Lecture 11 — Elementary Graph Algorithms (Part 1)

COMPSCI 308 — Design and Analysis of Algorithms

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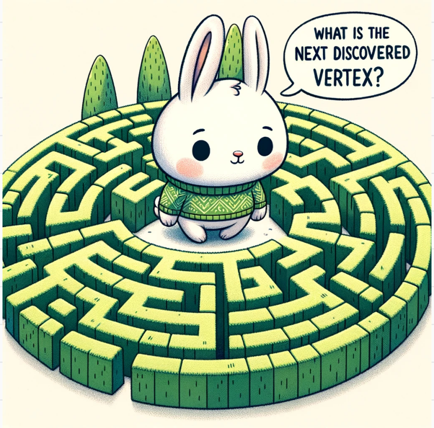
Summary

ASSIGNMENTS[1]



Practice makes perfect!

Introduction to Algorithms (ITA)

📖 Required Readings:

- Section 20.1.
- Section 20.2.

✏️ Required Exercises:

- Exercises 20.1 — 1, 3–6.
- Exercises 20.2 — 1–6.

[1] 💣 Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

ITA 20.1 Representation of Graphs

We can model friendships as a graph in which each vertex (node) represents a friend and each edge (line) represents a friendship.

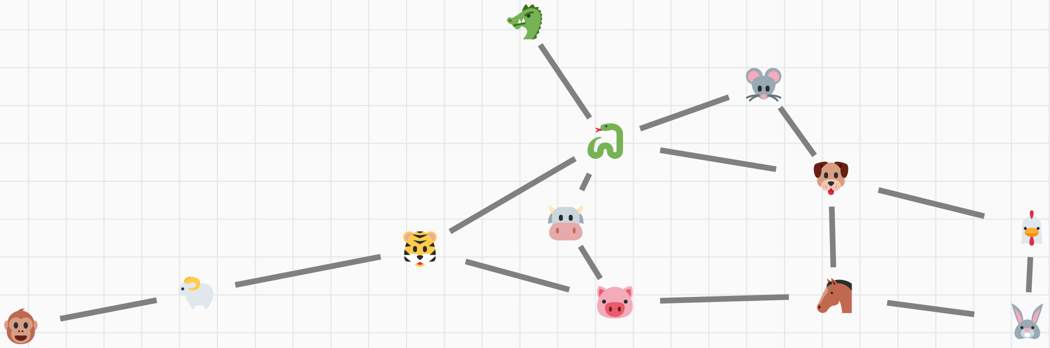


Figure 1: Friendships in COMPSCI 308

A metro network can also be modeled as a graph.

Figure 2: Mini Metro game screenshot. Source — Wikipedia.

Definition of Graph

A graph G is defined as a pair (V, E), where V is the vertex set, and E is the edge set.

An element in V is called a vertex or node.

An element in E is called an edge.

🍰 What is V and E of the following graph?

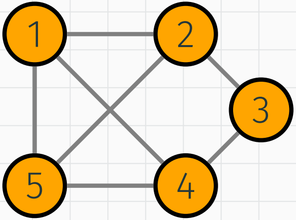


Figure 3: An undirected graph G

Definition of Graph

If $(x, y) \in E$, then x and y are adjacent, and the edge (x, y) is incident to both x and y.

If the edges are undirected, i.e., (x, y) and (y, x) are the same edge, the graph is said to be undirected.

🍰 What is the maximum number of edges a graph with $|V|$ vertices can have?

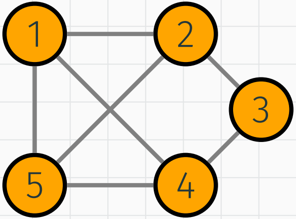


Figure 3: An undirected graph G

If the edges are directed, i.e., (x, y) and (y, x) are different edges, the graph is said to be directed or a digraph.

