HW4

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Problem 1

(Exercise 2.4)

Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the j~th toss results in a head and $X_j = -1$ if the j~th toss results in a tail. Consider the stochastic process $M_0, M_1, M_2, \ldots$ defined by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, \quad n \geq 1.$$

This is called a symmetric random walk, with each head, it steps up one, and with each tail, it steps down one.

(i) Using the properties of Theorem 2.3.2, show that $M_0, M_1, M_2, \ldots$ is a martingale.

(ii) Let σ be a positive constant and, for $n \geq 0$, define

$$S_n = e^{\sigma M_n} (\frac{2}{e^{\sigma} + e^{-\sigma}})^n$$

Show that $S_0, S_1, S_2, \ldots$ is a martingale. Note that even though the symmetric random walk M_n has no tendency to grow, the "geometric symmetric random walk" $e^{\sigma M_n}$ does have a tendency to grow. This is the result of putting a martingale into the (convex) exponential function (see Exercise 2.3). In order to again have a martingale, we must "discount" the geometric symmetric random walk, using the term $\frac{2}{e^{\sigma}+e^{-\sigma}}$ as the discount rate. This term is strictly less than one unless $\sigma = 0$.

> **Theorem 2.3.2 (Fundamental properties of conditional expectations)**
>
> Let N be a positive integer, and let X and Y be random variables depending on the first N coin tosses. Let $0 \leq n \leq N$ be given. The following properties hold.
>
> (i) **Linearity of conditional expectations.** For all constants c_1 and c_2, we have
>
> $$E_n[c_1 X + c_2 Y] = c_1 E_n[X] + c_2 E_n[Y]$$
>
> (ii) **Taking out what is known.** If X actually depends only on the first n coin tosses, then
>
> $$E_n[XY] = X \cdot E_n[Y]$$
>
> (iii) **Iterated conditioning.** If $0 < n < m \leq N$, then
>
> $$E_n[E_m[X]] = E_n[X]$$
>
> In particular, $E[E_m[X]] = E[X]$.

> (iv) **Independence.** If X depends only on tosses $n+1$ through N, then
>
> $$E_n[X] = E[X]$$
>
> (v) **Conditional Jensen's inequality.** If $\varphi(s)$ is a convex function of the dummy variable s, then
>
> $$E_n[\varphi(X)] \geq \varphi(E_n[X])$$

Solution

(i) To show that $M_0, M_1, M_2, \ldots$ is a martingale, we need to verify the three conditions:

1. $E[|M_n|] < \infty$ for all $n \geq 0$
2. M_n is measurable with respect to the information up to time n
3. $E[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n$ for all $n \geq 0$

For condition 1, since M_n is a sum of n random variables, each taking values ± 1, we have $|M_n| \leq n$, so $E[|M_n|] < \infty$.

For condition 2, M_n depends only on the first n tosses, so it is measurable with respect to the information up to time n.

For condition 3, we have:

$$E[M_{n+1}|M_0, M_1, \ldots, M_n] = E[M_n + X_{n+1}|M_0, M_1, \ldots, M_n]$$

By linearity of conditional expectation (Theorem 2.3.2(i)):

$$E[M_n + X_{n+1}|M_0, M_1, \ldots, M_n] = E[M_n|M_0, M_1, \ldots, M_n] + E[X_{n+1}|M_0, M_1, \ldots, M_n]$$

Since M_n is known given $M_0, M_1, \ldots, M_n$, we have $E[M_n|M_0, M_1, \ldots, M_n] = M_n$.

For $E[X_{n+1}|M_0, M_1, \ldots, M_n]$, since X_{n+1} is independent of the first n tosses, by Theorem 2.3.2(iv):

$$E[X_{n+1}|M_0, M_1, \ldots, M_n] = E[X_{n+1}] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$$

Therefore:

$$E[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n + 0 = M_n$$

Thus, $M_0, M_1, M_2, \ldots$ is a martingale.

