Analysis HW 5 - Luke Miles - October 14, 2015

Exercise 3.5 - 5: If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.

Solution: First, we will show the limit. Let $\varepsilon > 0$ and $n > 1/\varepsilon^2$ Then

$$\left|\sqrt{n+1} - \sqrt{n}\right| = \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon.$$

Now, for contradiction, suppose that (x_n) is Cauchy. Then for any $\varepsilon > 0$ there is some number H so that for any $n, m > H$, we have $|x_n - x_m| < \varepsilon$. To make things easier we will assume that $H > \varepsilon$ but note that this situation can be easily dealt with . Anyways, suppose such an H exists. Then let $m = H^2$ and $n = 4H^2$. So we have

$$|x_n - x_m| = |\sqrt{4H^2} - \sqrt{H^2}| = 2H - H = H > \varepsilon.$$

This contradiction shows that the sequence cannot be Cauchy. ■

Exercise 3.5 - 6: Let p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies $\lim |x_{n+p} - x_n| = 0$.

Solution: Let $x_n = n$ modulus p. Here, any two terms p apart in index are 0 apart in value, but otherwise they clearly do not get arbitrarily close. Let $\varepsilon := 1/2$ then for any $H(\varepsilon)$, and any $n > H(\varepsilon)$, $|x_{n+1} - x_n| = 1 > \varepsilon$. ■

Exercise 3.5 - 11: If $y_1 < y_2$ are arbitrary real numbers and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$ for $n > 2$, show that (y_n) is convergent. What is its limit?

Solution: This is a contractive sequence:

$$|y_{n+2} - y_{n+1}| = \left|\left(\frac{1}{3}y_{n+1} + \frac{2}{3}y_n\right) - y_{n+1}\right| = \frac{2}{3}|y_{n+1} - y_n|$$

Contractive sequences converge. This one converges to $(2/5)y_1 + (3/5)y_2$. This can be verified with monotone-ness and sup and inf on the odd and even terms of the sequence.

Exercise 3.6 - 5: Is the sequence $a_n := n \sin n$ properly divergent?

Solution: No, (a_n) is not properly divergent. Suppose that (a_n) trends towards $+\infty$. Then for any $\alpha \in \mathbb{R}$ there exists a $K(\alpha)$ so that for any $n > K(\alpha)$, $a_n > \alpha$. Let $\alpha > 0$. Then for at least one of $\{K+2, K+4, K+6\}$, we know $a_n = n \sin n$ is negative and therefore less than α. A similar argument shows (a_n) does not tend towards $-\infty$. ■

Exercise 3.6 - 8: Investigate the convergence or divergence of the following sequences:

(a) $(\sqrt{n^2+2})$: This sequence tends towards $+\infty$. Let $K(\alpha)$ be the smallest integer greater than α. Then for any $n \geq K(\alpha)$, $\sqrt{n^2+2} > \sqrt{n^2} = n > \alpha$.

(b) $(\sqrt{n}/(n^2+1))$: Clearly convergent towards 0. $0 < \sqrt{n}/(n^2+1) < n/(n^2+1) < n/n^2 = 1/n$.

(c) $(\sqrt{n^2+1}/\sqrt{n})$: Tends towards $+\infty$. Let $K(\alpha) > \alpha^2$. Then for any $n > K(\alpha)$, $\sqrt{n^2+1}/\sqrt{n} > \sqrt{n^2}/\sqrt{n} = \sqrt{n} > \sqrt{\alpha^2} = \alpha$.

(d) $(\sin\sqrt{n})$: This sequence is neither convergent nor properly divergent. It is not convergent because if $\varepsilon := 1$ then for any n there exists a k such that $|\sin\sqrt{n} - \sin\sqrt{n+k}| > 1$. It is not properly divergent because it never exceeds +1 or -1.

Exercise 3.7 - 3: Use partial fraction to show the following summations.

(a) $\sum\limits_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$:

$$\begin{aligned}
s_k &:= \sum_{n=0}^{k} \frac{1}{(n+1)(n+2)} \\
&= \sum_{n=0}^{k} \frac{1}{n+1} - \frac{1}{n+2} \\
&= \left(\sum_{n=0}^{k} \frac{1}{n+1}\right) - \left(\sum_{n=0}^{k} \frac{1}{n+2}\right) \\
&= \left(\sum_{n=0}^{k} \frac{1}{n+1}\right) - \left(\sum_{n=1}^{k+1} \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{k+2}
\end{aligned}$$

Clearly s_k converges to 1.

(b) $\sum\limits_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0$, if $\alpha > 0$.

$$\begin{aligned}
s_k &:= \sum_{n=0}^{k} \frac{1}{(\alpha+n)(\alpha+n+1)} \\
&= \sum_{n=0}^{k} \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1} \\
&= \left(\sum_{n=0}^{k} \frac{1}{\alpha+n}\right) - \left(\sum_{n=0}^{k} \frac{1}{\alpha+n+1}\right) \\
&= \left(\sum_{n=0}^{k} \frac{1}{\alpha+n}\right) - \left(\sum_{n=1}^{k+1} \frac{1}{\alpha+n}\right) \\
&= \frac{1}{\alpha} - \frac{1}{\alpha+k+1}
\end{aligned}$$

Clearly s_k converges to $1/\alpha$.

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

$$\begin{aligned}
s_k &:= \sum_1^k \frac{1}{n(n+1)(n+2)} \\
&= \sum_1^k \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)} \\
&= \frac{1}{2}\left(\sum_1^k \frac{1}{n}\right) - \left(\sum_1^k \frac{1}{n+1}\right) + \frac{1}{2}\left(\sum_1^k \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(\sum_1^k \frac{1}{n}\right) - \left(\sum_2^{k+1} \frac{1}{n}\right) + \frac{1}{2}\left(\sum_3^{k+2} \frac{1}{n}\right) \\
&= \left(\frac{1}{2}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{-1}{k+1} + \frac{1}{2(k+1)}\right) + \left(\frac{1}{2(k+2)}\right) \\
&= \frac{1}{4} - \frac{1}{2(k+1)} + \frac{1}{2(k+2)}
\end{aligned}$$

Clearly s_k converges to $1/4$.

