Analysis HW 3 - Luke Miles - September 21, 2015

Sorry, I'm still pretty new at $\LaTeX$.

Exercise 2.4 - 2: If $S := \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

<u>*Solution*</u>: First, we will show $\inf S = -1$. If $-1 = \inf S$, then (i.) -1 is a lower bound of S, and (ii.) there is no lower bound of S greater than -1.

(i) Suppose $\exists s < -1$ in S. Then for some $m, n \in \mathbb{N}$, $s = 1/n - 1/m < -1$. Multiplying through by nm yields $m - n < -nm$, or equivelantly, $n - m > nm$. Add m and factor to get $n > m(n+1)$, a clear contradiction.

(ii) Suppose $\exists x > -1$ which is less than every s in S. Then let $m = 1$ and let n be greater than $1/(x+1)$. Then $s = 1/n - 1/m = 1/n - 1 < \frac{1}{1/(x+1)} - 1 = x$. Contradiction.

This thoroughly shows that $\inf S = -1$. Now we will show $\sup S = 1$. Likewise, we must show that 1 is (i) an upper bound and (ii) there is no smaller upper bound.

(i) Suppose $\exists s > 1$ in S. Then for some $m, n \in \mathbb{N}$, $s = 1/n - 1/m > 1$. Multiplying through by nm yields $m - n > nm$. Adding n and factoring yields $m > n(m+1)$, a contradiction.

(ii) Now suppose $\exists x < 1$ which is greater than every s in S. Then let $n = 1$ and let m be greater than $1/(1-x)$. Then $s = 1/n - 1/m = 1 - 1/m > 1 - \frac{1}{1/(1-x)} = x$. Contradiction.

So $\sup S$ is necessarily 1. ■

Exercise 2.4 - 6: Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that [$\sup(a + S) = a + \sup S$] (for any bounded set S and any real number a) implies that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.$$

Also show that [$\inf(a + S) = a + \inf S$] implies

$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

<u>*Solution*</u>: Let $Y := \{f(x) : x \in X\}$. Note that $\sup Y$ must exist because f's range is bounded. Then $\sup\{a + f(x) : x \in X\} = \sup\{a + y : y \in Y\} = \sup(a + Y) = a + \sup Y = a + \sup\{f(x) : x \in X\}$. A similar argument proves the second equation. ■

Exercise 2.4 - 11: Let X and Y by nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by $f(x) := \sup\{h(x, y) : y \in Y\}$ and $g(y) := \inf\{h(x, y) : x \in X\}$. Prove that $\sup\{g(y) : y \in Y\} \le \inf\{f(x) : x \in X\}$.

<u>*Solution*</u>: (For simplicity, we treat the sets as if they include their sup and inf, but the arguments hold without this constraint.) Let $x \in X$ and $y \in Y$. Then

- $h(x, y) \le f(x)$ This is necessarily the case because f chooses the y that maximizes h.

- $g(y) \le h(x, y)$ This holds because g chooses the x that minimizes h.

Putting these inequalities together yields $g(y) \le f(x)$ for all $x \in X$ and $y \in Y$. Since any element in the range of g is less than every element in the range of f, this also holds for *the smallest element of the range of* g and *the largest element of the range of f*. An equivelant way of saying this:

$$\sup\{g(y) : y \in Y\} \le \inf\{f(x) : x \in X\} \blacksquare$$

Exercise 2.5 - 2: If $S \subseteq \mathbb{R}$ is nonempty, show that S is bounded if and only if there exists a bounded closed interval I such that $S \subseteq I$.

Solution: ($\Rightarrow$) Accept S is bounded. Let $I := [\inf S, \sup S]$. Now suppose (for contradiction) that $\exists s \in S$ where $s \notin I$. Since I is an interval, s can not be in a "hole" in I. So s must be to the left or right of I on the number line. This contradicts the definition of I, and hence I be a superset.

($\Leftarrow$) Accept S is a subset of some closed bounded interval I. Suppose S is not bounded above. Then for any real x, there exists some $s \in S$ where $s > x$. This includes $x := \sup I$. Furthermore, s is a member of I because S is a subset of I. But s can not be a member of a set I and greater than its sup!!! This contradiction shows that S is bounded above. A similar argument shows S is bounded below. $\blacksquare$

Exercise 2.5 - 12: Give the two binary representations of $\frac{3}{8}$ and $\frac{7}{16}$.

Solution: Every nonzero number has two binary expansions. One has finite length, the other has infinite length. This is due to the fact that

$$2^k = \sum_{i=k-1}^{-\infty} 2^i$$

for integer k.

$\frac{3}{8} = \frac{1}{4} + \frac{1}{8} = 2^{-2} + 2^{-3} = 0.011_2$

$\frac{3}{8} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots = 2^{-2} + 2^{-4} + 2^{-5} + 2^{-6} + \ldots = .010111\ldots_2$

$\frac{7}{16} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 2^{-2} + 2^{-3} + 2^{-4} = 0.0111_2$ and by the same process as above,

$\frac{7}{16} = 0.0110111111\ldots_2$

Exercise 2.5 - 17: What rationals are represented by the periodic decimals $1.25\overline{137}$ and $35.14\overline{653}$?

