Lecture V: Geometric Brownian Motion and Black-Scholes-Merton Model

Dangxing Chen

Duke Kunshan University

BM in practice

- Let $X_t = \ln(S_t)$, $X_t = \mu t + \sigma W_t$
- **Question**: How to estimate parameters in BM?

BM in practice

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- **Question**: How to estimate parameters in BM?
- MoM (MLE is the same here)
- Moment equations

$$\begin{aligned}
\mathbb{E}[X_{\Delta t}] &= \mu \Delta t, \\
\mathbb{E}[X_{\Delta t}^2] &= \mathbb{E}\left[\mu^2 \Delta t^2 + \sigma^2 \Delta W^2 + 2\mu \Delta t \sigma \Delta W\right] \\
&= \mu^2 \Delta t^2 + \sigma^2 \Delta t
\end{aligned}$$

- Estimations:

$$\begin{aligned}
\widehat{\mu} &= \frac{\overline{X_{\Delta t}}}{\Delta t}, \\
\widehat{\sigma}^2 &= \frac{\overline{X_{\Delta t}^2} - \overline{X_{\Delta t}}^2}{\Delta t}.
\end{aligned}$$

Unbiased estimator

- Before, given a random variable Y_i, we estimate variance as

$$S_n^2 = \frac{\sum_{i=1}^n \left(Y_i - \frac{1}{n}\sum_{i=1}^n Y_i\right)^2}{n-1} = \frac{\sum_{i=1}^n Y_i^2 - \frac{1}{n}(\sum_{i=1}^n Y_i)^2}{n-1}$$

- **Def**: $\widehat{\theta}$ is an **unbiased** estimator of θ if $\mathbb{E}[\widehat{\theta}] = \theta$.

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- **Def**: $\widehat{\theta}$ is an **unbiased** estimator of θ if $\mathbb{E}[\widehat{\theta}] = \theta$.
- S_n^2 is an unbiased estimator of variance

$$\begin{aligned}
\mathbb{E}[(n-1)S_n^2] &= \mathbb{E}\left[\sum_{i=1}^n Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^n Y_i\right)^2\right] \\
&= \sum_{i=1}^n \mathbb{E}[Y_i^2] - \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^n Y_i^2 + \sum_{i\neq j} Y_i Y_j\right] \\
&= n(\mu^2 + \sigma^2) - \frac{n(\mu^2+\sigma^2) + n(n-1)\mu^2}{n} \\
&= (n-1)\sigma^2.
\end{aligned}$$

Assess estimators

- We assume $Y_i \sim \mathcal{N}(\mu, \sigma^2)$
- **Q**: Now we have two choices, which one is better?
  - $S_n^2 = \frac{\sum_{i=1}^{n}\left(Y_i - \frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2}{n-1}$
  - $\widetilde{S}_n^2 = \frac{\sum_{i=1}^{n}\left(Y_i - \frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2}{n}$

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  - $\widetilde{S}_n^2 = \frac{\sum_{i=1}^n \left(Y_i - \frac{1}{n}\sum_{i=1}^n Y_i\right)^2}{n}$
- MSE: $\mathbb{E}[(\theta - \widehat{\theta})^2]$
- **Q**: How does Bias affect MSE?
- Bias-variance tradeoff

$$
\begin{aligned}
\mathbb{E}[(\theta - \widehat{\theta})^2] &= \mathbb{E}[(\theta - \mathbb{E}[\widehat{\theta}] + \mathbb{E}[\widehat{\theta}] - \widehat{\theta})^2] \\
&= \mathbb{E}[(\theta - \mathbb{E}[\widehat{\theta}])^2 + (\widehat{\theta} - \mathbb{E}[\widehat{\theta}])^2 - 2(\theta - \mathbb{E}[\widehat{\theta}])(\widehat{\theta} - \mathbb{E}[\widehat{\theta}])] \\
&= \mathbb{E}[(\theta - \mathbb{E}[\widehat{\theta}])^2] + \mathbb{E}[(\widehat{\theta} - \mathbb{E}[\widehat{\theta}])^2] \\
&= \text{Bias}^2(\widehat{\theta}) + \text{Var}(\widehat{\theta}).
\end{aligned}
$$

