# 302 HW3

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## Problem 1

In[644]:=

```
ClearAll["Global`*"]
(*Define the equispaced points for interpolation*)
xPoints = Range[0, 1, 1 / 5]; (*This creates the list {0,1/5,2/5,3/5,4/5,1}*)
y = Exp[xPoints]; (*This calculates e^x for each x in xPoints*)
(*Generate the Vandermonde matrix for interpolation points*)
A = Table[If[i == 0, 1, x^i], {x, xPoints}, {i, 0, 5}];
(*Solve for the polynomial coefficients*)
c = Inverse[A].y;
(*Define the 101 equispaced points for error calculation*)
xPointsFine = Range[0, 1, 1 / 100];
AFine = Table[If[i == 0, 1, x^i], {x, xPointsFine}, {i, 0, 5}];
(*Calculate the estimated Y values at the 101 points*)
yEstimatesFine = AFine.c;
(*Calculate the actual Y values and the error*)
yActualFine = Exp[xPointsFine];
maxError = Norm[N[yActualFine - yEstimatesFine], Infinity];
(*Output the maximum error*)
maxError
```

Out[654]=

$$2.65115 \times 10^{-6}$$

In[655]:=

```
(*matrix B for the original polynomial*)
B = DiagonalMatrix[Range[1, 5], 1, 6];
(*Coefficients of the derivative of the polynomial*)
d = B.c;
(*Evaluate the derivative polynomial at the original xPoints*)
y' = A.d;
MatrixForm[N[y']]
```

Out[658]//MatrixForm=

$$\begin{pmatrix} 1.00008 \\ 1.22139 \\ 1.49183 \\ 1.82211 \\ 2.22556 \\ 2.71819 \end{pmatrix}$$

In[785]:=
```
(*matrix B for the original polynomial*)
Bi = DiagonalMatrix[1 / Range[1, 6]];
(*Coefficients of the intergral of the polynomial*)
di = Bi.c;
(*Evaluate the intergral polynomial at the original xPoints*)
Atilde = Table[x^i, {x, xPoints}, {i, 1, 6}];
integraly = Atilde.di;
MatrixForm[N[integraly]]
```

Out[789]//MatrixForm=
$$\begin{pmatrix} 0. \\ 0.221403 \\ 0.491825 \\ 0.822119 \\ 1.22554 \\ 1.71828 \end{pmatrix}$$

So the max error is $2.65115 \times 10^{-6}$, the $p'_n(x)$ at $\{x_i\}$ is $\begin{pmatrix} 1.00008 \\ 1.22139 \\ 1.49183 \\ 1.82211 \\ 2.22556 \\ 2.71819 \end{pmatrix}$, and the $\int_0^{x_i} p_n(y)\,dy$ is

$$\begin{pmatrix} 0. \\ 0.221403 \\ 0.491825 \\ 0.822119 \\ 1.22554 \\ 1.71828 \end{pmatrix}.$$

# Problem 2

## To rewrite the construction:

Assume the function we're approximating with a Taylor series is f(x).

To prove the Taylor series around a point a is given by: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

Assume $p_n(x) = c_0 + c_1(x-a) + \ldots + c_n(x-a)^n$

In matrix language: $\begin{pmatrix} 1 & (x-a) & (x-a)^2 & \ldots & (x-a)^n \end{pmatrix} . \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (y)$

Take x = a in: $\begin{pmatrix} 1 & (a-a) & (a-a)^2 & \ldots & (a-a)^n \end{pmatrix} . \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (f(a))$

so $c_0 = f(a)$

$p_n'(x) = c_1 + 2c_2(x-a) + \ldots + nc_n(x-a)^{n-1}$

Suppose $f_n'(x) = d_0 + d_1(x-a) + \ldots + d_n(x-a)^n$

Then $d_n = 0$ and $d_i = (i+1)\,c_{i+1}$, 0≤i<n

Define $\boldsymbol{B} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$

$\mathbf{d} = \mathbf{Bc}$

$$(1\ (x-a)\ (x-a)^2\ \ldots\ (x-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (y')$$

take x = a in: $(1\ (a-a)\ (a-a)^2\ \ldots\ (a-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (f'(a))$

so $c_1 = f'(a)$

$$(1\ (x-a)\ (x-a)^2\ \ldots\ (x-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (y'')$$

