# HW3

**Problem 1** (Analytical, optional)**.** *2.1*

**Solution 1.** *1.* $\mathbb{P}(A) + \mathbb{P}(A^c) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in A^c} \mathbb{P}(\omega) = \sum_{w \in \Omega} \mathbb{P}(\Omega) = 1.$

*2. For* $N = 2$, $\mathbb{P}(A_1 \cup A_2) = \sum_{\omega \in A_1 \cup A_2} \mathbb{P}(\omega) \le \sum_{\omega \in A_1} \mathbb{P}(\omega) + \sum_{\omega \in A_2} \mathbb{P}(\omega) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$*. For disjoint* $A_1$ *and* $A_2$*, we have the equality. Then we can apply the result iteratively.*

**Problem 2** (Analytical)**.** *2.2*

**Solution 2.** *1.* $\widetilde{P}(S_3 = 32) = \frac{1}{8}, \widetilde{P}(S_3 = 8) = \frac{3}{8}, \widetilde{P}(S_3 = 2) = \frac{3}{8}, \widetilde{P}(S_3 = 0.5) = \frac{1}{8}$

*2.* $\widetilde{E}[S_1] = 5, \widetilde{E}[S_2] = 6.25, \widetilde{E}[S_3] = 7.8125$*. The average rates is 0.25.*

*3.* $P(S_3 = 32) = \frac{8}{27}, P(S_3 = 8) = \frac{4}{9}, P(S_3 = 2) = \frac{2}{9}, P(S_3 = 0.5) = \frac{1}{27}$*. The average rate is 0.5.*

**Problem 3** (Analytical)**.** *2.3*

**Solution 3.** *Apply conditional Jensen's inequality.*

**Problem 4** (Analytical)**.** *2.8*

**Solution 4.** *1.* $M_n = \mathbb{E}_n[M_N] = \mathbb{E}_n[M'_N] = M'_n$

*2. We have let* $X_n = V_n$ *when replicating the portfolio and we show that* $\frac{X_n}{(1+r)^n}$ *is a martingale under* $\widetilde{P}$*.*

*3. We know the right-hand side is a martingale.*

*4. Combine last three results.*

**Problem 5** (Analytical)**.** *2.13*

**Solution 5.** *1.* $\forall g$*, we have* $\mathbb{E}_n[g(S_{n+1}, Y_{n+1})] = \mathbb{E}_n\left[g\left(\frac{S_{n+1}}{S_n} S_n, Y_{n+1} + \frac{S_{n+1}}{S_n} S_n\right)\right] = pg(uS_n, Y_n + uS_n) + qg(dS_n, Y_n + dS_n)$*, where is a function of* $(S_n, Y_n)$*.*

*2. Let* $v_N(s, y) = f\left(\frac{y}{N+1}\right)$*. Then* $v_N(S_N, Y_N) = V_N$*. Suppose* $v_{n+1}$ *is given, then* $V_n = \mathbb{E}_n\left[\frac{V_{n+1}}{1+r}\right] = \frac{1}{1+r}[\widetilde{p} v_{n+1}(uS_n, Y_n + uS_n) + \widetilde{q} v_{n+1}(dS_n, Y_n + dS_n)] = v_n(S_n, Y_n)$*, where* $v_n(s, y) = \frac{\widetilde{v}_{n+1}(us, y+us) + \widetilde{v}_{n+1}(ds, y+ds)}{1+r}$*.*

**Problem 6** (Coding)**.** *Consider a* $N$*-period binomial model for a European call option with the initial stock price* $S_0$*, up factor* $u$*, down factor* $d$*, interest rate* $r$*, and strike price* $K$*. Implement the option price in two ways.*

1. *Direct formula using the binomial distribution, as we did in the class.*
2. *Recursively calculate the price backwards.*

*Use your code to calculate the option price for* $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5$*, and* $N = 10$*.*