MSE of the variance estimators

- To simplify calculation, $Y_i - \overline{Y} = (Y_i - \mu) - \overline{(Y - \mu)} = Z_i - \overline{Z}$
- Notation: $\mathbb{E}[Z_i] = 0, \mathbb{E}[Z_i^2] = \sigma^2, \mathbb{E}[Z_i^4] = \theta_4$

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$$
\begin{aligned}
\mathbb{E}\left[(n-1)^2 S_n^4\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} Z_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Z_i\right)^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n} Z_i^2\right)^2\right] - \frac{2}{n}\mathbb{E}\left[\left(\sum_{i=1}^{n} Z_i^2\right)\left(\sum_{i=1}^{n} Z_i\right)^2\right] \\
&+ \frac{1}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n} Z_i\right)^4\right]
\end{aligned}
$$

Calculation

- $\mathbb{E}\left[\left(\sum_{i=1}^n Z_i^2\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^n Z_i^4\right] + \mathbb{E}\left[\sum_{i\neq j} Z_i^2 Z_j^2\right] = n\theta_4 + n(n-1)\sigma^4$
- $\mathbb{E}\left[\left(\sum_{i=1}^n Z_i^2\right)\left(\sum_{i=1}^n Z_i\right)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^n Z_i^2\right)\left(\sum_{i=1}^n Z_i^2\right)\right] = n\theta_4 + n(n-1)\sigma^4$
- $\mathbb{E}\left[\left(\sum_{i=1}^n Z_i\right)^4\right] = n\theta_4 + \binom{n}{2}\binom{4}{2}\sigma^4 = n\theta_4 + 3n(n-1)\sigma^4$
- Combine the results

$$
\begin{aligned}
\mathbb{E}\left[(n-1)^2 S_n^4\right] &= n\theta_4 + n(n-1)\sigma^4 - \frac{2}{n}(n\theta_4 + n(n-1)\sigma^4) \\
&\quad + \frac{n\theta_4 + 3n(n-1)\sigma^4}{n^2} \\
&= \frac{(n-1)^2}{n}\theta_4 + \frac{n-1}{n}(n^2-2n+3)\sigma^4 \\
&\Rightarrow \mathbb{E}[S_n^4] = \frac{\theta_4}{n} + \frac{n^2-2n+3}{n(n-1)}\sigma^4
\end{aligned}
$$

Calculation

- $\mathbb{E}[S_n^2] = \sigma^2$, $\text{Bias}(S_n^2) = 0$
- $\mathbb{E}[S_n^4] = \frac{\theta_4}{n} + \frac{n^2-2n+3}{n(n-1)}\sigma^4$
- $\text{Var}(S_n^2) = \mathbb{E}[S_n^4] - \mathbb{E}^2[S_n^2] = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\sigma^4\right)$
- Suppose $Z_i \sim \mathcal{N}(0,\sigma^2)$, $\theta_4 = 3\sigma^4$
- $\text{Var}(S_n^2) = \frac{\sigma^4}{n}\left(3 - \frac{n-3}{n-1}\right) = \frac{\sigma^4}{n}\left(\frac{3n-3-n+3}{n-1}\right) = \frac{2\sigma^4}{n-1}$
- As a comparison, $\widetilde{S}_n^2 = \frac{n-1}{n}S_n^2$
- $\text{Bias}(\widetilde{S}_n^2) = \sigma^2 - \frac{n-1}{n}\sigma^2 = \frac{\sigma^2}{n}$
- $\text{Var}(\widetilde{S}_n^2) = \left(\frac{n-1}{n}\right)^2 \text{Var}(S_n^2) = \frac{2(n-1)}{n^2}\sigma^4$
- Large bias, smaller variance
- $\text{MSE}(\widetilde{S}_n^2) = \text{Var}(\widetilde{S}_n^2) + \text{Bias}^2(\widetilde{S}_n^2) = \frac{2n-1}{n^2}\sigma^4$
- $\frac{\text{MSE}(\widetilde{S}_n^2)}{\text{MSE}(S_n^2)} = \frac{(2n-1)(n-1)}{2n\cdot n} < 1$, the unbiased one is better!
- **Q**: Why not always use this?

