Problem 1.

$$\phi_0 = \frac{1}{(1,1)} = \frac{1}{\sqrt{\int_{-1}^{1}\sqrt{1-x^2}\,dx}} = \frac{1}{\sqrt{\int_0^{\pi}\sqrt{1-\cos^2\theta}\,\sin\theta\,d\theta}} = \frac{1}{\sqrt{\int_0^{\pi}\sin^2\theta\,d\theta}} = \frac{1}{\sqrt{\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta}} = \sqrt{\frac{2}{\pi}}$$

$$(x,\phi_0)\phi_0 = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\cdot x\,dx = \frac{2}{\pi}\left(-\frac{1}{2}\right)\int_{-1}^{1}\sqrt{1-x^2}\,(-2x)\,dx = \frac{2}{\pi}\left(-\frac{1}{3}\right)\left[(1-x^2)^{\frac{3}{2}}\right]_{x=-1}^{x=1} = 0$$

$$\|x-(x,\phi_0)\phi_0\|_2^2 = \int_{-1}^{1}\sqrt{1-x^2}\,x^2\,dx = \int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int_0^{\pi}\sin^2 2\theta\,d\theta = \frac{1}{4}\int_0^{\pi}\frac{1-\cos 4\theta}{2}\,d\theta = \frac{\pi}{8}$$

$$\phi_1 = \frac{x-(x,\phi_0)\phi_0}{\|x-(x,\phi_0)\phi_0\|_2} = \sqrt{\frac{8}{\pi}}\,x = \sqrt{\frac{2}{\pi}}\,2x$$

$$(x^2,\phi_0)\phi_0 = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,x^2\,dx = \frac{1}{4}$$

$$(x^2,\phi_1)\phi_1 = \frac{8}{\pi}x\int_{-1}^{1}\sqrt{1-x^2}\,x^3\,dx = \frac{8}{\pi}x\int_0^{\pi}\sin^2\theta\cos^3\theta\,d\theta = \frac{8}{\pi}x\int_0^{\pi}\frac{1}{4}\sin^2 2\theta\cos\theta\,d\theta$$

$$= \frac{1}{\pi}x\int_0^{\pi}(1-\cos 4\theta)\cos\theta\,d\theta = \frac{1}{\pi}x\int_0^{\pi}\left[\cos\theta - \frac{\cos(5\theta)+\cos(3\theta)}{2}\right]d\theta = 0$$

$$\|x^2-(x^2,\phi_0)\phi_0-(x^2,\phi_1)\phi_1\|_2^2 = \int_{-1}^{1}\sqrt{1-x^2}\left(x^2-\frac{1}{4}\right)^2dx = \int_0^{\pi}\sin^2\theta\left(\cos^2\theta-\frac{1}{4}\right)^2d\theta$$

$$= \int_0^{\pi}\left[\sin^2\theta\cos^4\theta - \frac{1}{2}\sin^2\theta\cos^2\theta + \frac{1}{16}\sin^2\theta\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{4}(1-\cos^2 2\theta)\cos^2\theta - \frac{1}{8}\sin^2 2\theta + \frac{1}{16}\sin^2\theta\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{1+\cos 2\theta}{8} - \frac{(\cos\theta+\cos 3\theta)^2}{16} - \frac{1}{8}\,\frac{1-\cos 4\theta}{2} + \frac{1}{16}\,\frac{1-\cos 2\theta}{2}\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{3}{32} - \frac{1}{16}\,\frac{1+\cos 2\theta}{2} - \frac{1}{8}\,\frac{\cos 2\theta+\cos 4\theta}{2} - \frac{1}{16}\,\frac{1+\cos 6\theta}{2}\right]d\theta = \frac{\pi}{32}$$

$$\phi_2 = \frac{x^2-(x^2,\phi_0)\phi_0-(x^2,\phi_1)\phi_1}{\|x^2-(x^2,\phi_0)\phi_0-(x^2,\phi_1)\phi_1\|_2} = \sqrt{\frac{32}{\pi}}\left(x^2-\frac{1}{4}\right) = \sqrt{\frac{2}{\pi}}\,(4x^2-1)$$

$$\boxed{\text{So. } \psi_0 = \sqrt{\frac{2}{\pi}},\quad \psi_1 = \sqrt{\frac{2}{\pi}}\,(2x),\quad \psi_2 = \sqrt{\frac{2}{\pi}}\,(4x^2-1)}$$

Problem 2.