take x = a in: $(1\ (a-a)\ (a-a)^2\ \ldots\ (a-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = (f''(a))$

so $c_2 = \frac{f''(a)}{2}$

Repeat this process, we have

$$(1\ (x-a)\ (x-a)^2\ \ldots\ (x-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^i \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = \left(y^{(i)}\right)$$

take x = a in: $(1\ (a-a)\ (a-a)^2\ \ldots\ (a-a)^n) \cdot \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}^i \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \ldots \\ c_n \end{pmatrix} = \left(f^{(i)}(a)\right)$

so $c_i = \frac{f^{(i)}(a)}{i!}$

So the Taylor series around a is proved to be f(x) = $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

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## To predict new points $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_m$:

Assume $y = p_n(x) = c_0 + c_1(x-a) + \ldots + c_n(x-a)^n$

Calculate coefficients $c_i = \frac{f^{(i)}(a)}{i!}$, 0≤i≤n, put in **c**

$$\tilde{\boldsymbol{A}} \in \mathbb{R}^{(m+1)\times(n+1)} = \begin{pmatrix} 1 & \tilde{x}_0 - a & \dots & (\tilde{x}_0 - a)^n \\ 1 & \tilde{x}_1 - a & \dots & (\tilde{x}_1 - a)^n \\ \dots & \dots & \dots & \dots \\ 1 & \tilde{x}_m - a & \dots & (\tilde{x}_m - a)^n \end{pmatrix}$$

Predict $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{A}}\mathbf{c}$

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## To calculate $f'_n(x)$ at $\{x_i\}$:

Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Assume $y = p_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$

Calculate coefficients $c_i = \frac{f^{(i)}(a)}{i!}$, 0≤i≤n, put in $\mathbf{c}$

$p_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots + nc_n(x-a)^{n-1}$

suppose $f_n'(x) = d_0 + d_1(x-a) + d_2(x-a)^2 + \dots + d_n(x-a)^n$

$d_n$=0 and $d_i$ = (i+1) $c_{i+1}$

Define $\tilde{\boldsymbol{B}} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$

$\mathbf{d} = \mathbf{B}\,\mathbf{c}$

Determine $p_n'$: $\mathbf{d} = \mathbf{Bc}$

$$\mathbf{A} = \begin{pmatrix} 1 & x_0 - a & \dots & (x_0 - a)^n \\ 1 & x_1 - a & \dots & (x_1 - a)^n \\ \dots & \dots & \dots & \dots \\ 1 & x_n - a & \dots & (x_n - a)^n \end{pmatrix}$$

Predict at $p_n'(x_i)$: $\mathbf{y}' = \mathbf{ABc}$

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## To calculate $\int_a^x p_n(z)\,dz$ at $\{x_i\}$:

Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Assume $y = p_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$

Calculate coefficients $c_i = \frac{f^{(i)}(a)}{i!}$, 0≤i≤n, put in $\mathbf{c}$

$\int p_n(x)\,dx = C + c_0 x + \frac{c_1}{2}\left(x^2 - 2ax\right) + \frac{c_2}{3}\left(x^3 - 3ax^2 + 3a^2 x\right) + \dots + \frac{c_n}{n+1}(x-a)^{n+1}$ (first three terms expanded before integral, otherwise the prediction is found to be abnormal)

Suppose $\int p_n(x)\,dx = d_0 + d_1(x-a) + d_2(x-a)^2 + \dots + d_{n+1}(x-a)^{n+1}$

$d_{i+1} = \frac{c_i}{i+1}$, 0≤i≤n

Define $\tilde{\boldsymbol{B}} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{n+1} \end{pmatrix}$

$\mathbf{d} = \mathbf{B}\,\mathbf{c}$ (without $d_0$)

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} x_0 & {x_0}^2 - 2\,a x_0 & {x_0}^3 - 3\,a {x_0}^2 + 3\,a^2 x_0 & (x_0 - a)^4 & \dots & (x_0 - a)^{n+1} \\ x_1 & {x_1}^2 - 2\,a x_1 & {x_1}^3 - 3\,a {x_1}^2 + 3\,a^2 x_1 & (x_1 - a)^4 & \dots & (x_1 - a)^{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & {x_n}^2 - 2\,a x_n & {x_n}^3 - 3\,a {x_n}^2 + 3\,a^2 x_n & (x_1 - a)^4 & \dots & (x_n - a)^{n+1} \end{pmatrix}$$

$$\int_a^x \mathbf{y}\, \mathbb{d}x = \tilde{\boldsymbol{A}}\, \mathbf{B}\, \mathbf{c}$$

