PHYS121 Integrated Science-Physics

W2T4 Linear Momentum

References:
[1] David Halliday, Jearl Walker, Resnick Jearl, ‘Fundamentals of Physics’, (Wiley, 2018)
[2] Doug Giancoli, ‘Physics for Scientists and Engineers with modern physics’, (Pearson, 2009)
[3] Hugh D. Young, Roger A. Freedman, ‘University Physics with Modern Physics’, (Pearson, 2012)
And others specified when needed.

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Learning Outcomes

- Calculate the center of mass (CoM) of an object/a system (of objects)
- Analyze the motion of the CoM.
- Find the linear momentum/impulse of a particle/an object.
- Apply conservation of linear momentum in 1- and 2-D.
- Solve problems of variable mass system.

Center of Mass Trajectory

MIT Department of Physics Technical Services Group

https://www.youtube.com/watch?v=DY3LYQv22qY&list=PL860B6886A47E5490&index=14&t=0s

Center of Mass

- The motion of rotating objects can be complicated (imagine flipping a baseball bat into the air)
- But there is a special point on the object for which the motion is simple
- The center of mass of the bat traces out a parabola, just as a tossed ball does
- All other points rotate around this point

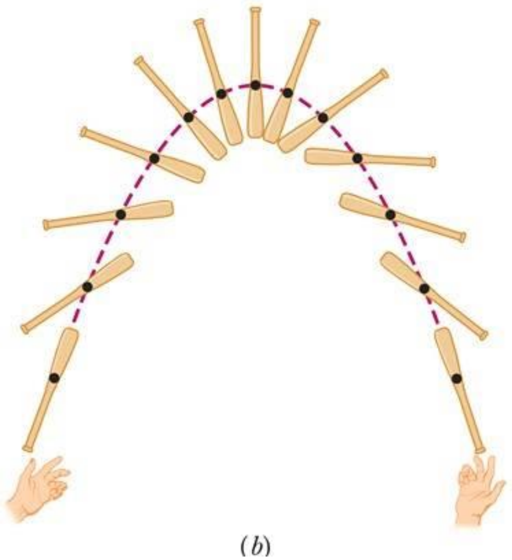
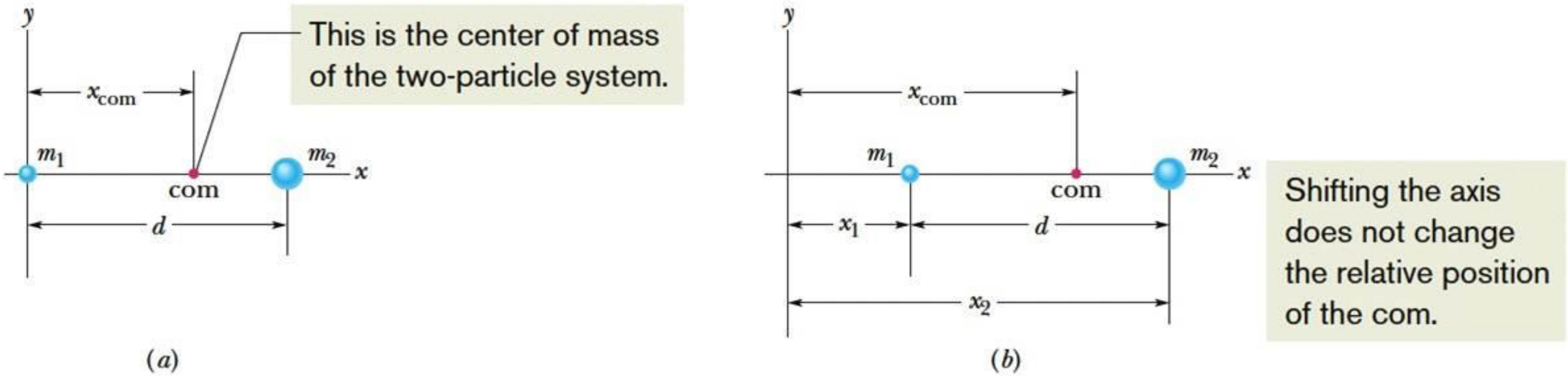


