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Lecture 05 — Divide and Conquer (Part 1)

COMPSCI 308 — Design and Analysis of Algorithms

Xing Shi Cai https://newptcai.gitlab.io

Duke Kunshan University, Kunshan, China

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Summary

ITA 2.3 Merge Sort

ITA 4.1 The Maximum-Subarray Problem

ITA 4.2 Strassen's Algorithm for Matrix Multiplication

Assignments[1]

Practice makes perfect!

📖 Required Readings:

- Section 2.3
- Section 4.1
- Section 4.2

✏️ Required Exercises:

- Exercises 2.3: 1–3.
- Exercises 4.1: 1–4
- Exercises 4.2: 1–5.

[1] 💣 Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

ITA 2.3 Merge Sort

Divide and Conquer

Divide and Conquer is a common technique in designing algorithms:

- 🔪 Divide: Break the problem into smaller sub-problems.
- ⚔️ Conquer: Solve the smaller sub-problems.
- 🥗 Combine: Merge the solutions to create a solution for the original problem.

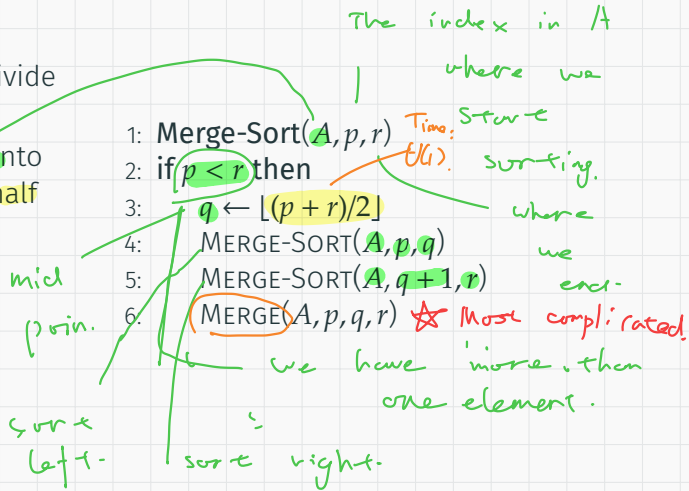
Figure 1: How to divide and conquer a big 🌱 pizza

Merge Sort

Merge Sort is a classic example of a divide and conquer algorithm.

- 🔪 Divide: Split the unsorted array into two subarrays, each containing half of the elements.
- ⚔️ Conquer: Sort each subarray.
- 🥗 Combine: Merge the two sorted subarrays.

```
1: Merge-Sort(A, p, r)
2: if p < r then
3:     q ← ⌊(p + r)/2⌋
4:     MERGE-SORT(A, p, q)
5:     MERGE-SORT(A, q + 1, r)
6:     MERGE(A, p, q, r)
```



