Lecture 03 — Asymptotic Notations

COMPSCI 308 — Design and Analysis of Algorithms

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Summary

Random Access Machine (RAM)

Order of Growth of Running Time

Asymptotic Analysis

Assignments[1]

Practice makes perfect!

Introduction to Algorithms (ITA)

📖 Read —

- Section 3.1

✏️ Practice —

- Exercises 3.1: 1–3.

[1] 💣 Assignments will not be collected; however, quiz problems will be selected from them. (This includes both Readings and Exercises.)

Random Access Machine (ram)

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We will design and analyze algorithms on an abstract computation model Random Access Machine (ram)[2] with the following properties —

- It is an abstract 🖥.

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- It is an abstract 💻.
- Each instruction is executed sequentially on RAM.
- Each simple operation ($+, \times, -, =$, `if`) takes exactly one time unit.

😵 Didn't we say that $\times$ might take more than one time unit?

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- Each simple operation ($+, \times, -, =,$ `if`) takes exactly one time unit.
- Loops and calls of subroutines are *not* simple operations.
- Each memory access takes exactly one time unit.
- It has infinitely many memory cells.

[2] This is in Section 2.2.

What ram Can Do

The ram model contains instructions commonly found on a real 💻, and each instruction takes a constant amount of time —

- Arithmetic: ➕, ➖, ✖, ➗, remainder, floor, ceiling.
- Data movement: load, store, copy
- Control: conditional and unconditional branch, loop, subroutine call and return

But ram does *not* contain unrealistic instructions such as `sort`.

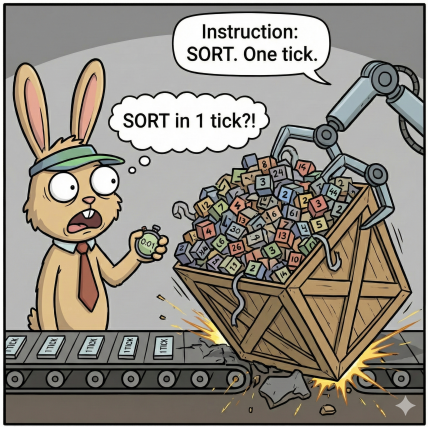

Figure 1: You cannot sort this mess instantly!

Running Time in RAM

In RAM, we count the number of instruction executions as running time, since most operations take constant time on a real 💻.

operation	example	nanoseconds
integer add	`a + b`	2.1
integer multiply	`a * b`	2.4
integer divide	`a / b`	5.4
floating-point add	`a + b`	4.6
floating-point multiply	`a * b`	4.2
floating-point divide	`a / b`	13.5
sine	`Math.sin(theta)`	91.3
arctangent	`Math.atan2(y, x)`	129.0
...	...	...

In ram, we count the number of instruction executions as running time, since most operations take constant time on a real 💻.

operation	example	nanoseconds
variable declaration	`int a`	c_1
assignment statement	`a = b`	c_2
integer compare	`a < b`	c_3
array element access	`a[i]`	c_4
array length	`a.length`	c_5
1D array allocation	`new int[N]`	$c_6 N$
2D array allocation	`new int[N][N]`	$c_7 N^2$

Figure 2: The running time of basic operations in Java on 💻

Gray Areas in the ram Model

Operations like exponentiation x^y may require multiple instructions on a real 💻.

Thus, they are *not* defined as instructions in the ram model.

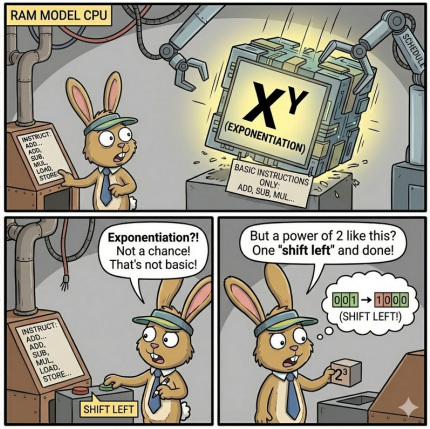

Figure 3: Is exponential an instruction?

