1 Completeness relations

By Anders J. E. Bjerrum (QPIT)

Prerequisites: Familiarity with quantum mechanics In this exercise we establish some important identities regarding the completeness of the coherent states and displacement operators.

1.1 Coherent states

We seek to establish the over completeness of the coherent states. We have the coherent states, given in the Fock basis,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{\sqrt{n!}} \alpha^n |n\rangle.$$

We introduce the operator P,

$$P = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha |\alpha\rangle\langle\alpha|.$$

The integral $\int_{\mathbb{C}} d^2\alpha$ should be understood to be over the complex plane, and we may for example perform the integral over the real and imaginary part of α, or over the polar decomposition of α. In this exercise we seek to establish that

$$P = I$$

where I is the identity.

1.1.1 a)

Show that the Fock state $|n\rangle$ satisfy,

$$P|n\rangle = |n\rangle$$

1.1.2 b)

By expanding an arbitrary pure state in the Fock basis, show that P is the identity.

1.1.3 c)

Show that the trace of an operator can be calculated using the coherent states as,

$$\mathrm{Tr}\,\{\rho\} = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \langle\alpha|\rho|\alpha\rangle$$

1.2 Displacement operators

In this exercise we seek to establish the completeness of the displacement operators. We have the displacement operators,

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.$$

We have $D^\dagger(\alpha) = D(-\alpha)$ and the coherent states satisfy,

$$\begin{aligned}|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger}|0\rangle &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{n!}\alpha^n \left(a^\dagger\right)^n |0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\end{aligned}$$

where $|0\rangle$ is the vacuum state.
We seek to establish the validity of the following expansion,

$$\rho = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha \chi(\alpha) D^\dagger(\alpha),$$

where ρ is an arbitrary operator, for example a density matrix. We call this equation Glauber's formula. We will also show how to calculate the expansion coefficient $\chi(\alpha)$ from ρ.

1.2.1 a)

Verify the following five relations,

I)

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

II)

$$D(\alpha)D(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} D(\alpha+\beta)$$

III)

$$D(\alpha)D(\beta) = e^{\beta^*\alpha - \beta\alpha^*} D(\beta)D(\alpha)$$

IV)

$$\langle\alpha|D(\gamma)|\beta\rangle = e^{-\frac{1}{2}\left(|\gamma|^2+|\alpha|^2+|\beta|^2\right)} e^{\alpha^*\beta} e^{\gamma\alpha^* - \gamma^*\beta}$$

V)

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}e^{\alpha^*\beta}$$

1.2.2 b)

Show that

$$\mathrm{Tr}\left\{D(\alpha)\right\} = \pi\delta^{(2)}(\alpha),$$

where when $\alpha = \alpha_R + i\alpha_I$, then $\delta^{(2)}(\alpha) = \delta(\alpha_R)\delta(\alpha_I)$.
Hint: The trace may be calculated in multiple ways, however it is fairly straightforward in the coherent state basis. See
https://mathworld.wolfram.com/DeltaFunction.html .

1.2.3 c)

Supposing that Glauber's formula is valid, find the expansion coefficient $\chi(\alpha)$ for the operator ρ using the trace identity from b. Argue that $\chi(\alpha)$ is in fact the (Wigner) characteristic function of ρ,

$$\chi(\alpha) = \mathrm{Tr}\left\{\rho D(\alpha)\right\}$$

1.2.4 d)

Show that Glauber's formula is correct for the operator $|\alpha\rangle\langle\beta|$, i.e. verify that,

$$|\alpha\rangle\langle\beta| = \int_{\mathbb{C}} \frac{d^2\gamma}{\pi}\mathrm{Tr}\left\{|\alpha\rangle\langle\beta|D(\gamma)\right\}D^\dagger(\gamma)$$

Hint: Expand the displacement operator in terms of coherent states as,

$$D^\dagger(\gamma) = \int_{\mathbb{C}}\frac{d^2\lambda}{\pi}\int_{\mathbb{C}}\frac{d^2\eta}{\pi}\langle\lambda|D^\dagger(\gamma)|\eta\rangle|\lambda\rangle\langle\eta|$$

and verify the following integral,

$$I = \int_{\mathbb{C}}\frac{d^2\gamma}{\pi}\langle\beta|D(\gamma)|\alpha\rangle\langle\lambda|D^\dagger(\gamma)|\eta\rangle = \langle\beta|\eta\rangle\langle\lambda|\alpha\rangle.$$

To this end you will need the following identity,

$$\int dx e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}}\exp\left(\frac{J^2}{2a}\right),$$

integrals such as this can be found at
https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory

1.2.5 e)

Using the result from d, verify that Glauber's formula holds in general for arbitrary operators ρ,

$$\rho = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \chi(\alpha) D^\dagger(\alpha).$$

Hint: Use that ρ can be expanded in the coherent state basis via $P = I$.

1.2.6 f)

Given two operators A, B which we can be expanded using Glauber's formula as,

$$\begin{aligned} A &= \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \chi_A(\alpha) D^\dagger(\alpha), \\ B &= \frac{1}{\pi} \int_{\mathbb{C}} d^2\beta \chi_B(\beta) D^\dagger(\beta). \end{aligned}$$

Show that,

$$\mathrm{Tr}\{AB\} = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \chi_A(\alpha) \chi_B(-\alpha)$$

1.2.7 g)

Generalize Glauber's formula to the n-mode case, i.e.

$$\rho = \frac{1}{\pi^n} \int_{\mathbb{C}^n} d^{2n}\alpha \chi(\alpha) D^\dagger(\alpha).$$

where α is a vector of complex numbers for the n modes,

$$\begin{aligned} \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}^T \\ D(\alpha) = \bigotimes_{k=1}^{n} D_k(\alpha_k) \\ \chi(\alpha) = \mathrm{Tr}\{D(\alpha)\rho\}. \end{aligned}$$

The integral $\int_{\mathbb{C}^n} d^{2n}\alpha$ means integration over each of the complex planes associated to the n complex numbers α_i.