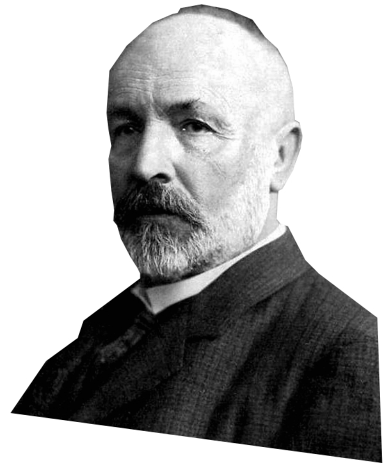
Analysis HW 9 - Luke Miles - December 6, 2015



Last homework!!!

Exercise 7.3 - 21: Let $f, g \in \mathcal{R}[a,b]$.

(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Use (a) to show that $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + (1/t)\int_a^b g^2$

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.

(d) Now prove that $\left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right)$. This is called the Cauchy-Bunyakovsky-Milkshake-Schwarz Inequality.

Solution: For brevity, we omit the limits on the integrals.

(a) Clearly $(tf(x) \pm g(x))^2 \geq 0$ for all $x \in [a,b]$. Integrating both sides gives the result.

(b) $0 \leq \int (tf \pm g)^2 = \int (t^2 f^2 \pm 2tfg + g^2) = t^2(\int f^2) \pm 2t(\int fg) + (\int g^2) \to \pm 2(\int fg) \leq t(\int f^2) + \frac{1}{t}(\int g^2)$ The conclusion follows.

(c) Since $f^2 \geq 0$, $\int f^2 = 0$ means that $f(x) = 0$ for all x. The conclusion follows.

(d) *(With help from book hint.)* If $\int f^2 = 0$ then everything is zero. Otherwise, letting $t = \sqrt{\int g^2 / \int f^2}$ in (b) yields $2|\int fg| \leq \sqrt{\int g^2 / \int f^2}\int f^2 + \sqrt{\int f^2 / \int g^2}\int g^2 = 2\sqrt{\int g^2 \int f^2}$. Dividing both sides by 2 and squaring shows the first is $\leq$ the third. The second is sandwhiched in their for semi-obvious reasons.

Exercise 7.4 - 5: Let f, g, h be bounded functions on $I := [a,b]$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. Show that if f and h are Darboux integrable and if $\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\int_a^b g = \int_a^b f$.

Solution: Let $\mathcal{P}$ be any partition of $[a,b]$. Then clearly $L(f,\mathcal{P}) \leq U(g,\mathcal{P})$ and $L(g,\mathcal{P}) \leq U(h,\mathcal{P})$. Hence, $L(f) \leq U(g)$ and $L(g) \leq U(h)$. And since $L(f) = U(h)^*$, we have $U(g) = L(g)$ and hence g is Darboux integrable.

Exercise 7.4 - 6: Let f be defined on $[0,2]$ by $f(x) := 1$ if $x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.

Solution: Let $\varepsilon > 0$ and define the interval $\mathcal{P} = (0, 1-\varepsilon, 1)$. Then
$L(f,\mathcal{P}) = 1 \cdot ((1-\varepsilon) - 0) + 0 \cdot (1 - (1-\varepsilon)) = 1 - \varepsilon$ and $U(f,\mathcal{P}) = 1 \cdot ((1-\varepsilon) - 0) + 1 \cdot (1 - (1-\varepsilon)) = 1$. Clearly L and U can get arbitrarily close to eachother (centering in on 1) by shrinking ε, so f is Darboux integrable and its value is 1.

Exercise 7.4 - 7:

(a) Prove that if $g(x) := 0$ for $0 \leq x \leq 1/2$ and $g(x) := 1$ for $1/2 < x \leq 1$, then the Darboux integral of g on $[0,1]$ is equal to $1/2$.

(b) Does the conclusion hold if we change the value of g at the point $1/2$ to 13?

Solution:

(a) Let $\varepsilon > 0$ and define $\mathcal{P} := (0, 1/2 - \varepsilon, 1/2 + \varepsilon, 1)$. Then $U(g,\mathcal{P}) = 1/2 + \varepsilon$ and $L(g,\mathcal{P}) = 1/2 - \varepsilon$. Then they get arbitrarily close and $L(g) = 1/2 = U(g)$.

(b) Yes. There is the difference is that $U(g,\mathcal{P}) = 1/2 + 25\varepsilon$, but you can choose an even smaller ε.

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*Because $\int_a^b g = \int_a^b f$