Example of Merge Sort

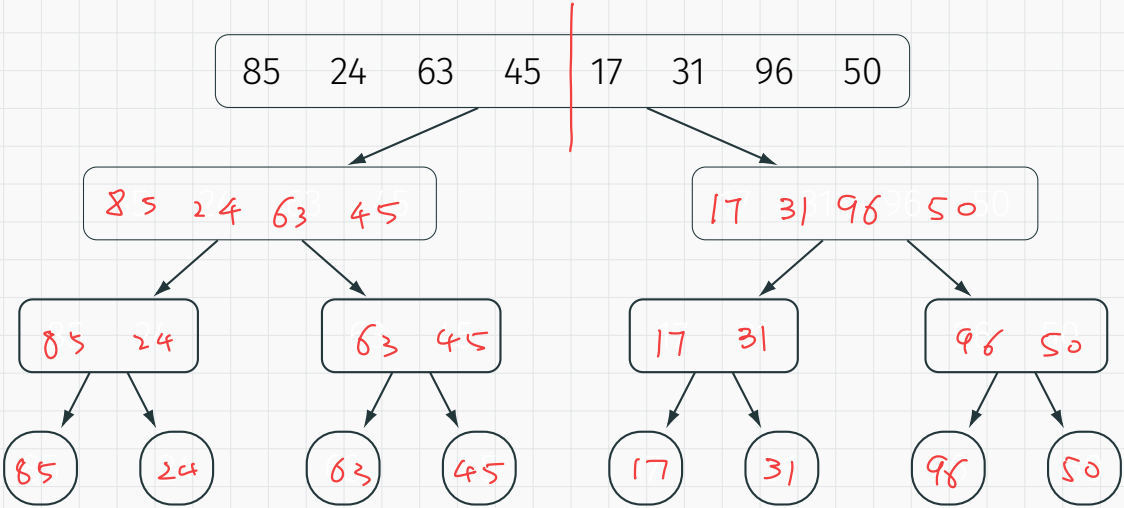


Figure 2: Divide

Example of Merge Sort

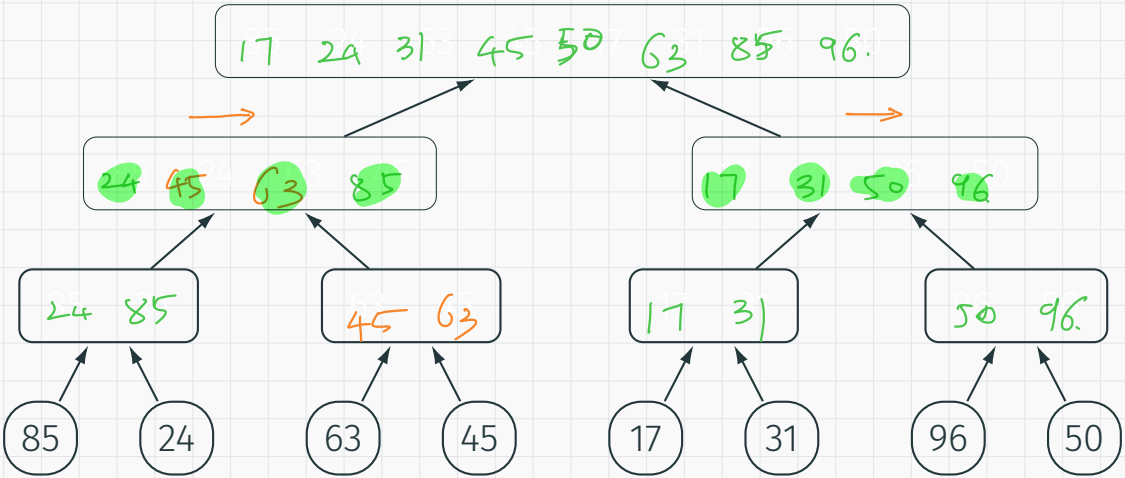


Figure 2: Combine

哨兵

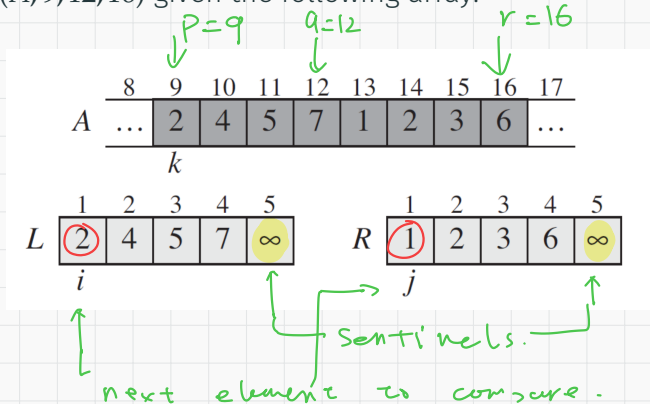
A sentinel is a soldier whose job is to guard something.

