Bell Non-locality from Wigner Negativity in Qudit Stabilizer States

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As a key concept that distinguishes quantum from classical models, we study non-locality in qudit systems. Although non-locality has been extensively studied for qubits, some findings from their two-level counterparts do not generalize to higher-dimensional systems. In contrast to qubits, qudit stabilizer states cannot display Bell non-locality with Clifford operators [5, 7]. As a further testimony of their semi-classical nature, they have non-negative Gross's Wigner functions [2]. Since Wigner negativity has been shown to be equivalent to contextuality, a generalization of non-locality, in qudit systems [1], it is a necessary prerequisite for non-locality, either present in the state or in the measurement.

We propose a family of Bell inequalities on two qudits for any finite odd prime dimension d by constructing a bipartite Bell operator that consists of stabilizer elements of the qudit Bell state under the local adjoint action of a non-Clifford unitary operator. The Bell state maximally violates the corresponding Bell inequality as a result of the difference between the 1-norm and maximum norm and the Wigner negativity from the non-Clifford operators. The Bell inequality is a natural extension of the Clauser-Horne-Shimony-Holt (CHSH) inequality for qubits, which is a linear combination of Pauli operators under local rotation of the T-gate. Moreover, the Bell operator not only serves as a measure for the singlet fraction, but also quantifies the volume of Wigner negativity. We are able to adapt the Bell operator on multiple qudits such that given stabilizer state maximally violates it with similar implications as for the bipartite case. Additionally, we demonstrate deterministic violations and violations with a constant number of measurements for the bipartite Bell state, relying on operators innate to higher-dimensional systems than the qudit at hand.

The unitary qudit operators X and Z are a natural generalization of the qubit Pauli operators σ_x and σ_z. They fulfill $X|k\rangle = |(k+1) \bmod d\rangle$ and $Z|k\rangle = \omega^k |k\rangle$, with the d^{th} root of unity $\omega = \exp(2\pi \mathrm{i}/d)$, and the relation $Z^z X^x = \omega^{xz} X^x Z^z$ for $x, z \in \mathbb{Z}_d$. The general qudit Pauli operators are the Heisenberg-Weyl displacement operators $T_{(x,z)} = \omega^{2^{-1}xz} X^x Z^z$. By applying the Fourier transform, we obtain positive Hermitian operators

$$A_{(u_x,u_z)} = \frac{1}{d} \sum_{(v_x,v_z)\in\mathbb{Z}_d^{2n}} \omega^{u_z v_x - u_x v_z} T_{(v_x,v_z)} \,, \tag{1}$$

which form an orthonormal basis in the vector space of operators equipped with the Hilbert-Schmidt inner product $(A,B) := \mathrm{tr}\big(A^\dagger B\big)$. Gross' Wigner function is then defined by

$$W_{(u_x,u_z)}(\rho) := \frac{1}{d} \mathrm{tr}\left(A_{(u_x,u_z)} \rho\right) . \tag{2}$$

The Bell state $|\Phi\rangle = \sum_{k=0}^{d-1} |k\,k\rangle/\sqrt{d}$ is stabilized by the Pauli operators $T_{(x,z)} \otimes T_{(x,-z)} |\Phi\rangle = |\Phi\rangle$, and has a non-negative Wigner function. To induce Wigner negativity, we use an extension of the qubit T-gate to qudits introduced by Howard and Vala [6] who provide an analytic expression for diagonal non-Clifford unitary operators that map Pauli to Clifford operators, which we call *unitary cube operators*,

Submitted to:
QPL 2024

$U_\nu = \sum_{k=0}^{d-1} \omega^{\nu_k} |k\rangle\langle k|$, with the third-order polynomial $\deg(\nu_k) = 3$ in the finite field $\mathbb{Z}_d$. The Bell state under the adjoint action of unitary cube operators has a Wigner function with negative values, $W^{\nu}_{v_1,v_2} := W_{v_1,v_2}\left(U_\nu \otimes \mathbb{1}\right)|\Phi\rangle\langle\Phi|\left(U_\nu^\dagger \otimes \mathbb{1}\right)$.

For $d > 3$, the Wigner function is determined by the character sum with a third-order odd polynomial. For general odd primes d, such polynomials are difficult to analyze analytically, but values a_1, a_3 such that $W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|) < 0$ always exist and can be found efficiently with an exhaustive brute-force search. Unitary operators beyond the unitary cube operators, for instance, from higher-degree polynomials over finite fields, can achieve larger Wigner negativity and enhance non-local violations. A measure of the amount of Wigner negativity of a state ρ is its volume $N[\rho] = (\sum_u |W_u(\rho)| - 1)/2$.

For the first Bell operator, we measure the operators that make up the Wigner function in (1) to exploit the Wigner negativity of the stabilizer states under unitary cube operators. To highlight the negative values, a favorable coefficient distinguishes the negative values from the positive ones, for which we use the Wigner function itself, resulting in the Bell operator

$$B_\nu = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} W^{\nu}_{v_1,v_2}\, U_\nu A_{v_1} U_\nu^\dagger \otimes A_{v_2}\,. \tag{3}$$

Since we measure a complete set of operators A_v, the lhv model assigns deterministic classical values $d\,\langle A_u\rangle_{\text{lhv}^*} = d^2 \delta_{u,a}$ for $a \in \mathbb{Z}_d^2$ to its local marginals (lhv*) and, hence, $\langle B_\nu\rangle_{\text{lhv}^*} = d^2\, W^{\nu}_{a_1,a_2}$. As a result, any lhv model can maximally achieve

$$\langle B_\nu\rangle_{\text{lhv}} \leq \max_{v_1,v_2 \in \mathbb{Z}_d^2} d^2 W^{\nu}_{v_1,v_2} =: B^{\max}_{\text{lhv}}\,, \quad \text{while} \quad \operatorname{tr}\left(B_\nu|\Phi\rangle\langle\Phi|\right) = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} d^2 (W^{\nu}_{v_1,v_2})^2 = 1\,,$$

is the expectation value for the Bell state. Since the eigenvalues of A_u are bounded by d^n (and here, $n = 2$), $B^{\max}_{\text{lhv}} < \operatorname{tr}\left(B_\nu|\Phi\rangle\langle\Phi|\right)$ from the Wigner negativity $N > 0$ and the norm inequality $\|\cdot\|_\infty \leq \|\cdot\|_1$ in compact spaces. The normalization of the Bell operator has been chosen such that $\operatorname{tr}(B_\nu \rho) = \langle\Phi|\rho|\Phi\rangle$, provides a measure for the singlet fraction with $\operatorname{tr}\left(B_\nu|\Phi\rangle\langle\Phi|\right) = 1$. Furthermore, it provides a lower bound for the volume of Wigner negativity $\operatorname{tr}(B_\nu\rho) \leq \operatorname{tr}(B_\nu\sigma) \leq B^{\max}_{\text{lhv}}(1 + 2N\left[(U_\nu \otimes \mathbb{1})\sigma(U_\nu^\dagger \otimes \mathbb{1})\right])$,

For a more compact Bell operator, we focus solely on the stabilizer elements, $T_{(x,z)} \otimes T_{(x,-z)}|\Phi\rangle = |\Phi\rangle$, which reduces the number of measurements from $(d+1)^2$ to $(d+1)$, and leads to

$$B'_\nu = \sum_{x,z,t \in \mathbb{Z}_d} W^{(\nu)}_{(x,z),(x,-z)}\, U_\nu A_{(x,z)} U_\nu^\dagger \otimes A_{(x,t-z)}\,. \tag{4}$$

The expectation values have the same form as for the full Bell operator, but the summation and maximum take only coefficients $x, z, t \in \mathbb{Z}_d$. The operator is also a measure of the singlet fraction and the volume of Wigner negativity.