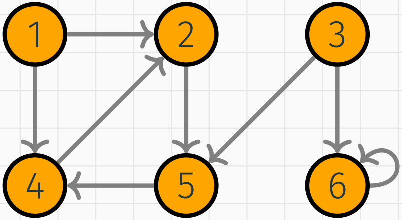


Figure 3: A directed graph G

Adjacency List Representation of a Graph

If G is sparse, i.e., $|E| = o(|V|^2)$, then we often represent G in adjacency-list form.

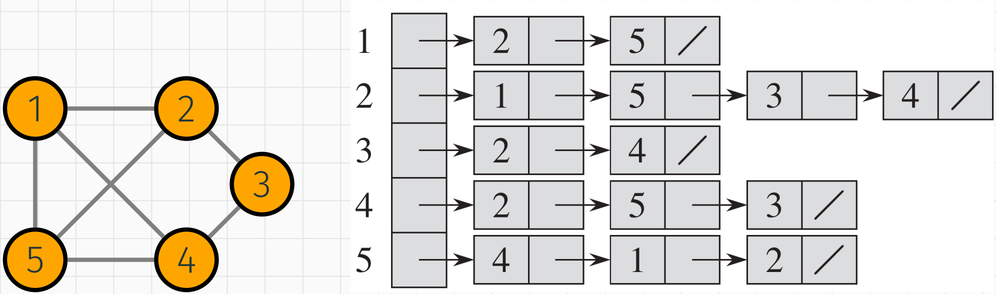
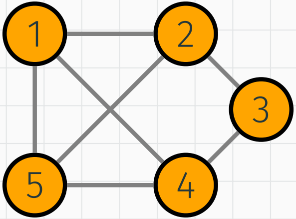


Figure 4: A graph and its adjacency-list representation

Adjacency Matrix Representation of a Graph

If G is dense, i.e., $|E| = \Theta(|V|^2)$, then we often represent G in adjacency-matrix form.



$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

Figure 5: A graph and its adjacency-matrix representation

🍰 Why is the adjacency-matrix for an undirected graph symmetric over the diagonal?

We can also represent a digraph in both adjacency-list and adjacency-matrix form.

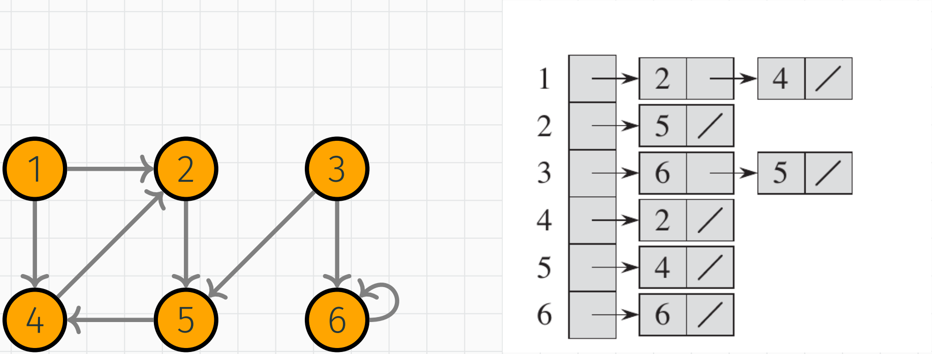


Figure 6: A digraph and its representations

Representation of Digraphs

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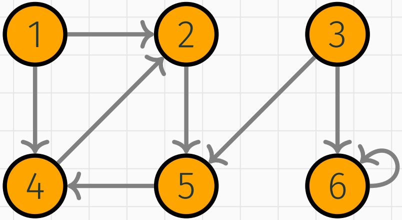

$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

Figure 6: A digraph and its representations

💡 Note that the adjacency matrix is not necessarily symmetric for a digraph.

We use the notations $u.f$ and $(u, v).f$ to indicate the attributes of a vertex or an edge.

These attributes, such as edge weight, can also be stored in adjacency-list or adjacency-matrix form.