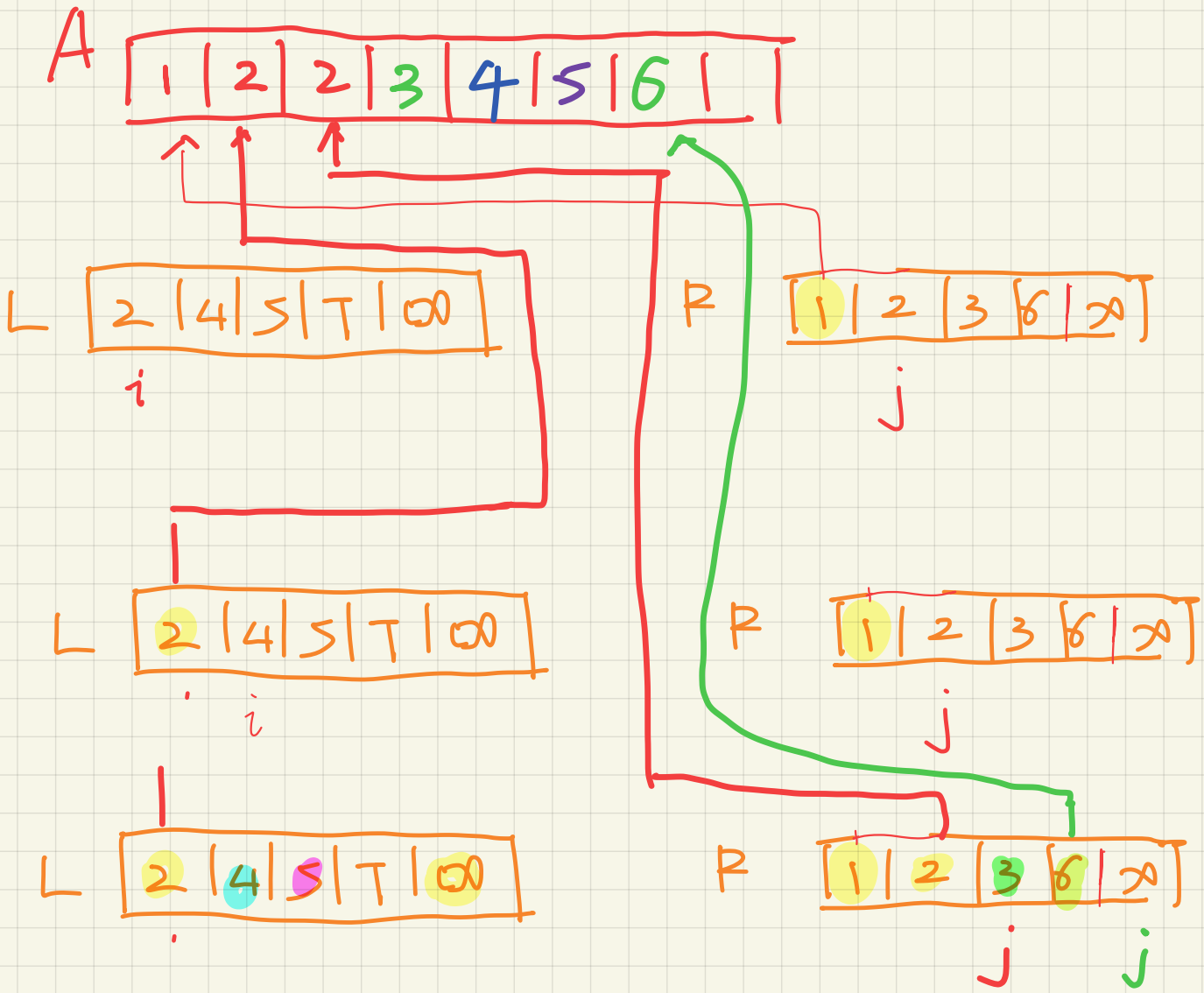
— Oxford Learner's Dictionary

Figure 3: The Carrot Sentinel!

Example of Merge

Let's run `Merge`$(A, 9, 12, 16)$ given the following array:




A
1 2 2 3 4 5 6
L
2 4 5 T ∞
i
R
1 2 3 6 ∞
j
L
2 4 5 T ∞
i
R
1 2 3 6 ∞
j
L
2 4 5 T ∞
i
R
1 2 3 6 ∞
j
j

Pseudocode of Merge Function

1: **Merge**(A, p, q, r)
2: $n_1 \leftarrow q - p + 1$
3: $n_2 \leftarrow r - q$
4: Let $L[1 \dots n_1 + 1]$ and $R[1 \dots n_2 + 1]$ be new arrays
5: **for** $i \leftarrow 1$ to n_1 **do**
6: $\quad L[i] \leftarrow A[p + i - 1]$
7: **for** $j \leftarrow 1$ to n_2 **do**
8: $\quad R[j] \leftarrow A[q + j]$
9: $L[n_1 + 1] \leftarrow \infty$ ▷ A sentinel
10: $R[n_2 + 1] \leftarrow \infty$ ▷ Another sentinel

begin of left array
end of left
end of right
len of left
len of right.
length n_1+1
len of n_2+1
for the ∞ (sentiel).
copy left and right in L and R.

Pseudocode of Merge Function

```
1:  Merge(A, p, q, r)
2:  n_1 ← q − p + 1
3:  n_2 ← r − q
4:  Let L[1 ... n_1 + 1] and R[1 ... n_2 + 1] be
    new arrays
5:  for i ← 1 to n_1 do
6:      L[i] ← A[p + i − 1]
7:  for j ← 1 to n_2 do
8:      R[j] ← A[q + j]
9:  L[n_1 + 1] ← ∞                ▷ A sentinel
10: R[n_2 + 1] ← ∞                ▷ Another sentinel
11: i ← 1
12: j ← 1
13: for k ← p to r do
14:     if L[i] ≤ R[j] then
15:         A[k] ← L[i]
16:         i ← i + 1
17:     else
18:         A[k] ← R[j]
19:         j ← j + 1
```

$n_1 + 1 = q - p + 2$.

