Lecture II: Option Pricing in Discrete-Time

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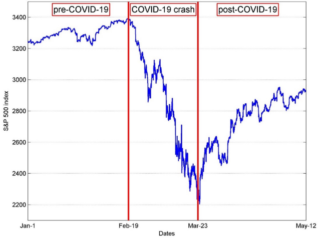
Option Pricing

Greeks

Multiperiod Binomial Model

Motivation

- Stock market is very volatile
- Reference: "S&P 500 index price spillovers around the COVID-19 market meltdown"



- Want "insurance"
- History goes back to ancient Greek
- **Question**: How would you design an insurance product?

European option

- **Def**: An **European option** is a contract which conveys to its owner the right, but not the obligation, to buy or sell a specific quantity of an underlying asset at a specified strike price on a specified date.
- Variables:
  - S_t - stock price
  - T - expiration date
  - K - strike price
- Types:
  - **Call option**: the option to buy, payoff $\max(S_T - K, 0)$
  - **Put option**: the option to sell, payoff $\max(K - S_T, 0)$

Examples of call options

- Motivation: Bet price will go up
- Suppose the current price of the stock is $\$30$
- Suppose a trader buys one call option contract on Amazon with a strike price $K = \$25$
- He pays $\$2$ for the call option
- On the expiration date, Amazon stock prices for $\$35$
- The holder of the option exercises his right to purchase Amazon at $\$25$ a share
- He immediately sells the shares at the current market price of $\$35$ per share
- His profit:

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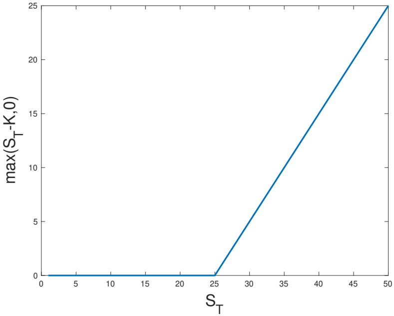
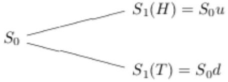


Binomial model

- Binomial model with up (H) and down (T)



- Probability of up and down: p and $q = 1 - p$
- Suppose the payoff of the option at $t = 1$ is $V_1(H)$ and $V_1(T)$
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- Suppose we have n i.i.d. return data with mean μ and variance σ^2
- $\widehat{\mu} = \overline{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$
- $\widehat{\sigma}^2 = S_n^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(R_i - \overline{R})^2$
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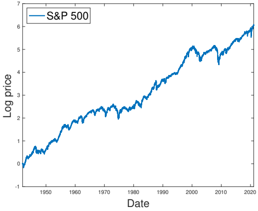
- $\text{Var}(\overline{R}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(R_i) = \frac{\sigma^2}{n}$
- $\text{Var}(S_n^2) = \frac{1}{4}\left[\theta_4 - \frac{n-3}{n-1}\sigma^4\right]$, where $\theta_4 = \mathbb{E}[(R_i - \mu)^4]$

Empirical results

- S&P 500 from 1942 to 2021:



- Daily result: $\overline{R} \approx 3 \times 10^{-4}, S_n^2 \approx 1 \times 10^{-4}$
- Annualized result: $252\overline{R} \approx 0.078, 252S_n^2 \approx 0.025$
- $\widehat{\mu} = 0.078$, $\widehat{\sigma} = 0.16$
- **Question**: for $n = 1000$ (about 40 years), how accurate?

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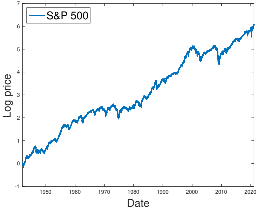

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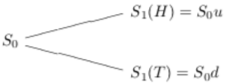
- $\widehat{\mu} = 0.078$, $\widehat{\sigma} = 0.16$
- **Question**: for $n = 1000$ (about 40 years), how accurate?
- $\sqrt{\mathrm{Var}(252\overline{R})} \approx 0.025$
- $\sqrt{\mathrm{Var}(252 S_n^2)} \approx 0.00036$
- Expected return is unreliable; volatility is reliable
- **Remark**: financial data is very noisy!

Binomial model

- Binomial model with up (H) and down (T)



- Another instrument: interest with rate r: $\$1$ at $t = 0$ will be $\$(1 + r)$ at $t = 1$
- **Question**: Relationship between u, d, r?

