PHYS 121 — HW3 Solutions

WRITTER

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Q1. Gravitation from a ball (8 pts)

Shell theorem: A uniform spherical shell of matter exerts no gravitational force on a particle inside it, and exerts the same gravitational force on a particle outside it as if all the shell's mass were concentrated at its center.

(2) Gravitational field $g(r)$: For a uniform solid sphere of radius R and mass M:

- **Outside** ($r \geq R$): The sphere acts as a point mass at the center, so

$$g(r) = \frac{GM}{r^2}, \quad r \geq R.$$

- **Inside** ($r < R$): Only the mass inside radius r contributes. The mass inside is $M(r) = M(r/R)^3$, so

$$g(r) = \frac{GM(r)}{r^2} = \frac{GM}{R^3}r, \quad r < R.$$

(3) Gravitational potential energy $U(r)$: With $U(\infty) = 0$:

- **Outside** ($r \geq R$):

$$U(r) = -\frac{GMm}{r}, \quad r \geq R.$$

- **Inside** ($r < R$): Integrate from infinity, ensuring continuity at $r = R$:

$$U(r) = U(R) + \int_R^r g(r')m\,dr' = -\frac{GMm}{R} + \frac{GMm}{2R^3}(r^2 - R^2),$$

$$U(r) = -\frac{GMm}{2R}\left(3 - \frac{r^2}{R^2}\right), \quad r < R.$$

At $r = R$, both expressions give $U(R) = -GMm/R$, ensuring continuity.

(4) Evaluations:

$$|U(R/2)| = \left|-\frac{GMm}{2R}\left(3 - \frac{1}{4}\right)\right| = \frac{11GMm}{8R},$$

$$|U(R)| = \frac{GMm}{R},$$

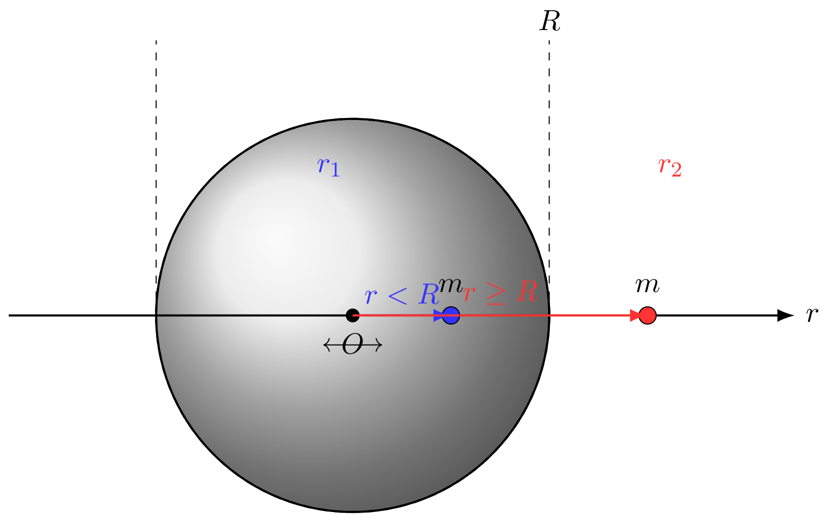
$$|U(2R)| = \frac{GMm}{2R}.$$

Ranking: $|U(R/2)| > |U(R)| > |U(2R)|$. The magnitude is largest at $r = R/2$ because the test mass is deeper in the potential well.

(i) Inside the sphere, $g(r) = (GM/R^3)r$ varies *linearly* with r.

(ii) As $r \to \infty$, $U(r) = -GMm/r \to 0^-$ (approaches zero from below).

(1) Drawing:



Q2. Explore Vesta (10 pts)

Given: diameter $D = 520\,\text{km}$, so radius $R = 260\,\text{km} = 2.60 \times 10^5\,\text{m}$, mass $M = 2.67 \times 10^{20}\,\text{kg}$.

