1 Wavefunction of a gaussian state

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Prerequisites: Gaussian transformations, thermal states, quadratures and wigner functions, completeness relations, some representation theory In this exercise we give an explicit formula for the wavefunction of a pure gaussian state. The direction will be to start from the characteristic function of a gaussian state, characterized by the covariance matrix Σ and mean $\bar{C}$, and then to state the corresponding wavefunction.

We use the following notation,

$$C_\lambda = \begin{pmatrix} \lambda \\ \lambda^* \end{pmatrix}, R_\lambda = \begin{pmatrix} \lambda_R \\ \lambda_I \end{pmatrix}, \bar{C} = \begin{pmatrix} z \\ z^* \end{pmatrix}$$

Central to the derivation will be Glauber's formula,

$$\rho = \frac{1}{\pi^n} \int_{\mathbb{C}^n} d^{2n}\alpha D^\dagger(\alpha)\chi(C_\alpha),$$

and the characteristic function of a general gaussian state,

$$\chi_G(C_\alpha) = \exp\left[\frac{1}{2} C_\alpha^T \Omega^T \Sigma \Omega C_\alpha\right] \exp\left[-\bar{C}^T \Omega C_\alpha\right].$$

We define the matrices,

$$T = \begin{pmatrix} \frac{1}{2}I, & \frac{1}{2}I \\ -\frac{1}{2}iI, & \frac{1}{2}iI \end{pmatrix}, X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

which have the following relations,

$$\begin{aligned} T^{-1} &= 2T^H \\ T^T \Omega T = T \Omega T^T &= \frac{i}{2}\Omega \\ T X T^T &= \frac{1}{2} I \\ |T| &= \left(\frac{1}{2} i\right)^n \\ R_\lambda &= T C_\lambda. \end{aligned}$$

Remember that Σ per its definition,

$$\Sigma = \frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a \otimes C_a^T + C_a^T \otimes C_a\right)\right\} - \mathrm{Tr}\left\{\rho C_a^T\right\} \otimes \mathrm{Tr}\left\{\rho C_a\right\},$$

can be written in block form as,

$$\Sigma = \begin{pmatrix} \Sigma_D & \Sigma_A \\ \Sigma_A^* & \Sigma_D^* \end{pmatrix},$$

and obeys the symmetries,

$$\begin{aligned} X\Sigma X &= \Sigma^* \\ \Sigma^T &= \Sigma \end{aligned}$$

1.1 Coherent wavefunction

1.1.1 a)

Given the gaussian state,

$$\rho_G = \frac{1}{\pi^n}\int_{\mathbb{C}^2} d^{2n}\alpha D^\dagger(\alpha)\chi_G(C_\alpha)$$

We seek to find an expression for the matrix element,

$$\rho_G(\alpha, \beta) = \langle\alpha|\rho_G|\beta\rangle$$

in terms of Σ and $\bar{C}$. Note that $|\alpha\rangle, |\beta\rangle$ are coherent states of amplitude α and β respectively.

I) Show that $\rho_G(\alpha, \beta)$ can be written as the integral,

$$\rho_G(\alpha, \beta) = e^{-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) + \beta^T\alpha^*}.$$

$$\frac{1}{\pi^n}\int d^{2n}\lambda \exp\left[\frac{1}{2}C_\lambda^T\Omega^T\left(\Sigma + \frac{1}{2}X\right)\Omega C_\lambda\right]\exp\left[-\left(\bar{C} - u_{\alpha\beta}\right)^T\Omega C_\lambda\right],$$

where we've defined,

$$u_{\alpha\beta} = \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix}.$$

II) Perform the integral in the quadratures $R_\lambda = TC_\lambda$ (over the real and imaginary parts of λ). Show that we have,

$$\rho_G(\alpha,\beta) = \sqrt{|iW|}e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}.$$
$$\exp\left[-\frac{1}{2}\left(\bar{C}-u_{\alpha\beta}\right)^T W\left(\bar{C}-u_{\alpha\beta}\right)\right].$$

where we've defined,

$$W = \left(\Sigma + \frac{1}{2}X\right)^{-1}$$

Verify that $|iW| = i^{2n}|W| = (-1)^n|W|$ is a real positive number.
Hint: Use the following reference
https://en.wikipedia.org/wiki/Common_integrals_in_quantum_field_theory.
Verify that the conditions for the validity of the integral formula are met.

1.1.2 b)

I) Using the symmetry properties of Σ, verify that W has the symmetry,

$$XWX = W^*$$

and that W can therefore be written in block form as,

$$W = \begin{pmatrix} W_D & W_A \\ W_A^* & W_D^* \end{pmatrix}$$

each block of dimension n.

II) Verify that W is symmetric.