Gray Areas in the ram Model

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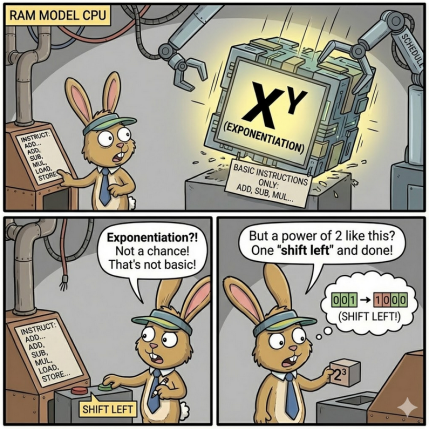
However, specific cases such as computing 2^k with a small k can be executed as a constant-time operation through a shift left (one bit) instruction.



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Gray Areas in the ram Model

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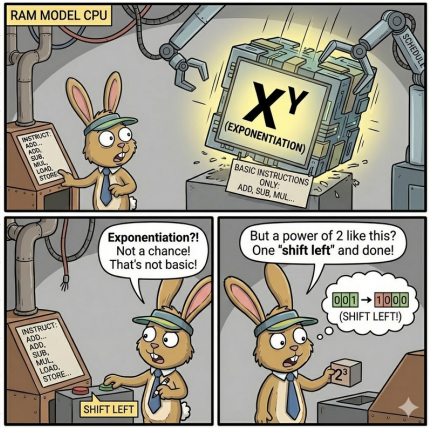
🍰 How can we compute 5^2 using *shift left*?

$$5 = 101_2 \xrightarrow{\text{shift left } ? \text{ bits}} \_\_\_\_\_ = 5^2$$



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Gray Areas in the ram Model

Operations like exponentiation x^y may require multiple instructions on a real 🖥️.

Thus, they are *not* defined as instructions in the RAM model.

💡 We will adhere to the set of instructions defined in the RAM model whenever possible and treat special cases, like 2^k, as constant-time operations.

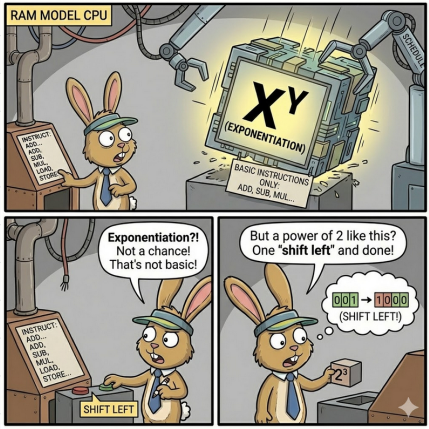

Figure 3: Is exponential an instruction?

Data Types in RAM

In the RAM model, two primary data types are used — integers and floating-point numbers.

While the focus is generally *not* on precision in floating-point arithmetic, there are scenarios where precision is critical.

Can you think of any?

What Is a Word?

In computing, a word refers to a fixed-sized piece of data that is processed as a unit by the CPU.

The size of a word varies by architecture —

- 8-bit — Zilog Z80
- 16-bit — Intel 8086
- 32-bit — Intel 80386
- 64-bit — AMD Athlon 64

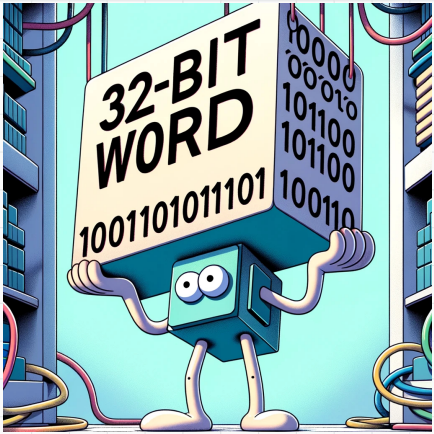

Figure 4: A 32-bit CPU

Assumptions on Word Size in ram

In ram, it is often assumed that there is a limit on the size of each word of data—

- When the input has size n, integers are typically represented by $c\lceil \lg n \rceil$ bits for some constant $c \geq 1$.

