1 Solutions for Quadratures and Wigner functions

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1.1 Gaussian transformations

1.1.1 a)

I)

$$e^{itH_G} C_a e^{-itH_G} = M_+ C_a + D_+$$

Then we have,

$$\begin{aligned} e^{itH_G} R_Q e^{-itH_G} &= e^{itH_G} T_k C_a e^{-itH_G} \\ &= T_k \left(e^{itH_G} C_a e^{-itH_G} \right) \\ &= T_k M_+ C_a + T_k D_+ \\ &= T_k M_+ T_k^{-1} T_k C_a + T_k D_+ \\ &= T_k M_+ T_k^{-1} R_Q + T_k D_+ . \end{aligned}$$

By direct calculation,

$$\begin{aligned} S_+ &= T_k M_+ T_k^{-1} \\ &= \frac{1}{2} \begin{pmatrix} I, & I \\ -iI, & iI \end{pmatrix} \begin{pmatrix} V & J \\ J^* & V^* \end{pmatrix} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} I, & I \\ -iI, & iI \end{pmatrix} \begin{pmatrix} V + J, & i\left(V - J\right) \\ J^* + V^*, & i\left(J^* - V^*\right) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \left(V + V^* + J + J^*\right), & i\left(V - J + J^* - V^*\right) \\ i\left(J^* + V^* - V - J\right), & \left(V + V^* - J - J^*\right) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}\left\{V\right\} + \mathrm{Re}\left\{J\right\}, & \mathrm{Im}\left\{J\right\} - \mathrm{Im}\left\{V\right\} \\ \mathrm{Im}\left\{V\right\} + \mathrm{Im}\left\{J\right\}, & \mathrm{Re}\left\{V\right\} - \mathrm{Re}\left\{J\right\} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}\left\{V + J\right\}, & \mathrm{Im}\left\{J - V\right\} \\ \mathrm{Im}\left\{V + J\right\}, & \mathrm{Re}\left\{V - J\right\} \end{pmatrix} \end{aligned}$$

We see that S_+does indeed only have real elements.
Likewise for the displacement μ_+,

$$\begin{aligned} \mu_+ = T_k D_+ &= k_{\mathrm{c}} \begin{pmatrix} \frac{1}{2} I, & \frac{1}{2} I \\ -\frac{1}{2} iI, & \frac{1}{2} iI \end{pmatrix} \begin{pmatrix} d \\ d^* \end{pmatrix} \\ &= k_{\mathrm{c}} \begin{pmatrix} \frac{d+d^*}{2} \\ -i\frac{d-d^*}{2} \end{pmatrix} = k_{\mathrm{c}} \begin{pmatrix} \mathrm{Re}\left\{d\right\} \\ \mathrm{Im}\left\{d\right\} \end{pmatrix} \end{aligned}$$

and we see that μ_+ only has real elements.

II) We have,

$$\begin{aligned}
&S_+\Omega S_+^T \\
&= TM_+T^{-1}\Omega T^{-T}M_+^T T^T \\
&= 4TM_+T^H\Omega T^* M_+^T T^T \\
&= 4T\left(-\frac{i}{2}\Omega\right)T^T \\
&= \Omega
\end{aligned}$$

1.1.2 b)

$$\begin{aligned}
e^{-itH_G}R_Q e^{itH_G} &= e^{-itH_G}T_k C_a e^{itH_G} \\
&= T_k\left(e^{-itH_G}C_a e^{itH_G}\right) \\
&= T_k M_- C_a + T_k D_- \\
&= T_k M_- T_k^{-1} T_k C_a + T_k D_- \\
&= TM_- T^{-1} R_Q + T_k D_- \\
&= TM_+^{-1}T^{-1}R_Q + \left(-T_k M_+^{-1}D_+\right),
\end{aligned}$$

and so

$$\begin{aligned}
S_- = TM_+^{-1}T^{-1} &= \left(TM_+T^{-1}\right)^{-1} \\
&= S_+^{-1}.
\end{aligned}$$

Likewise for the displacement,

$$\begin{aligned}
\mu_- &= -T_k M_+^{-1}D_+ \\
&= -T_k M_+^{-1}T_k^{-1}T_k D_+ \\
&= -S_+^{-1}T_k D_+ = -S_+^{-1}\mu_+
\end{aligned}$$

Note that since,

$$S_+\Omega S_+^T = \Omega$$

then,

$$\Omega S_+^T = S_+^{-1}\Omega$$

and finally $S_+^{-1} = \Omega S_+^T \Omega^T$.

