Interpolation Theory

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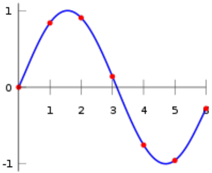
Introduction

- Suppose we want to study a very complicated function $f(x)$
- Don't know how to deal with $f(x)$
- $p_n(x)$ - polynomial
- If $f(x) \approx p_n(x)$, then we only need to focus on $p_n(x)$
- Assume data are generated according to $y_i = p_n(x_i)$
- **Question**: How to recover $p_n(x)$?
- **Def**: **Interpolation** is the selection of a function $p(x)$ from a given class of functions



Fitting problem

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, where x_i distinct
- Suppose y_i are generated by $p(x_i)$
- E.g., Suppose $p_1(x) = ax + b$ and we observe (x_0, y_0), (x_1, y_1)

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- $p_1(x) = \frac{y_1 - y_0}{x_1 - x_0} x + y_0 - x_0 \frac{y_1 - y_0}{x_1 - x_0}$
- **Question**: What if we have $n + 1$ points?

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- $p_1(x) = \frac{y_1 - y_0}{x_1 - x_0} x + y_0 - x_0 \frac{y_1 - y_0}{x_1 - x_0}$
- **Question**: What if we have $n+1$ points?
- Assume $p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Form the linear system:

$$
\begin{aligned}
c_0 + c_1 x_0 + c_2 x_0^2 + \cdots + c_n x_0^n &= y_0, \\
&\vdots \\
c_0 + c_1 x_n + c_2 x_n^2 + \cdots + c_n x_n^n &= y_n.
\end{aligned}
$$

- **Question**: What is the matrix form?

Vandermonde matrix

- $$\mathbf{A} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$$
- $$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$
- **Def**: $\mathbf{A}$ is called a **Vandermonde matrix**.
- $\mathbf{Ac} = \mathbf{y}$
- A polynomial is determined by its coefficients $\mathbf{c}$
- **Question**: Existence? Uniqueness?

Analysis

- **THM**: Given $n+1$ distinct x_i, there is a unique polynomial $p(x)$ of degree $\leq n$ that interpolates y_i at x_i.
- Proof from the linear algebra
  - $\det(\mathbf{X}) = \prod_{0 \leq j < i \leq n}(x_i - x_j)$ (HW)
  - $x_i \neq x_j$, so $\det(\mathbf{X}) \neq 0$
  - By linear algebra, the solutions exists and is unique
- Another proof
  - Suppose column vectors of $\mathbf{A}$ are linearly dependent
  - There exists a nontrivial $\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{0}$
  - There are $n+1$ roots
  - The only possibility is the zero polynomial
  - Matrix invertible, and therefore existence and uniqueness are guaranteed.

Polynomial interpolation theory

Application

Trigonometric interpolation

Error analysis and Runge phenomenon

Introduction

- Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$
- Suppose $\{y_i\}$ are generated by a polynomial $p(x)$
- Some applications:
  - How to predict new points $\widetilde{x}_0, \widetilde{x}_1, \dots, \widetilde{x}_m$
  - How to know derivatives at $\{x_i\}$, i.e., $f'(x_i)$
  - How to know integrals at $\{x_i\}$, i.e., $\int_a^{x_i} f(x)\, dx$
  - $\vdots$

Prediction

- Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Construct $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{bmatrix}$
- Solve $\mathbf{Ac} = \mathbf{y}$
- **Question**: How to predict new points $\widetilde{x}_0, \widetilde{x}_1, \dots, \widetilde{x}_m$?

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- Solve $\mathbf{Ac} = \mathbf{y}$
- **Question**: How to predict new points $\widetilde{x}_0, \widetilde{x}_1, \ldots, \widetilde{x}_m$?
- $\widetilde{\mathbf{A}} \in \mathbb{R}^{(m+1)\times(n+1)} = \begin{bmatrix} 1 & \widetilde{x}_0 & \widetilde{x}_0^2 & \ldots & \widetilde{x}_0^n \\ 1 & \widetilde{x}_1 & \widetilde{x}_1^2 & \ldots & \widetilde{x}_1^n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \widetilde{x}_m & \widetilde{x}_m^2 & \ldots & \widetilde{x}_m^n \end{bmatrix}$
- Predict $\widetilde{\mathbf{y}} = \widetilde{\mathbf{A}}\mathbf{c}$

Numerical differentiation

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Calculate coefficients $\mathbf{A}\mathbf{c} = \mathbf{y}$
- **Question**: What is $p_n'(x)$?

