HW4 solution

Problem 1. *3.1*

Solution 1. $\mathbb{E}[W^2(t) - t|\mathcal{F}_s] = \mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - W(s)^2|\mathcal{F}_s] - t = t - s + 2W(s)^2 - W(s)^2 - t = W(s)^2 - s.$

Problem 2. *3.3*

Solution 2. *We do the calculation and have*

$$\begin{aligned}
\varphi'(u) &= \sigma^2 u e^{\frac{\sigma^2 u}{2}}, \\
\varphi''(u) &= \left(\sigma^2 + \sigma^4 u^2\right) e^{\frac{\sigma^2 u^2}{2}}, \\
\varphi'''(u) &= \left(3\sigma^4 u + \sigma^6 u^3\right) e^{\frac{\sigma^2 u^2}{2}}, \\
\varphi''''(u) &= \left(3\sigma^4 + 6\sigma^6 u^2 + \sigma^8 u^4\right) e^{\frac{\sigma^2 u^2}{2}}.
\end{aligned}$$

Therefore, $\mathbb{E}[(X-\mu)^4] = \varphi''''(0) = 3\sigma^4$.

Problem 3. *3.6(i)*

Solution 3. *We calculate*

$$\mathbb{E}[f(X_t)|\mathcal{F}_t] = \mathbb{E}[f(W_t - W_s + (W_s + \mu t))|\mathcal{F}_s] = \int_{-\infty}^{\infty} f(x + W_s + \mu t)\frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\, dx.$$

We do change of variable, let $y = x + W_s + \mu t$, *then*

$$\mathbb{E}[f(X_t)|\mathcal{F}_t] = \int_{-\infty}^{\infty} f(y)\frac{e^{\frac{-(y-W_s-\mu t)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\, dy = \int_{-\infty}^{\infty} f(y)\frac{e^{\frac{-(y-W_s-\mu s-\mu(t-s))^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\, dy = g(X_s).$$

Problem 4. *Consider a normal random variable* $X \sim \mathcal{N}(0,t)$ *and a scaled random walk* $W^{(n)}(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{nt} X_j$, *whereas* $X_j = \begin{cases} 1, & p = 0.5, \\ -1, & p = 0.5 \end{cases}$. *Show the limit of the scaled random walk is the normal by comparing their moment-generating function.*

Solution 4. *In the class, we show that the moment-generating function for the scaled random walk is*

$$y = \left(\frac{1}{2}e^{\frac{2}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{s}{\sqrt{n}}}\right)^{nt}.$$

Taking the log, we have $\log(y) = nt\log\left(\frac{1}{2}e^{\frac{2}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{s}{\sqrt{n}}}\right)$. *Using the L'Hospital's rule,*

$$\lim_{n\to\infty} y = \frac{\frac{d}{dn}\log\left(\frac{1}{2}e^{\frac{2}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{s}{\sqrt{n}}}\right)}{\frac{d}{dn}\frac{1}{nt}} = \lim_{n\to\infty}\frac{st\sqrt{n}e^{\frac{s}{\sqrt{n}}} - st\sqrt{n}e^{-\frac{s}{\sqrt{n}}}}{2e^{\frac{s}{\sqrt{n}}} + 2e^{-\frac{s}{\sqrt{n}}}}.$$

Using the linear approximation, we have $\lim_{n\to\infty} = \frac{s^2 t}{2}$. *Therefore,* $\lim_{n\to\infty} e^y = e^{\frac{s^2 t}{2}}$, *which is the moment-generating function of* X.

Problem 5. *Numerically visualize the convergence of the random walk.*

Problem 6. *Simulate some trajectories of Brownian motions.*