1 Some representation theory

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1.0.1 a)

We will use the identities,

$$\begin{aligned}
\partial_{\alpha^*} e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} &= \left(a - \frac{1}{2}\alpha\right) e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} \\
\partial_{\alpha} e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha} &= \left(-a^\dagger + \frac{1}{2}\alpha^*\right) e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha}
\end{aligned}$$

which we can rewrite as,

$$\begin{aligned}
e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} a &= \left(\partial_{\alpha^*} + \frac{1}{2}\alpha\right) e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} \\
e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha} a^\dagger &= \left(\frac{1}{2}\alpha^* - \partial_{\alpha}\right) e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha}.
\end{aligned}$$

Then we get,

$$\begin{aligned}
\langle\alpha|a|\psi\rangle &= \langle 0|D(-\alpha)a|\psi\rangle \\
&= \langle 0|e^{-\alpha a^\dagger} e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*} a|\psi\rangle \\
&= \left(\partial_{\alpha^*} + \frac{1}{2}\alpha\right) \langle 0|e^{-\alpha a^\dagger} e^{\alpha^* a} e^{-\frac{1}{2}\alpha\alpha^*}|\psi\rangle \\
&= \left(\partial_{\alpha^*} + \frac{1}{2}\alpha\right) \langle\alpha|\psi\rangle
\end{aligned}$$

Likewise,

$$\begin{aligned}
\langle\alpha|a^\dagger|\psi\rangle &= \langle 0|D(-\alpha)a^\dagger|\psi\rangle \\
&= \langle 0|e^{\alpha^* a} e^{-\alpha a^\dagger} e^{\frac{1}{2}\alpha^*\alpha} a^\dagger|\psi\rangle \\
&= \left(\frac{1}{2}\alpha^* - \partial_{\alpha}\right) \langle\alpha|\psi\rangle
\end{aligned}$$

1.0.2 b)

$$
\begin{aligned}
\langle\psi|a^{\dagger n}a^m|\psi\rangle &= \langle\psi|\frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha|\alpha\rangle\langle\alpha|a^{\dagger n}a^m|\psi\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\langle\psi|\alpha\rangle\alpha^{*n}\langle\alpha|a^m|\psi\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\langle\psi|\alpha\rangle\alpha^{*n}\langle\alpha|a\left(a^{m-1}|\psi\rangle\right) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\langle\psi|\alpha\rangle\alpha^{*n}\left(\partial_{\alpha^*}+\frac{1}{2}\alpha\right)\langle\alpha|a\left(a^{m-2}|\psi\rangle\right) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\langle\psi|\alpha\rangle\alpha^{*n}\left(\partial_{\alpha^*}+\frac{1}{2}\alpha\right)^m\langle\alpha|\psi\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\psi^*(\alpha,\alpha^*)\alpha^{*n}\left(\partial_{\alpha^*}+\frac{1}{2}\alpha\right)^m\psi(\alpha,\alpha^*)
\end{aligned}
$$

1.0.3 c)

$$
\begin{aligned}
\langle x|a|\psi\rangle &= \langle x|k_{\mathrm{c}}^{-1}\left(q+ip\right)|\psi\rangle \\
&= k_{\mathrm{c}}^{-1}\left(x+\frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi(x)
\end{aligned}
$$

1.0.4 d)

We have the condition

$$
k_{\mathrm{c}}^{-1}\left(x+\frac{k_{\mathrm{c}}^2}{2}\partial_x\right)\psi_\alpha(x) = \alpha\psi_\alpha(x),
$$

rearranging we have the differential equation,

$$
\partial_x\psi_\alpha(x) = \left(\frac{2}{k_{\mathrm{c}}}\alpha - \frac{2}{k_{\mathrm{c}}^2}x\right)\psi_\alpha(x),
$$

which is a standard differential equation with solution,

$$
\psi_\alpha(x) = \mathcal{N}\exp\left[-\frac{x^2}{k_{\mathrm{c}}^2}+2\alpha\frac{x}{k_{\mathrm{c}}}\right].
$$

The normalization $\mathcal{N}$ is obtained by integration,

$$
\begin{aligned}
&\int_{\mathbb{R}} dx \, |\psi_\alpha(x)|^2 \\
&= |\mathcal{N}|^2 \int_{\mathbb{R}} dx \exp\left[-\frac{1}{2} x \left(\frac{4}{k_{\mathrm{c}}^2}\right) x + 4\alpha_R \frac{x}{k_{\mathrm{c}}}\right] \\
&= |\mathcal{N}|^2 \, k_{\mathrm{c}} \sqrt{\frac{\pi}{2}} \exp\left[2\alpha_R^2\right] = 1
\end{aligned}
$$

and so

$$
|\mathcal{N}| = \left(\frac{2}{\pi}\right)^{1/4} k_{\mathrm{c}}^{-1/2} \exp\left[-\alpha_R^2\right]
$$