Lecture 13 — Minimum Spanning Trees (Part 1)

COMPSCI 308 — Design and Analysis of Algorithms

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Summary

Assignments[1]

Practice makes perfect!

📖 Required Readings:

- Section 21.1.

✏️ Required Exercises:

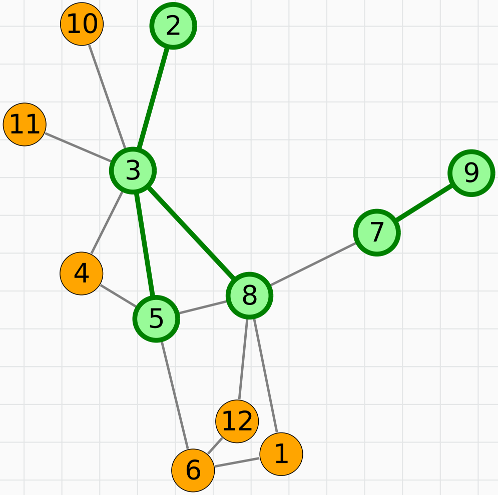
- Exercises 21.1 — 1–7.

[1] 💣 Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

ITA 23 Minimum Spanning Trees

A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a spanning subgraph when $W = V$.

🍰 Is the green part a spanning subgraph?



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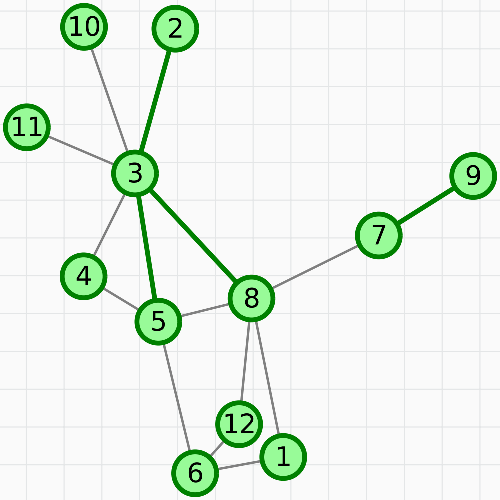

A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a spanning tree if H is both a *spanning subgraph* of G and a *tree*.

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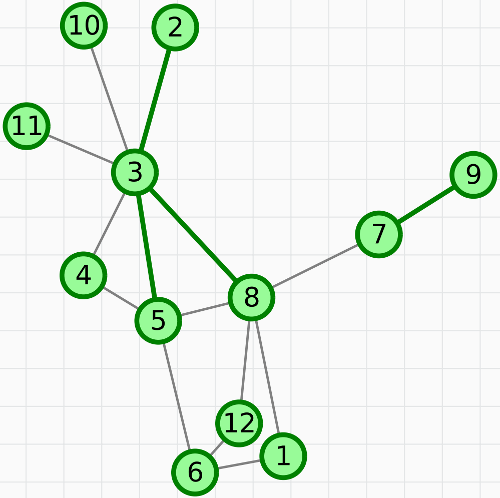

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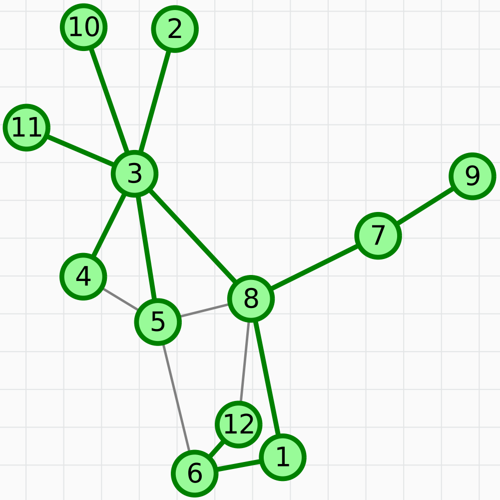

A weighted graph is a graph in which each edge has a numerical value, known as its weight.

🍰 What are some examples where networks can be modeled by a weighted graph?