1.1.3 c)

$$\begin{aligned}|S_+| &= \left|TM_+T^{-1}\right| \\ &= |T|\,|M_+|\,|T|^{-1} \\ &= |M_+| = 1\end{aligned}$$

See https://en.wikipedia.org/wiki/Determinant

1.2 Glauber's formula and the Wigner characteristic function

1.2.1 a)

$$\begin{aligned}D(\alpha) &= \exp\left[C_\alpha^T \Omega C_a\right] \\ &= \exp\left[C_\alpha^T \Omega T_k^{-1} R_Q\right] \\ &= \exp\left[2k_\mathrm{c}^{-1} C_\alpha^T \Omega T^H R_Q\right],\end{aligned}$$

we have the identity,

$$\Omega T^H = -iT^T\Omega,$$

and so,

$$\begin{aligned}D(\alpha) &= \exp\left[-i\left(2k_\mathrm{c}^{-1}TC_\alpha\right)^T \Omega R_Q\right] \\ &= \exp\left[-iR_\Lambda^T \Omega R_Q\right] = \exp\left[iR_Q^T \Omega R_\Lambda\right],\end{aligned}$$

where

$$R_\Lambda = 2k_\mathrm{c}^{-1}TC_\alpha = 2k_\mathrm{c}^{-1}\begin{pmatrix}\alpha_R \\ \alpha_I\end{pmatrix}.$$

1.2.2 b)

Using the result of a) we have,

$$\rho = \frac{1}{\pi^n}\int_{\mathbb{C}^n} d^{2n}\alpha \chi_\rho\left(C_\alpha\right)\exp\left[-iR_Q^T\Omega R_\Lambda\right].$$

We change integration variables to,

$$R_\Lambda = \begin{pmatrix}\Lambda_q \\ \Lambda_p\end{pmatrix} = 2k_\mathrm{c}^{-1}TC_\alpha = 2k_\mathrm{c}^{-1}\begin{pmatrix}\alpha_R \\ \alpha_I\end{pmatrix}.$$

Then we have,

$$d^n \alpha_R d^n \alpha_I = \left(\frac{k_{\mathrm{c}}}{2}\right)^{2n} d^n \Lambda_q d^n \Lambda_p = \left(\frac{k_{\mathrm{c}}}{2}\right)^{2n} d^{2n} R_\Lambda,$$

and

$$C_\alpha = \frac{k_{\mathrm{c}}}{2} T^{-1} R_\Lambda = k_{\mathrm{c}} T^H R_\Lambda.$$

Then we obtain the result,

$$\rho = \left(\frac{k_{\mathrm{c}}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n} R_\Lambda \chi_\rho \left(k_{\mathrm{c}} T^H R_\Lambda\right) \exp\left[-i R_Q^T \Omega R_\Lambda\right].$$

1.2.3 c)

$$\begin{aligned}
\chi_\rho^{(Q)}\left(R_\Lambda\right) &= \chi_\rho\left(k_{\mathrm{c}} T^H R_\Lambda\right) \\
&= \operatorname{Tr}\left\{\rho \exp\left[C_\alpha^T \Omega C_a\right]\right\} \\
&= \operatorname{Tr}\left\{\rho \exp\left[\left(k_{\mathrm{c}} T^H R_\Lambda\right)^T \Omega C_a\right]\right\} \\
&= \operatorname{Tr}\left\{\rho \exp\left[k_{\mathrm{c}} R_\Lambda^T T^* \Omega T_k^{-1} R_Q\right]\right\} \\
&= \operatorname{Tr}\left\{\rho \exp\left[2 R_\Lambda^T \left(\frac{i}{2}\Omega\right)^* R_Q\right]\right\} \\
&= \operatorname{Tr}\left\{\rho \exp\left[-i R_\Lambda^T \Omega R_Q\right]\right\} \\
&= \operatorname{Tr}\left\{\rho \exp\left[i R_Q^T \Omega R_\Lambda\right]\right\}.
\end{aligned}$$