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## Use Taylor series to solve Problem 1:

In[1080]:=

```
ClearAll["Global`*"]
a = 0.5;
```

$$\mathbf{c} = \begin{pmatrix} E^{0.5} \\ E^{0.5} \\ \frac{E^{0.5}}{2} \\ \frac{E^{0.5}}{3!} \\ \frac{E^{0.5}}{4!} \\ \frac{E^{0.5}}{5!} \end{pmatrix}$$

```
;(*As f^(i)(x) always equal to e^x*)

(*Define the 101 equispaced points for error calculation*)
xPointsFine = Range[0, 1, 1 / 100];
AFine = Table[If[i == 0, 1, x^i], {x, xPointsFine - a}, {i, 0, 5}];
(*Calculate the estimated Y values at the 101 points*)
yEstimatesFine = AFine.c;
(*Calculate the actual Y values and the error*)
yActualFine = Exp[xPointsFine];
maxError = Norm[N[yActualFine - yEstimatesFine], Infinity];
(*Output the maximum error*)
maxError
```

Out[1088]=

```
0.0000385043
```

In[1206]:=

```
xPoints = Range[0, 1, 1 / 5]; (*This creates the list {0,1/5,2/5,3/5,4/5,1}*)
(*Generate the Vandermonde matrix for interpolation points*)
A = Table[If[i == 0, 1, x^i], {x, xPoints - a}, {i, 0, 5}];
(*matrix B for the original polynomial*)
B = DiagonalMatrix[Range[1, 5], 1, 6];
(*Coefficients of the derivative of the polynomial*)
d = B.c;
(*Evaluate the derivative polynomial at the original xPoints*)
y' = A.d;
MatrixForm[N[y']]
```

Out[1211]//MatrixForm=

$$\begin{pmatrix} 1.0004 \\ 1.22143 \\ 1.49182 \\ 1.82212 \\ 2.22551 \\ 2.71781 \end{pmatrix}$$

In[1190]:=

```
(*matrix B for the original polynomial*)
Bi = DiagonalMatrix[1 / Range[1, 6]];
(*Coefficients of the intergral of the polynomial*)
di = Bi.c;
(*Evaluate the intergral polynomial at the original xPoints*)
Atilde = Table[
   If[i == 1, x, If[i == 2, x^2 - 2 * a * x, If[i == 3, x^3 - 3 * a * x^2 + 3 * a^2 * x, (x - a)^i]]],
   {x, xPoints}, {i, 1, 6}];
integraly = Atilde.di;
MatrixForm[N[integraly]]
```

Out[1194]//MatrixForm=

$$\begin{pmatrix} 0.00389997 \\ 0.2253 \\ 0.495722 \\ 0.826016 \\ 1.22944 \\ 1.72218 \end{pmatrix}$$

In[1216]:=
```
f[x_] := Exp[x];
MatrixForm[Transpose[N[f[xPoints]]]]
MatrixForm[Transpose[N[f[xPoints] - 1]]]
```

Out[1217]//MatrixForm=

$$\begin{pmatrix} 1. \\ 1.2214 \\ 1.49182 \\ 1.82212 \\ 2.22554 \\ 2.71828 \end{pmatrix}$$

Out[1218]//MatrixForm=

$$\begin{pmatrix} 0. \\ 0.221403 \\ 0.491825 \\ 0.822119 \\ 1.22554 \\ 1.71828 \end{pmatrix}$$

So the max error is 0.0000385043, the $p'_n(x)$ at $\{x_i\}$ is $\begin{pmatrix} 1.0004 \\ 1.22143 \\ 1.49182 \\ 1.82212 \\ 2.22551 \\ 2.71781 \end{pmatrix}$,

and the $\int_0^{x_i} p_n(y)\,dy$ is $\begin{pmatrix} 0.00389997 \\ 0.2253 \\ 0.495722 \\ 0.826016 \\ 1.22944 \\ 1.72218 \end{pmatrix}$.

