Lecture 14 — Minimum Spanning Trees (Part 2)

COMPSCI 308 — Design and Analysis of Algorithms

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Summary

Assignments[1]

Practice makes perfect!

📖 Required Readings:

- Section 21.2.

✏️ Required Exercises:

- Exercises 21.2 — 1–6.
- Problem 21.1.a.

[1] 💣 Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

ITA 21.2 The Algorithms of Kruskal and Prim

⏰ Review — Generic Algorithm for MST

The algorithms of Kruskal and Prim are based on the generic algorithm for Minimum Spanning Tree (MST) —

Generic-MST(G, w)

1: $A \leftarrow \emptyset$ ← Subset of some MST
2: **while** A does not form a spanning tree **do**
3: find an edge (u, v) that is *safe* for A ← How to find a safe edge for A.
4: $A \leftarrow A \cup \{(u, v)\}$ ← add (u, v).
5: **return** A

ITA 21.2 The Algorithms of Kruskal and Prim

Kruskal's Algorithm

👀 Review — Generic Algorithm for MST

Let A be a subset of edges in some MST of G.

An edge is a safe edge for A if adding it keeps A extendable to some minimum spanning tree; equivalently, $A \cup \{(u, v)\}$ remains a subset of an MST.

Line 3 maintains the following loop invariant — A is *always* a subset of some minimum spanning tree.

Generic-MST(G, w)

1: $A \leftarrow \varnothing$
2: **while** A does not form a spanning tree **do**
3: find an edge (u, v) that is safe for A
4: $A \leftarrow A \cup \{(u, v)\}$
5: **return** A

Overview of Kruskal's Algorithm

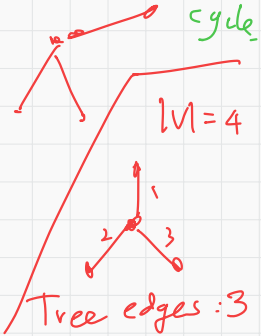
Kruskal's algorithm works as follows —

1. Sort all graph edges in non-decreasing order of weight.
2. Start with $A = \emptyset$ and each vertex in its own component.
3. Consider edges in order; add an edge (u, v) to A only if u and v are in different components.
4. Stop when $|A| = |V| - 1$.

🤔 How to show this algorithm is correct?

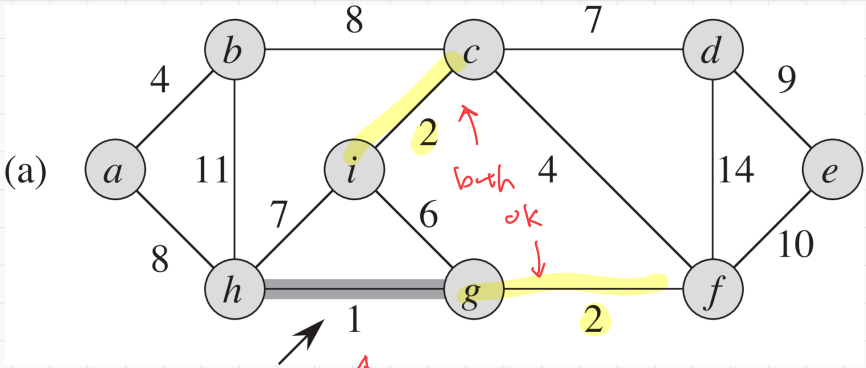
does not create a cycle

Not going to create cycle



This must be a tree since it does not have any cycles.

Example Run of Kruskal's Algorithm



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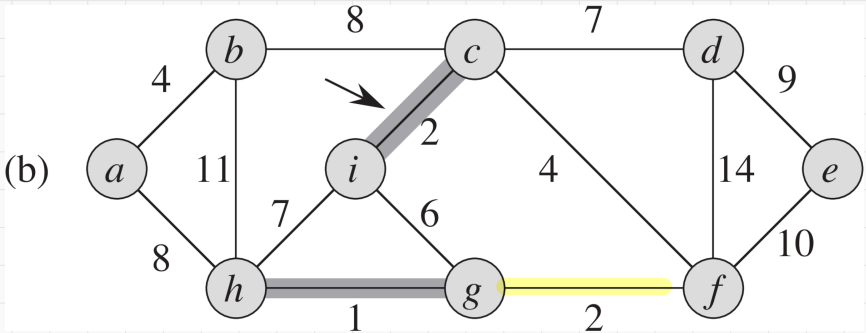

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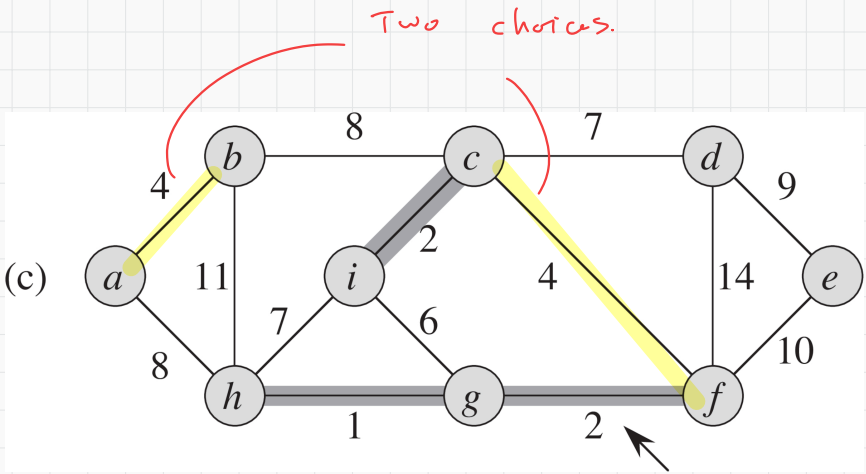

