In[639]:=
```
ResourceFunction["DarkMode"][]
```

HW2

Problem 1

1.

In[187]:=
```
ClearAll["Global`*"]
g[x_]:=-16+6x+12/x
gp[x_] = Evaluate[D[g[x],x]]
gp[2]
```

Out[189]=
$$6-\frac{12}{x^2}$$

Out[190]=
3

$g(x) = -16 + 6\,x + \frac{12}{x}$

$g'(x) = 6 - \frac{12}{x^2}$

$g'(\alpha) = g'(2) = 3 > 1$

So diverges.

2.

In[222]:=
```
ClearAll["Global`*"]
g[x_] := 2/3 x + 1/x^2
gp[x_] = Evaluate[D[g[x],x]]
gp[3^(1/3)]

gpp[x_] = Evaluate[D[%%,x]]
gpp[3^(1/3)]
```

Out[224]=
$$\frac{2}{3} - \frac{2}{x^3}$$

Out[225]=
0

Out[226]=
$$\frac{6}{x^4}$$

Out[227]=
$$\frac{2}{3^{1/3}}$$

$g(x) = \frac{2}{3}\,x + \frac{1}{x^2}$

$g'(x) = \frac{2}{3} - \frac{2}{x^3}$

$g'(\alpha) = g'(2) = 0$

$g''(x) = \frac{6}{x^4}$

$g''(\alpha) = g''(2) = \frac{2}{3^{1/3}} \neq 0$

So 2nd order converges.

3.

```
ClearAll["Global`*"]
g[x_] := 12/(1+x)
gp[x_] = Evaluate[D[g[x],x]]
gp[3]
Abs[%]
```

Out[268]=

$$-\frac{12}{(1+x)^2}$$

Out[269]=

$$-\frac{3}{4}$$

Out[270]=

$$\frac{3}{4}$$

$g(x) = \frac{12}{1+x}$

$g'(x) = -\frac{12}{(1+x)^2}$

$g'(\alpha) = g'(2) = -\frac{3}{4}$

$\lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n}\right| = |g'(\alpha)| = \frac{3}{4}$

So linear converge with rate of $\frac{3}{4}$

Problem 2

In[292]:=
```
ClearAll["Global`*"]
g[x_] := x(x^2+3a)/(3x^2+a)
gp[x_] = Evaluate[D[g[x],x]]
gp[√a]

gpp[x_] = Evaluate[D[%%,x]]
gpp[√a]

gppp[x_] = Evaluate[D[%%,x]]
gppp[√a]

%/3!
```

Out[294]=
$$-\frac{6x^2(3a+x^2)}{(a+3x^2)^2}+\frac{2x^2}{a+3x^2}+\frac{3a+x^2}{a+3x^2}$$

Out[295]=
$$0$$

Out[296]=
$$\frac{72x^3(3a+x^2)}{(a+3x^2)^3}-\frac{24x^3}{(a+3x^2)^2}-\frac{18x(3a+x^2)}{(a+3x^2)^2}+\frac{6x}{a+3x^2}$$

Out[297]=
$$0$$

Out[298]=
$$-\frac{1296x^4(3a+x^2)}{(a+3x^2)^4}+\frac{432x^4}{(a+3x^2)^3}+\frac{432x^2(3a+x^2)}{(a+3x^2)^3}-\frac{144x^2}{(a+3x^2)^2}-\frac{18(3a+x^2)}{(a+3x^2)^2}+\frac{6}{a+3x^2}$$

Out[299]=
$$\frac{3}{2a}$$

Out[300]=
$$\frac{1}{4a}$$

$$g(x)=\frac{x(x^2+3a)}{3x^2+a}$$

$$g'(x)=-\frac{6x^2(3a+x^2)}{(a+3x^2)^2}+\frac{2x^2}{a+3x^2}+\frac{3a+x^2}{a+3x^2}$$

$$g'(\alpha)=g'(\sqrt{a})=0$$

$$g''(x)=\frac{72x^3(3a+x^2)}{(a+3x^2)^3}-\frac{24x^3}{(a+3x^2)^2}-\frac{18x(3a+x^2)}{(a+3x^2)^2}+\frac{6x}{a+3x^2}$$

$$g''(\alpha)=g''(\sqrt{a})=0$$

$g'''(x) = -\frac{1296\,x^4\,(3\,a+x^2)}{(a+3\,x^2)^4} + \frac{432\,x^4}{(a+3\,x^2)^3} + \frac{432\,x^2\,(3\,a+x^2)}{(a+3\,x^2)^3} - \frac{144\,x^2}{(a+3\,x^2)^2} - \frac{18\,(3\,a+x^2)}{(a+3\,x^2)^2} + \frac{6}{a+3\,x^2}$

$g'''(\alpha) = g'''\left(\sqrt{a}\right) = \frac{3}{2\,a} \neq 0$

So 3nd order converges.

