Analysis HW 7 - Luke Miles - November 15, 2015

Exercise 5.4 - 2: Show that the function $f(x) := 1/x^2$ is uniformly continuous on $A := [1, \infty)$, but that it is not uniformly continuous on $B := (0, \infty)$.

Solution: First we will show that f is uniformly continuous on A. Let $\varepsilon > 0$ and choose $\delta := 2\varepsilon$. Then if $a, b \in A$ and $|a - b| < \delta$ and WLOG $a < b$, we have

$$|f(a) - f(b)| = \frac{1}{a^2} - \frac{1}{b^2} = \frac{b^2 - a^2}{a^2 b^2} = \frac{b + a}{a^2 b^2}(b - a) < 2(b - a) < 2\delta = \varepsilon.$$

Now we will show that f is not uniformly continuous on B. Choose $\varepsilon = 1$ and let $\delta > 0$ and further assume that $\delta < 1/3$. Choose $a := \delta, b := \frac{3}{2}\delta$, and we get

$$|f(a) - f(b)| = |f(\delta) - f(\frac{3}{2}\delta)| = \frac{1}{\delta^2} - \frac{1}{\frac{9}{4}\delta^2} = \frac{5}{9} \times \frac{1}{\delta^2} > 5 > \varepsilon.$$

Exercise 5.4 - 7: If $f(x) := x$ and $g(x) := \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

Solution:

- First, f is uniformly continuous: Let $\varepsilon > 0$ and choose $\delta := \varepsilon$. Then

$$|f(x) - f(u)| = |x - u| < \delta = \varepsilon.$$

- To see that g is uniformly continuous, notice that $|\sin a - \sin b| < |a - b|$ for all $a, b \in \mathbb{R}$. This holds because 2 triangles drawn in the unit circle always have a greater difference in arc length than in height.

- Finally, fg is not uniformly continuous. Choose $\varepsilon := 1$ and let $\delta > 0$. Now choose the smallest integer n where $n > |1/\sin(\delta/2)|$. Choose $u := 2n\pi$ and $x := u + \delta/2$. Then

$$|fg(x) - fg(u)| = |(2n\pi + \delta/2)\sin(2n\pi + \delta/2) - (2n\pi)\sin(2n\pi)|$$

$$= |(2n\pi + \delta/2)\sin(\delta/2)| > |2n\pi \sin(\delta/2)| > \left|\frac{2}{\sin(\delta/2)}\pi \sin(\delta/2)\right| = 2\pi > \varepsilon$$

Exercise 5.4 - 14: A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Solution: Let f be a continuous periodic function on $\mathbb{R}$ with period p and let $I := [0, p]$. Then

- the set $f(I)$ must be a closed interval. Now let $x \in \mathbb{R}$. Then $f(x) = f(a)$ for some $a \in I$, and hence f is bounded everywhere.
- f is uniformly continuous on I. By a similar argument, f is uniformly continuous everywhere.

Exercise 5.6 - 8: Let f, g be strictly increasing on an interval $I \subseteq \mathbb{R}$ and let $f(x) > g(x)$ for all $x \in I$. If $y \in f(I) \cap g(I)$, show that $f^{-1}(y) < g^{-1}(y)$.

Solution: Define $x_f := f^{-1}(y), x_g := g^{-1}(y)$. Suppose that $x_f \geq x_g$. Then, because g is strictly increasing, $g(x_f) \geq g(x_g) = y = f(x_f)$. Now we have the clear contradiction $g(x_f) \geq f(x_f)$. ∎

Exercise 5.6 - 10: Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If f has an absolute maximum [respectively, minimum] at an interior point c of I, show that f is not injective on I.

Solution: WLOG, assume c is an absolute maximum and $b > a$. Choose a small enough δ so that f is increasing over $I_1 := (c - \delta, c)$ and decreasing over $I_2 := (c, c + \delta)$. Then $S := f(I_1) \cap f(I_2)$ is either empty or nonempty. If it is nonempty, then there exists $a \in I_1, b \in I_2$ so that $f(a) = f(b)$, and hence f is not injective. If S is empty, then one of I_1 and I_2 are constant under f, and again f is not injective. ∎

Exercise 5.6 - 12: Let $f : [0, 1] \to \mathbb{R}$ be a continuous injective function with $f(0) < f(1)$. Show that f is strictly increasing on $[0, 1]$.