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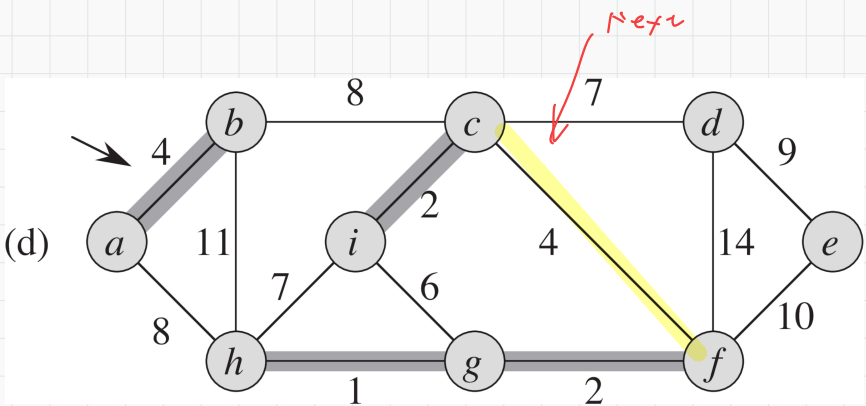
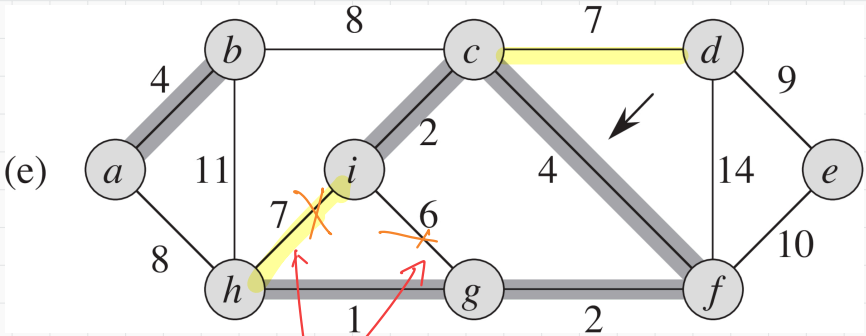


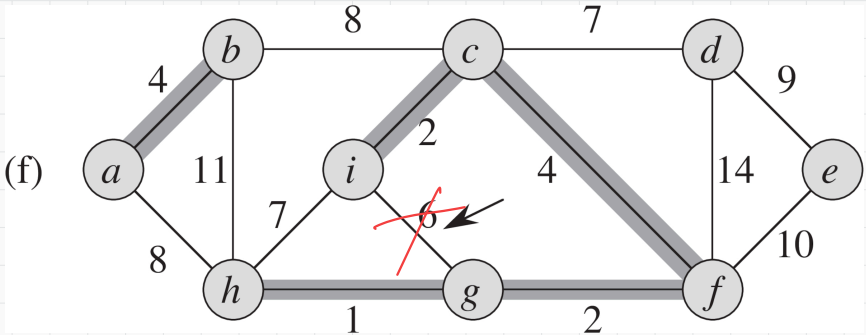
Example Run of Kruskal's Algorithm

ok.



No! This creates a cycle. Don't need to check this any more

📌 Example Run of Kruskal's Algorithm



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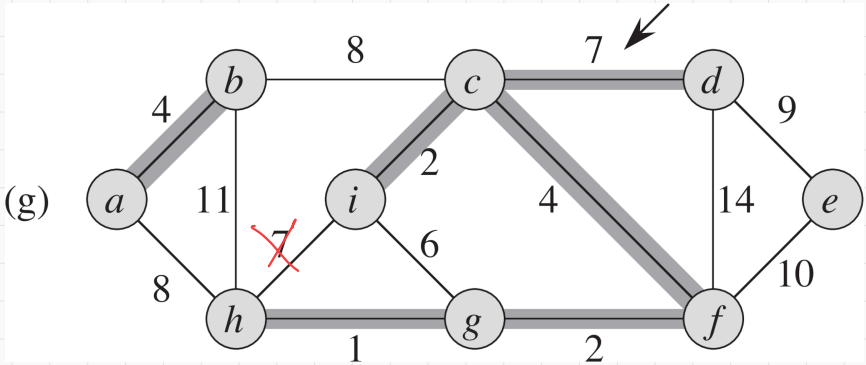

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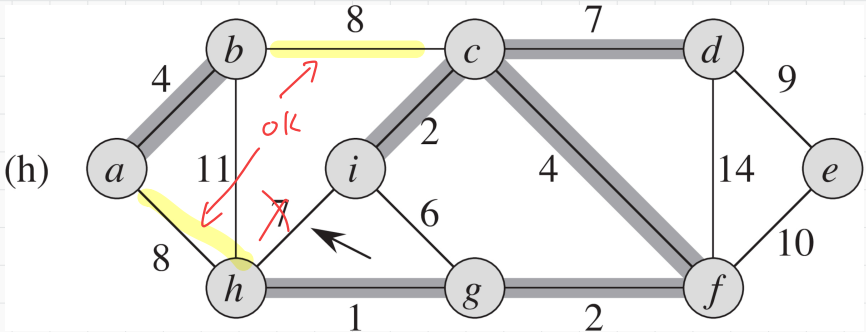

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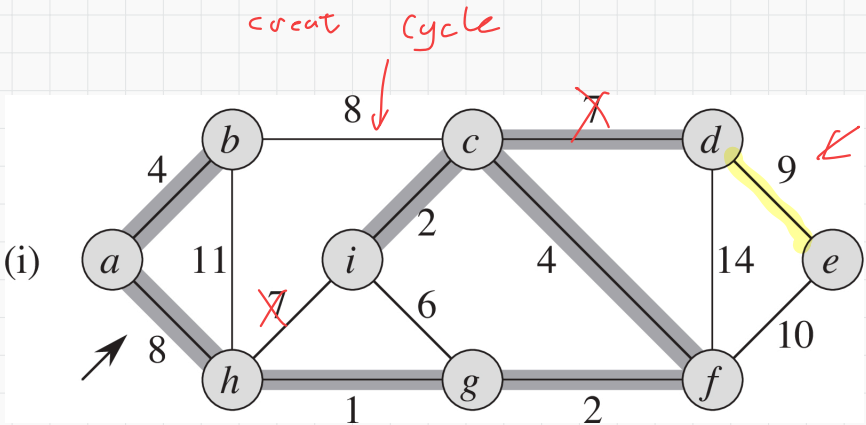

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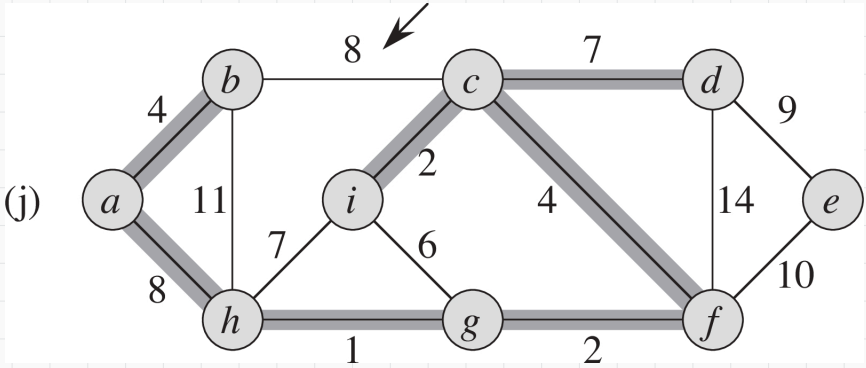
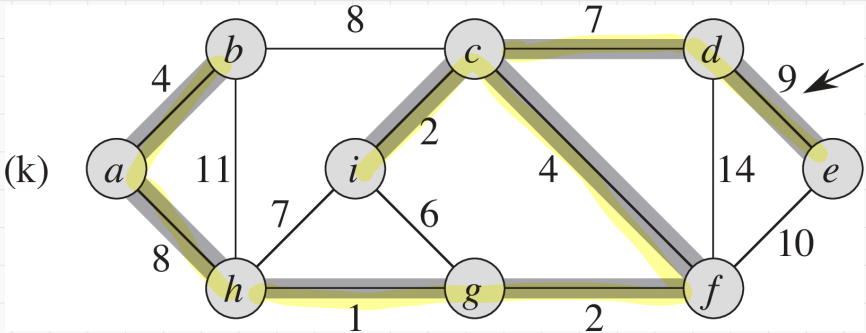
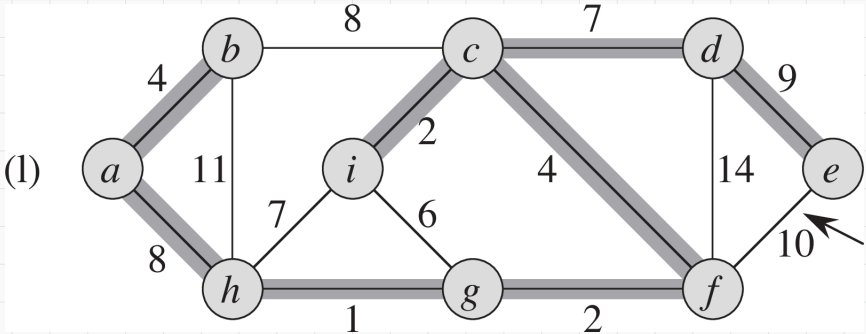


Example Run of Kruskal's Algorithm



This is a greedy algorithm

📌 Example Run of Kruskal's Algorithm



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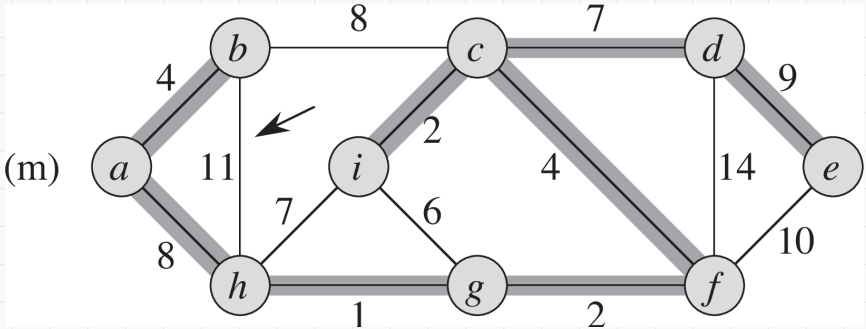
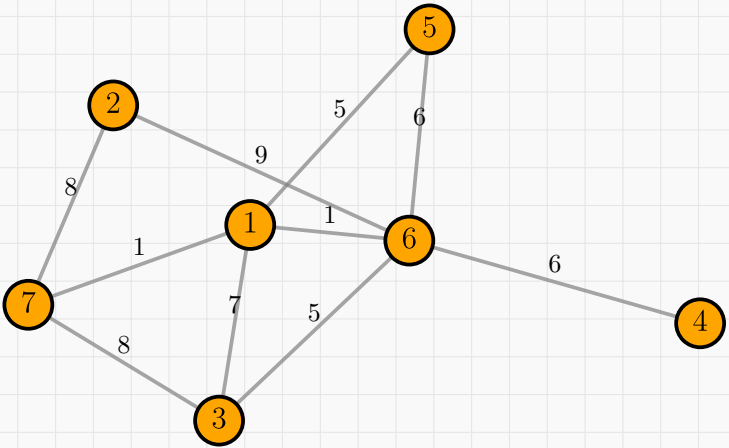


🍰 Run Kruskal's algorithm on the following graph to determine the weight of its MST?



⏰ Review — Corollary 21.2

Let A be a subset of the edges of a MST. Then A is a forest.

Take any component of A as a cut. A light edge for the cut is safe.

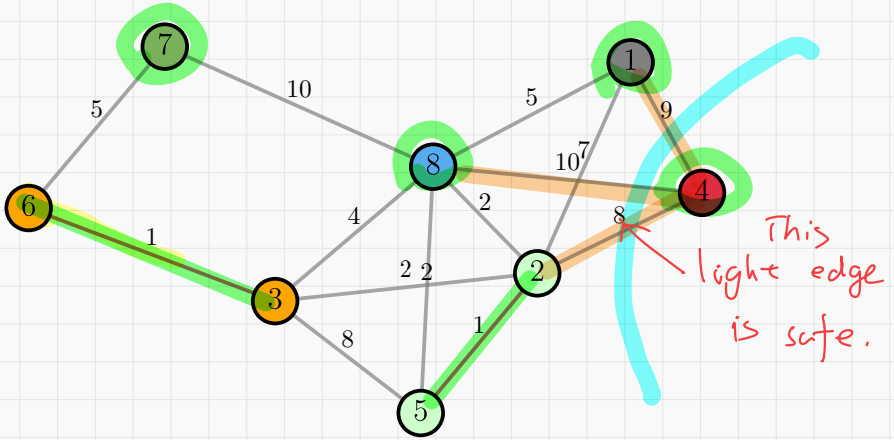


Figure 1: 🍰 What are the light edges for $S = \{5, 2\}$?

