HW6 solution

Problem 1 (Analytical Problem, Optional)**.** *3.8*

Solution 1. *We calculate*

$$\phi_n(u) = \mathbb{E}\left[e^{\frac{u}{\sqrt{n}}M_{nt,n}}\right] = \prod_{k=1}^{nt}\mathbb{E}\left[e^{\frac{u}{\sqrt{n}}X_{k,n}}\right] = \left[e^{\frac{u}{\sqrt{n}}}\left(\frac{\frac{r}{n}+1-e^{-\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}}-e^{-\frac{\sigma}{\sqrt{n}}}}\right) - e^{-\frac{u}{\sqrt{n}}}\left(\frac{\frac{r}{n}+1-e^{\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}}-e^{-\frac{\sigma}{\sqrt{n}}}}\right)\right]^{nt}.$$

Now let $x = \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}
\log(\varphi_{\frac{1}{x^2}})(u) &= \frac{t}{x^2}\log\left(e^{ux}\left(\frac{rx^2+1-e^{-\sigma x}}{e^{\sigma x}-e^{-\sigma x}}\right) - e^{-ux}\left(\frac{rx^2+1-e^{\sigma x}}{e^{\sigma x}-e^{-\sigma x}}\right)\right)\\
&= \frac{t}{x^2}\log\left(\frac{e^{ux}}{2}\frac{rx^2+1-e^{-\sigma x}}{\sinh(\sigma x)} - \frac{e^{-ux}}{2}\frac{rx^2+1-e^{\sigma x}}{\sinh(\sigma x)}\right)\\
&= \frac{t}{x^2}\log\left(\frac{\sinh(ux)}{\sinh(\sigma x)}(rx^2+1) - \frac{e^{ux}e^{-\sigma x}}{2\sinh(\sigma x)} + \frac{e^{-ux}e^{\sigma x}}{2\sinh(\sigma x)}\right)\\
&= \frac{t}{x^2}\log\left(\frac{\sinh(ux)}{\sinh(\sigma x)}(rx^2+1) - \frac{\sinh(ux-\sigma x)}{\sinh(\sigma x)}\right)\\
&= \frac{t}{x^2}\log\left(\cosh(ux) + \frac{(rx^2+1-\cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)}\right).
\end{aligned}$$

Then take the Taylor series approximations,

$$\begin{aligned}
\log(\varphi_{\frac{1}{x^2}})(u) &\sim \frac{t}{x^2}\log\left(1+\frac{u^2x^2}{2} + \frac{(rx^2+1-1-\frac{\sigma^2x^2}{2})ux}{\sigma x}\right) = \frac{t}{x^2}\log\left(1+\frac{u^2x^2}{2}+\frac{rux^2}{\sigma}-\frac{ux^2\sigma}{2}\right)\\
&\sim \frac{t}{x^2}\left(\frac{u^2x^2}{2}+\frac{rux^2}{\sigma}-\frac{ux^2\sigma}{2}\right) = \frac{u^2t}{2}+\frac{rut}{\sigma}-\frac{u\sigma t}{2}.
\end{aligned}$$

Adding the effect of σ, we have

$$\mathbb{E}\left[e^{\frac{\sigma u}{\sqrt{n}}M_{nt,n}}\right] = \frac{\sigma^2tu^2}{2} + rut - \frac{\sigma^2tu}{2},$$

which is the moment-generating function of normal $\mathcal{N}\left(\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2t\right)$.

Problem 2 (Analytical Problem)**.** *In class, we argue that when t is small, we have $e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t} \approx 1+\mu t+\sigma W_t$. Show that the linear approximation of $\mathbb{E}[S_t|S_0]$ and $Var(S_t|S_0)$ would match the conditional expectation and variance of the approximation $1+\mu t+\sigma W_t$.*

Solution 2. *We calculate that*

$$\begin{aligned}
\mathbb{E}[1+\mu t+\sigma W_t] &= \mu t,\\
Var(1+\mu t+\sigma W_t) &= \sigma^2 t,\\
\mathbb{E}\left[e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t}\middle|S_0\right] &= e^{\mu t} \approx 1+\mu t,\\
Var(S_t|S_0) &= e^{2\mu t}\left(e^{\sigma^2 t}-1\right) \approx (1+2\mu t)(1+\sigma^2 t-1) \approx \sigma^2 t.
\end{aligned}$$