Previously the max error is $2.651152 \times 10^{-7}$,

the $p'_n(x)$ at $\{x_i\}$ is $\begin{pmatrix} 1.00008 \\ 1.22139 \\ 1.49183 \\ 1.82211 \\ 2.22556 \\ 2.71819 \end{pmatrix}$, and the $\int_0^{x_i} p_n(y)\,dy$ is $\begin{pmatrix} 0. \\ 0.221403 \\ 0.491825 \\ 0.822119 \\ 1.22554 \\ 1.71828 \end{pmatrix}$.

The true solution shall be $\begin{pmatrix} 1. \\ 1.2214 \\ 1.49182 \\ 1.82212 \\ 2.22554 \\ 2.71828 \end{pmatrix}$ and $\begin{pmatrix} 0. \\ 0.221403 \\ 0.491825 \\ 0.822119 \\ 1.22554 \\ 1.71828 \end{pmatrix}$.

Though both results seems making approximation in some way, the results from polynomial are obviously

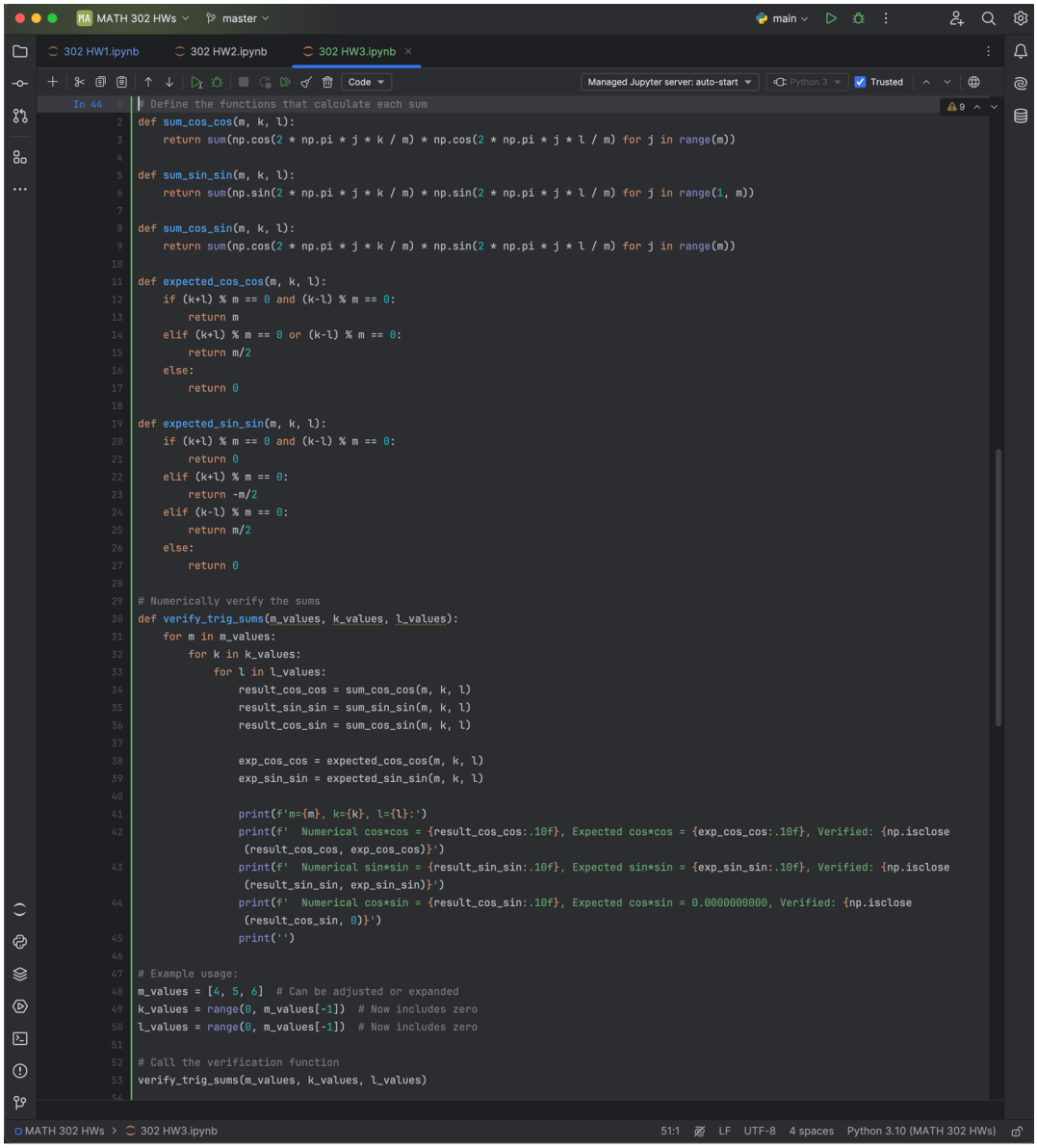
more accurate than the result from Taylor series, which conceed with the intuation since the Taylor is a approximation works for a small Neighborhood while the interpolation is made to work on the inteval.