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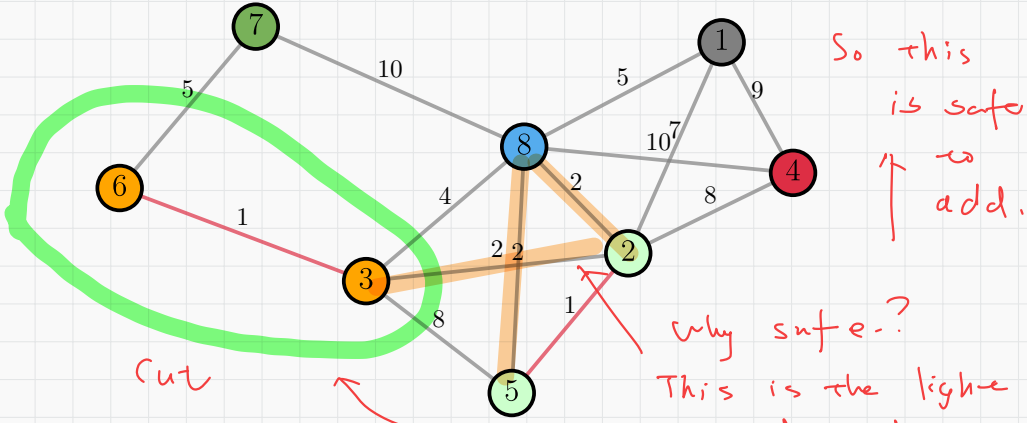

Figure 1: 🍦 What are the light edges for $S = \{3, 6\}$?

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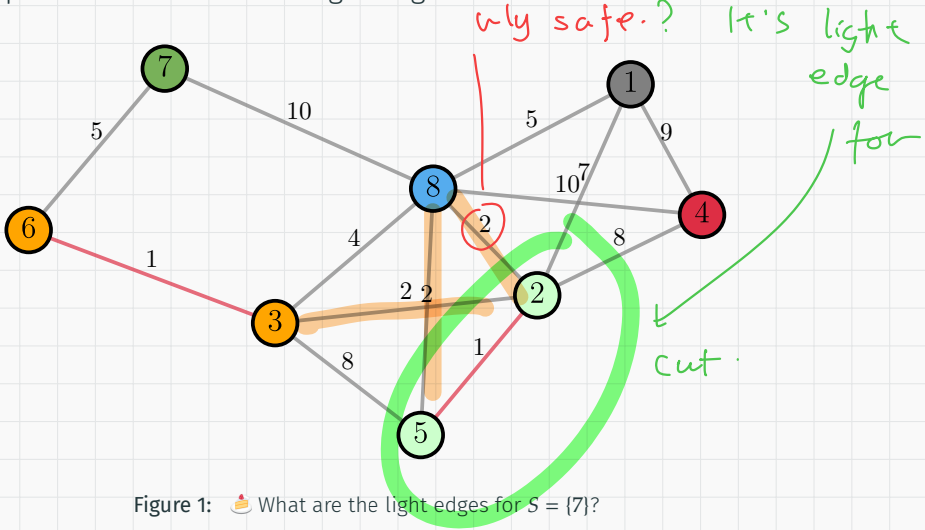

Figure 1: 🥧 What are the light edges for $S = \{7\}$?

✅ Correctness of Kruskal's Algorithm

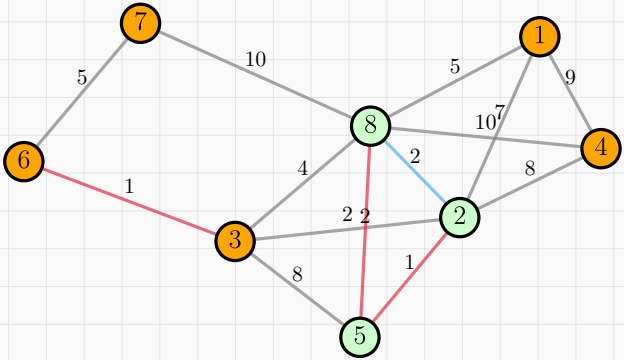

Figure 2: 🍰 Why is $(2, 8)$ *not* safe?

✅ Correctness of Kruskal's Algorithm

When an edge (u, v) added to A does not create a cycle in A, it must connect two different components in A.

If it is a *light edge* with this property, then by Corollary 21.2 it is safe.

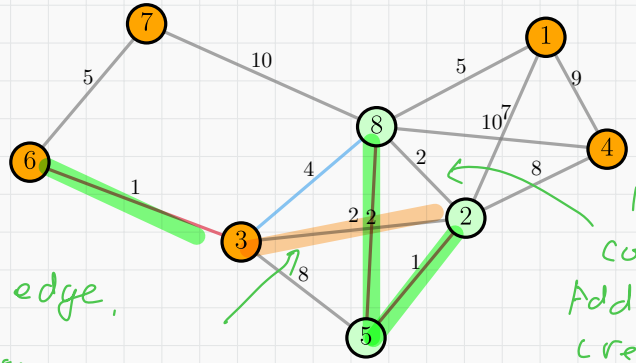

Figure 2: Why is (2, 3) safe?

Assuming G is connected, then we have

$$|V| - 1 \leq |E| \leq \binom{|V|}{2}.$$[2]

$\frac{|V|(|V|-1)}{2}$

[2] The textbook often uses V to mean $|V|$, E to mean $|E|$, etc., which should be clear from the context.

Assuming G is connected, then we have

$$|V| - 1 \le |E| \le \binom{|V|}{2}.^{2}$$

Let us assume that Kruskal's algorithm uses the disjoint-set data structures which provide —

- MAKE-SET(v),
- FIND-SET(v)
- UNION(u, v),

each of which takes $O(\log |V|) = O(\log |E|)$ time.