Calculation

- $\mathbb{E}[S_n^2] = \sigma^2$, $\text{Bias}(S_n^2) = 0$
- $\mathbb{E}[S_n^4] = \frac{\theta_4}{n} + \frac{n^2-2n+3}{n(n-1)}\sigma^4$
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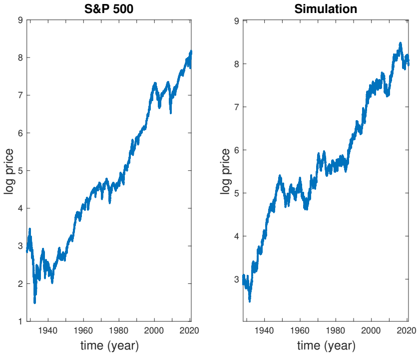
- **Q**: Why not always use this? We assume the normal

Inference of Brownian motion

- Let $X_t = \ln(S_t)$, $X_t = \mu t + \sigma W_t$
- Return: $X_{\Delta t} = \mu \Delta t + \sigma \Delta W \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t)$
- Mean: $\mathbb{E}[X_{\Delta t}] = \mu \Delta t$
- Mean estimator: $\sqrt{\text{Var}(\overline{X}_{\Delta t})} = \frac{\sigma \sqrt{\Delta t}}{\sqrt{n}}$
- Variance: $\text{Var}(X_{\Delta t}) = \sigma^2 \Delta t$
- Variance estimator: $\sqrt{\text{Var}(S_n^2)} = \frac{\sqrt{2}\sigma^2 \Delta t}{\sqrt{n-1}}$
- **Remark**: Under the BM assumption, mean is intrinsically difficult to be estimated accurately! On the other hand, variance could be estimated accurately.

Compare BM with data

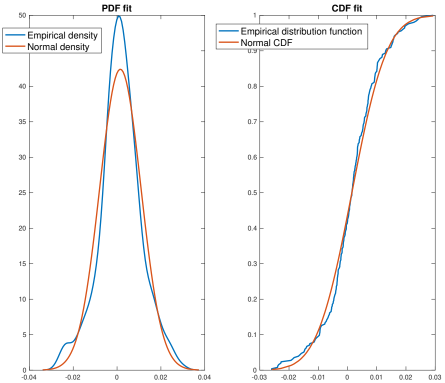
- **Question**: How good is BM?



- Looks good
- **Question**: Is there a better way to quantify the error?

Compare BM with data - continued

- Period: 1986-10-1 to 1987-10-1
- Under BM, compare $X_{\Delta t}$ with $\mathcal{N}\left(r\Delta t, \sigma^2 \Delta t\right)$



- **Question**: Intuitively seems OK, more rigorous measurement?

Kolmogorov-Smirnov test

- One way to check the error
- Empirical distribution function:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,x]}(X_i).$$

- **Def**: **Kolmogorov-Smirnov statistics** (K-S):

$$D_n = \sup_x |F_n(x) - F(x)|.$$

- If F is the truth, $\lim_{n\to\infty} D_n = 0$
- **Question**: Interpretation?

Kolmogorov-Smirnov test

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- If F is the truth, $\lim_{n\to\infty} D_n = 0$
- **Question**: Interpretation?
- If model is right, for infinitely amount of samples, the error goes to zero
- $\widehat{D}_n = 0.06$
- **Question**: For 1-year, is this statistic reasonable?

Hypothesis testing framework

- H_0: Model is right vs H_1: Model is wrong

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- **Question**: How small?

