Approximation of Functions

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Minimax problem

Least-squares Approximation

Orthogonal polynomials

Revisit Chebyshev polynomial

Comparison with Machine Learning Models

Weierstrass Theorem

- Worse-case scenario error measure: Inf norm
- **Inf norm**: $\|f(x) - p(x)\|_\infty = \max_{a \le x \le b} |f(x) - p(x)|$
- **Weierstrass THM**: Let $f(x)$ be continuous on $[a, b]$ and let $\epsilon > 0$. Then $\exists$ a polynomial $p(x)$ s.t.

$$\|f(x) - p(x)\|_\infty \le \epsilon, a \le x \le b.$$

- **Question**: Why the polynomial fails in the Runge phenomenon?

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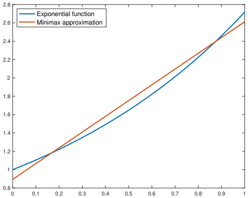
- **Question**: Why the polynomial fails in the Runge phenomenon?
- $\|f^{(n)}(x)\|_\infty$ increases fast w.r.t n
- Equispaced points error: $\|(x - x_0) \ldots (x - x_n)\|_\infty \leq n! \left(\frac{b-a}{n}\right)^{n+1}$
- Polynomial is very powerful!
- **Question**: What is and how to find the best polynomial?

Minimax problem

- Minimax problem $\min_{p_n} \|f - p_n\|_\infty$
  - Given p_n, maximize $|f(x) - p_n(x)|$
  - Find p_n, minimize $\|f - p_n\|_\infty$
- E.g., $f(x) = e^x$, $p_n(x) = a + bx$, $x \in [0, 1]$



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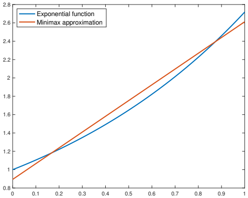

- arg $\max_x |f(x) - p_n(x)| = 0, 1, \log(b)$, $\|f - g\|_\infty = \rho$
  - $e_1(0) = 1 - a = \rho$
  - $e_1(\log(b)) = a + b\log(b) - b = \rho = 1 - a \Rightarrow a = \frac{e-(e-1)\log(e-1)}{2}$
  - $e_1(1) = e - a - b = \rho = 1 - a \Rightarrow b = e - 1$
- **THM**: p_n minimizes $\|f - p_n\|_\infty$ iff $\exists\ x_0 < \cdots < x_{n+1}$ s.t. $f(x_i) - p_n(x_i) = \sigma(-1)^i\|f - p_n\|_\infty$, where σ is 1 or -1.

Near Minimax problem

- Minimax is too difficult, depends on f
- Interpolation error: $e_n = \left| \frac{f^{(n+1)}(\eta)}{(n+1)!}(x - x_0) \dots (x - x_n) \right|$
- Cannot control $f^{(n+1)}(x)$

Near Minimax problem

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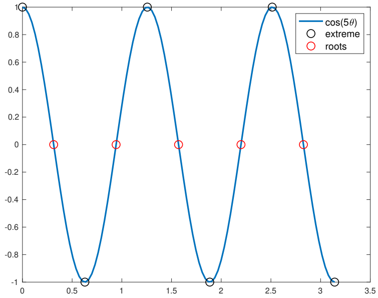
- Interpolation error: $e_n = \left| \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)\dots(x-x_n) \right|$
- Cannot control $f^{(n+1)}(x)$
- Near minimax: $\min_{x_0,\dots,x_n} |(x-x_0)\dots(x-x_n)|$
- Let $w_{n+1}(x) = (x-x_0)\dots(x-x_n)$
- **THM**: $w^*_{n+1}(x)$ is the solution iff $\exists\ a \le \widetilde{x}_0 < \cdots < \widetilde{x}_{n+1} \le b$ s.t. $w^*_{n+1}(\widetilde{x}_i) = \sigma(-1)^i \|w_{n+1}\|_\infty$, where σ is 1 or -1.

Proof

- **THM**: $w_{n+1}^*(x)$ is the solution iff $\exists\ a \leq \widetilde{x}_0 < \cdots < \widetilde{x}_{n+1} \leq b$ s.t. $w_{n+1}^*(\widetilde{x}_i) = \sigma(-1)^i \|w_{n+1}\|_\infty$, where σ is 1 or -1.
- $\Leftarrow$
  - Suppose q_n is a monic polynomial with $\|q_{n+1}\|_\infty < \|w_{n+1}^*\|_\infty$
  - WLOG, assume $\sigma = 1$
  - $q_{n+1}(\widetilde{x}_i) \begin{cases} < w_{n+1}^*(\widetilde{x}_i), & \text{if } i \text{ even,} \\ > w_{n+1}^*(\widetilde{x}_i), & \text{if } i \text{ odd.} \end{cases}$
  - $w_{n+1}^*(x) - q_{n+1}(x)$ flips sign at least $n+1$ times
  - $w_{n+1}^*(x) - q_{n+1}(x)$ is a degree n polynomial (dominant term cancels)
  - IVT $\Rightarrow n+1$ roots, contradiction
- **Question**: What function has such property?

Cosine function

- $\cos((n+1)\theta)$ for $\theta \in [0, \pi]$



- Extreme points: $\theta_k = \frac{k\pi}{n+1}$, $0 \le k \le n+1$, $|\cos(\theta_k)| = 1$
- Roots: $\theta_k = \frac{(2k+1)\pi}{2n+2}$, $0 \le k \le n$

Find the polynomial

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- Roots: $\theta_k = \frac{(2k+1)\pi}{2n+2}$, $0 \leq k \leq n$
- But $\cos(n\theta)$ is not a polynomial
- Suppose we focus on $f(x)$ with $x \in [-1, 1]$
- Goal: find a function $g(x)$ s.t.
  - $\cos(n\theta)$ is a polynomial
  - $\theta = g(x)$ s.t. $\cos(n\theta)$ has alternating extreme values
- Magic trick:

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- Magic trick: $g(x) = \cos^{-1}(x)$
- $T_n(x) = \cos(n \cos^{-1}(x))$
  - $T_0(x) = \cos(0) = 1$
  - $T_1(x) = x$
  - $T_{n\pm 1}(x) = \cos((n \pm 1)\theta) = \cos(n\theta)\cos(\theta) \mp \sin(n\theta)\sin(\theta)$
  - $T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\theta)\cos(\theta) = 2T_n(x)x$
  - $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

Chebyshev polynomial

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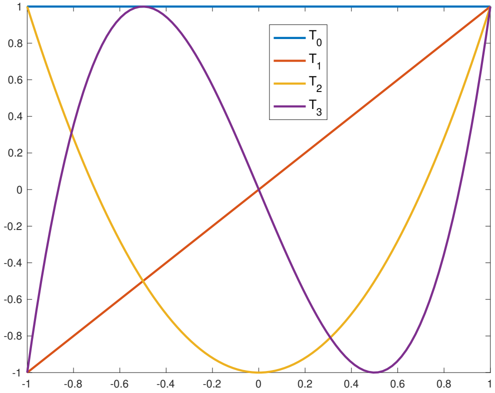

Comparison

- Chebyshev polynomial: $p_{n+1}(x) = \cos((n+1)\theta) = \cos((n+1)\cos^{-1}(x)), x \in [-1, 1]$
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- Error bound: $\|(x - x_0) \dots (x - x_n)\|_\infty \leq$

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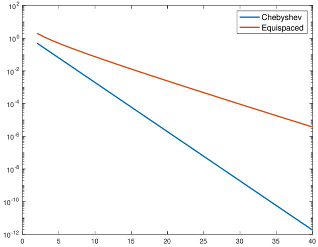
- Extreme points: $\theta_k = \frac{k\pi}{n+1}$, $0 \le k \le n+1$
- Error bound: $\|(x - x_0)\dots(x - x_n)\|_\infty \le \frac{1}{2^{n-1}}$
  - Extreme values: ± 1
  - $T_0(x) = 1$, $T_1(x) = x$
  - Recurrence relationship: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$
- Equispaced points: $\|(x - x_0)\dots(x - x_n)\|_\infty \le n! \left(\frac{2}{n}\right)^{n+1}$
- Comparison:



Interpolation

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- $f(x) \approx p_n(x) = c_0 + c_1 x + \cdots + c_n x^n$
- Roots: $\theta_k = \frac{(2k+1)\pi}{2n+2}$, $x_k = \cos(\theta_k)$, $0 \leq k \leq n$
- Same way with different points: $\mathbf{Ac} = \mathbf{f}$

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \ddots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

- Differentiation: same, $\mathbf{D} = \mathbf{ABA}^{-1}$
- Integration: same, $\mathbf{S} = \widetilde{\mathbf{A}}\mathbf{BA}^{-1}$

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The least squares approximation problem

- Average error measurement: 2 norm
- Notation: $\|g\|_2 = \sqrt{\int_a^b |g(x)|^2 \ dx}$
- Least squares approximation $\min_{p_n} \|f - p_n\|_2^2$
- E.g. $f(x) = e^x, -1 \leq x \leq 1$, let $p_1(x) = c_0 + c_1 x$
- **Question**: How to minimize $r = \|f - p_1(x)\|_2^2$?

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- **Question**: How to minimize $r = \|f - p_1(x)\|_2^2$?
- $\frac{\partial r}{\partial c_0} = \frac{\partial r}{\partial c_1} = 0$
- $0 = \frac{\partial r}{\partial c_0} = -2 \int_{-1}^{1} e^x - c_0 - c_1 x \; dx$
- $0 = \frac{\partial r}{\partial c_1} = -2 \int_{-1}^{1} (e^x - c_0 - c_1 x) \, x \; dx$

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- $\begin{bmatrix} 2 & \int_{-1}^{1} x \ dx \\ \int_{-1}^{1} x \ dx & \int_{-1}^{1} x^2 \ dx \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \int_{-1}^{1} e^x \ dx \\ \int_{-1}^{1} x e^x \ dx \end{bmatrix}$
- $c_0 = \frac{1}{2} \int_{-1}^{1} e^x \ dx$
- $c_1 = \frac{3}{2} \int_{-1}^{1} x e^x \ dx$

General least squares problem

- $r = \int_a^b w(x)[f(x) - p_n(x)]^2 \, dx$
- Examples:
  - $w(x) = 1, \; a \le x \le b$
  - $w(x) = \frac{1}{\sqrt{1-x^2}}, \; -1 \le x \le 1$
  - $w(x) = e^{-x^2}, \; -\infty < x < \infty$
- $w(x) = 1$ not necessarily always a good choice
- $w(x)$ allows different degrees of importance to different points
- **Question**: What are equations need to be solved?

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- $r(c_0, \ldots, , c_n) = \int_a^b w(x) \left[f(x) - \sum_{j=0}^n c_j x^j \right]^2 \ dx$
- $\frac{\partial r}{\partial c_k} = 0$
- $\int_a^b w(x) \left[f(x) - \sum_{j=0}^n c_j x^j \right] (-2) x^k \ dx = 0$
- $\sum_{j=0}^n c_j \int_a^b w(x) x^{k+j} \ dx = \int_a^b w(x) f(x) x^k \ dx$
- **Question**: How to solve this?

Solve the general least squares problem

- Form the linear system

$$\begin{bmatrix} \int_a^b w(x)\ dx & \int_a^b w(x)x\ dx & \dots & \int_a^b w(x)x^n\ dx \\ \int_a^b w(x)x\ dx & \int_a^b w(x)x^2\ dx & \dots & \int_a^b w(x)x^{n+1}\ dx \\ \vdots & \ddots & \ddots & \vdots \\ \int_a^b w(x)x^n\ dx & \int_a^b w(x)x^{n+1} & \dots & \int_a^b w(x)x^{2n}\ dx \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \int_a^b w(x)f(x)\ dx \\ \int_a^b w(x)f(x)x\ dx \\ \vdots \\ \int_a^b w(x)f(x)x^n\ dx \end{bmatrix}.$$

- Setting up the linear system requires a lot of integrals
- **Question**: Is there an alternative convenient way?

Inner product

- Problem: Basis $\{x^n\}$ are very linearly dependent
- **Def**: **Inner product** of f and g:

$$(f,g) = \int_a^b w(x)f(x)g(x)\ dx.$$

- Properties:
  - $(\alpha f, g) = (f, \alpha g) = \alpha(f,g),\ \forall \alpha \in \mathbb{R}$
  - $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$ and $(f, g_1 + g_2) = (f, g_1) + (f, g_2)$
  - $(f,g) = (g,f)$
  - $(f,f) \geq 0$ and $(f,f) = 0$ iff $f(x) = 0$
- **Def**: **Two norm**: $\|f\|_2 = \sqrt{\int_a^b w(x)[f(x)]^2\ dx} = \sqrt{(f,f)}$
- **Lemma (Cauchy-Schwartz)**: $|(f,g)| \leq \|f\|_2 \|g_2\|_2$
- Many properties in the textbook

Create orthogonal polynomials

- **Def**: f and g are **orthogonal** if $(f, g) = 0$
- **THM**: $\exists$ a sequence of polynomials $\{\phi_n(x) | n \geq 0\}$ with deg$(\phi_n) = n$, $\forall n$, and

$$(\phi_n, \phi_m) = 0, \quad \forall n \neq m, \quad n, m \geq 0.$$

  In addition, if we require $(\phi_n, \phi_n) = 1$ and coefficients of x^n is positive, then the sequence $\{\phi_n\}$ is unique.
- **Question**: Start with $1, x, \ldots, x^n$, how to calculate it?

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- **Question**: Start with $1, x, \ldots, x^n$, how to calculate it?
- **Gram-Schmidt process**:

$$\begin{aligned}
\phi_0 &= \frac{1}{\|1\|_2}, \\
\phi_1 &= \frac{x - (\phi_0, x)\phi_0}{\|x - (\phi_0, x)\phi_0\|_2}, \\
\phi_2 &= \frac{x^2 - (\phi_0, x^2)\phi_0 - (\phi_1, x^2)\phi_1}{\|x^2 - (\phi_0, x^2)\phi_0 - (\phi_1, x^2)\phi_1\|_2}, \\
&\vdots
\end{aligned}$$

Example

- Suppose $w(x) = 1, [a, b] = [-1, 1]$
- **Question**: How to calculate the first three orthogonal polynomials ?

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- Suppose $w(x) = 1, [a,b] = [-1,1]$
- **Question**: How to calculate the first three orthogonal polynomials ?
- $\phi_0 = \frac{1}{(1,1)} = \frac{1}{\sqrt{\int_{-1}^{1} 1 \ dx}} = \frac{1}{\sqrt{2}}$
- $(x, \phi_0)\phi_0 = \frac{1}{2} \int_{-1}^{1} x \ dx = 0$
- $\|x - (x, \phi_0)\phi_0\|_2^2 = \int_{-1}^{1} x^2 \ dx = \frac{2}{3}$
- $\phi_1 = \frac{x-(x,\phi_0)\phi_0}{\|x-(x,\phi_0)\phi_0\|_2} = \sqrt{\frac{3}{2}}x$
- $(x^2, \phi_0)\phi_0 = \frac{1}{2} \int_{-1}^{1} x^2 \ dx = \frac{1}{3}$
- $(x^2, \phi_1)\phi_1 = \frac{3}{2}x \int_{-1}^{1} x^3 \ dx = 0$
- $\|x^2 - (x^2, \phi_0)\phi_0 - (x^2, \phi_1)\phi_1\|_2^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \ dx = \frac{8}{45}$
- $\phi_2 = \frac{x^2-(x^2,\phi_0)\phi_0-(x^2,\phi_1)\phi_1}{\|x^2-(x^2,\phi_0)\phi_0-(x^2,\phi_1)\phi_1\|_2} = \frac{x^3-\frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{5}{8}}(3x^2-1)$

QR decomposition

- Function version of QR?

QR decomposition

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- Gram-Schmidt: $\{1, x, \dots, x^n\} \Rightarrow \{\phi_0, \phi_1, \dots, \phi_n\}$
- $\mathbf{x}^T(x) = \begin{bmatrix} 1 & \dots & x^n \end{bmatrix}$, $\mathbf{q}^T(x) = \begin{bmatrix} \phi_0(x) & \dots & \phi_n(x) \end{bmatrix}$
- $\mathbf{x}^T(x) = \begin{bmatrix} 1 & \dots & x^n \end{bmatrix} = \mathbf{q}^T(x)\mathbf{R}$
- $x^k = c_0\phi_0 + \dots + c_n\phi_n \Rightarrow c_i = (x^k, \phi_i)$
- $(x^k, \phi_i) = 0$ for $i > k$
- $\mathbf{R} = \begin{bmatrix} (1, \phi_0) & (x, \phi_0) & \dots & (x^n, \phi_0) \\ 0 & (x, \phi_1) & \dots & (x^n, \phi_1) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & (x^n, \phi_n) \end{bmatrix}$

Relationship between two polynomials

- **Question**: Relationships of coefficients?

Relationship between two polynomials

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- $$p_n(x) = c_0 + \cdots + c_n x^n = \begin{bmatrix} 1 & \dots & x^n \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{x}^T(x)\mathbf{c}$$
- $$q_n(x) = d_0\phi_0 + \cdots + d_n\phi_n = \begin{bmatrix} \phi_0 & \dots & \phi_n \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_n \end{bmatrix} = \mathbf{q}^T(x)\mathbf{d}$$
- $\mathbf{x}^T(x) = \mathbf{q}^T(x)\mathbf{R}$
- $p_n(x) = \mathbf{q}^T(x)\mathbf{R}\mathbf{c}$
- $q_n(x) = \mathbf{x}^T(x)\mathbf{R}^{-1}\mathbf{d}$

Chebyshev polynomial

- **Def**: **Chebyshev polynomial**, $-1 \le x \le 1$

$$T_n(x) = \cos(n\theta) = \cos(n\cos^{-1}(x)).$$

- Chebyshev polynomial is orthogonal with $w(x) = \frac{1}{\sqrt{1-x^2}}$
- $(T_m, T_n) = \int_{-1}^{1} \frac{\cos(m\cos^{-1}(x))\cos(n\cos^{-1}(x))}{\sqrt{1-x^2}}\, dx$

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$m \neq n$

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- $(T_m, T_n) = \int_{-1}^{1} \frac{\cos(m\cos^{-1}(x))\cos(n\cos^{-1}(x))}{\sqrt{1-x^2}}\ dx$
- Change of variable: $x = \cos(\theta), dx = -\sin(\theta)\ d\theta$
- $(T_m, T_n) = \int_{\pi}^{0} \frac{\cos(m\theta)\cos(n\theta)}{\sqrt{1-\cos^2(\theta)}}\ (-\sin(\theta))\ d\theta = \int_0^{\pi} \cos(m\theta)\cos(n\theta)\ d\theta$
- Recall: $2\cos(\theta)\cos(\eta) = \cos(\theta - \eta) + \cos(\theta + \eta)$
- $(T_m, T_n) = \begin{cases} 0, & n \neq m, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & n = m > 0. \end{cases}$

Chebyshev coefficients

- $f(x) = \frac{a_0}{2} + \sum_{i=1}^{n} a_i T_i(x)$
- **Question**: How are $\{a_i\}$ determined?

Chebyshev coefficients

- $f(x) = \frac{a_0}{2} + \sum_{i=1}^{n} a_i T_i(x)$
- **Question**: How are $\{a_i\}$ determined?
- $(T_j, f) = \frac{a_0}{2}(T_j, 1) + \sum_{i=1}^{n} a_i (T_j, T_i)$
- $j = 0$: $(T_0, f(x)) = \frac{a_0}{2}(T_0, T_0) = \frac{\pi}{2} \Rightarrow a_0 = \frac{2}{\pi}(T_0, f)$
- $j \neq 0$: $(T_j, f(x)) = a_j (T_j, T_j) \Rightarrow a_j = \frac{2}{\pi}(T_j, f)$
- $a_j = \frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x) f(x)}{\sqrt{1-x^2}}\, dx = \frac{2}{\pi}\int_{-1}^{1} \frac{\cos(j\cos^{-1}(x)) f(x)}{\sqrt{1-x^2}}\, dx$
- $x = \cos(\theta), 0 \le \theta \le \pi$
- **Cosine transform**: $a_j = \frac{2}{\pi}\int_{0}^{\pi} f(\cos(\theta))\cos(j\theta)\, d\theta$
- **Question**: How to discretize it?

Chebyshev coefficients

- $f(x) = \frac{a_0}{2} + \sum_{i=1}^{n} a_i T_i(x)$
- **Question**: How are $\{a_i\}$ determined?
- $(T_j, f) = \frac{a_0}{2}(T_j, 1) + \sum_{i=1}^{n} a_i (T_j, T_i)$
- $j = 0$: $(T_0, f(x)) = \frac{a_0}{2}(T_0, T_0) = \frac{\pi}{2} \Rightarrow a_0 = \frac{2}{\pi}(T_0, f)$
- $j \neq 0$: $(T_j, f(x)) = a_j (T_j, T_j) \Rightarrow a_j = \frac{2}{\pi}(T_j, f)$
- $a_j = \frac{2}{\pi}\int_{-1}^{1} \frac{T_j(x) f(x)}{\sqrt{1-x^2}}\, dx = \frac{2}{\pi}\int_{-1}^{1} \frac{\cos(j\cos^{-1}(x)) f(x)}{\sqrt{1-x^2}}\, dx$
- $x = \cos(\theta), 0 \leq \theta \leq \pi$
- **Cosine transform**: $a_j = \frac{2}{\pi}\int_0^{\pi} f(\cos(\theta))\cos(j\theta)\, d\theta$
- **Question**: How to discretize it?
- **Chebyshev nodes**: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$
- **Discrete orthogonality**: $\sum_{k=0}^{n} T_i(x_k) T_j(x_k) = K_i \delta_{i,j}$
- $K_0 = n + 1$ and $K_i = \frac{n+1}{2}$ for $1 \leq i \leq n$

Least squares

- Consider orthogonal polynomials $T_j(x)$
- $p_n(x) = a_0 T_0(x) + \cdots + a_n T_n(x)$
- $r = \|f - p_n\|_2^2 = \int_{-1}^{1} w(x) \left[f(x) - \sum_{j=0}^{n} a_j T_j(x) \right]^2 \, dx$
- $\min_{p_n} r \Rightarrow \frac{\partial r}{\partial a_k} = 0$
- $\frac{\partial r}{\partial a_k} = 0 \Rightarrow \int_{-1}^{1} w(x)(f(x) - \sum_{j=0}^{n} a_j T_j(x)) T_k(x) \, dx = 0 \Rightarrow$ $\sum_{j=0}^{n} a_j \int_{-1}^{1} w(x) T_j(x) T_k(x) \, dx = \int_{-1}^{1} w(x) f(x) T_k(x) \, dx$
- Inner product notation: $\sum_{j=0}^{n} a_j (T_j, T_k) = (f, T_k)$
- Orthogonality: $(T_j, T_k) = 0$ for $j \neq k$
- $a_k = \frac{(f, T_k)}{\|T_k\|_2^2}$
- $p_n^*(x) = \sum_{j=0}^{n} \frac{(f, T_j)}{\|T_j\|_2^2} T_j(x)$

Discrete cosine transform

- $p_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j T_j(x)$
- $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$
- $a_k = \frac{2}{n+1} \sum_{j=0}^{n} f(x_j) T_k(x_j)$
- $\theta_j = \frac{2j+1}{2N+2}\pi$
- **Discrete cosine transform**:
  $a_k = \frac{2}{n+1} \sum_{j=0}^{n} f(\cos(\theta_j)) \cos(n\theta_j)$
- **Question**: Matrix form
- $$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \frac{2}{n+1} \begin{bmatrix} 1 & 1 & \dots & 1 \\ \cos(\theta_0) & \cos(\theta_1) & \dots & \cos(\theta_n) \\ \vdots & \ddots & \ddots & \vdots \\ \cos(n\theta_0) & \cos(n\theta_1) & \dots & \cos(n\theta_n) \end{bmatrix}$$

Connection with polynomial interpolation approach

- $f(x) \approx p_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j T_j(x)$
- **Chebyshev nodes**: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right) = \cos(\theta_k)$, $0 \le k \le n$
- $f(x_k) = p_n(x_k)$, $\mathbf{A}\mathbf{a} = \mathbf{f}$

$$\begin{bmatrix} \frac{1}{2} & \cos(\theta_0) & \dots & \cos(n\theta_0) \\ \frac{1}{2} & \cos(\theta_1) & \dots & \cos(n\theta_1) \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2} & \cos(\theta_n) & \dots & \cos(n\theta_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(\cos(\theta_0)) \\ f(\cos(\theta_1)) \\ \vdots \\ f(\cos(\theta_n)) \end{bmatrix}$$

- Orthogonality:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \frac{n+1}{4} & 0 & \dots & 0 \\ 0 & \frac{n+1}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{n+1}{2} \end{bmatrix}.$$

Differentiation and integration

- $f(x) = \frac{a_0}{2} + \sum_n a_n T_n(x)$
- $f'(x) = \frac{b_0}{2} + \sum_n b_n T_n(x)$
- $\int_{-1}^{x} f(t)\, dt = \frac{c_0}{2} + \sum_n c_n T_n(x)$
- Analytical computation:
  - Differentiation:

$$b_k = 2 \sum_{\substack{j=k+1 \\ j+k \text{ odd}}} j \cdot a_j$$

  - Integration:

$$c_k = \frac{1}{2k}(a_{k-1} - a_{k+1}),\ k \geq 1,$$
$$c_0 = 2(c_1 - c_2 + c_3 - \dots)$$

- Matrix version:
  - $\mathbf{A}$ as the discrete cosine matrix
  - Differentiation matrix: $\mathbf{D} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}$
  - Integration matrix: $\mathbf{S} = \mathbf{A}^{-1}\mathbf{C}\mathbf{A}$

Comparison

- Theoretical supports: both have universal approximation property
- Advantages:
  - Chebyshev is a near minimax approximation and nice analytical properties
  - ML empirically work well for high-dimensional problem
- Disadvantages:
  - Polynomial interpolations face with curse of dimensionality
  - Polynomial not good with complex geometries
  - ML models convergence slow