$n_2 + 1 = r - q + 1$

← which two elements to compare.

$r - p + 2$.

← Put L[i] in A

← Put R[j] in A.

What is the running time of the above algorithm?

Upper bound:

$$c_1(n_1+1) + c_2(n_2+1) + c_3 \cdot (r-p+2)$$

$$= c_1(q-p+2) + c_2(r-q+1) + c_3(r-p+2)$$

$$\leq c_4 \cdot [q-p+2 + r-q+1 + r-p+2] \qquad (c_4 = \max(c_1, c_2, c_3))$$

$$= c_4[2 \cdot r - 2p + 5].$$

$$\leq O(r-p)$$

Lower bound

$$\Omega(r-p).$$

The run time $\Theta(r-p)$.

Time Complexity of Merge Sort

Let $T(n)$ denote the running time of the merge sort algorithm. Then $T(n)$ satisfies

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ D(n) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \Theta(n) & \text{otherwise.} \end{cases}$$

divide: $\Theta(1)$. conquer: combine (Merge)

Time Complexity of Merge Sort

Let $T(n)$ denote the running time of the merge sort algorithm. Then $T(n)$ satisfies

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ D(n) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \Theta(n) & \text{otherwise.} \end{cases}$$

Usually the order of growth of such an algorithm does not depend on

- the floor and ceiling functions,
- and the boundary conditions.

This leads to the simplified equation

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

Later we will see

$$T(n) = \Theta(n \log n).$$

Solve an Exact Recursive Equation

Suppose instead of asymptotic notations in the recursion, we have

$$T(n) = \begin{cases} c & \text{if } n = 1, \\ 2T\left(\frac{n}{2}\right) + cn & \text{otherwise.} \end{cases}$$

Moreover, assume that $n = 2^k$ for some integer $k \geq 0$.

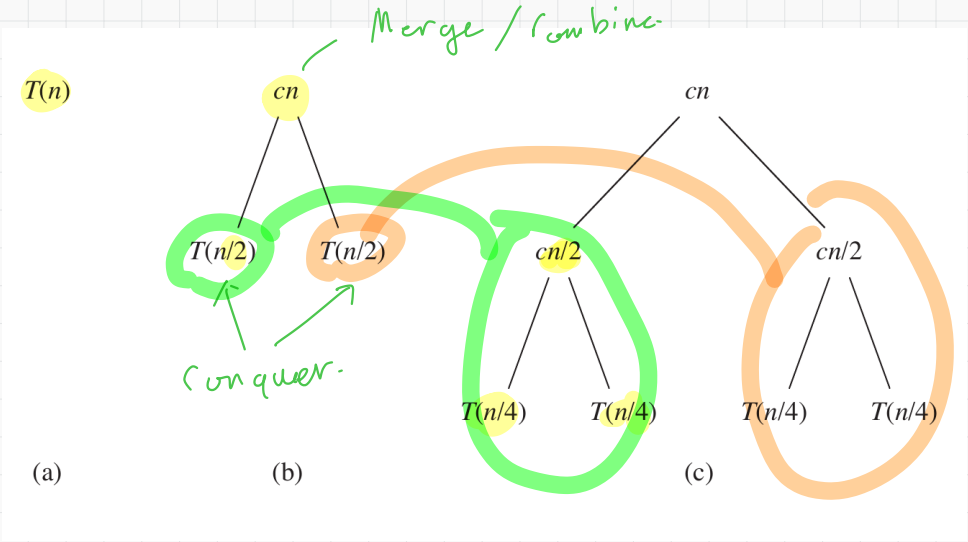
Solving this recursive equation, we get

$$T(n) = cn \log_2 n + cn = cn \lg n + cn.$$

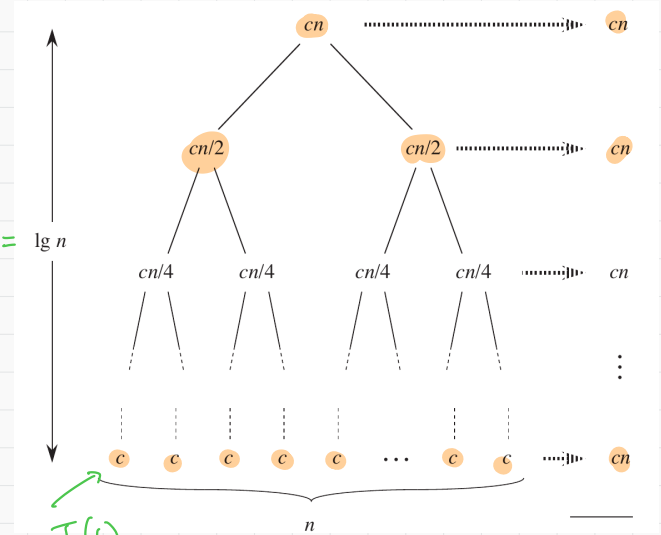
The argument uses a 🎄.