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Pseudocode for Kruskal's Algorithm

Let's show the complexity is $O(|E| \log |V|)$ for Kruskal's Algorithm.

```
Kruskal(G, w)
1:  A ← ∅                                                    ▷ O(1)
2:  for each vertex v ∈ G.V do
3:      MAKE-SET(v)                                          ▷ O(V + E)
4:  Create a single list of the edges in G.E                 ▷ O(E log E)
5:  Sort the edges of G.E into nondecreasing order by w      ▷ O(Eα(V))
6:  for each edge (u, v) ∈ G.E do
7:      if FIND-SET(u) ≠ FIND-SET(v) then
8:          A ← A ∪ {(u, v)}
9:          UNION(u, v)
10: return A                                                 ▷ Total: O(E log E)
```

Graph.

weight.

$O(|V| \cdot \log |V|) = O(|E| \log |V|)$.

At the beginning, each vertex is itself a component

$O(|E| \cdot \log |V|)$

u and v are in two different component.

The graph uses adjacent list.

$O(|E| \log |E|) = O(|E| \log |V|)$.

ITA 21.2 The Algorithms of Kruskal and Prim

Prim's Algorithm

Overview of Prim's Algorithm

Prim's algorithm constructs the MST as follows —

- Start with a single root vertex r and let T be the current tree.
- At each step, consider the cut $(T, V \setminus T)$ and add the minimum-weight edge that crosses the cut.
- The added edge brings a new vertex into T.
- Repeat until T spans all vertices.

Eventually a MST

light edge

safe edge

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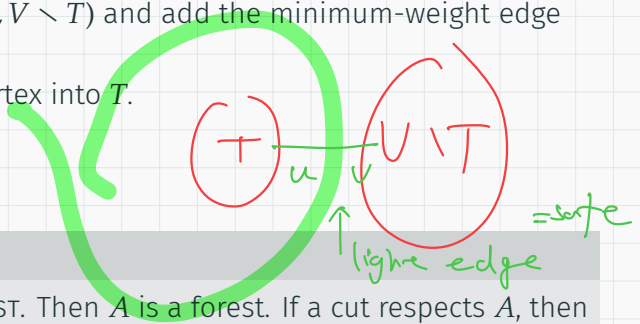
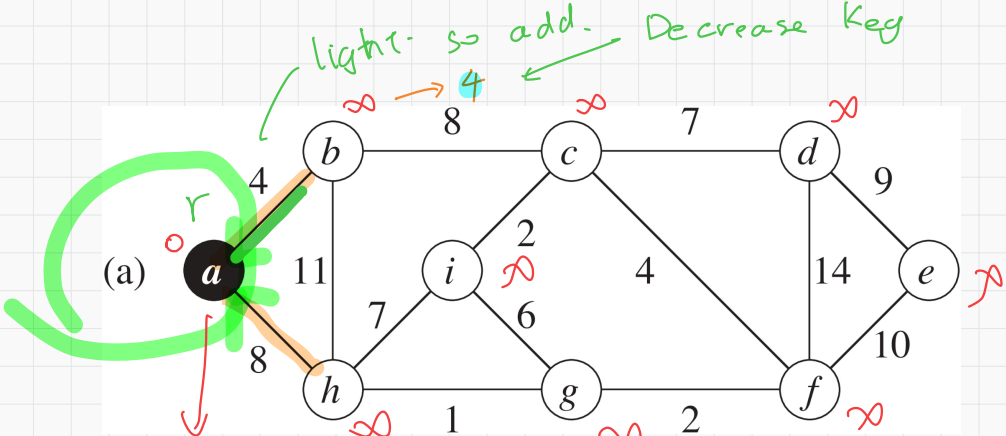


The correctness is guaranteed by —

Corollary 21.3

Let A be a subset of the edges of a MST. Then A is a forest. If a cut respects A, then a light edge crossing the cut is safe.

Example Run of Prim's Algorithm



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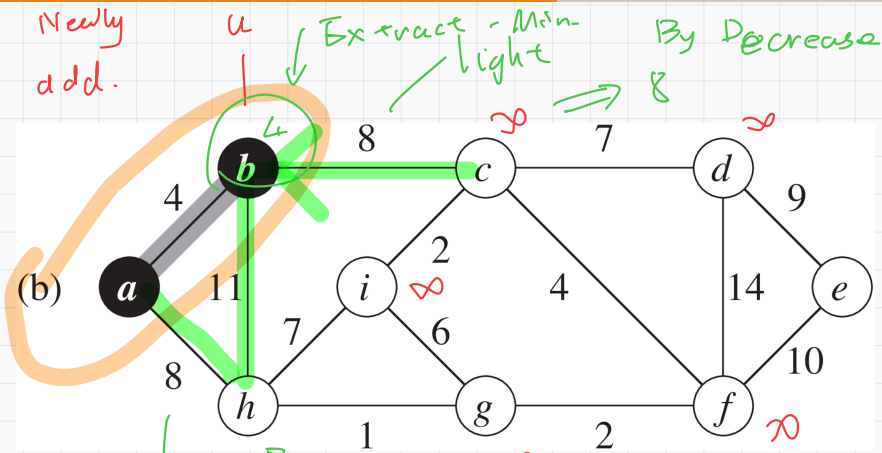

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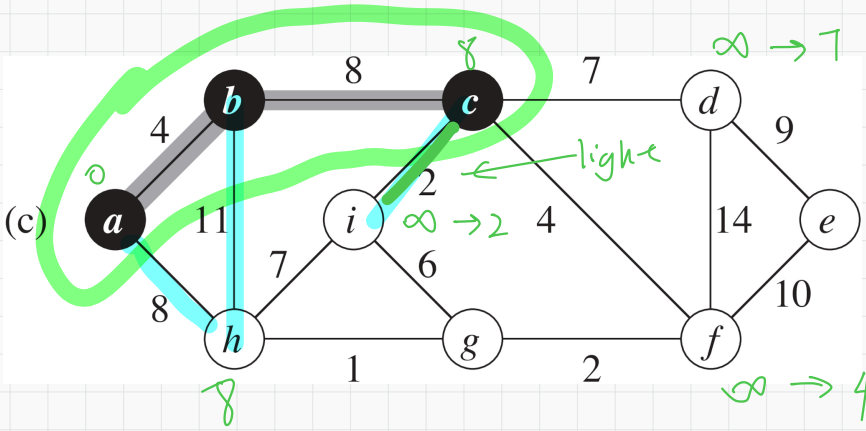

