Analysis HW 6 - Luke Miles - October 26, 2015

Exercise 4.2 - 4: Prove that $\lim_{x\to 0}\cos(1/x)$ does not exist but that $\lim_{x\to 0} x\cos(1/x) = 0$.
<u>*Solution*</u>: If $\cos(1/x)$ approaches L at 0, then for any sequence (x_n) that approaches 0 without reaching it, the sequence $(f(x_n)) := (\cos(1/x_n))$ should approach L. Define $x_n := 1/n$. Then $(f(x_n)) = (\cos n)$, which clearly does not converge! Hence, this limit can not exist.
Now we will show $\lim_{x\to 0} x\cos(1/x) = 0$. Let $\varepsilon > 0$ and choose $\delta := \varepsilon$. Then

$$|f(x)| = |x\cos(1/x)| \le |x| < \delta = \varepsilon$$

Exercise 4.2 - 12: Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Assume that $\lim_{x\to 0} f = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.
With assistance from Kathleen Bell
<u>*Solution*</u>:
By the definition of a limit, we have $\forall\varepsilon > 0 : \exists\delta > 0 : \forall x \in \mathbb{R} : |x| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Then
$0 < |x| < \delta/2 \to 0 < |2x| < \delta \to |f(2x) - L| < \varepsilon \to^{*} |2f(x) - L| < \varepsilon \to |f(x) - L/2| < \varepsilon/2 < \varepsilon$. So the function also approaches $L/2$. Hence, L must be 0.
The limit exists everywhere because f is continuous: $\lim_{x\to c}|f(x) - f(c)| = \lim_{x\to c}|f(x-c)| = \lim_{x\to 0}|f(x)| = 0$.

Exercise 4.3 - 5: Evaluate the following limits, or show that they do not exist.

(a) $\lim_{x\to 1+}\frac{x}{x-1}$: ∞. Let $\alpha > 1$. Then for any x where $1 < x < \alpha$,

$$f(x) = \frac{x}{x-1} <^{\dagger} \frac{\frac{\alpha}{\alpha-1}}{\frac{\alpha}{\alpha-1} - 1} = \alpha$$

So since $f(x)$ can get arbitrarily large, we call the limit ∞.

(b) $\lim_{x\to 1}\frac{x}{x-1}$: DNE. By the above argument, approaching 1 from below yields $+\infty$. A similar argument shows that approaching 1 from below yields $-\infty$, and hence the limit does not exist.

(c)

$$\lim_{x\to 0+}(x+2)/\sqrt{x} = \lim_{x\to 0+}\sqrt{x} + 2/\sqrt{x} > \lim_{x\to 0+} 2/\sqrt{x}$$

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*Because $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$.
†for any $1 < a < b$, we know $\frac{1}{a-1} > \frac{b}{b-1}$

The last term clearly tends toward $+\infty$ and so the first term must as well.

(d)
$$\lim_{x\to\infty} \frac{x+2}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{x} + \frac{2}{\sqrt{x}} = \left[\lim_{x\to\infty} \sqrt{x}\right] + \left[\lim_{x\to\infty} \frac{2}{\sqrt{x}}\right] = \left[\lim_{x\to\infty} \sqrt{x}\right] + 0$$

Again, the last term clearly tends towards $+\infty$ and so the first must as well.

(e) $\lim_{x\to 0} \sqrt{x+1}/x$: DNE. Bound x so $-1/2 < x < 1/2$. Then from the left, the fraction tends towards $-\infty$:

$$\frac{\sqrt{x+1}}{x} < \frac{\sqrt{x}}{x} = \frac{1}{x}$$

And likewise, from the right it tends towards $+\infty$. Hence, the limit does not exist.

(f) Assume $x > 1$. Then

$$\lim_{x\to\infty} \frac{\sqrt{x+1}}{x} \leq \lim_{x\to\infty} \frac{\sqrt{x+x}}{x} = \lim_{x\to\infty} \frac{\sqrt{2x}}{x} = \lim_{x\to\infty} \frac{\sqrt{2}}{\sqrt{x}} = 0$$

And since the function is strictly positive, the limit must be 0.

(g)
$$\lim_{x\to\infty} \frac{\sqrt{x}-5}{\sqrt{x}+3} = \lim_{x\to\infty} \frac{\sqrt{x}-5}{\sqrt{x}+3} \times \frac{\sqrt{x}-3}{\sqrt{x}-3} = \lim_{x\to\infty} \frac{x-8\sqrt{x}+15}{x-9}$$
$$= \left[\lim_{x\to\infty} \frac{x}{x-9}\right] - \left[\lim_{x\to\infty} \frac{8\sqrt{x}}{x-9}\right] + \left[\lim_{x\to\infty} \frac{15}{x-9}\right] = 1+0+0 = 0$$

(h) burble
$$\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = \lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} \times \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1/\sqrt{x}-1}{1/\sqrt{x}+1} = \frac{\lim_{x\to\infty} 1/\sqrt{x}-1}{\lim_{x\to\infty} 1/\sqrt{x}+1} = \frac{1}{1} = 1$$

Exercise 4.3 - 13: Let f and g be defined on (a, ∞) and suppose $\lim_{x\to\infty} f = L$ and $\lim_{x\to\infty} g = \infty$. Prove that $\lim_{x\to\infty} f \circ g = L$.

<u>*Solution*</u>: Expanding the definitions:

$$\forall \varepsilon > 0 : \exists K > 0 : \forall x > K : |f(x) - L| < \varepsilon$$

$$\forall \alpha > 0 : \exists K > 0 : \forall x > K : g(x) > \alpha$$

So say we need a specific value K_0 in order to get f within ε of L. Then clearly g can get that large because g can get larger than any α. ■

Exercise 5.1 - 4: Define $[x]$ as the floor of x. Determine the points of continuity of the following functions: *Every function of the form $[f(x)]$ is constant (and hence continuous) unless $f(x)$ "passes by" an integer. In each case, it is sufficient to show when this occurs.*

(a) $f(x) := [x]$? Continuous everywhere except integer x.

(b) $g(x) := x[x]$? Same as (a). Multiplying by a completely continuous function does not effect points of continuity.

(c) $h(x) := [\sin x]$? Continuous everywhere except when $\sin x$ is 0 or 1, which is when x is a number of the form $n\pi$ or $2n\pi + \pi/2$. Discontinuities do not occur at $\sin x = -1$ because $[-.999] = -1$.

(d) $k(x) := [1/x]$? Assuming $x \neq 0$, the inside is an integer whenever x is of the form $1/n$. Hence, $k(x)$ is continuous everywhere except $x = 1/n$.

Exercise 5.1 - 12: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$

Solution: Let $x \in \mathbb{R}$. If f is continuous, then for any sequence (x_n) that approaches x, whatever $(f(x_n))$ approaches must be $f(x)$. Consider an infinite decimal expansion $d_1, d_2, \dots$ of x and define $x_n := d_1, d_2, \dots, d_n$. Clearly every x_n is rational so $\lim f(x_n) = 0$. And since $\lim x_n = x$, $f(x) = 0$. ■

Exercise 5.1 - 14: Let $A := (0, \infty)$ and define $k : A \to \mathbb{R}$ as $k(x) = 0$ for irrational x and $k(a/b) = b$ for relatively prime a and b. Prove that k is unbounded on every open interval in A. Conclude that k is not continuous at any point of A.

Solution: Let $I := (c, d)$ be an arbitrary interval in A and let $\alpha > 0$ and choose $K > \alpha$. Choose an arbitrary rational r in I (we've shown in class that there is one) that also isn't an endpoint. Consider the quantity $r \times K/(K+1)$. Either this is in the interval and has denominator greater than α, is too small (in magnitude) for the interval, or has too small a denominator. In the latter two cases, one can clearly increase K until the problem goes away. ■

Exercise 5.2 - 8: Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

Solution: Yes. Let x be an irrational number and let (x_n) be a sequence that approaches x. Then since $f(x_n) = g(x_n)$ for all n, we know that $\lim f(x_n) = \lim g(x_n)$. And because g and f are continuous, $f(x) = g(x)$. ■

Exercise 5.2 - 13: Let f be a continuous additive function on $\mathbb{R}$ and let $c := f(1)$. Show that $f(x) = cx$ for all $x \in \mathbb{R}$.

Solution: For any natural number n, $f(n) = f(1 + 1 + \cdots + 1) = f(1) + f(1) + \cdots + f(1) = n \times f(1) = n \times c$.

For any rational number a/b, we have $f(a/b) = a \times f(1/b) =^{\ddagger} a/b \times f(1) = a/b \times c$. By continuity (define a sequence of rationals that approaches some irrational) we know $f(x) = x \times f(1)$ for all $x \in \mathbb{R}$ ■

Exercise 5.2 - 14: Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x + y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

Solution: Suppose that g is continuous at 0. Then for any sequence (x_n) that approaches 0, $(g(x_n))$ approaches $L := g(0)$. Let $y \in \mathbb{R}$ and let (y_n) be a sequence that approaches y. Then

$$g(-y) \times g(y) = g(-y + y) = g(0) = \lim g(y_n - y) = \lim(g(y_n) \times g(-y)) = g(-y) \times \lim g(y_n)$$

implies that $\lim g(y_n) = g(y)$. Hence, g is continuous everywhere in $\mathbb{R}$.

Now suppose that for some $a \in \mathbb{R}$, $g(a) = 0$. Then for any $x \in \mathbb{R}$,

$$g(x) = g(x + (a - a)) = g(x) \times g(a - a) = g(x) \times (g(a) \times g(-a)) = g(x) \times (0 \times g(-a)) = 0$$

Exercise 5.3 - 6: Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.

Solution: Consider $g(x) := f(x) - f(x + \frac{1}{2})$, defined on the interval $[0, \frac{1}{2}]$. Note that g cannot always be strictly positive because that would mean f is decreasing and $f(0) > f(1)$. Likewise, g cannot always be strictly negative. Hence, g must pass by zero (intermediate value theorem) and there is a point $c \in [0, \frac{1}{2}]$ where $g(c) = 0$. At that same point c, $f(c) = f(c + \frac{1}{2})$.

Now for the earth part, make a map between the circle and the interval [0,1] by multiplying by the appropriate constant. Temperature varies roughly continuously. The conclusion follows.

Exercise 5.3 - 13: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f = 0$ and $\lim\limits_{x\to\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

Solution: Because of the limits, we know that for any $\varepsilon > 0$ there exists an $M > 0$ where if $|x| > M$ then $|f(x)| < \varepsilon$. Choose $\varepsilon := 1$ and the appropriate M. Then f is clearly bounded when $|x| > M$. It is also bounded when $x \in [-M, M]$ because any continuous function from a closed interval is bounded. Hence, f is bounded everywhere. ■

Example function: $f(x) = 1/(x^2 + 1)$.

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‡because $f(1) = f(b/b) = b \times f(1/b) \to f(1/b) = f(1)/b$