💣 This is not a real proof since the recursion is not exact.

Solve an Exact Recursive Equation by Tree



Solve an Exact Recursive Equation by Tree



$\log_2 n =$

$T(1)$

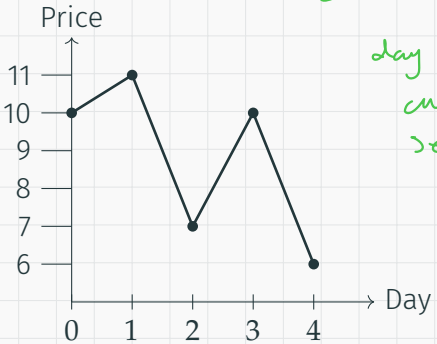
$T(n) = cn \cdot \log_2 n.$

ITA 4.1 The Maximum-Subarray Problem

📌 Example — Stock Price

The prices of a stock are given in the following plot.

🍰 If you buy once and sell once, when should you do so?



Common buy on day 4 and sell on day 1.

$S = \langle 10, 11, 7, 10, 6 \rangle$

Figure 4: What is the algorithm to maximize 💰!

$A = \langle 1, -4, 3, -4 \rangle$

The Maximum Subarray Problem

Let $S[0, \ldots, n]$ be the stock prices from day 0 to day n. Let

$$A[i] = S[i] - S[i-1], \text{ for } 1 \le i \le n.$$

Then if you buy on day $p-1$ and sell on day q, the profit is

$$S[q] - S[p-1] = \underline{S[q] - S[q-1] + S[q-1]} - S[q-2] + S[q-2] \cdots + S[p] - S[p-1].$$

$$= \underbrace{A[q] + A[q-1] \cdots + A[p]}$$

The sum of a subarray of A.

The Maximum Subarray Problem

Let $S[0, \ldots, n]$ be the stock prices from day 0 to day n. Let

$$A[i] = S[i] - S[i-1], \text{ for } 1 \leq i \leq n.$$

Then if you buy on day $p-1$ and sell on day q, the profit is

$$S[q] - S[p-1] = \sum_{i=p}^{q} A[i]$$

Given an array $A[1, \ldots, n]$, the maximum subarray problem aims to find indices p and q which maximizes

$$\sum_{i=p}^{q} A[i]$$

Divide of Conquer of Max-Subarray

L ← Left subarray of A

R ← Right subarray of A.

ml = the max subarray of L

mr = the max subarray of R.

m = max (ml, mr)

or

m = ml + mr

} Wrong

The Maximum Subarray Problem

Given an array $A[1, \ldots, n]$, the maximum subarray problem aims to find indices p and q which maximizes

$$\sum_{i=p}^{q} A[i]$$

```
1:  Max-Subarray-Brute-Force(A, n)
2:  max_profit ← −∞
3:  for p ← 1 to n do
4:      sum ← 0
5:      for q ← p to n do
6:          sum ← sum + A[q]
7:          if sum > max_profit then
8:              max_profit ← sum
9:              (buy, sell) ← (p − 1, q)
10: return max_profit
```

This runs in $O($_____$)$ time.

The Divide and Conquer Algorithm

🔪 **Divide**: Split the array into two halves.

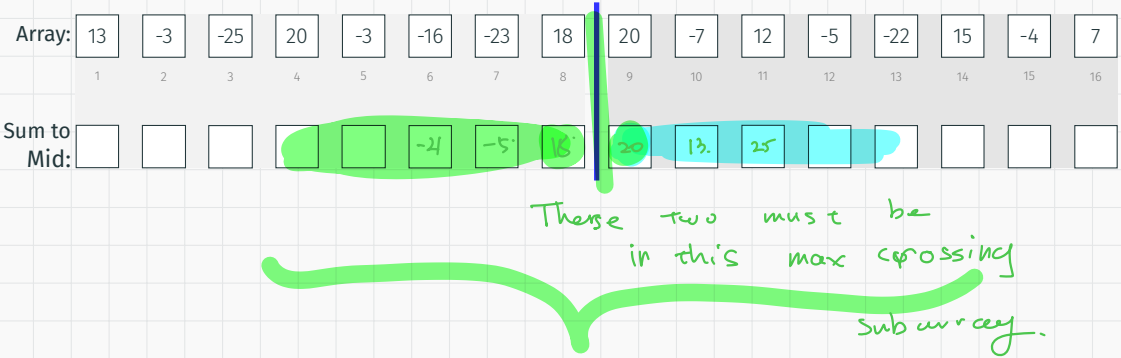
⚔️ **Conquer**: Recursively find the maximum subarray in each case:

- ⬅️ Maximum subarray entirely in the left half.
- ➡️ Maximum subarray entirely in the right half.
- ↔️ Maximum subarray crossing the midpoint.