Figure 9-1

- The **center of mass** (com) of a system of particles:

The center of mass of a system of particles is the point that moves *as though* (1) all of the system's mass were concentrated there and (2) all external forces were applied there.



$$x_{com} = \frac{m_2}{m_1 + m_2} d.$$

Figure 9-2

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

- The center of mass is in *the same location* regardless of the coordinate system used
- *It is a property of the particles, not the coordinates*

- For many particles, we can generalize the equation, where $M = m_1 + m_2 + \ldots + m_n$:

$$x_{\text{com}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i.$$

Equation (9-4)

- In three dimensions, we find the center of mass *along each axis separately*:

$$x_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i.$$

Equation (9-5)

Slide based on ref. [1]

- More concisely, we can write in terms of vectors:

$$\vec{r}_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i, \qquad \textbf{Equation (9-8)}$$

- For solid bodies, we take the limit of an infinite sum of infinitely small particles → integration!
- Coordinate-by-coordinate, we write:

$$x_{\text{com}} = \frac{1}{M}\int x\,dm, \quad y_{\text{com}} = \frac{1}{M}\int y\,dm, \quad z_{\text{com}} = \frac{1}{M}\int z\,dm, \quad \textbf{Equation (9-9)}$$

- Here M is the mass of the object

- We limit ourselves to objects of ***uniform density***, ρ, for the sake of simplicity

$$\rho = \frac{dm}{dV} = \frac{M}{V},$$ **Equation (9-10)**

- Substituting, we find the center of mass simplifies:

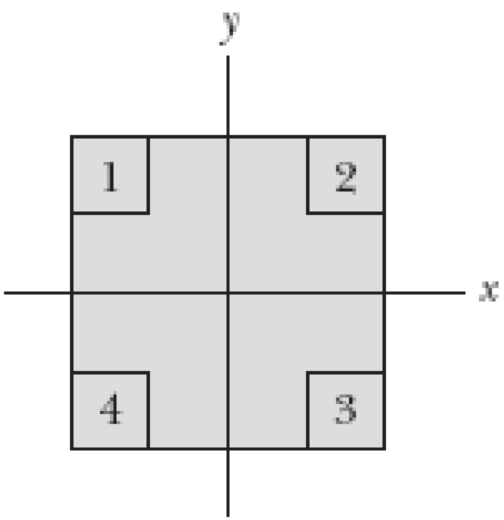
$$x_{\text{com}} = \frac{1}{V}\int x\, dV, \quad y_{\text{com}} = \frac{1}{V}\int y\, dV, \quad z_{\text{com}} = \frac{1}{V}\int z\, dV.$$ **Equation (9-11)**

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int \vec{\mathbf{r}}\, dm.$$

- You can bypass one or more of these integrals if the object has *symmetry*

Checkpoint 1

The figure shows a uniform square plate from which four identical squares at the corners will be removed. (a) Where is the center of mass of the plate originally? Where is it after the removal of (b) square 1; (c) squares 1 and 2; (d) squares 1 and 3; (e) squares 1, 2, and 3; (f) all four squares? Answer in terms of quadrants, axes, or points (without calculation, of course).



Answer:

(a) at the origin

(b) in Q_4, along $y = -x$

(c) along the $-y$ axis

(d) at the origin

(e) in Q_3, along $y = x$

(f) at the origin

Example Subtracting

- Task: find CoM of a disk with another disk taken out of it:
- Find the CoM of each individual disk (start from the bottom and work up)
- Find the CoM of the two individual CoM s (one for each disk), treating the cutout as having negative mass
- On the diagram, CoM $_C$ is the center of mass for Plate *P* and Disk *S* combined
- CoM $_P$ is the center of mass for the composite plate with Disk *S* removed

