

第十一 生成树

LECTURE 13 – MINIMUM SPANNING TREES (PART 1)

COMPSCI 308 – DESIGN AND ANALYSIS OF ALGORITHMS

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Jan-Mar 2026

ITA 23 Minimum Spanning Trees

ITA 21.1 Growing a Minimum Spanning Tree

Generic Algorithm for Minimum Spanning Tree (MST)

Exercise 21.1

ASSIGNMENTS¹



Practice makes perfect!

📘 Required Readings:

- Section 21.1.

✏️ Required Exercises:

- Exercises 21.1 — 1–7.

¹ ⚪ Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

ITA 23 MINIMUM SPANNING TREES

SPANNING SUBGRAPHS

$\subseteq V$ $\subseteq E$ vertex edge

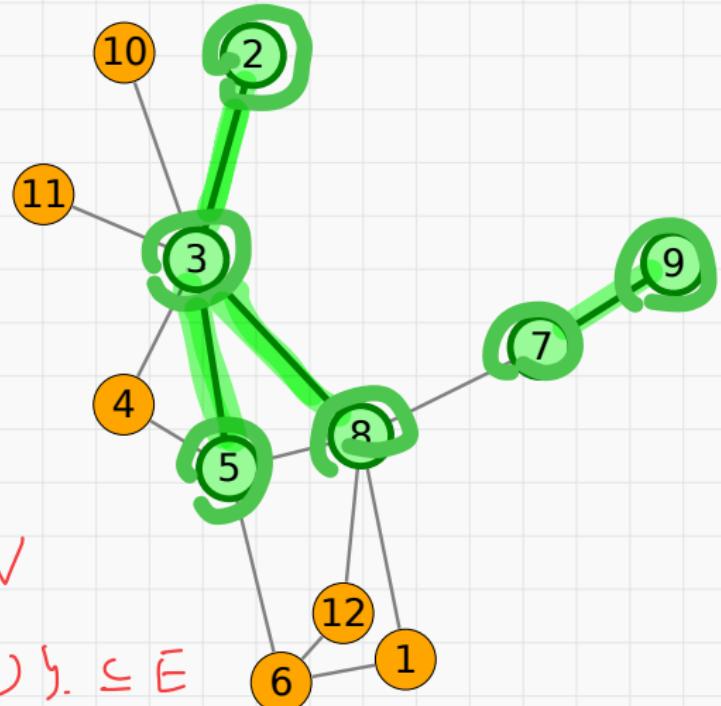
A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a *spanning subgraph* when $W = V$.

🎂 Is the green part a spanning subgraph?

$$W = \{ 2, 3, 5, 8, 7, 9 \} \subseteq V$$

$$F = \{ (3, 2), (3, 5), (3, 8), (7, 9) \} \subseteq E$$

⚠ Note that $(5, 8) \notin F$

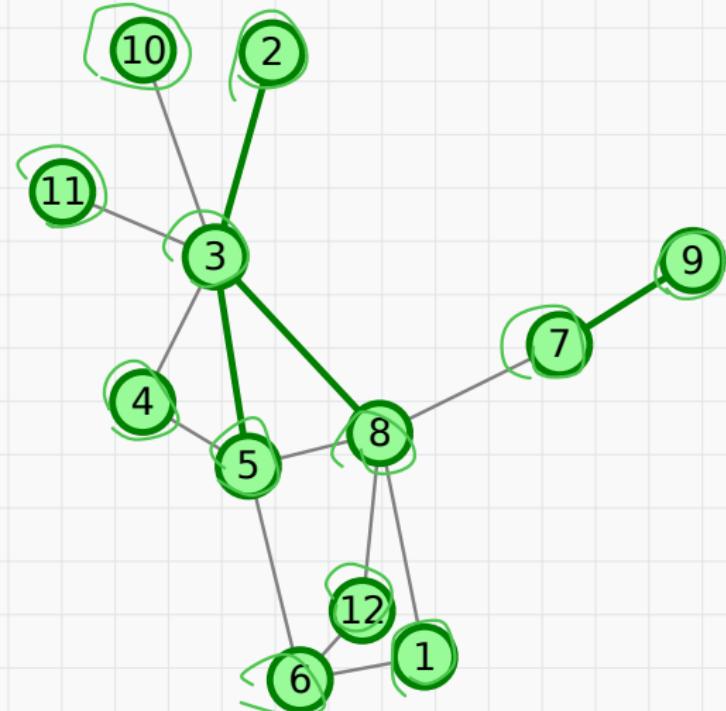


SPANNING SUBGRAPHS

A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a **spanning subgraph** when $W = V$.

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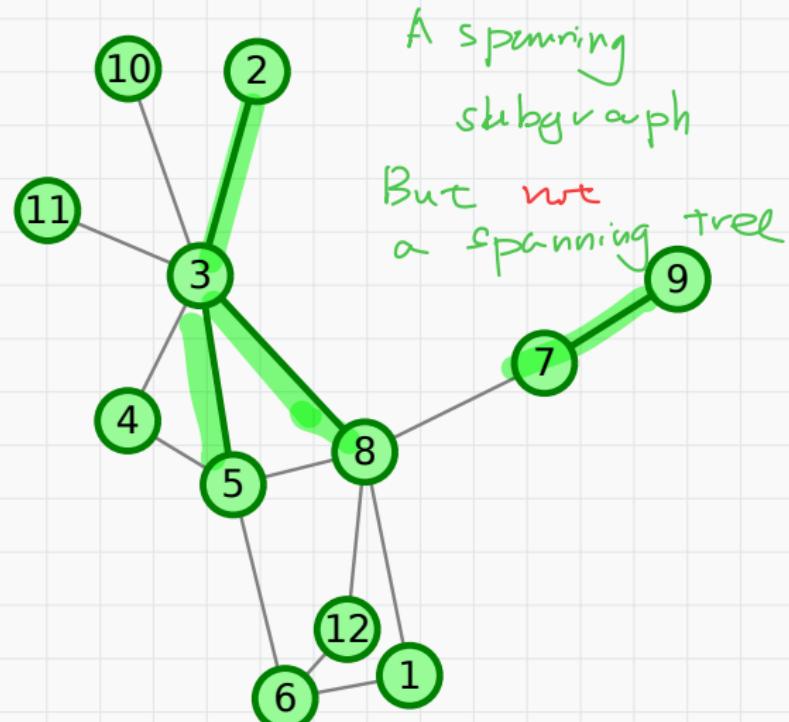
$$W = V = \{1, 2, 3, \dots, 10\}$$



SPANNING TREES

A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a *spanning tree* if H is both a *spanning subgraph* of G and a *tree*.

🍰 Is the green part a spanning tree?



SPANNING TREES

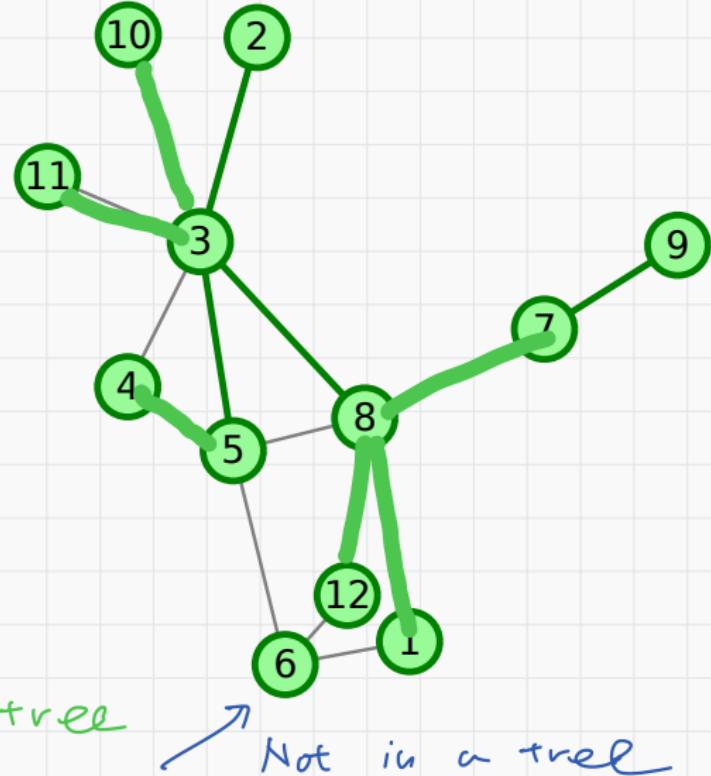
A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a **spanning tree** if H is both a *spanning subgraph* of G and a *tree*.

🎂 Is the green part a spanning tree?

This is spanning subgraph

$$W = V = \{1, 2, \dots, 10\}.$$

But it is not a spanning tree

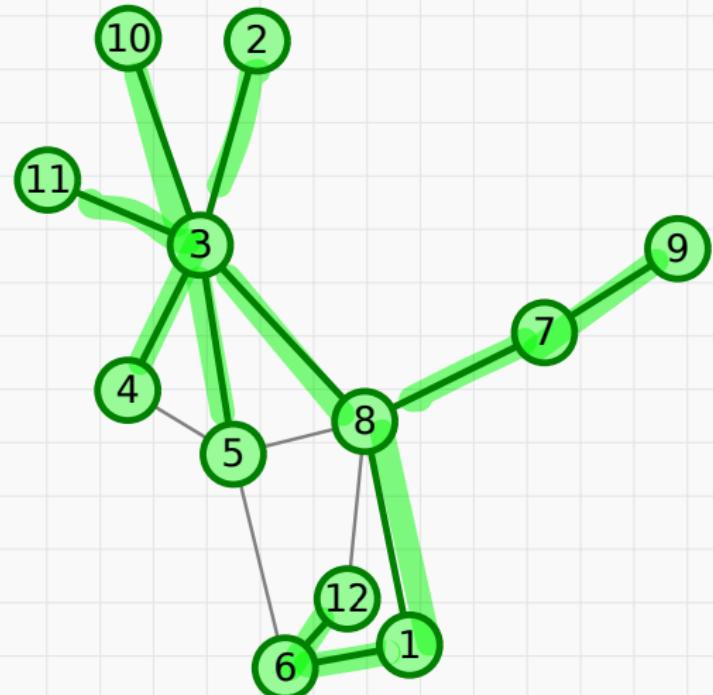


SPANNING TREES

A *subgraph* $H = (W, F)$ of $G = (V, E)$ is a **spanning tree** if H is both a *spanning subgraph* of G and a *tree*.

🍰 Is the green part a spanning tree?

Yes !



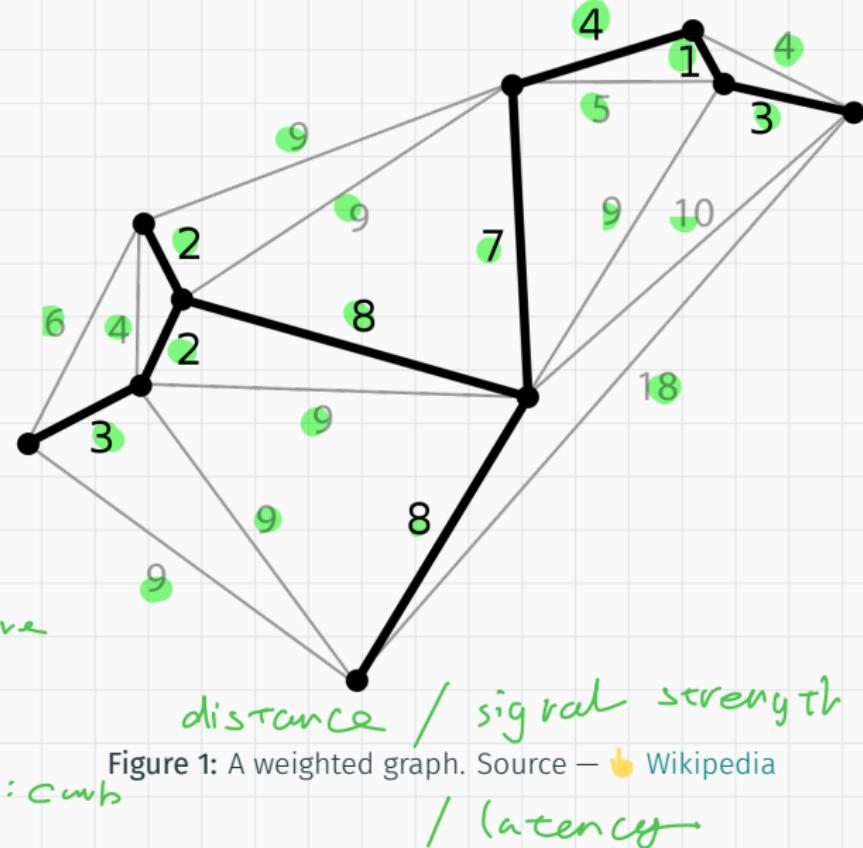
MINIMUM SPANNING TREE (MST)

A **weighted graph** is a graph in which each edge has a numerical value, known as its **weight**.

🎂 What are some examples where networks can be modeled by a weighted graph?

Road maps : The weights are length of road.

Cellphone towers : weight: curb



MST

A Minimum Spanning Tree (MST) is a spanning tree of a weighted graph that minimizes the sum of the weights of its edges.

🎂 Why does every connected weighted graph have at least one MST?

Finite graph has
finitely many ~~sub~~ spanning trees

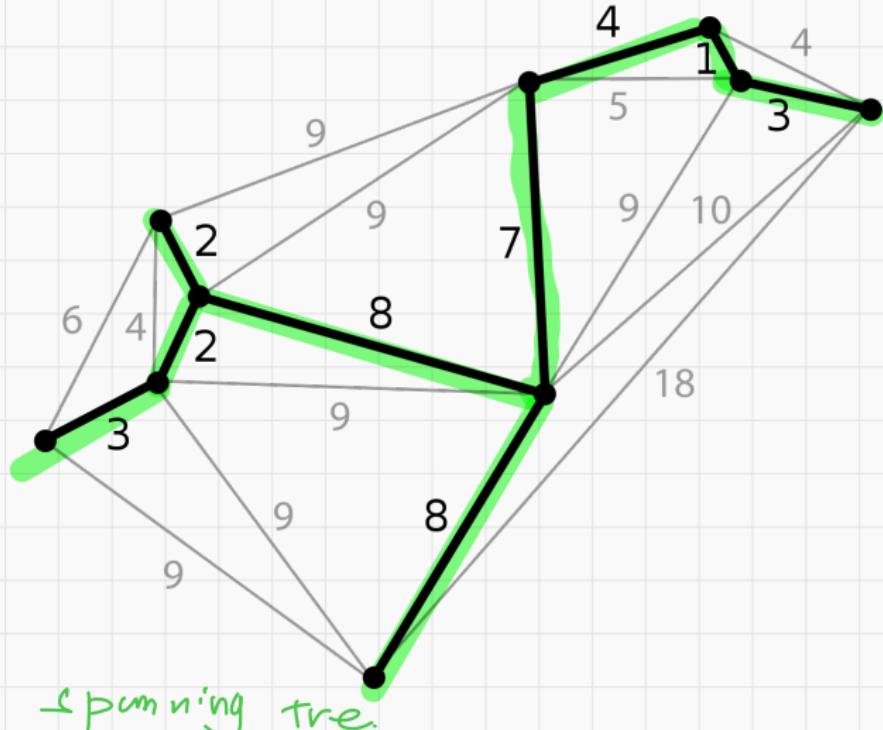
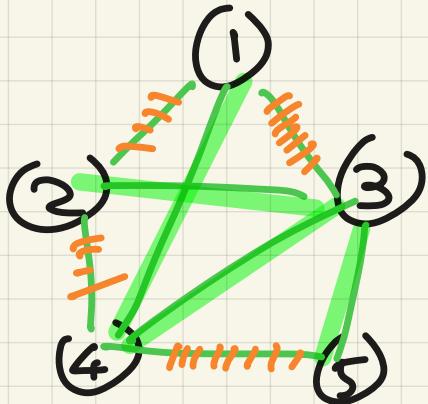


Figure 1: A MST of a weighted graph. Source — [Wikipedia](#)

Why every connected graph
has at least one spanning
tree



By deleting edges in cycles
repeatedly, we eventually
get a spanning tree.

 EXERCISE

🎂 Can you find the weight of the MST in the following graph?

7

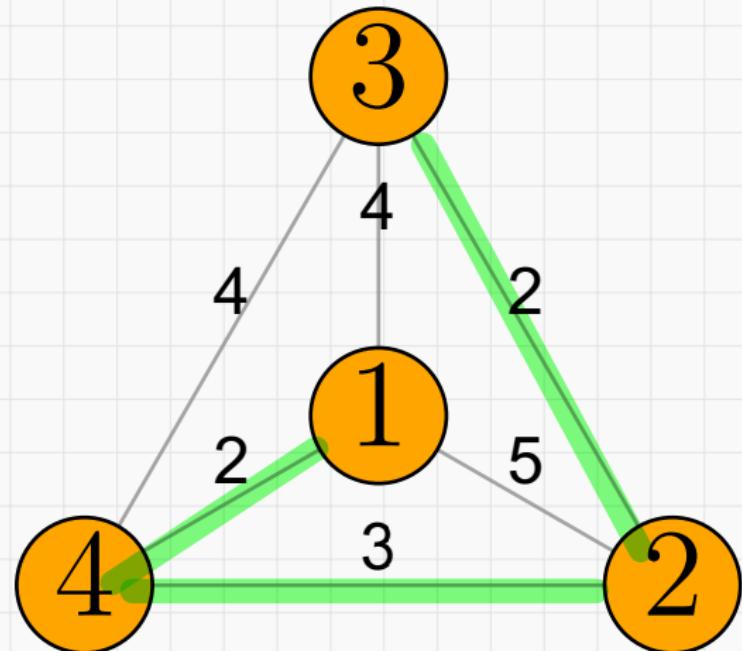


Figure 2: A weighted graph

THE IMPORTANCE OF MST

Reasons to study MSTS —

- Network Efficiency — Optical fiber networks.
- Cost Savings — Reduces \$ in node connections.
- Optimization — Travelling salesman problem.
- Infrastructure — 🌐 networks.



Figure 3: Fast Connections, Faster Trades

ITA 21.1 GROWING A MINIMUM SPANNING TREE

ITA 21.1 GROWING A MINIMUM SPANNING TREE

GENERIC ALGORITHM FOR MST

↑
Template

GENERIC ALGORITHM FOR MST

Let A be a subset of edges in some MST of G .

An edge is a **safe edge** for A if adding it keeps A extendable to some minimum spanning tree; equivalently, $A \cup \{(u, v)\}$ remains a **subset** of an **MST**.

Line 3 maintains the following **loop invariant** – A is **always** a subset of some minimum spanning tree.

1: **Generic-MST**(G, w)

2: $A \leftarrow \emptyset$ \triangleleft start with an empty set of edges

3: **while** A does **not** form a spanning tree **do**

4: find an edge (u, v) that is **safe** for A

5: $A \leftarrow A \cup \{(u, v)\}$

6: **return** A

Add (u, v) to A .

Empty set must be subset
of
a mst.

If A is
not
a mst,
add
edge.

CUTS IN AN UNDIRECTED GRAPH

Cut — For an undirected graph $G = (V, E)$, a cut $(S, V \setminus S)$ is a partition of V .

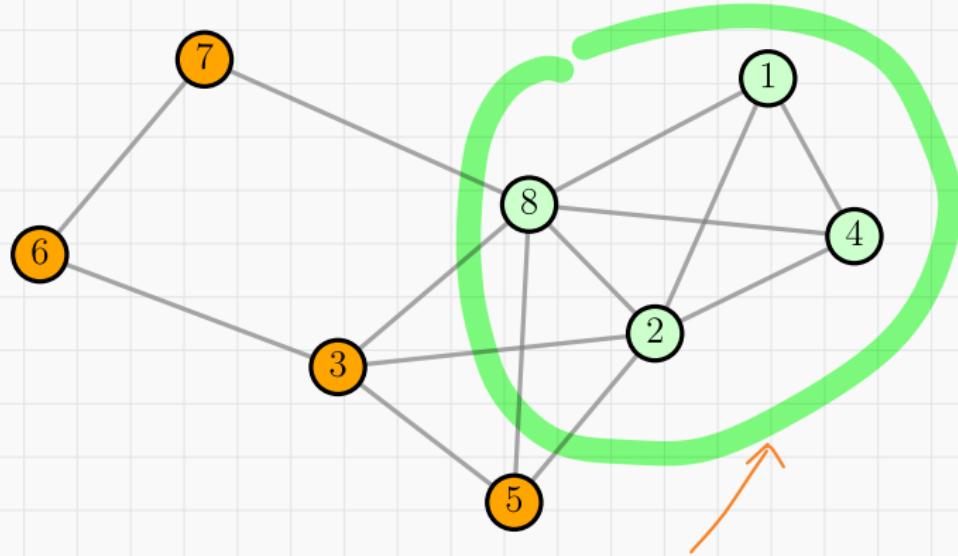


Figure 4: A cut of G with $S = \{1, 2, 4, 8\}$

CUTS IN AN UNDIRECTED GRAPH

Crossing the Cut – An edge $(u, v) \in E$ crosses the cut $(S, V \setminus S)$ if one of its endpoints is in S and the other is in $V \setminus S$.

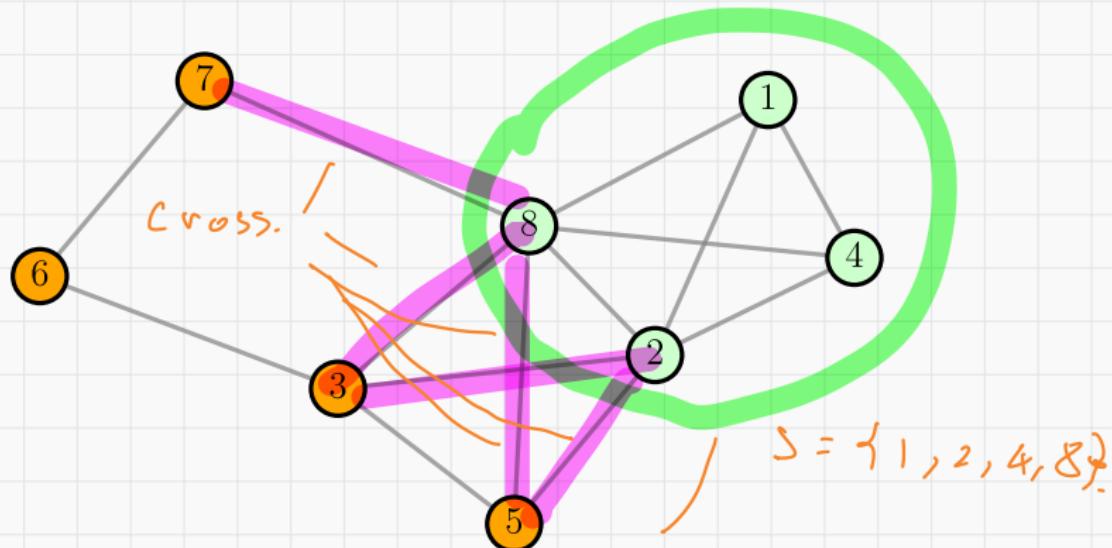


Figure 4: 🎉 Can you find edges crossing the cut?

CUTS IN AN UNDIRECTED GRAPH

Respecting ~~a set~~ — A cut respects a set A of edges if no edge in A crosses the cut.

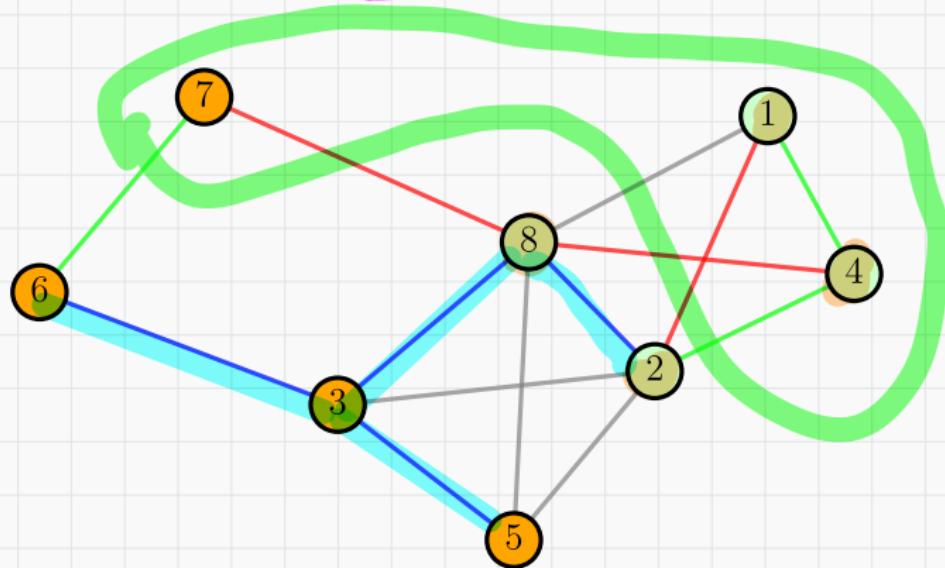


Figure 4: 🎂 Which set of edges does the cut respect? Blue, red or green?

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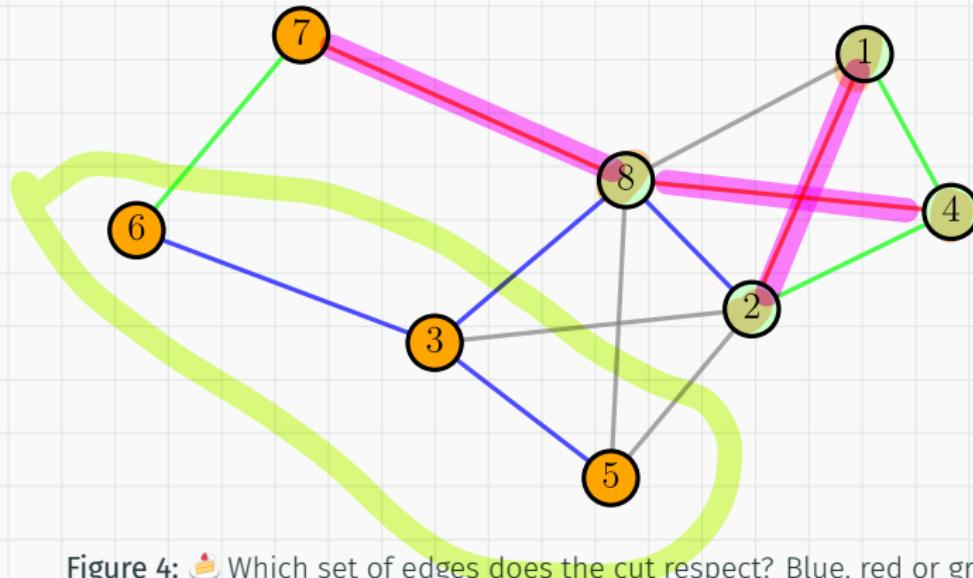


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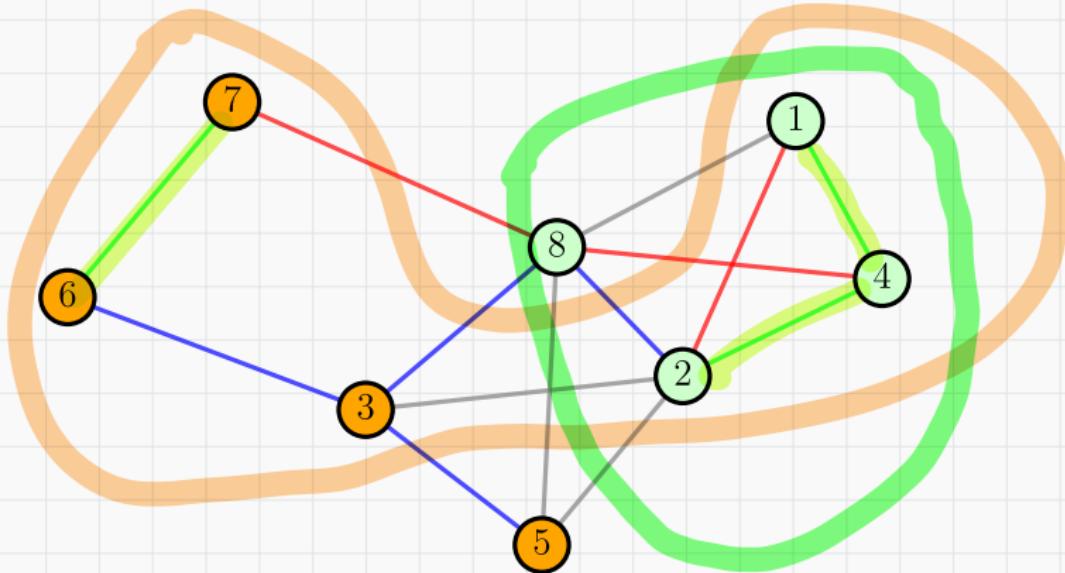


Figure 4: 🎂 Which set of edges does the cut respect? Blue, red or green?

CUTS IN AN UNDIRECTED GRAPH

Light Edge – An edge is a *light edge* crossing a cut if its weight is the minimum of any edge crossing the cut.

There may be multiple light edges crossing a cut in case of ties.

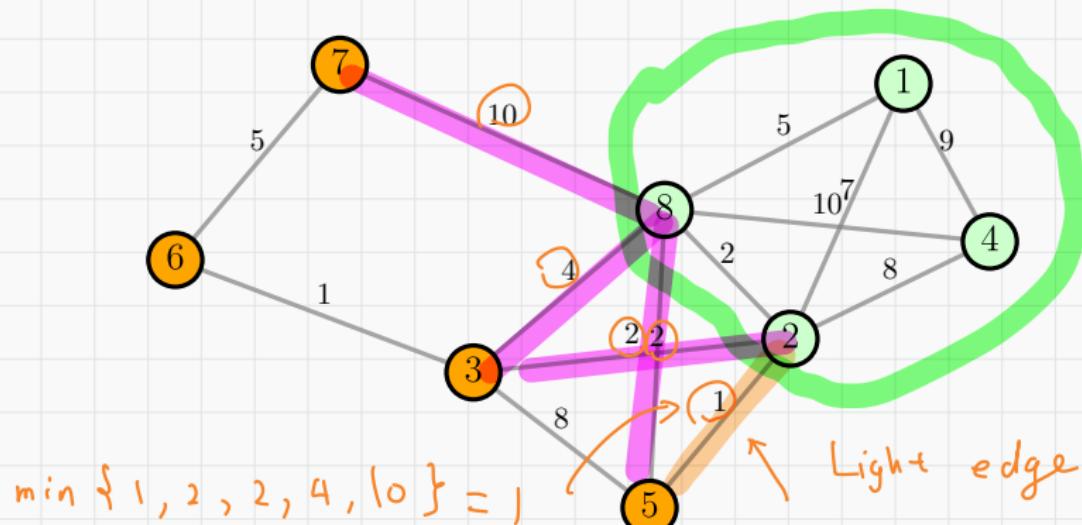


Figure 4: 🎂 Which edge is the light edge?

CUTS IN AN UNDIRECTED GRAPH

Light Edge with a Property – An edge is a *light edge* satisfying a given property if its weight is the minimum of any edge satisfying that property. *and it's a cross edge.*

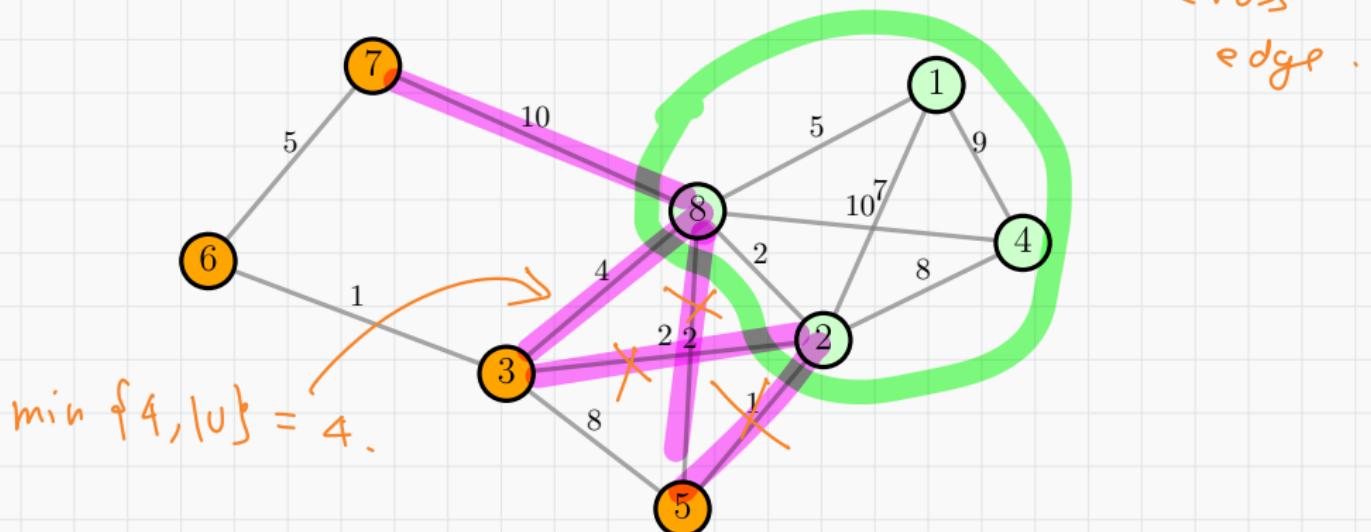


Figure 4: 🎉 Which edge is the light edge with the property “having weight at least 3”?

THEOREM 21.1

Let $G = (V, E)$ be a connected, undirected and weighted graph.

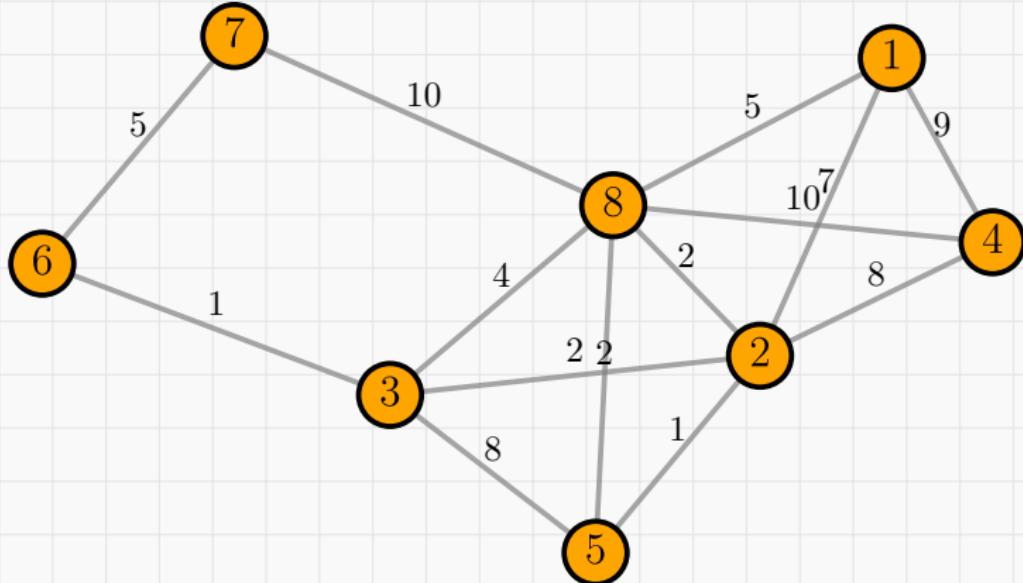
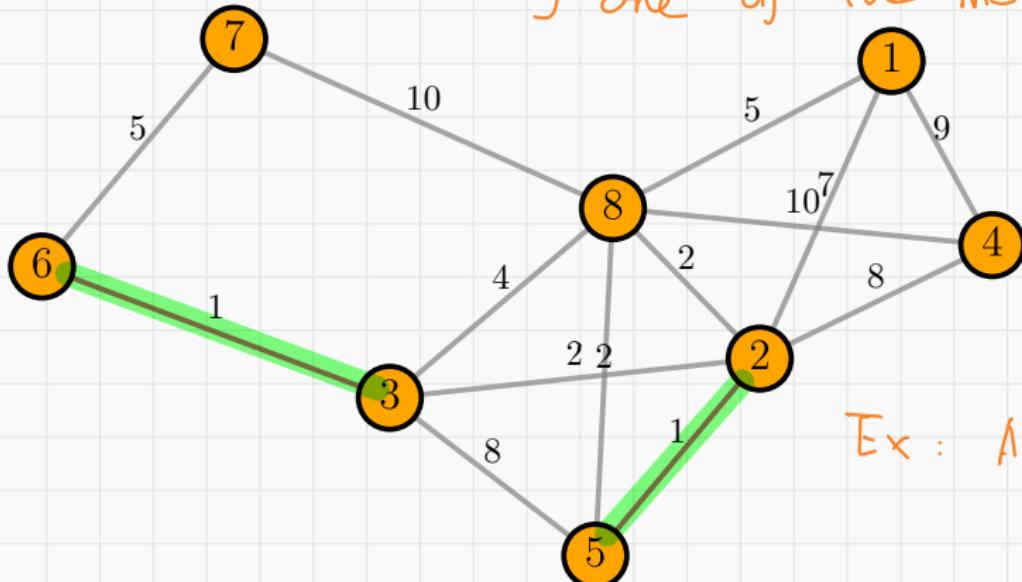


Figure 5: Example of — A weighted graph G

THEOREM 21.1

Let A be a subset of E that is included in some MST for G .

any one of the MSTs.



Ex : $A = \{ (6,3), (2,5) \}$.

Figure 5: Example of — A is some edges in a MST

THEOREM 21.1

Let $(S, V - S)$ be any cut of G that respects A .

$$S = \{1, 8\}$$

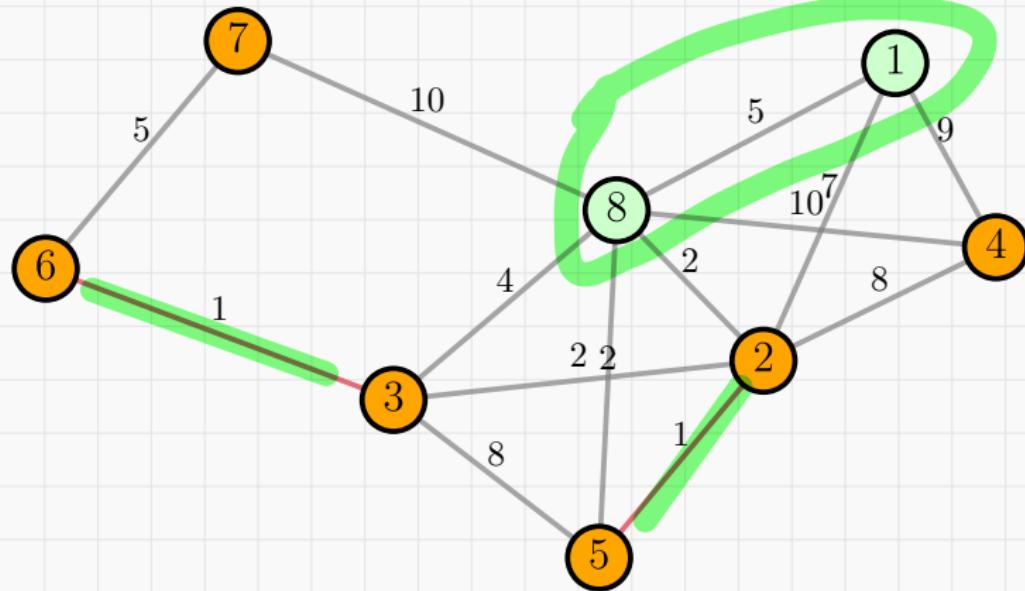


Figure 5: Example of — A cut respecting A .

THEOREM 21.1

Let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A .

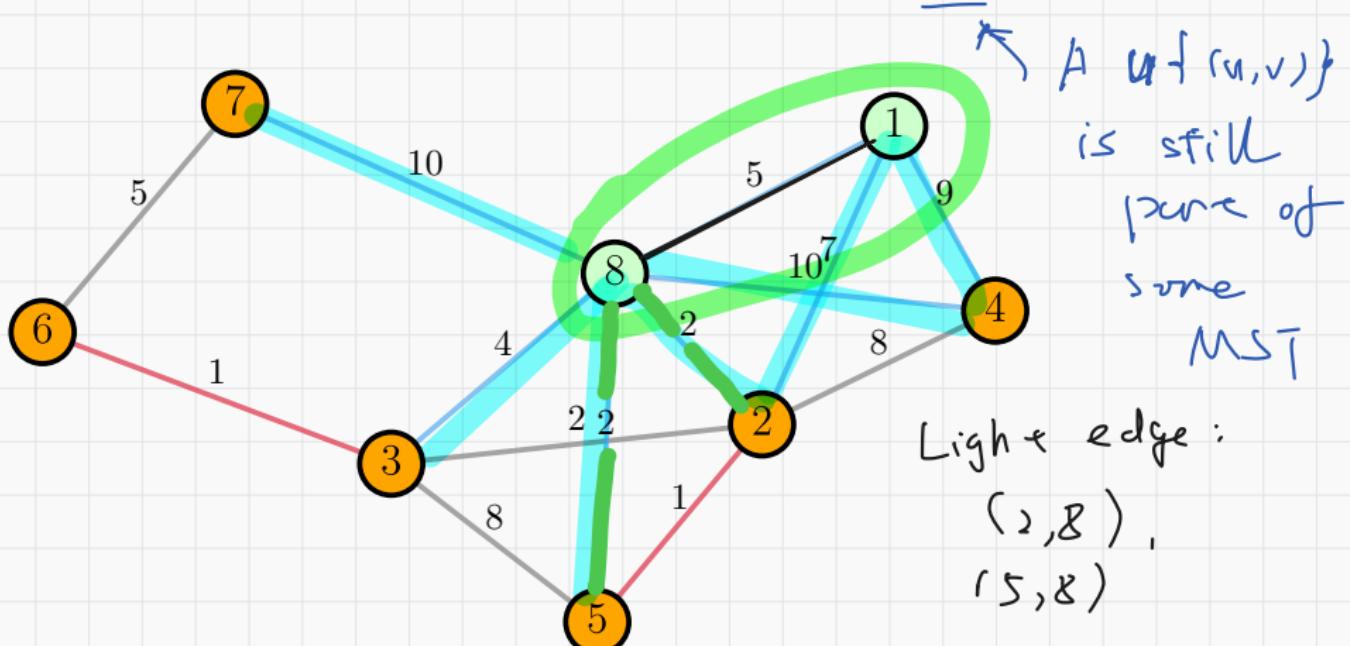


Figure 5: Example of —🎂 Which one is light among the crossing edges?

THEOREM 21.1

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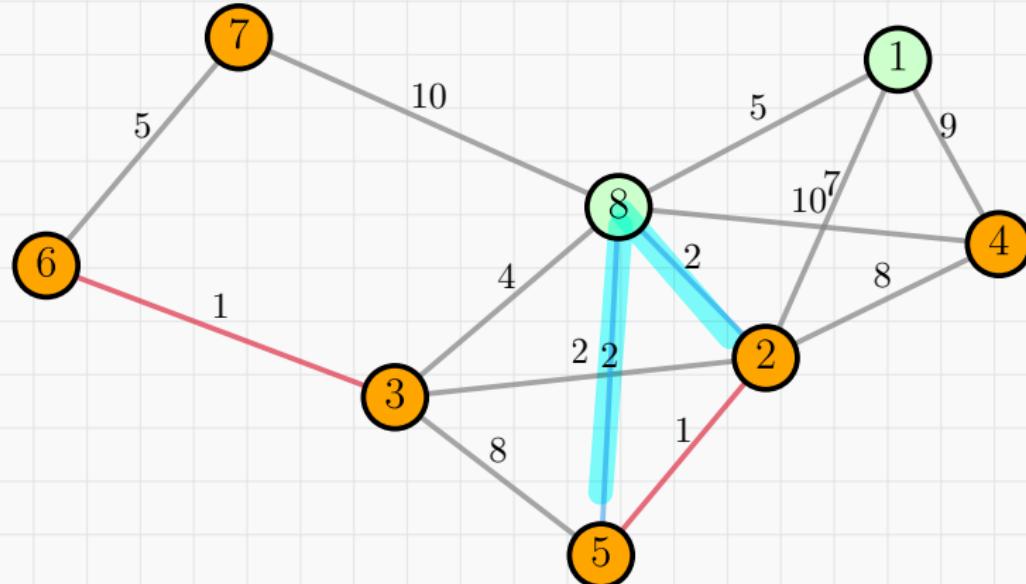


Figure 5: Example of — When there are multiple light edges, pick one arbitrarily to be (u, v) .

PROOF OF THEOREM 21.1

Proof — If T , the MST containing A , also contains (u, v) , then we are done.

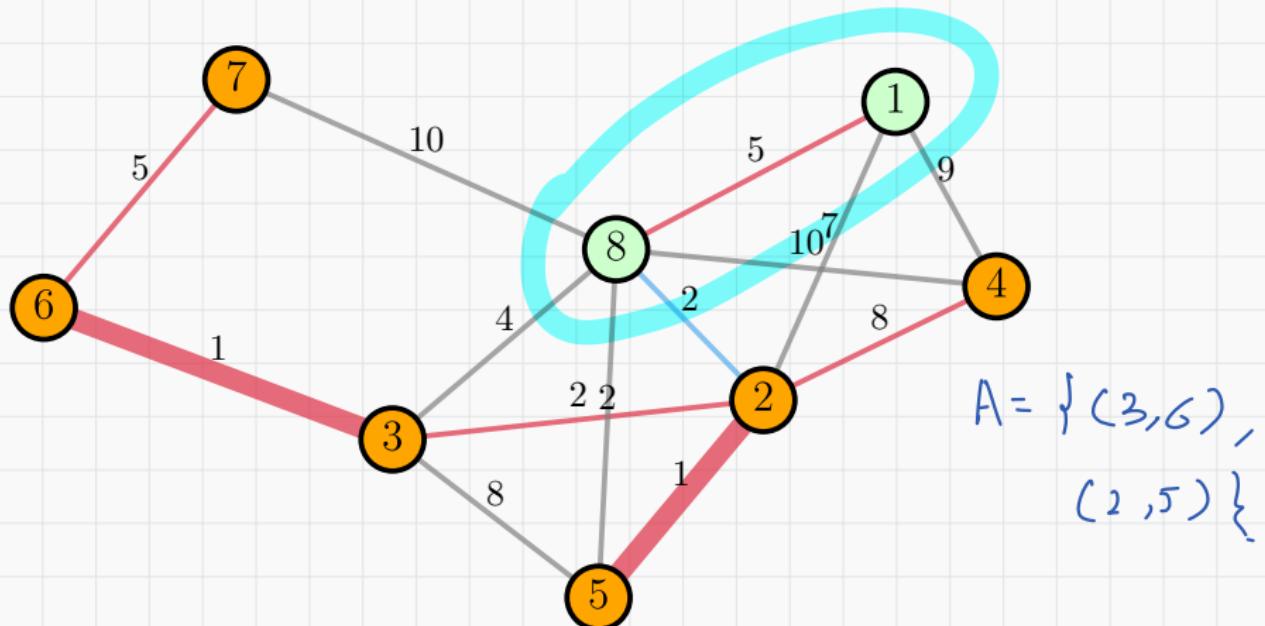


Figure 6: T contains (u, v)

PROOF OF THEOREM 21.1

Proof — If not, we build a new MST T' that contains (u, v) and A .

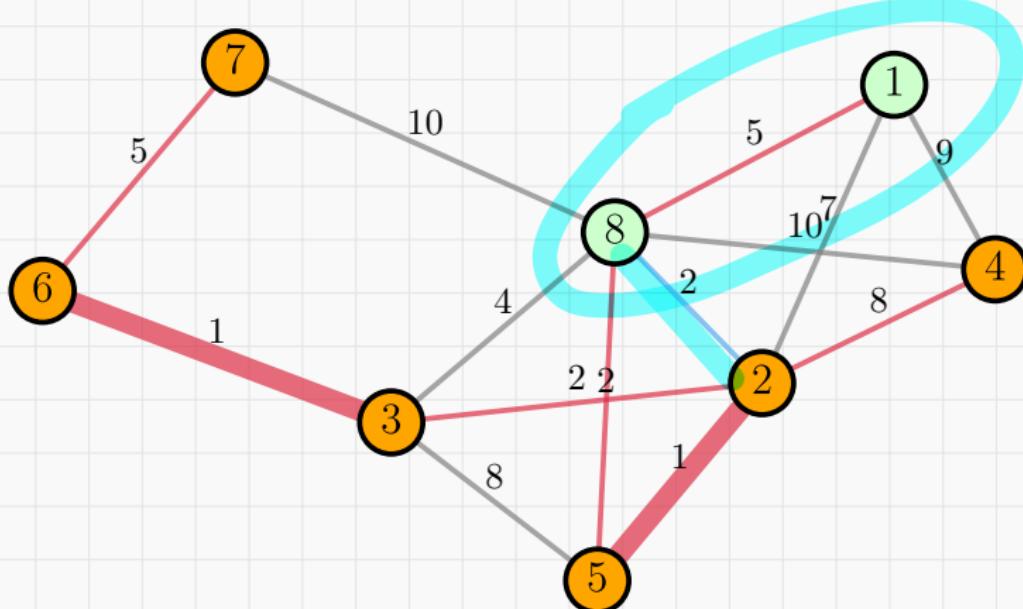


Figure 6: Since T does not contain $(u, v) = (2, 8)$, we replace $(x, y) = (8, 5)$ by (u, v) and get a different MST T'

PROOF OF THEOREM 21.1

Proof — If not, we build a new MST T' that contains (u, v) and A .

Replace xy
by uv in T ,
we get a
new MST

The
light-
edge.

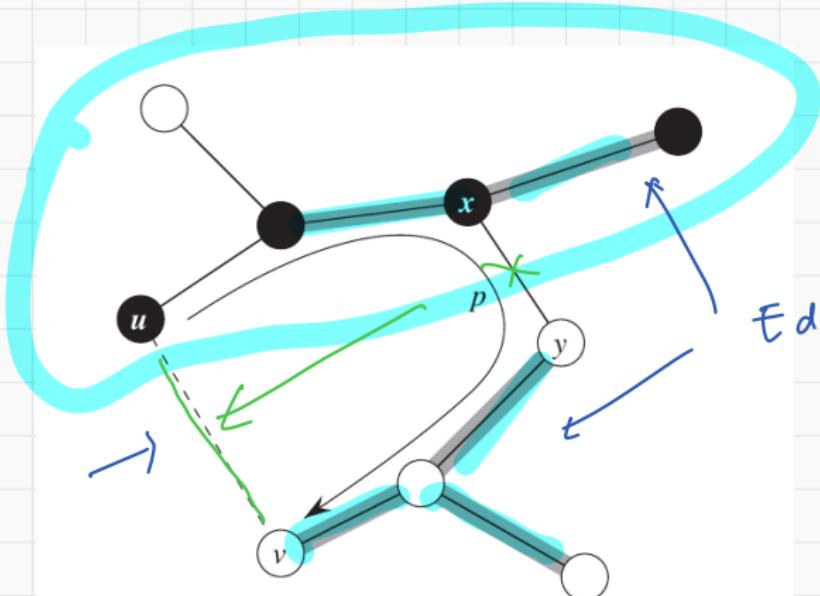


Figure 6: T does not contain (u, v)

Note that

$$w_{xy}$$

$$\geq w_{uv}$$

because uv is

Edges of A . ↑
light

Actually, we
must have
 $w_{xy} = w_{uv}$

COROLLARY 21.2

Let $G = (V, E)$ be a connected, undirected and weighted graph.

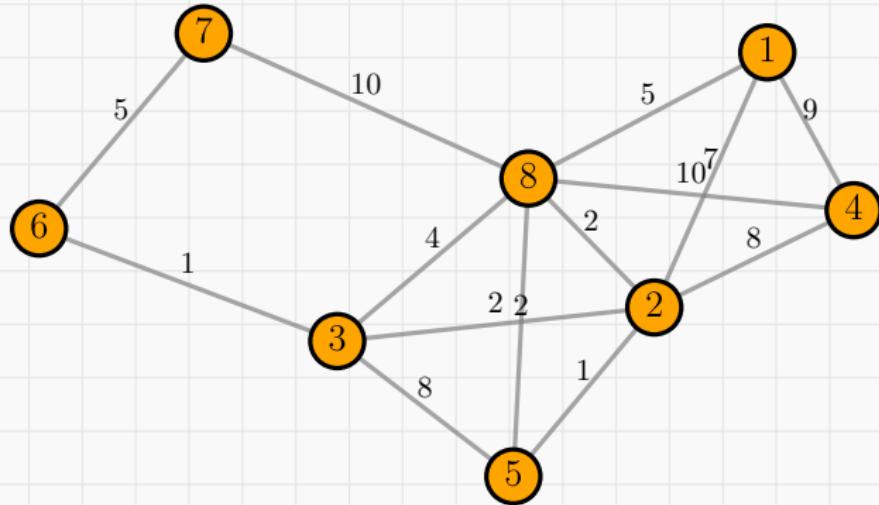
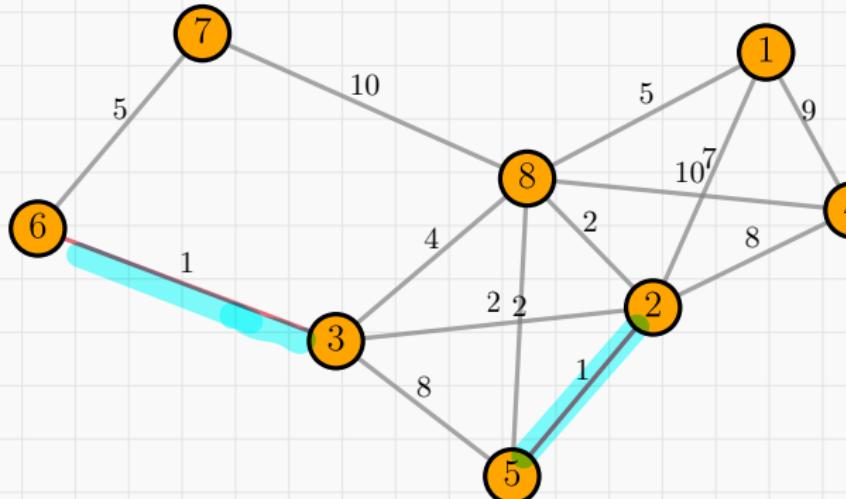


Figure 7: A weighted graph G

COROLLARY 21.2

Let A be a subset of E that is included in some MST for G .



$$A = \{ (3, 6), (2, 5) \}$$

Figure 7: Let A be some edges of a MST

COROLLARY 21.2

Let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$.

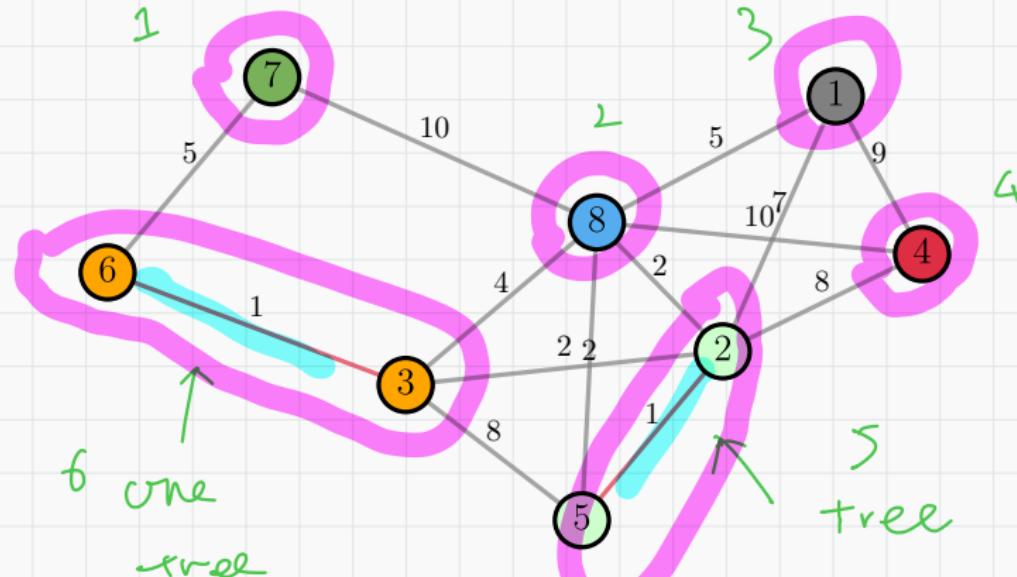


Figure 7: Let $C = (\{2, 5\}, \{(2, 5)\})$

COROLLARY 21.2

If (u, v) is a light edge connecting C to some other component in G_A , then (u, v) is safe for A .