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- **Question**: If H_0, what would we expect?
- D_n small and converges to zero eventually
- **Question**: How small?
- Theoretical experiment with true r and σ:
  - Draw n i.i.d. ΔX_i from $\mathcal{N}\left(\mu\Delta t, \sigma^2\Delta t\right)$
  - Estimate $\widehat{\mu}$ and $\widehat{\sigma}$ from samples
  - Collect $\widehat{D}_n$
- Lilliefors test
  - Estimate $\widehat{\mu}$ and $\widehat{\sigma}^2$ from empirical data, assume they are true parameters
  - Determine level of significance α
  - Generate $\Delta X_i \sim \mathcal{N}(\widehat{\mu}\Delta t, \widehat{\sigma}^2\Delta t)$ M times
  - Estimate $\widetilde{\mu}$ and $\widetilde{\sigma}^2$ from simulation
  - Collect $\{\widetilde{D}_n\}$
  - Calculate p-value $\mathbb{P}(\widetilde{D}_n > \widehat{D}_n | H_0)$ using Monte Carlo above
  - Carefully reject or "accept"
- In our experiment $\widehat{p} = 1.3\%$

Connection with continuous case

- n steps per unit time
- $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and $d_n = 1 - \frac{\sigma}{\sqrt{n}}$
- $\widetilde{p} = \frac{1}{2}$ and $\widetilde{q} = \frac{1}{2}$
- H_{nt} number of heads and T_{nt} number of tails, $nt = H_{nt} + T_{nt}$
- Random walk $M_{nt} = H_{nt} - T_{nt}$
- $H_{nt} = \frac{1}{2}(nt + M_{nt})$ and $T_{nt} = \frac{1}{2}(nt - M_{nt})$
- $S_n(t) = S_0 \left(1 + \frac{\mu}{n}\right)^{nt} u_n^{H_{nt}} d_n^{T_{nt}} =$
  $S_0 \left(1 + \frac{\mu}{n}\right)^{nt} \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+M_{nt})} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-M_{nt})}$
- **THM**: As $n \to \infty$, the distribution of $S_n(t)$ converges to

$$S(t) = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W(t)}.$$

- Log-normal distribution

Verification

- $S_n(t) = S_0 \left(1 + \frac{\mu}{n}\right)^{nt} \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+M_{nt})} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-M_{nt})}$

Verification

- $S_n(t) = S_0\left(1+\frac{\mu}{n}\right)^{nt}\left(1+\frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt+M_{nt})}\left(1-\frac{\sigma}{\sqrt{n}}\right)^{\frac{1}{2}(nt-M_{nt})}$
- $S_n(t) =$
  $S_0\left(1+\frac{\mu}{n}\right)^{nt}\left(1-\frac{\sigma^2}{n}\right)^{\frac{nt}{2}}\left(1+\frac{\sigma}{\sqrt{n}}\right)^{\frac{\sqrt{n}}{2}\frac{M_{nt}}{\sqrt{n}}}\left(1-\frac{\sigma}{\sqrt{n}}\right)^{-\frac{\sqrt{n}}{2}\frac{M_{nt}}{\sqrt{n}}}$
- $\lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^{nt} = e^{\mu t}$
  - Let $y = \left(1+\frac{\mu}{n}\right)^{nt} \Rightarrow \log(y) = nt\log\left(1+\frac{\mu}{n}\right)$
  - $\lim_{n\to\infty}\log(y) = \lim_{n\to\infty} t\frac{\log\left(1+\frac{\mu}{n}\right)}{1/n} = \lim_{n\to\infty} t\frac{\frac{-\frac{\mu}{n^2}}{1+\frac{\mu}{n}}}{-1/n^2} = \mu t$
- $\lim_{n\to\infty}\left(1-\frac{\sigma^2}{n}\right)^{\frac{nt}{2}} = e^{-\frac{\sigma^2 t}{2}}$
- $\lim_{n\to\infty}\left(1+\frac{\sigma}{\sqrt{n}}\right)^{\frac{\sqrt{n}}{2}\frac{M_{nt}}{\sqrt{n}}}\left(1-\frac{\sigma}{\sqrt{n}}\right)^{-\frac{\sqrt{n}}{2}\frac{M_{nt}}{\sqrt{n}}} = e^{\sigma W_t}$
  - $\lim_{n\to\infty}\left(1\pm\frac{\sigma}{\sqrt{n}}\right)^{\pm\frac{\sqrt{n}}{2}} = e^{\frac{\sigma}{2}}$
  - $\lim_{n\to\infty}\frac{M_{nt}}{\sqrt{n}} = W_t$