# Problem 3

```python
import numpy as np

def sum_of_sines(m, k):
    """Calculate the sum of sines for a given m and k."""
    return sum(np.sin(2 * np.pi * j * k / m) for j in range(1, m))

def sum_of_cosines(m, k):
    """Calculate the sum of cosines for a given m and k."""
    return sum(np.cos(2 * np.pi * j * k / m) for j in range(m))

def verify_sum(actual, expected, tolerance=1e-10):
    """Check if the actual sum is within the tolerance range of the expected result."""
    return abs(actual - expected) < tolerance

def verify_sums(m_values, k_values):
    """Verify the sums for a range of m and k values and print whether the verification is successful."""
    for m in m_values:
        for k in k_values:
            sine_sum = sum_of_sines(m, k)
            cosine_sum = sum_of_cosines(m, k)

            # Expected results based on the problem statement
            expected_sine_sum = 0
            expected_cosine_sum = m if k % m == 0 else 0

            sine_sum = abs(sine_sum) if verify_sum(sine_sum, expected_sine_sum) else sine_sum
            cosine_sum = abs(cosine_sum) if verify_sum(cosine_sum, expected_cosine_sum) else cosine_sum

            # Check if the sums are within the tolerance of the expected result
            sine_verified = verify_sum(sine_sum, expected_sine_sum)
            cosine_verified = verify_sum(cosine_sum, expected_cosine_sum)

            # Formatting the output
            sine_result = "verified" if sine_verified else "not verified"
            cosine_result = "verified" if cosine_verified else "not verified"

            print(f"m={m}, k={k}: Sum of sines = {sine_sum:.10f} ({sine_result}), "
                  f"Sum of cosines = {cosine_sum:.10f} ({cosine_result})")

# Expanded range of m values
m_values = [4, 5, 6, 10,]  # Adding a larger m value for broader coverage

# Expanded and more comprehensive range of k values
k_values = range(-10, 11)  # Covering negative k values, zero, and positive k values up to 10

# Call the verification function
verify_sums(m_values, k_values)
```

```
m = 4, k = -10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = -9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = -8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 4.0000000000 (verified)
m = 4, k = -7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = -6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = -5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
```

m = 4, k = −4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 4.0000000000 (verified)
m = 4, k = −3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = −2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = −1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 0 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 4.0000000000 (verified)
m = 4, k = 1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 4.0000000000 (verified)
m = 4, k = 5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 4.0000000000 (verified)
m = 4, k = 9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 4, k = 10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 5.0000000000 (verified)
m = 5, k = −9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 5.0000000000 (verified)
m = 5, k = −4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = −1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 0 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 5.0000000000 (verified)
m = 5, k = 1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 5.0000000000 (verified)
m = 5, k = 6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 5, k = 10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 5.0000000000 (verified)
m = 6, k = −10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 6.0000000000 (verified)
m = 6, k = −5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = −1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 0 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 6.0000000000 (verified)
m = 6, k = 1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 6.0000000000 (verified)
m = 6, k = 7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 6, k = 10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 10.0000000000 (verified)
m = 10, k = −9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = −1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 0 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 10.0000000000 (verified)
m = 10, k = 1 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 2 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 3 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 4 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 5 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 6 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 7 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 8 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 9 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 0.0000000000 (verified)
m = 10, k = 10 : Sum of sines = 0.0000000000 (verified), Sum of cosines = 10.0000000000 (verified)




```
m = 4, k = 0, l = 0 : Numerical cos * cos = 4.0000000000, Expected cos * cos = 4.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 0, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 0, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 0, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 0, l = 4 : Numerical cos * cos = 4.0000000000, Expected cos * cos = 4.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 0, l = 5 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 1, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 1, l = 1 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = 2.0000000000, Expected sin * sin = 2.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True
```