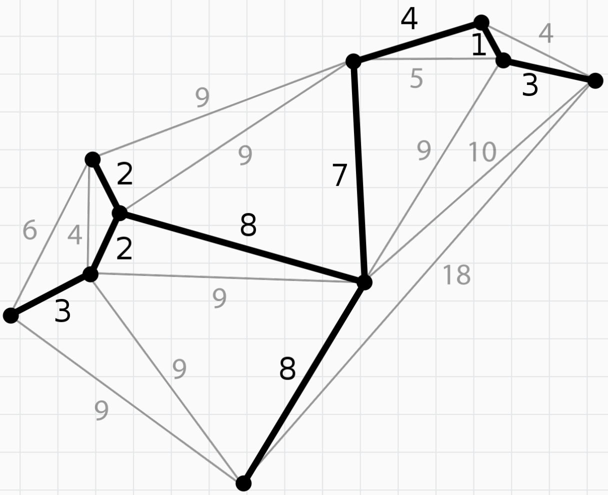

Figure 1: A weighted graph. Source — 👆 Wikipedia

A Minimum Spanning Tree (MST) is a spanning tree of a weighted graph that minimizes the sum of the weights of its edges.

🍰 Why does every connected weighted graph have at least one MST?

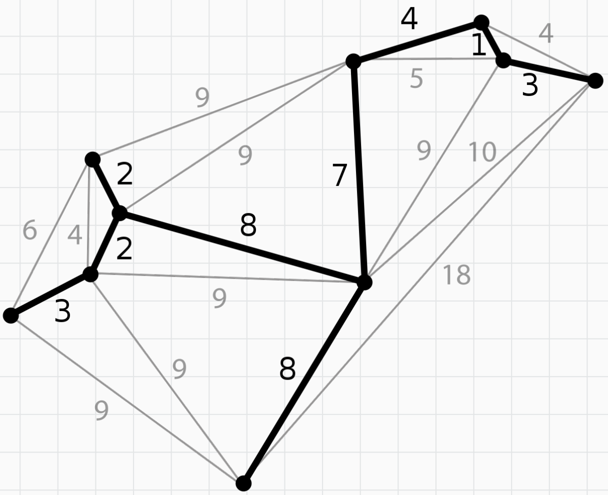

Figure 1: A MST of a weighted graph. Source — 👆 Wikipedia

Exercise

🍰 Can you find the weight of the MST in the following graph?

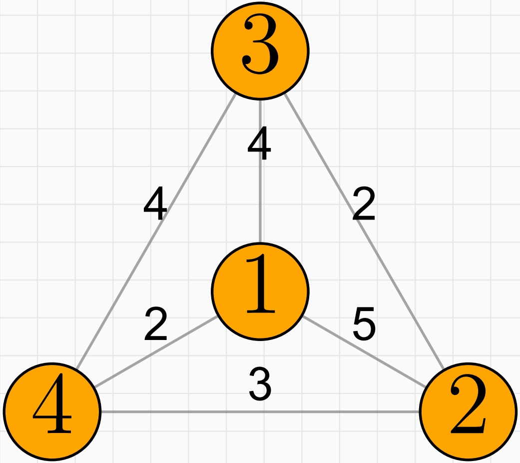

Figure 2: A weighted graph

The Importance of MST

Reasons to study MSTs —

- Network Efficiency — Optical fiber networks.
- Cost Savings — Reduces 💰 in node connections.
- Optimization — Travelling salesman problem.
- Infrastructure — 🚆 networks.

Figure 3: Fast Connections, Faster Trades

ITA 21.1 Growing a Minimum Spanning Tree

ITA 21.1 Growing a Minimum Spanning Tree

Generic Algorithm for MST

Generic Algorithm for MST

Let A be a subset of edges in some MST of G.

An edge is a safe edge for A if adding it keeps A extendable to some minimum spanning tree; equivalently, $A \cup \{(u,v)\}$ remains a subset of an MST.

Line 3 maintains the following loop invariant — A is *always* a subset of some minimum spanning tree.

```
1: Generic-MST(G, w)
2: A ← ∅
3: while A does not form a spanning tree do
4:     find an edge (u, v) that is safe for A
5:     A ← A ∪ {(u, v)}
6: return A
```

Cut — For an undirected graph $G = (V, E)$, a cut $(S, V \setminus S)$ is a partition of V.

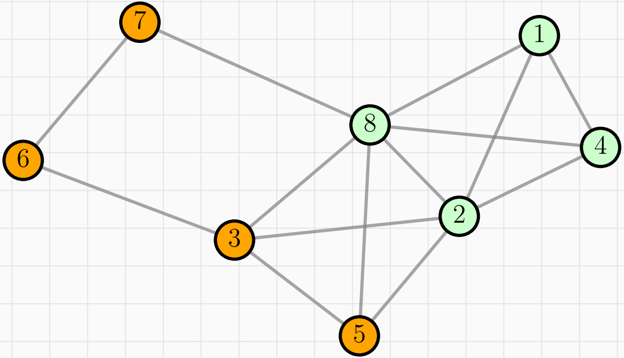

Figure 4: A cut of G with $S = \{1, 2, 4, 8\}$

Crossing the Cut — An edge $(u, v) \in E$ crosses the cut $(S, V \setminus S)$ if one of its endpoints is in S and the other is in $V \setminus S$.

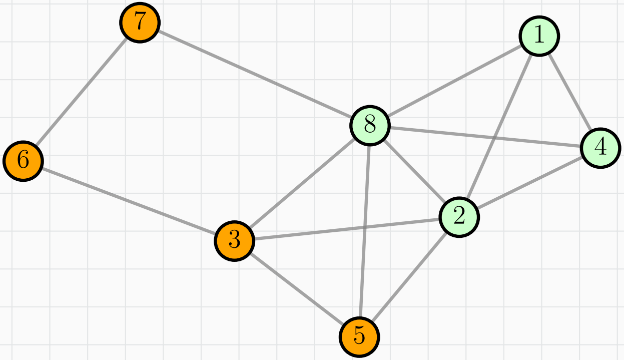

Figure 4: 🍰 Can you find edges crossing the cut?

Cuts in an Undirected Graph

Respecting a Cut — A cut *respects* a set A of edges if no edge in A crosses the cut.

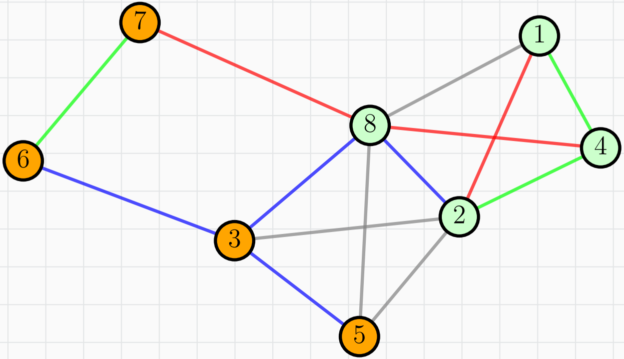

Figure 4: 🍰 Which set of edges does the cut respect? Blue, red or green?

Light Edge — An edge is a *light edge* crossing a cut if its weight is the minimum of any edge crossing the cut.

There may be multiple light edges crossing a cut in case of ties.

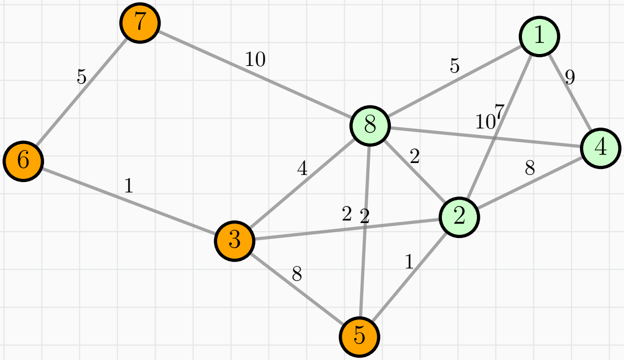

Figure 4: 🍰 Which edge is the light edge?

Light Edge with a Property — An edge is a *light edge* satisfying a given property if its weight is the minimum of any edge satisfying that property.

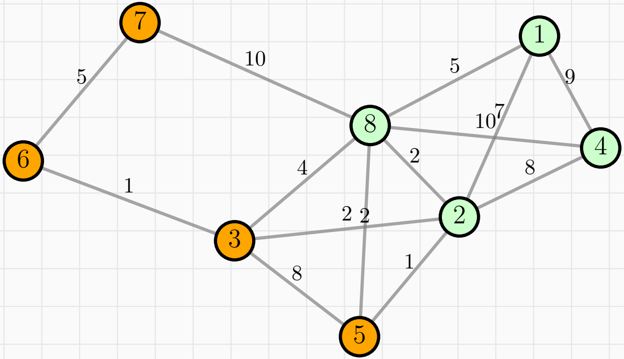

Figure 4: 🍰 Which edge is the light edge with the property "having weight at least 3"?

Theorem 21.1

Let $G = (V, E)$ be a connected, undirected and weighted graph.

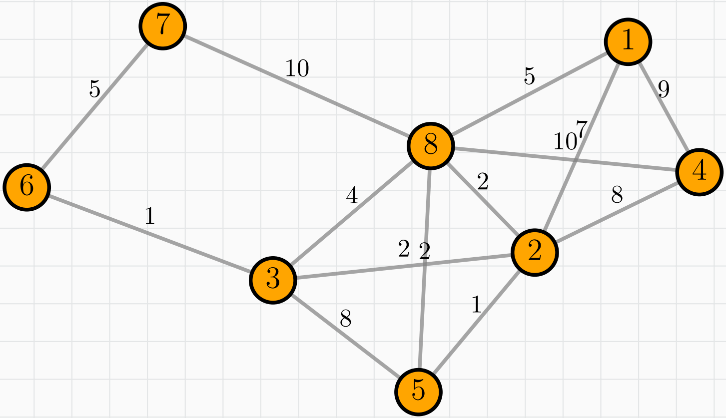

Figure 5: Example of — A weighted graph G

Let A be a subset of E that is included in some MST for G.

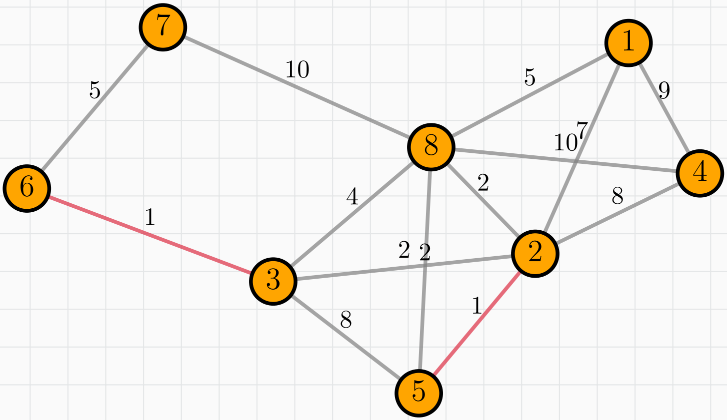

Figure 5: Example of — A is some edges in a MST

Let $(S, V - S)$ be any cut of G that respects A.

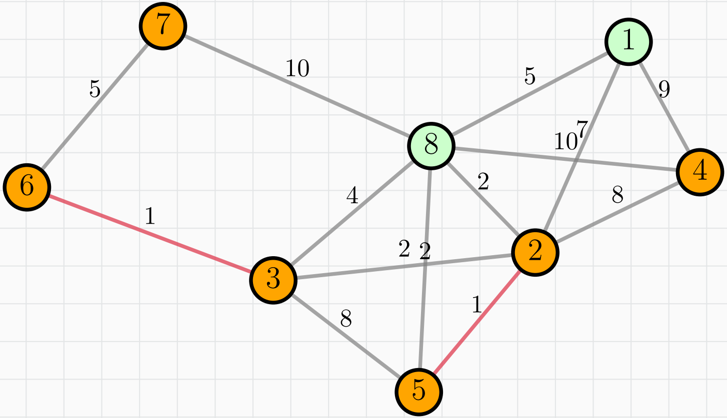

Figure 5: Example of — A cut respecting A.

