HW6 solution

Problem 1 (Analytical Problem, Optional)**.** *3.8*

Problem 2 (Analytical Problem)**.** *In class, we argue that when t is small, we have $e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t} \approx 1+\mu t+\sigma W_t$. Show that the linear approximation of $\mathbb{E}[S_t|S_0]$ and $Var(S_t|S_0)$ would match the conditional expectation and variance of the approximation $1+\mu t+\sigma W_t$.*

Problem 3 (Analytical Problem)**.** *Verify the Black-Scholes formula for the put option by evaluating $\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\max(K-S_T,0)|S_t\right]$ by integral.*

Problem 4 (Analytical Problem)**.** *Show that Vega of the call option is $S_0 N'(d_+)\sqrt{T-t}$.*

Problem 5 (Coding Problem)**.** *Denote X_t and $V(t,S_t)$ the portfolio and call option price at time t. Numerically verify that we can let $X_T=V_T$ through Delta hedging. Suppose the expiration date $T=1$, strike price $K=100$, interest rate $r=0$, initial stock price $S_0=100$, and the stock price follows the geometric Brownian motion $S_t=S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t}$, whereas W_t is the Brownian motion, $\mu=0.08$ and $\sigma=0.16$. By Delta hedging, if we let $\Delta(t,x)=\frac{\partial}{\partial x}V(t,x)$, where $V(t,x)$ is the Black-Scholes formula and then $dX_t=\Delta(t,S_t)dS_t$, then we have $X_T=V_T$. To verify, generate M sequence of stock prices. For each sequence, do the simulation as follows:*

- *Generate $S_0,\ldots,S_n$ for M (e.g., M=1000) times for a large choice of n (e.g., $n=2520$). For each sequence,*
  - *Simulate the geometric Brownian motion $S_i=S_0e^{\left(\mu-\frac{\sigma^2}{2}\right)t_i+\sigma W_i}$*
  - *Correspondingly, generate the portfolio by $X_{i+1}=X_i+\Delta(t_i,S_i)(S_{i+1}-S_i)$*
  - *At the expiration date T, verify that $X_T\approx V_T$*