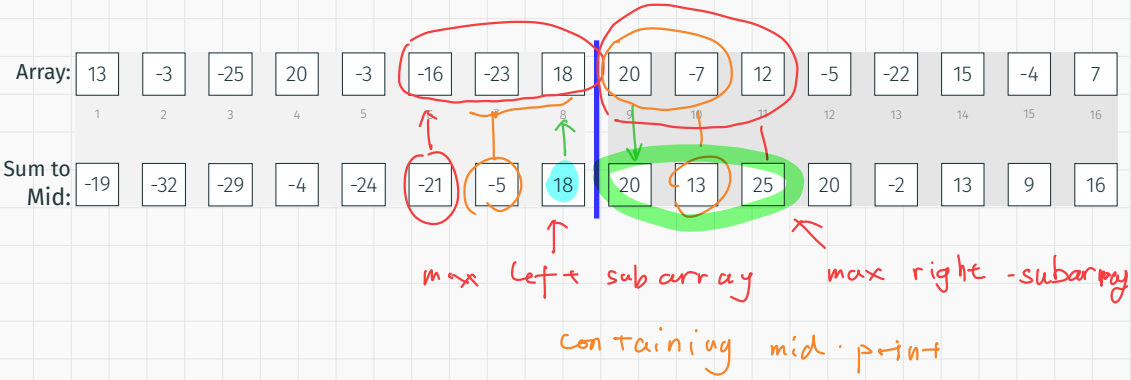
🥗 **Combine**: Pick the maximum subarray from each of the three possibilities.

🍰 What is the maximum crossing subarray in the following array?

What is the maximum crossing subarray in the following array?



What is the maximum crossing subarray in the following array?



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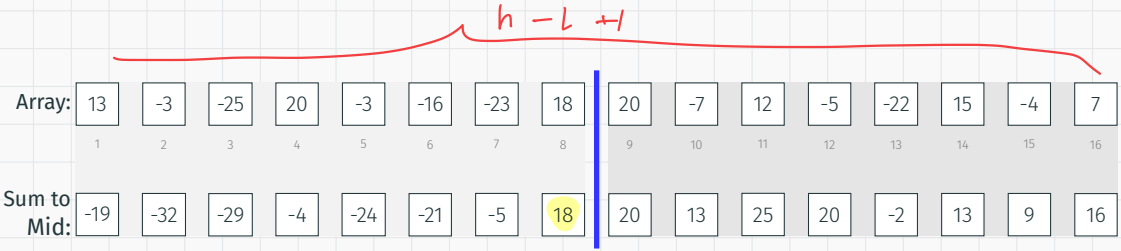

The maximum crossing subarray contains two parts:

- The maximum subarray entirely in the left half including the midpoint.
- The maximum subarray entirely in the right half including the midpoint.

Find Maximum Crossing Midpoint Subarray

```
1: Max-Crossing-Subarr(A, l, m, h)
2: left-sum ← −∞
3: sum ← 0
4: for i ← m downto l do
5:     sum ← sum + A[i]
6:     if sum > left-sum then
7:         left-sum ← sum
8:         max-left ← i
```

The array

low mid high

Find the max subarray containing mid point.

Find Maximum Crossing Midpoint Subarray

```
1:  Max-Crossing-Subarr(A, l, m, h)
2:  left-sum ← −∞
3:  sum ← 0
4:  for i ← m downto l do
5:      sum ← sum + A[i]
6:      if sum > left-sum then
7:          left-sum ← sum
8:          max-left ← i
9:  right-sum ← −∞
10: sum ← 0
11: for j ← m + 1 to h do
12:     sum ← sum + A[j]
13:     if sum > right-sum then
14:         right-sum ← sum
15:         max-right ← j
16: return
    (max-left, max-right, left-sum +
    right-sum)
```

Find max right sub array containing mid point.

