1 Wave function resulting from teleportation

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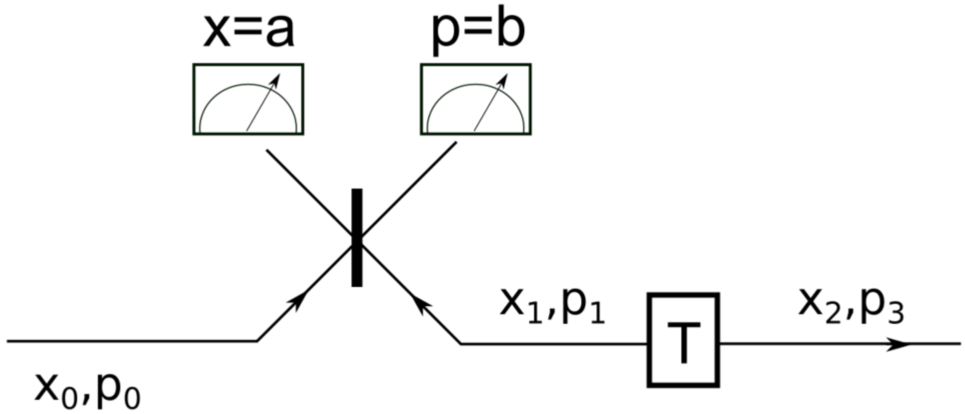

From the measurement outcomes we obtain the relations,

$$\frac{1}{\sqrt{2}}\left(\hat{x}_0 + \hat{x}_1\right) = a$$
$$\frac{1}{\sqrt{2}}\left(\hat{p}_0 - \hat{p}_1\right) = b$$

The result of the measurement is then to project the system into the subspace satisfying,

$$\frac{1}{\sqrt{2}}\left(\hat{x}_0 + \hat{x}_1\right)|\phi_{ab}\rangle = a|\phi_{ab}\rangle$$
$$\frac{1}{\sqrt{2}}\left(\hat{p}_0 - \hat{p}_1\right)|\phi_{ab}\rangle = b|\phi_{ab}\rangle$$

i.e. we project the state onto the subspace given by the span of all solutions $|\phi_{ab}\rangle$ to the above two equations. Using the differential representation,

$$\frac{1}{\sqrt{2}}\left(q_0 + q_1\right)\phi_{ab}(q_0, q_1) = a\phi_{ab}(q_0, q_1)$$
$$i\frac{1}{\sqrt{2}}\left(-\partial_{q_0} + \partial_{q_1}\right)\phi_{ab}(q_0, q_1) = b\phi_{ab}(q_0, q_1)$$

We note that the sum of derivatives is simply a derivative in a new direction when normalized appropriately. We introduce the orthogonal coordinates,

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$$

Then we have the system,

$$\begin{aligned} y_0 \phi_{ab}(y_0, y_1) &= a\phi_{ab}(y_0, y_1) \\ i\partial_{y_1} \phi_{ab}(y_0, y_1) &= b\phi_{ab}(y_0, y_1) \end{aligned}$$

The first equation requires up to a normalization,

$$\phi_{ab}(y_0, y_1) = \delta(y_0 - a)\phi_{ab}(a, y_1) = \delta(y_0 - a)\phi_{ab}(y_1)$$

The second equation requires that,

$$\partial_{y_1} \phi_{ab}(y_1) = -ib\phi_{ab}(y_1)$$

or

$$\phi_{ab}(y_1) = e^{-iby_1}$$

So the subspace consistent with the measurement outcome is spanned by the vector

$$\phi_{ab}(y_0, y_1) = \delta(y_0 - a)e^{-iby_1}$$

NB It is not a problem that ϕ_{ab} is not normalizable, as long as we don't consider it a quantum state, but instead a linear functional acting on states (a bra). If we seek to calculate probabilities, we must however normalize ϕ_{ab} so that the measurement is complete, i.e. we seek to find N such that

$$\psi(z_0, z_1) = \int dadbN\phi_{ab}(z_0, z_1) \int dy_0 dy_1 N^* \phi^*_{ab}(y_0, y_1)\psi(y_0, y_1) = *$$

We calculate the integrals,

$$\begin{aligned} * = |N|^2 \int dadb\delta(z_0 - a)e^{-ibz_1} \int dy_1 e^{iby_1} \psi(a, y_1) \\ = |N|^2 \int dy_1 \int dbe^{ib(y_1 - z_1)} \psi(z_0, y_1) \\ = |N|^2 \int dy_1 \delta\left(\frac{y_1 - z_1}{2\pi}\right) \psi(z_0, y_1) \\ = 2\pi|N|^2 \psi(z_0, z_1) \end{aligned}$$

and so we obtain,

$$\phi_{ab}(y_0, y_1) = \frac{1}{\sqrt{2\pi}}\delta(y_0 - a)e^{-iby_1}$$

This normalization likewise ensures that the sum of probabilities of getting any outcome is 1.