(ii) To show that $S_0, S_1, S_2, \ldots$ is a martingale, we need to verify the same three conditions.

First, let's compute S_0:

$$S_0 = e^{\sigma M_0} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^0 = e^0 \cdot 1 = 1$$

Now, let's verify the martingale property:

$$E[S_{n+1}|S_0, S_1, \ldots, S_n] = E\left[e^{\sigma M_{n+1}} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \middle| S_0, S_1, \ldots, S_n \right]$$

Since $M_{n+1} = M_n + X_{n+1}$, we have:

$$E[S_{n+1}|S_0, S_1, \dots, S_n] = E\left[e^{\sigma(M_n+X_{n+1})}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^{n+1}\Bigg|S_0, S_1, \dots, S_n\right]$$

$$= E\left[e^{\sigma M_n} \cdot e^{\sigma X_{n+1}}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n \cdot \frac{2}{e^\sigma + e^{-\sigma}}\Bigg|S_0, S_1, \dots, S_n\right]$$

Since M_n is known given $S_0, S_1, \dots, S_n$, by Theorem 2.3.2(ii):

$$= e^{\sigma M_n}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n \cdot \frac{2}{e^\sigma + e^{-\sigma}} \cdot E\left[e^{\sigma X_{n+1}}\Big|S_0, S_1, \dots, S_n\right]$$

Since X_{n+1} is independent of the first n tosses, by Theorem 2.3.2(iv):

$$= e^{\sigma M_n}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n \cdot \frac{2}{e^\sigma + e^{-\sigma}} \cdot E\left[e^{\sigma X_{n+1}}\right]$$

Now, $E\left[e^{\sigma X_{n+1}}\right] = \frac{1}{2}e^\sigma + \frac{1}{2}e^{-\sigma} = \frac{e^\sigma + e^{-\sigma}}{2}$

Therefore:

$$= e^{\sigma M_n}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n \cdot \frac{2}{e^\sigma + e^{-\sigma}} \cdot \frac{e^\sigma + e^{-\sigma}}{2}$$

$$= e^{\sigma M_n}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n = S_n$$

Thus, $S_0, S_1, S_2, \dots$ is a martingale.

Problem 2

(Exercise 2.5)

Let $M_0, M_1, M_2, \dots$ be the symmetric random walk of Exercise 2.4, and define $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j), \quad n = 1, 2, \dots$$

(i) Show that

$$I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$$

(ii) Let n be an arbitrary nonnegative integer, and let $f(i)$ be an arbitrary function of a variable i. In terms of n and f, define another function $g(i)$ satisfying

$$E_n[f(I_{n+1})] = g(I_n)$$

Note that although the function $g(I_n)$ on the right-hand side of this equation may depend on n, the only random variable that may appear in its argument is I_n; the random variable M_n may not appear. You will need to use the formula in part (i). The conclusion of part (ii) is that the process $I_0, I_1, I_2, \dots$ is a Markov process.

Solution:

(i)

We need to show that $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$.

Let's use induction. For the base case, $n = 0$: $I_0 = 0$ and $\frac{1}{2}M_0^2 - \frac{0}{2} = 0$, so the formula holds.

Now assume the formula holds for n, i.e., $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$.

For $n+1$, we have:

$$I_{n+1} = \sum_{j=0}^{n} M_j(M_{j+1} - M_j)$$

$$= \sum_{j=0}^{n-1} M_j(M_{j+1} - M_j) + M_n(M_{n+1} - M_n)$$

$$= I_n + M_n(M_{n+1} - M_n)$$

Using our induction hypothesis:

$$I_{n+1} = \frac{1}{2}M_n^2 - \frac{n}{2} + M_n(M_{n+1} - M_n)$$

$$= \frac{1}{2}M_n^2 - \frac{n}{2} + M_nM_{n+1} - M_n^2$$

$$= -\frac{1}{2}M_n^2 - \frac{n}{2} + M_nM_{n+1}$$

Since $M_{n+1} = M_n + X_{n+1}$ where $X_{n+1} = \pm 1$, we have:

$$I_{n+1} = -\frac{1}{2}M_n^2 - \frac{n}{2} + M_n(M_n + X_{n+1})$$

$$= -\frac{1}{2}M_n^2 - \frac{n}{2} + M_n^2 + M_nX_{n+1}$$

$$= \frac{1}{2}M_n^2 - \frac{n}{2} + M_nX_{n+1}$$

Now, $M_{n+1}^2 = (M_n + X_{n+1})^2 = M_n^2 + 2M_nX_{n+1} + X_{n+1}^2$. Since $X_{n+1}^2 = 1$, we have:

$$M_{n+1}^2 = M_n^2 + 2M_nX_{n+1} + 1$$

Solving for M_nX_{n+1}:

$$M_nX_{n+1} = \frac{M_{n+1}^2 - M_n^2 - 1}{2}$$

Substituting this into our expression for I_{n+1}:

$$I_{n+1} = \frac{1}{2}M_n^2 - \frac{n}{2} + \frac{M_{n+1}^2 - M_n^2 - 1}{2}$$

$$= \frac{1}{2}M_n^2 - \frac{n}{2} + \frac{M_{n+1}^2}{2} - \frac{M_n^2}{2} - \frac{1}{2}$$

$$= -\frac{n}{2} + \frac{M_{n+1}^2}{2} - \frac{1}{2}$$

$$= \frac{1}{2}M_{n+1}^2 - \frac{n+1}{2}$$

This completes the induction, proving that $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$ for all $n \geq 0$.

(ii)

We need to find a function $g(I_n)$ such that $E_n[f(I_{n+1})] = g(I_n)$.

From part (i), we know that $I_{n+1} = \frac{1}{2}M_{n+1}^2 - \frac{n+1}{2}$.

Since $M_{n+1} = M_n + X_{n+1}$ where $X_{n+1} = \pm 1$ with equal probability, we have:

$$M_{n+1}^2 = (M_n + X_{n+1})^2 = M_n^2 + 2M_nX_{n+1} + 1$$

Therefore:

$$I_{n+1} = \frac{1}{2}(M_n^2 + 2M_nX_{n+1} + 1) - \frac{n+1}{2}$$

$$= \frac{M_n^2}{2} + M_nX_{n+1} + \frac{1}{2} - \frac{n+1}{2}$$

$$= \frac{M_n^2}{2} + M_nX_{n+1} - \frac{n}{2}$$

Using the formula from part (i), we can substitute $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$, which gives us:

$$I_{n+1} = I_n + M_nX_{n+1}$$

Now, $E_n[f(I_{n+1})]$ means the expected value of $f(I_{n+1})$ given the information up to time n. Since X_{n+1} is independent of the past and takes values ± 1 with equal probability:

$$E_n[f(I_{n+1})] = E_n[f(I_n + M_nX_{n+1})]$$

$$= \frac{1}{2}f(I_n + M_n) + \frac{1}{2}f(I_n - M_n)$$

Therefore, we can define:

$$g(I_n) = \frac{1}{2}f(I_n + M_n) + \frac{1}{2}f(I_n - M_n)$$

However, this still contains M_n, which is not allowed. We need to express M_n in terms of I_n.

From part (i), we have $I_n = \frac{1}{2}M_n^2 - \frac{n}{2}$, which gives us:

$$M_n^2 = 2I_n + n$$

Since M_n can be either positive or negative, we have $M_n = \pm\sqrt{2I_n + n}$.

Given the symmetry of the random walk, both values of M_n are equally likely given I_n. Therefore:

$$g(I_n) = \frac{1}{2}f(I_n + \sqrt{2I_n + n}) + \frac{1}{2}f(I_n - \sqrt{2I_n + n})$$

This function $g(I_n)$ depends only on I_n and n, not on M_n directly, satisfying the requirement. The fact that $E_n[f(I_{n+1})] = g(I_n)$ for any function f confirms that $I_0, I_1, I_2, \ldots$ is a Markov process.