1.2.4 d)

We have for a single mode

$$\operatorname{Tr}\left\{D(\alpha)\right\} = \pi \delta\left(\alpha_I\right) \delta\left(\alpha_R\right)$$

Generalizing to n-modes we must have,

$$\operatorname{Tr}\left\{D(C_\alpha)\right\} = \pi^n \delta^{(2n)}\left(T C_\alpha\right)$$

Then,

$$
\begin{aligned}
\mathrm{Tr}\left\{\exp\left[iR_Q^T\Omega R_\Lambda\right]\right\} &= \mathrm{Tr}\left\{D(C_\alpha)\right\} \\
= \pi^n\delta^{(2n)}\left(TC_\alpha\right) &= \pi^n\delta^{(2n)}\left(\frac{k_\mathrm{c}}{2}R_\Lambda\right) \\
&= \pi^n\left(\frac{2}{k_\mathrm{c}}\right)^{2n}\delta^{(2n)}\left(R_\Lambda\right) \\
&= \left(\frac{4\pi}{k_\mathrm{c}^2}\right)^n\delta^{(n)}\left(\Lambda_q\right)\delta^{(n)}\left(\Lambda_p\right)
\end{aligned}
$$

1.2.5 e)

$$
\bar{R} = \mathrm{Tr}\left\{\rho R_Q\right\} = T_k\mathrm{Tr}\left\{\rho C_a\right\} = T_k\bar{C}
$$

and

$$
\begin{aligned}
Q &= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(R_Q\otimes R_Q^T + R_Q^T\otimes R_Q\right)\right\} - \mathrm{Tr}\left\{\rho R_Q^T\right\}\otimes\mathrm{Tr}\left\{\rho R_Q\right\} \\
&= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(T_kC_a\otimes C_a^TT_k^T + C_a^TT_k^T\otimes T_kC_a\right)\right\} - \mathrm{Tr}\left\{\rho C_a^TT_k^T\right\}\otimes\mathrm{Tr}\left\{\rho T_kC_a\right\} \\
&= \frac{1}{2}\mathrm{Tr}\left\{\rho\left(T_kC_a\otimes C_a^TT_k^T + \left(T_kC_a\otimes C_a^TT_k^T\right)^T\right)\right\} - \left(T_k\mathrm{Tr}\left\{\rho C_a\right\}\otimes\mathrm{Tr}\left\{\rho C_a^T\right\}T_k^T\right)^T \\
&= T_k\frac{1}{2}\mathrm{Tr}\left\{\rho\left(C_a\otimes C_a^T + C_a^T\otimes C_a\right)\right\}T_k^T - T_k\mathrm{Tr}\left\{\rho C_a^T\right\}\otimes\mathrm{Tr}\left\{\rho C_a\right\}T_k^T \\
&= T_k\Sigma T_k^T
\end{aligned}
$$

1.2.6 f)

$$
\begin{aligned}
Q &= k_\mathrm{c}^2T\Sigma T^T = k_\mathrm{c}^2TM_+\Sigma_\mathrm{Th}M_+^TT^T \\
&= k_\mathrm{c}^2TM_+T^{-1}T\Sigma_\mathrm{Th}T^T\left(TM_+T^{-1}\right)^T \\
&= k_\mathrm{c}^2S_+T\Sigma_\mathrm{Th}T^TS_+^T
\end{aligned}
$$

and

$$
\begin{aligned}
T\Sigma_\mathrm{Th}T^T &= \begin{pmatrix} \frac{1}{2}I & \frac{1}{2}I \\ -i\frac{1}{2}I & i\frac{1}{2}I \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & \nu_\mathrm{th} \\ \nu_\mathrm{th} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}I & -i\frac{1}{2}I \\ \frac{1}{2}I & i\frac{1}{2}I \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} \nu_\mathrm{th} & 0 \\ 0 & \nu_\mathrm{th} \end{pmatrix}
\end{aligned}
$$

So we have the claimed identity,

$$
Q = \frac{k_\mathrm{c}^2}{4}S_+\begin{pmatrix} \nu_\mathrm{th} & 0 \\ 0 & \nu_\mathrm{th} \end{pmatrix}S_+^T.
$$

