Causal influence versus signalling for interacting quantum channels

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1 Brief summary

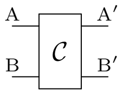
The literature on causal networks, and in particular quantum causal networks, is now flourishing, and attracting increasing interest in the quantum information and computation community, as well as in the foundation community. Up to now, the focus of the analysis of causal relations in quantum networks has been to establish whether a causal influence from a node A to a node B holds or not [1–29]. In quantum theory, this is equivalent to establishing whether A is an input subsystem and B an output subsystem of a channel that allows for signalling from A to B. In a wider context of general theories—and specifically in classical information theory—however, one might have causal influences that manifest themselves in the creation of correlations that do not allow for signalling, thus making the latter strictly stronger than causal influence. In this respect, a question of interest that is largely unexplored concerns the relation between the strength of correlations and the amount of signalling. We address the last question by defining quantifiers of signalling and causal influence and studying relations between them. In particular:

- we prove that a unitary interaction allows for a small amount of signaling if, and only if, it induces a small causal influence;
- we explicitly compute the signaling and the causal influence in the case of the quantum Cnot: the signalling is strictly smaller than the causal influence, while the latter attains its maximal value, thus indicating that the "extra" causal effect beyond signalling has to be sought in the leverage that it enables on correlations.

These results provide new tools for the study of quantum causal networks, allowing both the extension of fundamental results, e.g. in the form of stability analysis, and practical applications of quantum causal networks that might require a precise estimate of the quantities discussed above, e.g. in the security analysis of cryptographic protocols.

2 Methods and Results

In quantum theory an interaction between two systems, say A (controlled by Alice) and B (controlled by Bob) is represented by a bipartite channel $\mathcal{C}$ (i.e., a completely positive trace preserving map) sending quantum states of the Hilbert space $\mathrm{A} \otimes \mathrm{B}$ to quantum states of the Hilbert space $\mathrm{A}' \otimes \mathrm{B}'$, with $\otimes$ denoting the usual Hilbert spaces tensor product. We will write $\mathfrak{C}(\mathrm{C}, \mathrm{C}')$ for the set of channels from C to C' (shortened to $\mathfrak{C}(\mathrm{C})$ when $\mathrm{C}' = \mathrm{C}$), and $\mathfrak{U}(\mathrm{C})$ for that of *unitary* channels on system C. We will adopt the following graphical representation for bipartite quantum channels



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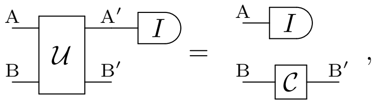
A bipartite channel $\mathcal{C} \in \mathfrak{C}(\mathrm{AB}, \mathrm{A}'\mathrm{B}')$ models an interaction between the quantum systems of the two users, Alice and Bob, and we will analyse it in terms of causal relations that it produces between Alice's input and Bob's output. As noticed by several authors [29, 30], the study of causal relations generated by non reversible channels may be ambiguous. Indeed, any channel $\mathcal{C}$ can be realized in a non unique way as a reversible channel $\mathcal{U}$ by discarding an appropriate environment system. It happens that the occurrence of causal relations between agents actually depends of the specific initial state of the environment involved in a reversible dilation. Accordingly, causal relations are unambiguously identified once the description is expanded such that all relevant systems are included, thus dealing with an "isolated system". For this reason we focus on the evolution of isolated quantum systems, thus exploring the causal relations mediated by unitary channels.

An extreme case is that where systems A and B are separately isolated, thus non interacting. Clearly, in this case the evolution channel cannot produce any causal relation between Alice and Bob. On the other hand, if one e.g. swaps systems A and B, the result is that the swap channel mediates as much causal influence as one can possibly expect. Now, still on the same line of thought, one can expect that a "little" interaction induces "little" causal effects. In order to prove this intuition we start by introducing two functions on the set of quantum unitary channels, denoted by $S(\mathcal{U})$ and $C(\mathcal{U})$, that quantify the amount of signalling and that of causal influence from Alice to Bob for the channel $\mathcal{U}$, respectively.