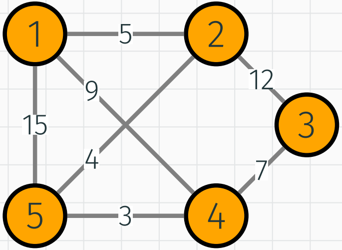

Figure 7: A weighted graph

$$\begin{bmatrix} 0 & 5 & 0 & 9 & 15 \\ 5 & 0 & 12 & 0 & 4 \\ 0 & 12 & 0 & 7 & 0 \\ 9 & 0 & 7 & 0 & 3 \\ 15 & 4 & 0 & 3 & 0 \end{bmatrix}$$

ITA 20.2 Breadth-First Search

🍰 Can you design an algorithm which moves the tiles to their correct positions using the empty space?

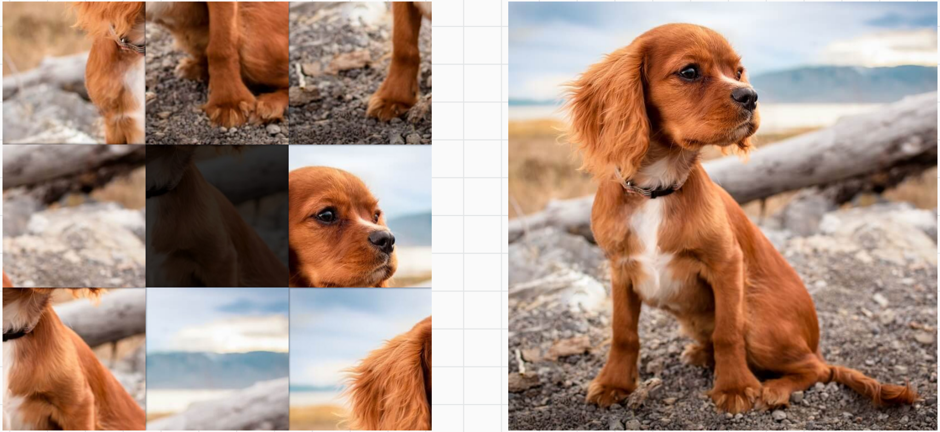

Figure 8: An 8-puzzle

Assume that a 🤧 is spreading on a graph $G = (V, E)$.

Vertex 1 is infected first on day 0.

Each day, an infected vertex v infects all its neighbours.

How many days will it take for all vertices to become 🤧?

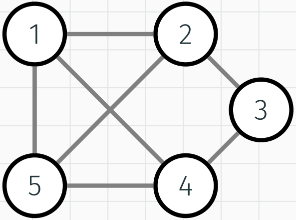

Figure 9: A graph G

Breadth First Search (BFS) essentially works like the spread of 😧.

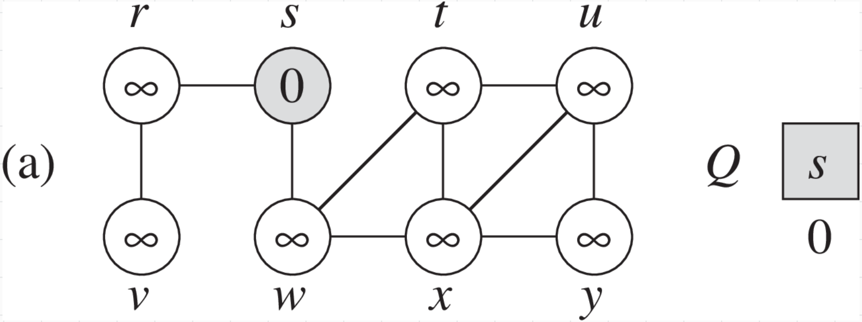

Figure 10: 📌 Run BFS on this graph

Example of BFS

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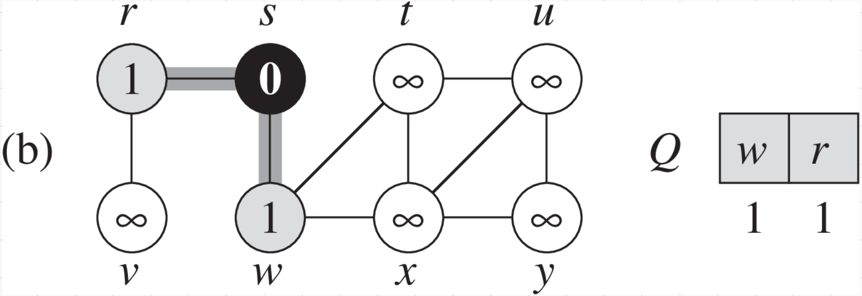