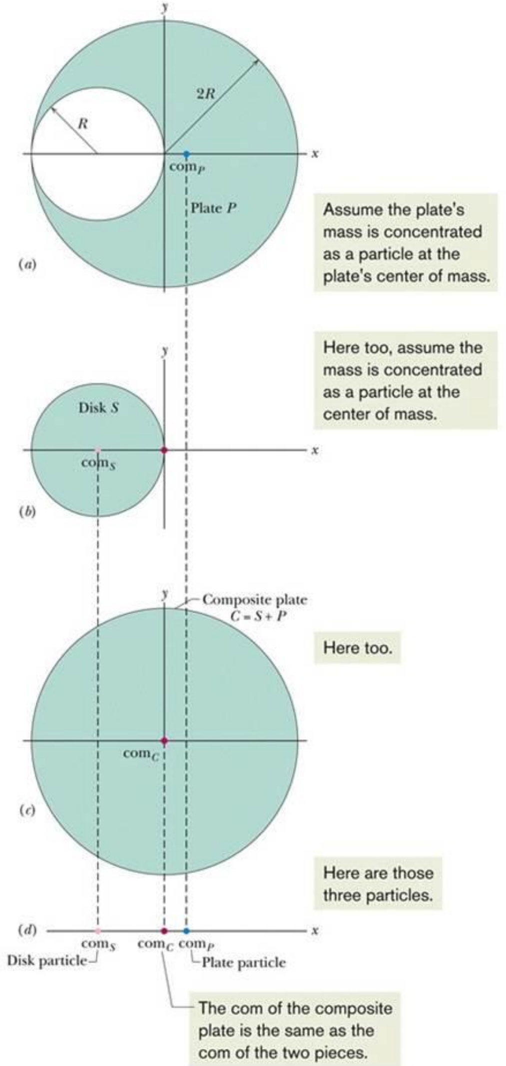


Figure 9-4

https://www.youtube.com/watch?v=gPDjaMsG_rg&list=PL860B6886A47E5490&index=18

Newton's Second Law for a System of Particles

- Center of mass motion continues unaffected by forces internal to a system (collisions between billiard balls) [*why?*]
- Motion of a system's center of mass:

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}} \quad (\text{system of particles}).$$ **Equation (9-14)**

$$F_{\text{net},\,x} = Ma_{\text{com},\,x} \quad F_{\text{net},\,y} = Ma_{\text{com},\,y} \quad F_{\text{net},\,z} = Ma_{\text{com},\,z}.$$ **Equation (9-15)**

- Reminders:
  1. F_{net} is the sum of all *external forces*
  2. M is the total, constant, mass of the **closed** system
  3. a_{com} is the center of mass acceleration

Examples Using the center of mass motion equation:

- Billiard collision: forces are only internal, $F = 0$ so $a = 0$
- Baseball bat: $a = g$, so com follows gravitational trajectory
- Exploding rocket: explosion forces are internal, so only the gravitational force acts on the system, and the CoM follows a gravitational trajectory as long as air resistance can be ignored for the fragments.

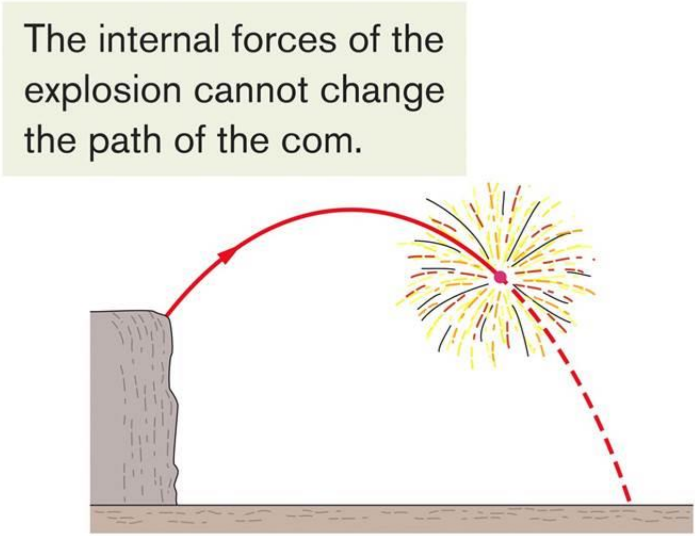
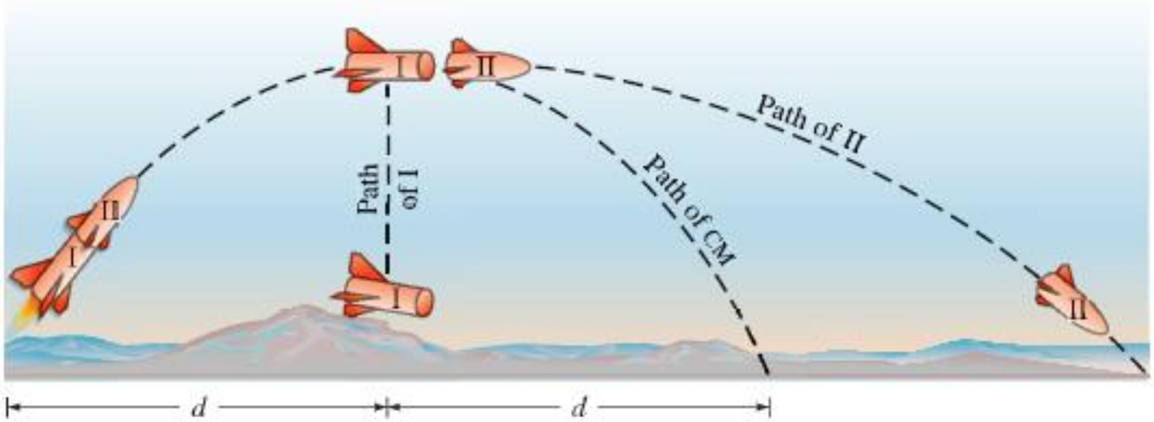


