HW 3

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Problem 1

Using Definition 2.1.1, show the following.

(i) If A is an event and A^c denotes its complement, then $P(A) = 1 - P(A^c)$

(ii) If $A_1, A_2, \ldots, A_n$ is a finite set of events, then

$$P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n).$$

If the events $A_1, A_2, \ldots, A_n$ are disjoint, then equality holds.

> **Definition 2.1.1.** A finite probability space consists of a sample space Ω and a probability measure P. The sample space Ω is a nonempty finite set and the probability measure P is a function that assigns to each element ω of Ω a number in $[0, 1]$ so that
>
> $$\sum_{\omega \in \Omega} P(\omega) = 1.$$
>
> An event is a subset of Ω, and we define the probability of an event A to be
>
> $$P(A) = \sum_{\omega \in A} P(\omega).$$
>
> As mentioned before, this is a model for some random experiment. The set Ω is the set of all possible outcomes of the experiment, $P(\omega)$ is the probability that the particular outcome ω occurs, and $P(A)$ is the probability that the outcome that occurs is in the set A. If $P(A) = 0$, then the outcome of the experiment is sure not to be in A; if $P(A) = 1$, then the outcome is sure to be in A. Because of $(2.1.4)$, we have the equation
>
> $$P(\Omega) = 1.$$
>
> i.e., the outcome that occurs is sure to be in the set Ω. Because $P(\omega)$ can be zero for some values of ω, we are permitted to put in Ω even some outcomes of the experiment that are sure not to occur. It is clear from $(2.1.5)$ that if A and B are disjoint subsets of Ω, then
>
> $$P(A \cup B) = P(A) + P(B)$$

Solution:

(i) We need to show that $P(A) = 1 - P(A^c)$ using Definition 2.1.1.

From the definition, we know that Ω is the sample space and $A \subset \Omega$ is an event. The complement $A^c = \Omega \setminus A$ consists of all outcomes that are not in A. Since A and A^c are disjoint and $A \cup A^c = \Omega$, we have:

$P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

From the definition, we know that $P(\Omega) = 1$, so: $1 = P(A) + P(A^c)$

Rearranging, we get: $P(A) = 1 - P(A^c)$

(ii) We need to show that $P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n)$.

Let's denote $B = \bigcup_{n=1}^{N} A_n$. Then B is the set of all outcomes that belong to at least one of the events $A_1, A_2, \ldots, A_N$.

For any outcome $\omega \in B$, let $k(\omega)$ be the number of sets A_n that contain ω. Clearly, $k(\omega) \geq 1$ for all $\omega \in B$.

Now, we can write:

$$P(B) = \sum_{\omega \in B} P(\omega)$$

$$\sum_{n=1}^{N} P(A_n) = \sum_{n=1}^{N} \sum_{\omega \in A_n} P(\omega) = \sum_{\omega \in \Omega} k(\omega) \cdot P(\omega)$$

Since $k(\omega) \geq 1$ for all $\omega \in B$ and $k(\omega) = 0$ for all $\omega \notin B$, we have: $\sum_{\omega \in \Omega} k(\omega) \cdot P(\omega) \geq \sum_{\omega \in B} P(\omega) = P(B)$

Therefore, $P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n)$.

If the events $A_1, A_2, \ldots, A_N$ are disjoint, then for each $\omega \in B$, there is exactly one set A_n that contains ω, so $k(\omega) = 1$ for all $\omega \in B$. In this case: $\sum_{\omega \in \Omega} k(\omega) \cdot P(\omega) = \sum_{\omega \in B} P(\omega) = P(B)$

Therefore, if the events are disjoint, equality holds: $P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$.

Problem 2

Consider the stock price S_3 in Figure 2.3.1.

(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}, \tilde{q} = \frac{1}{2}$?

(ii) Compute $\tilde{\mathbb{E}}[S_1]$, $\tilde{\mathbb{E}}[S_2]$, and $\tilde{\mathbb{E}}[S_3]$. What is the average rate of growth of the stock price under $\tilde{\mathbb{P}}$?

(iii) Answer (i) and (ii) again under the actual probabilities $p = \frac{2}{3}, q = \frac{1}{3}$.

> **Description of Figure 2.3.1:** Figure 2.3.1 depicts a **three-period binomial model** representing the evolution of a stock price S_t across a sequence of coin tosses. Each time step corresponds to a fair coin toss (Heads or Tails), and the stock price evolves accordingly. The model begins at time $t = 0$ with a stock price of:

> - $S_0 = 4$
>
> Each outcome in the tree represents a possible path of coin flips up to time $t = 3$. The paths and stock prices are defined as follows:
>
> ---
>
> ### ⏱️ Time Step 1:
>
> - $S_1(H) = 8$
> - $S_1(T) = 2$
>
> ---
>
> ### ⏱️ Time Step 2:
>
> - $S_2(HH) = 16$
> - $S_2(HT) = S_2(TH) = 4$
> - $S_2(TT) = 1$
>
> ---
>
> ### ⏱️ Time Step 3:
>
> - $S_3(HHH) = 32$
> - $S_3(HHT) = S_3(HTH) = S_3(THH) = 8$
> - $S_3(HTT) = S_3(THT) = S_3(TTH) = 2$
> - $S_3(TTT) = 0.5$

Solution:

(i) Under the risk-neutral probabilities $\tilde{p} = \frac{1}{2}, \tilde{q} = \frac{1}{2}$, the distribution of S_3 is:

- $P(S_3 = 32) = \tilde{p}^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
- $P(S_3 = 8) = \tilde{p}^2\tilde{q} + \tilde{p}\tilde{q}\tilde{p} + \tilde{q}\tilde{p}^2 = 3\tilde{p}^2\tilde{q} = 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$
- $P(S_3 = 2) = \tilde{p}\tilde{q}^2 + \tilde{q}\tilde{p}\tilde{q} + \tilde{q}^2\tilde{p} = 3\tilde{p}\tilde{q}^2 = 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$
- $P(S_3 = 0.5) = \tilde{q}^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(ii) Computing the expected values under risk-neutral probabilities:

$$\tilde{\mathbb{E}}[S_1] = \tilde{p} \cdot S_1(H) + \tilde{q} \cdot S_1(T) = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 2 = 4 + 1 = 5$$

$$\tilde{\mathbb{E}}[S_2] = \tilde{p}^2 \cdot S_2(HH) + \tilde{p}\tilde{q} \cdot S_2(HT) + \tilde{q}\tilde{p} \cdot S_2(TH) + \tilde{q}^2 \cdot S_2(TT)$$

$$= \frac{1}{4} \cdot 16 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 1 = 4 + 1 + 1 + 0.25 = 6.25$$

$$\tilde{\mathbb{E}}[S_3] = \frac{1}{8} \cdot 32 + \frac{3}{8} \cdot 8 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 0.5 = 4 + 3 + 0.75 + 0.0625 = 7.8125$$

The average rate of growth under risk-neutral probabilities $\tilde{\mathbb{P}}$ is:

$$\frac{\tilde{\mathbb{E}}[S_t]}{S_0} = \frac{S_0 \cdot (1+r)^t}{S_0} = (1+r)^t$$

For $t=1$: $\frac{\tilde{\mathbb{E}}[S_1]}{S_0} = \frac{5}{4} = 1.25 = (1+r)^1$, so $r = 0.25$ or 25%

For $t=2$: $\frac{\tilde{\mathbb{E}}[S_2]}{S_0} = \frac{6.25}{4} = 1.5625 = (1+r)^2 = (1.25)^2$, confirming $r = 0.25$

For $t=3$: $\frac{\tilde{\mathbb{E}}[S_3]}{S_0} = \frac{7.8125}{4} = 1.953125 = (1+r)^3 = (1.25)^3$, confirming $r = 0.25$

Therefore, the average rate of growth under risk-neutral probabilities is 25% per period.

(iii) Under the actual probabilities $p = \frac{2}{3}, q = \frac{1}{3}$:

Distribution of S_3:

- $P(S_3 = 32) = p^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
- $P(S_3 = 8) = 3p^2q = 3\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3 \cdot \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{9}$
- $P(S_3 = 2) = 3pq^2 = 3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = 3 \cdot \frac{2}{3} \cdot \frac{1}{9} = \frac{2}{9}$
- $P(S_3 = 0.5) = q^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

Expected values:

$$\mathbb{E}[S_1] = p \cdot S_1(H) + q \cdot S_1(T) = \frac{2}{3} \cdot 8 + \frac{1}{3} \cdot 2 = \frac{16}{3} + \frac{2}{3} = 6$$

$$\mathbb{E}[S_2] = p^2 \cdot S_2(HH) + pq \cdot S_2(HT) + qp \cdot S_2(TH) + q^2 \cdot S_2(TT)$$

$$= \frac{4}{9} \cdot 16 + \frac{2}{9} \cdot 4 + \frac{2}{9} \cdot 4 + \frac{1}{9} \cdot 1 = \frac{64}{9} + \frac{8}{9} + \frac{8}{9} + \frac{1}{9} = \frac{81}{9} = 9$$

$$\mathbb{E}[S_3] = \frac{8}{27} \cdot 32 + \frac{4}{9} \cdot 8 + \frac{2}{9} \cdot 2 + \frac{1}{27} \cdot 0.5$$

$$= \frac{256}{27} + \frac{32}{9} + \frac{4}{9} + \frac{0.5}{27} = \frac{256}{27} + \frac{96}{27} + \frac{12}{27} + \frac{0.5}{27} = \frac{364.5}{27} = 13.5$$

The average rate of growth under actual probabilities is: For $t=1$: $\frac{\mathbb{E}[S_1]}{S_0} = \frac{6}{4} = 1.5 = (1+r)^1$, so $r = 0.5$ or 50%

For $t=2$: $\frac{\mathbb{E}[S_2]}{S_0} = \frac{9}{4} = 2.25 = (1+r)^2 = (1.5)^2$, confirming $r = 0.5$

For $t=3$: $\frac{\mathbb{E}[S_3]}{S_0} = \frac{13.5}{4} = 3.375 = (1+r)^3 = (1.5)^3$, confirming $r = 0.5$

Therefore, the average rate of growth under actual probabilities is 50% per period.

Problem 3

Show that a convex function of a martingale is a submartingale. In other words, let $M_0, M_1, \ldots, M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.

Solution:

To show that a convex function of a martingale is a submartingale, I need to prove that $\mathbb{E}[\varphi(M_{t+1})|\mathcal{F}_t] \geq \varphi(M_t)$ for all $t = 0, 1, \ldots, N-1$.

Let's first clarify that $\mathcal{F}_t$ represents the filtration (information available) up to time t. A martingale M_t is adapted to this filtration, meaning M_t is $\mathcal{F}_t$-measurable.

Starting with the definition of a martingale, we know that:

$$\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t$$

Since φ is a convex function, by Jensen's inequality, for any random variable X and σ-algebra $\mathcal{G}$:

$$\varphi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\varphi(X)|\mathcal{G}]$$

Here, $\mathcal{G}$ represents any σ-algebra with respect to which we are taking the conditional expectation. In our specific application below, we will use $\mathcal{G} = \mathcal{F}_t$, which is the σ-algebra representing all information available up to time t.

Applying Jensen's inequality with $X = M_{t+1}$ and $\mathcal{G} = \mathcal{F}_t$:

$$\varphi(\mathbb{E}[M_{t+1}|\mathcal{F}_t]) \leq \mathbb{E}[\varphi(M_{t+1})|\mathcal{F}_t]$$

Since $\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t$ from the martingale property, we have:

$$\varphi(M_t) \leq \mathbb{E}[\varphi(M_{t+1})|\mathcal{F}_t]$$

This is precisely the definition of a submartingale. Therefore, $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_N)$ is a submartingale.

Problem 4

Consider an N-period binomial model.

(i) Let $M_0, M_1, \ldots, M_N$ and $M'_0, M'_1, \ldots, M'_N$ be martingales under the risk-neutral measure $\tilde{\mathbb{P}}$. Show that if $M_N = M'_N$ (for every possible outcome of the sequence of coin tosses), then, for each n between 0 and N, we have $M_n = M'_n$ (for every possible outcome of the sequence of coin tosses).

(ii) Let V_N be the payoff at time N of some derivative security. This is a random variable that can depend on all N coin tosses. Define recursively $V_{N-1}, V_{N-2}, \ldots, V_0$ by the algorithm $(1.2.16)$ of Chapter 1. Show that

$$\frac{V_0}{(1+r)^0}, \frac{V_1}{(1+r)^1}, \ldots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\tilde{\mathbb{P}}$.

(iii) Using the risk-neutral pricing formula $(2.4.11)$ of this chapter, define

$$V'_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right], \quad n = 0, 1, \ldots, N-1.$$

Show that

$$V_0', \frac{V_1'}{(1+r)^1}, \ldots, \frac{V_{N-1}'}{(1+r)^{N-1}}, \frac{V_N'}{(1+r)^N}$$

is a martingale.

(iv) Conclude that $V_n = V_n'$ for every n (i.e., the algorithm $(1.2.16)$ of Theorem $1.2.2$ of Chapter 1 gives the same derivative security prices as the risk-neutral pricing formula $(2.4.11)$ of Chapter 2).

> Theorem 1.2.2 (Replication in the multiperiod binomial model) Consider an N-period binomial asset-pricing model, with $0 < d < 1+r < u, and with$
> $\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-1-r}{u-d} Description of$(1.2.16)$of Chapter 1: Let$V_n
> $be a random variable (a derivative security paying off at time$N
> $) depending on the first$N$coin tosses$w_1 w_2 \ldots w_N
> $. Define recursively backward in time the sequence of random variables$V_{N-1},
> V_{N-2}, \ldots, V_0by
> $V_n(w_1 w_2 \ldots w_n) = \frac{1}{1+r}[\tilde{p} V_{n+1}(w_1 w_2 \ldots w_n H) + \tilde{q} V_{n+1}(w_1 w_2 \ldots w_n T)]$,$

Solution:

(i) Let's prove that if $M_N = M_N'$ for every possible outcome, then $M_n = M_n'$ for all n between 0 and N.

Consider the martingales $M_0, M_1, \ldots, M_N$ and $M_0', M_1', \ldots, M_N'$ under the risk-neutral measure $\tilde{\mathbb{P}}$. We'll use backward induction to prove this.

By assumption, we have $M_N = M_N'$ for all possible outcomes.

For the inductive step, assume that $M_{n+1} = M_{n+1}'$ for all possible outcomes at time $n+1$. We need to show that $M_n = M_n'$ for all possible outcomes at time n.

Since both M and M' are martingales under $\tilde{\mathbb{P}}$, we have:

$$\tilde{\mathbb{E}}[M_{n+1}|\mathcal{F}_n] = M_n$$

$$\tilde{\mathbb{E}}[M_{n+1}'|\mathcal{F}_n] = M_n'$$

By our inductive hypothesis, $M_{n+1} = M_{n+1}'$, so:

$$\tilde{\mathbb{E}}[M_{n+1}|\mathcal{F}_n] = \tilde{\mathbb{E}}[M_{n+1}'|\mathcal{F}_n]$$

Therefore, $M_n = M_n'$ for all possible outcomes at time n.

By induction, we conclude that $M_n = M_n'$ for all n between 0 and N and for all possible outcomes.

(ii) Let's define $M_n = \frac{V_n}{(1+r)^n}$ for $n = 0, 1, \ldots, N$. We need to show that M_n is a martingale under $\tilde{\mathbb{P}}$.

From the recursive algorithm (1.2.16), we know that:

$$V_n = \frac{1}{1+r}[\tilde{p} V_{n+1}(H) + (1-\tilde{p}) V_{n+1}(T)]$$

where $V_{n+1}(H)$ and $V_{n+1}(T)$ are the values at time $n+1$ if the coin toss at time $n+1$ is heads or tails, respectively, and $\tilde{p}$ is the risk-neutral probability.

Now, let's compute $\tilde{\mathbb{E}}[M_{n+1}|\mathcal{F}_n]$:

$$\tilde{\mathbb{E}}[M_{n+1}|\mathcal{F}_n] = \tilde{\mathbb{E}}\left[\frac{V_{n+1}}{(1+r)^{n+1}}|\mathcal{F}_n\right] = \frac{1}{(1+r)^{n+1}}[\tilde{p}V_{n+1}(H) + (1-\tilde{p})V_{n+1}(T)]$$

Using the recursive formula for V_n:

$$\tilde{\mathbb{E}}[M_{n+1}|\mathcal{F}_n] = \frac{1}{(1+r)^{n+1}} \cdot (1+r) \cdot V_n = \frac{V_n}{(1+r)^n} = M_n$$

Therefore, M_n is a martingale under $\tilde{\mathbb{P}}$.

(iii) Let's define $M_n' = \frac{V_n'}{(1+r)^n}$ for $n = 0, 1, \ldots, N$, where $V_N' = V_N$. We need to show that M_n' is a martingale under $\tilde{\mathbb{P}}$.

For $n < N$, we have:

$$V_n' = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]$$

Now, let's compute $\tilde{\mathbb{E}}[M_{n+1}'|\mathcal{F}_n]$:

$$\tilde{\mathbb{E}}[M_{n+1}'|\mathcal{F}_n] = \tilde{\mathbb{E}}\left[\frac{V_{n+1}'}{(1+r)^{n+1}}|\mathcal{F}_n\right] = \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^{n+1}} \cdot \tilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right]|\mathcal{F}_n\right]$$

Using the tower property of conditional expectation:

$$\tilde{\mathbb{E}}[M_{n+1}'|\mathcal{F}_n] = \frac{1}{(1+r)^{n+1}} \cdot \tilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^{N-(n+1)}}|\mathcal{F}_n\right] = \frac{1}{(1+r)^n} \cdot \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] = \frac{V_n'}{(1+r)^n}$$

Therefore, M_n' is a martingale under $\tilde{\mathbb{P}}$.

(iv) From parts (ii) and (iii), we have shown that both $\frac{V_n}{(1+r)^n}$ and $\frac{V_n'}{(1+r)^n}$ are martingales under $\tilde{\mathbb{P}}$. Additionally, we know that $V_N = V_N'$ (the payoff at maturity).

Applying the result from part (i), since these are two martingales with the same terminal value, they must be equal at all times. Therefore, $V_n = V_n'$ for all n between 0 and N.

This confirms that the algorithm (1.2.16) of Theorem 1.2.2 of Chapter 1 gives the same derivative security prices as the risk-neutral pricing formula (2.4.11) of Chapter 2.

Problem 5

(Asian option). Consider an N-period binomial model. An Asian option has a payoff based on the average stock price, i.e.,

$$V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right)$$

where the function f is determined by the contractual details of the option.

(i) Define $Y_n = \sum_{k=0}^{n} S_k$ and use the Independence Lemma 2.5.3 to show that the two-dimensional process (S_n, Y_n), $n = 0, 1, \ldots, N$ is Markov.

(ii) According to Theorem 2.5.8, the price V_n of the Asian option at time n is some function v_n of S_n and Y_n; i.e.,

$$V_n = v_n(S_n, Y_n), \quad n = 0, 1, \ldots, N$$

Give a formula for $v_N(s, y)$, and provide an algorithm for computing $v_n(s, y)$ in terms of v_{n+1}.

> **Lemma 2.5.3 (Independence Lemma)**
>
> In the N-period binomial asset pricing model, let n be an integer between 0 and N. Suppose the random variables $X^1, \ldots, X^K$ depend only on coin tosses 1 through n and the random variables $Y^1, \ldots, Y^L$ depend only on coin tosses $n+1$ through N. (The superscripts $1, \ldots, K$ on X and $1, \ldots, L$ on Y are superscripts, not exponents.) Let $f(x^1, \ldots, x^K, y^1, \ldots, y^L)$ be a function of dummy variables $x^1, \ldots, x^K$ and $y^1, \ldots, y^L$, and define
>
> $$g(x^1, \ldots, x^K) = \mathbb{E}[f(x^1, \ldots, x^K, Y^1, \ldots, Y^L)]. \tag{2.5.3}$$
>
> Then
>
> $$\mathbb{E}_n[f(X^1, \ldots, X^K, Y^1, \ldots, Y^L)] = g(X^1, \ldots, X^K). \tag{2.5.4}$$
>
> For the following discussion and proof of the lemma, we assume that $K = L = 1$. Then (2.5.3) takes the form
>
> $$g(x) = \mathbb{E}[f(x, Y)] \tag{2.5.3'}$$
>
> and (2.5.4) takes the form
>
> $$\mathbb{E}_n[f(X, Y)] = g(X), \tag{2.5.4'}$$
>
> where the random variable X is assumed to depend only on the first n coin tosses, and the random variable Y depends only on coin tosses $n+1$ through N.

> **Theorem 2.5.8**
>
> Let $X_0, X_1, \ldots, X_N$ be a Markov process under the risk-neutral probability measure $\tilde{P}$ in the binomial model. Let $v_N(x)$ be a function of the dummy variable x, and consider a derivative security whose payoff at time N is $v_N(X_N)$. Then, for each n between 0 and N, the price V_n of this derivative security is some function v_n of X_n, i.e.,
>
> $$V_n = v_n(X_n), \quad n = 0, 1, \ldots, N$$
>
> There is a recursive algorithm for computing u_n whose exact formula depends on the underlying Markov process $X_0, X_1, \ldots, X_N$. Analogous results hold if the underlying Markov process is multidimensional.

Solution:

(i) To show that (S_n, Y_n) is a Markov process, we need to demonstrate that the future values depend only on the current state, not on the past history.

Let's consider the transition from time n to time $n+1$. We know that:

- $S_{n+1} = S_n \cdot Z_{n+1}$, where Z_{n+1} denotes either u or d depending on the $(n+1)$-th coin toss
- $Y_{n+1} = Y_n + S_{n+1} = Y_n + S_n \cdot Z_{n+1}$

Since Z_{n+1} depends only on the $(n+1)$-th coin toss, and both S_{n+1} and Y_{n+1} are determined by S_n, Y_n, and Z_{n+1}, we can apply the Independence Lemma 2.5.3.

Let $X^1 = S_n$ and $X^2 = Y_n$, which depend only on the first n coin tosses. Let $Y^1 = Z_{n+1}$, which depends only on the $(n+1)$-th coin toss.

By the Independence Lemma, the conditional expectation of any function of (S_{n+1}, Y_{n+1}) given the history up to time n depends only on the current values (S_n, Y_n). This confirms that (S_n, Y_n) is a Markov process.

(ii) At maturity, the Asian option payoff is:

$$V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right) = f\left(\frac{Y_N}{N+1}\right)$$

Therefore, the function $v_N(s, y)$ is given by:

$$v_N(s, y) = f\left(\frac{y}{N+1}\right)$$

For $n < N$, we can compute $v_n(s, y)$ recursively using the risk-neutral pricing formula:

$$v_n(s, y) = \frac{1}{1+r}\tilde{\mathbb{E}}_n[v_{n+1}(S_{n+1}, Y_{n+1})]$$

Since (S_n, Y_n) is Markov, and under the risk-neutral measure $\tilde{\mathbb{P}}$, S_{n+1} equals uS_n with probability $\tilde{p}$ and dS_n with probability $1-\tilde{p}$, we have:

$$v_n(s, y) = \frac{1}{1+r}[\tilde{p} \cdot v_{n+1}(us, y+us) + (1-\tilde{p}) \cdot v_{n+1}(ds, y+ds)]$$

where $\tilde{p} = \frac{1+r-d}{u-d}$ is the risk-neutral probability.

This recursive formula allows us to compute the option price at any time n by working backward from maturity.

Problem 6

Consider a N-period binomial model for a European call option with the initial stock price S_0, up factor u, down factor d, interest rate r, and strike price K. Implement the option price in two ways.

1. Direct formula using the binomial distribution, as we did in the class.

2. Recursively calculate the price backwards.

Use your code to calculate the option price for $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $K = 5$, and $N = 10$

In [9]:
```python
import numpy as np
from scipy.stats import binom

S0 = 4
u = 2
d = 1/2
r = 1/4
K = 5
N = 10

p_tilde = (1 + r - d) / (u - d) # risk-neutral p

# Method 1: Direct formula using binomial distribution
def option_price_direct(S0, u, d, r, K, N):
    j_values = np.arange(N + 1)
    price = np.sum(np.maximum(S0 * (u ** j_values) * (d ** (N - j_values)) - K, 0) *
    return price

# Method 2: Recursive calculation (backward induction)
def option_price_recursive(S0, u, d, r, K, N):
    # Define a helper function for the recursive calculation
    def calculate_value(s, n):
        # Base case: at maturity
        if n == N:
            return max(s - K, 0)

        # Recursive case: calculate expected value of future payoffs
        up_value = calculate_value(s * u, n + 1)
        down_value = calculate_value(s * d, n + 1)

        # Apply risk-neutral pricing formula
        return (p_tilde * up_value + (1 - p_tilde) * down_value) / (1 + r)

    # Start the recursion from initial state
    return calculate_value(S0, 0)

price_direct = option_price_direct(S0, u, d, r, K, N)
price_recursive = option_price_recursive(S0, u, d, r, K, N)

print(f"Option price using direct formula: {price_direct:.6f}")
print(f"Option price using recursive calculation: {price_recursive:.6f}")

# Verify the implementations are correct by checking if they give the same result
print(f"Are the results equal? {np.isclose(price_direct, price_recursive)}")
```

```
Option price using direct formula: 3.666451
Option price using recursive calculation: 3.666451
Are the results equal? True
```