1 Wavefunction of a gaussian state

By Anders J. E. Bjerrum (QPIT)

Solution

1.1 Coherent wavefunction

1.1.1 a)

I) We take the overlap with a pair of coherent states,

$$\rho_G(\alpha,\beta) = \langle 0|D^\dagger(\alpha)\rho_G D(\beta)|0\rangle$$
$$= \frac{1}{\pi^n}\int_{\mathbb{C}^n} d^{2n}\lambda \chi_G(C_\lambda)\langle 0|D^\dagger(\alpha)D^\dagger(\lambda)D(\beta)|0\rangle$$

where

$$\langle 0|D^\dagger(\alpha)D^\dagger(\lambda)D(\beta)|0\rangle$$
$$= \exp\left[-\frac{1}{2}\left(|\lambda|^2+|\beta|^2+|\alpha|^2\right)+\beta^T\lambda^*+\beta^T\alpha^*-\lambda^T\alpha^*\right]$$
$$= e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\exp\left[\frac{1}{2}C_\lambda^T\Omega^T\left(\frac{1}{2}X\right)\Omega C_\lambda + u_{\alpha\beta}^T\Omega C_\lambda\right]$$

Then we have,

$$\rho_G(\alpha,\beta) = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\frac{1}{\pi^n}\cdot$$
$$\int_{\mathbb{C}^n} d^{2n}\lambda \exp\left[\frac{1}{2}C_\lambda^T\Omega^T\left(\Sigma+\frac{1}{2}X\right)\Omega C_\lambda\right]\exp\left[-\left(\bar{C}-u_{\alpha\beta}\right)^T\Omega C_\lambda\right]$$

II) We perform the integral in R_λ, we have $C_\lambda = 2T^H R_\lambda$, and so,

$$\frac{1}{2}C_\lambda^T\Omega^T\left(\Sigma+\frac{1}{2}X\right)\Omega C_\lambda$$
$$= 2R_\lambda^T T^*\Omega^T\left(\Sigma+\frac{1}{2}X\right)\Omega T^H R_\lambda$$
$$= -2R_\lambda^T\Omega^T T\left(\Sigma+\frac{1}{2}X\right)T^T\Omega R_\lambda$$

Likewise we have,

$$\left(\bar{C}-u_{\alpha\beta}\right)^{T}\Omega C_{\lambda}=2\left(\bar{C}-u_{\alpha\beta}\right)^{T}\Omega T^{H}R_{\lambda}$$

Then we have the matrix element,

$$\rho_G(\alpha,\beta)=e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\frac{1}{\pi^n}\cdot$$
$$\int_{\mathbb{C}^{2n}}d^{2n}\lambda\exp\left[-\frac{1}{2}R_\lambda^T\left(4\Omega^T T\left(\Sigma+\frac{1}{2}X\right)T^T\Omega\right)R_\lambda\right]\exp\left[-2\left(\bar{C}-u_{\alpha\beta}\right)^T\Omega T^H R_\lambda\right]$$
$$=e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\frac{1}{\pi^n}\sqrt{\frac{(2\pi)^{2n}}{\left|4T\left(\Sigma+\frac{1}{2}X\right)T^T\right|}}\cdot$$
$$\exp\left[2\left(\bar{C}-u_{\alpha\beta}\right)^T\Omega T^H\Omega^T T^*\left(\Sigma+\frac{1}{2}X\right)^{-1}T^H\Omega T^*\Omega^T\left(\bar{C}-u_{\alpha\beta}\right)\right]$$
$$=\frac{1}{\sqrt{\left|i\left(\Sigma+\frac{1}{2}X\right)\right|}}e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\cdot$$
$$\exp\left[-\frac{1}{2}\left(u_{\alpha\beta}-\bar{C}\right)^T\left(\Sigma+\frac{1}{2}X\right)^{-1}\left(u_{\alpha\beta}-\bar{C}\right)\right]$$
$$=\sqrt{|iW|}e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^T\alpha^*}\exp\left[-\frac{1}{2}\left(\bar{C}-u_{\alpha\beta}\right)^T W\left(\bar{C}-u_{\alpha\beta}\right)\right].$$

We now argue that $|iW|$ is real and positive. Notice that,

$$|iW|=\frac{1}{2^{2n}\left|T\left(\Sigma+\frac{1}{2}X\right)T^T\right|},$$

we know from the exercise on quadratures and wigner functions that,

$$T\Sigma T^T=k_{\mathrm{c}}^{-2}Q,$$

where Q is the covariance matrix. We know Q is positive definite. We likewise know that $TXT^T=\frac{1}{2}I$, and so we get,

$$|iW|=\frac{1}{2^{2n}\left|k_{\mathrm{c}}^{-2}Q+\frac{1}{4}I\right|}.$$

Since Q is positive definite, so too is $k_{\mathrm{c}}^{-2}Q+\frac{1}{4}I$. It follows that the determinant $\left|k_{\mathrm{c}}^{-2}Q+\frac{1}{4}I\right|$is positive, and so too is $|iW|$.

1.1.2 b)

I) We define,

$$W = \left(\Sigma + \frac{1}{2}X\right)^{-1} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

and so,

$$\begin{aligned} XWX = X\left(\Sigma + \frac{1}{2}X\right)^{-1} X \\ = \left(X\Sigma X + \frac{1}{2}X\right)^{-1} = \left(\Sigma^{*} + \frac{1}{2}X\right)^{-1} \\ = W^{*}. \end{aligned}$$

It then follows that,

$$XWX = \begin{pmatrix} W_{22} & W_{21} \\ W_{12} & W_{11} \end{pmatrix} = \begin{pmatrix} W_{11}^{*} & W_{12}^{*} \\ W_{21}^{*} & W_{22}^{*} \end{pmatrix},$$

implying that,

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^{*} & W_{11}^{*} \end{pmatrix}$$

as desired.

II) Secondly,

$$W^{T} = \left(\Sigma^{T} + \frac{1}{2}X^{T}\right)^{-1} = W$$

1.1.3 c)

Note that W is symmetric. Then we can rewrite as,

$$\begin{aligned} \left(u_{\alpha\beta} - \bar{C}\right)^{T} W \left(u_{\alpha\beta} - \bar{C}\right) = \\ = u_{\alpha\beta}^{T} W u_{\alpha\beta} - 2\bar{C}^{T} W u_{\alpha\beta} + \bar{C}^{T} W \bar{C} \end{aligned}$$

where

$$
\begin{aligned}
u_{\alpha\beta}^T W u_{\alpha\beta} &= \begin{pmatrix} \beta^T & \alpha^{*T} \end{pmatrix} \begin{pmatrix} W_D & W_A \\ W_A^* & W_D^* \end{pmatrix} \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix} \\
&= \beta^T W_D \beta + \beta^T W_A \alpha^* + \alpha^{*T} W_A^* \beta + \alpha^{*T} W_D^* \alpha^*
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{C}^T W u_{\alpha\beta} &= \begin{pmatrix} z^T & z^{*T} \end{pmatrix} \begin{pmatrix} W_D & W_A \\ W_A^* & W_D^* \end{pmatrix} \begin{pmatrix} \beta \\ \alpha^* \end{pmatrix} \\
&= \left(z^T W_D + z^{*T} W_A^*\right) \beta + \left(z^{*T} W_D^* + z^T W_A\right) \alpha^*
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{C}^T W \bar{C} &= \begin{pmatrix} z^T & z^{*T} \end{pmatrix} \begin{pmatrix} W_D & W_A \\ W_A^* & W_D^* \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix} \\
&= z^T W_D z + z^T W_A z^* + z^{*T} W_A^* z + z^{*T} W_D^* z^*
\end{aligned}
$$

Notice that $\left(\bar{C}^T W \bar{C}\right)^* = \bar{C}^T W \bar{C}$.