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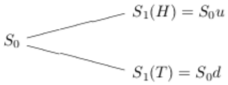

- Another instrument: interest with rate r: $\$1$ at $t = 0$ will be $\$(1 + r)$ at $t = 1$
- **Question**: Relationship between u, d, r? $0 < d < 1 + r < u$
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  - Call: $\max(S_1 - K, 0) = (S_1 - K)^+$
  - Put: $\max(K - S_1, 0) = (K - S_1)^+$
- Probability of up and down: p and $q = 1 - p$
- Remark:
  - u, r, d known
  - p, q unknown

Binomial model - continued

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Binomial model - continued

- We often work with $d = \frac{1}{u}$
- $R_i = \begin{cases} \log(u), & \text{up} \\ \log(d) = -\log(u), & \text{down} \end{cases}$
- $\mathbb{E}[R_i] = p\log(u) - (1-p)\log(u)$ unknown
- $\mathbb{E}[R_i^2] = p\log^2(u) + (1-p)\log^2(u) = \log^2(u)$ known
- Similar to the true market behavior

Example

- $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5$
- Call option payoff: $V_1(H) = 3$ and $V_1(T) = 0$
- **Question**: Replicate the result of option?

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- $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5$
- Call option payoff: $V_1(H) = 3$ and $V_1(T) = 0$
- **Question**: Replicate the result of option?
- $t = 0$: initial wealth $X_0 = 1.2$ and borrow 0.8, buy $\Delta_0 = \frac{1}{2}$ shares of stock
- Cash position at $t = 0$: $X_0 - \Delta_0 S_0 = -0.8$
- Cash position at $t = 1$: $(1 + r)(X_0 - \Delta_0 S_0) = -1$
- Stock value: $\frac{1}{2} S_1(H) = 4$ or $\frac{1}{2} S_1(T) = 1$
- Portfolio value at $t = 1$:

$$X_1(H) = \frac{1}{2} S_1(H) + (1 + r)(X_0 - \Delta_0 S_0) = 3,$$
$$X_1(T) = \frac{1}{2} S_1(T) + (1 + r)(X_0 - \Delta_0 S_0) = 0.$$

- **Question**: What should be the price of the option?

Example - continued

- X_0 is the **no-arbitrage price of the option at time zero**.
- Suppose price is 1.21

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- Suppose price is 1.21
  - Seller sells the option, get 1.21
  - $t = 0$: The seller invest the 0.01 in the money market and use remaining 1.2 to replicate the option
  - $t = 1$: get 0.0125
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  - $t = 1$: get 0.0125
- Suppose price is 1.19
  - Reverse the replicating strategy
  - Buyer buys the option, get -1.19
  - Sell short $1/2$ share of stock, get 0.81
  - Put 0.81 in the money market
  - If H, option 3, stock -4, money market 1.0125
  - If T, option 0, stock -1, money market 1.0125
  - $t = 1$: get 0.0125

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  - The interest rate for investing is the same as the interest rate for borrowing
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- **Question**: Which assumption we concern the most?
- The last one is most critical in practice
- **Question**: What is the price in the general binomial model?

Replication

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V_0 + \Delta_0 \left(\frac{1}{1+r} S_1(H) - S_0 \right) &= \frac{1}{1+r} V_1(H), \\
V_0 + \Delta_0 \left(\frac{1}{1+r} S_1(T) - S_0 \right) &= \frac{1}{1+r} V_1(T).
\end{aligned}
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- We can exactly replicate with $\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$.
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- Two variables Δ_0, V_0, two equations

Risk-neutral probability

- From supply and demand
- V_0 should be the discount expected value by market
- $V_0 = \frac{1}{1+r}(\widetilde{p}V_1(H) + \widetilde{q}V_1(T))$
- Recall

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- **Question**: What is $\widetilde{p}$ and $\widetilde{q}$?
- $V_0 + \Delta_0 \left(\frac{1}{1+r}[\widetilde{p}S_1(H) + \widetilde{q}S_1(T)] - S_0 \right) = \frac{1}{1+r}[\widetilde{p}V_1(H) + \widetilde{q}V_1(T)]$
- If $S_0 = \frac{1}{1+r}[\widetilde{p}S_1(H) + \widetilde{q}S_1(T)]$, then we are done

Risk-neural probability - continued

- $S_0 = \frac{1}{1+r}[\widetilde{p}uS_0 + (1-\widetilde{p})dS_0]$
- $1 + r = (u - d)\widetilde{p} + d$
- Solve this gives us

$$\widetilde{p} = \frac{1+r-d}{u-d},$$
$$\widetilde{q} = \frac{u-(1+r)}{u-d}.$$

- $0 < \widetilde{p} < 1$
- **Question**: Is this the "true" probability?

