Math 302 HW2
Section IV 2024
Duke Kunshan University

Problem 1 (2.24)**.** *Which of the following iterations will converge to the indicated fixed point α (provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.*

1. $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $\alpha = 2$
2. $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $\alpha = 3^{1/3}$
3. $x_{n+1} = \frac{12}{1+x_n}$, $\alpha = 3$

Problem 2 (2.25)**.** *Show that*

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a}, \ n \geq 0$$

is a third-order method for computing $\sqrt{a}$. Calculate

$$\lim_{n\to\infty} \frac{\sqrt{a} - x_{n+1}}{(\sqrt{a} - x_n)^3}$$

assuming x_0 has been chosen sufficient close to α.

Problem 3 (2.49)**.** *Prove that the iteration $\{\mathbf{x}_n\}$ in THM 2.9 will converge to a solution of $\mathbf{x} = \mathbf{g}(\mathbf{x})$.*

Problem 4 (2.50)**.** *Using Newton's method for nonlinear systems, solve the nonlinear system*

$$\begin{aligned} x^2 + y^2 &= 4, \\ x^2 - y^2 &= 1. \end{aligned}$$

The true solutions are easily determined to be $\left(\pm\sqrt{2.5}, \pm\sqrt{1.5}\right)$. As an initial guess, use $(x_0, y_0) = (1.6, 1.2)$.

Problem 5. *Derive the least square solution to the linear models $y_i = a + bx_i + \epsilon_i$.*

Problem 6. *Consider the linear system*

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{b},$$

where

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

and the true solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Implement the Neumann series (in slides) to solve the linear system and prove the convergence.

Problem 7 (3.1)**.** *Prove the determinant formula of the Vandermonde matrix.*

Problem 8 (3.11)**.** *Let $x_0, \ldots, x_n$ be distinct real points, and consider the following interpolation problem. Consider a function*

$$p_n(x) = \sum_{j=0}^{n} c_j e^{jx}$$

such that

$$p_n(x_i) = y_i, \quad i = 0, 1, \ldots, n$$

with the $\{y_i\}$ given data. Show there is a unique choice of $c_0, \ldots, c_n$.