Let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A.

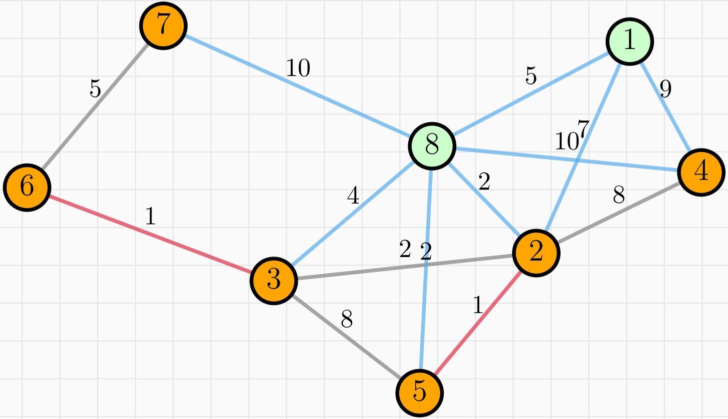

Figure 5: Example of — 🍰 Which one is light among the crossing edges?

Let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A.

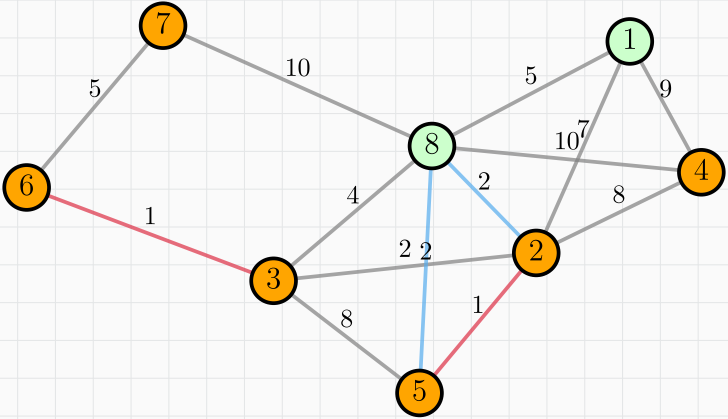

Figure 5: Example of — When there are multiple light edges, pick one arbitrarily to be (u, v).

Proof — If T, the MST containing A, also contains (u, v), then we are done.

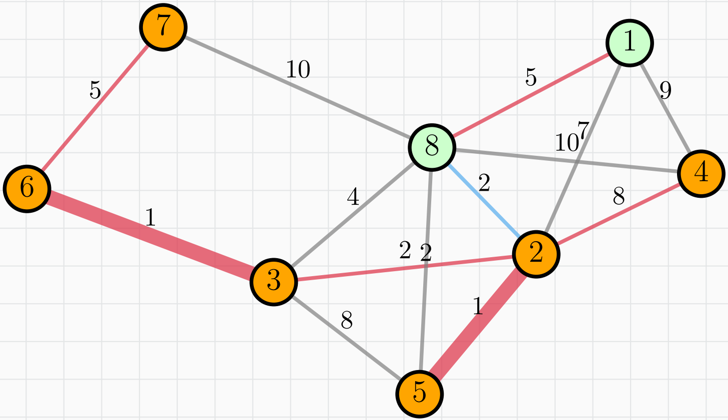

Figure 6: T contains (u, v)

Proof — If not, we build a new MST T' that contains (u, v) and A.