1.1.3 c)

Show that we can rewrite the matrix element as,

$$\rho_G(\alpha,\beta) = \psi(\alpha,\alpha^*)\psi^*(\beta,\beta^*)\exp\left[-\beta^T\left(W_A - I\right)\alpha^*\right]$$

where

$$\psi(\alpha,\alpha^*) = \left[(-1)^n|W|\right]^{1/4}\exp\left[-\frac{1}{4}\bar{C}^TW\bar{C}\right]\exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*T}W_D^*\alpha^* + \left(z^{*T}W_D^* + z^TW_A\right)\alpha^*\right]$$

1.1.4 d)

We now consider the structure of W when ρ_G is pure. From the exercise 'Gaussian transformations', we know that we can write the covariance matrix as,

$$\Sigma = M_+ \Sigma_{\text{th}} M_+^T.$$

$$M = \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix}$$

where

$$\begin{aligned} VJ^T - JV^T &= 0 \\ VV^H - JJ^H &= I, \end{aligned}$$

and V is always invertible. Furthermore,

$$\Sigma_{\text{th}} = \frac{1}{2} \begin{pmatrix} 0 & \nu_{\text{th}} \\ \nu_{\text{th}} & 0 \end{pmatrix}.$$

ρ_G is pure when it is obtained from gaussian transformations acting on a thermal state with temperature zero. From the exercise on thermal states, we know that this will correspond to,

$$\Sigma_{\text{th}} = \frac{1}{2} X$$

I) Verify that for a pure state we can write,

$$\Sigma + \frac{1}{2} X = \begin{pmatrix} VJ^T, & VV^H \\ V^*V^T, & V^*J^H \end{pmatrix}.$$

II) Verify by calculation that for a pure state we have the inverse,

$$W = \left(\Sigma + \frac{1}{2} X \right)^{-1} = \begin{pmatrix} -J^* V^{-1} & I \\ I & -JV^{-*} \end{pmatrix}$$

1.1.5 e)

Using c) and d) verify that the coherent wavefunction of a pure gaussian state $\rho_G = |\psi_G\rangle\langle\psi_G|$ is $\psi(\alpha, \alpha^*)$. I.e. verify that,

$$\langle \alpha | \psi_G \rangle = \psi(\alpha, \alpha^*).$$

1.2 Quadrature wavefunction

We now seek to determine the $q-$quadrature wavefunction in a similar way as for the coherent wave function in the previous exercise. We define quadratures

$$
\begin{aligned}
q &= \frac{k_c}{2}\left(a^\dagger + a\right) \\
p &= \frac{k_c}{2} i\left(a^\dagger - a\right)
\end{aligned}
$$

with q-quadrature eigenstates,

$$
\begin{aligned}
q|x\rangle &= x|x\rangle \\
q|y\rangle &= y|y\rangle
\end{aligned}
$$

where $q, p, a, a^\dagger$ are vectors of operators and x is a vector of numbers,

$$
\begin{aligned}
q &= \left(\begin{array}{cccc} q_1 & q_2 & \cdots & q_n \end{array}\right)^T \\
x &= \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}\right)^T .
\end{aligned}
$$

Before proceeding, we note a short-cut to obtain the quadrature wavefunction.

Suppose we have a gaussian state, obtained by a gaussian unitary acting on vacuum,

$$
|G\rangle = e^{-itH_G}|0\rangle.
$$

Then we have the relation,

$$
e^{-itH_G} a e^{itH_G}|G\rangle = 0,
$$

where the right hand side is the zero vector.

Using relations from the exercise on gaussian transformations, we have,

$$
\begin{aligned}
e^{-itH_G}\left(\begin{array}{c} a \\ a^\dagger \end{array}\right) e^{itH_G} &= M_- \left(\begin{array}{c} a \\ a^\dagger \end{array}\right) + D_- \\
&= \left(\begin{array}{cc} V^H & -J^T \\ -J^H & V^T \end{array}\right)\left(\begin{array}{c} a - z \\ a^\dagger - z^* \end{array}\right),
\end{aligned}
$$

where $D_+^T = \left(\begin{array}{cc} z^T & z^{*T} \end{array}\right)$.

In the following we set $z = 0$ for the sake of compactness, although no great complications are introduced by keeping z non-zero.