What is the time complexity of this procedure? $\Theta(h - l)$

Find Maximum Subarray Pseudocode

```
1:  Max-Subarr(A, l, h)
2:  if h == l then
3:      return (l, h, A[l])
4:  m ← ⌊(l+h)/2⌋
5:  (ll, lh, ls) ← Max-Subarr(A, l, m)
6:  (rl, rh, rs) ← Max-Subarr(A, m + 1, h)
7:  (cl, ch, cs) ← Max-Crossing-Subarr(A, l, m, h)
8:  if ls ≥ rs and ls ≥ cs then
9:      return (ll, lh, ls)
10: else if rs ≥ ls and rs ≥ cs then
11:     return (rl, rh, rs)
12: else
13:     return (cl, ch, cs)
```

divide: $O(1)$

conquer:

$\rightarrow \Theta(h - l) = \Theta(n)$ if A has length n.

Combine: $O(1)$.

Time Complexity of Maximum Subarray

Let $T(n)$ denote the running time of the previous algorithm. Then $T(n)$ satisfies

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ D(n) + 2T\left(\frac{n}{2}\right) + \Theta(n) + C(n) & \text{otherwise.} \end{cases}$$

divide: $O(1)$ conquer: combine: $O(1)$

Time Complexity of Maximum Subarray

Let $T(n)$ denote the running time of the previous algorithm. Then $T(n)$ satisfies

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This simplifies to

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

As the recursion is the same as merge sort, the solution is also

$$T(n) = \Theta(n \log n).$$

What is the *sum* of the maximum subarray that crosses the midpoint in the following array?

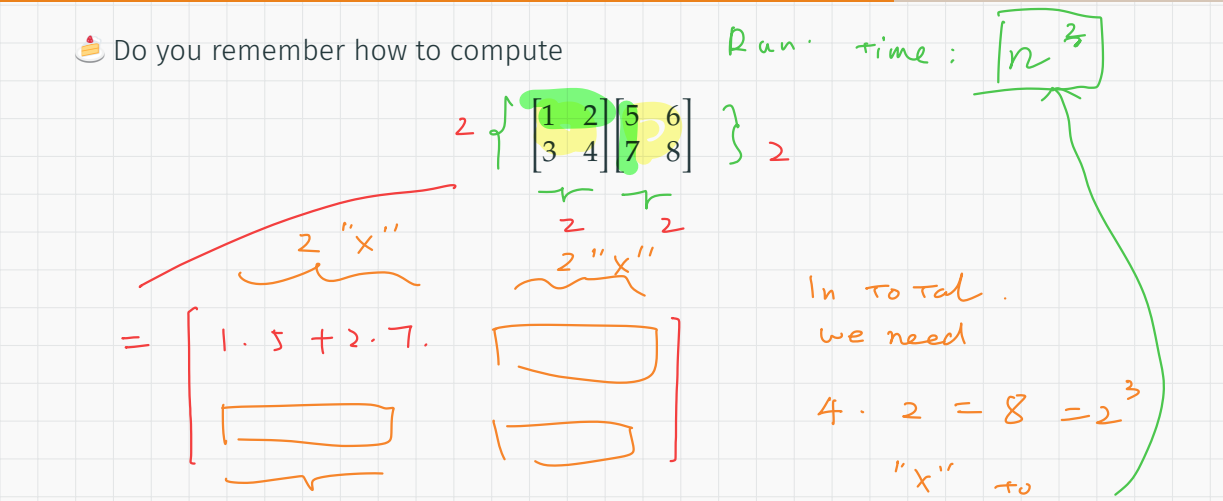
$$A = \langle 0, 3, 2, 2, -5, 1, -2, 2, 3, -1, -5, -2, 9, -4, -2, -2 \rangle.$$

Answer: 9.

ITA 4.2 STRASSEN'S ALGORITHM FOR MATRIX MULTIPLICATION

Review: Matrix Multiplication

Do you remember how to compute



Square Matrix Multiplication by Brute Force

Let A and B be two $n \times n$ matrices.

Let a_{ij} and b_{ij} be the entries of A and B respectively.

Let $C = A \cdot B$ and c_{ij} be the entries of C.

Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

This is called the row-column rule.

Square Matrix Multiplication by Brute Force

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Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

This is called the row-column rule.

1: **Square-Matrix-Multiply**(A, B)
2: $n \leftarrow A$.rows
3: let C be a new $n \times n$ matrix
4: **for** $i = 1$ to n **do**
5: **for** $j = 1$ to n **do**
6: $c_{ij} \leftarrow 0$
7: **for** $k = 1$ to n **do**
8: $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
9: **return** C

🍰 What is the time complexity of this algorithm?

Matrix Partitioning

Given $n \times n$ matrices A, B, and $C = AB$, we can partition them

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

n/2 × n/2.

where the sub-matrices are of size $n/2 \times n/2$.