Solution: Let $a, b \in [0, 1]$ with $a < b$. If $f(a) < f(b)$ we are done, $f(a) = f(b)$ is impossible because f is injective, and so we consider $f(a) > f(b)$. Define $I := [a, b]$ and consider $m := \max f(I)$. Either m is inside I, or $m = a$. If m is internal then exercise 5.6 - 10 shows that f is not injective and we have a contradiction. If $m = a$ then slide a backwards until $\max I$ is not an endpoint.

Exercise 6.1 - 1:

Use the definition to find the derivative of each of the following functions:

(a) $f(x) := x^3$ for $x \in \mathbb{R}$? Let $c \in \mathbb{R}$ and define $L := 3c^2$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon/(4c)$. Then

$$\left|\frac{f(x) - f(c)}{x - c} - L\right| = \left|\frac{x^3 - c^3}{x - c} - 3c^2\right| = |x^2 + cx + c^2 - 3c^2| = |x^2 + cx - 2c^2|$$
$$< |(c + \delta)^2 + c(c + \delta) - 2c^2| = |\delta^2 + 3c\delta| < |4c\delta| = |4c\frac{\varepsilon}{4c}| = \varepsilon$$

(b) $g(x) := 1/x$ for $x \in \mathbb{R}, x \neq 0$? Let $c \in \mathbb{R}$ and define $L := -1/c^2$. Let $\varepsilon > 0$ and choose $\delta = (c^3\varepsilon)/(1 + c^2\varepsilon)$. Then

$$\left|\frac{g(x) - g(c)}{x - c} - L\right| = \left|\frac{1/x - 1/c}{x - c} + 1/c^2\right| = \left|\frac{x - c}{c^2 x}\right| < \left|\frac{(c + \delta) - c}{c^2 x}\right|$$

$$= \left|\frac{\delta}{c^2x}\right| <^* \left|\frac{\delta}{c^2(c-\delta)}\right| = \left|\frac{(c^3\varepsilon)/(1+c^2\varepsilon)}{c^2(c-(c^3\varepsilon)/(1+c^2\varepsilon))}\right| = \varepsilon$$

Parts c and d were proven unsolvable by Gauss.

Exercise 6.1 - 4: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$.

Solution: *With help from hint in back of book.* Two in one! The function is differentiable because the following limit exists.

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} =^\dagger \lim_{x\to 0} x = 0$$

Exercise 6.1 - 9: Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function [that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an odd function [that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$]. Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.

Solution: Let $c \in \mathbb{R}$.

$$f'(-c) = \lim_{x\to -c}\frac{f(x)-f(-c)}{x-(-c)} = \lim_{x\to -c}\frac{f(x)-f(c)}{x+c} = \lim_{x\to c}\frac{f(-x)-f(c)}{-x+c} = \lim_{x\to c}\frac{f(x)-f(c)}{-x+c} = -f'(c)$$

$$g'(-c) = \lim_{x\to -c}\frac{f(x)-f(-c)}{x-(-c)} = \lim_{x\to -c}\frac{f(x)+f(c)}{x+c} = \lim_{x\to c}\frac{f(-x)+f(c)}{(-x)+c} = \lim_{x\to c}\frac{-f(x)+f(c)}{-x+c} = g'(c)$$

Exercise 6.1 - 15: Given that the restriction of the cosine function cos to $I := [0, \pi]$ is strictly decreasing and that $\cos 0 = 1, \cos\pi = -1$, let $J := [-1, 1]$, and let $\arccos : J \to \mathbb{R}$ be the function inverse to the restriction of cos to I. Show that the arccos is differentiable on $(-1, 1)$ and $D \arccos y = -1/\sqrt{1-y^2}$ for $y \in (-1, 1)$. Show that arccos is not differentiable at -1 and 1.

