Using the finite element method (FEM) to solve 2D steady flow with quadratic triangular element

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Abstract

A small note about FEM.

1 2D Poisson equation

The 2D Poisson equation is

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) = 0, \tag{1}$$

where h is pressure, k_x and k_y are conductivity parameters, and here $k_x = k_y = k$. Therefore, Eq. (1) can be rewritten as

$$k\nabla^2 h = 0, \tag{2}$$

In addition, the flux rates, q_x and q_y, can be determined as

$$q_x = -k\frac{\partial h}{\partial x}, \tag{3}$$

$$q_y = -k\frac{\partial h}{\partial y}, \tag{4}$$

2 Boundary condition (BC)

The first kind of BC is

$$h|_\Gamma = const, \tag{5}$$

and the second kind of BC is

$$k\frac{\partial h}{\partial n} = k\boldsymbol{q}\cdot\boldsymbol{n} = k(q_x\cos\theta_x + q_y\cos\theta_y) = k(q_x l + q_y m), \tag{6}$$

where $\boldsymbol{q}$ is flux rate vector, and $\boldsymbol{n}$ is the direction cosines of the normal to the boundary, and they are demonstrated in Fig. 1.

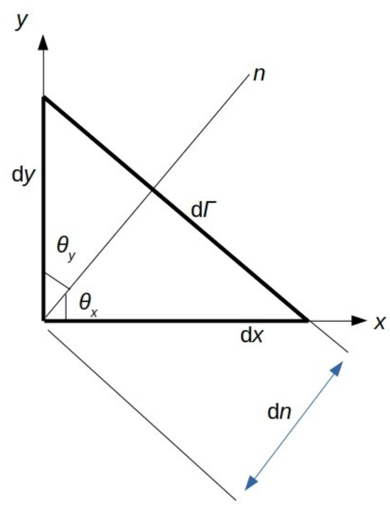

Figure 1: The direction cosines of the normal to the boundary

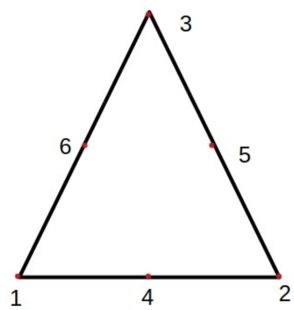

Figure 2: Quadratic triangular element

3 Quadratic triangular element

A quadratic triangular element is shown in Fig. 2. The corresponding shape function is

$$\boldsymbol{N} = [N_1, N_2, N_3, N_4, N_5, N_6]^T, \tag{7}$$

and the six components of $\boldsymbol{N}$ are

$$N_1 = (1 - \xi - \eta)(1 - 2\xi - 2\eta), \tag{8}$$

$$N_2 = \xi(2\xi - 1), \tag{9}$$

$$N_3 = \eta(2\eta - 1), \tag{10}$$

$$N_4 = 4\xi(1 - \xi - \eta), \tag{11}$$

$$N_5 = 4\xi\eta, \tag{12}$$

$$N_6 = 4\eta(1-\xi-\eta), \tag{13}$$

where ξ and η are two components of the local coordinate system, which can be converted to the global coordinates, i.e., (x, y) values of nodes belonging to a triangle (more details are given in the following section).

4 Weighted residual equation

According to the variational principle, the Eq. (1) can be transformed to:

$$k \iint_{\Omega} \boldsymbol{N} \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \mathrm{d}x\mathrm{d}y = 0, \tag{14}$$

By using the Green's theorem, the Eq. (14) can be rewritten as:

$$k \iint_{\Omega} \left(\frac{\partial \boldsymbol{N}}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \boldsymbol{N}}{\partial y} \frac{\partial h}{\partial y} \right) \mathrm{d}x\mathrm{d}y = k \int_{\Gamma} \boldsymbol{N} \frac{\partial h}{\partial n} \mathrm{d}\Gamma, \tag{15}$$

For **an arbitrary point** inside an internal triangular elements which are **not affected** by the second kind of BC, we can use the shape function (i.e., Eq. (7)) to approximate its pressure, namely, $h_{arb} = \boldsymbol{N}^T \boldsymbol{h_e}$, if we know the local coordinates of this point, (ξ_{arb}, η_{arb}), and the pressure values of the six nodes of the triangular element, $\boldsymbol{h_e}$. Therefore, for an internal element, the Eq. (15) can be rewritten as:

$$k \iint_{\Omega} \left(\frac{\partial \boldsymbol{N}}{\partial x} \frac{\partial \boldsymbol{N}^T}{\partial x} + \frac{\partial \boldsymbol{N}}{\partial y} \frac{\partial \boldsymbol{N}^T}{\partial y} \right) \mathrm{d}x\mathrm{d}y \boldsymbol{h_e} = 0, \tag{16}$$

However, as can be seen in Eqs. (8) to (13), the shape functions relate to local coordinate system but not global coordinate system. So, we can use the Jacobian matrix, $\boldsymbol{J}$, to convert global coordinate system to local coordinate system:

$$\begin{pmatrix} \partial \boldsymbol{N}/\partial x \\ \partial \boldsymbol{N}/\partial y \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \partial \boldsymbol{N}/\partial \xi \\ \partial \boldsymbol{N}/\partial \eta \end{pmatrix}, \tag{17}$$

The Jacobian matrix is

$$\boldsymbol{J} = \begin{pmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{pmatrix}, \tag{18}$$

where $\partial x/\partial \xi = (\partial \boldsymbol{N}^T/\partial \xi)\boldsymbol{x_e}$ (note that $\boldsymbol{x_e}$ is the x values of the six nodes of an element), because $x_{arb} = \boldsymbol{N}^T \boldsymbol{x_e}$. The same goes for other components of $\boldsymbol{J}$.

By introducing $\boldsymbol{J}$, the Eq. (16) can be rewritten as:

$$k \int_{-1}^{1} \int_{-1}^{1} |\boldsymbol{J}| \left(\frac{\partial \boldsymbol{N}}{\partial \xi} \frac{\partial \boldsymbol{N}^T}{\partial \xi} \boldsymbol{J}^{-1} \boldsymbol{J}^{-1} + \frac{\partial \boldsymbol{N}}{\partial \eta} \frac{\partial \boldsymbol{N}^T}{\partial \eta} \boldsymbol{J}^{-1} \boldsymbol{J}^{-1} \right) \mathrm{d}\xi\mathrm{d}\eta \boldsymbol{h_e} = 0, \tag{19}$$

and we can find that the determinant of $\boldsymbol{J}$ is in the left-hand side, because $\int_{x_1}^{x_2} G(x)\mathrm{d}x = \int_{-1}^{1} G^*(\xi)|\boldsymbol{J}|\mathrm{d}\xi$.

It is still seen that the Eq. (19) is a definite integration, and we have to transform it into a numerical integration. According to the Gaussian quadrature rule, the weight of six orders, $w_i(1 \leq w \leq 6)$, and the two coordinate components of the six Gaussian points are listed in Table 1. Finally, we get

$$k\sum_{i=1}^{6}\sum_{j=1}^{6}\left[w_i w_j|\boldsymbol{J}|\left(\frac{\partial \boldsymbol{N}}{\partial \xi_i}\frac{\partial \boldsymbol{N}^T}{\partial \xi_i}\boldsymbol{J}^{-1}\boldsymbol{J}^{-1} + \frac{\partial \boldsymbol{N}}{\partial \eta_j}\frac{\partial \boldsymbol{N}^T}{\partial \eta_j}\boldsymbol{J}^{-1}\boldsymbol{J}^{-1}\right)\right]\boldsymbol{h_e} = 0, \tag{20}$$

For simplicity, the Eq. (20) can be rewritten as

$$\boldsymbol{K}_e\boldsymbol{h}_e = 0, \tag{21}$$

so, $\boldsymbol{K}_e$ is a 6×6 matrix, and $\boldsymbol{h}_e$ is a 6×1 matrix.

Table 1: Nodes and weights of Gaussian quadrature formulas (six orders)

w_i	ξ_i	η_i
0.1713244923791700	0.9324695142031520	0.9324695142031520
0.3607615730481380	0.6612093864662640	0.6612093864662640
0.4679139345726910	0.2386191860831960	0.2386191860831960
0.1713244923791700	-0.9324695142031520	-0.9324695142031520
0.3607615730481380	-0.6612093864662640	-0.6612093864662640
0.4679139345726910	-0.2386191860831960	-0.2386191860831960

5 Influence of the second kind of BC

For an element adjacent to the second kind of BC, the right-hand side of the weighted residual equation (Eq. (15)) cannot be removed. There is an approximation (for an element)

$$\frac{\partial h}{\partial n} = \boldsymbol{N}_\Gamma^T\left(\frac{\partial \boldsymbol{h}}{\partial n}\right)_\Gamma^e, \tag{22}$$

So, the right-hand side of Eq. (15) is rewritten as:

$$k\int_\Gamma \boldsymbol{N}\frac{\partial h}{\partial n}\mathrm{d}\Gamma = k\int_{\Gamma_e}\boldsymbol{N}_\Gamma\left[\boldsymbol{N}_\Gamma^T\left(\frac{\partial \boldsymbol{h}}{\partial n}\right)_\Gamma^e\right]\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi}\mathrm{d}\xi, \tag{23}$$

An example is shown in Fig. 3: the edge NO. 1 is the second kind of BC, and the edge length is L_1. Thus, we have

$$\xi = 2 \times \frac{\Gamma}{L_1} - 1 \Rightarrow \Gamma = L_1\left(\frac{\xi+1}{2}\right), \tag{24}$$

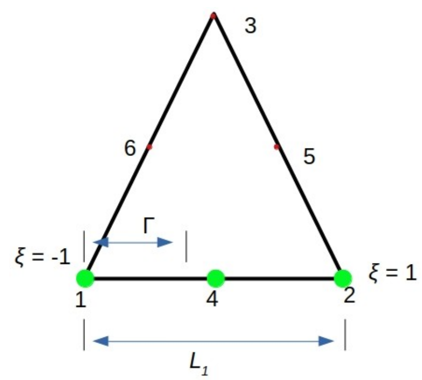

Figure 3: Edge NO. 1 of an element is the second king of BC

and

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} = \frac{L_1}{2}, \tag{25}$$

For the nodes NO. 1, 4 and 2, we defined their local IDs are NO. 1, 2 and 3, and the second kind of BC values are known

$$\left(\frac{\partial h}{\partial n}\right)^e_{\Gamma_1} = q_1, \left(\frac{\partial h}{\partial n}\right)^e_{\Gamma_2} = q_2, \left(\frac{\partial h}{\partial n}\right)^e_{\Gamma_3} = q_3, \tag{26}$$

The corresponding right-hand side of Eq. (15) is

$$\boldsymbol{F}_e = k \int_{\Gamma_e} \boldsymbol{N}_\Gamma \left[\boldsymbol{N}_\Gamma^T \left(\frac{\partial \boldsymbol{h}}{\partial n}\right)^e_\Gamma\right] \frac{\mathrm{d}\Gamma}{\mathrm{d}\xi} \mathrm{d}\xi, \tag{27}$$

where $\boldsymbol{F}_e$ is a 3×1 vector. Always note that $\left(\frac{\partial \boldsymbol{h}}{\partial n}\right)^e_\Gamma$ is a column vector, i.e., $[q_1, q_2, q_3]^T$, because the left-hand side of Eq. (23) is a number. $\boldsymbol{N}_\Gamma$ is

$$\boldsymbol{N}_\Gamma = \begin{bmatrix} 0.5\xi(\xi-1) \\ (1-\xi)(1+\xi) \\ 0.5\xi(\xi+1) \end{bmatrix}, \tag{28}$$

To be more specific, the three components of $\boldsymbol{F}_e$ after integration are:

$$F_{e1} = k\frac{L_1}{2}\int_{-1}^{1} 0.5\xi(\xi-1)[0.5\xi(\xi-1)q_1+(1-\xi)(1+\xi)q_2+0.5\xi(\xi+1)q_3]\mathrm{d}\xi, \tag{29}$$

$$F_{e2} = k\frac{L_1}{2}\int_{-1}^{1} (1-\xi)(1+\xi)[0.5\xi(\xi-1)q_1+(1-\xi)(1+\xi)q_2+0.5\xi(\xi+1)q_3]\mathrm{d}\xi, \tag{30}$$

$$F_{e3} = k\frac{L_1}{2}\int_{-1}^{1} 0.5\xi(\xi+1)[0.5\xi(\xi-1)q_1+(1-\xi)(1+\xi)q_2+0.5\xi(\xi+1)q_3]\mathrm{d}\xi, \tag{31}$$

respectively. Finally, for an element adjacent to boundary, we have

$$\boldsymbol{K}_e\boldsymbol{h}_e = 0 + \boldsymbol{F}_e, \tag{32}$$

6 Remaining steps

Before the solution, we have to assemble a global matrix. Briefly, we develop elementary matrices one by one, and we should label each row and column of an elementary matrix with the local and global node IDs. Based on these labels, we can transfer the elements in a local matrix to the right positions of a global matrix. Finally, we get

$$\boldsymbol{K}\boldsymbol{h} = \boldsymbol{F}, \tag{33}$$

However, the first kind of BC is not considered yet. This is important because the pressures of some boundary nodes are known. Here, the so-called Method of Large Numbers (*Programming Finite Elements in Java* by *Gennadiy Nikishkov*) are used to apply it into the global matrix. For instance, if node NO. 1 is a first kind of BC node, then we can find the element $K_{(1,1)}$ in global matrix and replace it with $K_{(1,1)} \times 10^{16}$, besides, we should find F_1 and similarly replace it with $K_{(1,1)} \times h_1 \times 10^{16}$.

Now, we can use a solver to handle Eq. (33). Furthermore, after solving the equation, we can determine flux rate of an arbitrary point in an element. Supposed that we have known $\boldsymbol{h}_e$ of an element, the pressure of an arbitrary point in an element, h_{arb}, is

$$h_{arb} = \boldsymbol{N}^T \boldsymbol{h}_e, \tag{34}$$

and therefore,

$$\begin{pmatrix} q_x^{arb} \\ q_y^{arb} \end{pmatrix} = \begin{pmatrix} -k \frac{\partial \boldsymbol{N}^T}{\partial x} \boldsymbol{h}_e \\ -k \frac{\partial \boldsymbol{N}^T}{\partial y} \boldsymbol{h}_e \end{pmatrix}, \tag{35}$$

The calculation of $\partial \boldsymbol{N}^T / \partial x$ can be found in Section 4. To determine inlet and outlet fluxes, we have to set points along the inlet and outlet boundaries, and then calculated the flux rates. Note that q_{inlet} or $q_{outlet} = \boldsymbol{q} \cdot \boldsymbol{n}$ ($\boldsymbol{n}$ is the normal to boundary), although the q of a point adjoining to an inlet or outlet boundary is usually parallel to the normal of inlet or outlet boundary.