Moments of GBM

- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- $\mathbb{E}[S_t | S_0]$

Moments of GBM

- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- $\mathbb{E}[S_t | S_0] = S_0 e^{\mu t}$

$$\begin{aligned}
\mathbb{E}\left[e^{\sigma W_t} \middle| W_0\right] &= \int_{-\infty}^{\infty} e^{\sigma x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\ dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-\sigma t)^2}{2t}} e^{\frac{\sigma^2 t}{2}}\ dx \\
&= e^{\frac{\sigma^2 t}{2}}
\end{aligned}$$

- $\mathrm{Var}(S_t | S_0) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

SDE for GBM

- **Q**: Perturbation of stock prices w.r.t time?
- $X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$
- $S_t = e^{X_t} = f(t, W_t)$

SDE for GBM

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- $X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$
- $S_t = e^{X_t} = f(t, W_t)$
- For deterministic f, linear approximation: $f(t,x) \approx f(0,0) + f_t(0,0)t + f_x(0,0)x$
- Better approximation: $f(t,x) \approx f(0,0) + f_t(0,0)t + f_x(0,0)x + f_{tt}(0,0)t^2 + f_{tx}(0,0)tx + \frac{1}{2}f_{xx}(0,0)x^2$
- **Q**: What if we replace x with W_t?

SDE for GBM

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- **Q**: What if we replace x with W_t?
- $\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t^2] = \text{Var}(X_t) = t$
- Linear approximation (Ito's Lemma): $f(t, W_t) \approx f(0,0) + f_t(0,0)t + f_x(0,0)W_t + \frac{f_{xx}}{2}(0,0)t$
- SDE: $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$

Transition probability density function

- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- **Q:** What is the distribution of $S_t | S_0$?

Transition probability density function

- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- **Q:** What is the distribution of $S_t | S_0$?
- Write $g(X) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}X}$, where $X \sim \mathcal{N}(0,1)$
- $\mathbb{P}(g(X) \le x) = \mathbb{P}\left(X \le \frac{\ln(x) - \ln(S_0) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right) = \mathbb{P}(X \le g^{-1}(x))$
- PDF:

$$f_g(x) = \frac{d}{dx}\mathbb{P}(X \le g^{-1}(x)) = f_{\mathcal{N}}(g^{-1}(x))(g^{-1}(x))'$$

$$= \frac{1}{x\sqrt{2\pi\sigma^2 t}} e^{-\frac{\left(\ln(x) - \ln(S_0) - \left(\mu - \frac{\sigma^2}{2}\right)t\right)^2}{2\sigma^2 t}}.$$

Discrete vs continuous

- Consider stock price S_t and the log return $R_t = \ln\left(\frac{S_t}{S_0}\right)$
- Empirical observation: $\widehat{\mathbb{E}}[R_t]$ inaccurate, $\widehat{\mathsf{Var}}(R_t)$ accurate

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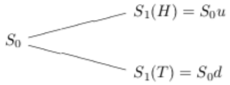
- Consider stock price S_t and the log return $R_t = \ln\left(\frac{S_t}{S_0}\right)$
- Empirical observation: $\widehat{\mathbb{E}}[R_t]$ inaccurate, $\widehat{\mathrm{Var}}(R_t)$ accurate
- Discrete Binomial model
  - Binomial model with up (H) and down (T)



  - For simplicity, let $d = \frac{1}{u}$
  - u known, p unknown
- Continuous geometric Brownian motion
  - $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
  - $R_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$
  - σ known, μ unknown
- **Q**: price of a call option with payoff $\max(S_T - K, 0)$?