Numerical differentiation

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Calculate coefficients $\mathbf{A}\mathbf{c} = \mathbf{y}$
- **Question**: What is $p_n'(x)$?
- $p_n'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1}$
- Suppose $p_n'(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n$
- $d_n = 0$ and $d_i = (i+1)c_{i+1}, 0 \le i < n$
- Define $\mathbf{B} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & n \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$
- $\mathbf{d} = \mathbf{B}\mathbf{c}$

Numerical differentiation - continued

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- **Question**: How to calculate $p_n'(x)$ at $\{x_i\}$?

Numerical differentiation - continued

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- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- **Question**: How to calculate $p_n'(x)$ at $\{x_i\}$?
- Determine p_n: $\mathbf{c} = \mathbf{A}^{-1}\mathbf{y}$
- Determine p_n': $\mathbf{d} = \mathbf{B}\mathbf{A}^{-1}\mathbf{y}$
- Predict at $p_n'(x_i)$: $\mathbf{y}' = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}\mathbf{y}$
- Differentiation matrix: $\mathbf{D} \in \mathbb{R}^{(n+1)\times(n+1)} = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}$
- Rank$(\mathbf{D}) =$

Numerical differentiation - continued

- Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$
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- Predict at $p_n'(x_i)$: $\mathbf{y}' = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}\mathbf{y}$
- Differentiation matrix: $\mathbf{D} \in \mathbb{R}^{(n+1)\times(n+1)} = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}$
- Rank$(\mathbf{D}) = n$.
- **Remark**: Matrix representation useful in differential equations, e.g. $y''(x) = f(x)$

Numerical integration

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Calculate coefficients $\mathbf{Ac} = \mathbf{y}$
- **Question**: What is $\int p_n(x)\ dx$?

Numerical integration

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- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Calculate coefficients $\mathbf{A}\mathbf{c} = \mathbf{y}$
- **Question**: What is $\int p_n(x)\ dx$?
- $\int p_n(x)\ dx = C + c_0 x + \frac{c_1}{2} x^2 + \cdots + \frac{c_n}{n+1} x^{n+1}$
- Suppose $\int p_n(x)\ dx = d_0 + d_1 x + d_2 x^2 + \cdots + d_{n+1} x^{n+1}$
- $d_{i+1} = \frac{c_i}{i+1}, 0 \le i \le n$
- Define $\mathbf{B} \in \mathbb{R}^{(n+1)\times(n+1)} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{n+1} \end{bmatrix}$
- $\mathbf{d} = \mathbf{B}\mathbf{c}$ (without d_0)

Numerical integration

- Observe $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- **Question**: How to calculate $\int_0^x p_n(z)\ dz$ at $\{x_i\}$?

Numerical integration

- Observe $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$
- Assume $y = p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- **Question**: How to calculate $\int_0^x p_n(z)\ dz$ at $\{x_i\}$?
- d_0 is no longer needed
- Determine p_n: $\mathbf{c} = \mathbf{A}^{-1}\mathbf{y}$
- Determine $\int p_n(x)\ dx$: $\mathbf{d} = \mathbf{B}\mathbf{A}^{-1}\mathbf{y}$
- Predict at $\int_0^{x_i} p_n(z)\ dz$:
- $\widetilde{\mathbf{A}} = \begin{bmatrix} x_0 & x_0^2 & \dots & x_0^{n+1} \\ x_1 & x_1^2 & \dots & x_1^{n+1} \\ \vdots & \ddots & \ddots & \vdots \\ x_n & x_n^2 & \dots & x_n^{n+1} \end{bmatrix}$
- $\int_0^x \mathbf{y}\ dx = \widetilde{\mathbf{A}}\mathbf{B}\mathbf{A}^{-1}\mathbf{y}$
- Integration matrix: $\mathbf{S} \in \mathbb{R}^{(n+1)\times(n+1)} = \widetilde{\mathbf{A}}\mathbf{B}\mathbf{A}^{-1}$

Change of variables

- Suppose fix $\{x_i\}_{i=0}^n$, where $x_i = \frac{i}{n}$
- Calculate interpolation matrix $\mathbf{A}$, differentiation matrix $\mathbf{D}$, and integration matrix $\mathbf{S}$
- Suppose now for $y = p_n(x)$ with $x \in [a, b]$
- **Question**: Can we avoid repeated calculation?