In the original coordinates

$$\phi_{ab}(q_0, q_1) = \frac{1}{\sqrt{2\pi}}\delta\left(\frac{1}{\sqrt{2}}\left[q_0 + q_1\right] - a\right) e^{-i\frac{b}{\sqrt{2}}(q_1 - q_0)}$$

We have the initial state

$$\psi(q_0, q_1, q_2) = \psi_0(q_0)\psi_{12}(q_1, q_2)$$

$\psi_{12}(q_1, q_2)$ is a two-mode squeezed vacuum state, obtained by mixing two squeezed states,

$$\psi(z_1, z_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{4\sigma^2}z_1^2}e^{-\frac{\sigma^2}{4}z_2^2}$$

on a beamsplitter, i.e. $z_1 = \frac{1}{\sqrt{2}}(q_1 + q_2)$ and $z_2 = \frac{1}{\sqrt{2}}(-q_1 + q_2)$, giving the state,

$$\psi_{12}(q_1, q_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8\sigma^2}(q_1+q_2)^2}e^{-\frac{\sigma^2}{8}(-q_1+q_2)^2}$$

Projecting the state $\psi(q_0, q_1, q_2)$ into the subspace associated to the measurement outcome (spanned by ϕ_{ab}) we obtain,

$$\begin{aligned}\psi_2(q_2) &= \int dq_0 dq_1 \phi^*_{ab}(q_0, q_1)\psi(q_0, q_1, q_2)\\ &= \frac{1}{\sqrt{\pi}}\int dq_0 dq_1 \delta\left(q_0 + q_1 - \sqrt{2}a\right) e^{i\frac{b}{\sqrt{2}}(q_1 - q_0)}\psi_0(q_0)\psi_{12}(q_1, q_2)\\ &= e^{-iba}\frac{1}{\sqrt{\pi}}\int dq_1 e^{i\sqrt{2}bq_1}\psi_0(-q_1 + \sqrt{2}a)\psi_{12}(q_1, q_2)\end{aligned}$$

Inserting $\psi_{12}(q_1, q_2)$,

$$\psi_2(q_2) = e^{-iba}\frac{1}{\sqrt{2}\pi}\int dq_1 e^{i\sqrt{2}bq_1}\psi_0(-q_1 + \sqrt{2}a)e^{-\frac{1}{8\sigma^2}(q_1+q_2)^2}e^{-\frac{\sigma^2}{8}(-q_1+q_2)^2}$$

We change the integration variable to $z_1 = q_1 + q_2$,

$$\psi_2(q_2) = \frac{1}{\sqrt{2\pi}} e^{-iba} e^{-i\sqrt{2}bq_2} \int dz_1 \psi_0(q_2 + \sqrt{2}a - z_1) e^{i\sqrt{2}bz_1} e^{-\frac{1}{8\sigma^2} z_1^2} e^{-\frac{\sigma^2}{8}(2q_2 - z_1)^2}$$

So the teleported state is smoothed displaced version of the input state.

In particular, we will consider the limit $\sigma \to 0$. We normalize the gaussian kernel,

$$e^{-\frac{1}{8\sigma^2} z_1^2} = \sqrt{2\pi \left(2\sigma\right)^2} N\left[2\sigma\right]\left(z_1\right)$$

where $N\left[2\sigma\right]\left(z_1\right)$ is a normal distribution of standard deviation 2σ. Then in the limit of $2\sigma \to 0$ we obtain a δ function,

$$\begin{aligned}\psi_2(q_2) = \frac{2\sigma}{\sqrt{\pi}} e^{-iba} e^{-i\sqrt{2}bq_2} \int dz_1 \psi_0(q_2 + \sqrt{2}a - z_1) e^{i\sqrt{2}bz_1} N\left[2\sigma\right]\left(z_1\right) e^{-\frac{\sigma^2}{8}(2q_2 - z_1)^2} \\ \to e^{-iba} \frac{2\sigma}{\sqrt{\pi}} e^{-i\sqrt{2}bq_2} \psi_0(q_2 + \sqrt{2}a) e^{-\frac{\sigma^2}{2} q_2^2}\end{aligned}$$

For all relevant values of q_2, we have $e^{-\frac{\sigma^2}{2} q_2^2} \approx 1$ and so we obtain,

$$\psi_2(q_2) = \frac{2\sigma}{\sqrt{\pi}} e^{-i\sqrt{2}bq_2} \psi_0(q_2 + \sqrt{2}a)$$

So the output is a displaced version of the input state. Note that $e^{-i\sqrt{2}bq_2}$ is a momentum displacement. Finally note that the norm of the state tends to zero.

This is not surprising, as σ tends to zero more quadrature measurement outcomes become probable because of the increased spread in phase space, as a result the probability of obtaining exactly the outcome a, b tends to zero.

2 Comment on the Heisenberg picture

Suppose we seek the expectation value of an arbitrary operator G. Suppose the system evolves under a string of unitaries and projectors (measurements).

The unitaries can be taken as functions of two non-commuting operators x, p (these two operators define the whole algebra),

$$U_n = U_n(x, p)$$

The projectors can likewise be taken as functions of x, p,

$$P^{(n)} = P^{(n)}(x,p)$$

Furthermore we will be using the labels,

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= U_1^\dagger(x_{0,}p_0)x_0U_1(x_0,p_0) \\ x_2 &= U_2^\dagger(x_{1,}p_1)x_1U_2(x_1,p_1) \end{aligned}$$

I.e. x_1 is the operator at time t_1, that is, after the application of the unitary evolution U_1. Note that x_2 is obtained by evolving using U_2 but with the argument (x_1,p_1) rather than (x_0,p_0).

Then, given an initial state $|\psi\rangle$, we somewhat arbitrarily, consider the time evolution of $|\psi\rangle$ in the Schrödinger picture,

$$|\psi_f\rangle = U_3(x_0,p_0)P^{(2)}(x_0,p_0)U_2(x_0,p_0)P^{(1)}(x_0,p_0)U_1(x_0,p_0)|\psi\rangle$$

Then we have the expectation value,

$$\langle G(x_0,p_0)\rangle = \langle\psi_f|G(x_0,p_0)|\psi_f\rangle$$

We now seek to move the time-evolution onto the operators x_0, p_0. We note that the unitaries and projectors behave differently, owing to the difference,

$$\begin{aligned} U_nU_n^\dagger &= 1 \\ P^{(n)}P^{(n)\dagger} &= P^{(n)}P^{(n)} = P^{(n)} \end{aligned}$$

Rearranging the time evolution,

$$\begin{aligned} |\psi_f\rangle &= U_3(x_0,p_0)P^{(2)}(x_0,p_0)U_2(x_0,p_0)P^{(1)}(x_0,p_0)U_1(x_0,p_0)|\psi\rangle \\ &= U_1(x_0,p_0)U_1^\dagger(x_0,p_0)U_3(x_0,p_0)P^{(2)}(x_0,p_0)U_2(x_0,p_0)P^{(1)}(x_0,p_0)U_1(x_0,p_0)|\psi\rangle \\ &= U_1(x_0,p_0)U_3(x_1,p_1)P^{(2)}(x_1,p_1)U_2(x_1,p_1)P^{(1)}(x_1,p_1)|\psi\rangle \\ &= U_1(x_0,p_0)U_2(x_1,p_1)U_2^\dagger(x_1,p_1)U_3(x_1,p_1)P^{(2)}(x_1,p_1)U_2(x_1,p_1)P^{(1)}(x_1,p_1)|\psi\rangle \\ &= U_1(x_0,p_0)U_2(x_1,p_1)U_3(x_2,p_2)P^{(2)}(x_2,p_2)P^{(1)}(x_1,p_1)|\psi\rangle \\ &= U_1(x_0,p_0)U_2(x_1,p_1)U_3(x_2,p_2)P^{(2)}(x_2,p_2)P^{(1)}(x_1,p_1)|\psi\rangle \end{aligned}$$

The expectation value is then,

$$\begin{aligned}
\langle G(x_0,p_0)\rangle &= \langle \psi_f | G(x_0,p_0) | \psi_f \rangle \\
= \langle \psi | P^{(1)}(x_1,p_1) P^{(2)}(x_2,p_2) U_3^\dagger(x_2,p_2) U_2^\dagger(x_1,p_1) U_1^\dagger(x_0,p_0) G(x_0,p_0) \\
U_1(x_0,p_0) U_2(x_1,p_1) U_3(x_2,p_2) P^{(2)}(x_2,p_2) P^{(1)}(x_1,p_1) |\psi\rangle
\end{aligned}$$

So we may instead evolve the operator as,

$$\begin{aligned}
G(x_0,p_0) &\to G(x_1,p_1) = U_1^\dagger(x_0,p_0) G(x_0,p_0) U_1(x_0,p_0) \\
&\to G(x_2,p_2) = U_2^\dagger(x_1,p_1) G(x_1,p_1) U_2(x_1,p_1) \\
&\to G(x_3,p_3) = U_3^\dagger(x_2,p_2) G(x_2,p_2) U_3(x_2,p_2)
\end{aligned}$$

where this unitary evolution is generated from a Heisenberg equation,

$$\partial_t G\left(x(t),p(t)\right) = i[H\left(x(t),p(t),t\right), G\left(x(t),p(t)\right)]$$

which is the Heisenberg picture.
The expectation value is then,

$$\langle G(x_0,p_0)\rangle = \langle \psi | P^{(1)}(x_1,p_1) P^{(2)}(x_2,p_2) G(x_3,p_3) P^{(2)}(x_2,p_2) P^{(1)}(x_1,p_1) |\psi\rangle$$

So we evolve the operators according to the unitary evolution, but we retain a set of projectors that act on the quantum state. In fact the evolution of the operators is completely unperturbed by the measurements.