For qutrits, all unitary operators defined by character polynomials are Clifford operators. Therefore, the third root of the characters, $(-1)^{1/9} = \omega^{1/3}$, is necessary in [6] but we can derive a Wigner function that yields equivalent results when applied to the Bell operators in Eqs. (3)-(6). However, operators with a spectrum beyond the qudit characters, in contrast to those whose eigenvalues are ω^a for some integer a, appear to achieve stronger Bell violations as showcased in [8] for qutrit GHZ states.

Likewise, we present a Bell operator with diagonal unitary operators $V_q = \sum_{k \in \mathbb{Z}_d} \omega^{kq}|k\rangle\langle k|$, where q is a non-integer rational number and $X_q = V_q X (V_q)^\dagger$. These operators have an obvious advantage, which is resorting to phases ω^q that are beyond the description of any local value assignment, which can only resort to characters ω^k. Then, the Bell operator

$$B_{(1/2)} = \frac{1}{d}\sum_{k\in\mathbb{Z}_d} X^k_{(1/2)} \otimes \left(X^k_{(1/2)} + \omega^{-k} X^k_{(-1/2)}\right) + X^k_{(-1/2)} \otimes \left(X^k_{(-1/2)} + \omega^{-k} X^k_{(1/2)}\right), \tag{5}$$

leads to a Bell inequality $\langle B_{(1/2)}\rangle_{\text{lhv}} \le 3$. In contrast, the Bell state has an expectation value $\langle\Phi|B|\Phi\rangle = 4$ that achieves a deterministic violation. One can even reduce the number of operators and only consider $k = 1, d-1$ in Eq. (5). Then, an lhv model can achieve $\langle B'_{(1/2)}\rangle_{\text{lhv}} \le 3 + \cos(4\pi/d) < 4$. As a trade-off, the separation between the classical and quantum models grows smaller with increasing d.

Lastly, we generalize the Bell operator in Eq. (4) to any n-qudit stabilizer state $|S\rangle$ with elements $S_{\boldsymbol{u}} = \omega^{[\boldsymbol{a},\boldsymbol{u}]} T_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \Sigma \subset \mathbb{Z}_d^{2n}$. For a unitary cube operator U_ν acting on the first qudit,

$$B_S = \sum_{\boldsymbol{u}\in\Sigma, q\in\mathbb{Z}_d} W^\nu_{\boldsymbol{u}} \left(U_\nu A_{u_1+(0,t)} U^\dagger_\nu\right) \bigotimes_{i=2}^{n} A_{u_i}\,. \tag{6}$$

The operator B_S is a measure for the overlap with the given stabilizer state under the condition that $((0,\mathbb{Z}_d)_1 \otimes (0,0)^{\otimes n-1}) \subset \Sigma$, which is exactly the case if $|S\rangle$ is entangled over the cut of the first qudit. Then, $\langle S|B_S|S\rangle = 1$, while $\langle B_S\rangle_{\text{lhv}} \le d^n \max_{\boldsymbol{u}\in\Sigma} W^\nu_{\boldsymbol{u}} < 1$. A family of Bell operators, where the unitary cube operator U_ν acts on a different qudit, can detect if all qudits are entangled with any other qudit but does not expose genuine multipartite entanglement.

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(Dated: June 14, 2024)

Nonlocality is an essential concept that distinguishes quantum from classical models and has been extensively studied in systems of qubits. For higher-dimensional systems, certain results for their two-level counterpart, like Bell violations with stabilizer states and Clifford operators, do not generalize. On the other hand, similar to continuous variable systems, Wigner negativity is necessary for nonlocality in qudit systems. We propose a family of Bell inequalities that inquire correlations related to the Wigner negativity of stabilizer states under the adjoint action of a generalization of the qubit $\pi/8$ gate, which, in the bipartite case, is an abstraction of the CHSH inequality. The classical bound is simple to compute, and a specified stabilizer state maximally violates the inequality among all qudit states based on the Wigner negativity and an inequality between the 1-norm and the maximum norm. The Bell operator not only serves as a measure for the singlet fraction but also quantifies the volume of Wigner negativity. Furthermore, we give deterministic Bell violations, as well as violations with a constant number of measurements, for the Bell state relying on operators innate to higher-dimensional systems than the qudit at hand.

As a canonical generalization of the classical bit as a unit of information, quantum information theory extensively studied two-level systems, now called qubits. A generalization to higher dimensions, systems of d levels are called qu$d$its. Although qubits are typically easier to describe, qudits can be more efficient for some quantum information tasks: apart from the theoretical interest in deriving no-go theorems in quantum information [1–3], qudit-based algorithms for Byzantine agreement [4] and secret sharing in graph states [5–7] appear to surpass schemes known for qubits [8]. Despite any advantage over qubit systems, quantum information with qudits broadens the range of physical platforms where quantum information tasks can be implemented as many quantum systems are naturally higher dimensional.

We study nonlocality in systems of two or more qudits, referring to quantum correlations that cannot be explained by any local hidden variable model [9, 10]. Previous studies have been conducted to expose nonlocality in high-dimensional systems with Bell states [11–14], Greenberger-Horne-Zeilinger (GHZ) states [15], and graph states [16]. However, the studied Bell inequalities are hard to analyze, and it is often unclear if they expose genuinely high-dimensional properties. Instead, they resort to properties that are inherent to systems of lower, higher or composite dimension by splicing qudits into subsystems, e.g. subsets of levels, and exploiting effects inherent to the subsystems. In three-dimensional systems, studies have been conducted in [17–23]. Notably, Kaniewski et al. [24] investigate Bell nonlocality based on intrinsic properties of d-level systems and even introduce self-testing mutually unbiased bases for the qutrit case. In [25], the author uses a known Bell operator [13] to find non-locality with magic states using tools from [26], which we also employ in this work.

This work focuses on qudit stabilizer states, which are uniquely defined by the Abelian subgroup of the qudit Pauli operators or Heisenberg-Weyl displacement operators, and encompass Bell states, GHZ states, and graph states. Contrary to qubits, Pauli operators on stabilizer states cannot invoke contextual violations —a prerequisite to nonlocality— due to their semi-classical nature exposed by a positive semi-definite Wigner function [27–29]. Similar to continuous variable systems [30], the negativity of Gross's Wigner function is equivalent to the contextuality of projective Pauli measurements in odd dimensions [31, 32]. Therefore, Wigner-negativity, whether located in the state or the measurement, is a necessary requirement for nonlocality.

For any odd prime dimension d, we construct a family of Bell operators on two qudits in Eq. (10) that relates to Gross's Wigner function and provides a measure for the singlet fraction as well as for the volume of Wigner negativity. The Bell operators consist of qudit Pauli operators rotated by a generalization of the qubit T- or $\pi/8$-gate introduced in [26], which we call *unitary cube operators*. Mapping the Pauli group to the qudit Clifford group [33, 34], *unitary cube operators* cause Wigner negativity in stabilizer states when these states are rotated by them. A corresponding Bell inequality is violated by the Bell state and is a natural extension of the Clauser-Horne-Shimony-Holt (CHSH) inequality for qubits [35], which is similarly a linear combination of Pauli operators under the local rotation of the T-gate. We can then reduce the number of measurement settings constituting the Bell operator by only testing the stabilizer elements of the Bell state with the same implications as before. While the underlying mechanisms for Bell violations are equivalent to the ones [25], our main contribution is a broader generalization of the qubit CHSH inequality adaptable to operators beyond the *unitary cube operators*, as well as a general insight into the origins of non-locality in qudit systems. Lastly, this Bell operator can be adapted to any entangled qudit stabilizer state that maximally violates a corresponding Bell inequality, though it does not detect genuine multipartite entanglement. For $d > 3$, unitary cube operators are inherently defined in the qudit system, that is, their eigenvalues are d^{th} roots of unity, called characters of the finite field [36]. Alternatively, we show strong nonlocal violations for Bell states in d dimension with operators whose spectrum contains characters of yet higher-dimensional systems.

