Analysis HW 8 - Luke Miles - November 22, 2015

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**Exercise 7.1 - 6**:

(a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

(b) Let $h(x) := 2$ if $0 \le x \le 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

Solution:

(a) Let $\varepsilon > 0$, choose $\delta = \varepsilon/3$, and let $\dot{\mathcal{P}}$ be a tagged partition of $[0,2]$ with $\|\dot{\mathcal{P}}\| < \delta$. Now break $\dot{\mathcal{P}}$ into two pieces, $\dot{\mathcal{P}}_1$ with its tags in $[0,1)$ and $\dot{\mathcal{P}}_2$ with its tags in $[1,2]$. Then the value of any $x$ in any interval in $\dot{\mathcal{P}}_1$ is less than $1+\delta$ and hence the value of $S(f,\dot{\mathcal{P}}_1)$ is less than $2(1+\delta)$. Likewise, $S(f,\dot{\mathcal{P}}_2) \le 1+\delta$. Putting this all together:

$$|S(f,\dot{\mathcal{P}}) - 3| = |S(f,\dot{\mathcal{P}}_1) + S(f,\dot{\mathcal{P}}_2) - 3| < |2(1+\delta) + (1+\delta) - 3| = 3\delta = \varepsilon \qquad \blacksquare$$

(b) Do the same argument as for (a), but have 3 partitions. One whose tags are in $[0,1)$, one who just has the tag 1, and one whose tags are in $(1,2]$. Then $S$ of these three chunks can't exceed $2(1+\delta)$, $3\delta$, and $1+\delta$ respectively. So their sum cannot exceed $3+5\delta$, and if we choose $\delta = \varepsilon/5$, then $S$ of the whole is within $\varepsilon$ of 3. Hence the integral exists and is 3. $\blacksquare$

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**Exercise 7.1 - 9**: If $f \in \mathcal{R}[a,b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$, prove that $\int_a^b f = \lim_n S(f,\dot{\mathcal{P}}_n)$.
Solution: Suppose $\int_a^b f = L$. Then

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall \dot{\mathcal{P}} : (\|\dot{\mathcal{P}}\| < \delta \longrightarrow |S(f,\dot{\mathcal{P}}) - L| < \varepsilon),$$

where $\dot{\mathcal{P}}$ is a tagged partition over $[a,b]$. And since $\|\dot{\mathcal{P}}_n\| \to 0$, for any $\delta > 0$ we can choose a large enough $k$ so that $\|\dot{\mathcal{P}}_k\| < \delta$ and hence $S(f,\dot{\mathcal{P}})$ is within $\varepsilon$ of $L$. Taking the limit as $n \to \infty$, we reach equality.

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**Exercise 7.1 - 10**: Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim_n S(g, \dot{\mathcal{P}}_n)$ exists.

Solution:

- Regardless of a chosen $L$, $\varepsilon$, and $\delta$, we know $S(g, \dot{\mathcal{P}}) = 0$ if rational tags are chosen for $\dot{\mathcal{P}}$ and $S(g, \dot{\mathcal{P}}) \geq 1$ if irrational tags are chosen. Since the Riemman integral is unique, we have that $g \notin \mathcal{R}[0,1]$.
- Define $\dot{\mathcal{P}}_n$ to evenly split the interval $[a,b]$ into $n$ pieces of size $1/n$, with all rational tags. Then clearly $\|\dot{\mathcal{P}}\| = 1/n$ goes to zero as $n$ goes to infinity. Furthermore, for any $n$, $S(g, \dot{\mathcal{P}}) = 0$.

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**Exercise 7.1 - 14**: Let $0 \leq a < b$, let $Q(x) := x^2$ for $x \in [a,b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a,b]$. For each $i$, let $q_i$ be the positive square root of $\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$.

(a) Show that $q_i$ satisfies $0 \leq x_{i-1} \leq q_i \leq x_i$.

(b) Show that $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

(c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags $q_i$, show that $S(Q, \dot{\mathcal{Q}}) = \frac{1}{3}(b^3 - a^3)$.

(d) Show that $Q \in \mathcal{R}[a,b]$ and $\int_a^b Q = \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3)$.

Solution:

(a) Expanding $q_i$, we have $0 \leq x_{i-1} \leq \sqrt{\frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)} \leq x_i$. The first inequality clearly holds because $0 \leq a$. To get the second and third inequalities, we square* and multiply by three: $3x_{i-1} \leq x_i^2 + x_i x_{i-1} + x_{i-1}^2 \leq 3x_i^2$. And this is clear enough if we keep in mind that $x_{i-1} < x_i$.

(b) Just expand and simplify:

$$Q(q_i) \cdot (x_i - x_{i-1}) = (x_i^2 + x_i x_{i-1} + x_{i-1}^2)/3 \cdot (x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$$

(c) We get a telexcoping sum.

$$S(Q, \dot{\mathcal{Q}}) = \sum Q(q_i)(x_i - x_{i-1}) = \sum (x_i^3 - x_{i-1}^3)/3 = (b^3 - a^3)/3$$

(d) For any $\varepsilon$, $\delta$, and $\dot{\mathcal{P}}$, $S$ is constant and evaluates to $(b^3 - a^3)/3$.