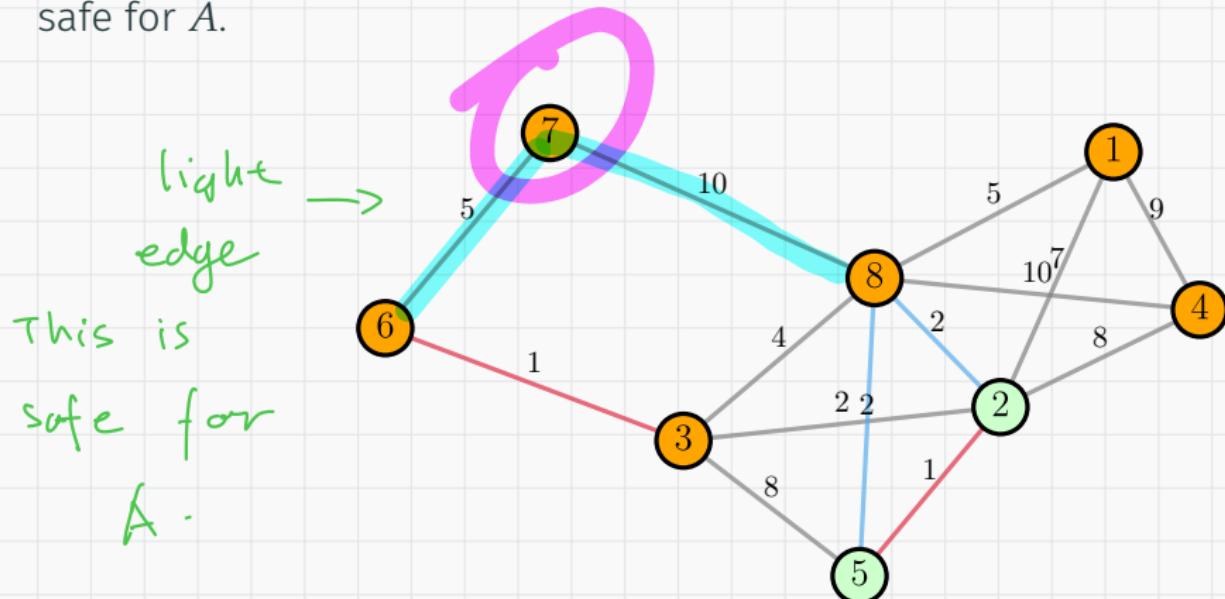


Figure 7: Light crossing edges – Let $(u, v) = (2, 8)$

🎂 Is the edge $(4, 2)$ safe for $A = \{(3, 6), (2, 5)\}$?

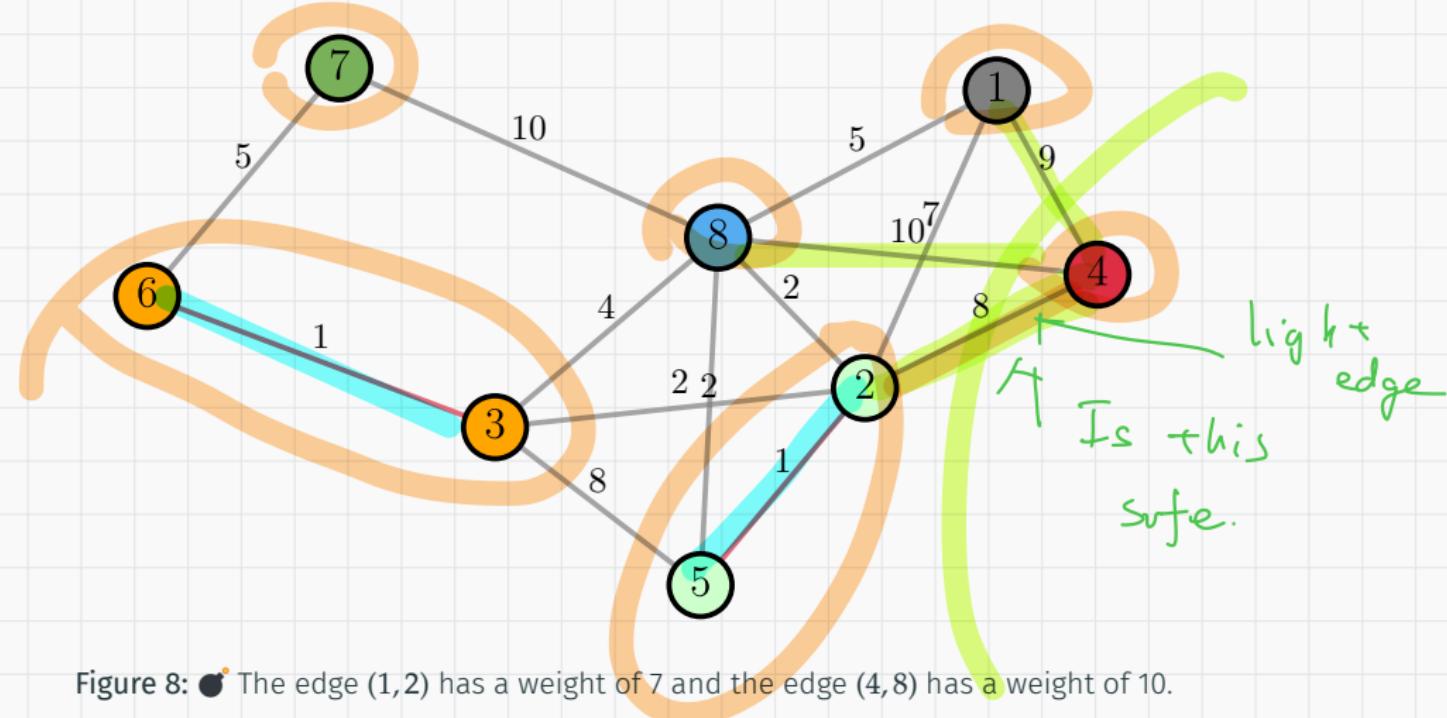


Figure 8: ⚡ The edge $(1, 2)$ has a weight of 7 and the edge $(4, 8)$ has a weight of 10.

In 2007, trader Brad Katsuyama noticed that when he split an order across multiple exchanges, the best prices often disappeared.

Sending the same order to just one exchange worked, which hinted that someone was reacting faster than his order could reach all venues.

The culprit was high-frequency trading exploiting tiny **latency** differences between exchanges — a speed advantage that looks like a shortest-path race in a weighted network.

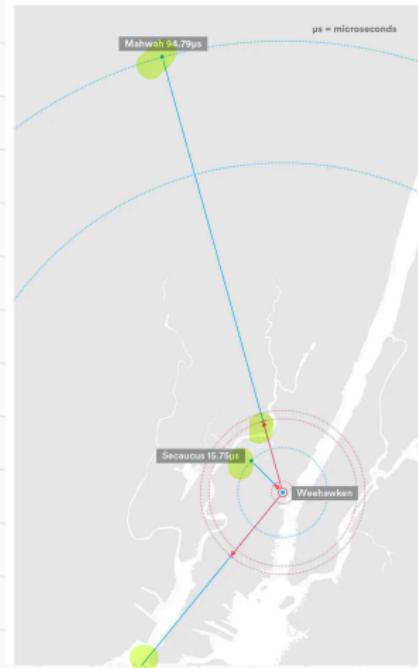


Figure 9: The distance from Katsuyama to the exchanges.

How to Stop the 🐺?

The solution was paradoxical — adding a delay to synchronize order arrivals at all exchanges.

IEX (Investors Exchange) launched in 2012, using a 350-microsecond delay to blunt the speed advantage of high-frequency traders.

Today, IEX's market share is about 2.5%.

Book — *Flash Boys*

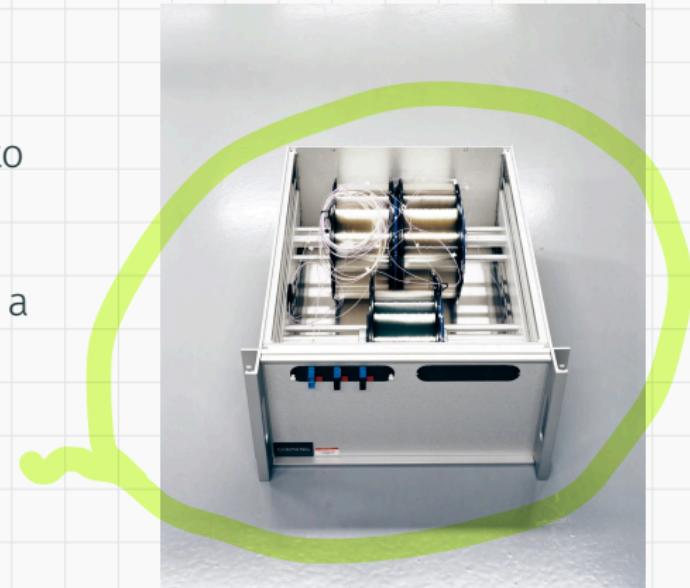


Figure 9: 32 miles of fiber optic cable for 350 microseconds delay.

ITA 21.1 GROWING A MINIMUM SPANNING TREE

EXERCISE 21.1

EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — Suppose that (for contradiction) the graph has 2 distinct spanning trees T and T' .

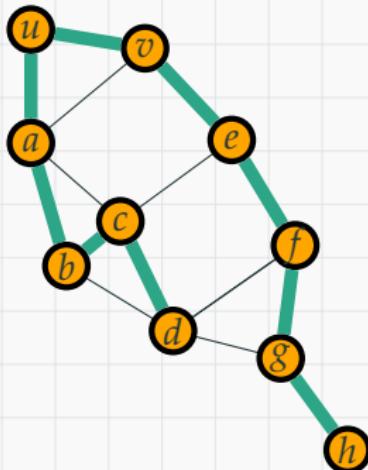


Figure 10: T

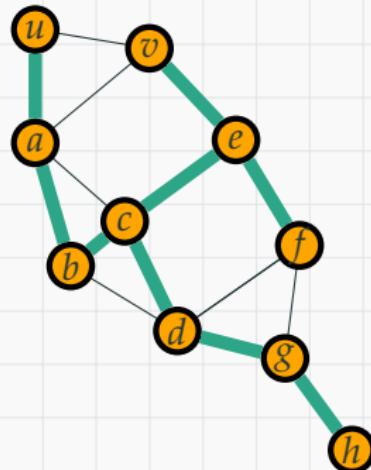


Figure 11: T'

EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — There must exist an edge (u, v) in T but not T' .

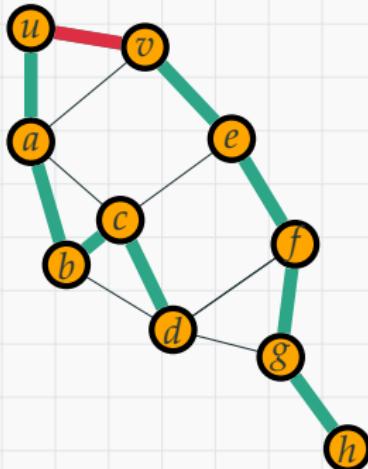


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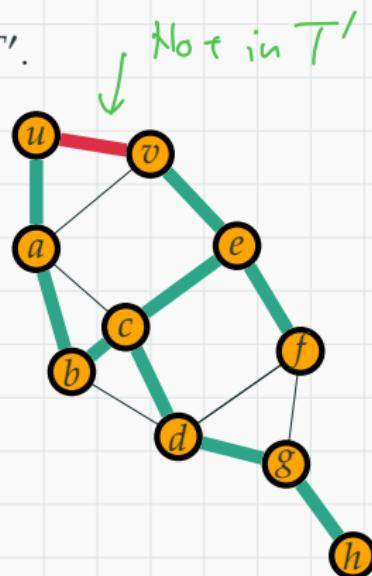


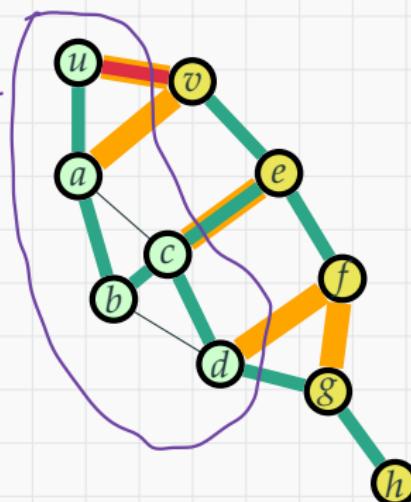
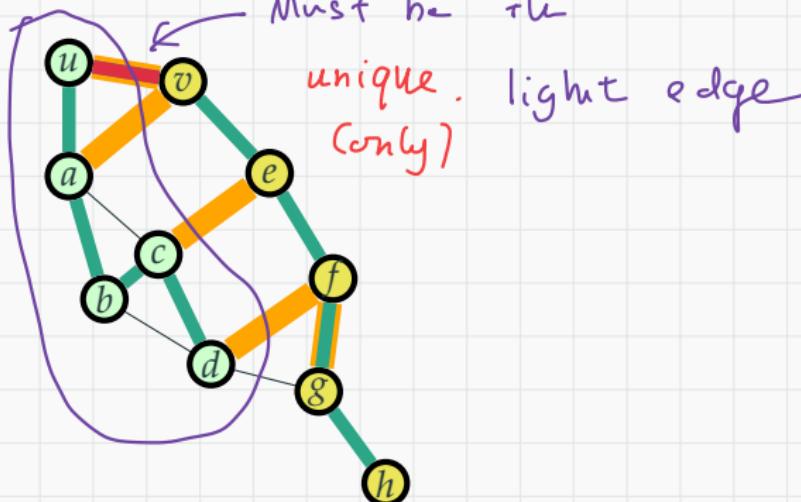
Figure 11: T'

EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — Removing (u, v) from T gives us a cut.

The edge (u, v) must be the unique light edge of this cut. Otherwise T cannot be a MST.



EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

Proof — Then replacing the crossing edge in T' with (u, v) gives a tree with smaller weight, which is a contradiction.

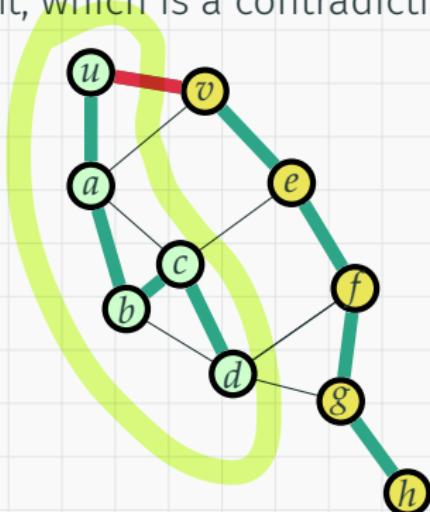


Figure 10: T

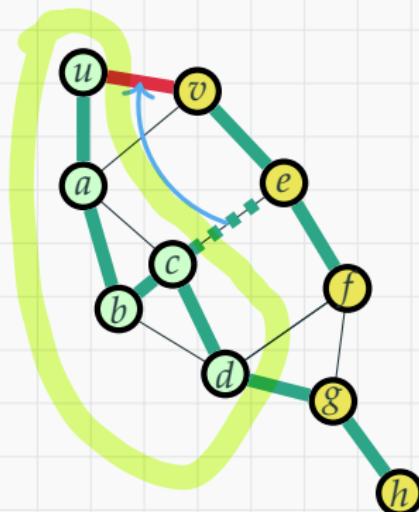


Figure 11: T'

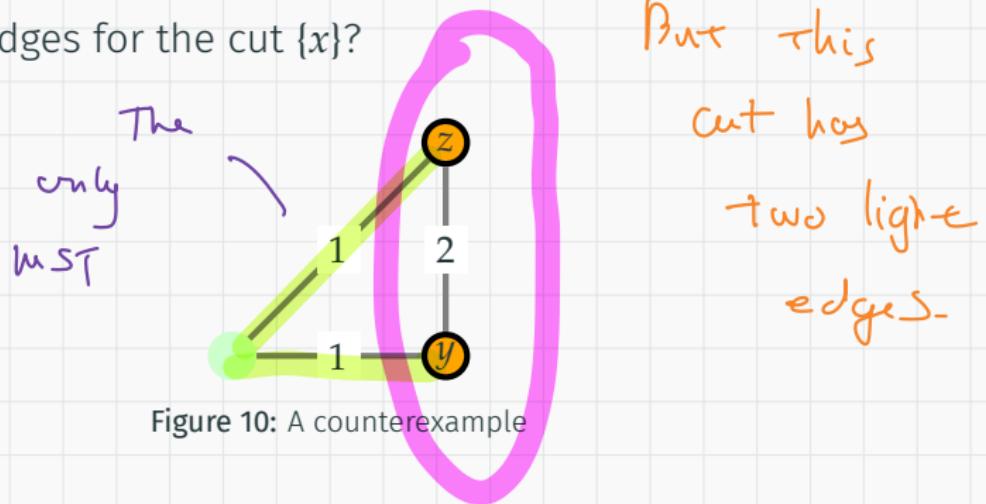
EXERCISE 21.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut.

✳️ Exercise — Show that the converse is not true.

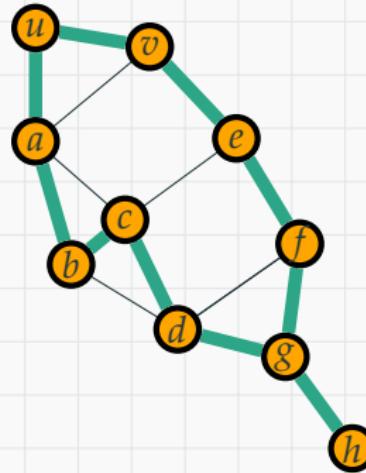
🍰 What is the unique MST of the following graph?

🎂 What are the light edges for the cut $\{x\}$?



EXERCISE 21.1-7

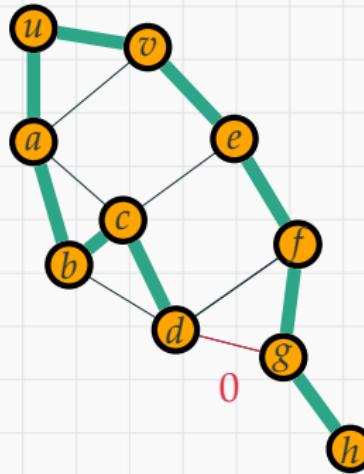
Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree.



EXERCISE 21.1-7

Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree.

Give an example to show that the same conclusion does not follow if we allow some weights to be nonpositive.



WHAT ARE YOUR MAIN TAKEAWAYS TODAY? ANY QUESTIONS?