m = 4, k = 1, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 1, l = 3 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = -2.0000000000, Expected sin * sin = -2.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 1, l = 4 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 1, l = 5 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = 2.0000000000, Expected sin * sin = 2.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 1 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 2 : Numerical cos * cos = 4.0000000000, Expected cos * cos = 4.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 3 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 2, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 1 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = -2.0000000000, Expected sin * sin = -2.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 2 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 3 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = 2.0000000000, Expected sin * sin = 2.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 3, l = 5 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = -2.0000000000, Expected sin * sin = -2.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 0 : Numerical cos * cos = 4.0000000000, Expected cos * cos = 4.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 4 : Numerical cos * cos = 4.0000000000, Expected cos * cos = 4.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 4, l = 5 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 1 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = 2.0000000000, Expected sin * sin = 2.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 3 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = -2.0000000000, Expected sin * sin = -2.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 4 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 4, k = 5, l = 5 : Numerical cos * cos = 2.0000000000, Expected cos * cos = 2.0000000000, Verified : True
Numerical sin * sin = 2.0000000000, Expected sin * sin = 2.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 0 : Numerical cos * cos = 5.0000000000, Expected cos * cos = 5.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 0, l = 5 : Numerical cos * cos = 5.0000000000, Expected cos * cos = 5.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 1 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = 2.5000000000, Expected sin * sin = 2.5000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 4 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = -2.5000000000, Expected sin * sin = -2.5000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 1, l = 5 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 1 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 2 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = 2.5000000000, Expected sin * sin = 2.5000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 3 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = -2.5000000000, Expected sin * sin = -2.5000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 4 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 2, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 1 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 2 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = -2.5000000000, Expected sin * sin = -2.5000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 3 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = 2.5000000000, Expected sin * sin = 2.5000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 4 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 3, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 1 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = -2.5000000000, Expected sin * sin = -2.5000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 2 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 3 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 4 : Numerical cos * cos = 2.5000000000, Expected cos * cos = 2.5000000000, Verified : True
Numerical sin * sin = 2.5000000000, Expected sin * sin = 2.5000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 4, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 0 : Numerical cos * cos = 5.0000000000, Expected cos * cos = 5.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 5, k = 5, l = 5 : Numerical cos * cos = 5.0000000000, Expected cos * cos = 5.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 0 : Numerical cos * cos = 6.0000000000, Expected cos * cos = 6.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 4 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 0, l = 5 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 1 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = 3.0000000000, Expected sin * sin = 3.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 2 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 3 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 1, l = 5 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = -3.0000000000, Expected sin * sin = -3.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 2 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = 3.0000000000, Expected sin * sin = 3.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 3 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 4 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = -3.0000000000, Expected sin * sin = -3.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 2, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 0 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 1 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 2 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 3 : Numerical cos * cos = 6.0000000000, Expected cos * cos = 6.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 3, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

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m = 6, k = 4, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 4, l = 1 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 4, l = 2 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = -3.0000000000, Expected sin * sin = -3.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 4, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 4, l = 4 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = 3.0000000000, Expected sin * sin = 3.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 4, l = 5 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 0 : Numerical cos * cos = -0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 1 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = -3.0000000000, Expected sin * sin = -3.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 2 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = -0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 3 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = 0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 4 : Numerical cos * cos = 0.0000000000, Expected cos * cos = 0.0000000000, Verified : True
Numerical sin * sin = 0.0000000000, Expected sin * sin = 0.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True

m = 6, k = 5, l = 5 : Numerical cos * cos = 3.0000000000, Expected cos * cos = 3.0000000000, Verified : True
Numerical sin * sin = 3.0000000000, Expected sin * sin = 3.0000000000, Verified : True
Numerical cos * sin = -0.0000000000, Expected cos * sin = 0.0000000000, Verified : True
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