Replicating

Replicating

- Wealth in discrete: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$
- Wealth in continuous: $dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t) \, dt$
  - Δ_t: buy and sell at continuous time
  - Neglect details (need PDE and SDE theory)
- PDE's approach
- SDE's approach:
  - Known that X_t to replicate V_t
  - Let $X_N = V_N$
  - Calculate $V_0 = \widetilde{\mathbb{E}} \left[e^{-rT} \max(S_T - K, 0) \middle| S_0 \right]$

PDE's approach for BSM

- For simplicity, assume $r = 0$
- Suppose $c(t, S_t)$ is the option price
- Perturbation of $c(t, S_t)$?

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- Perturbation of $c(t, S_t)$? By Ito's Lemma
- $dc(t, S_t) \approx c_t(t, S_t)dt + c_x(t, S_t)dS_t + \frac{c_{xx}(t, S_t)}{2} dS_t^2$
- $X_{n+1} = \Delta_n S_{n+1} + X_n - \Delta_n S_n$
- Perturbation: $dX_t = \Delta_t dS_t$
- Replicating strategy: $\Delta_t = c_x(t, S_t)$
- **Q**: What else do we need?

PDE's approach for BSM

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- Suppose $c(t, S_t)$ is the option price
- Perturbation of $c(t, S_t)$? By Ito's Lemma
- $dc(t, S_t) \approx c_t(t, S_t)dt + c_x(t, S_t)dS_t + \frac{c_{xx}(t,S_t)}{2} dS_t^2$
- $X_{n+1} = \Delta_n S_{n+1} + X_n - \Delta_n S_n$
- Perturbation: $dX_t = \Delta_t dS_t$
- Replicating strategy: $\Delta_t = c_x(t, S_t)$
- **Q**: What else do we need?
- $dc(t, S_t) \approx c_t(t, S_t)dt + c_x(t, S_t)dS_t + c_{xx}(t, S_t)\frac{\sigma^2}{2} S_t^2 \; dt$
- $c_t(t, S_t) + \frac{\sigma^2}{2} S_t^2 c_{xx}(t, S_t) = 0$
- Black-Scholes-Merton PDE:

$$c_t(t, x) + \frac{\sigma^2}{2} x^2 c_{xx}(t, x) = 0.$$

SDE's approach for BSM

- Suppose can be replicated (can be proved)
- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- $\mathbb{E}[S_t | S_0] = S_0 e^{\mu t}$
- Risk-neutral measure: $S_t = \widetilde{\mathbb{E}}[e^{-r(T-t)} S_T | S_t]$

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- Risk-neutral measure: $S_t = \widetilde{\mathbb{E}}[e^{-r(T-t)} S_T | S_t]$
- Let $\mu = r$: $S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$
- **Black-Scholes-Merton (BSM) formula:**

$$C(t, S_t) = \widetilde{\mathbb{E}} \left[e^{-r(T-t)} \max(S_T - K, 0) \middle| S_t \right],$$
$$P(t, S_t) = \widetilde{\mathbb{E}} \left[e^{-r(T-t)} \max(K - S_T, 0) \middle| S_t \right].$$

Call option formula

- GBM: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t - \sigma W_t}$
- BSM for call: $C(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \max(S_T - K, 0) \middle| S_t\right]$

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- BSM for call: $C(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \max(S_T - K, 0) \middle| S_t\right]$
- Let $\tau = T - t$, $c(t, x) = \int_{-\infty}^{\infty} e^{-r\tau} \max\left(xe^{\left(r - \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y} - K, 0\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$
- $xe^{\left(r - \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y} = K \Rightarrow \ln(x) + \left(r - \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}y = \ln(K)$
- Let $d_- = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{e^{r\tau}x}{K}\right) - \frac{\sigma^2\tau}{2}\right]$
- $c(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}y + \left(r - \frac{\sigma^2}{2}\right)\tau} - K\right) e^{-\frac{y^2}{2}}\, dy$
- Second term: $-Ke^{-r\tau}N(d_-)$

