HW2

Problem 1 (Analytical)**.** *1.1*

Solution 1. *We have the condition that*

$$0 < d < 1 + r < u.$$

When X_0, we have

$$X_1 = \Delta_0(S_1 - (1+r)S_0) = \Delta_0 \Delta S.$$

We want $X_1 > 0$ with probability 1. If H, we have $\Delta S(H) > 0$, then $\Delta_0 < 0$; if T, we have $\Delta S(T) < 0$, then $\Delta_0 > 0$. This cannot happen at the same time.

Problem 2 (Analytical)**.** *1.4*

Solution 2. *With the similar calculation, we have*

$$\begin{aligned} X_{n+1}(T) &= (1+r)X_n + \Delta_n(d-(1+r))S_n = (1+r)V_n + \frac{(V_{n+1}(H) - V_{n+1}(T))(d-(1+r))}{u-d} \\ &= (1+r)V_n - \widetilde{p}V_{n+1}(H) + \widetilde{p}V_{n+1}(T) = \widetilde{p}V_{n+1}(H) + q^*V_{n+1}(T) - \widetilde{p}V_{n+1}(H) + \widetilde{p}V_{n+1}(T) = V_{n+1}(T). \end{aligned}$$

Problem 3 (Analytical)**.** *1.8*

Solution 3.

1. $V_n(s,y) = \frac{2}{5}\left(V_{n+1}(2s, y+2s) + V_{n+1}\left(\frac{s}{2}, y + \frac{s}{2}\right)\right)$
2. *1.216*
3. $\Delta_n(s,y) = \frac{V_{n+1}(us, y+us) - V_{n+1}(ds, y+ds)}{(u-d)s}$

Problem 4 (Analytical, optional)**.** *1.9*

Solution 4.

1. $V_n = \frac{\widetilde{p}_n V_{n+1}(H) + q_n^* V_{n+1}(T)}{1+r_n}$
2. $\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u_n - d_s)S_n}$
3. *9.375*

Problem 5 (Analytical)**.** *Consider a binomial model with $d = \frac{1}{u}$. Suppose we are concerned with the fluctuations of a call option price due to underlying stock price and scale factor u. We are not satisfied with the linear approximation and would like to do better. Therefore, we consider the quadratic approximation of the form:*

$$V(S,u) = a + b \cdot S + c \cdot u + d \cdot S^2 + e \cdot u^2 + f \cdot S \cdot u.$$

1. *Extend the analysis of the linear approximation to quadratic approximation by further using second-order derivatives*
2. *Calculate the second-order derivatives of the call option and plug into the quadratic approximation (For complicated ones, you can use the software).*
3. *Suppose $S_0 = 4, u = 2, K = 5, r = \frac{1}{4}$. Consider $\widetilde{S}_0 = 5$ and $\widetilde{u} = 2.5$, compare the linear approximation, quadratic approximation with the true option price, and comment.*

Solution 5.

1. *We have*

$$\begin{aligned} V(S,u) \approx V(S_0,u_0) &+ \frac{\partial V}{\partial S}(S_0,u_0)(S-S_0) + \frac{\partial V}{\partial u}(S_0,u_0)(u-u_0) \\ &+ \frac{\frac{\partial^2 V}{\partial S^2}(S_0,u_0)}{2}(S-S_0)^2 + \frac{\frac{\partial^2 V}{\partial u^2}(S_0,u_0)}{2}(u-u_0)^2 + \frac{\partial^2 V}{\partial S \partial u}(S_0,u_0)(S-S_0)(u-u_0). \end{aligned}$$

2. *We calculate*

$$
\begin{aligned}
\frac{\partial V}{\partial S} &= \frac{\widetilde{p}u}{1+r}, \\
\frac{\partial^2 V}{\partial S^2} &= 0, \\
\frac{\partial V}{\partial u} &= \frac{(uS-K)(u-(1+r))}{(1+r)(u^2-1)^2} + \frac{[u(1+r)-1](uK-S)}{(1+r)(u^2-1)^2}, \\
\frac{\partial^2 V}{\partial u^2} &= \frac{2S(3ru^2+r-(u-1)^3) - 2K((r+1)u^3+3(r+1)u-3u^2-1)}{(r+1)(u^2-1)^3}, \\
\frac{\partial^2 V}{\partial u \partial S} &= \frac{-2(r+1)u+u^2+1}{(r+1)(u^2-1)^2}.
\end{aligned}
$$

3. *True price is about 2.43. Linear approximation gives 2.5. Quadratic approximation give 2.4. The quadratic approximation is more accurate.*

Problem 6 (Coding)**.** *Download the S&P 500 data (use close prices) online (e.g., from Nasdaq) between 2020.1.1 and 2024.12.31 (excluding all weekends and holidays). Denote the stock price in order as S_i and daily log return as $R_i = log(S_{i+1}) - log(S_i)$ (assuming nothing happens during the weekends and holidays). Denote the total number of daily return as n.*

1. *Visualize the trajectories of log prices.*
2. *Estimate the mean and variance of R_i as $\widehat{\mu}$ and $\widehat{\sigma}^2$. Assume the estimated variance $\widehat{\sigma}^2$ is the true variance σ^2, calculate the accuracy of the mean estimator by $\sqrt{\mathbb{E}[(\mu - \frac{1}{n}\sum_{i=1}^n R_i)^2]} = \frac{\sigma}{\sqrt{n}}$. Assume return data is i.i.d., provide a 95% confidence interval of $\widehat{\mu}$.*
3. *Assume the log return follows the normal distribution with the estimated mean and variance that $R_i \sim \mathcal{N}(\widehat{\mu}, \widehat{\sigma}^2)$. Numerically verify the accuracy of the mean estimator above by Monte Carlo simulation. To do this, generate M (e.g., 10000) sequence with n i.i.d normal random variables $\mathcal{N}(\widehat{\mu}, \widehat{\sigma}^2)$. For each sequence, estimate its mean. Then Calculate the standard deviation of the mean estimator and compare with the formula above. Convince yourself that the mean estimation of log return for S&P 500 is inaccurate.*