Math 302 HW1
Section IV 2024
Duke Kunshan University

HW questions: 2, 4, 6, 9, 10, 11, 15 ,16, 17

Problem 1 (2.2). *Write a program implementing the algorithm Bisect given in Section 2.1. Use the program to calculate the real roots of the following equations. Use an error tolerance of* $\epsilon = 10^{-5}$.

1. $e^x - 3x^2 = 0$
2. $x^3 = x^2 + x + 1$
3. $e^x = \frac{1}{0.1+x^2}$
4. $x = 1 + 0.3\cos(x)$

Solution 1. *See codes on Canvas.*

Problem 2 (2.4). *Implement the algorithm Newton given in Section 2.2. Use it to solve the equation in Problem 1.*

Solution 2. *See codes on Canvas.*

Problem 3 (2.6). *Use Newton's method to calculate the unique root of*

$$x + e^{-Bx^2}\cos(x) = 0$$

with $B > 0$ *a parameter to be set. Use a variety of increasing values of* B*, for example,* $B = 1, 5, 10, 25, 50$*. Among the choices of* x_0 *used, choose* $x_0 = 0$ *and explain any anomalous behavior. Theoretically, the Newton method will converge for any value of* x_0 *and* B*. Compare this with actual computations for large values of* B*.*

Solution 3. *For large* B*, we do not have convergence. Note for very large* B*,* e^{-Bx^2} *is very close to zero except when* x *is very close to zero. Therefore, for each iteration, we have*

$$x_{n+1} \begin{cases} \approx 0, & x \neq 0, \\ = -1, & x = 0. \end{cases}$$

The problem is that the computer can only do finite digits calculation. As an example, $1 + \epsilon = 1$ *in the computer for very small* ϵ*, say* $\epsilon = 10^{-22}$.

Problem 4 (2.9). *Use the secant method to solve the equations given in Problem 1.*

Solution 4. *See the code on Canvas.*

Problem 5 (2.10). *Ue the secant method to solve the equation of Problem 3.*

Solution 5. *See the code on Canvas.*

Problem 6 (2.11). *Show the error formula (2.3.2) for the secant method,*

$$\alpha - c = -(\alpha - b)(\alpha - a)\frac{f[a, b, \alpha]}{f[a, b]}.$$

Solution 6. *The secant iteration is defined as*

$$\alpha - x_{n+1} = \alpha - x_n + f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

We assume the root is defined as α. By definition, we have

$$f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}},$$
$$f[x_{n-1}, x_n, \alpha] = \frac{\frac{f(\alpha)-f(x_n)}{\alpha - x_n} - \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}}{\alpha - x_{n-1}}.$$

Basically, we want to show that

$$\alpha - x_n + f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} = -(\alpha - x_n)(\alpha - x_{n-1}) \frac{f[x_{n-1}, x_n, \alpha]}{f[x_{n-1}, x_n]}.$$

It is much easier to simplify the right-hand side to obtain the left-hand side. Using the fact $f(\alpha) = 0$, we calculate

$$\begin{aligned}
-(\alpha - x_n)(\alpha - x_{n-1}) \frac{f[x_{n-1}, x_n, \alpha]}{f[x_{n-1}, x_n]} &= -(\alpha - x_n)(\alpha - x_{n-1}) \frac{x_n - x_{n-1}}{\alpha - x_{n-1}} \frac{\frac{f(\alpha)-f(x_n)}{\alpha - x_n} - \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}}{f(x_n) - f(x_{n-1})} \\
&= \frac{-(x_n - x_{n-1})(f(\alpha) - f(x_n)) + (\alpha - x_n)(f(x_n) - f(x_{n-1}))}{f(x_n) - f(x_{n-1})} \\
&= f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} + (\alpha - x_n).
\end{aligned}$$

Here, we conclude.

Problem 7 (2.15)**.** *Apply Newton's method*

1. *to the function*

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0 \end{cases}$$

with the root $\alpha = 0$. What is the behavior of the iterates? Do they converge, and if so, at what rate?

2. *Do the same as in (1), but with*

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0, \\ -\sqrt[3]{x^2}, & x < 0. \end{cases}$$

Solution 7. *The code is attached on Canvas.*

1. *The Newton's iteration is $x_{n+1} = -x_n$. Therefore, x_n only changes the sign in each iteration and never converge.*

2. *From the code, we can see the iteration does converge, but not second order. To see why, calculate*

$$f'(x) = \begin{cases} \frac{2x}{3(x^2)^{2/3}}, & x > 0 \\ \frac{-2x}{3(x^2)^{2/3}}, & x < 0. \end{cases}$$

Note the limit blows up around the root zero. Therefore, we cannot apply Taylor series around the zero, and convergence analysis for the Newton no longer hold. Now the Newton's iteration is

$$x_{n+1} = -\frac{x_n}{2}.$$

So the rate is

$$\lim_{n \to \infty} \frac{|x_{n+1} - 0|}{|x_n - 0|} = \frac{1}{2},$$

and it becomes the first-order method.

Problem 8 (2.16)**.** *A sequence $\{x_n\}$ is said to converge superlinearly to α if*

$$|\alpha - x_{n+1}| \leq c_n |\alpha - x_n|, \quad n \geq 0$$

with $c_n \to 0$ as $n \to \infty$. Show that in this case,

$$\lim_{n \to \infty} \frac{|\alpha - x_n|}{|x_{n+1} - x_n|} = 1.$$

Thus $|\alpha - x_n| \approx |x_{n+1} - x_n|$ is increasingly valid as $n \to \infty$.

Solution 8. *Apply the triangle inequality,*

$$\begin{aligned}\frac{|\alpha - x_n|}{|x_{n+1}-x_n|} &\le \frac{|\alpha - x_{n+1}| + |x_{n+1}-x_n|}{|x_{n+1}-x_n|} \le \frac{c_n|\alpha - x_n| + |x_{n+1}-x_n|}{|x_{n+1}-x_n|} \le c_n\frac{|\alpha - x_n|}{|x_{n+1}-x_n|} + 1 \\ \Rightarrow (1-c_n)\frac{|\alpha - x_n|}{|x_{n+1}-x_n|} &\le 1 \Rightarrow \frac{|\alpha - x_n|}{|x_{n+1}-x_n|} \le \frac{1}{1-c_n}.\end{aligned}$$

Similarly,

$$\frac{|\alpha - x_n|}{|x_{n+1}-x_n|} \ge \frac{|\alpha - x_n|}{|\alpha - x_{n+1}| + |\alpha - x_n|} \ge \frac{|\alpha - x_n|}{(1+c_n)|\alpha - x_n|} \ge \frac{1}{1+c_n}.$$

Taking the limit, we have the desired result.

Problem 9 (2.17)**.** *Newton's method for finding a root α of $f(x) = 0$ sometimes requires the initial guess x_0 to be quite close to α in order to obtain convergence. Verify that this is the case of the root $\alpha = \frac{\pi}{2}$ of*

$$f(x) = \cos(x) + \sin^2(50x).$$

Give a rough estimate of how small $|x_0 - \alpha|$ should be in order to obtain convergence to α. (Hint: Consider (2.2.6))

Solution 9. *We first calculate*

$$\begin{aligned}f'(x) &= 50\sin(100x) - \sin(x),\\ f''(x) &= 5000\cos(100x) - \cos(x).\end{aligned}$$

According to the hint, we want to have

$$\frac{\max_{x\in I}|f''(x)|}{2\min_{x\in I}|f'(x)|}|\alpha - x_0| < 1.$$

We know the true solution $\alpha = \frac{\pi}{2}$ and we want to get a rough estimation of I. Denote ϵ as the threshold such that we have the equality, i.e., we want to have $|\alpha - x_0| < \epsilon$. In this case, $I = \left[\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right]$. First, note $5000\cos(100x) \gg \cos(x)$ in the neighborhood of $\frac{\pi}{2}$ and it already obtains its maximum, therefore, we use $\max_{x\in I}|f''(x)| = 5000$. Note $f''\left(\frac{\pi}{2}\right) > 0$, so $f'(x)$ is increasing in the neighborhood of $\frac{\pi}{2}$. Note $f'\left(\frac{\pi}{2}\right) = -1$, therefore, to determine ϵ, the $\min_{x\in I}|f'(x)|$ will takes the place at $x_0 + \epsilon$. To simplify $f'(x)$, we use the linear approximation around $\frac{\pi}{2}$ to obtain $f'(x) \approx -1 + 5000\left(x - \frac{\pi}{2}\right)$. So, $\min_{x\in I}|f'(x)| \approx 1 - 5000\epsilon$. Calculate

$$\frac{5000}{2(1-5000\epsilon)}\epsilon < 1 \Rightarrow \epsilon < \frac{1}{7500}.$$

So we require that $|x_0 - \alpha| < \frac{1}{7500}$.