*This is allowed because everything is positive.

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**Exercise 7.2 - 8**: Suppose that $f$ is continuous on $[a,b]$, that $f(x) \geq 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution: Suppose that, for some $c \in [a,b]$, we have $f(c) > 0$. If, for some $\alpha > 0$, $\inf f([c-\alpha, c+\alpha]) > 0$, then clearly the integral would be nonzero.[†] So no such $\alpha$ exists and hence $c$ is a point of discontinuity and we have a contradiction.

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**Exercise 7.2 - 10**: If $f$ and $g$ are continuous on $[a,b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Solution: If no such $c$ existed, then one function would have to be strictly greater than the other[‡] and their integrals would be different. Hence, $c$ must exist.

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**Exercise 7.2 - 12**: Show that $g(x) := \sin(1/x)$ for $x \in (0,1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0,1]$.

Solution: *With help from book hint.* Note that $g$ is bounded by 1 on $[0,1]$ and that $\int_a^1 g$ exists for all $a \in (0,1)$[§]. Exercise 7.2 - 11 applies and we get that $g \in \mathcal{R}[a,b]$.

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**Exercise 7.2 - 13**: Give an example of a function $f : [a,b] \to \mathbb{R}$ that is in $\mathcal{R}[c,b]$ for every $c \in (a,b)$ but which is not in $\mathcal{R}[a,b]$.

Solution: $f(x) := 1/x$ with $a := 0, b := 1$ works. $f$ is clearly in $\mathcal{R}[c,1]$ for all $c \in (0,1)$ because it is a continuous function over a closed interval. And since $\int_c^1 f$ gets arbitrarily large as $c$ approaches 0 from the right[¶], $\int_0^1 f$ can not exist.

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**Exercise 7.2 - 18**: Let $f$ be continuous on $[a,b]$, let $f(x) \geq 0$ for $x \in [a,b]$, and let $M_n := (\int_a^b f^n)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a,b]\}$.

Solution: skipped

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**Exercise 7.3 - 8**: Let $F(x)$ be defined for $x \geq 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n), n \in \mathbb{N}$. Show that $F$ is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [x]dx$ for $0 \leq a < b$, where $[x]$ is the floor of $x$.

Solution: If $x \in (n-1, n)$ then $F$ is linear and hence continuous.

[†]Because $\int_{c-\alpha}^{c+\alpha} f > 0$ and there are no negative chunks to cancel it out.

[‡]intermediate value theorem

[§]continuous on closed interval

[¶]This can be seen with a geometric argument where you draw rectangles of width $1/(n-1) - 1/n$ and height $n$, with a combined area as big as you want. Alternatively, for any $\alpha > 0$, choose $a = e^{-\alpha}$.

We will show $f$ is also continuous at $x = n$:

$$\lim_{x \to n-} F(x) = \lim_{x \to n-} ((n-1)x - (n-1)n/2) = (n-1)n - (n-1)n/2 = ((n+1)-1)n - ((n+1)-1)(n+1)/2 = F(n)$$

$F'(x) = n - 1 = [x]$ and exists only when $x \in (n-1, n)$. By the fundamental theorem of calculus etc:

$$\int_a^b [x]dx = \int_a^b F'(x) = F(b) - F(a)$$

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**Exercise 7.3 - 10**: Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and let $v : [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$ with $v([c,d]) \subseteq [a,b]$. If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c,d]$.
_Solution_: Since $f \in \mathcal{R}[a,b]$, we can choose a function $F$ so that $F'(x) = f(x)$ for all $x \in [a,b]$. Then

$$\frac{d}{dx} \int_a^{v(x)} f(x)dx = \frac{d}{dx}(F(v(x)) - F(a)) = \left(\frac{d}{dx}F(v(x))\right) - \left(\frac{d}{dx}F(a)\right) =^{\|} f(v(x)) \cdot v'(x)$$

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**Exercise 7.3 - 14**: Show there does not exist a continuously differentiable function $f$ on $[0,2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \le 2$ for $0 \le x \le 2$.
_Solution_: In general, if $g(x) \le h(x)$ for all $x$ in $[a,b]$, then $\int_a^b g \le \int_a^b h$. But if the described $f$ existed then we'd have $2 \ge f'(x)$ and $\int_0^2 2 = 4 < 5 = \int_0^2 f'(x)$ which is ridiculous.

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**Exercise 4.3 - 16**: If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0,1]$, show that $f(x) = 0$ for all $x \in [0,1]$.
_Solution_: If $\int_0^x f = \int_x^1 f$ then there exists an $F$ so that $F(x) - F(0) = F(1) - F(x)$ and hence
$2F(x) = F(1) - F(0)$. Taking $d/dx$ of both sides, we have $2f(x) = 0$ and hence $f(x) = 0$ for all $x \in [0,1]$.

$\|$Chain rule on first term, second term is zero.