|\nu_\mathrm{th}\right|^2.$$

1.2.7 g)

A general gaussian state has the characteristic function,

$$\chi_G(C_\alpha) = \exp\left[\frac{1}{2}C_\alpha^T\Omega^T\Sigma\Omega C_\alpha\right]\exp\left[-\bar{C}^T\Omega C_\alpha\right].$$

Show that the quadrature characteristic function for a gaussian state is,

$$\chi_G^{(Q)}\left(R_\Lambda\right) = \exp\left[-\frac{1}{2}R_\Lambda^T\Omega^T Q\Omega R_\Lambda\right]\exp\left[i\bar{R}^T\Omega R_\Lambda\right],$$

where Q is the covariance matrix of the state and $\bar{R}$ is the displacement.
Hint: Use that

$$\chi_\rho^{(Q)}\left(R_\Lambda\right) = \chi_\rho\left(k_c T^H R_\Lambda\right).$$

1.2.8 h)

Show that applying a gaussian transformation,

$$\rho_f = e^{-itH_G}\rho_i e^{itH_G},$$

transforms the quadrature characteristic function as,

$$\chi_f^{(Q)}\left(R_\Lambda\right) = \exp\left[i\mu_+^T\Omega R_\Lambda\right]\chi_i^{(Q)}\left(S_+^{-1}R_\Lambda\right),$$

where ρ_i has the quadrature characteristic function $\chi_i^{(Q)}\left(R_\Lambda\right)$.
Hint: We previously showed that,

$$\chi_f(C_\alpha) = \chi_i\left(M_+^{-1}C_\alpha\right)\exp\left[C_\alpha^T\Omega D_+\right]$$

1.2.9 i)

We apply a gaussian unitary to a gaussian state ρ_i,

$$\rho_f = e^{-itH_G}\rho_i e^{itH_G}.$$

Using the results from e) argue that the final state ρ_f has the new covariance matrix and displacement,

$$\begin{aligned} Q_f &= S_+ Q_i S_+^T \\ \bar{R}_f &= S_+\bar{R}_i + \mu_+. \end{aligned}$$

Using g) and h), verify that the quadrature characteristic function of the final state is,

$$\begin{aligned}\chi_f^{(Q)}(R_\Lambda) &= \exp\left[-\frac{1}{2}R_\Lambda^T\Omega^T\left(S_+Q_iS_+^T\right)\Omega R_\Lambda\right]\exp\left[i\left(S_+\bar{R}_i+\mu_+\right)^T\Omega R_\Lambda\right]\\ &= \exp\left[-\frac{1}{2}R_\Lambda^T\Omega^T Q_f\Omega R_\Lambda\right]\exp\left[i\bar{R}_f^T\Omega R_\Lambda\right].\end{aligned}$$

Hint: The following identities are useful,

$$\begin{aligned}\bar{C}_f &= M_+\bar{C}_i + D_+\\ \Sigma_f &= M_+\Sigma_i M_+^T\\ S_+^{-1} &= \Omega S_+^T\Omega^T.\end{aligned}$$

1.3 Wigner function

We now motivate the Wigner function and relate it to the quadrature characteristic function. We define arrays of real numbers,

$$R_X = \begin{pmatrix} X_q \\ X_p \end{pmatrix}, R_Y = \begin{pmatrix} Y_q \\ Y_p \end{pmatrix}$$

We may identify $\chi_\rho^{(Q)}(R_\Lambda)$ as an expectation value,

$$\chi_\rho^{(Q)}(R_\Lambda) = \mathrm{Tr}\left\{\rho\exp\left[iR_Q^T\Omega R_\Lambda\right]\right\} = \left\langle e^{iR_Q^T\Omega R_\Lambda}\right\rangle.$$

The Wigner function is introduced by assuming that this expectation value can be calculated as an average over a distribution, $W_\rho(R_X)$,

$$\left\langle e^{iR_Q^T\Omega R_\Lambda}\right\rangle = \int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho(R_X)\, e^{iR_X^T\Omega R_\Lambda}.$$

1.3.1 a)

Verify that if $W_\rho(R_X)$ is given by,

$$W_\rho(R_X) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_\rho^{(Q)}(R_\Lambda)\, e^{-iR_X^T\Omega R_\Lambda},$$

then we have the desired result, i.e. $\left\langle e^{iR_Q^T\Omega R_\Lambda}\right\rangle = \int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho(R_X)\, e^{iR_X^T\Omega R_\Lambda}$.

1.3.2 b)

We now prove a few properties of the Wigner function,

I) Show that the Wigner function $W_\rho(R_X)$ of a quantum state is normalized,

$$\int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho(R_X) = 1$$

II) Show that the expectation value of an operator A can be computed from the Wigner function W_ρ of the state ρ and the Wigner function of A, which we label W_A, as,

$$\langle A\rangle = \mathrm{Tr}\left\{\rho A\right\} = \left(\pi k_\mathrm{c}^2\right)^n \int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho\left(R_X\right) W_A(R_X),$$

where,

$$W_A(R_X) = \frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_A^{(Q)}\left(R_\Lambda\right) e^{-iR_X^T\Omega R_\Lambda}$$

Hint:
Use that we can write A as,

$$A = \left(\frac{k_\mathrm{c}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_A^{(Q)}\left(R_\Lambda\right) \exp\left[-iR_Q^T\Omega R_\Lambda\right],$$

1.3.3 c)

A formula for calculating the Wigner function directly from the operator ρ exists. Using the relation,

$$W_\rho(R_X) = \frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \mathrm{Tr}\left\{\rho \exp\left[iR_Q^T\Omega R_\Lambda\right]\right\} e^{-iR_X^T\Omega R_\Lambda},$$

show that the Wigner function can also be calculated as,

$$W_\rho(R_X) = \left(\frac{2}{\pi k_\mathrm{c}^2}\right)^n \int_{\mathbb{R}^n} d^n y e^{i4k_\mathrm{c}^{-2}X_p^T y}\langle X_q - y|\rho|X_q + y\rangle$$

where $q|X_q + y\rangle = \left(X_q + y\right)|X_q + y\rangle$, i.e. they are q-quadrature eigenstates.
Hint: Perform the trace in the q-quadrature eigenstates $|x\rangle$.