APPENDIX D Gravitational Force due to a Spherical Mass Distribution

In Chapter 6 we stated that the gravitational force exerted by or on a uniform sphere acts as if all the mass of the sphere were concentrated at its center, if the other object (exerting or feeling the force) is outside the sphere. In other words, the gravitational force that a uniform sphere exerts on a particle outside it is

$$F = G\frac{mM}{r^2}, \qquad [m \text{ outside sphere of mass } M]$$

where m is the mass of the particle, M the mass of the sphere, and r the distance of m from the center of the sphere. Now we will derive this result. We will use the concepts of infinitesimally small quantities and integration.

First we consider a very thin, uniform spherical shell (like a thin-walled basketball) of mass M whose thickness t is small compared to its radius R (Fig. D–1). The force on a particle of mass m at a distance r from the center of the shell can be calculated as the vector sum of the forces due to all the particles of the shell. We imagine the shell divided up into thin (infinitesimal) circular strips so that all points on a strip are equidistant from our particle m. One of these circular strips, labeled AB, is shown in Fig. D–1. It is $R\,d\theta$ wide, t thick, and has a radius $R\sin\theta$. The force on our particle m due to a tiny piece of the strip at point A is represented by the vector $\vec{\mathbf{F}}_A$ shown. The force due to a tiny piece of the strip at point B, which is diametrically opposite A, is the force $\vec{\mathbf{F}}_B$. We take the two pieces at A and B to be of equal mass, so $F_A = F_B$. The horizontal components of $\vec{\mathbf{F}}_A$ and $\vec{\mathbf{F}}_B$ are each equal to

$$F_A \cos\phi$$

and point toward the center of the shell. The vertical components of $\vec{\mathbf{F}}_A$ and $\vec{\mathbf{F}}_B$ are of equal magnitude and point in opposite directions, and so cancel. Since for every point on the strip there is a corresponding point diametrically opposite (as with A and B), we see that the net force due to the entire strip points toward the center of the shell. Its magnitude will be

$$dF = G\frac{m\,dM}{\ell^2}\cos\phi,$$

where dM is the mass of the entire circular strip and ℓ is the distance from all points on the strip to m, as shown. We write dM in terms of the density ρ; by density we mean the mass per unit volume (Section 13–2). Hence, $dM = \rho\,dV$, where dV is the volume of the strip and equals $(2\pi R\sin\theta)(t)(R\,d\theta)$. Then the force dF due to the circular strip shown is

$$dF = G\frac{m\rho 2\pi R^2 t\sin\theta\,d\theta}{\ell^2}\cos\phi. \qquad \textbf{(D–1)}$$

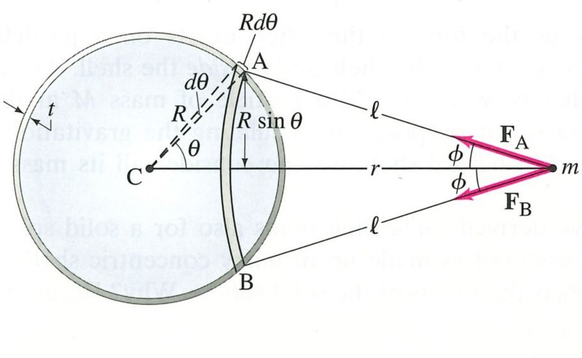

FIGURE D–1 Calculating the gravitational force on a particle of mass m due to a uniform spherical shell of radius R and mass M.

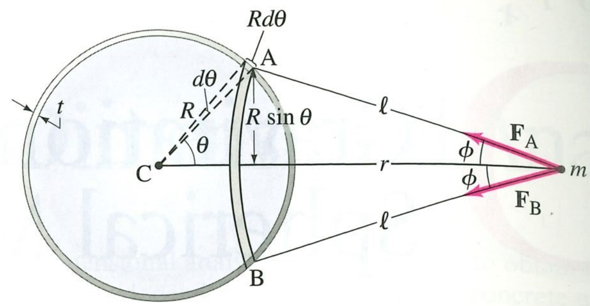

FIGURE D–1 (repeated) Calculating the gravitational force on a particle of mass m due to a uniform spherical shell of radius R and mass M.

To get the total force F that the entire shell exerts on the particle m, we must integrate over all the circular strips: that is, we integrate

$$dF = G\frac{m\rho 2\pi R^2 t \sin\theta\, d\theta}{\ell^2}\cos\phi \tag{D–1}$$

from $\theta = 0°$ to $\theta = 180°$. But our expression for dF contains ℓ and ϕ, which are functions of θ. From Fig. D–1 we can see that

$$\ell\cos\phi = r - R\cos\theta.$$

Furthermore, we can write the law of cosines for triangle CmA:

$$\cos\theta = \frac{r^2 + R^2 - \ell^2}{2rR}. \tag{D–2}$$

With these two expressions we can reduce our three variables (ℓ, θ, ϕ) to only one, which we take to be ℓ. We do two things with Eq. D–2: (1) We put it into the equation for $\ell\cos\phi$ above:

$$\cos\phi = \frac{1}{\ell}(r - R\cos\theta) = \frac{r^2 + \ell^2 - R^2}{2r\ell}.$$

and (2) we take the differential of both sides of Eq. D–2 (because $\sin\theta\, d\theta$ appears in the expression for dF, Eq. D–1), considering r and R to be constants when summing over the strips:

$$-\sin\theta\, d\theta = -\frac{2\ell\, d\ell}{2rR} \quad \text{or} \quad \sin\theta\, d\theta = \frac{\ell\, d\ell}{rR}.$$

We insert these into Eq. D–1 for dF and find

$$dF = Gm\rho\pi t\frac{R}{r^2}\left(1 + \frac{r^2 - R^2}{\ell^2}\right)d\ell.$$

Now we integrate to get the net force on our thin shell of radius R. To integrate over all the strips ($\theta = 0°$ to $180°$), we must go from $\ell = r - R$ to $\ell = r + R$ (see Fig. D–1). Thus,

$$F = Gm\rho\pi t\frac{R}{r^2}\left[\ell - \frac{r^2 - R^2}{\ell}\right]_{\ell=r-R}^{\ell=r+R}$$

$$= Gm\rho\pi t\frac{R}{r^2}(4R).$$

The volume V of the spherical shell is its area $(4\pi R^2)$ times the thickness t. Hence the mass $M = \rho V = \rho 4\pi R^2 t$, and finally

$$F = G\frac{mM}{r^2}. \qquad \left[\begin{array}{c}\text{particle of mass } m \text{ outside a}\\ \text{thin uniform spherical shell of mass } M\end{array}\right]$$

This result gives us the force a thin shell exerts on a particle of mass m a distance r from the center of the shell, and *outside* the shell. We see that the force is the same as that between m and a particle of mass M at the center of the shell. In other words, for purposes of calculating the gravitational force exerted on or by a uniform spherical shell, we can consider all its mass concentrated at its center.

What we have derived for a shell holds also for a solid sphere, since a solid sphere can be considered as made up of many concentric shells, from $R = 0$ to $R = R_0$, where R_0 is the radius of the solid sphere. Why? Because if each shell has mass dM, we write for each shell, $dF = Gm\,dM/r^2$, where r is the distance from the center C to mass m and is the same for all shells. Then the total force equals the sum or integral over dM, which gives the total mass M. Thus the result

$$F = G\frac{mM}{r^2} \qquad \begin{bmatrix}\text{particle of mass } m \text{ outside} \\ \text{solid sphere of mass } M\end{bmatrix} \qquad \textbf{(D–3)}$$

is valid for a solid sphere of mass M even if the density varies with distance from the center. (It is not valid if the density varies within each shell—that is, depends not only on R.) Thus the gravitational force exerted on or by spherical objects, including nearly spherical objects like the Earth, Sun, and Moon, can be considered to act as if the objects were point particles.

This result, Eq. D–3, is true only if the mass m is outside the sphere. Let us next consider a point mass m that is located inside the spherical shell of Fig. D–1. Here, r would be less than R, and the integration over ℓ would be from $\ell = R - r$ to $\ell = R + r$, so

$$\left[\ell - \frac{r^2 - R^2}{\ell}\right]_{R-r}^{R+r} = 0.$$

Thus the force on any mass inside the shell would be zero. This result has particular importance for the electrostatic force, which is also an inverse square law. For the gravitational situation, we see that at points within a solid sphere, say 1000 km below the Earth's surface, only the mass up to that radius contributes to the net force. The outer shells beyond the point in question contribute zero net gravitational effect.

The results we have obtained here can also be reached using the gravitational analog of Gauss's law for electrostatics (Chapter 22).