$x_{n+1} = g(x_n) = g\left(\sqrt{\alpha}\right) + (x_n - \sqrt{\alpha})g'(\alpha) + (x_n - \sqrt{\alpha})$^2 / 2 * $g''(\alpha) + (x_n - \sqrt{\alpha})$^3 / 3! * $g'''(\alpha)$

$g\left(\sqrt{\alpha}\right) - x_{n+1} = \left(\sqrt{\alpha} - x_n\right)$^3 / 3! * $g'''(\alpha)$

$\lim_{n\to\infty} \frac{\sqrt{a} - x_{n+1}}{\left(\sqrt{a} - x_n\right)^3} = \frac{g'''(\alpha)}{3!} = \frac{1}{4\,a}$

Problem 3

$\boldsymbol{x}_{n+1} = \boldsymbol{g}(\boldsymbol{x}_n)$

Given $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{x})$ has a unique solution $\alpha \in$ D

By MVT, $\alpha_i - x_{i,\,n+1} = \partial\, g_i(\delta_n^{(i)}) \,/\, \partial\, x_1 * (\alpha_i - x_{1,\,n}) + \ldots + \partial\, g_i(\delta_n^{(i)}) \,/\, \partial\, x_m * (\alpha_i - x_{m,\,n})$

where $\delta_n{}^{(i)}$ denotes a point on the line segment between α and $\boldsymbol{x}_n$

$$\alpha - \boldsymbol{x}_{n+1} = \begin{pmatrix} \partial\, g_1\,(\delta_n^{(i)})\,/\partial x_1 & \ldots & \partial\, g_1\,(\delta_n^{(i)})\,/\partial x_m \\ \ldots & \ldots & \ldots \\ \partial\, g_o\,(\delta_n^{(i)})\,/\partial x_1 & \ldots & \partial\, g_o\,(\delta_n^{(i)})\,/\partial x_m \end{pmatrix} (\alpha - \boldsymbol{x}_n)$$

Let G_n denotes the matrix above where $\alpha - \boldsymbol{x}_{n+1} = \boldsymbol{G}_n(\alpha - \boldsymbol{x}_n)$

$\|\,\alpha - \boldsymbol{x}_{n+1}\,\|_\infty = \|\,\mathbf{g}(\alpha) - \mathbf{g}(\boldsymbol{x}_n)\,\|_\infty = \|\boldsymbol{G}_n(\alpha - \boldsymbol{x}_n)\,\|_\infty \leq \|\boldsymbol{G}_n\|_\infty \,\|\,\alpha - \boldsymbol{x}_n\,\|_\infty \leq \max_{\boldsymbol{x}\in D}\|\boldsymbol{G}(\boldsymbol{x})\|_\infty \|\,\alpha - \boldsymbol{x}_n\,\|_\infty$

i.e. $\|\,\alpha - \boldsymbol{x}_{n+1}\,\|_\infty \leq \lambda\, \|\,\alpha - \boldsymbol{x}_n\,\|_\infty$

induction made that $\|\,\alpha - \boldsymbol{x}_n\,\|_\infty \leq \lambda^n\, \|\,\alpha - \boldsymbol{x}_0\,\|_\infty$

as $\lambda < 1$, $\lim_{n\to\infty}\|\,\alpha - \boldsymbol{x}_n\,\|_\infty \leq 0$

so as $n\to\infty$, $\boldsymbol{x}_n \to \alpha$

the iterates converge to the solution to $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{x})$

Problem 4

In[594]:=
```
ClearAll["Global`*"]
f1[x_, y_] := x^2 + y^2 - 4
f2[x_, y_] := x^2 - y^2 - 1
f1x[x_, y_] = Evaluate[D[f1[x,y],x]]
f1y[x_, y_] = Evaluate[D[f1[x,y],y]]
f2x[x_, y_] = Evaluate[D[f2[x,y],x]]
f2y[x_, y_] = Evaluate[D[f2[x,y],y]]

J[x_,y_] := (f1x[x, y] f1y[x, y]
             f2x[x, y] f2y[x, y])
MatrixForm[J[x,y]]
Jinv[x_,y_] := Inverse[J[x,y]]
MatrixForm[Jinv[x,y]]
```