Since S_+ and ν_{th} are real, then Q is real. Q is clearly symmetric. To see that V is positive definite, we note that S_+ is invertible, and so an arbitrary real vector u, can be written in the form,

$$u = S_+^{-T} v$$

where v is another real vector.
Then,

$$\begin{aligned} u^T V u &= \frac{k_c^2}{4} u^T S_+ \begin{pmatrix} \nu_{\text{th}} & 0 \\ 0 & \nu_{\text{th}} \end{pmatrix} S_+^T u \\ &= \frac{k_c^2}{4} v^T \begin{pmatrix} \nu_{\text{th}} & 0 \\ 0 & \nu_{\text{th}} \end{pmatrix} v, \end{aligned}$$

since ν_{th} is diagonal with all entries greater than 1, it follows that Q is positive definite.

1.2.7 g)

We have,

$$\chi_G(C_\alpha) = \exp\left[\frac{1}{2} C_\alpha^T \Omega^T \Sigma \Omega C_\alpha\right] \exp\left[-\bar{C}^T \Omega C_\alpha\right]$$

and then,

$$\begin{aligned} \chi_G^{(Q)}\left(R_\Lambda\right) &= \chi_G\left(k_c T^H R_\Lambda\right) \\ &= \exp\left[\frac{1}{2} k_c^2 R_\Lambda^T T^* \Omega^T \Sigma \Omega T^H R_\Lambda\right] \exp\left[-k_c \bar{C}^T \Omega T^H R_\Lambda\right] \end{aligned}$$

we have,

$$\begin{aligned} \Omega T^H &= -i T^T \Omega \\ T^* \Omega^T &= -i \Omega^T T \\ T_k &= k_c T \end{aligned}$$

and so

$$\begin{aligned} \chi_G^{(Q)}\left(R_\Lambda\right) &= \exp\left[-\frac{1}{2} R_\Lambda^T \Omega^T T_k \Sigma T_k^T \Omega R_\Lambda\right] \exp\left[i\left(T_k \bar{C}\right)^T \Omega R_\Lambda\right] \\ &= \exp\left[-\frac{1}{2} R_\Lambda^T \Omega^T Q \Omega R_\Lambda\right] \exp\left[i \bar{R}^T \Omega R_\Lambda\right] \end{aligned}$$

1.2.8 h)

We have the action of a gaussian transformation,

$$\chi_f(C_\alpha) = \chi_i\left(M_+^{-1}C_\alpha\right)\exp\left[C_\alpha^T\Omega D_+\right].$$

We have the relations,

$$\begin{aligned}\chi_f^{(Q)}\left(R_\Lambda\right) &= \chi_f\left(k_c T^H R_\Lambda\right)\\ \chi_i^{(Q)}\left(k_c^{-1}T^{-H}C_\alpha\right) &= \chi_i\left(C_\alpha\right),\end{aligned}$$

and so,

$$\begin{aligned}\chi_f^{(Q)}(R_\Lambda) &= \chi_i\left(M_+^{-1}k_c T^H R_\Lambda\right)\exp\left[k_c R_\Lambda^T T^*\Omega D_+\right]\\ &= \chi_i^{(Q)}\left(T^{-H}M_+^{-1}T^H R_\Lambda\right)\exp\left[k_c R_\Lambda^T T^*\Omega D_+\right].\end{aligned}$$

We have,

$$\begin{aligned}T^H &= \frac{1}{2}T^{-1}\\ T^*\Omega &= -i\Omega T\end{aligned}$$

and so,

$$\begin{aligned}\chi_f^{(Q)}(R_\Lambda) = \chi_i^{(Q)}&\left(\left(T_k M_+ T_k^{-1}\right)^{-1}R_\Lambda\right)\exp\left[-iR_\Lambda^T\Omega T_k D_+\right]\\ &= \chi_i^{(Q)}\left(S_+^{-1}R_\Lambda\right)\exp\left[-iR_\Lambda^T\Omega\mu_+\right]\end{aligned}$$

1.2.9 i)

Given that the initial state had covariance matrix Σ_i and displacement $\bar{C}_i$, then we have from the exercise on gaussian transformations, that the final displacement and covariance matrix are,

$$\begin{aligned}\bar{C}_f &= M_+\bar{C}_i + D_+\\ \Sigma_f &= M_+\Sigma_i M_+^T.\end{aligned}$$

Using the result from e) we can convert this to the quadrature representation,

$$\begin{aligned}\bar{R}_f &= T_k M_+ T_k^{-1}\bar{R}_i + T_k D_+\\ &= S_+\bar{R}_i + \mu_+\end{aligned}$$

and

$$\begin{aligned} Q_f &= T_k M_+ T_k^{-1} T_k \Sigma_i T_k^T T_k^{-T} M_+^T T_k^T \\ &= S_+ Q_i S_+^T. \end{aligned}$$

The quadrature characteristic function of a gaussian state ρ_i is given in g),

$$\chi_i^{(Q)}\left(R_\Lambda\right) = \exp\left[-\frac{1}{2}R_\Lambda^T \Omega^T Q_i \Omega R_\Lambda\right] \exp\left[i\bar{R}_i{}^T \Omega R_\Lambda\right].$$

Transforming the state as,

$$\rho_f = e^{-itH_G}\rho_i e^{itH_G},$$

we know from h) that the quadrature characteristic function of the final state is,

$$\begin{aligned} \chi_f^{(Q)}\left(R_\Lambda\right) &= \exp\left[i\mu_+^T \Omega R_\Lambda\right] \chi_i^{(Q)}\left(S_+^{-1} R_\Lambda\right) \\ = \exp\left[i\mu_+^T \Omega R_\Lambda\right] \exp\left[-\frac{1}{2}\left(S_+^{-1}R_\Lambda\right)^T \Omega^T Q_i \Omega S_+^{-1} R_\Lambda\right] &\exp\left[i\bar{R}_i{}^T \Omega S_+^{-1} R_\Lambda\right] \\ = \exp\left[i\mu_+^T \Omega R_\Lambda\right] \exp\left[-\frac{1}{2}R_\Lambda^T S_+^{-T} \Omega^T Q_i \Omega S_+^{-1} R_\Lambda\right] &\exp\left[i\bar{R}_i{}^T \Omega S_+^{-1} R_\Lambda\right] \\ = \exp\left[-\frac{1}{2}R_\Lambda^T \Omega^T S_+ Q_i S_+^T \Omega R_\Lambda\right] &\exp\left[i\left(\mu_+ + S_+\bar{R}_i\right)^T \Omega R_\Lambda\right] \\ = \exp\left[-\frac{1}{2}R_\Lambda^T \Omega^T \left(S_+ Q_i S_+^T\right) \Omega R_\Lambda\right] &\exp\left[i\left(\mu_+ + S_+\bar{R}_i\right)^T \Omega R_\Lambda\right]. \end{aligned}$$

1.3 Wigner function

1.3.1 a)

We seek to verify that with the given definition of $W_\rho(R_X)$ we have,

$$\left\langle e^{iR_Q^T \Omega R_\Lambda}\right\rangle = \int_{\mathbb{R}^{2n}} d^{2n} R_X W_\rho(R_X) e^{iR_X^T \Omega R_\Lambda}.$$

By explicit calculation and resolving a delta-function, we get,

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho(R_X) e^{iR_X^T\Omega R_\Lambda} \\
&= \int_{\mathbb{R}^{2n}} d^{2n}R_Y \chi_\rho^{(Q)}\left(R_Y\right) \frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_X e^{iR_X^T\Omega(R_\Lambda - R_Y)} \\
&= \int_{\mathbb{R}^{2n}} d^{2n}R_Y \chi_\rho^{(Q)}\left(R_Y\right) \delta\left(R_\Lambda - R_Y\right) \\
&= \chi_\rho^{(Q)}\left(R_\Lambda\right) = \left\langle e^{iR_Q^T\Omega R_\Lambda}\right\rangle.
\end{aligned}
$$

1.3.2 b)

I)

$$
\begin{aligned}
&\int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho(R_X) \\
&= \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_\rho^{(Q)}\left(R_\Lambda\right) \frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_X e^{-iR_X^T\Omega R_\Lambda} \\
&= \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_\rho^{(Q)}\left(R_\Lambda\right) \delta\left(R_\Lambda\right) \\
&= \chi_\rho^{(Q)}\left(0\right) = \mathrm{Tr}\left\{\rho\right\} = 1
\end{aligned}
$$