Defining $\psi_G(x) = \langle x|G\rangle$, and using that,

$$\langle x|a|G\rangle = k_{\mathrm{c}}^{-1}\left(x + \frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi_G(x)$$
$$\langle x|a^\dagger|G\rangle = k_{\mathrm{c}}^{-1}\left(x - \frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi_G(x),$$

where ∂_x is the gradient,

$$\partial_x = \begin{pmatrix} \partial_{x_1} & \partial_{x_2} & \cdots & \partial_{x_n} \end{pmatrix}^T,$$

we find that we can rewrite $\langle x|e^{-itH_G}ae^{itH_G}|G\rangle = 0$ as,

$$\left[\left(V^H - J^T\right)x + \frac{k_{\mathrm{c}}^2}{2}\left(V^H + J^T\right)\partial_x\right]\psi_G(x) = 0.$$

Upon multiplying with V^{-H} from the left,

$$\left[\left(I - V^{-H}J^T\right)x + \frac{k_{\mathrm{c}}^2}{2}\left(I + V^{-H}J^T\right)\partial_x\right]\psi_G(x) = 0.$$

We will now assume without proof that $\left(I + V^{-H}J^T\right)$ is invertible, then we find,

$$\partial_x\psi_G(x) = -\frac{2}{k_{\mathrm{c}}^2}\left(I + V^{-H}J^T\right)^{-1}\left(I - V^{-H}J^T\right)x\psi_G(x).$$

We note the commutator,

$$\left[\left(I + V^{-H}J^T\right), \left(I - V^{-H}J^T\right)\right] = 0,$$

and so it follows that,

$$\left[\left(I + V^{-H}J^T\right)^{-1}, \left(I - V^{-H}J^T\right)\right] = 0.$$

We also note that $V^{-H}J^T$ is symmetric, as follows from the result of exercise e) in the exercise set on gaussian transformations.

As a consequence, we find that the matrix,

$$\Gamma_x = \left(I + V^{-H}J^T\right)^{-1}\left(I - V^{-H}J^T\right)$$

is symmetric. Therefore the above equation has the solution,

$$\psi_G(x) \propto \exp\left[-\frac{1}{k_{\mathrm{c}}^2} x^T \Gamma_x x\right]$$

which is the q-quadrature wavefunction.

We now turn to the exercise, if the reader is willing to forego concerns about the invertibility of $\left(I + V^{-H} J^T\right)$, and not interested in the structure of Γ_x, then this exercise can be skipped.

We have the quadrature characteristic function of a gaussian state,

$$\chi_G^{(Q)}\left(R_\Lambda\right) = \exp\left[-\frac{1}{2} R_\Lambda^T \Omega^T Q \Omega R_\Lambda\right] \exp\left[i \bar{R}^T \Omega R_\Lambda\right]$$

$$R_\Lambda = \left(\begin{array}{cc} \Lambda_q^T & \Lambda_p^T \end{array}\right)^T, \bar{R} = \left(\begin{array}{cc} \bar{q}^T & \bar{p}^T \end{array}\right)^T$$

and the appropriate version of Glauber's formula,

$$\rho = \left(\frac{k_{\mathrm{c}}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n} R_\Lambda \chi_G^{(Q)}\left(R_\Lambda\right) \exp\left[-i R_Q^T \Omega R_\Lambda\right].$$

1.2.1 a)

Let $|x\rangle, |y\rangle$ be q-quadrature eigenstates. Show that we can write,

$$\begin{aligned} &\langle x| \exp\left[-i R_Q^T \Omega R_\Lambda\right] |y\rangle \\ &= \left(\frac{2}{k_{\mathrm{c}}^2}\right)^n e^{i \frac{k_{\mathrm{c}}^2}{4} \Lambda_q^T \Lambda_p} e^{-i y^T \Lambda_p} \delta\left(\Lambda_q - \frac{2}{k_{\mathrm{c}}^2}(y - x)\right) \end{aligned}$$

1.2.2 b)

Show that we can write the quadrature density matrix as the integral,

$$\begin{aligned} &\rho(x, y) = \langle x|\rho|y\rangle \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} d^n \Lambda_p \chi_G^{(Q)}\left(\frac{2}{k_{\mathrm{c}}^2}(y - x), \Lambda_p\right) e^{-i \frac{1}{2}(x+y)^T \Lambda_p} \end{aligned}$$

1.2.3 c)

We write the covariance matrix in block form,

$$Q = \left(\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array}\right).$$

Using the vector,

$$v_{xy} = \begin{pmatrix} x \\ y \end{pmatrix},$$

show that we can expand as,

$$\begin{aligned} &\chi_G^{(Q)}\left(\frac{2}{k_c^2}(y-x), \Lambda_p\right) e^{-i\frac{1}{2}(x+y)^T\Lambda_p} \\ &= \exp\left[v_{xy}^T U_1 v_{xy} + v_{xy}^T k_p\right] \exp\left[-\frac{1}{2}\Lambda_p^T Q_{11}\Lambda_p + \mu_{xy}^T \Lambda_p\right] \end{aligned}$$

where

$$\begin{aligned} \mu_{xy} &= \phi x - \phi^* y + i\bar{q} \\ \phi &= -\frac{2}{k_c^2} Q_{12} - i\frac{1}{2} I \\ U_1 = \frac{2}{k_c^4} &\begin{pmatrix} -Q_{22} & Q_{22} \\ Q_{22} & -Q_{22} \end{pmatrix} \\ k_p &= i\frac{2}{k_c^2}\begin{pmatrix} \bar{p} \\ -\bar{p} \end{pmatrix} \end{aligned}$$