$\vec{s} = \vec{_3h}^T B A^{-1}$, s.t. $I \approx I_h = \vec{s}\,\vec{y}$

where $\vec{_3h}^T = [3h \;\; 9h^2 \;\; 27h^3]$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & h & h^2 \\ 1 & 2h & 4h^2 \end{bmatrix}^{-1} = \frac{1}{2h^3}\begin{bmatrix} 2h^3 & 0 & 0 \\ -3h^2 & 4h^2 & -h^2 \\ h & -2h & h \end{bmatrix}$

$$\boxed{\text{So } I_h = \vec{s}\,\vec{y} = [3h \;\; 9h^2 \;\; 27h^3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2h} & \frac{2}{h} & -\frac{1}{2h} \\ \frac{1}{2h^2} & -\frac{1}{h^2} & \frac{1}{2h^2} \end{bmatrix}\begin{bmatrix} f(0) \\ f(h) \\ f(2h) \end{bmatrix} = \frac{3}{4}h\left[f(0) + 3f(2h)\right]}$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^5(\xi)}{120}x^5$$

$$I = \int_0^{3h} f(x)\,dx = f(0)3h + f'(0)\frac{9h^2}{2} + \frac{f''(0)}{2}\frac{27h^3}{3} + \frac{f^{(3)}(0)}{6}\frac{81h^4}{4} + \frac{f^{(4)}(\xi)}{24}\frac{243h^5}{5}$$

where ξ is some num. $[0, 3h]$

$$I_h = \frac{3}{4}h[f(0)+3f(2h)] = \frac{3}{4}f(0)h + \frac{9}{4}f(0)h + \frac{9}{4}f'(0)2h^2 + \frac{9}{4}\frac{f''(0)}{2}4h^3 + \frac{9}{4}\frac{f^{(3)}(0)}{6}8h^4 + \frac{9}{4}\frac{f^{(4)}(\eta)}{24}16h^5$$

where η is some num. $[0, 2h]$

$$I - I_h = \cancel{f(0)3h} + \cancel{f'(0)\frac{9h^2}{2}} + \cancel{\frac{f''(0)}{2}\frac{27h^3}{3}} + \frac{f^{(3)}(0)}{6}\frac{81h^4}{4} + \frac{f^{(4)}(\xi)}{24}\frac{243h^5}{5}$$

$$- \left[\cancel{\frac{3}{4}f(0)h} + \cancel{\frac{9}{4}f(0)h} + \cancel{\frac{9}{4}f'(0)2h^2} + \cancel{\frac{9}{4}\frac{f''(0)}{2}4h^3} + \frac{9}{4}\frac{f^{(3)}(0)}{6}8h^4 + \frac{9}{4}\frac{f^{(4)}(\eta)}{24}16h^5\right]$$

$$= \frac{3}{8}h^4 f^{(3)}(0) + \left(\frac{f^{(4)}(\xi)}{24}\frac{243}{5} - \frac{f^{(4)}(\eta)}{24}\frac{144}{4}\right)h^5$$

Assume $|f^{(4)}(\xi)| \le M$ and $|f^{(4)}(\eta)| \le N$. for some constant M, N, on interval

$$\left|\left(\frac{f^{(4)}(\xi)}{24}\frac{243}{5} - \frac{f^{(4)}(\eta)}{24}\frac{144}{4}\right)h^5\right| \le \left|\frac{M}{24}\frac{243}{5} - \frac{N}{24}\frac{144}{4}\right| |h^5| \quad \text{for arbitrary } h$$

so $\left(\frac{f^{(4)}(\xi)}{24}\frac{243}{5} - \frac{f^{(4)}(\eta)}{24}\frac{144}{4}\right)h^5$ is said to be $O(h^5)$

$$\boxed{\text{So } I - I_h = \frac{3}{8}h^4 f^{(3)}(0) + O(h^5) \text{. Q.E.D.}}$$

