Lecture VI: Limitation of Geometric Brownian Motion

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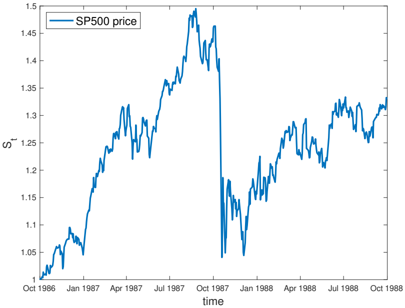
Physical Measure

Risk-neural Measure

1987 Stock market crash

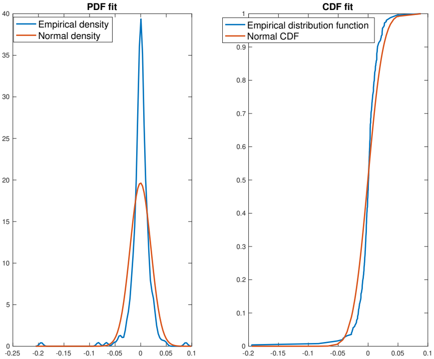
- **Question**: Is GBM always a good assumption?
- 1987 Stock market crash
- WLOG, let $S_0 = 1$



- Crash date: October 19, 1987
- **Question**: Is this reasonable for GBM?

Hypothesis testing

- H_0: Model is right vs H_1: Model is wrong



- $\widehat{p} < 10^{-3}$
- **Question**: At a global level, what causes the failure of the GBM?

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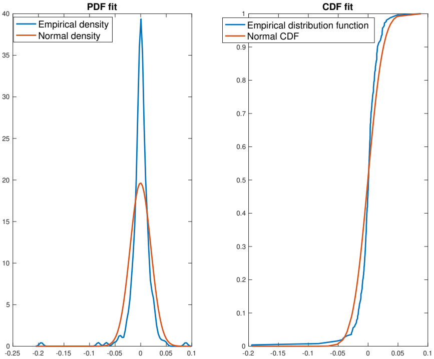

- $\widehat{p} < 10^{-3}$
- **Question**: At a global level, what causes the failure of the GBM? Fat tail

Failure of the GBM

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- **Question**: Locally, is that possible for the GBM to simulate such a sharp drawdown?
- **Question**: How to quantitative test it?
- $\mathbb{P}(\min(\{\Delta X_i\}) \geq x) = \prod_{i=1}^{n} \mathbb{P}(\Delta X_i \geq x) = (\mathbb{P}(\Delta X_1 \geq x))^n$
- $\mathbb{P}(\min(\{\Delta X_i\}) \geq x | H_0) = (1 - F(x; \mu, \sigma))^n$
- $\mathbb{P}(\min(\{\Delta X_i\}) \leq x | H_0) = 1 - (1 - F(x; \mu, \sigma))^n$
- $Z_n = \min(\{\Delta X_i\})$
- H_0: The fitted GBM generates the observed sharp drawdown
- During crash: $\mathbb{P}(Z_n \leq \widehat{z}_n | \widehat{\mu}, \widehat{\sigma}) = 0$
- Before crash: $\mathbb{P}(Z_n \leq \widehat{z}_n | \widehat{\mu}, \widehat{\sigma}) = 36\%$

Failure of the GBM - continued

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Failure of the GBM - continued

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- $\overline{X_{\Delta t}} = 4.2 \times 10^{-4}$, $S_n(X_{\Delta t}) = 0.011$
- Suppose we are interested in $\min(\{\Delta X_i\}) = -5\%$ and $\alpha = 5\%$

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- Suppose we are interested in $\min(\{\Delta X_i\}) = -5\%$ and $\alpha = 5\%$
- Want $1 - (1 - F(-0.05; \widehat{r}, \widehat{\sigma}^2))^n \ge \alpha$
- $n \ge \frac{\log(1-\alpha)}{\log(1-F(-0.05;\widehat{r},\widehat{\sigma}^2))} \approx 46.6$ years
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- $n \approx$ 0.5 year

Stochastic volatility

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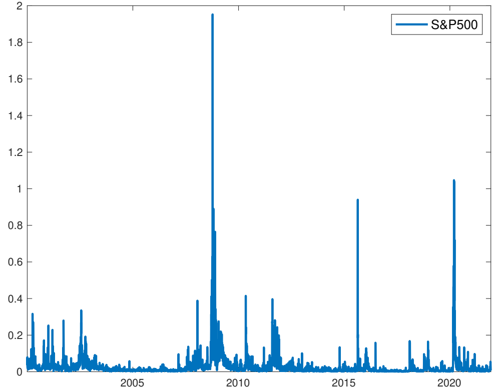
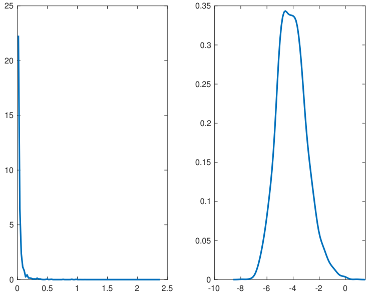
Stochastic volatility

- **Question**: If V_t is stochastic, how to calculate it?
- Using high-frequency data (thanks to technologies)
- $\lim_{n\to\infty}\sum_{j=1}^{n}\Delta X_{i,j}^2 = \int_{i\Delta t}^{(i+1)\Delta t} V_s\, ds$
- Reference: `https://realized.oxford-man.ox.ac.uk/`
- Realized variance of S&P 500



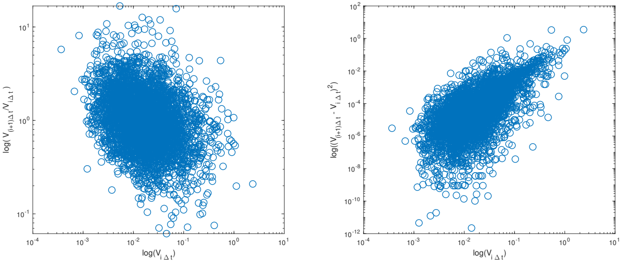
Density function

- PDF for S&P 500



Stylized pattern

- Pattern of realized variance for S&P 500



- Mean-reversion effect
- Variance of variance monotonically increasing w.r.t the magnitude of variance

Physical Measure

Risk-neural Measure

Option pricing

- Call option: $C(S_t, \sigma, r, K, T)$
  - S_t: underlying stock price at time t
  - σ: volatility
  - r: risk-free interest rate
  - K: strike price
  - T: maturity date
- At date t, we observe $C(S_t, \sigma, r, K_i, T_i)$
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- Suppose we fix T =1-month and observe $C(S_t, \sigma, r, K_i, T = \frac{30}{365})$
- What should we observe for σ vs K_i?

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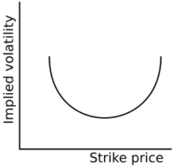

More evidence

- Calculate VIX data $\mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+\Delta t} V_s\ ds \middle| V_t\right]$ at different horizons, $\Delta t = 1,3,6,9,12$ months
- Smooth structure: