Homework 1

Luke Miles

August 28, 2015

Sorry I'm new at $\LaTeX$

Exercise 1.1-14
$f^{-1}(A \cup B)$
$= \{y : \exists\ x \in (A \cup B) \text{ s.t. } f^{-1}(x) = y\}$
$= \{y : \exists\ x \in A \text{ s.t. } f^{-1}(x) = y \text{ or } \exists\ x \in B \text{ s.t. } f^{-1}(x) = y\}$
$= \{y : \exists\ x \in A \text{ s.t. } f^{-1}(x) = y\} \cup \{\exists\ x \in B \text{ s.t. } f^{-1}(x) = y\}$
$= f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(A \cap B)$
$= \{y : \exists\ x \in (A \cap B) \text{ s.t. } f^{-1}(x) = y\}$
$= \{y : \exists\ x \in A \text{ s.t. } f^{-1}(x) = y \text{ and } \exists\ x \in B \text{ s.t. } f^{-1}(x) = y\}$
$= \{y : \exists\ x \in A \text{ s.t. } f^{-1}(x) = y\} \cap \{\exists\ x \in B \text{ s.t. } f^{-1}(x) = y\}$
$= f^{-1}(A) \cap f^{-1}(B)$

Exercise 1.1-16
In order to show a bijection, we will show that f is (I) one-to-one and (II) onto.
I.
We will show that f is one-to-one with the existence of an inverse in the appropriate interval.
$y = x/\sqrt{x^2 + 1}$
$y^2 = x^2/(x^2 + 1)$
$y^2(x^2 + 1) = x^2$
$y^2x^2 + y^2 - x^2 = 0$
$x^2(y^2 - 1) + y^2 = 0$
$x^2 = -y^2/(y^2 - 1)$
$x^2 = y^2/(1 - y^2)$
$x = y/\sqrt{1 - y^2}$

II.
Since any two inputs that produce the same output are equal, f is also onto:

$f(x_1) = f(x_2)$
$x_1/\sqrt{x_1^2+1} = x_2/\sqrt{x_2^2+1}$
$x_1^2/(x_1^2+1) = x_2^2/(x_2^2+1)$
$x_1^2(x_2^2+1) = x_2^2(x_1^2+1)$
$x_1^2x_2^2 + x_1^2 = x_2^2x_1^2 + x_2^2$
$x_1^2 = x_2^2$
$x_1 = x_2$

Exercise 1.2-6
We will use induction.

Define $f(x) := x^3 + 5x$.
Clearly, $f(1) = 1^3 + 5 * 1 = 1 + 5 = 6$ is divisible by 6.
Now suppose that $f(i)$ is divisible by 6 for all $i \leq k$.
Since
$f(k+1) - f(k)$
$= (k+1)^3 + 5(k+1) - (k^3 + 5k)$
$= k^3 + 3k^2 + 3k + 1 + 5k + 5 - k^3 - 5k$
$= 3k^2 + 3k + 6$
$= 3(k^2 + k + 2)$
is divisible by 6 ($k^2 + k + 2$ is always even), $f(k+1)$ is also divisible by 6
By induction, 6 divides $f(n)$ for all natural n.

Exercise 1.2-10
We will show that $1/(1\times 3) + 1/(3\times 5) + \ldots + 1/((2n-1)(2n+1)) = n(2n+1)$.
The base case is trivial.
Now suppose the relation is true for $i \leq k$. We will look at $k+1$.
$k/(2k+1) + 1/((2(k+1)-1)(2(k+1)+1))$
$= k/(2k+1) + 1/((2k+1)(2k+3))$
$= k(2k+3)/((2k+1)(2k+3)) + 1/((2k+1)(2k+3))$
$= (2k^2 + 3k + 1)/((2k+1)(2k+3))$
$= (2k+1)(k+1)/((2k+1)(2k+3))$
$= (k+1)/(2(k+1)+1)$
Exercise 1.2-18
Using induction. Base case is trivial.
Showing that the LHS increases more than the RHS when you add a term:
$\sqrt{n(n+1)} > \sqrt{n \times n} = n$
$\sqrt{n(n+1)} + 1 > n + 1$
$\sqrt{n} + 1/\sqrt{n+1} > \sqrt{n+1}$