Out[597]=

$2x$

Out[598]=

$2y$

Out[599]=

$2x$

Out[600]=

$-2y$

Out[602]//MatrixForm=

$$\begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

Out[604]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4y} & -\frac{1}{4y} \end{pmatrix}$$

In 1

```python
import numpy as np

# Define the functions of the system
def F(x):
    return [x[0]**2 + x[1]**2 - 4, x[0]**2 - x[1]**2 - 1]

# Define the Jacobian matrix of the system
def J_inv(x):
    return [[1/4/x[0], 1/4/x[0]], [1/4/x[1], -1/4/x[1]]]

# Initial guess
x0 = [1.6, 1.2]

# Newton's method iteration function
def newton_method(F, J_inv, x0, tol=1e-10, max_iter=100):
    x = np.array(x0)
    for _ in range(max_iter):
        Fx = np.array(F(x))
        Jx_inv = np.array(J_inv(x))
        # Update the guess using the inverse Jacobian
        x = x - np.dot(Jx_inv, Fx)
        # Check for convergence
        if np.linalg.norm(Fx, ord=np.inf) < tol:
            return x
    raise ValueError('Newton method did not converge')

# Perform Newton's method
solution = newton_method(F, J_inv, x0)

solution
```

Executed at 2024.04.02 14:38:04 in 75ms

Out 1

2 rows × 1 columns

	0
0	1.581139
1	1.224745

Check Results using Mathematica built-in:

In[615]:= `FindRoot[{f1[x,y],f2[x,y]},{x,1.6},{y,1.2},WorkingPrecision→10]`

Out[615]= {x → 1.581138830, y → 1.224744871}

Check Results using true solutions:

In[636]:=
"x^*="N[$\pm\sqrt{2.5}$]
"y^*="N[$\pm\sqrt{1.5}$]

Out[636]= $x^* = (\pm 1.58114)$

Out[637]= $y^* = (\pm 1.22474)$

$$J = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

det(J) = -8xy

$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} -2\,y & -2\,y \\ -2\,x & 2\,x \end{pmatrix} = \begin{pmatrix} \frac{1}{4\,x} & \frac{1}{4\,x} \\ \frac{1}{4\,y} & -\frac{1}{4\,y} \end{pmatrix}$

using python, $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \mathbf{J}^{-1}(\boldsymbol{x}_n)\,\boldsymbol{F}(\boldsymbol{x}_n)$, where $\boldsymbol{F}(x) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix}$,

iterations stars with $\boldsymbol{x}_0 = \begin{pmatrix} 1.6 \\ 1.2 \end{pmatrix}$,

The output is $\mathbf{x}^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 1.581139 \\ 1.224745 \end{pmatrix}$, (shown above)

so the ans is (1.581139, 1.224745), which coincides with the given true solutions.

Problem 5

$\epsilon_i = y_i - (a + bx_i)$

$e = \frac{1}{n}\sum_{i=1}^{n} \epsilon_i^{\,2} = \frac{1}{n}\sum_{i=1}^{n} [y_i - (a + bx_i)]^2$

To calculate $\min_{a,b} e$

$\frac{\partial e}{\partial a} = \frac{\partial e}{\partial b} = 0$

$\frac{\partial e}{\partial a} = -\frac{2}{n}\sum_{i=1}^{n} [y_i - (a + bx_i)] = 0$ (eq1)

$\frac{\partial e}{\partial b} = -\frac{2}{n}\sum_{i=1}^{n} x_i [y_i - (a + bx_i)] = 0$ (eq2)

introduce the averages $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

from eq1, $-\bar{y} + \hat{a} + \hat{b}\,\bar{x} = 0$

which is $\hat{a} = \bar{y} - \hat{b}\,\bar{x}$

from eq2, $\sum_{i=1}^{n} x_i\,y_i - \hat{a}\sum_{i=1}^{n} x_i - \hat{b}\sum_{i=1}^{n} x_i^2 = 0$

take $\hat{a}$ in, $\sum_{i=1}^{n} x_i\,y_i - \left(\bar{y} - \hat{b}\,\bar{x}\right)\sum_{i=1}^{n} x_i - \hat{b}\sum_{i=1}^{n} x_i^2 = 0$

$$\sum_{i=1}^{n} x_i y_i - (\bar{y} - \hat{b}\bar{x}) \sum_{i=1}^{n} x_i - \hat{b} \sum_{i=1}^{n} x_i^2 = 0$$

$$\hat{b} = \frac{\sum_{i=1}^{n} x_i y_i - \bar{y} \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2 - \bar{x} \sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$= \frac{\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \bar{y} - \sum_{i=1}^{n} \bar{x} y_i + \sum_{i=1}^{n} \bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i \bar{x} + \sum_{i=1}^{n} \bar{x}^2} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

so the answer is:

$$\hat{b} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\hat{a} = \bar{y} - \hat{b}\,\bar{x}$$