Exercise 3.7 - 13: If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n a_{n+1}}$ always convergent? Either prove it or give a counterexample.

<u>*Solution*</u>: Proving it by contrapositive. Suppose that $\sum \sqrt{a_n a_{n+1}}$ does not converge. Then neither does $\sum a_n a_{n+1}$, and by the Cauchy-Schwarz inequality, neither does $\left(\sum a_n\right)\left(\sum a_{n+1}\right)$. But that last expression must be finite because it is the product of two finite numbers! Hence we have a contradiction and $\sum \sqrt{a_n a_{n+1}}$ must converge. ■

Exercise 3.7 - 15: Let $(a(n))$ be a decreasing sequence of strictly positive numbers and let $s(k) := \sum_{n=1}^{k} a(n)$. First show that

$$\frac{1}{2}(a(1) + 2a(2) + \cdots + 2^n a(2^n)) \leq s(2^n) \leq (a(1) + 2a(2) + \cdots + 2^{n-1}a(2^{n-1})) + a(2^n),$$

then use this result to show that $\sum_{n=1}^{\infty} a(n)$ converges if and only if $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges.

<u>*Solution*</u>: TODO

Exercise 3.7-18: Show that if $c > 1$, then the following series are convergent:

(a) $\sum \frac{1}{n(\ln n)^c}$: *I looked at the hint in the back of the book.* The Cauchy Condensation Test states that $\sum x_n$ converges if and only if $\sum 2^n x_{2^n}$ converges.

$$\sum 2^n x_{2^n} = \sum \frac{2^n}{2^n(\ln 2^n)^c} = \sum \frac{1}{n^c}\frac{1}{(\ln 2)^c} = \frac{1}{(\ln 2)^c}\sum \frac{1}{n^c}.$$

The last expression clearly only converges for $c > 1$.

(b) $\sum \frac{1}{n(\ln n)(\ln \ln n)^c}$: We will apply the cauchy condensation test twice.

Once:
$$\begin{aligned}
y_n := 2^n x_{2^n} &= \sum \frac{2^n}{2^n(\ln 2^n)(\ln \ln 2^n)^c} \\
&= \sum \frac{1}{n \times \ln 2 \times (\ln(n \times \ln 2))^c} \\
&= \sum \frac{1}{n \times \ln 2 \times (\ln n + \ln \ln 2)^c}
\end{aligned}$$

Twice:
$$\begin{aligned}
z_n := 2^n y_{2^n} &= \sum \frac{2^n}{2^n \times \ln 2 \times (\ln 2^n + \ln \ln 2)^c} \\
&= \sum \frac{1}{\ln 2 \times (n \times \ln 2 + \ln \ln 2)^c} \\
&= \frac{1}{\ln 2} \sum \frac{1}{(n \times \ln 2 + \ln \ln 2)^c} \\
&< \frac{1}{\ln 2} \sum \frac{1}{(n \times \ln 2)^c} \\
&= \frac{1}{(\ln 2)^{c+1}} \sum \frac{1}{n^c}
\end{aligned}$$

Again, this converges for any $c > 1$.

Exercise 4.1 - 7: Show that $\lim\limits_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

Solution: Let $\varepsilon > 0$. Choose $\delta := \min\{1, \varepsilon/(2c^2 + 4|c| + 1)\}$. Then

$$|x^3 - c^3| = |x^2 + cx + c| \times |x - c| < ((|c|+1)^2 + c^2 + |c|)|x - c| = (2c^2 + 4|c| + 1)|x - c| < (2c^2 + 4|c| + 1)\frac{\varepsilon}{2c^2 + 4|c| + 1} = \varepsilon.$$

And since we have a way of choosing δ for any arbitrary assigned ε, we know the limit is correct. ■

Exercise 4.1 - 11: Use the definition of limit to prove the following:

(a) $\lim\limits_{x \to 3} \frac{2x+3}{4x-9} = 3$: Let $\varepsilon > 0$, define $f(x) := (2x + 3)/(4x - 9)$, choose $\delta := \min\{1, 17\varepsilon/10\}$, and bound $2 < x < 4$. Then

$$|f(x) - 3| = \left|\frac{2x+3}{4x+9} - 3\right| = \left|\frac{-10x + 30}{4x + 9}\right| = |x - 3| \times \left|\frac{10}{4x + 9}\right| < |x - 3| \times \frac{10}{17} < \frac{17\varepsilon}{10} \times \frac{10}{4 \times 2 + 9} < \frac{17\varepsilon}{10} \times \frac{10}{17} = \varepsilon$$

So since $f(x)$ can get arbitrarily close to 3 as x gets close to 3, we can be sure this limit is correct. ■

(b) $\lim\limits_{x \to 6} \frac{x^2 - 3x}{x+3} = 2$: Let $\varepsilon > 0$, define $f(x) := (x^2 - 3x)/(x + 3)$, choose $\delta := \min\{1, \epsilon\}$, and bound $5 < x < 7$. Then

$$|f(x) - 2| = \left|\frac{x^2 - 3x}{x + 3} - 2\right| = \left|\frac{x^2 - 5x - 6}{x + 3}\right| = |x - 6| \times \left|\frac{x + 1}{x + 3}\right| < |x - 6| \times \frac{7 + 1}{5 + 3} = |x - 6| < \varepsilon$$

Again, this shows the limit. ■