Proof that gaussian states together with quadrature measurements result in local statistics

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1 Problem statement

We seek to prove that gaussian states together with quadrature measurements and displacements are local. Suppose N participants perform quadrature measurements on a shared state, obtaining outcomes $a = (a_1, a_2, \cdots, a_N)$ for settings $\theta = (\theta_1, \theta_2, \cdots, \theta_N)$. The settings θ will correspond to the phase of the local oscillator (LO). Then the probability density $\rho(a|\theta)$ will be local, provided we can make the decomposition,

$$\rho(a|\theta) = \int d\lambda \rho(a|\theta, \lambda)\rho(\lambda) \tag{1}$$

where the probability densities satisfy,

$$\int d\lambda \rho(\lambda) = 1 \tag{2}$$

$$\rho(\lambda) \geq 0 \tag{3}$$

$$\int da \rho(a|\theta, \lambda) = 1 \tag{4}$$

$$\rho(a|\theta, \lambda) \geq 0 \tag{5}$$

and the conditional probability density factorizes

$$\rho(a|\theta, \lambda) = \prod_{k=1}^{N} \rho_k(a_k|\theta_k, \lambda) \tag{6}$$

2 Analysis

2.1 Definitions

The Wigner function of a general N-partite gaussian state can be written as,

$$W(X) = \frac{\exp\left[-\frac{1}{2}\left(X - \langle R\rangle\right)^T V^{-1}\left(X - \langle R\rangle\right)\right]}{(2\pi)^N\sqrt{|V|}} \tag{7}$$

We use the operator ordering $(q_1, p_1, q_2, p_2 \cdots q_N, p_N)$ and X is a vector of coordinates $X = \left(X_q^{(1)}, X_p^{(1)}, X_q^{(2)}, X_p^{(2)}, \cdots, X_q^{(N)}, X_p^{(N)}\right)$. We also construct the vectors,

$$X_q = \left(X_q^{(1)}, X_q^{(2)}, \cdots, X_q^{(N)}\right) \tag{8}$$

$$X_p = \left(X_p^{(1)}, X_p^{(2)}, \cdots, X_p^{(N)}\right) \tag{9}$$

$$X^{(k)} = \left(X_q^{(k)}, X_p^{(k)}\right) \tag{10}$$

Rather than changing the phase of the LO to perform different measurements, the participants can apply a phase rotation to each of their parts of the N-partite state. The symplectic transformation corresponding to such a phase rotation is written as,

$$S_\theta = \bigoplus_n R(\theta_n) = \left(\begin{array}{ccc} R(\theta_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R(\theta_N) \end{array}\right) \tag{11}$$

where we have defined the phase rotation sympletic,

$$R(\theta_n) = \left(\begin{array}{cc} \cos(\theta_n), & \sin(\theta_n) \\ -\sin(\theta_n), & \cos(\theta_n) \end{array}\right) \tag{12}$$

The resulting state then has the the Wigner function,

$$W(X;\theta) = \frac{\exp\left[-\frac{1}{2}\left(X - S_\theta\langle R\rangle\right)^T \left(S_\theta V S_\theta^T\right)^{-1}\left(X - S_\theta\langle R\rangle\right)\right]}{(2\pi)^N\sqrt{|V|}} \tag{13}$$

$$= \frac{\exp\left[-\frac{1}{2}\left(S_\theta^{-1}X - \langle R\rangle\right)^T S_\theta^T \left(S_\theta V S_\theta^T\right)^{-1} S_\theta \left(S_\theta^{-1}X - \langle R\rangle\right)\right]}{(2\pi)^n\sqrt{|V|}} \tag{14}$$

$$= \frac{\exp\left[-\frac{1}{2}\left(S_\theta^{-1}X - \langle R\rangle\right)^T V^{-1}\left(S_\theta^{-1}X - \langle R\rangle\right)\right]}{(2\pi)^N\sqrt{|V|}} \tag{15}$$

$$= W(S_\theta^{-1}X) \tag{16}$$

The probability density of obtaining a set of measurement outcomes X_q, can be obtained by integrating out X_p,

$$\rho(X_q|\theta) = \int dX_p W(X;\theta) \tag{17}$$

We assume that our measurement has some finite measurement precision σ, which we may take to zero in the end.

This measurement precision results in a blurring of $\rho(X_q|\theta)$. This blurring is modelled by convolution with a normalized positive distribution G_σ parametrized by σ,

$$\rho_{\text{obs}}(X_q|\theta) = \int dY_q \rho(Y_q|\theta) G_\sigma\left(X_q - Y_q\right) \tag{18}$$

where $G_\sigma\left(X_q - Y_q\right)$ is symmetric and a nascent delta function for $\sigma \to 0$. Note that $\rho_{\text{obs}}(X_q|\theta)$ is normalized. $G_\sigma\left(X_q - Y_q\right)$ blurs each participants measurement individually, without correlation, therefore we may factorize the distribution as,

$$G_\sigma\left(X_q - Y_q\right) = \prod_{k=1}^{N} G_\sigma\left(X_q^{(k)} - Y_q^{(k)}\right) \tag{19}$$

2.2 Decomposition

Then we find the probability density over the quadrature measurements,

$$\rho_{\text{obs}}(X_q|\theta) = \int dY W(S_\theta^{-1}Y) \prod_{k=1}^{N} G_\sigma\left(X_q^{(k)} - Y_q^{(k)}\right) \tag{20}$$

where $\int dY = \int dY_p \int dY_q$. Since S_θ^{-1} is a linear transformation with determinant 1, we can make a simple coordinate change to,

$$Z = S_\theta^{-1} Y \tag{21}$$

$$Y_q^{(k)} = \cos(\theta_k) Z_q^{(k)} + \sin(\theta_k) Z_p^{(k)} \tag{22}$$

Then we have a decomposition of the observed probability distribution,

$$\rho_{\text{obs}}(X_q|\theta) = \int dZ W(Z) \prod_{k=1}^{N} G_\sigma\left(X_q^{(k)} - \left[\cos(\theta_k) Z_q^{(k)} + \sin(\theta_k) Z_p^{(k)}\right]\right) \tag{23}$$

which after some interpretation, proves that the measurement results will be local.

2.3 Identification

We identify $Z = \lambda$ and $Z^{(k)}$as being $\lambda^{(k)}$ and

$$\rho(\lambda) = W(\lambda) \tag{24}$$

as being a proper probability density, since,

$$\int d\lambda W(\lambda) = 1 \tag{25}$$

$$W(\lambda) \geq 0 \tag{26}$$

We likewise identify $X_q^{(k)} = a_k$ and $X_q = a$ and,

$$\rho_k(a_k|\theta_k, \lambda) = G_\sigma \left(a_k - \left[\cos(\theta_k)\lambda_q^{(k)} + \sin(\theta_k)\lambda_p^{(k)} \right] \right) \tag{27}$$

Then by the assumptions made on G_σ, we have,

$$\int da_k \rho_k(a_k|\theta_k, \lambda) = 1 \tag{28}$$

$$\rho_k(a_k|\theta_k, \lambda) \geq 0 \tag{29}$$

and we may write $\rho_{\text{obs}}(a|\theta)$ in local form,

$$\rho_{\text{obs}}(a|\theta) = \int d\lambda W(\lambda) \prod_{k=1}^{N} \rho_k(a_k|\theta_k, \lambda) \tag{30}$$

All these considerations appear to remain valid as we let $\sigma \to 0$.

2.4 Comment

Note that the derivation hinged on the assumption $W(\lambda) \geq 0$.