Problem 3

$$I(f) = \int_a^b f(x)\,dx \simeq I_M(f) := (b-a)\,f\left(\frac{a+b}{2}\right)$$

for $x \in [a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b]$

$$f(x) = f\left(\frac{a+b}{2}\right) + \left(x - \frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right) + \left(x - \frac{a+b}{2}\right)^2\frac{f''(\xi)}{2} \quad \text{some } \xi \in [a,b]$$

$$ME_M(f) = I(f) - I_M(f) = \int_a^b \left[f(x) - f\left(\frac{a+b}{2}\right)\right]dx$$

$$= \int_a^b \left[\left(x - \frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right) + \left(x - \frac{a+b}{2}\right)^2\frac{f''(\xi)}{2}\right]dx$$

Assume $\max_{x\in(a,b)} |f''(x)| \le K$

$$ME_M(f) \le f'\left(\frac{a+b}{2}\right)\left[\frac{x^2}{2} - \left(\frac{a+b}{2}\right)x\right]_{x=a}^{x=b} + \frac{K}{2}\left[\frac{(x-\frac{a+b}{2})^3}{3}\right]_{x=a}^{x=b}$$

$$= f'\left(\frac{a+b}{2}\right)\left[\cancel{\frac{b^2}{2}} - \cancel{\frac{ab}{2}} - \cancel{\frac{b^2}{2}} - \cancel{\frac{a^2}{2}} + \cancel{\frac{a^2}{2}} + \cancel{\frac{ab}{2}}\right] + \frac{K}{6}\left[\frac{(b-a)^3}{8} + \frac{(b-a)^3}{8}\right]$$

So $\boxed{ME_M(f) = \frac{K}{24}(b-a)^3 = \frac{(b-a)^3}{24}f''(\eta)}$ for some η (where $f''(x)$ max)

Problem 4

Let. $f_{xy}(x, y) = a f(x+\Delta x, y+\Delta y) + b f(x+\Delta x, y-\Delta y) + c f(x-\Delta x, y+\Delta y) + d f(x-\Delta x, y-\Delta y)$

when $f(x,y) = 1,\ x,\ y,\ x^2,\ y^2,\ xy$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x+\Delta x & x+\Delta x & x-\Delta x & x-\Delta x \\ y+\Delta y & y-\Delta y & y+\Delta y & y-\Delta y \\ (x+\Delta x)^2 & (x+\Delta x)^2 & (x-\Delta x)^2 & (x-\Delta x)^2 \\ (y+\Delta y)^2 & (y-\Delta y)^2 & (y+\Delta y)^2 & (y-\Delta y)^2 \\ (x+\Delta x)(y+\Delta y) & (x+\Delta x)(y-\Delta y) & (x-\Delta x)(y+\Delta y) & (x-\Delta x)(y-\Delta y) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} a + b + c + d = 0 & ① \\ (x+\Delta x)a + (x+\Delta x)b + (x-\Delta x)c + (x-\Delta x)d = 0 & ② \\ (y+\Delta y)a + (y-\Delta y)b + (y+\Delta y)c + (y-\Delta y)d = 0 & ③ \\ (x+\Delta x)^2 a + (x+\Delta x)^2 b + (x-\Delta x)^2 c + (x-\Delta x)^2 d = 0 & ④ \\ (y+\Delta y)^2 a + (y-\Delta y)^2 b + (y+\Delta y)^2 c + (y-\Delta y)^2 d = 0 & ⑤ \\ (x+\Delta x)(y+\Delta y)a + (x+\Delta x)(y-\Delta y)b + (x-\Delta x)(y+\Delta y)c + (x-\Delta x)(y-\Delta y)d = 1 & ⑥ \end{cases}$$

$(x+\Delta x)$② − ④ : $2\Delta x(x-\Delta x)(c+d) = 0$ ⑦

④ − $(x-\Delta x)$② : $2\Delta x(x+\Delta x)(a+b) = 0$ ⑧

① : $a+b = -(c+d)$ ⑨

⑦ ⑧ ⑨ & $x-\Delta x$ and $x+\Delta x$ not 0 same time

$\Rightarrow a+b = c+d = 0$ ⑩

$(y+\Delta y)$③ − ⑤ : $2\Delta y(y-\Delta y)(b+d) = 0$ ⑪

⑤ − $(y-\Delta y)$③ : $2\Delta y(y+\Delta y)(a+c) = 0$ ⑫

① : $b+d = -(a+c)$ ⑬

⑪ ⑫ ⑬ & $y-\Delta y$ and $y+\Delta y$ not 0 sametime

$\Rightarrow a+c = b+d = 0$ ⑭

⑩ ⑭ $\Rightarrow a = -b = -c = d$ ⑮

⑥ ⑮ : $(\cancel{xy} + \cancel{\Delta xy} + \cancel{x\Delta y} + \Delta x\Delta y)a - (\cancel{xy} + \cancel{\Delta xy} - \cancel{x\Delta y} - \Delta x\Delta y)a$
$- (\cancel{xy} - \cancel{\Delta xy} + \cancel{x\Delta y} - \Delta x\Delta y)a + (\cancel{xy} - \cancel{\Delta xy} - \cancel{x\Delta y} + \Delta x\Delta y)a = 1$

$\Rightarrow \boxed{a = \frac{1}{4\Delta x\Delta y},\ b = -\frac{1}{4\Delta x\Delta y},\ c = -\frac{1}{4\Delta x\Delta y},\ d = \frac{1}{4\Delta x\Delta y}}$

Problem 5

```python
import numpy as np
import scipy.special as special
import matplotlib.pyplot as plt
import pandas as pd
```
Executed at 2024.05.02 01:43:18 in 338ms

```python
# Function for the trapezoidal rule
def trapezoidal_rule(f, a, b, n):
    h = (b - a) / n
    x = np.linspace(a, b, n+1)
    y = f(x)
    return h * (0.5 * y[0] + np.sum(y[1:-1]) + 0.5 * y[-1])

# Function definitions for the integrals
def f1(x):
    return np.exp(-x**2)

def f2(x):
    return np.exp(x) * np.cos(4 * x)

# Exact solutions
I_exact_1 = np.sqrt(np.pi) / 2 * special.erf(1)
I_exact_2 = (1/17) * (-1 + np.exp(np.pi))
# Evaluate the integrals for n = 4, 8, 16
n_values = [4, 8, 16]
results_1 = []
results_2 = []
errors_1 = []
errors_2 = []

for n in n_values:
    I_n_1 = trapezoidal_rule(f1, 0, 1, n)
    I_n_2 = trapezoidal_rule(f2, 0, np.pi, n)
    results_1.append(I_n_1)
    results_2.append(I_n_2)
    errors_1.append(abs(I_n_1 - I_exact_1))
    errors_2.append(abs(I_n_2 - I_exact_2))

# Create a DataFrame for each integral
df1 = pd.DataFrame({
    'n': n_values,
    'I_n for 1': results_1,
    'I for 1': [I_exact_1] * len(n_values),
    'Error': errors_1
})

df2 = pd.DataFrame({
    'n': n_values,
    'I_n for 2': results_2,
    'I for 2': [I_exact_2] * len(n_values),
    'Error': errors_2
})

# Display the DataFrames
print("Results for the integral of e^-x^2 from 0 to 1")
display(df1)
print("Results for the integral of e^x cos(4x) from 0 to π")
display(df2)
```
Executed at 2024.05.02 01:43:18 in 9ms

Results for the integral of e^-x^2 from 0 to 1

3 rows × 4 columns

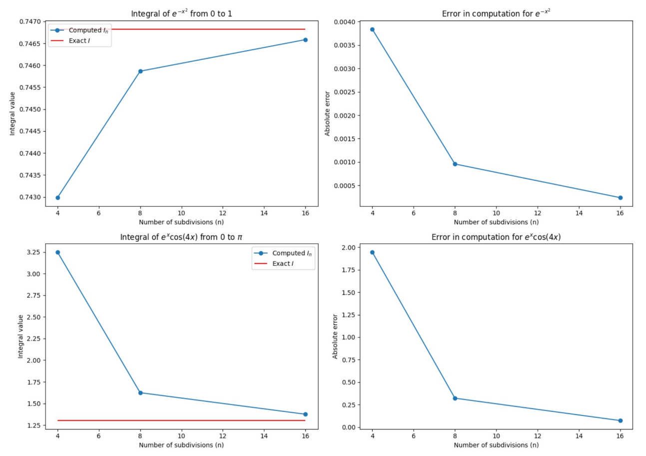
	n	I_n for 1	I for 1	Error
0	4	0.742984	0.746824	0.003840
1	8	0.745866	0.746824	0.000959
2	16	0.746585	0.746824	0.000240

3 rows × 4 columns

	n	I_n for 2	I for 2	Error
0	4	3.249050	1.302394	1.946657
1	8	1.624525	1.302394	0.322132
2	16	1.375723	1.302394	0.073329

```python
# Plotting results
```

```python
# Plotting results
fig, ax = plt.subplots(2, 2, figsize=(14, 10))

ax[0, 0].plot(n_values, results_1, 'o-', label='Computed $I_n$')
ax[0, 0].hlines(I_exact_1, n_values[0], n_values[-1], colors='r', label='Exact $I$')
ax[0, 0].set_title('Integral of $e^{-x^2}$ from 0 to 1')
ax[0, 0].set_xlabel('Number of subdivisions (n)')
ax[0, 0].set_ylabel('Integral value')
ax[0, 0].legend()

ax[0, 1].plot(n_values, errors_1, 'o-')
ax[0, 1].set_title('Error in computation for $e^{-x^2}$')
ax[0, 1].set_xlabel('Number of subdivisions (n)')
ax[0, 1].set_ylabel('Absolute error')

ax[1, 0].plot(n_values, results_2, 'o-', label='Computed $I_n$')
ax[1, 0].hlines(I_exact_2, n_values[0], n_values[-1], colors='r', label='Exact $I$')
ax[1, 0].set_title('Integral of $e^x \cos(4x)$ from 0 to $\pi$')
ax[1, 0].set_xlabel('Number of subdivisions (n)')
ax[1, 0].set_ylabel('Integral value')
ax[1, 0].legend()

ax[1, 1].plot(n_values, errors_2, 'o-')
ax[1, 1].set_title('Error in computation for $e^x \cos(4x)$')
ax[1, 1].set_xlabel('Number of subdivisions (n)')
ax[1, 1].set_ylabel('Absolute error')

plt.tight_layout()
plt.show()
```
Executed at 2024.05.02 01:43:19 in 435ms




```python
# Calculate the rate of convergence based on the textbook method
R_n1 = (results_1[1] - results_1[0]) / (results_1[2] - results_1[1])
R_n2 = (results_2[1] - results_2[0]) / (results_2[2] - results_2[1])

# Assuming the rate is calculated as the logarithm of the ratio
p1 = np.log2(np.abs(R_n1))
p2 = np.log2(np.abs(R_n2))

print("Rate of convergence for f1:", p1)
print("Rate of convergence for f2:", p2)
```
Executed at 2024.05.02 01:43:19 in 2ms

```
Rate of convergence for f1: 2.002801111401687
Rate of convergence for f2: 2.7069439468761702
```

```python
# Define the functions from Problem 5
```

In class we have $|I - I_n| \le O\left(\frac{b-a}{n^3}\right)$

i.e. $I - I_n = \frac{c}{n^p} \le \frac{c}{n^3}$, $p > 0$

To verify this, from text book

$$R_n = \frac{I_{2n} - I_n}{I_{4n} - I_{2n}} = 2^p$$

From above, for

1. $p = 2.0028$
2. $p = 2.7069$

each of them $\in (2,3]$. Verified

Problem 6

```python
# Define the functions from Problem 5
def f1(x):
    return np.exp(-x**2)

def f2(x):
    return np.exp(x) * np.cos(4*x)

# Gauss-Legendre Quadrature
def gauss_legendre_quadrature(f, a, b, n):
    [x, w] = np.polynomial.legendre.leggauss(n+1)
    # Transform the x values from [-1, 1] to [a, b]
    t = 0.5 * (x + 1) * (b - a) + a
    return np.sum(w * f(t)) * 0.5 * (b - a)

# Exact values from Problem 5
I_exact_1 = np.sqrt(np.pi) / 2 * special.erf(1)
I_exact_2 = 1/17 * (-1 + np.exp(np.pi))

# Evaluate the integrals for n = 4, 8, 16
n_values = [4, 8, 16]
results_1 = []
results_2 = []
errors_1 = []
errors_2 = []

for n in n_values:
    # Apply Gauss-Legendre quadrature
    I_gl_1 = gauss_legendre_quadrature(f1, 0, 1, n)
    I_gl_2 = gauss_legendre_quadrature(f2, 0, np.pi, n)
    results_1.append(I_gl_1)
    results_2.append(I_gl_2)
    errors_1.append(abs(I_gl_1 - I_exact_1))
    errors_2.append(abs(I_gl_2 - I_exact_2))

# Create a DataFrame for each integral
df1 = pd.DataFrame({
    'n': n_values,
    'I_n for 1': results_1,
    'I for 1': [I_exact_1] * len(n_values),
    'Error': errors_1
})

df2 = pd.DataFrame({
    'n': n_values,
    'I_n for 2': results_2,
    'I for 2': [I_exact_2] * len(n_values),
    'Error': errors_2
})

# Display the DataFrames
print("Results for the integral of e^-x^2 from 0 to 1")
display(df1)
print("Results for the integral of e^x cos(4x) from 0 to π")
display(df2)
```

Executed at 2024.05.02 01:43:19 in 8ms

Results for the integral of e^-x^2 from 0 to 1

3 rows × 4 columns

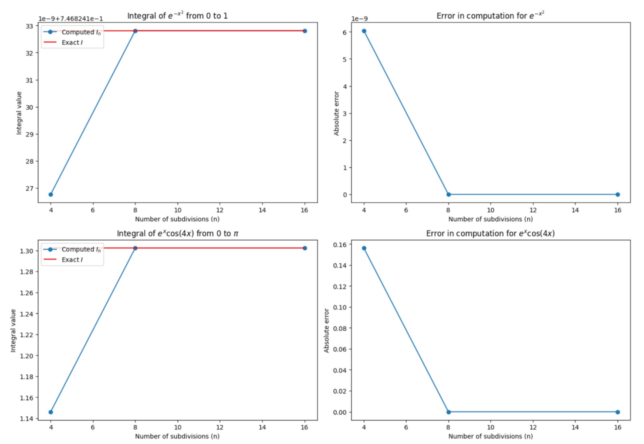
	n	I_n for 1	I for 1	Error
0	4	0.746824	0.746824	6.046179e-09
1	8	0.746824	0.746824	0.000000e+00
2	16	0.746824	0.746824	1.110223e-16

3 rows × 4 columns

	n	I_n for 2	I for 2	Error
0	4	1.146206	1.302394	1.561873e-01
1	8	1.302392	1.302394	1.831402e-06
2	16	1.302394	1.302394	2.176037e-14

```python
# Plotting results
fig, ax = plt.subplots(2, 2, figsize=(14, 10))

ax[0, 0].plot(n_values, results_1, 'o-', label='Computed $I_n$')
ax[0, 0].hlines(I_exact_1, n_values[0], n_values[-1], colors='r', label='Exact $I$')
ax[0, 0].set_title('Integral of $e^{-x^2}$ from 0 to 1')
ax[0, 0].set_xlabel('Number of subdivisions (n)')
ax[0, 0].set_ylabel('Integral value')
ax[0, 0].legend()

ax[0, 1].plot(n_values, errors_1, 'o-')
ax[0, 1].set_title('Error in computation for $e^{-x^2}$')
ax[0, 1].set_xlabel('Number of subdivisions (n)')
ax[0, 1].set_ylabel('Absolute error')

ax[1, 0].plot(n_values, results_2, 'o-', label='Computed $I_n$')
ax[1, 0].hlines(I_exact_2, n_values[0], n_values[-1], colors='r', label='Exact $I$')
ax[1, 0].set_title('Integral of $e^x \cos(4x)$ from 0 to $\pi$')
ax[1, 0].set_xlabel('Number of subdivisions (n)')
ax[1, 0].set_ylabel('Integral value')
ax[1, 0].legend()

ax[1, 1].plot(n_values, errors_2, 'o-')
ax[1, 1].set_title('Error in computation for $e^x \cos(4x)$')
ax[1, 1].set_xlabel('Number of subdivisions (n)')
ax[1, 1].set_ylabel('Absolute error')

plt.tight_layout()
plt.show()
```

Executed at 2024.05.02 01:43:19 in 282ms



Gauss-Legendre gives results with smaller error & faster converge in Comparison to Trapozoidal.