Problem 3

(Exercise 2.6 Discrete-time stochastic integral).

Suppose $M_0, M_1, \ldots, M_N$ is a martingale, and let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted process. Define the discrete-time stochastic integral (sometimes called a martingale transform) $I_0, I_1, \ldots, I_N$ by setting $I_0 = 0$ and

$$I_n = \sum_{k=0}^{n-1} \Delta_k(M_{k+1} - M_k), \quad 1 \le n \le N$$

Show that $I_0, I_1, \ldots, I_N$ is a martingale.

Solution

To show that $I_0, I_1, \ldots, I_N$ is a martingale, we need to verify three conditions:

1. I_n is adapted to the filtration
2. $E[|I_n|] < \infty$ for all n
3. $E[I_{n+1}|\mathcal{F}_n] = I_n$ for all $0 \le n < N$

First, since Δ_k is adapted and M_k is adapted, the product $\Delta_k(M_{k+1} - M_k)$ is adapted. Therefore, I_n as a sum of adapted random variables is also adapted.

Second, assuming that $E[|M_n|] < \infty$ for all n (which is true for a martingale) and that Δ_k is bounded or at least satisfies $E[|\Delta_k(M_{k+1} - M_k)|] < \infty$, we have $E[|I_n|] < \infty$ by the triangle inequality.

Third, for the martingale property, we need to show that $E[I_{n+1}|\mathcal{F}_n] = I_n$:

$$E[I_{n+1}|\mathcal{F}_n] = E\left[\sum_{k=0}^{n} \Delta_k(M_{k+1} - M_k)|\mathcal{F}_n\right]$$

We can split this sum into two parts:

$$E[I_{n+1}|\mathcal{F}_n] = E\left[\sum_{k=0}^{n-1} \Delta_k(M_{k+1} - M_k)|\mathcal{F}_n\right] + E[\Delta_n(M_{n+1} - M_n)|\mathcal{F}_n]$$

The first term equals I_n since it's $\mathcal{F}_n$-measurable. For the second term:

$$E[\Delta_n(M_{n+1} - M_n)|\mathcal{F}_n] = \Delta_n \cdot E[(M_{n+1} - M_n)|\mathcal{F}_n]$$

Since Δ_n is $\mathcal{F}_n$-measurable, we can take it outside the conditional expectation. And since M_n is a martingale, we have $E[M_{n+1}|\mathcal{F}_n] = M_n$, which implies:

$$E[(M_{n+1} - M_n)|\mathcal{F}_n] = E[M_{n+1}|\mathcal{F}_n] - M_n = M_n - M_n = 0$$

Therefore:

$$E[\Delta_n(M_{n+1} - M_n)|\mathcal{F}_n] = \Delta_n \cdot 0 = 0$$

This gives us:

$$E[I_{n+1}|\mathcal{F}_n] = I_n + 0 = I_n$$

Thus, $I_0, I_1, \ldots, I_N$ is indeed a martingale.

Problem 4

(Exercise 2.11) Put-call parity

Consider a stock that pays no dividend in an N-period binomial model. A European call has payoff $C_N = (S_N - K)^+$ at time N. The price C_n of this call at earlier times is given by the risk-neutral pricing formula (2.4.11):

$$C_n = \tilde{\mathbb{E}}_n[\frac{C_N}{(1+r)^{N-n}}], \quad n = 0, 1, \ldots, N-1.$$

Consider also a put with payoff $P_N = (K - S_N)^+$ at time N, whose price at earlier times is:

$$P_n = \tilde{\mathbb{E}}_n[\frac{P_N}{(1+r)^{N-n}}], \quad n = 0, 1, \ldots, N-1.$$

Finally, consider a forward contract to buy one share of stock at time N for K dollars. The price of this contract at time N is $F_N = S_N - K$, and its price at earlier times is:

$$F_n = \tilde{\mathbb{E}}_n[\frac{F_N}{(1+r)^{N-n}}], \quad n = 0, 1, \ldots, N-1.$$

(Note that, unlike the call, the forward contract requires that the stock be purchased at time N for K dollars and has a negative payoff if $S_N < K$.)