Signalling, that is communication from Alice to Bob (or vice-versa), is based on the dependence of the local output system B' of Bob's on the choice of the local input system A of Alice's: in general, Alice can influence the outcome probabilities for Bob's local measurements on B', by varying her choice of intervention on system A. If Bob's output at B' does not depend on the state of Alice's input at A, then we say that $\mathcal{U}$ is *no-signalling* from Alice to Bob. One can straightforwardly prove that this condition corresponds to the following identity



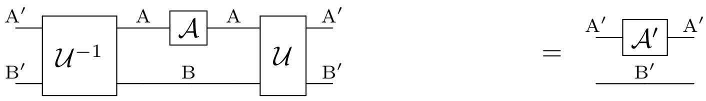
(1)

for some channel $\mathcal{C} \in \mathfrak{C}(\mathrm{B}, \mathrm{B}')$, where the trivial POVM I on system A (or A') in the diagram represents the partial trace operator Tr_{A} (or $\mathrm{Tr}_{\mathrm{A}'}$) that describes discarding A (or A'). On this basis, given a channel $\mathcal{U}$, we quantify its signalling from A to B' via the function

$$S(\mathcal{U}) := \inf_{\mathcal{C} \in \mathfrak{C}(\mathrm{B}, \mathrm{B}')} \|(\mathrm{Tr}_{\mathrm{A}'} \otimes \mathcal{I}_{\mathrm{B}'})\mathcal{U} - \mathrm{Tr}_{\mathrm{A}} \otimes \mathcal{C}\|_{\diamond}, \tag{2}$$

where $\mathcal{I}_{\mathrm{B}'}$ denotes the identity channel on system B', $\|\mathcal{X}\|_{\diamond} := \sup_{\mathrm{E}} \sup_{\rho \in \mathsf{St}(\mathrm{EA})} \|(\mathcal{I}_{\mathrm{E}} \otimes \mathcal{X})(\rho)\|_1$ is the *diamond norm* of the hermitian-preserving map $\mathcal{X}$ in the real span of $\mathfrak{C}(\mathrm{A}, \mathrm{A}')$, and $\|\cdot\|_1$ denotes the trace-norm on the space of operators on the Hilbert space $\mathrm{E} \otimes \mathrm{A}'$, i.e. $\|X\|_1 := \mathrm{Tr}[(X^\dagger X)^{1/2}]$.

The signalling condition thus boils down to the possibility of using $\mathcal{U}$ to send a message from Alice to Bob, but in a general theory of information processing this does not exhaust the ways in which an intervention on system A can causally affect the system B'. Indeed a local operation involving only system A before the reversible transformation $\mathcal{U}$ can influence the output correlations between Alice and Bob. This possibility has been extensively explored in Refs. [29, 31] and encompassed in the notion of *causal influence* of system A on system B'. The definition (by negation) of causal influence is the following. Given the unitary $\mathcal{U} \in \mathfrak{C}(\mathrm{AB}, \mathrm{A}'\mathrm{B}')$, system A has *no causal influence* on B' if for every $\mathcal{A} \in \mathfrak{C}(\mathrm{A})$ one has



(3)

for a suitable local operation $\mathcal{A}' \in \mathfrak{C}(\mathrm{A}')$. The above condition has been proved [29] to be strictly stronger than no-signalling for a general information theory. Indeed on one hand it prevents Alice to signal to Bob, but it also ensures that the evolution $\mathcal{U}$ cannot "propagate" the effect of any local operation of Alice (on system A) to alter the correlations with the output system of Bob's created by $\mathcal{U}$. Remarkably, in Ref. [29] it was also proved that in quantum theory no-causal influence coincides with no-signalling, while in classical information theory there exist examples of channels that cannot be used for transmitting signals to a given subsystem but still can be used to influence its correlations. In other words, there exist no-signalling gates that have causal influence. As proved in Ref. [29], to verify if a channel has causal influence from A to B$'$ it is not necessary to check the factorization on the rhs of Eq. (3) for every local map $\mathcal{A}$, but it is sufficient to do it on a single probe corresponding to the swap operator between two copies of Alice's input system A: in formula, $\mathcal{U}$ has no causal influence from A to B$'$ if and only if

$$\mathcal{T}(\mathcal{U}) = \mathcal{T}' \otimes \mathcal{I}_{\mathrm{B}'}, \qquad \mathcal{T}(\mathcal{U}) := (\mathcal{I}_{\mathrm{A}} \otimes \mathcal{U})(\mathcal{S} \otimes \mathcal{I}_{\mathrm{B}})(\mathcal{I}_{\mathrm{A}} \otimes \mathcal{U}^{-1}), \tag{4}$$

where $\mathcal{S} \in \mathfrak{C}(\mathrm{AA})$ is the swap channel given by $\mathcal{S}(\rho) := S\rho S$, with $S\left|\psi\right\rangle \otimes \left|\phi\right\rangle = \left|\phi\right\rangle \otimes \left|\psi\right\rangle$ for any pair $\left|\phi\right\rangle, \left|\psi\right\rangle \in \mathrm{A}$, and $\mathcal{T}'$ is a suitable channel in $\mathfrak{C}(\mathrm{AA}')$. We exploit this criterion to define a quantifier for the causal influence from A to B$'$ via the following function

$$C(\mathcal{U}) := \inf_{\mathcal{T}' \in \mathfrak{C}(\mathrm{AA}')} \|\mathcal{T}(\mathcal{U}) - \mathcal{T}' \otimes \mathcal{I}_{\mathrm{B}'}\|_{\diamond}. \tag{5}$$

As we mentioned earlier, a non trivial fact about quantum theory is the equivalence between no-signalling and no-causal influence, that can now be expressed as

$$S(\mathcal{U}) = 0 \Leftrightarrow C(\mathcal{U}) = 0. \tag{6}$$

It is interesting to observe a striking consequence of Eq. (6). We know that causal influence includes signalling as a special case, keeping track also of the correlations that the channel $\mathcal{U}$ can generate between Bob's and Alice's systems at its outcome. On one side it is possible to have signalling without inducing any correlations, an elementary example being $\mathcal{U} \in \mathfrak{U}(\mathrm{AB})$ with $\mathrm{A} \equiv \mathrm{B}$ and $\mathcal{U} = \mathcal{S}$ coinciding with the swap gate: while signalling from Alice to Bob (and viceversa) is obvious, since $\mathcal{U}$ exchanges their systems, if A and B are uncorrelated at the input they will remain uncorrelated after the swap. On the other hand, a channel $\mathcal{U}$ cannot generate correlations between Alice and Bob without allowing also for signalling: it is impossible to have $C(\mathcal{U}) > 0$ and $S(\mathcal{U}) = 0$ simultaneously.

The first question answered is whether the above equivalence (6) between no-signalling and no-causal influence is robust to perturbations of the ideal case where the channel does not mediate causal relations. State of the art knowledge on this subject is null as, in principle, the relative magnitude of the two quantities may arbitrarily fluctuate as one departs from the condition expressed in Eq. (6).

This is indeed not the case, as our result is the bound

$$S(\mathcal{U}) \leq C(\mathcal{U}) \leq 2\sqrt{2}S(\mathcal{U})^{\frac{1}{2}}. \tag{7}$$

These inequalities, proved in the following, establish the robustness of the equivalence between signalling and causal influence, that can be summarised in the sentence "little signalling is equivalent to little causal influence".

Notice however that, due to the singularity of the derivative of $x^{1/2}$ in $x = 0$, in a neighbourhood of $S(\mathcal{U}) = 0$, one can have a large increase in causal influence with a negligible increase in signalling. This observation can be seen as spotlighting the remnant of the non-equivalence of the two notions that we remarked in the classical case. The second main result is indeed the proof that causal influence and signalling are not equal. The mismatch between the two quantities is definitely established by their analytical computation, provided in the following, for the quantum Cnot unitary channel:

$$C(\mathrm{Cnot}) = 2 > 1 \geq S(\mathrm{Cnot}). \tag{8}$$

This states that there exist interactions where Alice's local operations have effects on Bob's system that "exceed" those on Bob's local states. Such effects are not to be sought in communication capacity but in the perturbation of Bob's system correlations.

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