Call option formula - continued

- First term:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-r\tau}\left(xe^{-\sigma\sqrt{\tau}y+\left(r-\frac{\sigma^2}{2}\right)\tau}\right)e^{-\frac{y^2}{2}}\;dy$$

$$=e^{-r\tau}xe^{\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-\sigma\sqrt{\tau}y-\frac{y^2}{2}}\;dy$$

$$=e^{-r\tau}xe^{\left(r-\frac{\sigma^2}{2}\right)\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}e^{\frac{\sigma^2\tau}{2}}\;dy$$

- Let $z = y + \sigma\sqrt{\tau}$, $dz = dy$, $y = d_- \Rightarrow z = d_- + \sigma\sqrt{\tau} = d_+$
- First term: $xN(d_+)$
- BSM call formula: $C(t, S_t) = N(d_+)S_t - N(d_-)Ke^{-r(T-t)}$

Put option formula

- **Q**: A convenient way to calculate the put option?
- Call: $C(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \max(S_T - K, 0) \middle| S_t\right]$
- Put: $P(t, S_t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \max(K - S_T, 0) \middle| S_t\right]$

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- Put-Call parity:

$$\begin{aligned} C(t, S_t) - P(t, S_t) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S_T - K) \middle| S_t\right] \\ &= S_t - e^{-r(T-t)}K. \end{aligned}$$

- BSM put formula:

$$\begin{aligned} P(t, S_t) &= Ke^{-r(T-t)} - S_t + C(t, S_t) \\ &= Ke^{-r(T-t)}(1 - N(d_-)) + S_t(N(d_+) - 1) \\ &= Ke^{-r(T-t)}N(-d_-) - S_tN(-d_+). \end{aligned}$$

Greeks

- Call: $c(t,x) = N(d_+)x - N(d_-)Ke^{-r(T-t)}$
- Put: $p(t,x) = Ke^{-r(T-t)}N(-d_-) - S_t N(-d_+)$
- $d_- = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{e^{r\tau}x}{K}\right) - \frac{\sigma^2\tau}{2}\right]$
- $d_+ = d_- + \sigma\sqrt{\tau}$
- Common Greeks:
  - Delta: $\frac{\partial V}{\partial S} \geq 0$
  - Vega: $\frac{\partial V}{\partial \sigma} \geq 0$
  - Theta: $\frac{\partial V}{\partial T} \geq 0$
  - Rho: $\frac{\partial V}{\partial r} \geq 0$
  - Gamma: $\frac{\partial^2 V}{\partial S^2} \geq 0$
  - Vomma: $\frac{\partial^2 V}{\partial \sigma^2} \geq 0$

Delta

- Focus on call: $c(0,x) = N(d_+)x - N(d_-)Ke^{-rT}$
- $d_- = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{e^{rT}x}{K}\right) - \frac{\sigma^2 T}{2}\right], d_+ = d_- + \sigma\sqrt{T}$
- Delta: $\frac{\partial V}{\partial S}$

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- Delta: $\frac{\partial V}{\partial S}$

$$
\begin{aligned}
\frac{\partial c}{\partial x} &= x\frac{d}{dx}N(d_+) + N(d_+) - \frac{d}{dx}N(d_-)Ke^{-rT} \\
&= N(d_+) + x\frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}}\frac{\partial}{\partial x}d_+ - \frac{e^{-\frac{d_-^2}{2}}}{\sqrt{2\pi}}\frac{\partial}{\partial x}d_- Ke^{-rT} \\
&= N(d_+) + \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{d_-^2}{2} - \frac{\sigma^2 T}{2} - d_-\sigma\sqrt{T}} - \frac{Ke^{-rT}}{\sqrt{2\pi}\sigma\sqrt{T}x}e^{-\frac{d_-^2}{2}} \\
&= N(d_+) + \frac{1}{\sqrt{2\pi}\sigma\sqrt{T}}e^{-\frac{d_-^2}{2}}\left(e^{-\frac{\sigma^2 T}{2} - d_-\sigma\sqrt{T}} - \frac{Ke^{-rT}}{x}\right)
\end{aligned}
$$