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- **Question**: Is this the "true" probability? No!
- $\widetilde{p}$ is called the **risk-neutral probability**.

Option Pricing

Greeks

Multiperiod Binomial Model

Symmetric case

- Call option: $V_1(S) = \max(S - K, 0)$
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- Let $d = \frac{1}{u}$
- $\widetilde{p} = \frac{1+r-\frac{1}{u}}{u-\frac{1}{u}} = \frac{u+ur-1}{u^2-1}$
- $V_0 = \frac{\widetilde{p}}{1+r}(uS_0 - K)$
- **Question**: How sensitive is the option?

Delta

- Pricing formula:$V_0 = \frac{\widetilde{p}}{1+r}(uS_0 - K)$ with $\widetilde{p} = \frac{u+ur-1}{u^2-1}$
- Delta: $\frac{\partial V_0}{\partial S_0}$

Delta

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- Delta: $\frac{\partial V_0}{\partial S_0} = \frac{\widetilde{p}u}{1+r} > 0$

Vega

- Pricing formula:$V_0 = \frac{\widetilde{p}}{1+r}(uS_0 - K)$ with $\widetilde{p} = \frac{u+ur-1}{u^2-1}$
- $V_0 = \frac{1}{1+r}\frac{(u(1+r)-1)(uS_0-K)}{u^2-1}$
- Vega: $\frac{\partial V_0}{\partial u}$

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- Vega: $\frac{\partial V_0}{\partial u}$

$$\begin{aligned}
\frac{\partial V_0}{\partial u} &= \frac{\left((1+r)(uS_0-K) + [u(1+r)-1]S_0\right)(u^2-1)}{(1+r)(u^2-1)^2} \\
&- \frac{2u[u(1+r)-1](uS_0-K)}{(1+r)(u^2-1)^2} \\
&= \frac{(uS_0-K)\left[(1+r)u^2-(1+r)-u^2(1+r)+u\right]}{(1+r)(u^2-1)^2} \\
&- \frac{[u(1+r)-1](S_0u^2-S_0-u^2S_0+uK)}{(1+r)(u^2-1)^2} \\
&= \frac{(uS_0-K)(u-(1+r))}{(1+r)(u^2-1)^2} + \frac{[u(1+r)-1](uK-S_0)}{(1+r)(u^2-1)^2} \geq 0
\end{aligned}$$

Linear approximation

- **Q:** How to take advantages of Greeks?
- Suppose we worry about the fluctuations of option prices

Linear approximation

- **Q:** How to take advantages of Greeks?
- Suppose we worry about the fluctuations of option prices
- Use a linear approximation $V(S,u,r) = a + b \cdot S + c \cdot u$
- Know $V(S_0,u_0), \frac{\partial V}{\partial S}(S_0,u_0), \frac{\partial V}{\partial u}(S_0,u_0)$

Linear approximation

- **Q:** How to take advantages of Greeks?
- Suppose we worry about the fluctuations of option prices
- Use a linear approximation $V(S, u, r) = a + b \cdot S + c \cdot u$
- Know $V(S_0, u_0)$, $\frac{\partial V}{\partial S}(S_0, u_0)$, $\frac{\partial V}{\partial u}(S_0, u_0)$
- Linear approximation:

$$V(S, u) \approx V(S_0, u_0) + \frac{\partial V}{\partial S}(S_0, u_0)(S - S_0) + \frac{\partial V}{\partial u}(S_0, u_0)(u - u_0).$$

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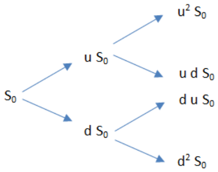
- Two-period binomial model:



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Multiperiod Binomial model

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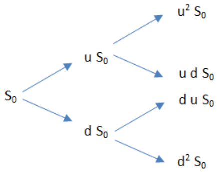
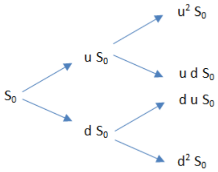


- Equations to solve for $\Delta_0, \Delta_1(H), \Delta_1(T), X_1(H), X_1(T), V_0$

- Two-period binomial model:



- Equations to solve for $\Delta_0, \Delta_1(H), \Delta_1(T), X_1(H), X_1(T), V_0$

$$
\begin{aligned}
X_1(H) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0), \\
X_1(T) &= \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0), \\
V_2(HH) &= \Delta_1(H) S_2(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)), \\
V_2(HT) &= \Delta_1(H) S_2(HT) + (1+r)(X_1(H) - \Delta_1(H) S_1(H)), \\
V_2(TH) &= \Delta_1(T) S_2(TH) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)), \\
V_2(TT) &= \Delta_1(T) S_2(TT) + (1+r)(X_1(T) - \Delta_1(T) S_1(T)).
\end{aligned}
$$

Replication formula

- $0 < d < 1 + r < u$
- Risk-neural probability:

$$\widetilde{p} = \frac{1+r-d}{u-d},$$
$$\widetilde{q} = \frac{u-1-r}{u-d}.$$

- Toss results $\omega_1 \dots \omega_N$
- $V_n(\omega_1 \dots \omega_n) = \frac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1 \dots \omega_n H) + \widetilde{q}V_{n+1}(\omega_1 \dots \omega_n T)]$
- $\Delta_n(\omega_1 \dots \omega_n) = \frac{V_{n+1}(\omega_1 \dots \omega_n H) - V_{n+1}(\omega_1 \dots \omega_n T)}{S_{n+1}(\omega_1 \dots \omega_n H) - S_{n+1}(\omega_1 \dots \omega_n T)}.$
- $X_N(\omega_1 \dots \omega_n) = V_N(\omega_1 \dots \omega_n)$
- We have the general formula!

Example

- $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5, N = 2$
- $\widetilde{p} = \widetilde{q} = \frac{1}{2}$
- $V_1(HH) = 11, V_1(HT) = V_1(TH) = 0, V_1(TT) = 0$

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- $V_1(HH) = 11, V_1(HT) = V_1(TH) = 0, V_1(TT) = 0$

$$
\begin{aligned}
V_1(H) &= \frac{\widetilde{p}V_1(HH) + \widetilde{q}V_1(HT)}{1+r} = 4.4, \\
V_1(T) &= \frac{\widetilde{p}V_1(TH) + \widetilde{q}V_1(TT)}{1+r} = 0, \\
V_0 &= \frac{\widetilde{p}V_1(H) + \widetilde{q}V_1(T)}{1+r} = 1.76.
\end{aligned}
$$

Binomial distribution

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- Binomial random variable $X \sim B(n,p)$
- Probability Mass Function (PMF): $\mathbb{P}(X=k) = \binom{N}{k} p^k (1-p)^{N-k}$
- Under the risk-neutral probability:

$$V_0 = \frac{\widetilde{\mathbb{E}}[V_N]}{(1+r)^N}$$
$$= \frac{\sum_{k=0}^{N} \binom{N}{k} \widetilde{p}^k (1-\widetilde{p})^{N-k} \max(u^k d^{N-k} S_0 - K, 0)}{(1+r)^N}.$$

- E.g. $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5, N = 2$

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$$V_0 = \frac{\binom{2}{2} 0.5^2 11}{1.25^2} = 1.76.$$

Lookback option

- Consider path-dependent options
- Lookback option $V_T = \max_{t\in[0,T]}(S_t - S_T)$
- E.g., $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, n = 2$
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$$
\begin{aligned}
V_2(HH) &= S_2(HH) - S_2(HH) = 0,\\
V_2(HT) &= S_1(H) - S_2(HT) = 4,\\
V_2(TH) &= S_0 - S_2(TH) = 0,\\
V_2(TT) &= S_0 - S_2(TT) = 3.
\end{aligned}
$$

- Price formula?

Lookback option

- Consider path-dependent options
- Lookback option $V_T = \max_{t \in [0,T]}(S_t - S_T)$
- E.g., $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, n = 2$
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\begin{aligned}
V_2(HH) &= S_2(HH) - S_2(HH) = 0, \\
V_2(HT) &= S_1(H) - S_2(HT) = 4, \\
V_2(TH) &= S_0 - S_2(TH) = 0, \\
V_2(TT) &= S_0 - S_2(TT) = 3.
\end{aligned}
$$

- Price formula? $\widetilde{p} = \widetilde{q} = \frac{1}{2}$

$$
\begin{aligned}
V_n(\omega_1 \dots \omega_n) &= \frac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1 \dots \omega_n H) + \widetilde{q}V_{n+1}(\omega_1 \dots \omega_n T)] \\
&= \frac{2}{5}[V_{n+1}(\omega_1 \dots \omega_n H) + V_{n+1}(\omega_1 \dots \omega_n T)]
\end{aligned}
$$

Lookback option - continued

- At $t = 1$:

$$V_1(H) = \frac{2}{5}(V_2(HH) + V_2(HT)) = 1.6,$$
$$V_1(T) = \frac{2}{5}(V_2(TH) + V_2(TT)) = 1.2.$$

- At $t = 0$:

$$V_0 = \frac{2}{5}(V_1(H) + V_1(T)) = 1.12.$$