(1) Escape speed:

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 2.67 \times 10^{20}}{2.60 \times 10^5}} = \boxed{370\,\text{m/s}}.$$

(2) Orbital period: For a circular orbit at altitude $h = 15\,\text{km}$ above the surface, the orbital radius is $r = R + h = 2.75 \times 10^5\,\text{m}$. Using Kepler's third law:

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(2.75 \times 10^5)^3}{6.674 \times 10^{-11} \times 2.67 \times 10^{20}}} = 2\pi\sqrt{\frac{2.08 \times 10^{16}}{1.78 \times 10^{10}}} = 2\pi \times 1080 = \boxed{6.79 \times 10^3\,\text{s} = 1.89\,\text{h}}.$$

(3) Reflection: A spherical model is only marginally useful because Vesta is not spherical—it's an irregular asteroid with significant deviations from sphericity. Real orbits are tricky due to:

- Non-uniform mass distribution (density variations)
- Irregular shape causing non-central gravitational field
- Rotational effects and potential tumbling
- Surface features (craters, ridges) that create gravitational anomalies

These factors cause orbital perturbations, precession, and instability that a simple spherical model cannot capture.

Q3. Kepler's law—Pluto's small moons (8 pts)

Given: Charon orbits at $r_C = 19\,600\,\text{km}$ with period $T_C = 6.39\,\text{d}$. Two small satellites at $r_1 = 48\,000\,\text{km}$ and $r_2 = 64\,000\,\text{km}$.

Kepler's third law: $T^2 \propto r^3$, so $T \propto r^{3/2}$. Scaling from Charon:

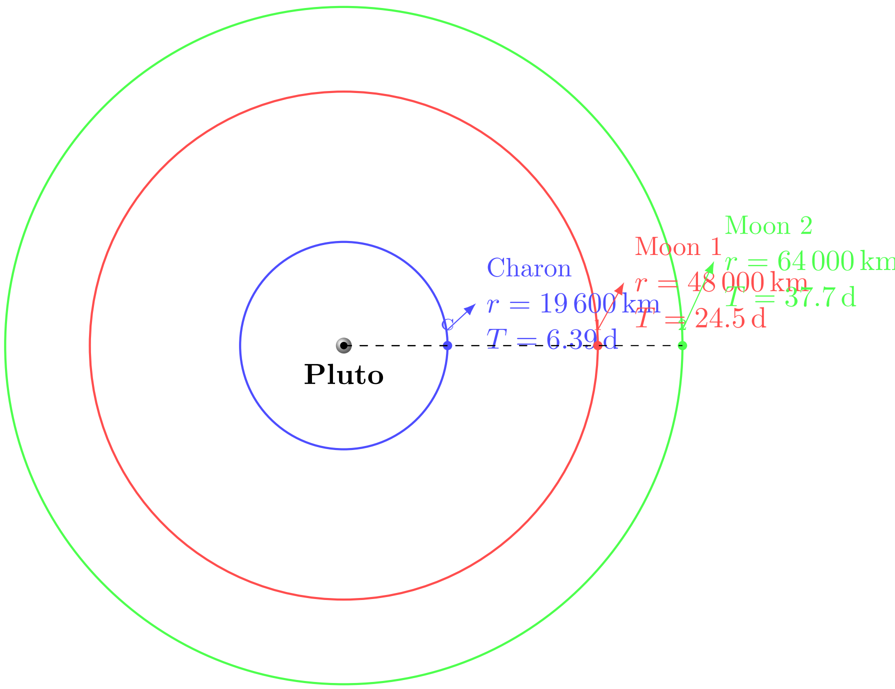
$$\frac{T}{T_C} = \left(\frac{r}{r_C}\right)^{3/2} \quad \Rightarrow \quad T = T_C\left(\frac{r}{r_C}\right)^{3/2}.$$

For $r_1 = 48\,000\,\text{km}$:

$$T_1 = 6.39 \times \left(\frac{48000}{19600}\right)^{3/2} = 6.39 \times (2.449)^{3/2} = 6.39 \times 3.83 = \boxed{24.5\,\text{d} = 588\,\text{h}}.$$

For $r_2 = 64\,000\,\text{km}$:

$$T_2 = 6.39 \times \left(\frac{64000}{19600}\right)^{3/2} = 6.39 \times (3.265)^{3/2} = 6.39 \times 5.90 = \boxed{37.7\,\text{d} = 905\,\text{h}}.$$



Q4. Gravity and SHM in a straight tunnel (14 pts)

For a uniform spherical planet of radius R and density ρ, the mass inside radius r is $M(r) = \frac{4}{3}\pi r^3 \rho$.

(2) Gravitational force: At distance r from the center ($r \leq R$), only the mass inside r contributes:

$$F(r) = -\frac{GM(r)m}{r^2} = -\frac{G \cdot \frac{4}{3}\pi r^3 \rho \cdot m}{r^2} = -\frac{4\pi G\rho m}{3} r.$$

The force is proportional to r and directed toward the center (restoring).

(3) Simple harmonic motion: Newton's second law gives

$$m\frac{d^2r}{dt^2} = -\frac{4\pi G\rho m}{3} r \quad \Rightarrow \quad \frac{d^2r}{dt^2} = -\frac{4\pi G\rho}{3} r.$$

This is SHM with angular frequency

$$\omega = \sqrt{\frac{4\pi G\rho}{3}}.$$

(4) Period:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{G\rho}}.$$

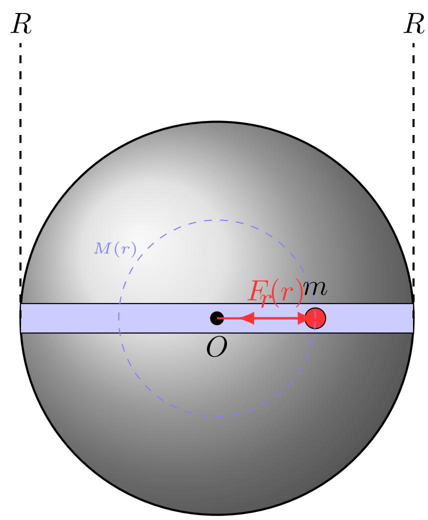
(5) Comparison with circular orbit: For a circular orbit skimming the surface ($r = R$), the orbital speed is $v = \sqrt{GM/R}$ where $M = \frac{4}{3}\pi R^3 \rho$. The period is

$$T_{\text{orbit}} = \frac{2\pi R}{v} = 2\pi R\sqrt{\frac{R}{GM}} = 2\pi\sqrt{\frac{R^3}{G \cdot \frac{4}{3}\pi R^3 \rho}} = \sqrt{\frac{3\pi}{G\rho}}.$$

Thus $T = T_{\text{orbit}}$: the tunnel period equals the surface-skimming orbital period.

(6) Check: If density doubles ($\rho \to 2\rho$), then $T = \sqrt{3\pi/(G\rho)} \propto 1/\sqrt{\rho}$, so the period decreases by a factor of $\sqrt{2}$.

(1) Drawing:



Q5. Pendulum in an elevator (10 pts)

For a pendulum of length ℓ, the period is $T = 2\pi\sqrt{\ell/g_{\text{eff}}}$, where g_{eff} is the effective gravitational acceleration in the elevator's reference frame.

(1) Cases:

- **Rest:** $g_{\text{eff}} = g$, so $T_0 = 2\pi\sqrt{\ell/g}$.
- **Accelerating upward at a:** In the elevator frame, effective gravity is $g_{\text{eff}} = g + a$, so $T_{\text{up}} = 2\pi\sqrt{\ell/(g+a)}$.
- **Accelerating downward at a (with $a < g$):** Effective gravity is $g_{\text{eff}} = g - a$, so $T_{\text{down}} = 2\pi\sqrt{\ell/(g-a)}$.

(3) Numerical values: With $\ell = 0.90\,\text{m}$ and $a = 2.0\,\text{m/s}^2$:

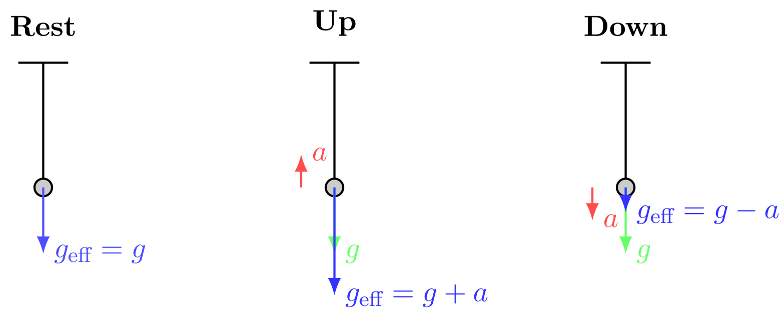
$$T_0 = 2\pi\sqrt{\frac{0.90}{9.8}} = \boxed{1.90\,\text{s}},$$

$$T_{\text{up}} = 2\pi\sqrt{\frac{0.90}{9.8+2.0}} = 2\pi\sqrt{\frac{0.90}{11.8}} = \boxed{1.74\,\text{s}},$$

$$T_{\text{down}} = 2\pi\sqrt{\frac{0.90}{9.8-2.0}} = 2\pi\sqrt{\frac{0.90}{7.8}} = \boxed{2.13\,\text{s}}.$$

Ranking: $T_{\text{down}} > T_0 > T_{\text{up}}$. The period is longest when accelerating downward (smallest effective g).