Solution: If an n-digit natural number $d_1 d_2 \ldots d_n$ is divided by n "9's in a row," then the repeating decimal $0.\overline{d_1 d_2 \ldots d_n}$ is produced. This holds even if a reduction is possible. For example, $\frac{39}{99} = \frac{13}{33} = 0.\overline{39}$. To produce a fraction from an arbitrary decimal, extract the repeating part and add it over n 9's in a row to the nonrepeating part. Applying this to the first piece of the problem:

$$1.25\overline{137} = 1.25 + 0.00\overline{137} = \frac{5}{4} + \frac{137}{999 \times 100} = \frac{31253}{24975}$$

And in the second piece:

$$35.14\overline{653} = 35.14 + .00\overline{653} = \frac{3514}{100} + \frac{653}{999 \times 100} = \frac{3511139}{99900}$$

Exercise 3.1 - 5 *Use the definition of the limit of a sequence to establish the following limits.* It suffices to show in each case that, given any positive ε, one can produce an n such that the n'th term in the sequence is less than ε away from the limit.

(a) $\lim(\frac{n}{n^2+1}) = 0$. *pf*: Let $\varepsilon > 0$ and $n > 1/\varepsilon$. Then

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{1/\varepsilon} = \varepsilon$$

(b) $\lim(\frac{2n}{n+1}) = 2$. *pf*: Let $\varepsilon > 0$. Let $n > \frac{2}{\varepsilon} - 1$. Then

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n}{n+1} - \frac{2(n+1)}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{(2/\varepsilon - 1) + 1} = \varepsilon$$

(c) $\lim(\frac{3n+1}{2n+5}) = 3/2$. *pf*: Let $\varepsilon > 0$ and let $n > \frac{1}{4}(13\frac{1}{\varepsilon} - 5)$. Then

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{3n+1}{2n+5}\frac{2}{2} - \frac{3}{2}\frac{2n+5}{2n+5}\right| = \left|\frac{-13}{4n+5}\right| = \frac{13}{4n+5} < \frac{13}{4(\frac{1}{4}(13\frac{1}{\varepsilon} - 5)) + 5} = \varepsilon$$

(d) $\lim(\frac{n^2-1}{2n^2+3}) = \frac{1}{2}$. *pf*: Let $\varepsilon > 0$ and $n > \sqrt{\frac{1}{4}(1/\varepsilon - 6)}$. Then

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{-5}{6+4n^2}\right| = \frac{5}{4n^2+6} < \frac{1}{4n^2+6} < \frac{1}{4\sqrt{\frac{1}{4}(1/\varepsilon - 6)}^2 + 6} = \varepsilon$$

Exercise 3.1 - 7: Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.

(a) Use the definition of a limit to show that $\lim(x_n) = 0$.

Solution: Let $\varepsilon > 0$ and let $n > e^{1/\varepsilon}$. Then

$$|x_n - 0| = \left|\frac{1}{\ln(n+1)}\right| = \frac{1}{\ln(n+1)} < \frac{1}{\ln n} < \frac{1}{\ln(e^{1/\varepsilon})} = \frac{1}{1/\varepsilon} = \varepsilon \blacksquare$$

(b) Find a specific value of $n(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.

Solution: If $\varepsilon = 1/2$, then $e^{\frac{1}{\varepsilon}} = e^{\frac{1}{1/2}} = e^2 \approx 7.4$. So $n = 8$ should work. Indeed, $1/(\ln(8+1)) \approx 0.455 < 1/2$. Likewise, if $\varepsilon = 1/10$, then $e^{\frac{1}{\varepsilon}} = e^{\frac{1}{1/10}} \approx 22026.5$. So here we can choose $n = 22027$. $\blacksquare$

Exercise 3.1-11: Show that $\lim(1/n - 1/(n+1)) = 0$.

<u>*Solution*</u>: We've shown in class that for any functions f and g, $\lim(f(x) - g(x)) = \lim f(x) - \lim g(x)$. Hence, $\lim(1/n - 1/(n+1)) = \lim(1/n) - \lim(1/(n+1))$. And since $\lim f(x) = \lim f(x+1)$ for any function f, $\lim(1/n) = \lim(1/(n+1))$. So finally we have

$$\lim\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim\frac{1}{n} - \lim\frac{1}{n+1} = \lim\frac{1}{n} - \lim\frac{1}{n} = 0 \blacksquare$$

Exercise 3.1 - 17: Show that $\lim(2^n/n!) = 0$.

<u>*Solution*</u>: Define a sequence $x_n = 2^n/n!$. Note that $x_4 = 2^4/4! = 16/24 = 2/3$. Consider x_k for $k > 4$.

$$x_k = \frac{2^k}{k!} = \left(\frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\right)\frac{2}{5}\frac{2}{6}\frac{2}{7}\cdots\frac{2}{k} = \frac{2}{3}\frac{2}{5}\frac{2}{6}\frac{2}{7}\cdots\frac{2}{k} < \frac{2}{3}\frac{2}{4}\frac{2}{4}\frac{2}{4}\cdots\frac{2}{4_{k-4}} = \frac{2}{3}\frac{1}{2^{k-4}}$$

Define the sequence $y_n := 2/(3 \times 2^{n-4})$ with domain $n > 4$. The above shows that $x_n < y_n$. Furthermore, y_n clearly converges to 0. We will define another sequence $z_n = 0$. This sequence converges to 0 and is strictly less than x_n (x_n is always positive). We now have two sequences that surround x_n and both converge to 0, so by the squeeze theorem (covered in class), $\lim x_n = 0$. $\blacksquare$