Setting. The prime setting for nonlocality, in a Bell scenario for n parties, each party performs a local measurement O_{m_i} depending on the local measurement setting m_i. Physical models can pose restrictions on the general probability $p(\boldsymbol{a}|\boldsymbol{m}) = p((a_i)_i|(m_i)_i)$ of the outcome $\boldsymbol{a}$ for the measurement settings $\boldsymbol{m}$. In the quantum model, the probability distribution is derived from Born's rule $p(\boldsymbol{a}|\boldsymbol{m}) = \mathrm{Tr}\,(\Pi^{\boldsymbol{m}}_{a^*}\rho)$ for a n-partite quantum state ρ and projectors $\Pi^{\boldsymbol{m}}_{a^*}$ of the joint measurement $\bigotimes_{i=1}^n O_{m_i}$ with outcome $a^* = \prod_{i=1}^n a_i$. The (classical) local hidden variable (lhv) model is more restrictive with probability distributions of the form $p(\boldsymbol{a}|\boldsymbol{m}) = \sum_\lambda \mu(\lambda)\prod_{i=1}^n p_i(a_i|m_i,\lambda)$, for probability distributions μ and p_i. The separation can be proven with a Bell inequality for a Bell operator $\mathcal{B} = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}} \bigotimes_{i=1}^n O_{m_i}$, with coefficients $c_{\boldsymbol{m}}$. If an lhv model describes the probability distribution p, then

$$\langle\mathcal{B}\rangle_{\mathrm{lhv}} = \sum_{\boldsymbol{a},\boldsymbol{m},\lambda} \mu(\lambda)\, c_{\boldsymbol{m}} \prod_{i=1}^n a_i\, p(a_i|m_i,\lambda)\,, \tag{1}$$

and $\langle\mathcal{B}\rangle_{\mathrm{lhv}} \le \max_{p\in\mathrm{lhv}} \langle\mathcal{B}\rangle_{\mathrm{lhv}} := B_{\mathrm{lhv}}$ is a Bell inequality. Any distribution $p(\boldsymbol{a}|\boldsymbol{m})$ such as $\sum_{\boldsymbol{a},\boldsymbol{m}} c_{\boldsymbol{m}} \boldsymbol{a}^* p(\boldsymbol{a}|\boldsymbol{m}) > B_{\mathrm{lhv}}$ defies classical local models. In this work, the measurements that define the Bell experiment are projective measurements on qudits with non-degenerate eigenvalues as outcomes.

To prevent studying properties inherent to subsystems, which is contiguous in systems of composite dimensions, we restrict our study to qudit systems of prime dimension d. For example, consider the n-party GHZ state $|\mathrm{GHZ}^d\rangle = \sum_{k=0}^{d-1} |k\rangle^{\otimes n}/\sqrt{d}$. If $d = pq$ for two integers q, p, the GHZ state is a product state over two composite systems with dimensions p and q, $|\mathrm{GHZ}^p\rangle \otimes |\mathrm{GHZ}^q\rangle = \sum_{j=0}^{p-1} |j\rangle^{\otimes n} \otimes \sum_{k=0}^{q-1} |k\rangle^{\otimes n}/\sqrt{pq} = \sum_{j=0}^{p-1}\sum_{k=0}^{q-1} |j\otimes k\rangle^{\otimes n}/\sqrt{d} = |\mathrm{GHZ}^d\rangle$ with the representation of $j \otimes k \sim k + jp$. As the case $d = 2$ corresponds to qubits, this leaves us with odd prime dimension, where all numbers are integers modulo d forming a finite field, in particular 2^{-1} is unambiguous.

Pauli group and Wigner function. The qudit operators X and Z are a natural generalization of the qubit Pauli operators σ_x and σ_z as they act as $X|k\rangle = |(k+1) \bmod d\,\rangle$ and $Z|k\rangle = \omega^k|k\rangle$ where $\omega = \exp(2\pi\mathrm{i}/d)$ is the first d^{th} root of unity. They obey the relation $Z^z X^x = \omega^{xz} X^x Z^z$ for $x, z \in \mathbb{Z}_d$ and are unitary with $X^{-1} = X^{d-1} = X^\dagger$ and $Z^{-1} = Z^{d-1} = Z^\dagger$. For the general qudit Pauli operators, we use the Heisenberg-Weyl displacement operators

$$T_{(x,z)} = \omega^{2^{-1}xz} X^x Z^z\,. \tag{2}$$

The qudit Pauli group $\mathcal{P}$ can be decomposed into $d+1$ subgroups of commuting operators $\mathcal{G}_r = \{T_{(s,sr)}\,, s\in\mathbb{Z}_d\} \subset \mathcal{P}$ for $r = 0,\dots,d-1$ and $\mathcal{G}_d = \{T_{(0,z)}\,, z\in\mathbb{Z}_d\}$. The eigenbases of two elements of distinct groups are mutually unbiased, i.e., their respective eigenvector's inner product is of magnitude $1/\sqrt{d}$. Taking an element from each commutative group defines a complete set of mutually unbiased bases through their eigenvectors.

The set of Hermitian operators obtained by applying the Fourier transform to Pauli operators forms an orthonormal basis in the vector space of operators equipped with the Hilbert-Schmidt inner product $(A,B) := d^{-1}\mathrm{Tr}(A^\dagger B)$. Writing $u = (u_x, u_z) \in \mathbb{Z}_d^2$ and using the symplectic form $[u,v] := u_z v_x - u_x v_z$, the operators

$$A_u = \frac{1}{d}\sum_{v\in\mathbb{Z}_d^2} \omega^{[u,v]} T_v\,, \tag{3}$$

generate Gross's Wigner function

$$W_u(\rho) := d^{-1}\mathrm{Tr}\,(A_u\rho)\,, \tag{4}$$

a quasi-probability distribution that is real-valued and normalized, but can have negative values [29].

For a system of n qudits, we denote $\boldsymbol{u} = (\boldsymbol{u}_x, \boldsymbol{u}_z) \in \mathbb{Z}_d^{2n}$, and the operator $T_{\boldsymbol{u}} = \bigotimes_{i=1}^n T_{((u_i)_x,(u_i)_z)}$, which holds similarly for $A_{\boldsymbol{u}}$ as well as for the Wigner function $W_{\boldsymbol{u}}$for multiple qudits. In this context, we also use the inner product $\boldsymbol{u}\boldsymbol{v} = \sum_{i=1}^n u_i v_i$, which also extends to the symplectic form $[\boldsymbol{u},\boldsymbol{v}] = \boldsymbol{u}_z\boldsymbol{v}_x - \boldsymbol{u}_x\boldsymbol{v}_z$.

Stabilizer states encompass a large class of states, including Bell states, GHZ states, and graph states. Each stabilizer state $|S\rangle$ is defined by an Abelian subgroup $\mathcal{S}$ of the Pauli group. We define the set $\boldsymbol{u} \in \Sigma \subset \mathbb{Z}_d^{2n}$ so that $\omega^{t(\boldsymbol{u})}T_{\boldsymbol{u}} \in \mathcal{S}$ for a linear function t, which can be written as $t(\boldsymbol{u}) = [\boldsymbol{a},\boldsymbol{u}]$ for an $\boldsymbol{a} \in \mathbb{Z}_d^{2n}$. The stabilizer elements act trivially on the stabilizer state $\omega^{t(\boldsymbol{u})}T_{\boldsymbol{u}}|S\rangle = |S\rangle$, while for any other $\omega^{t(\boldsymbol{v})}T_{\boldsymbol{v}} \notin \mathcal{S}$, the commutation relation leads to $\langle S|T_{\boldsymbol{v}}|S\rangle = 0$. The Wigner function of stabilizer states is non-negative and remains so under the action of Clifford operators, as their group is the normalizer of the Pauli group. Appleby [33, 34] finds a unique representation for every Clifford operator with a decomposition into a Pauli operator and a unitary evolution under quadratic Hamiltonians.