(i) If at time zero you buy a forward contract and a put, and hold them until expiration, explain why the payoff you receive is the same as the payoff of a call; i.e., explain why $C_N = F_N + P_N$.

(ii) Using the risk-neutral pricing formulas given above for C_n, P_n, and F_n and the linearity of conditional expectations, show that $C_n = F_n + P_n$ for every n.

(iii) Using the fact that the discounted stock price is a martingale under the risk-neutral measure, show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

(iv) Suppose you begin at time zero with F_0, buy one share of stock, borrowing money as necessary to do that, and make no further trades. Show that at time N you have a portfolio valued at F_N. (This is called a static replication of the forward contract. If you sell the forward contract for F_0 at time zero, you can use this static replication to hedge your short position in the forward contract.)

(v) The forward price of the stock at time zero is defined to be that value of K that causes the forward contract to have price zero at time zero. The forward price in this model is $(1+r)^N S_0$. Show that, at time zero, the price of a call struck at the forward price is the same as the price of a put struck at the forward price. This fact is called put-call parity.

(vi) If we choose $K = (1+r)^N S_0$, we just saw in (v) that $C_0 = P_0$. Do we have $C_n = P_n$ for every n?

Solution

(i) At time N, the payoff of the forward contract is $F_N = S_N - K$, and the payoff of the put is $P_N = (K - S_N)^+$. When we combine these:

$F_N + P_N = (S_N - K) + (K - S_N)^+$

If $S_N \geq K$, then $(K - S_N)^+ = 0$, so $F_N + P_N = S_N - K = (S_N - K)^+ = C_N$.

If $S_N < K$, then $(K - S_N)^+ = K - S_N$, so
$F_N + P_N = (S_N - K) + (K - S_N) = 0 = (S_N - K)^+ = C_N$.

Therefore, $F_N + P_N = C_N$ in all cases.

(ii) Using the risk-neutral pricing formulas and the linearity of conditional expectations:

$$C_n = \tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \tilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] = F_n +$$

(iii) Using the fact that the discounted stock price is a martingale under the risk-neutral measure:

$$F_0 = \tilde{\mathbb{E}}_0\left[\frac{F_N}{(1+r)^N}\right] = \tilde{\mathbb{E}}_0\left[\frac{S_N - K}{(1+r)^N}\right] = \tilde{\mathbb{E}}_0\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N}$$

Since the discounted stock price is a martingale under the risk-neutral measure, we have:

$$\tilde{\mathbb{E}}_0\left[\frac{S_N}{(1+r)^N}\right] = S_0$$

Therefore:

$$F_0 = S_0 - \frac{K}{(1+r)^N}$$

(iv) At time 0, we start with $F_0 = S_0 - \frac{K}{(1+r)^N}$. We buy one share of stock for S_0, borrowing $S_0 - F_0 = \frac{K}{(1+r)^N}$.

At time N, we need to repay the loan with interest, which amounts to $\frac{K}{(1+r)^N} \cdot (1+r)^N = K$. Our portfolio consists of one share of stock worth S_N, minus the debt repayment of K, giving us $S_N - K = F_N$.

(v) Using the put-call parity from (ii), we have $C_0 = F_0 + P_0$. If $K = (1+r)^N S_0$, then from (iii):

$$F_0 = S_0 - \frac{K}{(1+r)^N} = S_0 - \frac{(1+r)^N S_0}{(1+r)^N} = S_0 - S_0 = 0$$

Therefore, $C_0 = 0 + P_0 = P_0$, which means the price of a call struck at the forward price equals the price of a put struck at the forward price.