Problem 3 (Analytical Problem)**.** *Verify the Black-Scholes formula for the put option by evaluating $\widetilde{\mathbb{E}}\left[e^{-r(T-t)}\max(K-S_T,0)|S_t\right]$ by integral.*

Solution 3. *The calculation is similar to the slide. First, to simplify the calculation, let $\tau = T - t$. We wish to calculate the integral*

$$p(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} \max\left(K - xe^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y}, 0\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\ dy.$$

We wish to determine the integral range, we calculate

$$K = xe^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y} \Rightarrow y = \frac{\ln\left(\frac{K}{e^{r\tau}x}\right) + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}} = -d_-.$$

Then we have

$$p(t,x) = e^{-r\tau} \int_{-\infty}^{-d_-} \left(K - xe^{\left(\mu-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\ dy.$$

Note the first term is $Ke^{-rT}N(-d_-)$. For the second term, we calculate

$$\begin{aligned}
-e^{-r\tau} \int_{-\infty}^{-d_-} xe^{\left(\mu-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\ dy &= -e^{-r\tau} xe^{\left(r-\frac{\sigma^2}{2}\right)\tau} \int_{-\infty}^{-d_-} e^{\sigma\sqrt{\tau}y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\ dy \\
&= -e^{-r\tau} xe^{\left(r-\frac{\sigma^2}{2}\right)} \int_{-\infty}^{-d_-} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma\sqrt{\tau})^2}{2}} e^{\frac{\sigma^2\tau}{2}}\ dy, \text{ Let } z = y - \sigma\sqrt{\tau} \\
&= -xN(-d_+).
\end{aligned}$$

So in total, we have $p(t,x) = Ke^{-rT}N(-d_-) - xN(-d_+)$.

Problem 4 (Analytical Problem)**.** *Show that Vega of the call option is $S_0 N'(d_+)\sqrt{T-t}$.*

Solution 4. *We have*

$$\begin{aligned}
C(x,t) &= xN(d_+) - N(d_-)Ke^{-r\tau}, \\
d_+ &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right], \\
d_- &= d_+ - \sigma\sqrt{\tau}.
\end{aligned}$$

We calculate

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= xN'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-r\tau}N'(d_-)\frac{\partial d_-}{\partial \sigma} \\
&= xN'(d_+)\left(\sqrt{\tau} - \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma^3\tau}\right) - Ke^{-r\tau}N'(d_-)\left(-\frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma^3\tau}\right).
\end{aligned}$$

It is then sufficient to show that

$$-xN'(d_+)\frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma^3\tau} + Ke^{-r\tau}N'(d_-)\frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma^3\tau} = 0.$$

We calculate

$$N'(d_+) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(d_- + \sigma\sqrt{\tau})^2}{2}} = N'(d_-)e^{-d_-\sigma\sqrt{\tau}}e^{-\frac{\sigma^2\tau}{2}} = N'(d_-)\frac{K}{x}e^{-\left(r+\frac{\sigma^2}{2}\right)\tau}e^{-\frac{\sigma^2\tau}{2}}.$$

Therefore,

$$-xN'(d_+) = -N'(d_-)Ke^{-r\tau}.$$

We conclude.

Problem 5 (Coding Problem)**.** *Denote X_t and $V(t, S_t)$ the portfolio and call option price at time t. Numerically verify that we can let $X_T = V_T$ through Delta hedging. Suppose the expiration date $T = 1$, strike price $K = 100$, interest rate $r = 0$, initial stock price $S_0 = 100$, and the stock price follows the geometric Brownian motion $S_t = S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t + \sigma W_t}$, whereas W_t is the Brownian motion, $\mu = 0.08$ and $\sigma = 0.16$. By Delta hedging, if we let $\Delta(t,x) = \frac{\partial}{\partial x}V(t,x)$, where $V(t,x)$ is the Black-Scholes formula and then $dX_t = \Delta(t, S_t)dS_t$, then we have $X_T = V_T$. To verify, generate M sequence of stock prices. For each sequence, do the simulation as follows:*

- *Generate* $S_0, \ldots, S_n$ *for* M *(e.g., M=1000) times for a large choice of* n *(e.g.,* $n = 2520$*). For each sequence,*
  - *Simulate the geometric Brownian motion* $S_i = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t_i + \sigma W_i}$
  - *Correspondingly, generate the portfolio by* $X_{i+1} = X_i + \Delta(t_i, S_i)(S_{i+1} - S_i)$
  - *At the expiration date* T*, verify that* $X_T \approx V_T$