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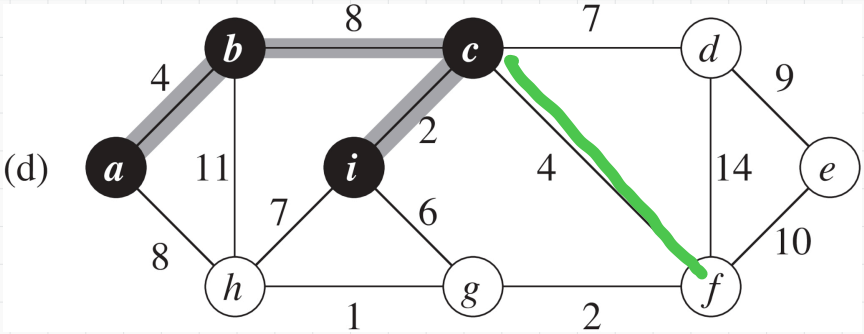

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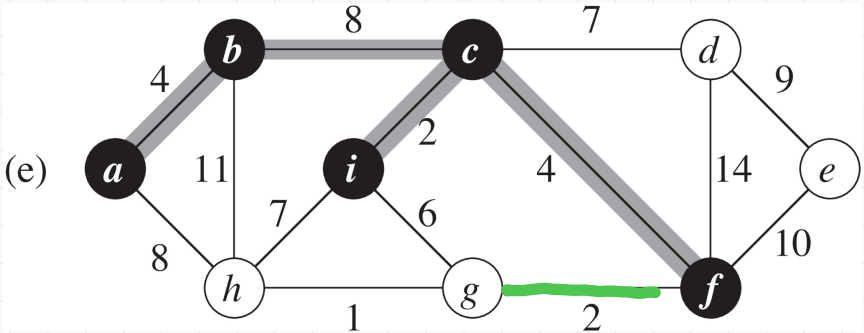

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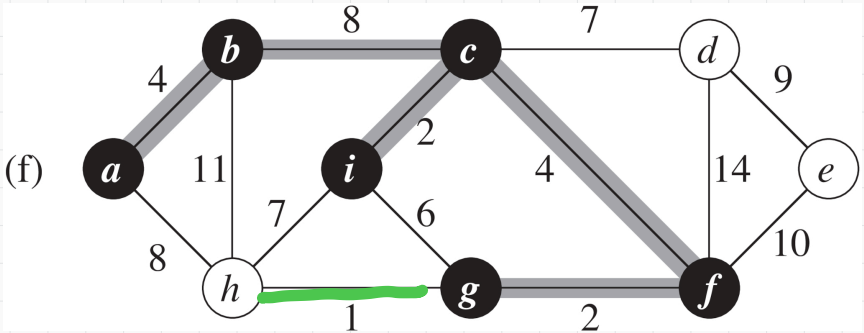

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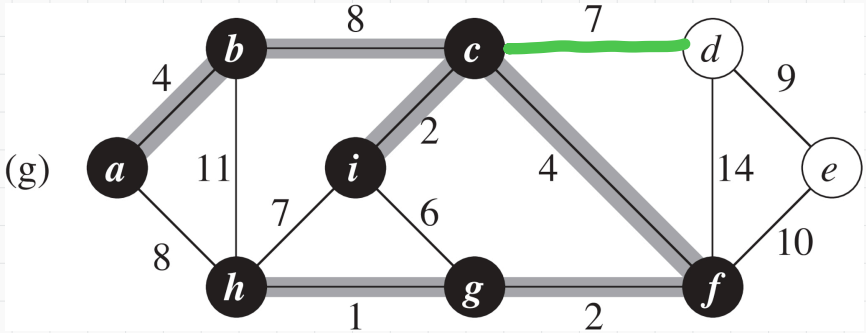

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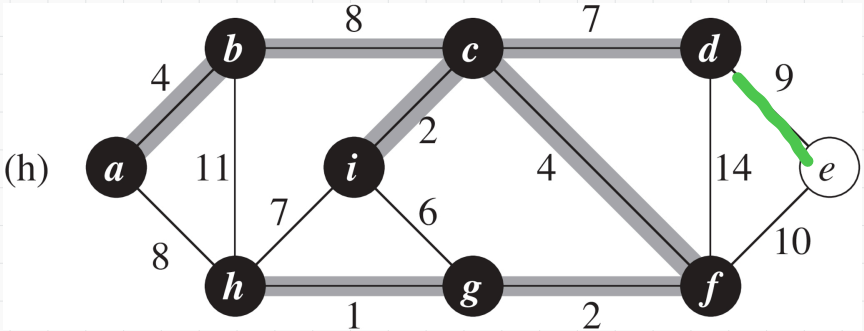

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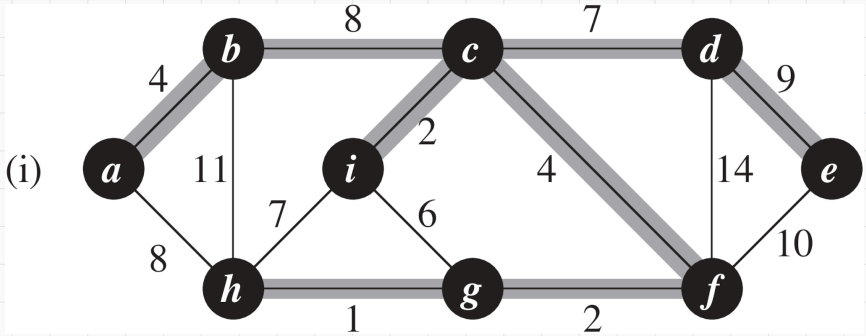
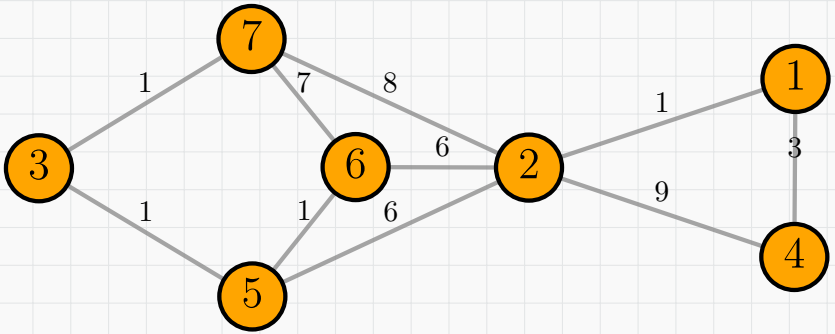