Problem 6

In[683]:=
$$A = \frac{1}{4}\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

$$\text{Dot}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - A, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right]$$

```
MatrixForm[%%]
MatrixForm[%%]
```

Out[683]=
$$\left\{\left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right\}, \left\{-\frac{3}{4}, -\frac{5}{4}, -\frac{3}{4}\right\}, \left\{\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right\}\right\}$$

Out[684]=
$$\left\{\left\{-\frac{3}{4}\right\}, \left\{\frac{15}{4}\right\}, \left\{-\frac{3}{4}\right\}\right\}$$

Out[685]//MatrixForm=
$$\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{5}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Out[686]//MatrixForm=
$$\begin{pmatrix} -\frac{3}{4} \\ \frac{15}{4} \\ -\frac{3}{4} \end{pmatrix}$$

Take **I** - **A** = $\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{5}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ and **x** = $\begin{pmatrix} -\frac{3}{4} \\ \frac{15}{4} \\ -\frac{3}{4} \end{pmatrix}$ in,

b = $\begin{pmatrix} -\frac{3}{4} \\ \frac{15}{4} \\ -\frac{3}{4} \end{pmatrix}$

By Neumann series,

$\mathbf{x} = \mathbf{b} + \mathbf{A}\mathbf{b} + A^2\mathbf{b} + \ldots$

To implement,

$x_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{b}$

In 2

```python
# Given matrix A and vector x from the problem
A = np.array([[1, 3, 3],
              [-3, -5, -3],
              [3, 3, 1]]) / 4.0
x_true = np.array([1, 1, 1])


# Initial guess x0 for the iteration
x0 = np.array([1.1,1.1,1.1])

# Since (I - A)x = b, we find b given the true solution x
b = np.array([-3/4,15/4,-3/4])

# Iteration method to solve x_{n+1} = A*x_n + b
def iterative_solve(A, b, x0, tol=1e-5, max_iter=1000):
    x_n = x0
    for i in range(max_iter):
        x_n_plus_1 = np.dot(A, x_n) + b
        # Check for convergence
        if np.linalg.norm(x_n_plus_1 - x_n, ord=np.inf) < tol:
            return x_n_plus_1
        x_n = x_n_plus_1
    raise ValueError("The method did not converge")

# Perform the iteration
x_approx = iterative_solve(A, b, x0)

x_approx
```

Executed at 2024.04.03 01:57:09 in 7ms

Out 2

3 rows　3 rows × 1 columns

	0
0	1.000002
1	0.999997
2	1.000002

With Python codes above, the result is (1.000002, 0.999997, 1.000002).

To prove the convergence,
let α denote the true solution, s.t. $\alpha = \mathbf{g}(\alpha) = \boldsymbol{A}\alpha + \mathbf{b}$
$\boldsymbol{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{b}$
$\mathbf{e}_{n+1} = \alpha - \boldsymbol{x}_{n+1} = \boldsymbol{A}\alpha + \mathbf{b} - (\mathbf{A}\mathbf{x}_n + \mathbf{b}) = \boldsymbol{A}(\alpha - \boldsymbol{x}_n) = \mathbf{A}\mathbf{e}_n$

In[753]:=
```
{eigenvalues, eigenvectors} = Eigensystem[A];

{eigenvalues, eigenvectors}

V = Transpose[eigenvectors];  (* Mathematica returns eigenvectors as a list of
Dia = DiagonalMatrix[eigenvalues];
VInverse = Inverse[V];

Det[V]

(* Display V, D, and V^-1 *)
{MatrixForm[V], MatrixForm[Dia], MatrixForm[VInverse]}

(* Verify the decomposition *)
MatrixForm[A_reconstructed = V.Dia.VInverse]
```

Out[754]=

$$\left\{\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}\right\}, \{\{-1, 0, 1\}, \{-1, 1, 0\}, \{1, -1, 1\}\}\right\}$$

Out[758]=

$$-1$$

Out[759]=

$$\left\{\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}$$

Out[760]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{5}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

To derive $\boldsymbol{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$

$\det(\mathbf{A}\text{-}\lambda\mathbf{I}) = 0 \Rightarrow \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{4}$

take $\lambda_1 = -\frac{1}{2}$ in $(\mathbf{A}\text{-}\lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

take $\lambda_2 = \frac{1}{4}$ in $(\mathbf{A}\text{-}\lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

s.t. $\mathbf{V} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$\det(\mathbf{V}) = -1 \neq 0$

$\mathbf{D} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

$$\boldsymbol{V}^{-1} = \frac{1}{\det(\boldsymbol{V})}\text{adj}(\mathbf{V}) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\boldsymbol{A} = \mathbf{VDV}^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} . \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} . \begin{pmatrix} -1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\boldsymbol{e}_n = \boldsymbol{A}^n \boldsymbol{e}_1 = \mathbf{VD}^n \boldsymbol{V}^{-1} \mathbf{e}_0$
$\| \boldsymbol{D} \|_\infty = \frac{1}{4} < 1$

$\lim_{n\to\infty} \boldsymbol{e}_n = \lim_{n\to\infty} \mathbf{VD}^n \boldsymbol{V}^{-1} \mathbf{e}_0 = \mathbf{0}$
i.e. as n→∞, $\boldsymbol{e}_n \to \mathbf{0}$
$\boldsymbol{x}_n \to \alpha$

So, convergence to α

Problem 7

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{pmatrix}$$

Subtracting a multiple of one column from another column in a matrix A is equivalent to subtracting the corresponding multiple of one row from another row in the transpose of A, denoted A^T. Since the determinant of a matrix is equal to the determinant of its transpose, that is, $\det(A) = \det(A^T)$, these operations do not change the determinant's value.

Do column operations, $C_i = C_i - x_0\, C_{i-1}$ except i=1, starting from last column

$$\det(A) = \det\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & x_1^2(x_1 - x_0) & \ldots & x_1^{n-1}(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) & x_2^2(x_2 - x_0) & \ldots & x_2^{n-1}(x_2 - x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n - x_0 & x_n(x_n - x_0) & x_n^2(x_n - x_0) & \ldots & x_n^{n-1}(x_n - x_0) \end{pmatrix}$$

with co-factor expansion

$$\det(A) = \det\begin{pmatrix} x_1 - x_0 & x_1(x_1 - x_0) & x_1^2(x_1 - x_0) & \ldots & x_1^{n-1}(x_1 - x_0) \\ x_2 - x_0 & x_2(x_2 - x_0) & x_2^2(x_2 - x_0) & \ldots & x_2^{n-1}(x_2 - x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n - x_0 & x_n(x_n - x_0) & x_n^2(x_n - x_0) & \ldots & x_n^{n-1}(x_n - x_0) \end{pmatrix} =$$

$$(x_1-x_0)(x_2-x_0)\ldots(x_n-x_0)\det\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix} = \prod_{1<i\le n}(x_i-x_0)\det(A')$$

where A′ is the Vandermonde matrix in $x_1, \ldots, x_n$. Iterating this process on this smaller Vandermonde matrix,
eventually get det(A) = $\prod_{1\le i\le n}(x_i - x_0)\prod_{2\le i\le n}(x_i - x_1)\ldots\prod_{n\le i\le n}(x_i - x_{n-1}) = \prod_{0\le j<i\le n}(x_i - x_j)$

Problem 8

let t = e^x
take x(t) = ln(t) in, $p_n(x(t)) = \sum_{j=0}^{n} c_j e^{j\ln(t)} = \sum_{j=0}^{n} c_j t^j = p(t)$

as t(x) = e^x singular increasing on (-∞, ∞),
and $x_0, \ldots, x_n$ are distinct real points,
so $t_0, \ldots, t_n$ are also distinct points.

as $p_n(x_i) = y_i$, i = 0, 1, ..., n
$p(t_i) = y_i$, i = 0, 1, ..., n

$$\mathbf{A} = \begin{pmatrix} 1 & t_0 & t_0^2 & \ldots & t_0^n \\ 1 & t_1 & t_1^2 & \ldots & t_1^n \\ 1 & t_2 & t_2^2 & \ldots & t_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & t_n & t_n^2 & \ldots & t_n^n \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix}$$

Ac = **y**

$$\begin{pmatrix} \det(A) = \prod_{0\le j<i\le n}(t_i - t_j) \\ t_i \ne t_j, \text{ so } \det(A) \ne 0 \\ \text{By linear algebra, the solution exists and is unique} \end{pmatrix}$$

By THM, given n+1 distinct t_i, there is a unique polynomial p(t) of degree ≤ n that interpolates y_i at x_i
i.e. there is a unique choice of $c_0, \ldots, c_n$.