Math 302 - Numerical Analysis

Recitation #3 Handout: Some Linear Al

Objective: This session is a review of essential linear algebra concepts, including:

- Linear independence
- Existence and uniqueness of solutions for linear systems
- Rank
- Eigenvalue decomposition (EVD)
- Orthogonal decomposition (QR decomposition)

Complete as much as you can during the recitation. If time runs out, please finish the remaining material as homework.

Note: **The auto-magic power of VS Code or JetBrains will not be here to help you.** But it does not mean you need to remember every detailed command line we used ---- LLMs always ready to help you. But what you do need is understanding from the top level (to give clear instructions as "leader").

There's lots of IMPORTANT explanation here, for this course and beyond. Actual tasks you're asked to do are colored cyan for your convenience, but read everything. Thanks!

0.What is Taken as Known

Some key topics from MATH 105 are assumed to be known. If these concepts feel unfamiliar, please revisit them:

- **Matrix basics**: Operations, transpose, and inverse
- **Determinants**: Properties and computation
- **Vector spaces**: Span, basis, and dimension

Go through the following terms as a quick check:

$$\begin{aligned}&\text{Matrix operations: } A+B,\ cA,\ AB,\ A^{-1},\ A^T\\&\text{Determinants: } \det(A), \text{ and the properties } \det(AB)=\det(A)\det(B),\ \det(\\&\text{Vector spaces: } \operatorname{span}(\mathbf{v}_1,\ldots,\mathbf{v}_n),\ \dim(V), \text{ basis of } V\end{aligned}$$

1.Linear Independence

Definition: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \implies c_1 = c_2 = \cdots = c_n = 0 \tag{2}$$

Otherwise, the vectors are linearly dependent.

Determine if the following vectors in $\mathbb{R}^3$ are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \tag{3}$$

Methods:

- Gaussian elimination on the linear system $V\mathbf{c} = \mathbf{0}$
- Determinant of V (for suqare V)

2.Existence and Uniqueness of Solutions

For a linear system $A\mathbf{x} = \mathbf{y}$:

- A solution exists if the system is **consistent**, i.e., $\text{rank}(A) = \text{rank}([A \mid \mathbf{y}])$.
- The solution is unique if $\text{rank}(A) = \text{number of variables.}$

For the matrix A and vector $\mathbf{y}$ below, determine if a unique solution exists:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 34 \\ 21 \end{bmatrix} \tag{4}$$

Methods:

- Gaussian elimination on the linear system $A\mathbf{x} = \mathbf{y}$
- Inverse of A (for square A)

3.Rank

Definition: The rank of a matrix A is the maximum number of linearly independent rows or columns. It determines the dimension of the column space (range) of A.

Compute the rank of the following matrix by reducing it to row-echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{5}$$

Methods:

- Reducing to row-echelon form on A

4.Eigenvalue Decomposition (EVD)

Definition: For a square matrix A, if there exists a scalar λ and a non-zero vector $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}, \tag{6}$$

then λ is an eigenvalue, and $\mathbf{v}$ is the corresponding eigenvector.

Steps to Compute EVD:

1. Solve $\det(A - \lambda I) = 0$ to find eigenvalues λ.
2. For each eigenvalue, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find eigenvectors.

Find the eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{7}$$

Methods:

- Solving $\det(A - \lambda I) = 0$
- Solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$

5.Orthogonal Decomposition (QR Decompositic

Definition: For a matrix A, the QR decomposition expresses it as:

$$A = QR \tag{8}$$

where:

- Q is an orthogonal matrix ($Q^T Q = I$).
- R is an upper triangular matrix.

Steps to Compute QR (Gram-Schmidt Process):

1. Start with the columns of A: $\mathbf{a}_1, \mathbf{a}_2, \ldots$.
2. Compute an orthonormal basis $\mathbf{q}_1, \mathbf{q}_2, \ldots$ using:

$$\mathbf{q}_k = \frac{\mathbf{a}_k - \mathrm{Proj}_{\mathbf{q}_1}(\mathbf{a}_k) - \cdots - \mathrm{Proj}_{\mathbf{q}_{k-1}}(\mathbf{a}_k)}{\|\mathbf{a}_k - \mathrm{Proj}_{\mathbf{q}_1}(\mathbf{a}_k) - \cdots - \mathrm{Proj}_{\mathbf{q}_{k-1}}(\mathbf{a}_k)\|} \tag{9}$$

3. Construct Q from $\mathbf{q}_1, \mathbf{q}_2, \ldots$, and R from the projections.

Compute the QR decomposition of:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{10}$$

Methods:

- Gram-Schmidt process on A

6.Note for Students

- This material is foundational for numerical methods in interpolation, root-finding, and optimization.
- Aim to understand the concepts at a high level so you can explain them clearly and confidently.
- Feel free to use computational tools, but focus on understanding the algorithms and theory behind the computations.

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