🍰 Run Prim's algorithm on the following graph with $r = 7$.



🔧 Min-Priority Queue

A min-priority queue (with a binary min-heap implementation) is a data structure that provides the following operations —

- Build-Queue(V) — build a queue in $O(|V|)$ time.
- Extract-Min(Q) — remove and return the element with the lowest `key` in $O(\log |V|)$ time.
- Decrease-Key($Q, u,$ `newKey`) — reduce the `key` of u to `newKey` in $O(\log |V|)$ time.

💡 The min-priority queue helps us to find the light edge to add.

Pseudocode for Prim's Algorithm

```
Prim(G, w, r)
1:  let B[1..|V|] be a new bit array
2:  for each u ∈ G.V do                                   ▷ O(V)
3:      u.key = ∞
4:      u.π = NIL
5:      B[u] = True
6:  r.key = 0
7:  Q = BUILD-QUEUE(G.V)                                  ▷ O(V)
8:  while Q ≠ ∅ do
9:      u = EXTRACT-MIN(Q)                                ▷ O(V log V)
10:     let B[u] = False
11:     for each v ∈ G.Adj[u] do
12:         if B[v] == True and w(u, v) < v.key then
13:             v.π = u
14:             DECREASE-KEY(Q, v, w(u, v))               ▷ O(E log V)
                                                          ▷ Total: O(E log V)
```

Stores if v is in T.

Initial estimate of dist to T.

u.π will be the vertex connect u to T.

u is not in T.

r is in T

All vertexes are in T

The light edge.

u is in T now.

for every neighbour of u, update estimate to T.

not in T

edge weight from u to

Time Complexity of Prim's Algorithm

The running time depends on the implementation of the priority queue Q.

- Using a binary min-heap:
  - Build-Queue: $O(V)$.
  - Extract-Min: Executed $|V|$ times, each takes $O(\log V)$. Total: $O(V \log V)$.
  - Decrease-Key: Executed $O(E)$ times, each takes $O(\log V)$. Total: $O(E \log V)$.
- Total time: $O(V \log V + E \log V) = O(E \log V)$.
- This is asymptotically the same as Kruskal's algorithm.

Implementation details:

- Map between vertices and heap elements.
- Membership check in Q takes $O(1)$ using a bit array.

✅ Correctness of Prim's Algorithm

Prior to each iteration of the `while` loop, three things are always true —

1. $A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$.

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   - $v.key < \infty$

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3. For all vertices v in Q, if $v.\pi \neq$ `NIL`, then
   - $v.key < \infty$
   - and $v.key$ is the weight of $(v, v.\pi)$ connecting v to a vertex which is in the MST,
   - and $(v, v.\pi)$ has the lightest weight among all such edges.

✅ Correctness of Prim's Algorithm

Prior to each iteration of the `while` loop, three things are always true —

1. $A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$.
2. The vertices already placed into the MST are those in $V - Q$.
3. For all vertices v in Q, if $v.\pi \neq$ `NIL`, then
   - $v.key < \infty$
   - and $v.key$ is the weight of $(v, v.\pi)$ connecting v to a vertex which is in the MST,
   - and $(v, v.\pi)$ has the lightest weight among all such edges.

💡 So each time a vertex u is dequeued, it is guaranteed that $(u.\pi, u)$ is a safe edge.

ITA 21.2 The Algorithms of Kruskal and Prim

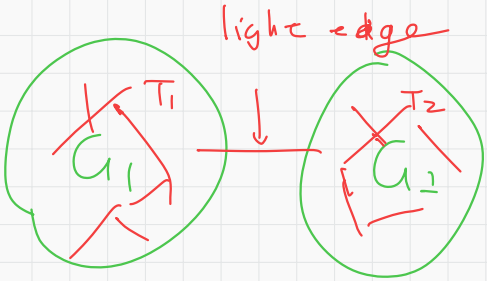
Exercise 21.2

Exercise 21.2-6

Consider a divide-and-conquer algorithm for MST

1. Partition vertex set V of graph $G = (V, E)$ into V_1 and V_2 such that $|V_1|$ and $|V_2|$ differ by at most 1.
2. Define E_1 as edges incident only on V_1, and E_2 as edges incident only on V_2.
3. Recursively compute MST for subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.
4. Select the lightest edge in E that connects V_1 and V_2 to merge two MSTs.

🍰 Do you think this algorithm is correct?



Exercise 21.2-6

- What are the MSTs for the green and orange subgraphs?
- What is the MSTs for the whole graph?