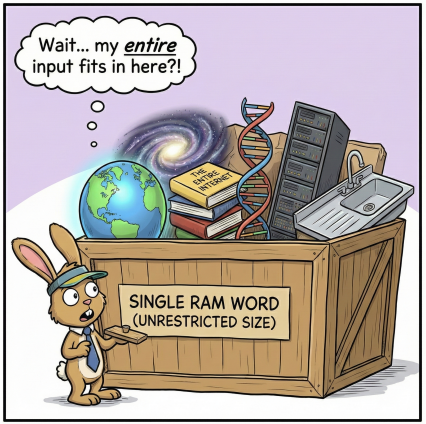

Figure 5: The universe in a single word

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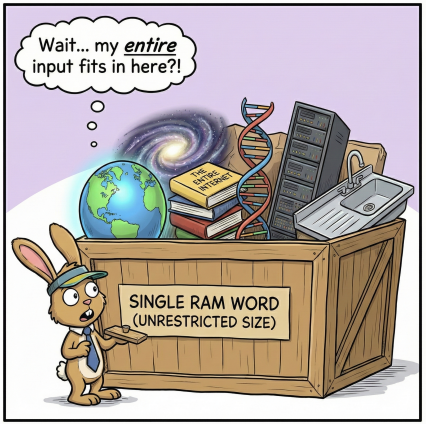
- The constant $c \geq 1$ ensures that each word can hold any integer in $\{0, 1, \ldots, n\}$.

[a] In this course, $\lg x \log_2 x$.



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- The constant $c \geq 1$ ensures that each word can hold any integer in $\{0, 1, \ldots, n\}$.
- c is restricted to be a constant to prevent the word size from growing arbitrarily.

😆 Without any restriction, we can put any data in a single word!

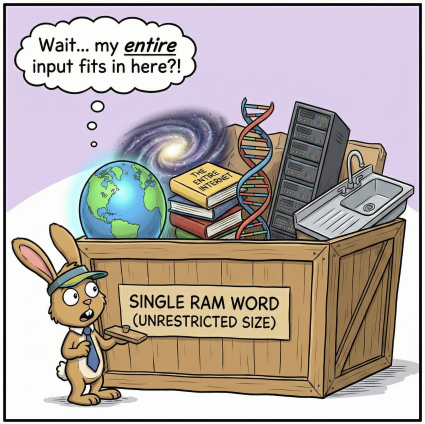


Figure 5: The universe in a single word

The Unrealistic ram Model

The ram model used in algorithm analysis is unrealistic in many ways.

For example, on a 💻:

- ✖ takes much longer than ➕.
- The compiler may unroll loops for efficiency.
- Memory access is not at all constant time, as assumed.

💬 If the ram model is unrealistic, why do we still use it in algorithm analysis?

🤪 All Models Are Wrong, Some Are Useful

Since all models are wrong, the scientist cannot obtain a "correct" one by excessive elaboration. … the ability to devise simple but evocative models is the signature of the great scientist …

— George Box

Figure 6: The flat earth model is good enough for building a house.

Order of Growth of Running Time

Order of Growth

Let $T(n)$ be the worst-case/average running time of an algorithm with input size n on RAM.

The order/rate of growth of $T(n)$ describes how fast $T(n)$ grows as n increases.

For example, if

$$T(n) = an^2 + bn + c,$$

we say its *order of growth* is

$$\Theta(n^2)$$

Typical Order of Growth

Some Functions Ordered by Growth Rate	Common Name
$\log n$	logarithmic
$\log^2 n$	polylogarithmic
$\sqrt{n}$	square root
n	linear
$n \log n$	linearithmic
n^2	quadratic
n^3	cubic
2^n	exponential

Table 1: Some Functions Ordered by Growth Rate

Why Order of Growth?

We care about the order of growth because —

- It is a simple characterization of the algorithm's efficiency.
- There are enormous differences between different orders.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Figure 7: Running Time of Algorithms with Different Orders of Growth on a Computer with 10^6 Instructions Per Second

Concerns with Precise Running Time

Why not aim for a precise running time like:

$$1.63n^2 + 3.5n + 8$$

Because —

- Achieving precise bounds can be exhausting.
- Coarser granularity reveals clearer patterns.
- Overly detailed measurement can be misleading.

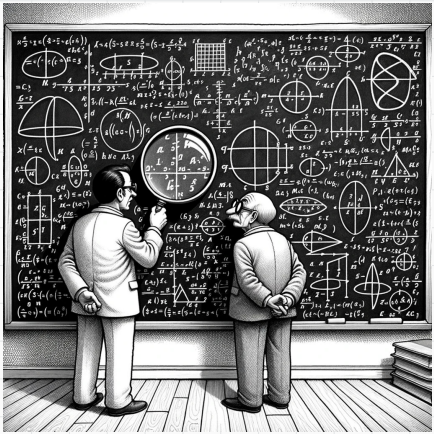


Figure 8: Too many details can be misleading

Asymptotic Analysis

In mathematics, asymptotic analysis, or asymptotics, is a method of describing the limiting behaviour of a function.

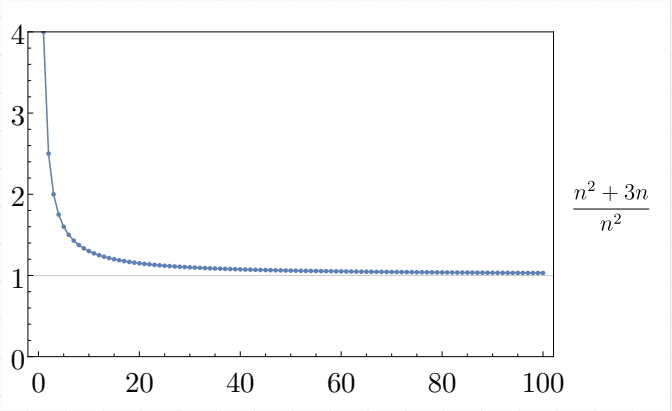


Figure 9: $(n^2 + 3)/n^2$

Asymptotic Analysis

In mathematics, asymptotic analysis, or asymptotics, is a method of describing the limiting behaviour of a function.

For example, when n is large,

$$n^2 + 3n \approx n^2$$

In other words, the term $3n$ becomes insignificant.

Asymptotic notation makes $\approx$ more precise by defining equations like

$$n^2 + 3n = \Theta(n^2).$$

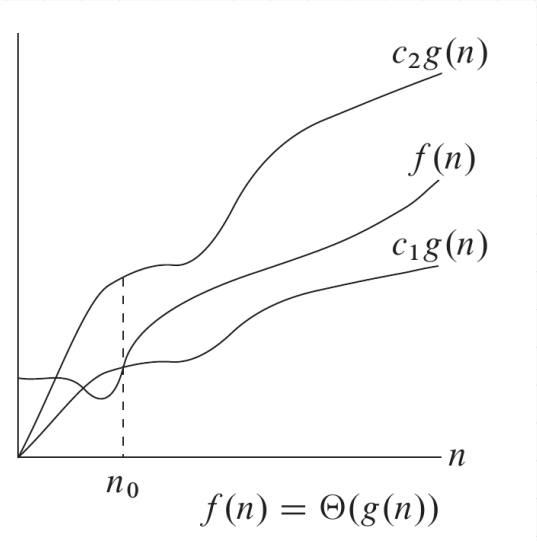
The Θ Notation

We say that $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1 and c_2 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n),$$

for all $n \geq n_0$.

💡 Normally, we pick a $g(n)$ that is a bit simpler than $f(n)$.



📌 Example of Θ Notation

🍰 How can we show that

$$\frac{1}{2}n^2 - 3n = \Theta(n^2).$$

Quadratic Functions

For $a > 0$, we have

$$an^2 + bn + c = \Theta(n^2).$$

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✏️ *Proof.*—For any n,

$$an^2 - |b|n - |c| \leq an^2 + bn + c \leq an^2 + |b|n + |c|.$$

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Let

$$n_0 = 2 \max\left((|b|)/a, \sqrt{(|c|)/a}\right).$$

If $n \geq n_0$, then $n \geq 2|b|/a$ and $n \geq 2\sqrt{|c|/a}$, so

$$|b|n \leq \frac{a}{2}n^2 \qquad \text{and} \qquad |c| \leq \frac{a}{4}n^2.$$

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Thus,

$$\frac{a}{4}n^2 \leq an^2 + bn + c \leq \frac{7a}{4}n^2, \qquad \text{for all } n \geq n_0.$$

Given that

$$p(n) = \sum_{i=0}^{d} a_i n^i,$$

where $a_d > 0$, we have

$$p(n) = \Theta(n^d).$$

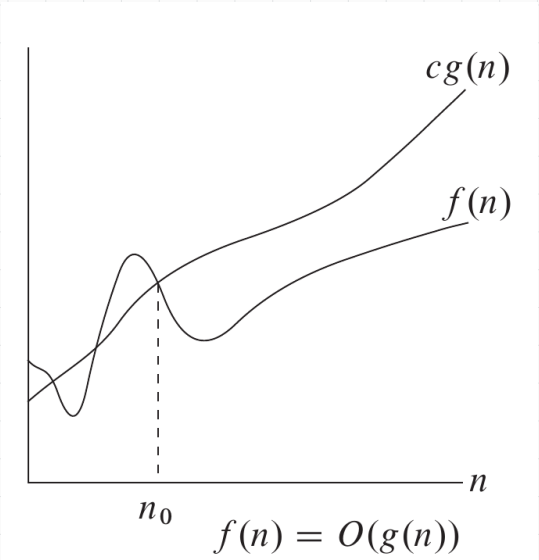
The O Notation

We say that $f(n) = O(g(n))$ if there exist positive constants n_0, c such that

$$0 \leq f(n) \leq cg(n),$$

for all $n \geq n_0$.

💡 This is to say that $f(n)$ increases at most as fast as $g(n)$.



📌 Example of O Notation

🍰 How can we show that

$$an^2 + bn + c = O(n^2),$$

if $a > 0$?

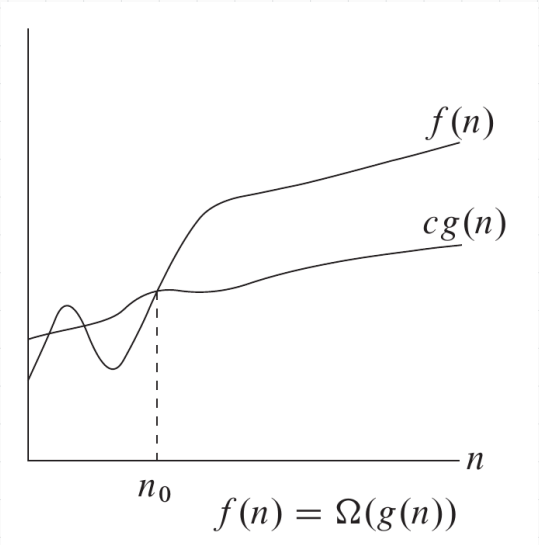
The Ω Notation

We say that $f(n) = \Omega(g(n))$ if there exist positive constants n_0, c such that

$$f(n) \geq cg(n),$$

for all $n \geq n_0$.

💡 This is to say that $f(n)$ increases at least as fast as $g(n)$.



🍰 Which of the following are correct?

1. $\log(n) + \log(n)^2 = O(n^2)$
2. $n \log(n) = O(n^2)$
3. $\log\left(n^k\right) = O(\log(n))$ for any constant $k > 0$
4. $2^n = O(n!)$
5. $\sqrt{n} + n = O(n \log n)$
6. $n^{0.5} = O(\log(n))$

📌 Example of Ω Notation

🍰 How can we show that

$$an^2 + bn + c = \Omega(n^2),$$

if $a > 0$?

Theorem 3.1

For any two functions $f(n)$ and $g(n)$, we have

$$f(n) = \Theta(g(n))$$

if and only if

$$f(n) = O(g(n)) \qquad \text{and} \qquad f(n) = \Omega(g(n)).$$

What are your main takeaways today? Any questions?