Figure 9-5

Picture from ref. [2]



Checkpoint 2

Two skaters on frictionless ice hold opposite ends of a pole of negligible mass. An axis runs along it, with the origin at the center of mass of the two-skater system. One skater, Fred, weighs twice as much as the other skater, Ethel. Where do the skaters meet if (a) Fred pulls hand over hand along the pole so as to draw himself to Ethel, (b) Ethel pulls hand over hand to draw herself to Fred, and (c) both skaters pull hand over hand?

Answer: The system consists of Fred, Ethel and the pole. All forces are internal. Therefore the CoM will remain in the same place. Since the origin is the CoM, they will meet at the origin in all three cases! (Of course the origin where the com is located is closer to Fred than to Ethel.)

Linear Momentum

- The **linear momentum** is *defined* as:

$$\vec{p} = m\vec{v}$$ **Equation (9-22)**

- The momentum:
  - Points in the same direction as the velocity
  - Can only be changed by a net external force

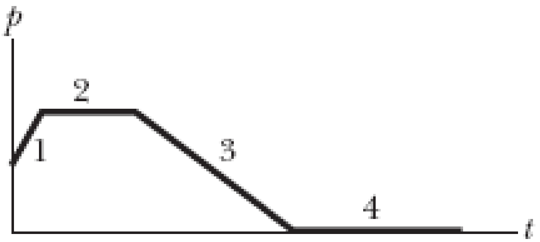
The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

- We can write Newton's second law thus:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$ **Equation (9-23)**

Checkpoint 3

The figure gives the magnitude *p* of the linear momentum versus time *t* for a particle moving along an axis. A force directed along the axis acts on the particle. (a) Rank the four regions indicated according to the magnitude of the force, greatest first. (b) In which region is the particle slowing?



Answer:

(a) 1, 3, 2 & 4

(b) region 3

- We can sum momenta for a system of particles to find:

$$\vec{P} = M\vec{v}_{\text{com}} \quad \text{(linear momentum, system of particles)},$$ **Equation (9-25)**

The linear momentum of a system of particles is equal to the product of the total mass M of the system and *the velocity of the center of mass*.