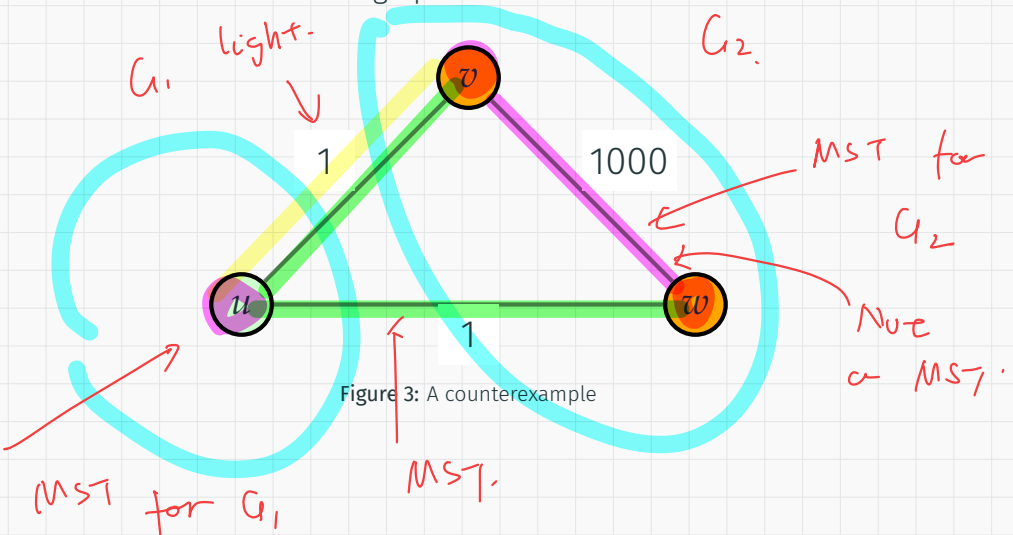


Figure 3: A counterexample

Exercise 21.2-1

Kruskal's algorithm can return different spanning trees for the same graph G when the edge weights tie. Show that for every minimum spanning tree T of G, there is a way to order the edges so Kruskal's algorithm returns T.

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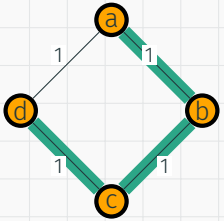
Kruskal's algorithm can return different spanning trees for the same graph G when the edge weights tie. Show that for every minimum spanning tree T of G, there is a way to order the edges so Kruskal's algorithm returns T.

Answer: Order the edges by nondecreasing weight, breaking ties so that edges of T are considered before any other edges with the same weight.



(a,b), (b,c), (c,d), (a,d)

One order to get this tree.

Any order with (a,d) in the end works

Figure 4: How to order the edges in this graph to get the highlighted MST?

ITA 23 Problems

Problem 23-1 — Uniqueness of mst

Let $G = (V, E)$ be a connected graph with distinct edge weights. Show that T, the mst of G, is unique.

Proof: Recall that —

Exercise 21.1-8

Let T and T' be the mst of some graph G. Let L and L' be *sorted* list of the weights of the edges of T and T', respectively. Then $L = L'$.

If G has two distinct mst T and T', then they must differ by at least one edge.

As the edge weights are unique, it is not possible that L and L' are the same. This is a contradiction.

Problem 23-1: Non-uniqueness of second-best mst

Let $\mathcal{T}$ be the set spanning trees of G.

The second-best MST of G is the spanning tree T' with

$$w(T') = \min_{T'' \in \mathcal{T} - \{T\}} w(T'')$$

where T is the unique MST of G.

Show that the second-best MST of G is not necessarily unique.

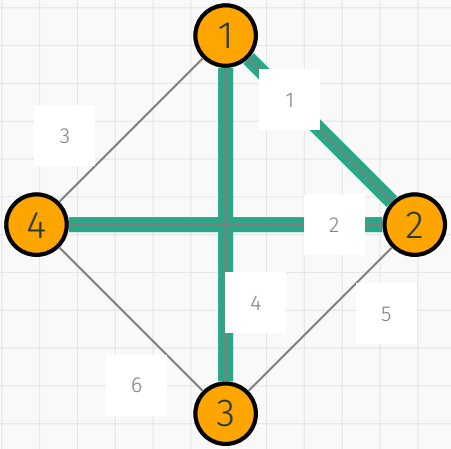


Figure 5: 🍰

Problem 23-1: Non-uniqueness of second-best MST

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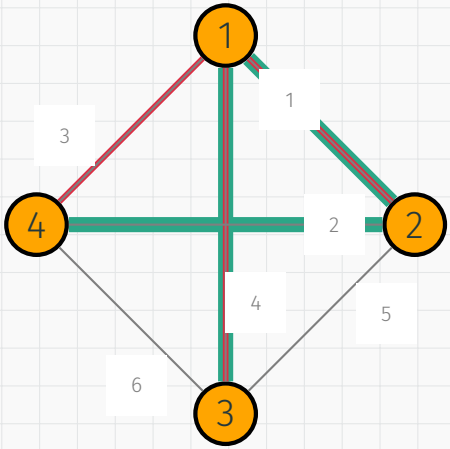

Figure 5: What is the MST of G?

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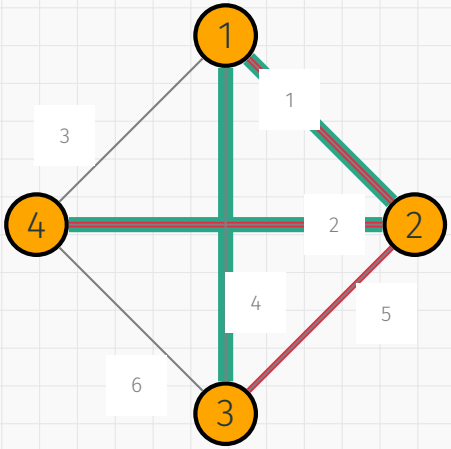

Figure 5: What is one second-best MST of G?

Problem 23-1: Non-uniqueness of second-best MST

Let $\mathcal{T}$ be the set spanning trees of G.

The second-best MST of G is the spanning tree T' with

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Show that the second-best MST of G is not necessarily unique.

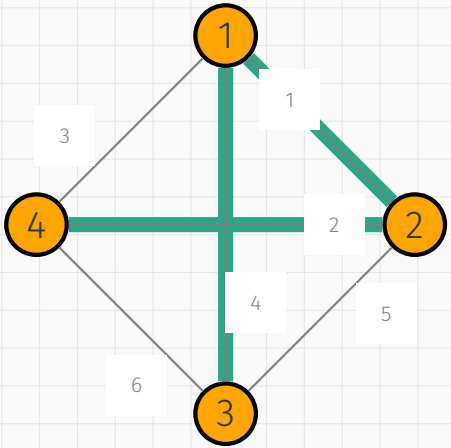

Figure 5: What is another second-best MST of G?

What are your main takeaways today? Any questions?