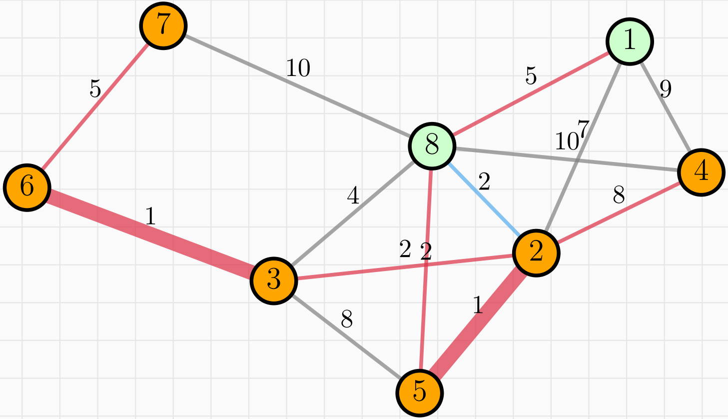

Figure 6: Since T does not contain $(u, v) = (2, 8)$, we replace $(x, y) = (8, 5)$ by (u, v) and get a different MST T'

Proof — If not, we build a new MST T' that contains (u, v) and A.

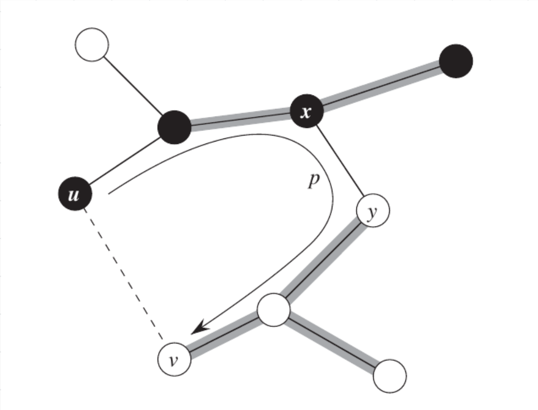

Figure 6: T does not contain (u, v)

Let $G = (V, E)$ be a connected, undirected and weighted graph.

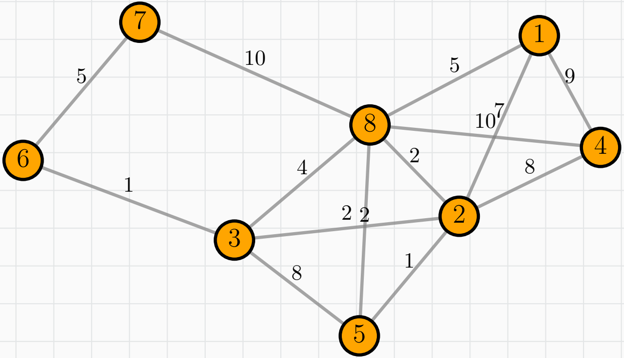

Figure 7: A weighted graph G

Let A be a subset of E that is included in some MST for G.

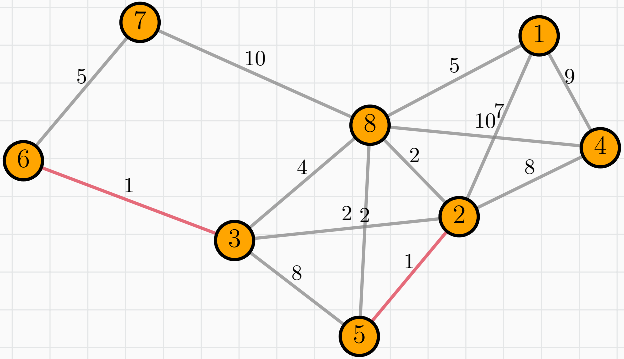

Figure 7: Let A be some edges of a MST

Let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$.

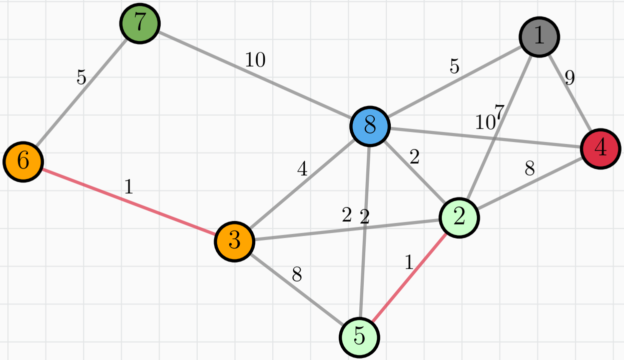

Figure 7: Let $C = (\{2, 5\}, \{(2, 5)\})$

Corollary 21.2

If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A.

