Analysis HW 4 - Luke Miles - October 5, 2015

Exercise 3.2 - 6: Find the limits of the following sequences:

(a) $(2 + 1/n)^2$

(b) $(-1)^n/(n + 2)$

(c) $(\sqrt{n} - 1)/(\sqrt{n} + 1)$

(d) $(n + 1)/(n\sqrt{n})$

<u>*Solution*</u>: ε is an arbitrary positive number. For each limit, it is sufficient to show that there exists an n such that the n'th element of the sequence is within ε of the limit.

(a) The limit is 4. Let $n > 3/\varepsilon$. Then $\left|(2 + 1/n)^2 - 4\right| = \left|2/n + 1/n^2\right| = 2/n + 1/n^2 < 2/n + 1/n = 3/n < 3/(3/\varepsilon) = \varepsilon$.

(b) The limit is 0. Let $n > 1/\varepsilon$. Then $|(-1)^n/(n + 2) - 0| = 1/(n + 2) < 1/n < 1/(1/\varepsilon) = \varepsilon$.

(c) The limit is 1. Let $n > 4/\varepsilon^2$. Then $|(\sqrt{n} - 1)/(\sqrt{n} + 1) - 1| = |-2/(\sqrt{n} + 1)| = 2/(\sqrt{n} + 1) < 2/\sqrt{n} < 2/\sqrt{4/\varepsilon^2} = \varepsilon$.

(d) The limit is 0. Let $n > 4/\varepsilon^2$. Then
$|(n+1)/(n\sqrt{n})| = (n+1)/(n\sqrt{n}) = n/(n\sqrt{n})+1/(n\sqrt{n}) = 1/\sqrt{n}+1/(n\sqrt{n}) < 1/\sqrt{n}+1/\sqrt{n} = 2/\sqrt{n} < 2/\sqrt{4/\varepsilon^2} = \varepsilon$.

Exercise 3.2 - 9: Let $y_n := \sqrt{n + 1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.
<u>*Solution*</u>: *(With a hint from Alex Malone.)* Define $x_n := \sqrt{n}y_n = \sqrt{n^2 + n} - n$. For any n,

$$n < n + 1/4$$

hence,
$$n^2 + n < n^2 + n + 1/4$$

hence,
$$\sqrt{n^2 + n} < n + 1/2$$

hence,
$$\sqrt{n^2 + n} - n < 1/2.$$

So x_n is bounded by above by $1/2$. Also, for any $n \in \mathbb{N}$ (naturals start with 1, right?) we have

$$0 < 7n^2 + 4n - 4$$

$\div - 4n^2$
$$0 > -7/4 - 1/n + 1/n^2$$

$$n^2 + n > n^2 + n - 7/4 - 1/n + 1/n^2 = (n + 1/2 - 1/n)^2$$

$$\sqrt{n^2 + n} > n + 1/2 - 1/n$$

$$\sqrt{n^2 + n} - n > 1/2 - 1/n$$

So x_n is bounded below by $1/2 - 1/n$, which clearly converges to $1/2$. By the squeeze theorem, x_n converges to $1/2$ as well. ■

Exercise 3.2 - 13: *(With assistance from Niven Achenjang)* If $a > 0, b > 0$, show that $\lim(\sqrt{(n+a)(n+b)} - n) = (a+b)/2$

<u>*Solution*</u>: WLOG, suppose $a \leq b$. Then by the harmonic-mean geometric-mean arithmetic-mean inequality, we have

$$n + a \leq \frac{2}{\frac{1}{n+a} + \frac{1}{n+b}} \leq \sqrt{(n+a)(n+b)} \leq \frac{(n+a)+(n+b)}{2} \leq n + b$$
$$a \leq \frac{2}{\frac{1}{n+a} + \frac{1}{n+b}} - n \leq \sqrt{(n+a)(n+b)} - n \leq \frac{a+b}{2} \leq b$$

Looking at the second and third terms, we get a lower bound:

$$\begin{aligned}
&\lim\left(\frac{2}{\frac{1}{n+a} + \frac{1}{n+b}} - n\right) \\
&= \lim\left(\frac{2(n+a)(n+b)}{(n+b)+(n+a)} - n\right) \\
&= \lim\left(\frac{2n^2 + 2n(a+b) + 2ab}{2n+a+b} - n\right) \\
&= \lim\left(\left(\frac{2n^2}{2n+a+b} - n\right) + \frac{2n(a+b)}{2n+a+b} + \frac{2ab}{2n+a+b}\right) \\
&= \lim\left(\frac{2n^2}{2n+a+b} - n\right) + \lim\frac{2n(a+b)}{2n+a+b} + \lim\frac{2ab}{2n+a+b} \\
&= \lim\frac{2n^2 - n(2n+a+b)}{2n+a+b} + \lim\frac{2n(a+b)}{2n+a+b} + \lim\frac{2ab}{2n+a+b} \\
&= \lim\frac{-n(a+b)}{2n+a+b} + \lim\frac{2n(a+b)}{2n+a+b} + \lim\frac{2ab}{2n+a+b} \\
&= \frac{-(a+b)}{2} + (a+b) + 0 \\
&= \frac{a+b}{2}
\end{aligned}$$

Looking at the third and fourth terms, we get an upper bound of $(a+b)/2$ as well. So, by the squeeze theorem, the limit of $(\sqrt{(n+a)(n+b)} - n)$ is $(a+b)/2$. ■

Exercise 3.2 - 14: Use the squeeze theorem to determine the limits of (a) n^{1/n^2} and (b) $(n!)^{1/n^2}$.

<u>*Solution*</u>:

(a) Define $y_n := n^{1/n^2}$. Clearly y_n is at least 1 for any n. So y_n is bounded below by the constant sequence $x_n := 1$. Now define the sequence $z_n := 1 + 1/n$. Note that

$$z_n^{n^2} = (1 + \frac{1}{n})^{n^2} = 1 + \binom{n^2}{1}\frac{1}{n} + [\text{POSITIVE JUNK}] > 1 + \binom{n^2}{1}\frac{1}{n} = 1 + n > n$$

It follows that $z_n = 1 + 1/n > n^{1/n^2} = y_n$ for all n. And since $\lim x_n = 1 = \lim z_n$, it is also true that $\lim y_n = 1$.

(b) Let $y_n := (n!)^{1/n^2}$. Again, y_n is bounded below by $x_n := 1$. This time let $z_n := n^{1/n}$. Note that

$$z_n^n = (n^{1/n})^n = n^{1/n} \times n^{1/n} \times \cdots \times n^{1/n} > 1^{1/n} \times 2^{1/n} \times \cdots \times n^{1/n} = (n!)^{1/n} = y_n^n$$

Hence $z_n > y_n$ for all n. And since we've shown in class that $\lim z_n = \lim n^{1/n} = 1 = \lim x_n$, we can say that $\lim y_n = \lim(n!)^{1/n^2} = 1$.

Exercise 3.3 - 4: Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

Solution: We can show that (x_n) converges to 2 by showing it is (i) bounded above by 2, and (ii) monotone increasing. Once we know it converges, 2 is the limit if there is (iii) no lesser upper bound.

(i) $x_1 = 1 < 2$. Now suppose that for all $n \le k$, we have $x_n < 2$. Then

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + 2} = 2$$

(ii) In general, $\sqrt{m + a} > a$ for all $0 < a < m^2 - m$. In this case $\sqrt{2 + x_n} > x_n$ holds for any $0 < x_n < 2^2 - 2 = 2$, which is all x_n by (i). So $x_{n+1} > x_n$ for all n.

(iii) Suppose $L := \sup(x_n)$ is less than 2. Then $L = 2 - \varepsilon_1$ for some $\varepsilon_1 > 0$. We also know that for any $\varepsilon_2 > 0$, there exists a k such that $x_k > L - \varepsilon_2$. Let $\varepsilon_2 := \varepsilon_1/10$. Then $x_k > L - \varepsilon_1/10$ and $x_{k+1} > \sqrt{2 + L - \varepsilon_1/10} = \sqrt{(2 - \varepsilon_1/10) + L} > \sqrt{L + L} > L$. Contradiction! The least upper bound of (x_n) must be 2.

Exercise 3.3 - 5: Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$. Show that (y_n) converges and find the limit.

Solution: *We've shown in class that the limit of a recursive sequence is a fixed point under it. I don't think this occurs in the book at all, so I'm going to use it.*

If y_n converges, then $\lim y_n = y$ and y is a fixed point in the sequence:

$$y = \sqrt{p + y}$$

$$y^2 = p + y$$

$$y^2 - y - p = 0$$

by the quadratic equation
$$y = \frac{1}{2}(1 \pm \sqrt{1 + 4p})$$

Rejecting the negative result, we have $y = \frac{1}{2}(1 + \sqrt{1 + 4p})$. ■

Exercise 3.3 - 11: Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges.

Solution: x_n is obviously increasing. Now note that for $k \ge 2$, $1/k^2 \le 1/(k(k-1)) = 1/(k-1) - 1/k$. Then

$$\begin{aligned}
x_n &= 1/1^2 + 1/2^2 + 1/3^2 + \cdots + 1/(n-1)^2 + 1/n^2 \\
&\le 1 + (1/1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-2) - 1/(n-1)) + (1/(n-1) - 1/n) \\
&= 1 + 1 + (-1/2 + 1/2) + (-1/3 + 1/3) + \cdots + (-1/(n-2) + 1/(n-2)) + (-1/(n-1) + 1/(n-1)) + 1/n \\
&= 2 + 1/n
\end{aligned}$$

and hence x_n is certainly at most 3. ■

Exercise 3.4 - 10: Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence that converges to S.

Solution: First note that (s_n) is a decreasing sequence and hence it must converge to S. Now remember the definition of peaks we used in class. Let P_k be the number of peaks in (x_n) for $n \geq k$. Choose a k large enough so that P_k is either 0 or ∞. If $P_k = 0$, then all the s_n for $n \geq k$ are equal and for $\varepsilon > 0$, there are infinitely many x_n less than ε below S. Let those x_n be the subsequence which converges to S. If P_k is infinite, then look at the subsequence of just peaks. This is a subseq of s_n and hence converges to the same limit S. ■

Exercise 3.4 - 12: Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution: Suppose that (x_n) is unbounded because it gets arbitrarily large positives. Then let x_{n_k} be the first x_n at least k. Then for $\varepsilon > 0$, let $k > 1/\varepsilon$. Then $1/x_{n_k} \leq 1/k < 1/(1/\varepsilon) = \varepsilon$. A similar argument shows the limit for (x_n) arbitrarily large negative. ■

Exercise 3.4 - 17: Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4 \ldots).$$

Determine (a) $\limsup(x_n)$, (b) $\liminf(x_n)$, (c) $\sup\{x_n\}$, and (d) $\inf\{x_n\}$.

Solution:

(a) $\limsup(x_n) = 1$. $\inf V \leq 1$ because for any $\varepsilon > 0$, there are only finitely many values of the form $1 + 1/n$ greater than $1 + \varepsilon$. $\inf V \geq 1$ because there are infinitely many values of the form $1 + 1/n$ greater than any number less than 1.

(b) $\liminf(x_n) = 0$. $\sup W \geq 0$ because for any $\varepsilon > 0$, there are only finitely many values of the form $-1/n$ less than $-\varepsilon$. $\sup W \leq 0$ because any positive number has infinitely many numbers of the form $-1/n$ less than it.

(c) $\sup\{x_n\} = 2$ because $1 + 1/n$ is decreasing and for any n, m we know $1 + 1/n > -1/m$.

(d) $\inf\{x_n\} = -1$ because $-1/n$ is increasing and for any n, m we know $-1/n < 1 + 1/m$.