Non-Clifford operations and Wigner negativity. Howard and Vala [26] provide a generalization of the qubit T-gate to qudits. They start from a diagonal unitary operator

$$U_f = \sum_{k=0}^{d-1} \omega^{f(k)}|k\rangle\langle k|\,, \tag{5}$$

with a function $f(k)$ and derive an analytic expression that map Pauli to Clifford operators, where $f(k)$ turns out to be a polynomial of degree 3. In the following, we refer to their generalized T-gates as *unitary cube operators.*

For $d > 3$, it is $f(k) = \nu_k = 12^{-1}k(\gamma + k(6z + (2k + 3)\gamma)) + \epsilon k$ with $z, \epsilon \in \mathbb{Z}_d$ and $\gamma \in \mathbb{Z}_d^* = \mathbb{Z}_d \setminus \{0\}$. We discuss the case $d = 3$ later together with non-character unitary operators. The coefficients are such that $U_\nu X U_\nu^\dagger = \omega^\epsilon X Z^z \sum_{k\in\mathbb{Z}_d} \omega^{\gamma/2}|k\rangle\langle k|$.

Stabilizer states transformed by the adjoint action of unitary cube operators can then showcase Wigner negativity. A measure of the amount of Wigner negativity is its volume defined as minus the sum of all negative values, or, since the Wigner function is normalized, as $N[\rho] = (\sum_u |W_u(\rho)| - 1)/2$. For the Bell state $|\Phi\rangle = \sum_{k=0}^{d-1} |k\,k\rangle/\sqrt{d}$, this leads to $|\Phi_\nu\rangle := (U_\nu \otimes \mathbb{1})\,|\Phi\rangle$, with the Wigner function, for $d > 3$,

$$W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|) = \frac{1}{d^3}\delta_{(u_1)_x,(u_2)_x} \sum_{k=0}^{d-1} \omega^{a_3 k^3 + a_1 k} . \quad (6)$$

For completeness, note that $a_1(u_1, u_2) = \epsilon + (u_1)_z + (u_2)_z + z(u_1)_x + 2^{-1}\gamma((u_1)_x^2 - (u_1)_x + 6^{-1})$ and $a_3 = 24^{-1}\gamma$, such that, independently of u_1 and u_2, a_1 can take any value in $\mathbb{Z}_d$ and a_3 any value in $\mathbb{Z}_d^*$. For $(u_1)_x = (u_2)_x$, the value of $W^\nu_{u_1,u_2} := W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|)$ is determined by the sum of characters with a third-order odd polynomial as an argument. Although such polynomials are difficult to analyze for general odd prime d, a simple brute-force search finds values a_1, a_3 such that $W^\nu_{u_1,u_2} < 0$ for any u_1 and u_2. The negativity of $W^\nu_{u_1,u_2}$ is pivotal in achieving a Bell violation with the Bell operator in Eq. (10).

In the following, we discuss some technical features of the Wigner function $W^\nu_{u_1,u_2}$ and optimizations of its Wigner negativity with different polynomials f in Eq. (5). The polynomial ν_k is proportional to the third Dickson polynomial [37], whose value set has been studied in [38]. From [39, Weil's Theorem 5.38, p. 223], the value of the corresponding character sum is bound by $2\sqrt{d}$ such that

$$|W^\nu_{u_1,u_2}| \le \frac{2}{d^2\sqrt{d}} . \quad (7)$$

The variance of the character sum with Dickson polynomials over uniformly random values a_3/a_1^3 if $a_1 \neq 0$, otherwise over values of a_3, is of order $O(\sqrt{d})$ [40, Lemma 2], while its mean value is of order $O(1)$. Table I lists minimum values of $W^\nu_{v_1,v_2}$ for small d.

To find unitary operators that achieve larger Wigner negativity, consider diagonal operators from Eq. (5) for some polynomial f with $3 < \deg(f) < d$. From Eq. (A1), the Wigner function of the Bell state under the adjoint action of U is proportional to $\chi(f) = \sum_{k=0}^{d-1} \omega^{g_k}$, for the polynomial $g_k \sim f_k - f_{-k}$. The lower bound of $\chi(g)$ is

$$\min_g \chi(g) \ge C_{\min} = 1 - (d-1)\cos(\pi/d) \quad (8)$$

since $g_0 = 0$ and $a = (d \pm 1)/2$ minimizes $\Re(\omega^a)$. For comparison, numerical values of $\min_g \chi(g)$ for small odd prime numbers d are in Tab. I which differ from $W_{\min} = \min_{u_1,u_2 \in \mathbb{Z}_d^2} W^\nu_{v_1,v_2}$ for all $d > 7$. For $d \to \infty$, $C_{\min} \to 2 - d$ while $W_{\min} \ge -2/d^2\sqrt{d}$ from Eq. (7). To estimate if $C_{\min}$ is achievable by a polynomial g, we study the value set $g(\mathbb{Z}_d)$ of a polynomial over finite fields. Specifically, $C_{\min}$ requires g to have a value set of three elements. The value set of any polynomial has the lower bound

$$\left\lceil \frac{d-1}{\deg(g)} \right\rceil + 1 \le |g(\mathbb{Z}_d)| , \quad (9)$$

such that, for $|g(\mathbb{Z}_d)| = 3$ to achieve the $C_{\min}$ in Eq. (8), it is necessary that $\deg(g) \ge (d-1)/2$, [42].

d	$d^2 \max W^\nu_{v_1,v_2}$	$\max N[\Phi_\nu]$	$d^3 \min W^\nu_{v_1,v_2}$	$C_{\min}$
2**	0.213	0.354	−0.707	− 0.707
3*	0.844	0.293	−0.879	− 0.879
5	0.724	0.447	−2.236	− 2.236
7	0.677	0.725	−4.406	− 4.406
11	0.535	0.914	−4.211	− 8.595
13	0.442	1.102	−6.953	−10.651
17	0.437	1.251	−7.030	−14.728
19	0.449	1.437	−6.438	−16.755
23	0.371	1.531	−8.654	−20.795

TABLE I: Extremal values of the Wigner function $W^\nu_{v_1,v_2}$ in Eq. (6) and Eq. (17), maximal values of its negativity volume $N[\Phi_\nu]$, all found through an exhaustive search, as well as values of $C_{\min}$ (Eq. (8)) for small prime dimension d. For $d \le 7$, we saturate the bound of Eq. (9) and achieve $C_{\min} = d^3 \min W^\nu_{v_1,v_2}$. * For $d = 3$, we instead use $\chi(f) = 1 + 2\cos(8\pi/9)$ as we allow for cube roots of the character. ** For $d = 2$, we use the Wigner function adapted to qubits for the Bell state and the T-gate in Eq. (A4). Its minimum value is $-1/\sqrt{2} \approx -0.707$, corresponding to the Tsirelson bound [41]. Anyway, the Bell operator (10) and analysis thereof is not apt to qubits.