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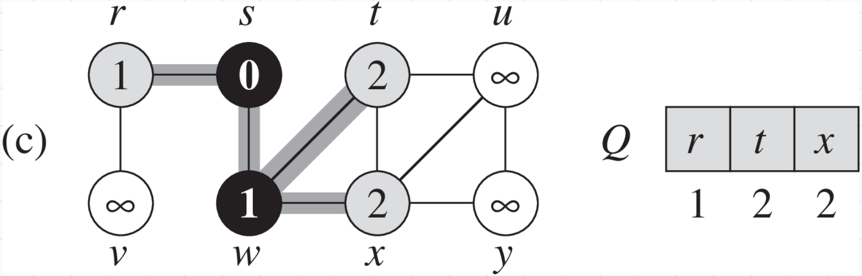

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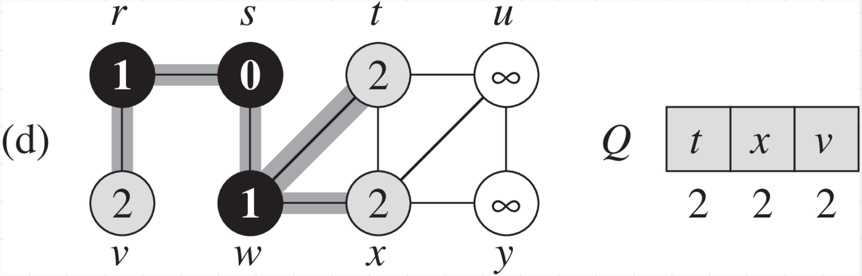

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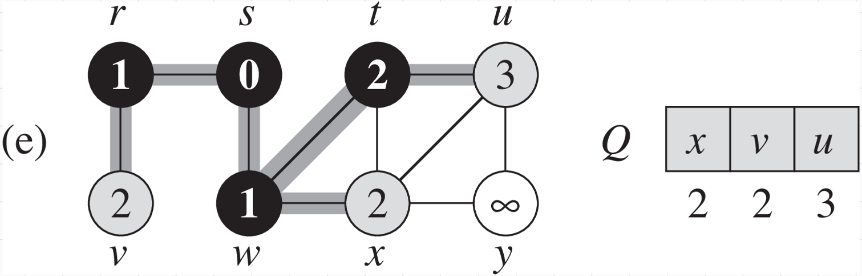

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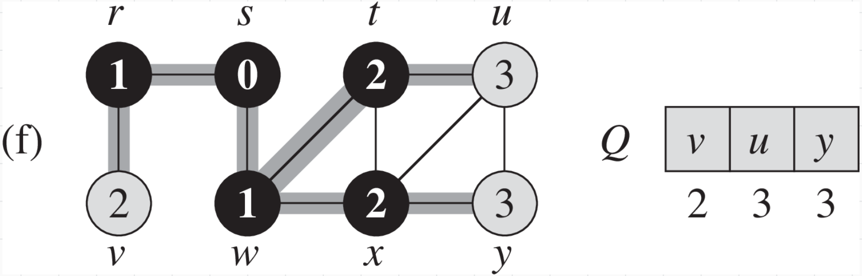

