Math 411 Midterm
Section IV 2025
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Problem 1 (5+5+5=15 points). *True or False. Consider a single-period binomial model with stock prices S_0, S_1, an up factor u with probability p, a down factor d with probability $1-p$, and an interest rate r.*

1. *Denote $R = \log\left(\frac{S_1}{S_0}\right)$ as the log return. If $d = \frac{1}{u}$, then we know $\mathbb{E}[R^2]$ even if we don't know p.*
2. *For a European call option V with $d = \frac{1}{u}$, we have $\frac{\partial V}{\partial S_0} \geq 0$.*
3. *In the binomial model, we need $0 < d < 1 + r < u$ to exclude arbitrage opportunity.*

Solution 1. *TTT*

Problem 2 (5 points). *Which of the following is not an implication from the efficient market hypothesis.*

a) *Stock prices process is a stochastic process.* b) *Stock prices process is a Markovian process.* c) *We only need to look at stock price data when we build the math model.* d) *Stock prices follows a binomial model.*

Solution 2. *D.*

Problem 3 (5 points). *Given the function $f(x) = \max(K - x, 0)$ and a discrete random variable X, which of the following statement is true.*

a) $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$ b) $\mathbb{E}[f(X)] = f(\mathbb{E}[X])$ c) $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$ d) *None of the above*

Solution 3. *C.*

Problem 4 (5 points). *Given a stock A, we estimate its expected yearly log return by taking the average with $\overline{R} = 0.1$. Suppose we know the true variance of the yearly return is 0.025. For how many years of data I can trust this estimation such that $\sqrt{\mathbb{E}[(\mu - \overline{R})^2]} \approx 0.01$?*

a) *2.5 years* b) *25 years* c) *250 years* d) *2500 years*

Solution 4. *C.*

Problem 5 (5+5+5+5=20 points). *Consider a simple class of model of the form*

$$y_t = r + y_{t-1} + c_0\epsilon_t + c_{-1}\epsilon_{t-1} + \cdots + c_{-p}\epsilon_{t-p}, \tag{1}$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$. We want to use this model to mimic the stock market. Here, y_t represent the log-price, r represents the expected return, and ϵ_t represents the impact from the news and are independent from each other. We define the residual as $z_t = y_t - r - y_{t-1}$.

1. *Calculate the expectation of log return $\mathbb{E}[y_t - y_{t-1}]$.*
2. *Suppose $y_t = r + y_{t-1} + \epsilon_t$ and $y_0 = 0$. Suppose we are interested in the event when $y_1 < \eta$ for a threshold η to represent the scenario that the company has received bad news. Calculate the conditional expectation of $\mathbb{E}[z_2 | y_1 < \eta]$.*
3. *Suppose now $y_t = r + y_{t-1} + \epsilon_t + c_{-1}\epsilon_{t-1}$, $y_0 = 0$, and $\epsilon_0 = 0$. Calculate the conditional expectation of $\mathbb{E}[z_2 | y_1 < \eta]$ (You can use the $\phi(x)$ to denote the cumulative distribution function of a standard normal distribution in the answer).*
4. *Suppose now $y_t = r + y_{t-1} + \epsilon_t + c_{-1}\epsilon_{t-1}$, $y_0 = 0$, and $\epsilon_0 = 0$. Suppose now we are interested in how volatile is the market if the stock prices has been volatile. Calculate the conditional expectation of $\mathbb{E}[z_2^2 | z_1^2 > \kappa]$.*

Solution 5. 1. $\mathbb{E}[y_t - y_{t-1}] = \mathbb{E}[r + c_0\epsilon_t + c_{-1}\epsilon_{t-1} + \ldots] = r$

2. $\mathbb{E}[z_2|y_1 < \eta] = \mathbb{E}[\epsilon_2|y_1 < \eta] = 0$ *due to independency*

3. *Let* $\mathcal{Z}$ *be the standard normal, then* $\epsilon_1 = \sigma\mathcal{Z}$.

$$\mathbb{E}[z_2|y_1 < \eta] = \mathbb{E}[\epsilon_2 + c_{-1}\epsilon_1|r + \epsilon_1 < \eta] = \mathbb{E}[\epsilon_2|r + \epsilon_1 < \eta] + c_{-1}\mathbb{E}[\epsilon_1|r + \epsilon_1 < \eta] = c_{-1}\sigma\mathbb{E}\left[\mathcal{Z}\middle|\mathcal{Z} < \frac{\eta - r}{\sigma}\right]$$

$$= c_{-1}\sigma\frac{\int_{-\infty}^{\frac{\eta-r}{\sigma}} \frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx}{\int_{-\infty}^{\frac{\eta-r}{\sigma}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx} = c_{-1}\sigma\frac{-\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\frac{\eta-r}{\sigma}}}{\phi\left(\frac{\eta-r}{\sigma}\right)} = -\frac{\frac{c_{-1}\sigma}{\sqrt{2\pi}}e^{-\frac{(\eta-r)^2}{2\sigma^2}}}{\phi\left(\frac{\eta-r}{\sigma}\right)}.$$

4. *First, we calculate that*

$$\mathbb{E}[z_2^2|z_1^2 > \kappa] = \mathbb{E}[(\epsilon_2 + c_{-1}\epsilon_1)^2|\epsilon_1^2 > \kappa] = \sigma^2 + c_{-1}^2\mathbb{E}[\epsilon_1^2|\epsilon_1^2 > \kappa].$$

Then we calculate that

$$\mathbb{E}[\epsilon_1^2|\epsilon_1^2 > \kappa] = \mathbb{E}[\sigma^2\mathcal{Z}^2|\sigma^2\mathcal{Z}^2 > \kappa] = \sigma^2\mathbb{E}\left[\mathcal{Z}^2\middle|\mathcal{Z}^2 > \frac{\kappa}{\sigma^2}\right]$$

$$= \sigma^2\frac{\int_{-\infty}^{-\frac{\sqrt{\kappa}}{\sigma}} \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx + \int_{\frac{\sqrt{\kappa}}{\sigma}}^{\infty} \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx}{\int_{-\infty}^{-\frac{\sqrt{\kappa}}{\sigma}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx + \int_{\frac{\sqrt{\kappa}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx} = \sigma^2\frac{2\int_{\frac{\sqrt{\kappa}}{\sigma}}^{\infty} \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx}{2\phi\left(-\frac{\sqrt{\kappa}}{\sigma}\right)}$$

$$= \sigma^2\frac{-\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\Big|_{\frac{\sqrt{\kappa}}{\sigma}}^{\infty} - \int_{\frac{\sqrt{\kappa}}{\sigma}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx}{\phi\left(-\frac{\sqrt{\kappa}}{\sigma}\right)} = \sigma^2\frac{\frac{\sqrt{\kappa}}{\sqrt{2\pi}\sigma}e^{-\frac{\kappa}{2\sigma^2}} + 1 - \phi\left(\frac{\sqrt{\kappa}}{\sigma}\right)}{\phi\left(-\frac{\sqrt{\kappa}}{\sigma}\right)}.$$

Problem 6 (5+5+5+5 = 20 points). *Consider a single-period binomial model with the initial stock price* $S_0 = 4$, *up factor* $u = 2$ *with probability* p, *down factor* $d = \frac{1}{2}$, *interest rate* $r = \frac{1}{4}$, *and the strike price* $K = 6$.

1. *Write down the payoff of a European call option at the expiration date.*

2. *Set up the equations to replicate the European call option.*

3. *From question 2, solve the equations to give the no-arbitrage price of a European call option.*

4. *Now consider a one-period model with three potential outcomes,* $S_1(H) = 8, S_1(T) = 2, S_1(E) = 4$. *Can we find the no-arbitrary price of this option? If yes, provide the value; if not, provide the detailed explanation.*

Solution 6. 1. $\max(S_1 - K, 0)$.

2. *We want*

$$V_0 + \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right) = \frac{V_1(H)}{(1+r)},$$
$$V_0 + \Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right) = \frac{V_1(T)}{(1+r)}.$$

3. *The risk-neutral probability can be calculated as*

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{1}{2}.$$

Therefore,

$$V_0 = \frac{\widetilde{p}V_1(H)}{1+r} = 0.8.$$

4. *We will end up with three equations with two variables, therefore no solutions exist to solve it.*

Problem 7 (5+5=10 points). *Consider a multi-period binomial model with the initial stock price* S_0, *up factor* u, *down factor* d, *interest rate* r, *and the strike price* K. *Suppose* $S_0 = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 6$.

1. *Calculate the two-period call option.*

2. *Denote the N as the number of periods, what is the relationship between V_0 and N? Is V_0 always monotonic increasing or decreasing with respect to N? If yes, prove it; if no, explain the reason.*

Solution 7. 1. *We do this recursively. We calculate*

$$V_1(H) = \frac{\widetilde{p}V_1(HH) + \widetilde{q}V_1(HT)}{1+r} = 4,$$
$$V_1(T) = \frac{\widetilde{p}V_1(TH) + \widetilde{q}V_1(TT)}{1+r} = 0,$$
$$V_0 = \frac{\widetilde{p}V_1(H) + \widetilde{q}V_1(T)}{1+r} = 1.6.$$

2. *We compare the case at stage N and $N+1$. Due to the backward property, we just need to compare $V_N(\omega_1,\dots,\omega_N)$ at stage N and $\widetilde{\mathbb{E}}_N(V_N(\omega_1,\dots,\omega_N))$. We consider the following cases:*

- $2S_N(\omega_1,\dots,\omega_N), \frac{1}{2}S_N(\omega_1,\dots,\omega_N) \le K$: *both are zero.*
- $2S_N(\omega_1,\dots,\omega_N), \frac{1}{2}S_N(\omega_1,\dots,\omega_N) \ge K$: $\widetilde{\mathbb{E}}_N(V_N(\omega_1,\dots,\omega_N)) = \frac{\widetilde{\mathbb{E}}[S_N(\omega_1,\dots,\omega_N)]-K}{1+r} = S_N(\omega_1,\dots,\omega_N) - \frac{K}{1+r} \ge S_N(\omega_1,\dots,\omega_N) - K$
- $2S_N(\omega_1,\dots,\omega_N) \ge K, \frac{1}{2}S_N(\omega_1,\dots,\omega_N) \le K, S_N(\omega_1,\dots,\omega_N) \le K$: $\widetilde{\mathbb{E}}_N(V_N(\omega_1,\dots,\omega_N)) \ge 0 = V_N(\omega_1,\dots,\omega_N)$
- $2S_N(\omega_1,\dots,\omega_N) \ge K, \frac{1}{2}S_N(\omega_1,\dots,\omega_N) \le K, S_N(\omega_1,\dots,\omega_N) \ge K$: $\widetilde{\mathbb{E}}_N(V_N(\omega_1,\dots,\omega_N)) = \frac{\frac{1}{2}(2S_N(\omega_1,\dots,\omega_N)-K)}{\frac{5}{4}} = \frac{4}{5}S_N(\omega_1,\dots,\omega_N) - \frac{2K}{5}$. *We calculate* $\widetilde{\mathbb{E}}_N(V_N(\omega_1,\dots,\omega_N)) - (S_N(\omega_1,\dots,\omega_N) - K) = \frac{1}{5}(3K - S_N(\omega_1,\dots,\omega_N)) \ge 0$

Therefore, the price is monotonically nondecreasing w.r.t N.