(vi) No, we don't necessarily have $C_n = P_n$ for every n when $K = (1+r)^N S_0$. At time 0, we have $F_0 = 0$ which gives us $C_0 = P_0$. However, as time progresses, the stock price evolves, and the forward price changes. At time $n > 0$, we generally have $F_n \neq 0$, which means $C_n = F_n + P_n \neq P_n$ unless $F_n = 0$.

Problem 5

Exercise 2.12 (Chooser option)

Let $1 \le m \le N-1$ and $K > 0$ be given. A chooser option is a contract sold at time zero that confers on its owner the right to receive either a call or a put at time m. The owner of the chooser may wait until time m before choosing. The call or put chosen expires at time N with strike price K. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time N and having strike price K, and a call, expiring at time m and having strike price $\frac{K}{(1+r)^{N-m}}$. (Hint: Use put-call parity from Exercise 2.11).

Solution

Let's denote the time-zero price of the chooser option as CH_0. At time m, the owner will choose either a call or a put, whichever has the higher value at that time.

$$CH_0 = \tilde{E}_0\left[\frac{\max(C_m, P_m)}{(1+r)^m}\right]$$

From put-call parity (Problem 4, part ii), we know that $C_m = F_m + P_m$, where F_m is the forward price at time m.

Therefore:

$$\max(C_m, P_m) = \max(F_m + P_m, P_m) = P_m + \max(F_m, 0)$$

Since $\max(F_m, 0) = (F_m)^+$ is the payoff of a call option on the forward price with strike 0, we have:

$$CH_0 = \tilde{E}_0\left[\frac{P_m + (F_m)^+}{(1+r)^m}\right] = \tilde{E}_0\left[\frac{P_m}{(1+r)^m}\right] + \tilde{E}_0\left[\frac{(F_m)^+}{(1+r)^m}\right]$$

The first term is the time-zero price of a put option that expires at time m. However, we need a put that expires at time N. Using the risk-neutral pricing formula:

$$\tilde{E}_0\left[\frac{P_m}{(1+r)^m}\right] = \tilde{E}_0\left[\frac{1}{(1+r)^m}\tilde{E}_m\left[\frac{P_N}{(1+r)^{N-m}}\right]\right] = \tilde{E}_0\left[\frac{P_N}{(1+r)^N}\right] = P_0$$

For the second term, we need to evaluate $(F_m)^+$. From Problem 4, part iii, we know that $F_m = S_m - \frac{K}{(1+r)^{N-m}}$. Therefore:

$$(F_m)^+ = \max\left(S_m - \frac{K}{(1+r)^{N-m}}, 0\right)$$

This is the payoff of a call option on the stock with strike price $\frac{K}{(1+r)^{N-m}}$ at time m. Let's denote this call option as C_0^*.

Therefore, the time-zero price of the chooser option is:

$$CH_0 = P_0 + C_0^*$$

where P_0 is the time-zero price of a put option expiring at time N with strike K, and C_0^* is the time-zero price of a call option expiring at time m with strike $\frac{K}{(1+r)^{N-m}}$.

Problem 6

Simulate some trajectories of drifted and scaled Brownian motion and visualize them.

In [1]:

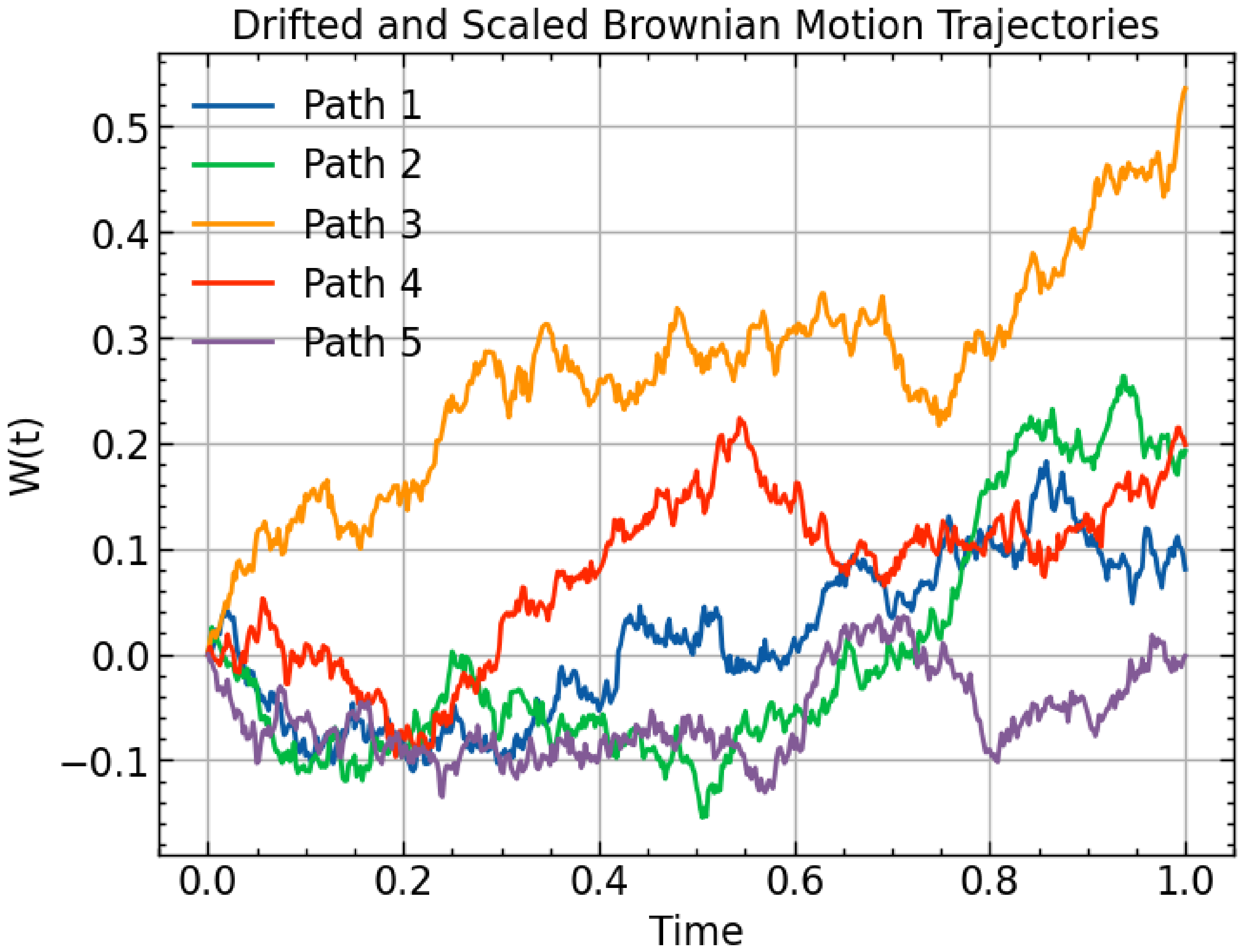
```
import numpy as np
import matplotlib.pyplot as plt
import scienceplots
plt.style.use(['science', 'notebook'])

T = 1.0
N = 500
dt = T / N  # Time step size
num_paths = 5
mu = 0.05  # Drift parameter
sigma = 0.2  # Volatility parameter

t = np.linspace(0, T, N+1)
np.random.seed(42)

plt.figure()
for i in range(num_paths):
    dB = np.random.normal(0, np.sqrt(dt), N)  # Standard Brownian increments
    W = np.concatenate(([0], np.cumsum(mu * dt + sigma * dB)))  # drift, scaled

    plt.plot(t, W, label=f'Path {i+1}')
plt.title('Drifted and Scaled Brownian Motion Trajectories')
plt.xlabel('Time')
plt.ylabel('W(t)')
plt.grid(True)
plt.legend()
plt.show()
```



Drifted and Scaled Brownian Motion Trajectories
Path 1
Path 2
Path 3
Path 4
Path 5
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
W(t)
0.0
0.2
0.4
0.6
0.8
1.0
Time

In [ ]: