Math 302 HW1
Section IV 2024
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Problem 1. *Write a program implementing the algorithm Bisection method. Use the program to calculate the real roots of the following equations. Use an error tolerance of* $\epsilon = 10^{-5}$.

1. $e^x - 3x^2 = 0$
2. $x^3 = x^2 + x + 1$
3. $e^x = \frac{1}{0.1+x^2}$
4. $x = 1 + 0.3\cos(x)$

Problem 2. *Implement the algorithm Newton method. Use it to solve the equation in Problem 1.*

Problem 3. *Use Newton's method to calculate the unique root of*

$$x + e^{-Bx^2}\cos(x) = 0$$

with $B > 0$ *a parameter to be set. Use a variety of increasing values of* B*, for example,* $B = 1, 5, 10, 25, 50$*. Among the choices of* x_0 *used, choose* $x_0 = 0$ *and explain any anomalous behavior. Theoretically, the Newton method will converge for any value of* x_0 *and* B*. Compare this with actual computations for large values of* B*.*

Problem 4. *Use the secant method to solve the equations given in Problem 1.*

Problem 5. *Ue the secant method to solve the equation of Problem 3.*

Problem 6. *Show the error formula (as in slides) for the secant method,*

$$\alpha - c = -(\alpha - b)(\alpha - a)\frac{f[a,b,\alpha]}{f[a,b]}.$$

Here α *is the root.*

Problem 7. *Apply Newton's method*

1. *to the function*

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0 \end{cases}$$

   with the root $\alpha = 0$*. What is the behavior of the iterates? Do they converge, and if so, at what rate?*

2. *Do the same as in (1), but with*

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0, \\ -\sqrt[3]{x^2}, & x < 0. \end{cases}$$

Problem 8. *A sequence* $\{x_n\}$ *is said to converge superlinearly to* α *if*

$$|\alpha - x_{n+1}| \leq c_n|\alpha - x_n|, \quad n \geq 0$$

with $c_n \to 0$ *as* $n \to \infty$*. Show that in this case,*

$$\lim_{n\to\infty} \frac{|\alpha - x_n|}{|x_{n+1} - x_n|} = 1.$$

Thus $|\alpha - x_n| \approx |x_{n+1} - x_n|$ *is increasingly valid as* $n \to \infty$*.*

Problem 9. *Newton's method for finding a root α of $f(x) = 0$ sometimes requires the initial guess x_0 to be quite close to α in order to obtain convergence. Verify that this is the case of the root $\alpha = \frac{\pi}{2}$ of*

$$f(x) = \cos(x) + \sin^2(50x).$$

Give a rough estimate of how small $|x_0 - \alpha|$ should be in order to obtain convergence to α.