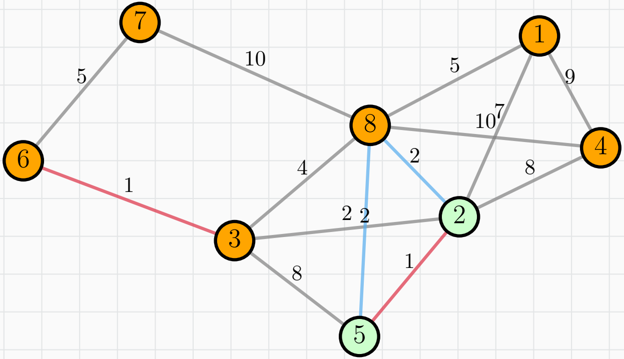

Figure 7: Light crossing edges — Let $(u, v) = (2, 8)$

🍰 Is the edge $(4, 2)$ safe for $A = \{(3, 6), (2, 5)\}$?

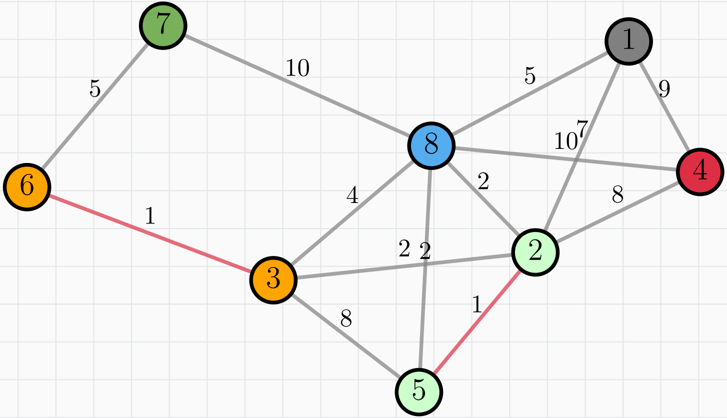

Figure 8: 💣 The edge $(1, 2)$ has a weight of 7 and the edge $(4, 8)$ has a weight of 10.

In 2007, trader Brad Katsuyama noticed that when he split an order across multiple exchanges, the best prices often disappeared.

Sending the same order to just one exchange worked, which hinted that someone was reacting faster than his order could reach all venues.

The culprit was high-frequency trading exploiting tiny latency differences between exchanges — a speed advantage that looks like a shortest-path race in a weighted network.

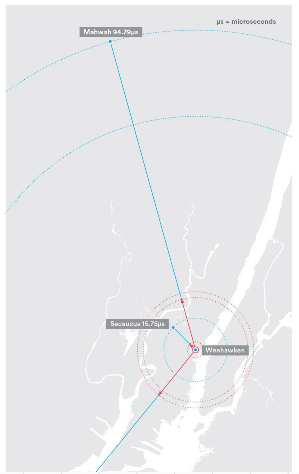

Figure 9: The distance from Katsuyama to the exchanges.

How to Stop the 🐺?

The solution was paradoxical — adding a delay to synchronize order arrivals at all exchanges.

IEX (Investors Exchange) launched in 2012, using a 350-microsecond delay to blunt the speed advantage of high-frequency traders.

😑 Today, IEX's market share is about 2.5%.

📖 Book — *Flash Boys*

Figure 9: 32 miles of fiber optic cable for 350 microseconds delay.

ita 21.1 Growing a Minimum Spanning Tree

Exercise 21.1

Exercise 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

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Proof — Suppose that (for contradiction) the graph has 2 distinct spanning trees T and T'.

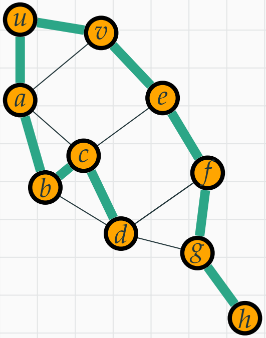


Figure 10: T

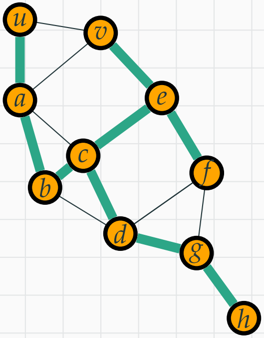


Figure 11: T'

Exercise 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — There must exist an edge (u, v) in T but not T'.

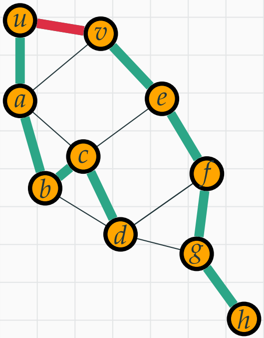


Figure 10: T

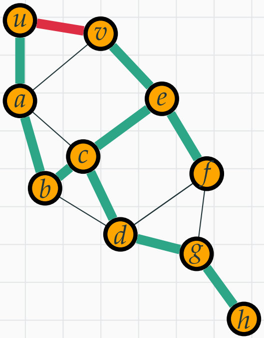


Figure 11: T'

Exercise 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — Removing (u, v) from T gives us a cut.

The edge (u, v) must be the unique light edge of this cut. Otherwise T cannot be a MST.

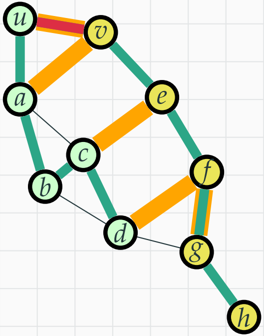


Figure 10: T

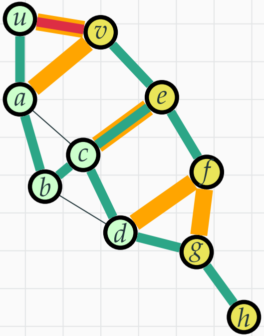


Figure 11: T'

Exercise 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — Then replacing the crossing edge in T' with (u, v) gives a tree with smaller weight, which is a contradiction.

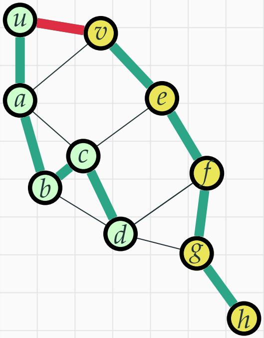


Figure 10: T

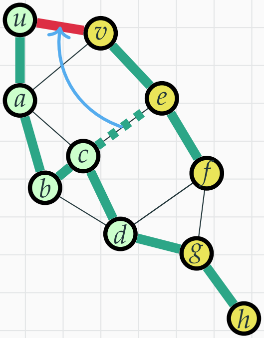


Figure 11: T'

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

🧩 Exercise — Show that the converse is not true.

🍰 What is the unique MST of the following graph?

🍰 What are the light edges for the cut $\{x\}$?

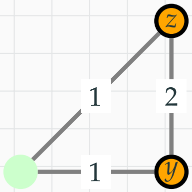
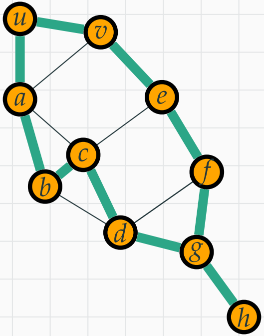


Figure 10: A counterexample

Exercise 21.1-7

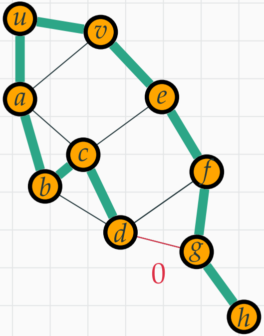
Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree.



Exercise 21.1-7

Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree.

Give an example to show that the same conclusion does not follow if we allow some weights to be nonpositive.



What are your main takeaways today? Any questions?