Bell inequality. To construct a Bell inequality for qudit stabilizer states, we measure the operators that make up the Wigner function $W^\nu_{v_1,v_2}$ to exploit the Wigner negativity of the stabilizer states under unitary cube operators. To highlight the negative values, a favorable coefficient distinguishes the negative values from the positive ones. The head-on candidate is the Wigner function itself, resulting in the Bell operator

$$\mathcal{B}_\nu = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} W^\nu_{v_1,v_2}\, U_\nu A_{v_1} U_\nu^\dagger \otimes A_{v_2} . \quad (10)$$

While we write the Bell operator using the operators A_v, the measurements are defined in the measurements in terms of the Pauli operators with projectors $\Pi_m^{(v)} = d^{-1}\sum_{k\in\mathbb{Z}_d} \omega^{mk} T_v^k$ of outcome $m \in \mathbb{Z}_d$ and $v \in \mathbb{Z}_d^2$. Although the operator comprises of d^4 local operators A_v, only $(d+1)^2$ measurements corresponding to mutually unbiased bases for each qudit are necessary. The Bell operator's expectation value for the Bell state is

$$\mathrm{Tr}\,(\mathcal{B}_\nu |\Phi\rangle\langle\Phi|) = \sum_{v_1,v_2 \in \mathbb{Z}_d^2} d^2 (W^\nu_{v_1,v_2})^2 = 1 , \quad (11)$$

since the rotated Bell state is a pure state. The expectation value of the Bell operator for any bipartite state σ is a measure for the singlet fraction

$$\langle \mathcal{B}_\nu \rangle_\sigma = \mathrm{Tr}\,(\mathcal{B}_\nu \sigma) = \langle\Phi|\sigma|\Phi\rangle . \quad (12)$$

The details of the evaluation are given in Eq. (A2).

Measuring a complete set of $\{A_u,\, u \in \mathbb{Z}_d^2\}$ restricts the lhv model, which needs to comply with the operators' algebra. The operators A_u are projectors and have an equivalent description in terms of the Pauli operators T_u and vice versa by means of the Fourier transform. Given a set of measurements described by Pauli operators whose eigenbases are mutually unbiased, their outcomes are multiplicative characters of the form $\omega^{[a,u]}$ for $a \in \mathbb{Z}_d$ due to the structure of the Pauli group, specifically its intersection into subgroups $\mathcal{G}_r$ for $r = 0, \ldots, d$. As a result, a deterministic classical value assignment (lhv*) can only assign deterministic values 0 and d to the operator's outcome, $d\,\langle A_u\rangle_{\mathrm{lhv}^*} = \sum_{v\in\mathbb{Z}_d^2} \omega^{[u,v]-[a,v]} = d^2\delta_{u,a}$ for $a \in \mathbb{Z}_d^2$ and, hence, $\langle \mathcal{B}_\nu\rangle_{\mathrm{lhv}^*} = d^2\, W^\nu_{a_1,a_2}$. As a result, any lhv model, a convex mixture of deterministic classical value assignment (lhv*), can maximally achieve

$$\langle \mathcal{B}_\nu\rangle_{\mathrm{lhv}} \le d^2 \max_{v_1,v_2\in\mathbb{Z}_d^2} W^\nu_{v_1,v_2} := B^{\max}_{\mathrm{lhv}} \,. \tag{13}$$

Since $|W^\nu_{v_1,v_2}| \le 1/d^2$, we obtain $B^{\max}_{\mathrm{lhv}} \le 1 = \langle\Phi|\mathcal{B}_\nu|\Phi\rangle$. On the other hand, from Eq. (A5), the expectation value with a quantum state σ can be bound by $B^{\max}_{\mathrm{lhv}}$ and the volume of Wigner negativity using Hölder's inequality with the 1-norm and the maximum norm,

$$\mathrm{Tr}\,(\mathcal{B}_\nu\sigma) \le B^{\max}_{\mathrm{lhv}}(1 + 2N[(U_\nu^\dagger \otimes \mathbb{1})\sigma(U_\nu \otimes \mathbb{1})])\,, \tag{14}$$

and in particular,

$$1 \le B^{\max}_{\mathrm{lhv}} \sum_{v_1,v_2\in\mathbb{Z}_d^2} \left|W^\nu_{v_1,v_2}\right| = B^{\max}_{\mathrm{lhv}}(1 + 2N[\Phi_\nu])\,. \tag{15}$$

Thus, a Bell violation $\mathrm{Tr}\,(\mathcal{B}_\nu|\Phi\rangle\langle\Phi|) = 1 > B^{\max}_{\mathrm{lhv}}$ is possible for a nontrivial negativity volume, $N[\Phi_\nu] > 0$. The previous analysis of character polynomials defining the Wigner function $W^\nu_{v_1,v_2}$, in particular Eq. (7), gives $B^{\max}_{\mathrm{lhv}} \le 2/\sqrt{d}$ for $d > 3$. For $d = 3$, an exhaustive search using Eq. (17) finds $B^{\max}_{\mathrm{lhv}} < 0.845$. As a result, we indeed obtain a Bell violation $1 > d^2 \max W^\nu_{v_1,v_2} = B^{\max}_{\mathrm{lhv}}$. Table I lists such values $d^2 \max W^\nu_{v_1,v_2}$ for $d \le 23$.

For a more compact Bell operator, we now focus solely on the stabilizer elements, $T_{(x,z)}\otimes T_{(x,-z)}|\Phi\rangle = |\Phi\rangle$, which reduces the number of measurements from $(d+1)^2$ to $(d+1)$, and leads to

$$\mathcal{B}'_\nu = \sum_{x,z,t\in\mathbb{Z}_d} W^\nu_{(x,z),(x,-z)}\, U_\nu A_{(x,z)} U_\nu^\dagger \otimes A_{(x,t-z)}\,. \tag{16}$$

The expectation values have the same form as for the full Bell operator, but the summation and the maximum take only coefficients $x, z, t \in \mathbb{Z}_d$. The Bell operator is a measure of the singlet fraction and the volume of Wigner negativity. The Bell inequality is structurally the same and Bell state achieves the same Bell violation.

Non-character operators. As mentioned before, the case of $d = 3$ differs from other odd prime dimensions in the sense that the approach with unitary cube operators does not work since all unitary operators defined by character polynomials are Clifford operators, since for $x \in \mathbb{Z}_d$ it is $x^d \sim x$. Therefore, Howard and Vala [26] use the third root of the characters, $(-1)^{1/9} = \omega^{1/3}$ by defining $\nu_k = (6zk^2 + 2\gamma k + 3k\epsilon)/3$ in Eq. (5). The Wigner function of the rotated Bell state is

$$W^\nu_{u_1,u_2} = 3^{-3}\delta_{(u_1)_x,(u_2)_x} \sum_{k=-1,0,1} \omega^{a_1k + a_3k/3}\,, \tag{17}$$

with $a_3 = \gamma \in \mathbb{Z}_d^*$ and $a_1 = (u_1)_z + (u_2)_z + z(u_1)_x + \epsilon$ for $z, \epsilon \in \mathbb{Z}_d$. This Wigner function is applicable to all Eqs. (10)–(16). We will see how to achieve stronger Bell violations using unitary operators with a spectrum beyond the qudit characters, in contrast to those whose eigenvalues are ω^a for some integer a. Lawrence [17] finds a deterministic nonlocal paradox with non-character operators for qutrit GHZ states shared by at least three parties. We construct a Bell operator for the bipartite case with such operators,

$$\begin{aligned}\mathcal{B}_3 = X \otimes X + \omega\, X_{(1/3)} \otimes X_{(1/3)} + \\ X \otimes X_{(1/3)} + X_{(1/3)} \otimes X + h.c.\,,\end{aligned} \tag{18}$$

where $X_{(1/3)} = UXU^\dagger$, and $U = \mathrm{diag}(1, \omega^{2/3}, \omega^{1/3})$. Note that these operators $O = X, X_{(1/3)}$ are not Hermitian but unitary yet represent valid measurements in terms of their Fourier transform with projectors $\Pi_m = d^{-1}\sum_{k\in\mathbb{Z}_d} \omega^{mk} O^k$ of outcome $m \in \mathbb{Z}_d$. The same holds for a deterministic value assignment. For operators that have the same algebra as the Pauli operators T_u, it resorts to multiplicative characters of the form $\omega^{[a,u]}$ for $a \in \mathbb{Z}_d$, since the results of two commuting operators are equal when measured sequentially. As a result, any lhv model attains $B^{\max}_{3,\mathrm{lhv}} = \max(\omega^{a_0+b_0} + \omega^{2a_0+2b_1} + \omega^{2a_1+2b_0} + \omega^{a_1+b_1+1} + h.c.) = 6 + \omega + \omega^2 = 5$, where the maximum is taken over $a_0, b_0, a_1, b_1 \in \mathbb{Z}_3$, and is attained for $b_1 = 2$ and $a_0, b_0, a_1 = 0$. On the other hand, the quantum expectation value, $\mathrm{Tr}\,(\mathcal{B}_3\,|\Phi\rangle\langle\Phi|) = 1 + 3(2\omega^{1/3} + \omega^{-2/3}) + h.c. \approx 5.412$, exceeds that of the classical model, $B^{\max}_{3,\mathrm{lhv}} < \mathrm{Tr}\,(\mathcal{B}_3\,|\Phi\rangle\langle\Phi|)$.

In fact, operators with a spectrum beyond the qudit characters can achieve stronger Bell violations for any odd prime dimension d. For example, the diagonal unitary operators $V_q = \sum_{k\in\mathbb{Z}_d} \omega^{kq}|k\rangle\langle k|$ where q is a non-integer rational number, rotate the Pauli operators $X_{(q)} = V_q\, X V_q^\dagger$, which, for $q = 1/2$, lead to the Bell operator

$$\begin{aligned}\mathcal{B}_{(1/2)} = \frac{1}{d}\sum_{k\in\mathbb{Z}_d} X^k_{(1/2)} \otimes \left(X^k_{(1/2)} + \omega^{-k} X^k_{(-1/2)}\right) \\ + X^k_{(-1/2)} \otimes \left(X^k_{(-1/2)} + \omega^{k} X^k_{(1/2)}\right)\,.\end{aligned} \tag{19}$$

The corresponding Bell inequality is $\langle\mathcal{B}_{(1/2)}\rangle_{\mathrm{lhv}} \le 3$. In contrast, the Bell state has an expectation value $\langle\Phi|\mathcal{B}_{(1/2)}|\Phi\rangle = 4$ achieving a deterministic violation. One can even reduce the number of operators and only consider $k = 1, d-1$ in Eq. (19). Then, an lhv model can achieve $\langle\mathcal{B}'_{(1/2)}\rangle_{\mathrm{lhv}} \le 3 + \cos(4\pi/d) < 4$. As a trade-off, the separation between the classical and quantum models grows smaller with increasing d.

Multipartite Bell inequality. Lastly, we generalize the Bell operator in Eq. (16) to any multipartite qudit stabilizer state. Consider the state $|S\rangle$ in n qudits with stabilizer elements $S_{\boldsymbol{u}} = \omega^{[\boldsymbol{a},\boldsymbol{u}]}T_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \Sigma \subset \mathbb{Z}_d^{2n}$. For a unitary cube operator U_ν acting on the first qudit,

$$\mathcal{B}_S = \sum_{\boldsymbol{u}\in\Sigma, t\in\mathbb{Z}_d} W^\nu_{\boldsymbol{u}} \left(U_\nu \, A_{u_1+(0,t)} \, U_\nu^\dagger\right) \bigotimes_{i=2}^{n} A_{u_i} \, . \qquad (20)$$

The operator $\mathcal{B}_S$ is a measure for the overlap with the given stabilizer state under the condition that $((0,\mathbb{Z}_d)_1 \otimes (0,0)^{\otimes n-1}) \subset \Sigma$, which is exactly the case if $|S\rangle$ is entangled over the cut of the first qudit. Then, $\langle S|\mathcal{B}_S|S\rangle = 1$, while $\langle \mathcal{B}_S\rangle_{\text{lhv}} \leq d^n \max_{\boldsymbol{u}\in\Sigma} W^\nu_{\boldsymbol{u}} < 1$, with the same arguments as for the bipartite case. A single Bell operator of the form (20) cannot detect genuine multipartite entanglement [43]. A family of Bell operators, where the unitary cube operator U_ν acts on a different qudit, can detect if all qudits are entangled with any other qudit, but fails to expose genuine multipartite entanglement.

Conclusion and outlook. We have constructed a family of qudit Bell inequalities for any odd prime dimension. Similarly to the $\pi/8$-rotation leading to the CHSH inequality, our Bell operator consists of stabilizer elements under local rotation of a generalization of the qubit T-gate characterized in [26]. The constituting measurements test correlations related to the rotated states' Wigner negativity, a witness of non-classicality. The maximally entangled Bell state violates the corresponding Bell inequality and attains the quantum bound due to the separation between the 1-norm and the maximum norm of its rotated Wigner function. The Bell operator measures the singlet fraction and bounds the extent of Wigner negativity. Our family of Bell operators, as well as its quantum and classical bound, is easy to analyze and can be adapted to any entangled stabilizer state, which also reduces the number of measurements. Furthermore, the measurements can be generalized to any non-Clifford unitary operator, and we specifically discuss those from higher-degree polynomials over finite fields, which we find to achieve larger Wigner negativity and thus increase the separation between quantum and classical models. Moreover, nonlocal correlations can emerge with qudit Pauli operators when rotated by unitary operators with a non-character spectrum in bipartite systems. These operators have an obvious advantage, which is resorting to phases ω^q that are beyond the description of any local value assignment that can only resort to characters ω^k. Such operations might arise from interactions with higher-dimensional systems, but the feasibility of the implementation of any non-Clifford operator present in this work remains an open question. Its study heavily depends on the platform of experimental realization with possible candidates being super-conducting circuits, cold atoms and optical systems involving GKP (Gottesman–Kitaev–Preskill) states [44].

Acknowledgements. We acknowledge funding from Horizon Europe Research and Innovation Actions under Grant Agreement 101080173 CLUSTEC and from the Plan France 2030 through the ANR-22-PETQ-0006NISQ2LSQ and ANR-22-PETQ-0007 EPiQ projects.

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Appendix A: Wigner Negativity

We evaluate the Wigner function of the state $|\Phi_f\rangle = U_f|\Phi\rangle$ for a U_f from Eq. (5) with a polynomial $f(k) = f_k$,

$$
\begin{aligned}
W_{u_1,u_2}(|\Phi_f\rangle\langle\Phi_f|) &= \frac{1}{d^2}\mathrm{Tr}\left(A_{u_1}A_{u_2}|\Phi_f\rangle\langle\Phi_f|\right) \\
&= \frac{1}{d^5}\sum_{\substack{j,k\in\mathbb{Z}_d\\ v_1,v_2\in\mathbb{Z}_d^2}} \omega^{[u_1,v_1]+[u_2,v_2]}\langle jj|U_f\,T_{v_1}U_f^\dagger\otimes T_{v_2}|kk\rangle \\
&= \frac{1}{d^5}\sum_{\substack{j,k\in\mathbb{Z}_d\\ v_1,v_2\in\mathbb{Z}_d^2}} \omega^{[u_1,v_1]+[u_2,v_2]+k((v_1)_z+(v_2)_z)+2^{-1}((v_1)_x(v_1)_z+(v_2)_x(v_2)_z)} \\
&\qquad\qquad\cdot\langle jj|U_f\,X^{(v_1)_x}U_f^\dagger\otimes X^{(v_2)_x}|kk\rangle \\
&= \frac{1}{d^5}\sum_{\substack{k\in\mathbb{Z}_d\\ v_1,v_2\in\mathbb{Z}_d^2}} \delta_{(v_1)_x,(v_2)_x}\,\omega^{[u_1,v_1]+[u_2,v_2]+k((v_1)_z+(v_2)_z)+2^{-1}((v_1)_x(v_1)_z+(v_2)_x(v_2)_z)+f_{k+(v_1)_x}-f_k} \\
{\scriptstyle (v_x:=(v_1)_x)}\quad &= \frac{1}{d^5}\sum_{\substack{v_x,k\in\mathbb{Z}_d\\ (v_1)_z,(v_2)_z\in\mathbb{Z}_d}} \omega^{((u_1)_z+(u_2)_z)v_x-(u_1)_x(v_1)_z-(u_2)_x(v_2)_z+k((v_1)_z+(v_2)_z)+2^{-1}v_x((v_1)_z+(v_2)_z)+f_{k+v_x}-f_k} \\
&= \frac{1}{d^3}\sum_{v_x,k\in\mathbb{Z}_d} \delta_{k,(u_1)_x-2^{-1}v_x}\delta_{k,(u_2)_x-2^{-1}v_x}\omega^{((u_1)_z+(u_2)_z)v_x+f_{k+v_x}-f_k} \\
&= \frac{1}{d^3}\delta_{(u_1)_x,(u_2)_x}\sum_{v_x\in\mathbb{Z}_d}\omega^{((u_1)_z+(u_2)_z)v_x+f_{(u_1)_x+2^{-1}v_x}-f_{(u_1)_x-2^{-1}v_x}}\,.
\end{aligned}
\tag{A1}
$$

For any bipartite state $\sigma = \sum_{i,j,k,l}\sigma_{ijkl}|i\rangle\langle j|\otimes|k\rangle\langle l|$, the Bell operator's expectation value from Eq. (10) is

$$
\begin{aligned}
\mathrm{Tr}\left(\mathcal{B}_\nu\sigma\right) &= \sum_{u_1,u_2\in\mathbb{Z}_d^2} W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|)\sum_{\substack{k,l,\\ m,n\in\mathbb{Z}_d}}\sigma_{klmn}\langle l|U_\nu A_{u_1}U_\nu^\dagger|i\rangle\langle n|A_{u_2}|m\rangle \\
&= \frac{1}{d^5}\sum_{\substack{(u_1)_z,(u_2)_z,\\ u_x,j\in\mathbb{Z}_d\\ v_1,v_2\in\mathbb{Z}_d^2}} \omega^{((u_1)_z+(u_2)_z)j+\nu_{u_x+2^{-1}j}-\nu_{u_x-2^{-1}j}+(u_1)_z(v_1)_x+(u_2)_z(v_2)_x-u_x((v_1)_z+(v_2)_z)} \\
&\qquad\cdot\sum_{\substack{k,l,\\ m,n\in\mathbb{Z}_d}}\sigma_{klmn}\,\omega^{2^{-1}(v_1)_x(v_1)_z+2^{-1}(v_2)_x(v_2)_z}\langle l|U_\nu X^{(v_1)_x}Z^{(v_1)_z}U_\nu^\dagger|k\rangle\langle n|X^{(v_2)_x}Z^{(v_2)_z}|m\rangle \\
&= \frac{1}{d^3}\sum_{\substack{u_x,j\in\mathbb{Z}_d\\ v_1,v_2\in\mathbb{Z}_d^2}} \delta_{-j,(v_1)_x}\delta_{-j,(v_2)_x}\omega^{\nu_{u_x+2^{-1}j}-\nu_{u_x-2^{-1}j}-u_x((v_1)_z+(v_2)_z)+2^{-1}(v_1)_x(v_1)_z+2^{-1}(v_2)_x(v_2)_z} \\
&\qquad\cdot\sum_{\substack{k,l,\\ m,n\in\mathbb{Z}_d}}\sigma_{klmn}\omega^{k(v_1)_z+m(v_2)_z+\nu_{k+(v_1)_x}-\nu_k}\langle l|k+(v_1)_x\rangle\langle n|m+(v_2)_x\rangle \\
&= \frac{1}{d^3}\sum_{\substack{u_x,j\in\mathbb{Z}_d\\ (v_1)_z,(v_2)_z\in\mathbb{Z}_d}} \omega^{\nu_{u_x+2^{-1}j}-\nu_{u_x-2^{-1}j}-u_x((v_1)_z+(v_2)_z)-2^{-1}j(v_1)_z-2^{-1}j(v_2)_z} \\
&\qquad\cdot\sum_{\substack{k,l,\\ m,n\in\mathbb{Z}_d}}\sigma_{klmn}\omega^{k(v_1)_z+m(v_2)_z+\nu_{k-j}-\nu_k}\langle l|k-j\rangle\langle n|m-j\rangle \\
&= \frac{1}{d}\sum_{u_x,j\in\mathbb{Z}_d}\sum_{\substack{k,l,\\ m,n\in\mathbb{Z}_d}}\sigma_{klmn}\omega^{\nu_{u_x+2^{-1}j}-\nu_{u_x-2^{-1}j}+\nu_{k-j}-\nu_k}\delta_{u_x+2^{-1}j,k}\delta_{u_x+2^{-1}j,m}\delta_{l,k-j}\delta_{n,m-j} \\
&= \frac{1}{d}\sum_{u_x,j\in\mathbb{Z}_d}\sigma_{u_x+2^{-1}j,u_x-2^{-1}j,u_x+2^{-1}j,u_x-2^{-1}j}\omega^{\nu_{u_x+2^{-1}j}-\nu_{u_x-2^{-1}j}+\nu_{u_x-2^{-1}j}-\nu_{u_x+2^{-1}j}} \\
&= \frac{1}{d}\sum_{n,m\in\mathbb{Z}_d}\sigma_{n,m,n,m} = \langle\Phi|\sigma|\Phi\rangle\,.
\end{aligned}
\tag{A2}
$$

For $d=3$, $\nu_k = (6zk^2 + 2\gamma k + 3k\epsilon)/3$ and

$$W_{u_1,u_2}(|\Phi_\nu\rangle\langle\Phi_\nu|) = \frac{1}{d^3}\delta_{(u_1)_x,(u_2)_x}\left(1 + \omega^{a_1((u_1)_z,(u_2)_z)-a_3/3} + \omega^{-a_1((u_1)_z,(u_2)_z)+a_3/3}\right), \tag{A3}$$

with $a_1((u_1)_z,(u_2)_z) = (u_1)_z + (u_2)_z + z(u_1)_x + \epsilon$ and $a_3 = \gamma$.

For qubits ($d=2$) an equivalent quantity can be characterized with the T-gate $T = |0\rangle\langle 0| + e^{i\pi/4}|1\rangle\langle 1|$, such that

$$\begin{aligned} W_{u_1,u_2}((T\otimes\mathbb{1})|\Phi\rangle\langle\Phi|(T^\dagger\otimes\mathbb{1})) &= \frac{1}{2^4}\sum_{\substack{(v_1)_x,(v_1)_z,\\(v_2)_x,(v_2)_z=0,1}} (-1)^{(u_1)_z(v_1)_x+(u_1)_x(v_1)_z+(u_2)_z(v_2)_x+(u_2)_x(v_2)_z} \\ &\qquad\cdot \mathrm{Tr}(T^\dagger\sigma_{((v_1)_x,(v_1)_z)}T\otimes\sigma_{((v_2)_x,(v_2)_z)}|\Phi\rangle\langle\Phi|) \\ &= \frac{1}{2^4}\Big(1 + (-1)^{(u_1)_x+(u_2)_x} \\ &\qquad + (-1)^{(u_1)_z+(u_2)_z}\big(1 - (-1)^{(u_1)_x+(u_2)_x} + (-1)^{(u_1)_x} + (-1)^{(u_2)_x}\big)/\sqrt{2}\Big), \end{aligned} \tag{A4}$$

with the Pauli operators $\sigma_{(0,0)} = \mathbb{1}$, $\sigma_{(1,0)} = X$, $\sigma_{(1,1)} = Y$, $\sigma_{(0,1)} = Z$. Note that for qubits the Bell operators in Eq. (10) and Eq. (16) do not lead to violation since any LHV model can assign independent variables to the qubit Pauli operators.

Lastly, for qudits, we derive Eq. (14) from

$$\begin{aligned} \mathrm{Tr}(\sigma\mathcal{B}) &= d^2 \sum_{u_1,u_2\in\mathbb{Z}_d^2} W^\nu_{u_1,u_2} W_{u_1,u_2}\left((U_\nu)^\dagger\sigma\, U_\nu\right) \\ &\le d^2 \max_{u_1,u_2\in\mathbb{Z}_d^2} |W^\nu_{u_1,u_2}| \sum_{u_1,u_2\in\mathbb{Z}_d^2} |W_{u_1,u_2}\left((U_\nu)^\dagger\sigma\, U_\nu\right)| \\ &= B^{\max}_{\mathrm{lhv}}(1 + 2N[(U_\nu)^\dagger\sigma\, U_\nu])\,. \end{aligned} \tag{A5}$$

Appendix B: Bell Inequality for Stabilizer States

For the Bell operator's expectation value in Eq. (20) of an arbitrary quantum state on n qudits, $\rho = \sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho)A_{\mathbf{r}}$, we determine

$$\begin{aligned} \sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho)\mathrm{Tr}\left(\mathcal{B}_S A_{\mathbf{r}}\right) &= \sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho) \sum_{\substack{\boldsymbol{u}\in\Sigma\\ z\in\mathbb{Z}_d}} W^\nu_{\boldsymbol{u}}\, \mathrm{Tr}\left(A_{\mathbf{r}}U_1\, A_{\boldsymbol{u}+(0,z)_1}\,(U_1)^\dagger\right) \\ &= d^{-2n}\sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho) \sum_{\substack{\mathbf{s},\boldsymbol{u}\in\Sigma\\ z\in\mathbb{Z}_d}} \mathrm{Tr}\left(U_1 A_{\mathbf{s}}(U_1)^\dagger A_{\boldsymbol{u}}\right)\mathrm{Tr}\left(A_{\mathbf{r}}U_1\, A_{\boldsymbol{u}+(0,z)_1}\,(U_1)^\dagger\right), \end{aligned}$$

with $\mathrm{Tr}\left(A_{\boldsymbol{u}}A_{\boldsymbol{v}}\right) = \delta_{\boldsymbol{u},\boldsymbol{v}}$, and

$$\begin{aligned} \mathrm{Tr}\left(UA_sU^\dagger A_u\right) &= \frac{1}{d^2}\sum_{t\in\mathbb{Z}_d^2}\omega^{[s,t]+2^{-1}t_xt_z}\sum_{k,l\in\mathbb{Z}_d}\omega^{\nu_{k+t_x}-\nu_k-kl}\mathrm{Tr}\left(X^{t_x}Z^{t_z+l}A_u\right) \\ &= \frac{1}{d^2}\sum_{t\in\mathbb{Z}_d^2}\omega^{[s,t]+2^{-1}t_xt_z}\sum_{k,l\in\mathbb{Z}_d}\omega^{\nu_{k+t_x}-\nu_k-kl+[(t_x,t_z+l),u]-2^{-1}t_x(t_z+l)} \\ &= \frac{1}{d^2}\sum_{t\in\mathbb{Z}_d^2}\sum_{k,l\in\mathbb{Z}_d}\omega^{[s,t]+\nu_{k+t_x}-\nu_k-kl+[(t_x,t_z+l),u]-2^{-1}t_xl} \\ &= \frac{1}{d}\sum_{t\in\mathbb{Z}_d^2}\sum_{k\in\mathbb{Z}_d}\omega^{[s,t]+\nu_{k+t_x}-\nu_k+[(t_x,t_z),u]}\delta_{k,u_x-2^{-1}t_x} \\ &= \delta_{s_x,u_x}\sum_{t_x\in\mathbb{Z}_d}\omega^{\nu_{u_x+2^{-1}t_x}-\nu_{u_x-2^{-1}t_x}+(s_z-u_z)t_x}\,, \end{aligned}$$

using $\mathrm{Tr}\left(X^x Z^z A_{(u_x,u_z)}\right) = \omega^{zu_x - xu_z - 2^{-1}xz}$. Then, it is

$$
\begin{aligned}
& d^{-2n} \sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho) \sum_{\substack{\mathbf{s},\boldsymbol{u}\in\Sigma\\ z\in\mathbb{Z}_d}} \mathrm{Tr}\left(UA_{\mathbf{s}}U^\dagger A_{\boldsymbol{u}}\right) \mathrm{Tr}\left(A_{\mathbf{r}}U\, A_{\boldsymbol{u}+(0,z)_1}\, U^\dagger\right) \\
&= \sum_{\mathbf{r}\in\mathbb{Z}_d^{2n}} W_{\mathbf{r}}(\rho) \sum_{\substack{\mathbf{s},\boldsymbol{u}\in\Sigma\\ z\in\mathbb{Z}_d}} \delta_{(s_1)_x,(u_1)_x}\delta_{(u_1)_x,(r_1)_x}\delta_{\mathbf{s}_{\setminus s_1},\boldsymbol{u}_{\setminus u_1}}\delta_{\mathbf{r}_{\setminus r_1},\boldsymbol{u}_{\setminus u_1}} \\
&\qquad\qquad \sum_{k,m\in\mathbb{Z}_d} \omega^{\nu_{(u_1)_x+2^{-1}k}-\nu_{(u_1)_x-2^{-1}k}+((s_1)_z-(u_1)_z)k+\nu_{(r_1)_x+2^{-1}m}-\nu_{(r_1)_x-2^{-1}m}+((u_1)_z+z-(r_1)_z)m} \\
&= d^{-1} \sum_{\substack{\mathbf{r}_{\setminus(r_1)_z}\in\Sigma\\ (r_1)_z\in\mathbb{Z}_d}} W_{\mathbf{r}}(\rho) \sum_{(s_1)_z,k,m,z\in\mathbb{Z}_d} \delta_{k,m}\, \omega^{\nu_{(r_1)_x+2^{-1}k}-\nu_{(r_1)_x-2^{-1}k}+(s_1)_z k+\nu_{(r_1)_x+2^{-1}m}-\nu_{(r_1)_x-2^{-1}m}+(z-(r_1)_z)m} \\
&= \sum_{\substack{\mathbf{r}_{\setminus(r_1)_z}\in\Sigma\\ (r_1)_z\in\mathbb{Z}_d}} W_{\mathbf{r}}(\rho) \sum_{(s_1)_z,k\in\mathbb{Z}_d} \omega^{\nu_{(r_1)_x+2^{-1}k}-\nu_{(r_1)_x-2^{-1}k}+\nu_{(r_1)_x+2^{-1}k}-\nu_{(r_1)_x-2^{-1}k}+((r_1)_z+(s_1)_z)k}\delta_k \\
&= d \sum_{\substack{\mathbf{r}_{\setminus(r_1)_z}\in\Sigma\\ (r_1)_z\in\mathbb{Z}_d}} W_{\mathbf{r}}(\rho)\,.
\end{aligned}
$$