1 Some representation theory

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Prerequisites: Familiarity with quantum mechanics Defining quadratures,

$$q = \frac{k_c}{2}\left(a^\dagger + a\right)$$
$$p = \frac{k_c}{2} i \left(a^\dagger - a\right)$$

where k_c is a real constant, chosen according to convention. Common conventions in the litterature are, $k_c = \left\{1, \sqrt{2}, 2\right\}$.

It is familiar that the action of the quadrature operators q and p on the wavefunction are respectively,

$$q\psi(x) = x\psi(x)$$
$$p\psi(x) = -i\frac{k_c^2}{2}\partial_x\psi(x).$$

The reason can be understood from bra-ket notation as follows, let $|x\rangle$ be a q-quadrature eigenstate such that,

$$q|x\rangle = x|x\rangle.$$

Then we have,

$$\langle x|q|\psi\rangle = x\langle x|\psi\rangle = x\psi(x)$$

where we've defined the wavefunction as $\psi(x) = \langle x|\psi\rangle$.

For the action of the p operator we consider the exponentiated form for real s,

$$\begin{aligned} qe^{-isp}|x\rangle &= e^{-isp}e^{isp}qe^{-isp}|x\rangle \\ &= e^{-isp}\left(q + s\frac{k_c^2}{2}\right)|x\rangle \\ &= \left(x + s\frac{k_c^2}{2}\right)e^{-isp}|x\rangle, \end{aligned}$$

where we used the BCH formula and $[q,p] = i\frac{k_c^2}{2}$. So we identify,

$$e^{-isp}|x\rangle = |x + s\frac{k_c^2}{2}\rangle.$$

For vanishing s we then obtain,

$$(1 - isp)\,|x\rangle = |x + s\frac{k_c^2}{2}\rangle,$$

or rearranging

$$p|x\rangle = i\frac{|x + s\frac{k_c^2}{2}\rangle - |x\rangle}{s}.$$

Then we obtain the action on the wavefunction,

$$\begin{aligned}
\langle x|p|\psi\rangle &= -i\frac{\langle x + s\frac{k_c^2}{2}| - \langle x|}{s}|\psi\rangle \\
&= -i\frac{\psi(x + s\frac{k_c^2}{2}) - \psi(x)}{s} \\
&= -i\frac{k_c^2}{2}\frac{\psi(x + s\frac{k_c^2}{2}) - \psi(x)}{s\frac{k_c^2}{2}} \\
&= -i\frac{k_c^2}{2}\partial_x\psi(x)
\end{aligned}$$

For the following exercise we use the Wirtinger derivatives for $\alpha = \alpha_R + i\alpha_I$,

$$\begin{aligned}
\partial_\alpha &= \frac{1}{2}\left(\frac{\partial}{\partial\alpha_R} - i\frac{\partial}{\partial\alpha_I}\right) \\
\partial_{\alpha^*} &= \frac{1}{2}\left(\frac{\partial}{\partial\alpha_R} + i\frac{\partial}{\partial\alpha_I}\right).
\end{aligned}$$

One may verify that for a polynomial in α, α^* these derivatives act like differentiating with respect to α and α^*respectively.

1.0.1 a)

We will refer to the overlap $\psi(\alpha, \alpha^*) = \langle\alpha|\psi\rangle$, where $|\alpha\rangle$ is a coherent state, as the coherent wave function.

Derive the representation of $a, a^\dagger$ on the coherent wavefunction, show that we can put the representation into a symmetric form,

$$\langle\alpha|a|\psi\rangle = \left(\partial_{\alpha^*} + \frac{1}{2}\alpha\right)\psi(\alpha,\alpha^*)$$

$$\langle\alpha|a^\dagger|\psi\rangle = \alpha^*\psi(\alpha,\alpha^*) = \left(-\partial_\alpha + \frac{1}{2}\alpha^*\right)\psi(\alpha,\alpha^*)$$

Hint: Verify and use the identities,

$$\begin{aligned}
\partial_{\alpha^*} e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} &= \left(a - \frac{1}{2}\alpha\right) e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} \\
\partial_\alpha e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha} &= \left(-a^\dagger + \frac{1}{2}\alpha^*\right) e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha},
\end{aligned}$$

together with $|\alpha\rangle = D(\alpha)|0\rangle$.

1.0.2 b)

Show that for a product of ladder operators $a^{\dagger n}a^m$, we can calculate the expectation value using the coherent wave function as,

$$\langle\psi|a^{\dagger n}a^m|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\,\psi^*(\alpha,\alpha^*)\alpha^{*n}\left(\partial_{\alpha^*} + \frac{1}{2}\alpha\right)^m\psi(\alpha,\alpha^*)$$

1.0.3 c)

Show that,

$$\begin{aligned}
\langle x|a|\psi\rangle &= k_{\mathrm{c}}^{-1}\left(x + \frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi(x) \\
\langle x|a^\dagger|\psi\rangle &= k_{\mathrm{c}}^{-1}\left(x - \frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi(x)
\end{aligned}$$

Hint: Use that

$$\begin{aligned}
a &= k_{\mathrm{c}}^{-1}\left(q + ip\right) \\
a^\dagger &= k_{\mathrm{c}}^{-1}\left(q - ip\right)
\end{aligned}$$

1.0.4 d)

Show that a coherent state $|\alpha\rangle$ has the q-quadrature wavefunction,

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \left(\frac{2}{\pi k_\mathrm{c}^2}\right)^{1/4} \exp\left[-\frac{x^2}{k_\mathrm{c}^2} + 2\alpha\frac{x}{k_\mathrm{c}} - \alpha_R^2\right]$$

Hint: Use that,

$$\langle x|a|\alpha\rangle = \alpha\psi_\alpha(x)$$