- $e^{-d_-\sigma\sqrt{T}} = e^{\frac{\sigma^2 T}{2}}\frac{Ke^{-rT}}{x}$
- Delta: $\frac{\partial c(0,x)}{\partial x} = N(d_+) \geq 0$

Gamma

- Call: $c(0,x) = N(d_+)x - N(d_-)Ke^{-rT}$
- $d_- = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{e^{rT}x}{K}\right) - \frac{\sigma^2 T}{2}\right], d_+ = d_- + \sigma\sqrt{T}$
- Gamma: $\frac{\partial^2 V}{\partial S^2}$
- $\frac{\partial c(0,x)}{\partial x} = N(d_+)$

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$$\frac{\partial^2 c(0,x)}{\partial x^2} = N'(d_+)\frac{\partial d_+}{\partial x} = \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{d_+^2}{2}}}{x\sigma\sqrt{T}} \geq 0.$$

- Takeaway:

Gamma

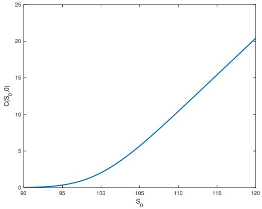
- Call: $c(0,x) = N(d_+)x - N(d_-)Ke^{-rT}$
- $d_- = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{e^{rT}x}{K}\right) - \frac{\sigma^2 T}{2}\right], d_+ = d_- + \sigma\sqrt{T}$
- Gamma: $\frac{\partial^2 V}{\partial S^2}$
- $\frac{\partial c(0,x)}{\partial x} = N(d_+)$

$$\frac{\partial^2 c(0,x)}{\partial x^2} = N'(d_+)\frac{\partial d_+}{\partial x} = \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{d_+^2}{2}}}{x\sigma\sqrt{T}} \geq 0.$$

- Takeaway:
  - Call price increases when underlying stock price increases
  - The speed of increase is faster and faster

Pattern

- Risk pattern



- Increasing marginal effect: $\forall \overline{x} \geq \widetilde{x}$, $\frac{c(\overline{x},\mathbf{x}_\neg)}{\overline{x}} \geq \frac{c(\widetilde{x},\mathbf{x}_\neg)}{\widetilde{x}}$

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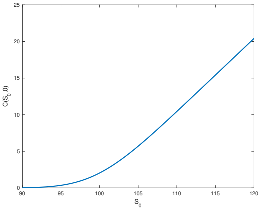

- Increasing marginal effect: $\forall \overline{x} \geq \widetilde{x}$, $\frac{c(\overline{x}, \mathbf{x}_\neg)}{\overline{x}} \geq \frac{c(\widetilde{x}, \mathbf{x}_\neg)}{\widetilde{x}}$

$$\begin{aligned}
\frac{\partial}{\partial x} \frac{c(\overline{x}, \mathbf{x}_\neg)}{\overline{x}} &= \frac{\frac{\partial}{\partial x} c(\overline{x}, \mathbf{x}_\neg)\overline{x} - c(\overline{x}, \mathbf{x}_\neg)}{\overline{x}^2} \\
&= \frac{\frac{\partial}{\partial x} c(\overline{x}, \mathbf{x}_\neg)\overline{x} - \int_0^{\overline{x}} \frac{\partial}{\partial x} c(y, \mathbf{x}_\neg)\ dy}{\overline{x}^2} \\
&\geq \frac{\frac{\partial}{\partial x} c(\overline{x}, \mathbf{x}_\neg)\overline{x} - \frac{\partial}{\partial x} c(\overline{x}, \mathbf{x}_\neg)\overline{x}}{\overline{x}^2} = 0.
\end{aligned}$$