1.2.4 d)

Perform the integral from b) and show that,

$$\rho(x,y) = \left[(2\pi)^n |Q_{11}|\right]^{-1/2} \exp\left[v_{xy}^T U_1 v_{xy} + v_{xy}^T k_p\right] \exp\left[\frac{1}{2}\mu_{xy}^T Q_{11}^{-1}\mu_{xy}\right]$$

and show that we can rewrite as,

$$\frac{1}{2}\mu_{xy}^T Q_{11}^{-1}\mu_{xy} = v_{xy}^T U_2 v_{xy} + v_{xy}^T k_q - \frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}$$

where

$$\begin{aligned} U_2 = \frac{1}{2} &\begin{pmatrix} \phi^T Q_{11}^{-1}\phi, & -\phi^T Q_{11}^{-1}\phi^* \\ -\phi^H Q_{11}^{-1}\phi, & \phi^H Q_{11}^{-1}\phi^* \end{pmatrix} \\ k_q &= \begin{pmatrix} \phi^T i Q_{11}^{-1}\bar{q} \\ -\phi^H i Q_{11}^{-1}\bar{q} \end{pmatrix} \end{aligned}$$

Note thatQ_{11} is symmetric and real. Q_{11} is centered on the diagonal of the positive definite matrix Q, and as a result Q_{11} is also positive definite. See
https://en.wikipedia.org/wiki/Normal_matrix
https://en.wikipedia.org/wiki/Definite_matrix
https://en.wikipedia.org/wiki/Invertible_matrix

1.2.5 e)

Using the notation from d), show that the quadrature density matrix can be rewritten as,

$$\rho(x,y) = \frac{\exp\left[-\frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/2}} \exp\left[v_{xy}^T U_{xy} v_{xy} + x^T \mu_{qp} + y^T \mu_{qp}^*\right]$$

where

$$U_{xy} = U_1 + U_2 = \begin{pmatrix} -\frac{2}{k_c^4} Q_{22} + \frac{1}{2}\phi^T Q_{11}^{-1}\phi, & \frac{2}{k_c^4} Q_{22} - \frac{1}{2}\phi^T Q_{11}^{-1}\phi^* \\ \frac{2}{k_c^4} Q_{22} - \frac{1}{2}\phi^H Q_{11}^{-1}\phi, & -\frac{2}{k_c^4} Q_{22} + \frac{1}{2}\phi^H Q_{11}^{-1}\phi^* \end{pmatrix}$$

and

$$\mu_{qp} = i\frac{2}{k_c^2}\bar{p} + \phi^T i Q_{11}^{-1}\bar{q}$$

1.2.6 f)

Argue that for a pure state we must have the equality,

$$\frac{4}{k_c^4} Q_{22} - \phi^T Q_{11}^{-1}\phi^* = 0.$$

Show that the quadrature-wavefunction is then,

$$\psi(x) = \frac{\exp\left[-\frac{1}{4}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/4}} \exp\left[-x^T\left(\Gamma_x/k_c^2\right)x + x^T\mu_{qp}\right]$$

where

$$\Gamma_x = \frac{2}{k_c^2} Q_{22} - \frac{k_c^2}{2}\phi^T Q_{11}^{-1}\phi$$

1.2.7 g)

We now rewrite the quadrature wavefunction a bit to make the structure more apparent.

I) Using the condition of purity,

$$\frac{2}{k_c^4} Q_{22} - \frac{1}{2} \phi^T Q_{11}^{-1} \phi^* = 0$$

Show that,

$$\Gamma_x = \frac{k_c^2}{4} Q_{11}^{-1} - i Q_{21} Q_{11}^{-1}.$$

II) We now argue that Γ_x is independent of k_c^2.
Define

$$\begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix} = \frac{1}{4} S_+ S_+^T$$

and show that,

$$\Gamma_x = \frac{1}{4} \tilde{Q}_{11}^{-1} - i \tilde{Q}_{21} \tilde{Q}_{11}^{-1}.$$

Hint: Remember that we can write the covariance matrix as,

$$Q = \frac{k_c^2}{4} S_+ S_+^T.$$

III) Show that the normalization becomes,

$$\frac{\exp\left[-\frac{1}{4}\bar{q}^T Q_{11}^{-1} \bar{q}\right]}{\left[(2\pi)^n \left|Q_{11}\right|\right]^{1/4}} = \frac{\exp\left[-\frac{1}{4k_c^2}\bar{q}^T \tilde{Q}_{11}^{-1} \bar{q}\right]}{\left[(2\pi k_c^2)^n \left|\tilde{Q}_{11}\right|\right]^{1/4}}$$