(4) Free fall: In free fall, $a = g$, so $g_{\text{eff}} = g - g = 0$. The period becomes infinite—the pendulum doesn't oscillate; it floats relative to the elevator.



Q6. Van der Waals ≈ Hooke near the minimum (8 pts)

Given potential: $U(r) = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right]$.

(1) Sketch: The minimum occurs when $dU/dr = 0$. At $r = R_0$, $U(R_0) = U_0(1-2) = -U_0$.

(2) Conservative force:

$$F(r) = -\frac{dU}{dr} = -U_0\left[-12\frac{R_0^{12}}{r^{13}} + 12\frac{R_0^6}{r^7}\right] = 12U_0R_0^6\left(\frac{R_0^6}{r^{13}} - \frac{1}{r^7}\right).$$

(3) Expansion near minimum: Let $r = R_0 + r'$ with $|r'| \ll R_0$. For small r':

$$\frac{R_0}{r} = \frac{R_0}{R_0 + r'} = \frac{1}{1 + r'/R_0} \approx 1 - \frac{r'}{R_0} + \cdots.$$

More precisely, expand $F(r)$:

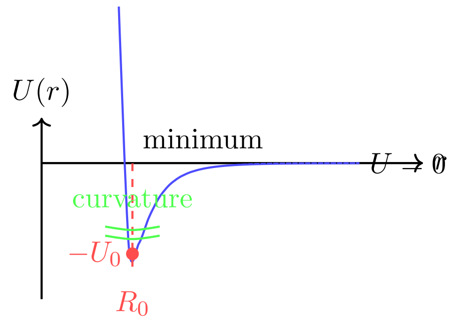
$$F(R_0 + r') = 12U_0R_0^6\left[\frac{R_0^6}{(R_0 + r')^{13}} - \frac{1}{(R_0 + r')^7}\right].$$

Using $(R_0 + r')^{-n} \approx R_0^{-n}(1 - nr'/R_0)$ for small r':

$$\begin{aligned} F(R_0 + r') &\approx 12U_0R_0^6\left[\frac{1}{R_0^7}\left(1 - \frac{13r'}{R_0}\right) - \frac{1}{R_0^7}\left(1 - \frac{7r'}{R_0}\right)\right] \\ &= 12U_0R_0^6 \cdot \frac{1}{R_0^7}\left(-\frac{13r'}{R_0} + \frac{7r'}{R_0}\right) \\ &= -\frac{72U_0}{R_0^2}r'. \end{aligned}$$

Thus $F \approx -kr'$ with $\boxed{k = \frac{72U_0}{R_0^2}}$.

(4) Check: Units: $[k] = [U_0]/[R_0^2] = \text{J/m}^2 = \text{N/m}$ (correct for spring constant). The force is restoring: $F = -kr'$ means F opposes displacement from equilibrium.



Q7. A moving pulse (9 pts)

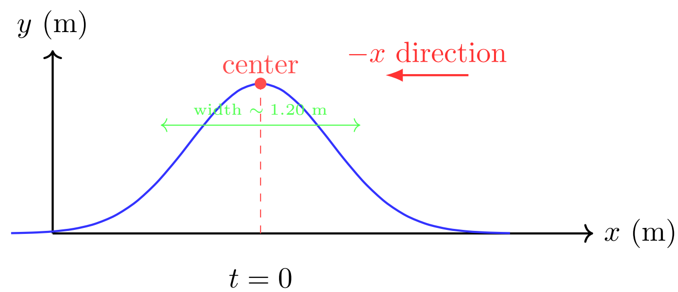
Given: $y(x,t) = 4.20\,\mathrm{m} \cdot e^{-(x+2.00t)^2/(1.20)^2}$.