Solution: Suppose that $x = \arccos y$. Then $\cos x = y$. Taking the derivative of both sides with respect to y yields $-\sin y \times \frac{dx}{dy} = 1$. Dividing through by $-\sin y$, we have our desired result:

$$\frac{d \arccos y}{dy} = \frac{dx}{dy} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 x}} = \frac{-1}{\sqrt{1-y^2}}$$

Clearly this is well defined for all $x \in (0, 1)$. The derivative does not exist at $x = 0$ or $x = 1$ because arccos is not continuous there (0 and 1 are endpoints).

Exercise 6.2 - 2: Find the points of relative extrema, the intervals on which the following functions are increasing, and those on which they are decreasing. *Since the problem asks to find the values, I provide minimal explanation.*

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*Assuming $c > 0$. Just switch to $c + \delta$ for $c < 0$.

†Because $0 \le |f(x)/x| \le |x|$ for all $x \in \mathbb{R}$

(a) $f(x) := x + 1/x$ for $x \neq 0$? f has a relative maximum of -2 at $x = -1$ and a relative minimum of 2 at $x = 1$, both holding inside of $\delta = 1/2$. f is increasing over $(\infty, -1)$ and $(1, \infty)$ and decreasing over $(-1, 0)$ and $(0, 1)$. Changes occur at -1 and 1 because $|1/x| > |x|$ only if $|x| < 1$.

(b) $g(x) := x/(x^2 + 1)$ for $x \in \mathbb{R}$? g has a relative minimum of -1/2 at $x = -1$ and a relative maximum of 1/2 at $x = 1$, both holding inside of $\delta = 1/2$. g is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$. Similar to f, -1 and 1 are critical points because $|x| < x^2$ only if $|x| < 1$.

(c) $h(x) := \sqrt{x} - 2\sqrt{2 + x}$ for $x > 0$? h has a relative (and absolute) maximum of $-\sqrt{6}$ at $x = 2/3$, again holding within $\delta = 1/2$. h has no relative minimums. h is increasing over $(0, 2/3)$ and decreasing over $(2/3, \infty)$.

(d) $k(x) := 2x + 1/x^2$ for $x \neq 0$? k has no relative maximums, but does have a relative minimum of 3 at $x = 1$. h increases over $(-\infty, 0)$ and $(1, \infty)$ and decreases over $(0, 1)$. You might expect $x = -1$ to be a critical point because of the $1/x^2$ term, but the curve is grabbed and pulled down by $2x$ and the function ends up being monotone through that point.

Exercise 6.2 - 4: Let $a_1, a_2, \ldots, a_n$ be real numbers and let f be defined on $\mathbb{R}$ by

$$f(x) := \sum_{i=1}^{n} (a_i - x)^2 \text{ for } x \in \mathbb{R}.$$

Find the unique point of relative minimum for f.

Solution: Since $f(x) = \sum(x - a_i)^2 = nx^2 - 2x \sum a_i + \sum a_i^2$ is a simple function of the form $ax^2 + bx + c$, it has an absolute minimum of $-b/(2a) = (2 \sum a_i)/(2n) = (\sum a_i)/n$. ■

Exercise 6.2 - 10: Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of 0 the derivative $g'(x)$ takes on both positive and negative values. Thus g is not monotonic in any neighborhood of 0.

Solution: *With help from book hint.* The derivative at 0:

$$\lim_{x \to 0} g'(x) = \lim_{x \to 0} D(x \times (1 + 2x \sin \frac{1}{x})) = \lim_{x \to 0} xD(1 + 2x \sin \frac{1}{x}) + 1 + 2x \sin \frac{1}{x} = \lim_{x \to 0} 1 + 2x \sin \frac{1}{x} = 1 + 0 = 1$$

The derivative elsewhere:

$$g'(x) = D(x + 2x^2 \sin \frac{1}{x}) = 1 + D(2x^2 \sin \frac{1}{x}) = 1 + 4x \sin \frac{1}{x} + 2x^2 D(\sin \frac{1}{x}) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$$

Let $\delta > 0$ and assume $\delta < 1/10$. Then choose an n such that $x := 1/(n\pi) < \delta$. Then clearly, depending on whether n is odd or even, $g'(x)$ can be positive or negative.

Exercise 6.2 - 15: Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I.