Matrix Partitioning

Given $n \times n$ matrices A, B, and $C = AB$, we can partition them

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where the sub-matrices are of size $n/2 \times n/2$.

For instance, consider

$$A = \begin{pmatrix} 2 & 3 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 3 & 3 & 2 \\ 1 & 0 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 3 & 3 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 15 & 12 & 9 & 13 \\ 9 & 6 & 9 & 13 \\ 19 & 14 & 9 & 9 \\ 17 & 9 & 11 & 13 \end{pmatrix}$$

Can you identify what A_{11}, ..., C_{22} are?

Then, $C = A \cdot B$ can be written as:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This leads to four corresponding equations.

$$\begin{cases} C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}, \\ C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}, \\ C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}, \\ C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}. \end{cases}$$

we still need $\frac{n^3}{4} \times 4 = n^3$ multiplication

$\left(\frac{n}{2}\right)^3$ "x" $\left(\frac{n}{2}\right)^3$ "x" $\frac{n^3}{4}$ "x"

Recursive Square Matrix Multiplication Algorithm

Let us abbreviate Square-Matrix-Multiply-Recursive as SMMR.

SMMR(A, B)
$n \leftarrow A$.rows
$C \leftarrow$ new $n \times n$ matrix
if $n = 1$ then
    $C_{11} \leftarrow a_{11} \cdot b_{11}$
else
    Partition A, B, and C as in equations (4.9)
    $C_{11} \leftarrow \text{SMMR}(A_{11}, B_{11}) + \text{SMMR}(A_{12}, B_{21})$
    $C_{12} \leftarrow \text{SMMR}(A_{11}, B_{12}) + \text{SMMR}(A_{12}, B_{22})$
    $C_{21} \leftarrow \text{SMMR}(A_{21}, B_{11}) + \text{SMMR}(A_{22}, B_{21})$
    $C_{22} \leftarrow \text{SMMR}(A_{21}, B_{12}) + \text{SMMR}(A_{22}, B_{22})$
return C

Time Complexity of Recursive Square Matrix Multiplication

Let $T(n)$ denote the running time of SMMR. Then $T(n)$ satisfies

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8T\left(\frac{n}{2}\right) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

Thus,

$$T(n) = 8T\left(\frac{n}{2}\right) + \Theta(n^2).$$

In the next lecture, we'll see that this implies $T(n) = \Theta(n^{\lg 8}) = \Theta(n^3)$.

😩 Nothing has changed!

Strassen's Algorithm

Strassen's method improves SMMR by computing

$$
\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22})B_{11} \\
M_3 &= A_{11}(B_{12} - B_{22}) \\
M_4 &= A_{22}(B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12})B_{22} \\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}
$$

which reduces the number of matrix-multiplications to 7.

The resulting product matrix $C = A \cdot B$ is then computed as: You can merge the given expressions into a single equation as follows:

$$C = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}.$$

Thus, the running time satisfies:

Num of "x" needed.

Combine.

$$T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2).$$

Conquer

Num of "x" needed for $\left(\frac{n}{2} \times \frac{n}{2}\right)$ times $\left(\frac{n}{2} \times \frac{n}{2}\right)$ matrices

In the next lecture, we'll see that this implies

$$T(n) = \Theta(n^{\lg 7}) \approx \Theta(n^{2.81}) = o(n^3)$$

$\log_2 7$

which is much better than $\Theta(n^3)$!

AlphaGo

- Treated the complex game of Go as a **search problem**.
- Beat the best human players of Go in 2017.

Figure 5: Demis Hassabis (1976-), co-founder of Google DeepMind.

🤪 One Logic, Two Breakthroughs

🏆 AlphaGo

- Treated the complex game of Go as a **search problem**.
- Beat the best human players of Go in 2017.

🔢 AlphaTensor

- Treats **matrix multiplication** as a game.
- Found algorithms faster than human limits to speed up all AI.

Figure 5: Demis Hassabis (1976-), co-founder of Google DeepMind.

🤪 One Logic, Two Breakthroughs

🏆 AlphaGo

- Treated the complex game of Go as a **search problem**.
- Beat the best human players of Go in 2017.

🔢 AlphaTensor

- Treats **matrix multiplication** as a game.
- Found algorithms faster than human limits to speed up all AI.

🧬 AlphaFold

- Treats **protein folding** as a search for the best shape.
- **Result:** 2024 Nobel Prize in Chemistry.

Figure 5: Demis Hassabis (1976-), co-founder of Google DeepMind.

What are your main takeaways today? Any questions?