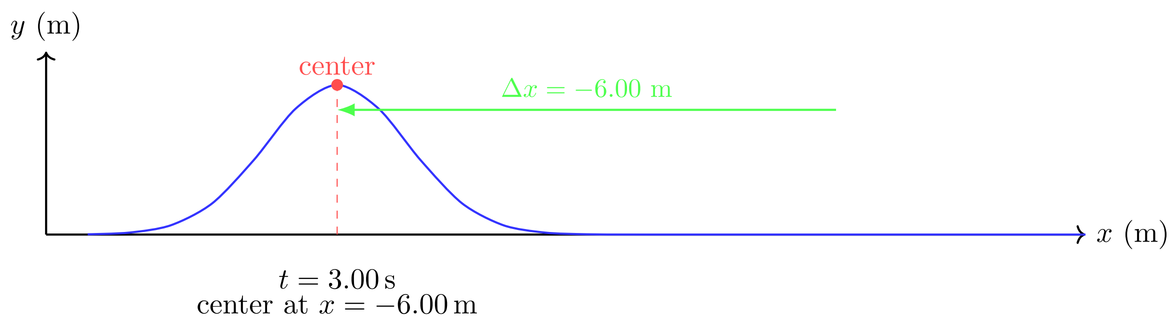
(1) At $t = 0$: $y(x,0) = 4.20\,\mathrm{m} \cdot e^{-(x/1.20)^2}$. This is a Gaussian pulse centered at $x = 0$ with width $\sim 1.20\,\mathrm{m}$.

(2) Rewrite: $y(x,t) = 4.20\,\mathrm{m} \cdot e^{-(x+2.00t)^2/(1.20)^2} = f(x + 2.00t)$ where $f(u) = 4.20\,\mathrm{m} \cdot e^{-u^2/(1.20)^2}$.

The form $f(x + vt)$ indicates motion in the $-x$ **direction** with speed $\boxed{\mathrm{v} = 2.00\,\mathrm{m/s}}$.

(3) Center movement: In $3.00\,\mathrm{s}$, the center moves $\Delta x = -v\Delta t = -2.00 \times 3.00 = \boxed{-6.00\,\mathrm{m}}$ (to the left).





Q8. Wave on a string (10 pts)

Given: $y(x,t) = A\cos(kx - \omega t + \varphi) = 0.15\,\mathrm{m}\cos\big(0.15\,\mathrm{m}^{-1} \cdot x + 1.50\,\mathrm{s}^{-1} \cdot t + 0.25\big)$.

Note: The signs are $+kx$ and $+\omega t$, so we rewrite as $y = A\cos(kx + \omega t + \varphi)$.

(1) Wave speed and direction:

$$v = \frac{\omega}{k} = \frac{1.50\,\mathrm{s}^{-1}}{0.15\,\mathrm{m}^{-1}} = \boxed{10.0\,\mathrm{m/s}}.$$

The form $\cos(kx+\omega t)$ indicates motion in the $-x$ **direction** (leftward).

(2) At $x = 0.40\,\mathrm{m}$, $t = 5.00\,\mathrm{s}$:

$$\begin{aligned} kx+\omega t+\varphi &= 0.15\times 0.40 + 1.50\times 5.00 + 0.25 \\ &= 0.06 + 7.50 + 0.25 = 7.810. \end{aligned}$$

$$\begin{aligned} y &= 0.15\,\mathrm{m}\cos(7.810) = \boxed{0.007\,\mathrm{m}}, \\ v_y &= \frac{\partial y}{\partial t} = -A\omega\sin(kx+\omega t+\varphi) \\ &= -0.15\,\mathrm{m}\times 1.50\,\mathrm{s}^{-1}\sin(7.810) = \boxed{-0.225\,\mathrm{m/s}}, \\ a_y &= \frac{\partial^2 y}{\partial t^2} = -A\omega^2\cos(kx+\omega t+\varphi) \\ &= -0.15\,\mathrm{m}\times(1.50\,\mathrm{s}^{-1})^2\cos(7.810) = \boxed{-0.015\,\mathrm{m/s^2}}. \end{aligned}$$

Bonus. Wave in different mediums (0–2 pts)

When you shout near a pond, sound waves in air (a compressional wave with speed $\sim 343\,\mathrm{m/s}$) encounter the water surface. At the air–water interface, most sound energy is *reflected* because of the large impedance mismatch (water's density and bulk modulus are much higher than air's). However, a small fraction *transmits* into water, where sound travels faster ($\sim 1500\,\mathrm{m/s}$) due to water's higher density and bulk modulus. The transmitted wave has a different wavelength ($\lambda_{\mathrm{water}} = v_{\mathrm{water}}/f$) but the same frequency as the incident wave. A detector in water would register this transmitted component, though it would be much weaker than the original sound in air due to the reflection loss at the interface.

Assistance notes:

LLMs were used as translators and concept explainers. No direct solutions were provided.