Change of variables

- Suppose fix $\{x_i\}_{i=0}^n$, where $x_i = \frac{i}{n}$
- Calculate interpolation matrix $\mathbf{A}$, differentiation matrix $\mathbf{D}$, and integration matrix $\mathbf{S}$
- Suppose now for $y = p_n(x)$ with $x \in [a, b]$
- **Question**: Can we avoid repeated calculation?
- We have $\widetilde{x} = a + (b - a)x$
- Change of variables: $p_n(\widetilde{x}) = p_n(a + (b - a)x) = q_n(x)$
- E.g., $p_n(\widetilde{x}) = \widetilde{x}^2$, $\widetilde{x} \in [0, 2]$; $q_n(x) = 4x^2$, $x \in [0, 1]$
- E.g., $p_n(\widetilde{x}) = \widetilde{x}^2$, $\widetilde{x} \in [1, 3]$: $q_n(x) = (1 + 2x)^2$, $x \in [0, 1]$
- Sufficient to just learn q_n

Prediction

- Observe $(\widetilde{x}_0, y_0), (\widetilde{x}_1, y_1), \ldots, (\widetilde{x}_n, y_n)$ on $\widetilde{x} \in [a, b]$
- $\widetilde{x} = a + (b - a)x$, $x \in [0, 1]$
- $p_n(\widetilde{x}) = q_n(x)$
- Assume $q_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Solve $\mathbf{Ac} = \mathbf{y}$
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- Assume $q_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Solve $\mathbf{A}\mathbf{c} = \mathbf{y}$
- **Question**: How to predict new points $\widehat{x}_0, \widehat{x}_1, \dots, \widehat{x}_m$?
- $\widetilde{\mathbf{A}} = \begin{bmatrix} 1 & \frac{\widehat{x}_0 - a}{b-a} & \left(\frac{\widehat{x}_0 - a}{b-a}\right)^2 & \dots & \left(\frac{\widehat{x}_0 - a}{b-a}\right)^n \\ 1 & \frac{\widehat{x}_1 - a}{b-a} & \left(\frac{\widehat{x}_1 - a}{b-a}\right)^2 & \dots & \left(\frac{\widehat{x}_1 - a}{b-a}\right)^n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \frac{\widehat{x}_m - a}{b-a} & \left(\frac{\widehat{x}_m - a}{b-a}\right)^2 & \dots & \left(\frac{\widehat{x}_m - a}{b-a}\right)^n \end{bmatrix}$
- Predict $\widetilde{\mathbf{y}} = \widetilde{\mathbf{A}}\mathbf{c}$

Differentiation

- Observe $(\widetilde{x}_0, y_0), (\widetilde{x}_1, y_1), \ldots, (\widetilde{x}_n, y_n)$ on $\widetilde{x} \in [a, b]$
- $\widetilde{x} = a + (b - a)x$, $x \in [0, 1]$
- $p_n(\widetilde{x}) = q_n(x)$
- Assume $q_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$
- Construct the differentiation matrix $\mathbf{D}$ on $x \in [0, 1]$
- **Question**: How to calculate $p_n'(\widetilde{x}_i)$?

Differentiation

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- Chain rule:

$$\begin{aligned}\frac{d}{d\widetilde{x}} p_n(\widetilde{x}) = \frac{d}{d\widetilde{x}} q_n(x) = \frac{d}{d\widetilde{x}} q_n\left(\frac{\widetilde{x} - a}{b - a}\right) \\ = \frac{1}{b-a} q_n'(x)\end{aligned}$$

- $\mathbf{y}' = \frac{1}{b-a}\mathbf{D}\mathbf{y}$

Integration

- Observe $(\widetilde{x}_0, y_0), (\widetilde{x}_1, y_1), \ldots, (\widetilde{x}_n, y_n)$ on $\widetilde{x} \in [a, b]$
- $\widetilde{x} = a + (b - a)x$, $x \in [0, 1]$
- $p_n(\widetilde{x}) = q_n(x)$
- Construct the integration matrix $\mathbf{S}$ on $x \in [0, 1]$
- **Question**: How to calculate $\int_a^{\widetilde{x}} p_n(z)\, dz$?

Integration

- Observe $(\widetilde{x}_0, y_0), (\widetilde{x}_1, y_1), \ldots, (\widetilde{x}_n, y_n)$ on $\widetilde{x} \in [a, b]$
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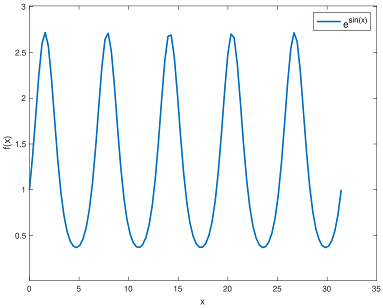
- $p_n(\widetilde{x}) = q_n(x)$
- Construct the integration matrix $\mathbf{S}$ on $x \in [0, 1]$
- **Question**: How to calculate $\int_a^{\widetilde{x}} p_n(z)\ dz$?
- Integration by substitution:

$$\begin{aligned}\int_a^{\widetilde{x}} p_n(z)\ dz &= (b-a)\int_0^x p_n(a + (b-a)z)\ dz \\ &= (b-a)\int_0^x q_n(z)\ dz.\end{aligned}$$

- $\int \mathbf{y}\ d\widetilde{x} = (b-a)\mathbf{S}\mathbf{y}$

Oscillatory function interpolation

- Some functions are oscillatory, e.g., signals



- For such functions, a polynomial is not necessarily the best choice
- **Question**: What might be a potential better choice?

Fourier series

- For simplicity, assumes $x \in [-\pi, \pi]$
- **Def**: **Fourier series**

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

- **Question**: What are a_k and b_k, theoretically?

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- **Question**: $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = ?$
- $2\cos(\theta)\cos(\eta) = \cos(\theta - \eta) + \cos(\theta + \eta)$

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- **Question**: $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = ?$
- $2\cos(\theta)\cos(\eta) = \cos(\theta - \eta) + \cos(\theta + \eta)$
- $j \neq k$: $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((j+k)x) + \cos((j-k)x)\ dx = \frac{1}{2}\left(\frac{\sin((j+k)x)}{j+k} + \frac{\sin((j-k)x)}{j-k}\right)\Bigg|_{-\pi}^{\pi} = 0$
- $j = k$: $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2kx)\ dx = \pi + \frac{\sin(2kx)}{4k}\Bigg|_{-\pi}^{\pi} = \pi$
- $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = \pi \cdot \delta_{j,k}$, where $\delta_{j,k}$ **Kronecker delta**

Fourier coefficients

- Integral identities:
  - $\int_{-\pi}^{\pi} \sin(jx)\sin(kx)\ dx = \pi\delta_{j,k}$
  - $\int_{-\pi}^{\pi} \cos(jx)\cos(kx)\ dx = \pi\delta_{j,k}$
  - $\int_{-\pi}^{\pi} \sin(jx)\cos(kx)\ dx = 0$
  - $\int_{-\pi}^{\pi} \sin(kx)\ dx = 0$
  - $\int_{-\pi}^{\pi} \cos(kx)\ dx = 0$
- $f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$.
- **Question**: What are a_k, b_k?

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- $f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx)$.
- **Question**: What are a_k, b_k?
- Taking integrals
  - $\int_{-\pi}^{\pi} f(x)\ dx = \pi a_0$
  - $\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx = a_k\int_{-\pi}^{\pi}\cos(kx)\cos(kx)\ dx = a_k\pi$
  - $\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx = b_k\int_{-\pi}^{\pi}\sin(kx)\sin(kx)\ dx = b_k\pi$
- Fourier coefficients:
  - $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$
  - $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx$
  - $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx$

Discrete integral

- Fourier coefficients:
  - $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx$
  - $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\ dx$
  - $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\ dx$
- $x_i = -\pi + i\Delta x$, $\Delta x = \frac{2\pi}{n}$
- Trapezoidal's rule: $\int_{x_i}^{x_{i+1}} f(x)\ dx \approx \frac{\Delta x}{2}(f(x_i) + f(x_{i+1}))$
- Periodic assumption: $f(x_0) = f(x_n)$
- $a_0 \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_i)$
- $a_k \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i) \cos(kx_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_i) \cos(kx_i)$
- $b_k \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i) \sin(kx_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_j) \sin(kx_i)$
- **Question**: How good are these integral approximations?

Discrete integral

- Fourier coefficients:
  - $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\ dx$
  - $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\ dx$
  - $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\ dx$
- $x_i = -\pi + i\Delta x$, $\Delta x = \frac{2\pi}{n}$
- Trapezoidal's rule: $\int_{x_i}^{x_{i+1}} f(x)\ dx \approx \frac{\Delta x}{2}(f(x_i) + f(x_{i+1}))$
- Periodic assumption: $f(x_0) = f(x_n)$
- $a_0 \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_i)$
- $a_k \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i) \cos(kx_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_i) \cos(kx_i)$
- $b_k \approx \frac{1}{\pi} \sum_{i=0}^{n-1} f(x_i) \sin(kx_i)\Delta x = \frac{2}{n} \sum_{i=0}^{n-1} f(x_j) \sin(kx_i)$
- **Question**: How good are these integral approximations?
- No errors in the interpolation!

Example

- E.g., $f(x) = \frac{a_0}{2} + a_1\cos(x) + b_1\sin(x)$
- $x_0 = -\pi, x_1 = -\frac{\pi}{3}, x_2 = \frac{\pi}{3}, x_3 = \pi,\ n = 3$
- Coefficients:
  - $a_0 = \frac{2}{3}\left(f(-\pi) + f\left(-\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)\right)$
  - $a_1 = \frac{2}{3}\left(f(-\pi)\cos(-\pi) + f\left(-\frac{\pi}{3}\right)\cos\left(-\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\right)$
  - $b_1 = \frac{2}{3}\left(f(-\pi)\sin(-\pi) + f\left(-\frac{\pi}{3}\right)\sin\left(-\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\right)$
- $\cos(-\pi) = -1, \cos\left(\pm\frac{\pi}{3}\right) = \frac{1}{2}$
- $\sin(-\pi) = 0, \sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}$
- Coefficients:
  - $a_0 = \frac{2}{3}\left(f(-\pi) + f\left(-\frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)\right)$
  - $a_1 = \frac{2}{3}\left(-f(-\pi) + \frac{1}{2}f\left(-\frac{\pi}{3}\right) + \frac{1}{2}f\left(\frac{\pi}{3}\right)\right)$
  - $b_1 = \frac{2}{3}\left(-\frac{\sqrt{3}}{2}f\left(-\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}f\left(\frac{\pi}{3}\right)\right)$
- **Question**: How to verify?

Example - continued

- $f(x) = \frac{a_0}{2} + a_1 \cos(x) + a_2 \sin(x)$
- Suppose $f(x) = 1$
  - $a_0 = \frac{2}{3}(1+1+1) = 2$
  - $a_1 = \frac{2}{3}\left(-1 + \frac{1}{2} + \frac{1}{2}\right) = 0$
  - $b_1 = \frac{2}{3}\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = 0$
- Suppose $f(x) = \cos(x)$
  - $a_0 = \frac{2}{3}\left(\cos(-\pi) + \cos\left(-\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right) = 0$
  - $a_1 = \frac{2}{3}\left(-\cos(-\pi) + \frac{1}{2}\cos\left(-\frac{\pi}{3}\right) + \frac{1}{2}\cos\left(\frac{\pi}{3}\right)\right) = 1$
  - $b_1 = \frac{2}{3}\left(-\frac{\sqrt{3}}{2}\cos\left(-\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{\pi}{3}\right)\right) = 0$
- Suppose $f(x) = \sin(x)$
  - $a_0 = \frac{2}{3}\left(\sin(-\pi) + \sin\left(-\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right) = 0$
  - $a_1 = \frac{2}{3}\left(-\sin(-\pi) + \frac{1}{2}\sin\left(-\frac{\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right) = 0$
  - $b_1 = \frac{2}{3}\left(-\frac{\sqrt{3}}{2}\sin\left(-\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{3}\right)\right) = 1$
- Exactly recovered!
- **Question**: Why different from the polynomial interpolation?

Connection with polynomial fitting

- Fourier series: $f(x) = \frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x)$
- $x_0 = -\pi, x_1 = -\frac{\pi}{3}, x_2 = \frac{\pi}{3}, x_3 = \pi$, $n = 3$
- Form the linear system $\mathbf{Ac} = \mathbf{f}$

$$\begin{bmatrix} \frac{1}{2} & \cos(x_0) & \sin(x_0) \\ \frac{1}{2} & \cos(x_1) & \sin(x_1) \\ \frac{1}{2} & \cos(x_2) & \sin(x_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}.$$

Connection with polynomial fitting

- Fourier series: $f(x) = \frac{a_0}{2} + a_1\cos(x) + b_1\sin(x)$
- $x_0 = -\pi, x_1 = -\frac{\pi}{3}, x_2 = \frac{\pi}{3}, x_3 = \pi,\ n = 3$
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$$\begin{bmatrix} \frac{1}{2} & \cos(x_0) & \sin(x_0) \\ \frac{1}{2} & \cos(x_1) & \sin(x_1) \\ \frac{1}{2} & \cos(x_2) & \sin(x_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}.$$

- Orthogonality:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \frac{n}{4} & 0 & 0 \\ 0 & \frac{n}{2} & 0 \\ 0 & 0 & \frac{n}{2} \end{bmatrix}$$

- Solves the linear system

$$\mathbf{c} = \begin{bmatrix} \frac{4}{n} & 0 & 0 \\ 0 & \frac{2}{n} & 0 \\ 0 & 0 & \frac{2}{n} \end{bmatrix} \mathbf{A}^T\mathbf{f}$$

- **Remark**: RHS is discrete cosine/sine transformation we did!

General form

- $f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$
- $x_i = -\pi + \frac{2\pi}{2n} i$, $N = 2n + 2$ points
- Linear system form: $\mathbf{Ac} = \mathbf{f}$
- $\mathbf{A} =$

$$\begin{bmatrix} \frac{1}{2} & \cos(x_0) & \dots & \cos(nx_0) & \sin(x_0) & \dots & \sin(nx_0) \\ \frac{1}{2} & \cos(x_1) & \dots & \cos(nx_1) & \sin(x_1) & \dots & \sin(nx_1) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{2} & \cos(x_{N-1}) & \dots & \cos(nx_{N-1}) & \sin(x_{N-1}) & \dots & \sin(nx_{N-1}) \end{bmatrix}$$

- $\mathbf{c} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$ 2n+1 , $\mathbf{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \end{bmatrix}$

Integral form

- $\widetilde{\mathbf{A}} =$
$$\begin{bmatrix} 1 & \cos(x_0) & \dots & \cos(nx_0) & \sin(x_0) & \dots & \sin(nx_0) \\ 1 & \cos(x_1) & \dots & \cos(nx_1) & \sin(x_1) & \dots & \sin(nx_1) \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & \cos(x_{N-\cancel{1}2}) & \dots & \cos(nx_{N-\cancel{1}2}) & \sin(x_{N-\cancel{1}2}) & \dots & \sin(nx_{N-\cancel{1}2}) \end{bmatrix}$$
- Discrete cosine/sine transformation:

$$\mathbf{c} = \frac{2}{N-1}\widetilde{\mathbf{A}}^T\mathbf{f}.$$

Polynomial interpolation theory

Application

Trigonometric interpolation

Error analysis and Runge phenomenon

Taylor series

- A simplified analysis
- $f(x) = p_{n+1}(x)$ and assume $f(x) \approx p_n(x)$ for $x \in [a, b]$
- Taylor series around x_0: $p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$
- **Question**: Error $e_n(x) = |f(x) - p_n(x)|$?

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- **Question**: Error $e_n(x) = |f(x) - p_n(x)|$?
- Match one more term the fit is exact
- $e_n(x) = \left| \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} \right|$
- General theorem: $e_n = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right|$, $\exists \xi$ between x_0 and x

Polynomial interpolation

- A simplified analysis
- $f(x) = p_{n+1}(x)$ and assume $f(x) \approx p_n(x)$ for $x \in [a, b]$
- Interpolation: $p_n(x_i) = p_{n+1}(x_i)$ for $0 \leq i \leq n$
- **Question**: Error $e_n(x) = |f(x) - p_n(x)|$?

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- x_i are roots of e_n
- $e_n = |c \prod_{i=0}^{n} (x - x_i)|$

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- x_i are roots of e_n
- $e_n = |c \prod_{i=0}^{n}(x - x_i)|$
- $c = \frac{f^{(n+1)}(x)}{(n+1)!}$
- General theorem: $e_n = \left| \frac{(x-x_0)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi) \right|$, $\exists \xi$ between $[a, b]$

Example

- Error formula: $f(x) - p_n(x) = \frac{(x-x_0)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi)$
- Suppose $f(x) = a_0 + a_1 x + a_2 x^2$ for $x \in [0, 1]$
- Use $p_1(x) = b_0 + b_1 x$ to fit at $[0, 1]$
- Error: $e_1(x) = |f(x) - p_1(x)|$
- 0,1 are two roots
- $R_1(x) = f(x) - p_1(x) = cx(x-1)$
- **Question**: What is c?

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- **Question**: What is c?
- $2c = R_1''(x) = f''(x) = 2a_2$
- $R_1(x) = a_2 x(x-1)$

Comparison

- Taylor series: $e_n = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1} \right|$
- Interpolation: $e_n = \left| \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)\dots(x-x_n) \right|$
- Consider the largest error $\|e_n\|_\infty = \max_{a \le x \le b} e_n$
- We have no control of $f^{(n+1)}$ and ξ, η
- Compare $\left|(x-c)^{n+1}\right|$ and $|(x-x_0)\dots(x-x_n)|$
- E.g., $[a,b] = [-1,1]$, $n=2$, $|(x-c)^3|$ vs $|(x-1)x(x+1)|$

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- Taylor: let $c=0$, $\|(x-c)^3\|_\infty = 1$
- Interpolation: $\frac{d}{dx}(x-1)x(x+1) = 3x^2-1$ with roots $\pm\frac{1}{\sqrt{3}}$
- Interpolation: $\|(x-1)x(x+1)\|_\infty = \frac{2}{3\sqrt{3}} \approx 0.4$
- Conclusion: Polynomial interpolation wins!

Runge Phenomenon

- So far the interpolation method (by polynomial) works for many cases, e.g., $e^x, \sin(x), \ldots$
- **Question**: $\lim_{n\to\infty} \max_{a\le x\le b} |f(x) - p_n(x)| = 0$?

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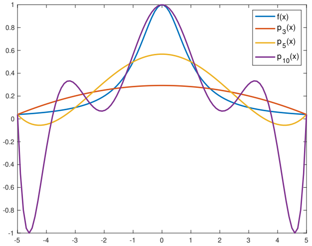
- **Question**: $\lim_{n\to\infty} \max_{a\le x\le b} |f(x) - p_n(x)| = 0$? Not necessarily
- Runge Phenomenon
- $f(x) = \frac{1}{1+x^2}, -5 \le x \le 5$



- **Question**: Why?

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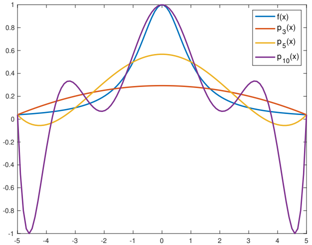

- **Question**: Why? $f^{(n+1)}$ grows too fast

Further analysis

- For polynomial interpolation in $[a,b]$ with $x_i = a + (b-a)\frac{i}{n}$
- $\|(x-x_0)\dots(x-x_n)\|_\infty \leq n!\left(\frac{b-a}{n}\right)^{n+1}$
- $e_n = \left|\frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)\dots(x-x_n)\right|$
- **Question**: How to avoid Runge phenomenon?

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- **Question**: How to avoid Runge phenomenon?
- Fix n to avoid high oscillations
- Reduce $b-a$
- Piecewise interpolation

Hermite interpolation

- $x_0, \dots, x_n$
- Piecewise interpolation
- Cubic spline: In $[x_i, x_{i+1}]$, $p_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$
- $s(x) = p_i(x)$ if $x \in [x_i, x_{i+1}]$
- **Question**: number of unknowns?

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- Impose conditions:
  - $s(x_i) = y_i$, $n+1$ equations
  - $p_i(x_i) = p_{i+1}(x_i)$, $n-1$ equations
  - $p_i'(x_i) = p_{i+1}'(x_i)$, $n-1$ equations
  - $p_i''(x_i) = p_{i+1}''(x_i)$, $n-1$ equations
  - Other conditions, e.g., $s'(x_0) = y_0'$ and $s'(x_n) = y_n'$, 2 equations
- **Question**: Matrix form?