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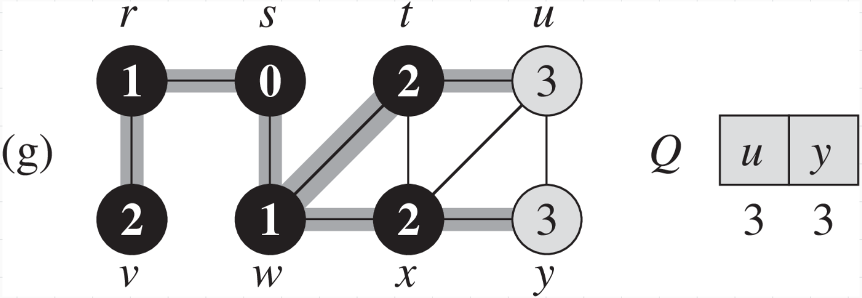

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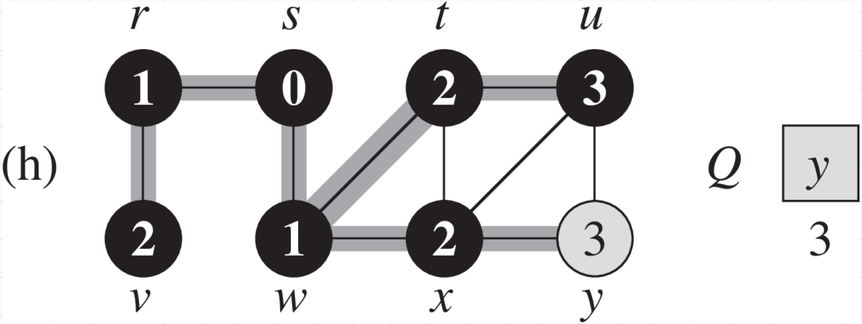

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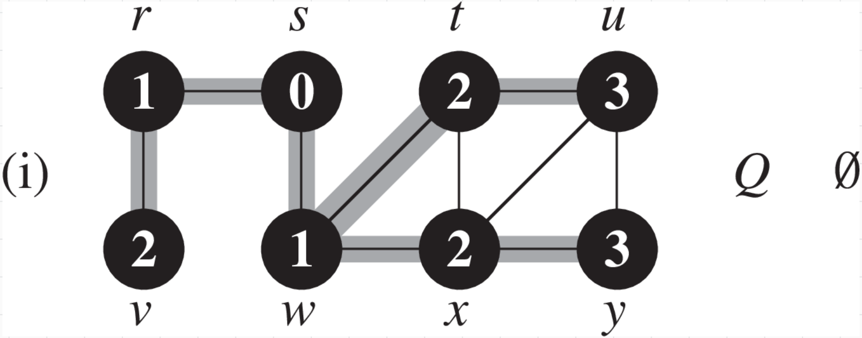

Figure 10: 📌 Run BFS on this graph

What Is BFS

BFS operates on a graph $G = (V, E)$ with a source vertex s.

The algorithm explores edges of G to find all vertices reachable from s.

BFS computes the distance from s to each vertex, constructing a breadth-first tree.

Paths in the tree represent the shortest paths in G.

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Paths in the tree represent the shortest paths in G.

Vertices are initially white, becoming grey upon discovery, and black once fully explored.

Discovery occurs upon first encounter in the search.

A grey vertex has been discovered but not fully explored.

A black vertex and all its adjacent vertices have been fully explored.

Definition — A queue is a linear data structure that follows the First-In, First-Out (FIFO) principle.

Operations —

- Enqueue — Add an element to the end of the queue.
- Dequeue — Remove an element from the front of the queue.

Real-World Analogy — A line of customers waiting their turn.

Figure 11: A line of patients waiting to see a doctor is a queue

BFS Pseudocode

BFS(G, s)

for each vertex $u \in G.V - \{s\}$ **do**

  $u.color$ = WHITE

  $u.d = \infty$

  $u.\pi$ = NIL

$s.color$ = GRAY

$s.d = 0$

$s.\pi$ = NIL

$Q = \emptyset$

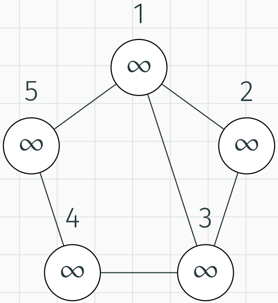
ENQUEUE(Q, s)

BFS(G, s)
for each vertex $u \in G.V - \{s\}$ **do**
    $u.color$ = WHITE
    $u.d = \infty$
    $u.\pi$ = NIL
$s.color$ = GRAY
$s.d = 0$
$s.\pi$ = NIL
$Q = \emptyset$
ENQUEUE(Q, s)
while $Q \neq \emptyset$ **do**
    u = DEQUEUE(Q)
    for each $v \in G.Adj[u]$ **do**
        if $v.color$ == WHITE **then**
            $v.color$ = GRAY
            $v.d = u.d + 1$
            $v.\pi = u$
            ENQUEUE(Q, v)
    $u.color$ = BLACK

🍰 What's the ⏱️ complexity?

🍰 Run BFS on the following graph with $s = 1$ and stops after three vertices become black. What is $4.d$ at this moment?

Assume that the neighbours of a vertex are stored in adjacency-list form, ordered by their labels in increasing order.

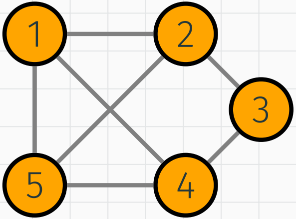


For two vertices u, v in G, the distance from u to v is the length of the shortest path from u to v.

We denote this distance by $\delta(u, v)$.

If there is no path from u to v, then $\delta(u, v) = \infty$.

🍰 What are $\delta(1, 2), \delta(1, 3), \delta(5, 3)$?



Let $G = (V, E)$ be a graph. Let $s \in V$ be an arbitrary vertex.

Then for every edge $(u, v) \in E$,

$$\delta(s, v) \le \delta(s, u) + 1.$$

Lemma 20.2

Upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$.

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Proof Sketch —

We use induction on the number of `ENQUEUE` operations.

Base Case — Initially, only the source s is enqueued with $s.d = 0 = \delta(s, s)$ and $v.d = \infty \geq \delta(s, v)$ for all other v.

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Base Case — Initially, only the source s is enqueued with $s.d = 0 = \delta(s, s)$ and $v.d = \infty \geq \delta(s, v)$ for all other v.

For the inductive step, when a white vertex v is discovered from u, we set

$$v.d = u.d + 1 \geq \delta(s, u) + 1 \geq \delta(s, v)$$

Since $v.d$ is set only once, this maintains the inductive hypothesis.

Lemma 20.3

Suppose that during the execution of BFS the queue Q contains the vertices

$$\langle v_1, v_2, \dots, v_r \rangle,$$

where v_1 is the head of Q and v_r is the tail. Then,

$$v_r.d \le v_1.d + 1$$

and

$$v_i.d \le v_{i+1}.d \qquad \text{for } i \in \{1, \dots, r-1\}.$$

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Corollary 20.4

Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \leq v_j.d$ at the time that v_j is enqueued.

Proof Outline of Lemma 20.3 (by Induction)

Base Case —

- Initially, only the source s is in Q, so the lemma holds.

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Inductive Step —

- *Dequeue Operation —*
    - If v_1 is dequeued, v_2 becomes the new head.
    - By induction, $v_1.d \le v_2.d$, and $v_r.d \le v_1.d + 1 \le v_2.d + 1$, preserving the inequalities.

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- *Dequeue Operation —*
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    - By induction, $v_1.d \leq v_2.d$, and $v_r.d \leq v_1.d + 1 \leq v_2.d + 1$, preserving the inequalities.
- *Enqueue Operation —*
    - New vertex v becomes v_{r+1} after u is processed.
    - By induction, $v_{r+1}.d = v.d = u.d + 1 \leq v_1.d + 1$.
    - This maintains $v_r.d \leq u.d + 1 = v_{r+1}.d$.

Theorem 20.5 — Correctness of BFS

During its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$.

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During its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$.

Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

💡 In short, BFS is correct.

Proof of bfs Correctness

Assume, for contradiction, that some vertex v has a d-value different from its shortest-path distance $\delta(s, v)$.

Let v be the vertex with minimum $\delta(s, v)$ that receives an incorrect d value; $v \neq s$.

Proof of bfs Correctness

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Let v be the vertex with minimum $\delta(s, v)$ that receives an incorrect d value; $v \neq s$.

By Lemma 20.2, $v.d \geq \delta(s, v)$, so

$$v.d > \delta(s, v).$$

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$$v.d > \delta(s, v).$$

Let u be the vertex immediately before v on a shortest path from s to v, meaning

$$\delta(s, v) = \delta(s, u) + 1.$$

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Let u be the vertex immediately before v on a shortest path from s to v, meaning

$$\delta(s, v) = \delta(s, u) + 1.$$

Thus, $u.d = \delta(s, u)$, so we have —

$$v.d > \delta(s, v) = \delta(s, u) + 1 = u.d + 1.$$

Proof of BFS Correctness

Thus, $u.d = \delta(s, u)$, so we have —

$$v.d > \delta(s, v) = \delta(s, u) + 1 = u.d + 1.$$

When BFS dequeues u, v is either white, gray, or black —

- If v is white, then $v.d = u.d + 1$, contradicting $v.d > u.d + 1$.
- If v is black, then $v.d \leq u.d$ by Corollary 20.4, again a contradiction.
- If v is gray, it was painted gray when dequeuing some vertex w with $w.d \leq u.d$. So

  $$v.d = w.d + 1 \leq u.d + 1,$$

  which also contradicts $v.d > u.d + 1$.

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Thus, $u.d = \delta(s,u)$, so we have —

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  which also contradicts $v.d > u.d + 1$.

Therefore, we have a contradiction and BFS is correct.

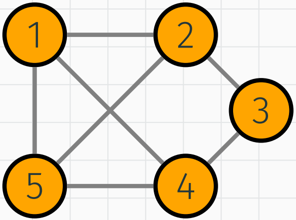
For a graph $G = (V, E)$ with source s, we define the predecessor subgraph of a BFS as $G_\pi = (V_\pi, E_\pi)$, where

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$$

and

$$E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}.$$

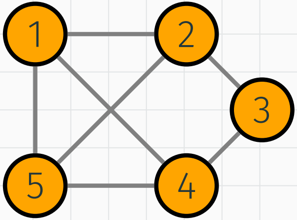
🍰 What is G_π for the following graph if $s = 1$?



A subgraph of G is called a breadth-first tree if

- it contains all vertices reachable from s,
- every path in it is also the shortest path between the two endpoints,
- it is a tree.

🍰 What is a breadth-first tree for the following graph if $s = 1$?



Breadth-First Tree

A subgraph of G is called a breadth-first tree if

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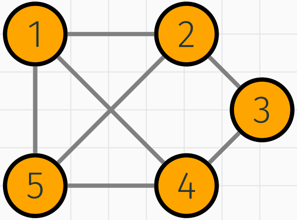
- it is a tree.

🍰 Can you think of another one?



The predecessor subgraph G_π of a graph G, as produced by BFS, is a breadth-first tree.

Proof Sketch —

- G_π is connected because it contains all vertices reachable from s, by Theorem 20.5.

Lemma 20.6

The predecessor subgraph G_π of a graph G, as produced by BFS, is a breadth-first tree.

Proof Sketch —

- G_π is connected because it contains all vertices reachable from s, by Theorem 20.5.
- The number of edges in G_π is one less than the number of vertices, as each vertex (except s) has a unique predecessor. Thus, G_π forms a tree.

🤪 Lemma 20.6

The predecessor subgraph G_π of a graph G, as produced by BFS, is a breadth-first tree.

Proof Sketch —

- G_π is connected because it contains all vertices reachable from s, by Theorem 20.5.
- The number of edges in G_π is one less than the number of vertices, as each vertex (except s) has a unique predecessor. Thus, G_π forms a tree.
- The distance from s to any vertex v in G_π is $v.d = \delta(s, v)$, by Theorem 20.5. Therefore, the unique path from s to v in G_π represents a shortest path from s to v in G.

Find the Shortest Path

Given a graph G whose breadth-first tree has been computed by BFS, we can print the shortest path from s to v using the following algorithm —

```
Print-Path(G, s, v)
if v == s then
    print s
else if v.π == NIL then
    print "no path from" s "to" v
else
    Print-Path(G, s, v.π)
    print v
```

🍰 How can we solve this maze using BFS?

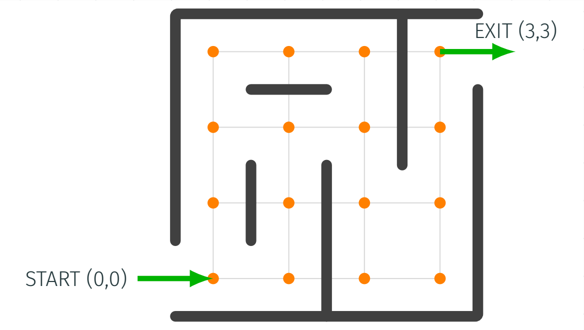


Figure 12: 4x4 Maze for BFS Visualization

Try it out at `https://brkwok.github.io/Maze-Solver/`.

🍰 How can we solve the 8-puzzle using BFS?

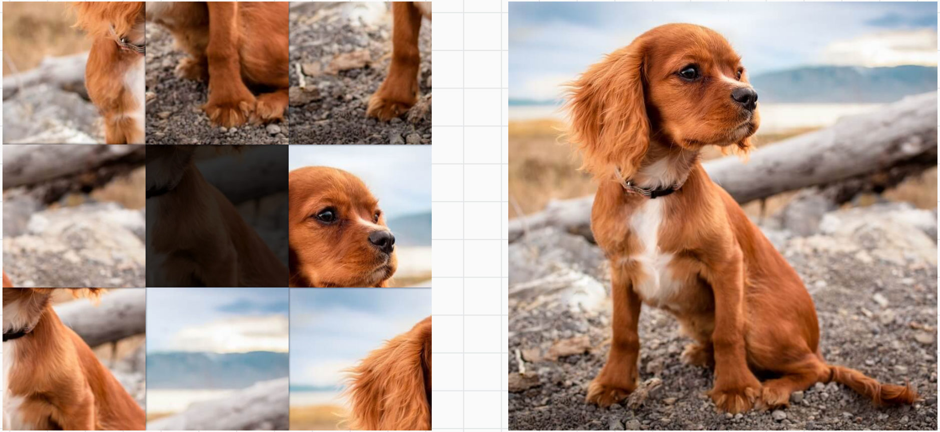


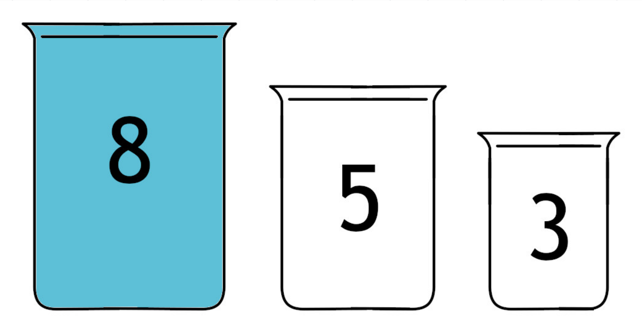
Figure 13: An 8-puzzle

Water Pouring Puzzle

We have a jug filled with 8 units of water, and two empty jugs of sizes 5 and 3.

How can we make the first and second jugs both contain 4 units, and the third is empty, without using any extra tools?

🧩 How to solve this puzzle using BFS?



What are your main takeaways today? Any questions?