When inserting these relations we find that we can rearrange the matrix element into the form,

$$
\rho_G(\alpha, \beta) = \psi(\alpha, \alpha^*) \psi^*(\beta, \beta^*) e^{-\beta^T (W_A - I) \alpha^*}
$$

where we have used that $W_A^{*T} = W_A$ and we've defined,

$$
\psi(\alpha, \alpha^*) = \left[(-1)^n |W|\right]^{1/4} \exp\left[-\frac{1}{4} \bar{C}^T W \bar{C}\right] \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*T} W_D^* \alpha^* + \left(z^{*T} W_D^* + z^T W_A\right) \alpha^*\right]
$$

1.1.4 d)

I)

$$
\begin{aligned}
\Sigma = \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} V^T & J^H \\ J^T & V^H \end{pmatrix} \\
= \frac{1}{2} \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix} \begin{pmatrix} J^T & V^H \\ V^T & J^H \end{pmatrix} \\
= \frac{1}{2} \begin{pmatrix} V J^T + J V^T, & V V^H + J J^H \\ J^* J^T + V^* V^T, & J^* V^H + V^* J^H \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma + \frac{1}{2}X = \frac{1}{2}\begin{pmatrix} VJ^T + JV^T, & VV^H + JJ^H + I \\ J^*J^T + V^*V^T + I, & J^*V^H + V^*J^H \end{pmatrix} \\
= \begin{pmatrix} VJ^T, & VV^H \\ V^*V^T, & V^*J^H \end{pmatrix}
\end{aligned}
$$

where we made repeated use of the identities,

$$
\begin{aligned}
VJ^T - JV^T &= 0 \\
VV^H - JJ^H &= I.
\end{aligned}
$$

II) We verify the inverse by explicit calculation,

$$
\begin{aligned}
W\left(\Sigma + \frac{1}{2}X\right) \\
= \begin{pmatrix} -J^*V^{-1} & I \\ I & -JV^{-*} \end{pmatrix}\begin{pmatrix} VJ^T, & VV^H \\ V^*V^T, & V^*J^H \end{pmatrix} \\
= \begin{pmatrix} -J^*V^{-1}VJ^T + V^*V^T, & -J^*V^{-1}VV^H + V^*J^H \\ VJ^T - JV^{-*}V^*V^T, & VV^H - JV^{-*}V^*J^H \end{pmatrix} \\
= \begin{pmatrix} V^*V^T - J^*J^T, & V^*J^H - J^*V^H \\ VJ^T - JV^T, & VV^H - JJ^H \end{pmatrix} \\
= \begin{pmatrix} I, & 0 \\ 0, & I \end{pmatrix}
\end{aligned}
$$

1.1.5 e)

For a pure state we have $W_A = I$. We then have,

$$
\rho_G(\alpha, \beta) = \psi(\alpha, \alpha^*)\psi^*(\beta, \beta^*) = \langle \alpha | \psi_G \rangle \langle \psi_G | \beta \rangle
$$

It follows that we can identify the wavefunction up to a global phase as,

$$
\langle \alpha | \psi_G \rangle = \psi(\alpha, \alpha^*)
$$

1.2 Quadrature wavefunction

1.2.1 a)

$$
\begin{aligned}
&\langle x|\exp\left[-iR_Q^T\Omega R_\Lambda\right]|y\rangle \\
&=\langle x|\exp\left[+ip^T\Lambda_q-iq^T\Lambda_p\right]|y\rangle \\
&=e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}\langle x|\exp\left[+ip^T\Lambda_q\right]\exp\left[-iq^T\Lambda_p\right]|y\rangle \\
&=e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}\langle x|\exp\left[+ip^T\Lambda_q\right]|y\rangle e^{-iy^T\Lambda_p} \\
&=e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}\langle x|y-\frac{k_c^2}{2}\Lambda_q\rangle e^{-iy^T\Lambda_p} \\
&=e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}e^{-iy^T\Lambda_p}\delta\left(x-y+\frac{k_c^2}{2}\Lambda_q\right) \\
&=\left(\frac{2}{k_c^2}\right)^n e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}e^{-iy^T\Lambda_p}\delta\left(\Lambda_q-\frac{2}{k_c^2}(y-x)\right)
\end{aligned}
$$

1.2.2 b)

$$
\begin{aligned}
\langle x|\rho|y\rangle &=\left(\frac{k_c}{2}\right)^{2n}\frac{1}{\pi^n}\int_{\mathbb{R}^{2n}}d^{2n}R_\Lambda\chi_G^{(Q)}(K_\Lambda)\langle x|\exp\left[-iR_Q^T\Omega R_\Lambda\right]|y\rangle \\
&=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^{2n}}d^{2n}R_\Lambda\chi_G^{(Q)}(\Lambda_q,\Lambda_p)\,e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}e^{-iy^T\Lambda_p}\delta\left(\Lambda_q-\frac{2}{k_c^2}(y-x)\right) \\
&=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}d^n\Lambda_p\chi_G^{(Q)}\left(\frac{2}{k_c^2}(y-x),\Lambda_p\right)e^{i\frac{1}{2}(y-x)^T\Lambda_p}e^{-iy^T\Lambda_p}
\end{aligned}
$$

We have,

$$
e^{i\frac{1}{2}(y-x)^T\Lambda_p}e^{-iy^T\Lambda_p}=e^{-i\frac{1}{2}(x+y)^T\Lambda_p}
$$

1.2.3 c)

and

$$
\begin{aligned}
&\chi_G^{(Q)}(\Lambda_q,\Lambda_p) \\
&=\exp\left[-\frac{1}{2}\begin{pmatrix}\Lambda_p^T, & -\Lambda_q^T\end{pmatrix}\begin{pmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}\begin{pmatrix}\Lambda_p\\ -\Lambda_q\end{pmatrix}\right]\exp\left[i\bar{q}^T\Lambda_p-i\bar{p}^T\Lambda_q\right] \\
&=\exp\left[-\frac{1}{2}\left(\Lambda_p^TQ_{11}\Lambda_p-\Lambda_p^TQ_{12}\Lambda_q-\Lambda_q^TQ_{21}\Lambda_p+\Lambda_q^TQ_{22}\Lambda_q\right)\right]\exp\left[i\bar{q}^T\Lambda_p-i\bar{p}^T\Lambda_q\right]
\end{aligned}
$$

Note that $Q_{21}=Q_{12}^T$ and so,

$$
\begin{aligned}
&\chi_G^{(Q)}\left(\frac{2}{k_c^2}(y-x),\Lambda_p\right)e^{-i\frac{1}{2}(x+y)^T\Lambda_p}\\
&= \exp\left[-\frac{1}{2}\Lambda_p^T Q_{11}\Lambda_p + \left(-\left[Q_{12}\frac{2}{k_c^2}+i\frac{1}{2}I\right]x + \left[Q_{12}\frac{2}{k_c^2}-i\frac{1}{2}I\right]y + i\bar{q}\right)^T\Lambda_p\right]\\
&\cdot\exp\left[-\frac{2}{k_c^4}(y-x)^T Q_{22}(y-x) - i\frac{2}{k_c^2}\bar{p}^T(y-x)\right]
\end{aligned}
$$

To shorten notation, we define,

$$
\begin{aligned}
\left(\left[-\frac{2}{k_c^2}Q_{12}-i\frac{1}{2}I\right]x + \left[\frac{2}{k_c^2}Q_{12}-i\frac{1}{2}I\right]y + i\bar{q}\right)^T\Lambda_p &= \mu_{xy}^T\Lambda_p\\
\mu_{xy} &= \phi x - \phi^* y + i\bar{q}\\
\phi &= -\frac{2}{k_c^2}Q_{12} - i\frac{1}{2}I
\end{aligned}
$$

and also

$$
\begin{aligned}
-\frac{2}{k_c^4}(y-x)^T Q_{22}(y-x) &= v_{xy}^T U_1 v_{xy}\\
U_1 = \frac{2}{k_c^4}\begin{pmatrix} -Q_{22} & Q_{22}\\ Q_{22} & -Q_{22}\end{pmatrix}&, v_{xy} = \begin{pmatrix} x\\ y\end{pmatrix}
\end{aligned}
$$

and also

$$
\begin{aligned}
-i\frac{2}{k_c^2}(y-x)^T\bar{p} &= v_{xy}^T k_p\\
k_p &= i\frac{2}{k_c^2}\begin{pmatrix}\bar{p}\\ -\bar{p}\end{pmatrix}
\end{aligned}
$$

So we have,

$$
\begin{aligned}
&\chi_G^{(Q)}\left(\frac{2}{k_c^2}(y-x),\Lambda_p\right)e^{-i\frac{1}{2}(x+y)^T\Lambda_p}\\
&= \exp\left[v_{xy}^T U_1 v_{xy} + v_{xy}^T k_p\right]\exp\left[-\frac{1}{2}\Lambda_p^T Q_{11}\Lambda_p + \mu_{xy}^T\Lambda_p\right]
\end{aligned}
$$

1.2.4 d)

$$
\begin{aligned}
&\rho(x,y) \\
&= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} d^n\Lambda_p \chi_G^{(Q)}\left(\frac{2}{k_c^2}(y-x), \Lambda_p\right) e^{-i\frac{1}{2}(x+y)^T\Lambda_p} \\
&= \left(\frac{1}{2\pi}\right)^n \exp\left[v_{xy}^T U_1 v_{xy} + v_{xy}^T k_p\right] \int_{\mathbb{R}^n} d^n\Lambda_p \exp\left[-\frac{1}{2}\Lambda_p^T Q_{11}\Lambda_p + \mu_{xy}^T\Lambda_p\right] \\
&= \left[(2\pi)^n |Q_{11}|\right]^{-1/2} \exp\left[v_{xy}^T U_1 v_{xy} + v_{xy}^T k_p\right] \exp\left[\frac{1}{2}\mu_{xy}^T Q_{11}^{-1}\mu_{xy}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{2}\mu_{xy}^T Q_{11}^{-1}\mu_{xy} &= \frac{1}{2}\left(x^T\phi^T - y^T\phi^H + i\bar{q}^T\right) Q_{11}^{-1}\left(\phi x - \phi^* y + i\bar{q}\right) \\
&= \frac{1}{2}\left(x^T\phi^T - y^T\phi^H\right) Q_{11}^{-1}\left(\phi x - \phi^* y\right) + \left(x^T\phi^T - y^T\phi^H\right) iQ_{11}^{-1}\bar{q} - \frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}
\end{aligned}
$$

again we introduce some notation,

$$
\frac{1}{2}\left(x^T\phi^T - y^T\phi^H\right) Q_{11}^{-1}\left(\phi x - \phi^* y\right) = v_{xy}^T U_2 v_{xy}
$$

where

$$
U_2 = \frac{1}{2}\begin{pmatrix} \phi^T Q_{11}^{-1}\phi, & -\phi^T Q_{11}^{-1}\phi^* \\ -\phi^H Q_{11}^{-1}\phi, & \phi^H Q_{11}^{-1}\phi^* \end{pmatrix}
$$

and also

$$
\left(x^T\phi^T - y^T\phi^H\right) iQ_{11}^{-1}\bar{q} = v_{xy}^T k_q
$$

where

$$
k_q = \begin{pmatrix} \phi^T iQ_{11}^{-1}\bar{q} \\ -\phi^H iQ_{11}^{-1}\bar{q} \end{pmatrix}
$$

1.2.5 e)

Inserting the results from d),

$$
\rho(x,y) = \frac{\exp\left[-\frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/2}} \exp\left[v_{xy}^T\left(U_1 + U_2\right)v_{xy} + v_{xy}^T\left(k_p + k_q\right)\right]
$$

we define

$$U_{xy} = U_1 + U_2$$

$$= \left(\begin{array}{cc} -\frac{2}{k_c^4}Q_{22} + \frac{1}{2}\phi^T Q_{11}^{-1}\phi, & \frac{2}{k_c^4}Q_{22} - \frac{1}{2}\phi^T Q_{11}^{-1}\phi^* \\ \frac{2}{k_c^4}Q_{22} - \frac{1}{2}\phi^H Q_{11}^{-1}\phi, & -\frac{2}{k_c^4}Q_{22} + \frac{1}{2}\phi^H Q_{11}^{-1}\phi^* \end{array} \right)$$

and also

$$k_p + k_q$$

$$= \left(\begin{array}{c} i\frac{2}{k_c^2}\bar{p} + i\phi^T Q_{11}^{-1}\bar{q} \\ -i\frac{2}{k_c^2}\bar{p} - i\phi^H Q_{11}^{-1}\bar{q} \end{array} \right) = \left(\begin{array}{c} \mu_{qp} \\ \mu_{qp}^* \end{array} \right)$$

where

$$\mu_{qp} = i\frac{2}{k_c^2}\bar{p} + \phi^T i Q_{11}^{-1}\bar{q}$$

and so,

$$\rho(x,y) = \frac{\exp\left[-\frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/2}} \exp\left[v_{xy}^T U_{xy} v_{xy} + v_{xy}^T \left(\begin{array}{c} \mu_{qp} \\ \mu_{qp}^* \end{array} \right)\right]$$