II) The expectation value A is,

$$
\begin{aligned}
\langle A\rangle = \mathrm{Tr}\left\{\rho A\right\} &= \left(\frac{k_\mathrm{c}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_A^{(Q)}\left(R_\Lambda\right) \mathrm{Tr}\left\{\rho \exp\left[-iR_Q^T\Omega R_\Lambda\right]\right\} \\
&= \left(\frac{k_\mathrm{c}}{2}\right)^{2n} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_A^{(Q)}\left(R_\Lambda\right) \left[\int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho\left(R_X\right) e^{-iR_X^T\Omega R_\Lambda}\right] \\
&= \left(\frac{k_\mathrm{c}}{2}\right)^{2n} \frac{\left(2\pi\right)^{2n}}{\pi^n} \int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho\left(R_X\right) \left[\frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \chi_A^{(Q)}\left(R_\Lambda\right) e^{-iR_X^T\Omega R_\Lambda}\right] \\
&= \left(\pi k_\mathrm{c}^2\right)^n \int_{\mathbb{R}^{2n}} d^{2n}R_X W_\rho\left(R_X\right) W_A(R_X)
\end{aligned}
$$

1.3.3 c)

$$
W_\rho(R_X) = \frac{1}{\left(2\pi\right)^{2n}} \int_{\mathbb{R}^{2n}} d^{2n}R_\Lambda \mathrm{Tr}\left\{\rho e^{iR_Q^T\Omega R_\Lambda}\right\} e^{-iR_X^T\Omega R_\Lambda}
$$

We perform the trace in the position basis,

$$
\begin{aligned}
&\mathrm{Tr}\left\{\rho e^{iR_Q^T\Omega R_\Lambda}\right\} \\
&= \int_{\mathbb{R}^n} d^n x \langle x|\rho e^{iR_Q^T\Omega R_\Lambda}|x\rangle \\
&= \int_{\mathbb{R}^n} d^n x \langle x|\rho e^{-i\frac{k_c^2}{4}\Lambda_p^T\Lambda_q} e^{iq^T\Lambda_p} e^{-ip^T\Lambda_q}|x\rangle
\end{aligned}
$$

We make the identification,

$$
e^{-ip^T\Lambda_q}|x\rangle = |x+\Lambda_q\frac{k_c^2}{2}\rangle
$$

and so,

$$
\mathrm{Tr}\left\{\rho e^{iR_Q^T\Omega R_\Lambda}\right\} = e^{i\frac{k_c^2}{4}\Lambda_q^T\Lambda_p}\int_{\mathbb{R}^n} d^n x e^{ix^T\Lambda_p}\langle x|\rho|x+\Lambda_q\frac{k_c^2}{2}\rangle
$$

and so we have the Wigner function,

$$
\begin{aligned}
W_\rho(R_X) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d^n\Lambda_q \int_{\mathbb{R}^n} d^n x e^{iX_p^T\Lambda_q}\langle x|\rho|x+\Lambda_q\frac{k_c^2}{2}\rangle \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d^n\Lambda_p e^{i\left(\frac{k_c^2}{4}\Lambda_q+x-X_q\right)^T\Lambda_p} \\
&= \left(\frac{4}{k_c^2}\right)^n \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d^n\Lambda_q \int_{\mathbb{R}^n} d^n x e^{iX_p^T\Lambda_q}\langle x|\rho|x+\Lambda_q\frac{k_c^2}{2}\rangle \delta^{(n)}\left(\Lambda_q - \frac{4}{k_c^2}\left[X_q-x\right]\right) \\
&= \left(\frac{4}{k_c^2}\right)^n \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} d^n x e^{i4k_c^{-2}X_p^T[X_q-x]}\langle x|\rho|2X_q-x\rangle
\end{aligned}
$$

We shift to the integration variable $y = X_q - x$, then,

$$
W_\rho(R_X) = \left(\frac{2}{\pi k_c^2}\right)^n \int_{\mathbb{R}^n} d^n y e^{i4k_c^{-2}X_p^T y}\langle X_q - y|\rho|X_q + y\rangle
$$