Solution: Let $x, c \in I$. Then, because the derivative is bounded, there exists a natural number K so that $|\frac{f(x)-f(c)}{x-c}| < K$. A little algebra proves the result:

$$\left|\frac{f(x)-f(x)}{x-c}\right| = \frac{|f(x)-f(c)|}{|x-c|} < K \Rightarrow |f(x)-f(c)| < K|x-c|$$

Exercise 6.3 - 8: Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{\arctan x}{x}$ $(-\infty, \infty)$? Applying L'Hospital's rule, we get $\lim_{x\to 0} \frac{1}{1+x^2} = 1$.

(b) $\lim_{x\to 0} \frac{1}{x(\ln x)^2}$ $(0,1)$? We can rewrite it as $\frac{1/x}{(\ln x)^2}$ and apply L'Hospital to get $\frac{-1/x^2}{2\ln x/x} = \frac{1}{2}\frac{1/x}{-\ln x}$. Applying L'Hospital again, we have $\frac{1}{2}\frac{-1/x^2}{-1/x} = \frac{1}{2x}$. Finally, we get $\lim_{x\to 0+} \frac{1}{2x} = \infty$.

(c) $\lim_{x\to 0+} x^3 \ln x$ $(0,\infty)$? Rewrite as $\frac{\ln x}{1/x^3}$ and apply L'Hospital's rule to get $\frac{1/x}{-3/x^4} = \frac{-x^3}{3}$ and we have $\lim_{x\to 0+} \frac{-x^3}{3} = 0$.

(d) $\lim_{x\to\infty} \frac{x^3}{e^3}$ $(0,\infty)$? ∞.

Exercise 6.4 - 3: Use induction to prove Leibniz's rule for the nth derivative of a product:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

Solution: Equality clearly holds for $n = 1$. Now suppose that the equation is true for all $n \leq j$. We will show it also holds for $n = j+1$. For brevity, we omit the "of x" (x) and express differentiation with normal looking exponents.

$$\begin{aligned}
(fg)^{j+1} &= \left((fg)^j\right)^1 \\
&= \frac{d}{dx}\sum_{k=0}^{j}\binom{j}{k} f^{j-k}g^k \\
&= \sum_{k=0}^{j}\frac{d}{dx}\binom{j}{k} f^{j-k}g^k \\
&= \sum_{k=0}^{j}\binom{j}{k}\left(f^{j-k+1}g^k + f^{j-k}g^{k+1}\right) \\
&= \sum_{k=0}^{j+1}\binom{j+1}{k} f^{j-k+1}g^k
\end{aligned}$$

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Exercise 6.4 - 10: Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does *not* converge to zero as $n \to \infty$ for $x \neq 0$. (hint in book)

Solution:

- Note the following:

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x^k} =^{\ddagger} \lim_{x\to 0} \frac{\frac{2}{x^3}e^{-1/x^2}}{kx^{k-1}} = \frac{2}{k}\lim_{x\to 0} \frac{e^{-1/x^2}}{x^{k+2}} \Rightarrow \lim_{x\to 0} \frac{e^{-1/x^2}}{x^k} = 0$$

  And since every $h^{(n)}(x)$ is some composition of products and additions of $\frac{h(x)}{x^k}$, we know $h^{(n)}(0) = 0$.

- Let $n \in \mathbb{N}$. If we choose $x_0 = 0$, then

$$h(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + \cdots + \frac{h^{(n)}(0)}{n!}x^n + \frac{h^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{h^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

  In fact, the constant term is constant as n increases, and hence clearly does not converge to 0.

Exercise 6.4 - 22: The equation $\ln x = x - 2$ has two solutions. Approximate them using Newton's Method. What happens if $x_1 := \frac{1}{2}$ is the initial point?

Solution: Put the equality in the form $\ln x - x + 2 = 0$ and define $f(x) := \ln x - x + 2$. Also define the recurrence

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n} = x_n - \frac{\ln x - x + 2}{1/x - 1}.$$

Choosing $x_1 = 1/6$ gives the first four terms of $(0.166667, 0.158352, 0.158594, 0.158594)$. Choosing $x_1 = 3$ gives $(3, 3.14792, 3.14619, 3.14619)$. Hence, the two solutions are roughly $x = 0.158954$ and $x = 3.14619$. If $x_1 = 1/2$, then x_2 is negative and x_3 is complex.

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‡L'Hospital's rule applies.