1.2.6 f)

In order that $\rho(x,y)$ factors as,

$$\rho(x,y) = \psi(x)\psi^*(y)$$

i.e. the state is pure. Then all terms coupling x and y must vanish. This can only happen if the sum of cross terms yield zero,

$$x^T \left(\frac{2}{k_c^4}Q_{22} - \frac{1}{2}\phi^T Q_{11}^{-1}\phi^* \right) y + y^T \left(\frac{2}{k_c^4}Q_{22} - \frac{1}{2}\phi^H Q_{11}^{-1}\phi \right) x$$
$$= x^T \left(\frac{4}{k_c^4}Q_{22} - \phi^T Q_{11}^{-1}\phi^* \right) y = 0,$$

and so the requirement for a pure state is,

$$\frac{4}{k_c^4}Q_{22} - \phi^T Q_{11}^{-1}\phi^* = 0.$$

We define,

$$\Gamma_x = \frac{2}{k_{\rm c}^2} Q_{22} - \frac{k_{\rm c}^2}{2} \phi^T Q_{11}^{-1} \phi.$$

In the case that the above requirement is satisfied, we can write $\rho(x, y)$ as,

$$\rho(x, y) = \frac{\exp\left[-\frac{1}{2}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/2}} \exp\left[-x^T \frac{1}{k_{\rm c}^2}\Gamma_x x + x^T \mu_{qp} - y^T \frac{1}{k_{\rm c}^2}\Gamma_x^* y + y^T \mu_{qp}^*\right]$$

We can then identify the wavefunction as,

$$\psi(x) = \frac{\exp\left[-\frac{1}{4}\bar{q}^T Q_{11}^{-1}\bar{q}\right]}{\left[(2\pi)^n |Q_{11}|\right]^{1/4}} \exp\left[-x^T \frac{1}{k_{\rm c}^2}\Gamma_x x + x^T \mu_{qp}\right]$$

1.2.7 g)

I)

$$Q_{22} = \frac{k_{\rm c}^4}{4} \phi^T Q_{11}^{-1} \phi^*,$$

then,

$$\Gamma_x = \frac{k_{\rm c}^2}{2} \phi^T Q_{11}^{-1} \left(\phi^* - \phi\right)$$

where

$$\phi = -\frac{2}{k_{\rm c}^2} Q_{12} - i\frac{1}{2} I$$
$$\phi^* - \phi = iI$$

and so,

$$\Gamma_x = \frac{k_{\rm c}^2}{4} Q_{11}^{-1} - iQ_{21}Q_{11}^{-1}$$

II) We define,

$$\begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix} = \frac{1}{4} S_+ S_+^T$$

where each submatrix is clearly independent of k_c^2.
Then,

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = k_\mathrm{c}^2 \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix}$$

Examining Γ_x,

$$\begin{aligned} \Gamma_x = \frac{k_\mathrm{c}^2}{4} \left(k_\mathrm{c}^2 \tilde{Q}_{11} \right)^{-1} - i k_\mathrm{c}^2 \tilde{Q}_{21} \left(k_\mathrm{c}^2 \tilde{Q}_{11} \right)^{-1} \\ = \frac{1}{4} \tilde{Q}_{11}^{-1} - i \tilde{Q}_{21} \tilde{Q}_{11}^{-1} \end{aligned}$$

III)

$$|Q_{11}| = \left| k_\mathrm{c}^2 \tilde{Q}_{11} \right| = \left(k_\mathrm{c}^2 \right)^n \left| \tilde{Q}_{11} \right|$$