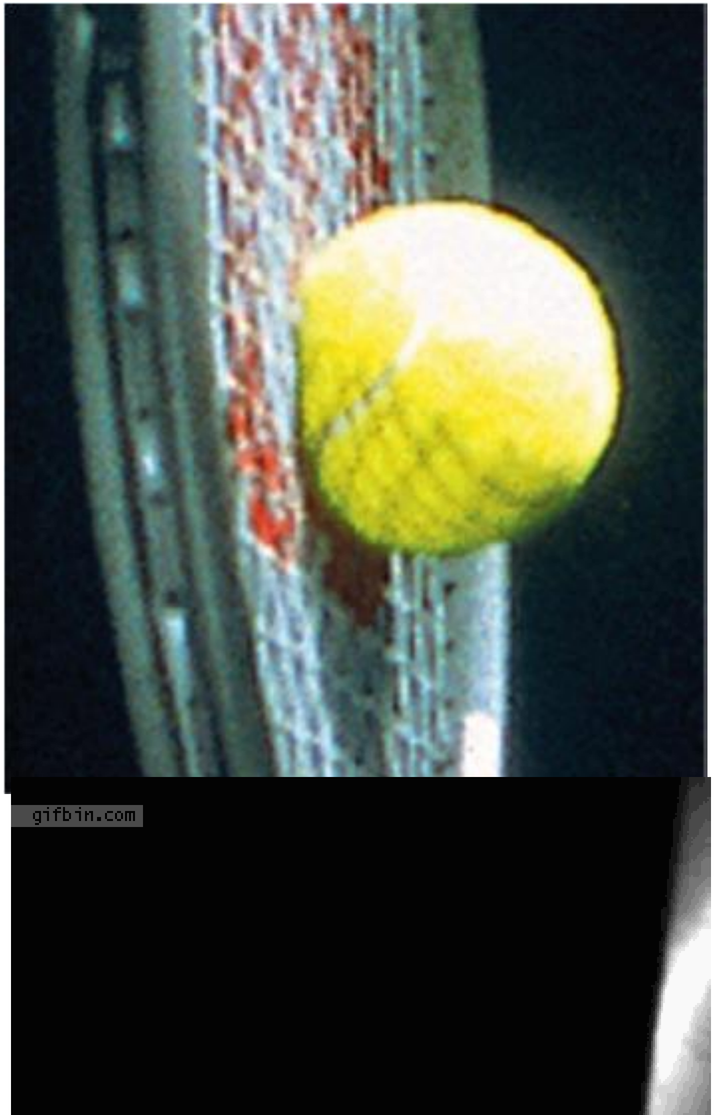
- Taking the time derivative we can write Newton's second law *for a system of particles* as:

$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt} \quad \text{(system of particles)}, \vec{F}_{\text{net}} = M\vec{a}_{\text{com}} \quad \text{(system of particles)}.$$

Equation (9-27)

- The net external force on a system changes linear momentum
- <u>*Without a net external force, the total linear momentum of a system of particles cannot change*</u>

Collisions and Impulse



During a collision, objects are deformed due to the large forces involved.

Since $\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt}$**, we can write** $d\vec{\mathbf{p}} = \vec{\mathbf{F}}\,dt.$

Integrating,

$$\int_{\mathrm{i}}^{\mathrm{f}} d\vec{\mathbf{p}} = \vec{\mathbf{p}}_{\mathrm{f}} - \vec{\mathbf{p}}_{\mathrm{i}} = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \vec{\mathbf{F}}\,dt.$$

Slide based on ref. [2]

This quantity is defined as the impulse 冲量, J:

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}\, dt.$$

The impulse is equal to the change in momentum (*for one particle/object*):

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \int_{t_i}^{t_f} \vec{\mathbf{F}}\, dt = \vec{\mathbf{J}}.$$

Slide based on ref. [2]

Collision and Impulse

- Given F_{avg} and duration:

$$J = F_{avg}\Delta t.$$ **Equation (9-35)**

- We are integrating: we only need to know the area under the force curve

$$\vec{\mathbf{F}}_{avg}\,\Delta t = \int_{t_i}^{t_f} \vec{\mathbf{F}}\,dt.$$

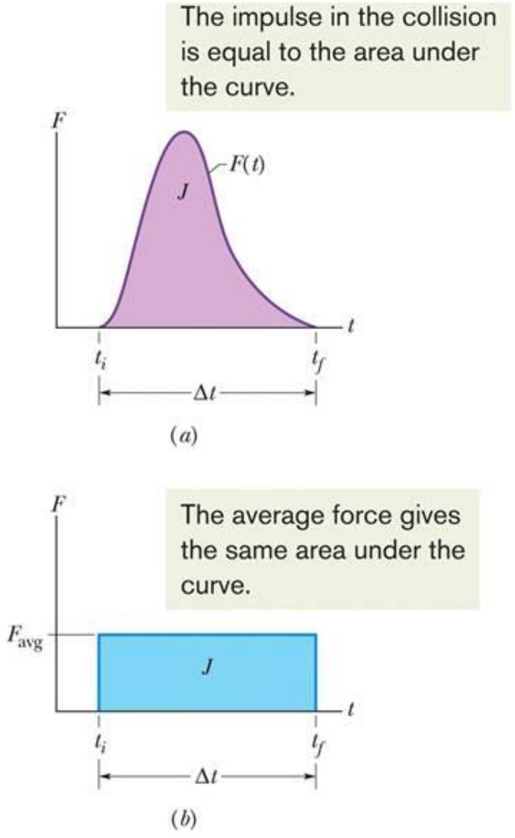


Figure 9-9

Checkpoint 4

A paratrooper whose chute fails to open lands in snow; he is hurt slightly. Had he landed on bare ground, the stopping time would have been 10 times shorter and the collision lethal. Does the presence of the snow increase, decrease, or leave unchanged the values of (a) the paratrooper's change in momentum, (b) the impulse stopping the paratrooper, and (c) the force stopping the paratrooper?

Answer:

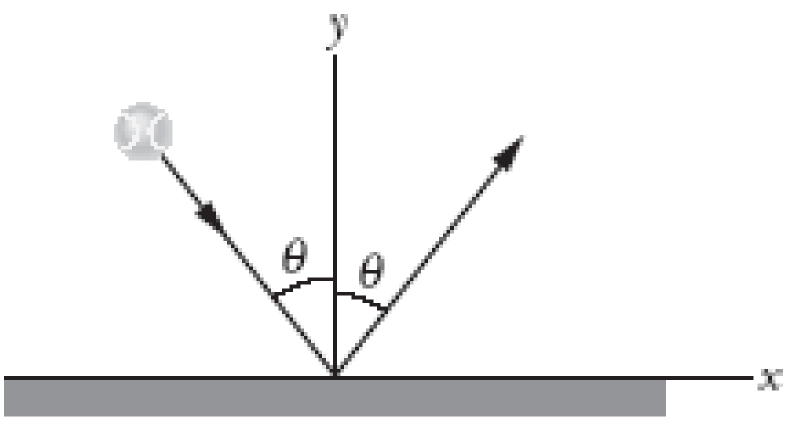
(a) unchanged

(b) unchanged

(c) decreased

Checkpoint 5

The figure shows an overhead view of a ball bouncing from a vertical wall without any change in its speed. Consider the change $\Delta\vec{p}$ in the ball's linear momentum. (a) Is Δp_x positive, negative, or zero? (b) Is Δp_y positive, negative, or zero? (c) What is the direction of $\Delta\vec{p}$?



Answer:

(a) zero

(b) positive

(c) along the positive *y*-axis (normal force)

Conservation of Linear Momentum

- For an impulse of zero we find:

$$\vec{P} = \text{constant} \quad (\text{closed, isolated system}).$$ **Equation (9-42)**

- Which says that:

If no net external force acts on a system of particles, the total linear momentum $\vec{P}$ *of the system cannot change.*

- This is called the **law of conservation of linear momentum**
- Check the *<u>components</u>* of the net external force to know if you should apply this

Checkpoint 6

An initially stationary device lying on a frictionless floor explodes into two pieces, which then slide across the floor, one of them in the positive x direction. (a) What is the sum of the momenta of the two pieces after the explosion? (b) Can the second piece move at an angle to the x axis? (c) What is the direction of the momentum of the second piece?

Answer:

(a) zero

(b) no

(c) the negative x direction

Momentum and Kinetic Energy in Collisions

- Types of collisions:
- **Elastic collisions**:
  - Total *kinetic energy* is unchanged (conserved)
  - A useful approximation for common situations
  - In *real* collisions, some energy is always transferred
- **Inelastic collisions**: some energy is transferred
- **Completely inelastic collisions**:
  - *The objects stick together*
  - Greatest loss of kinetic energy

- For one dimension:
- Inelastic collision $m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$
- Completely inelastic collision, for target at rest:

$$m_1 v_{1i} = \left(m_1 + m_2\right) V$$

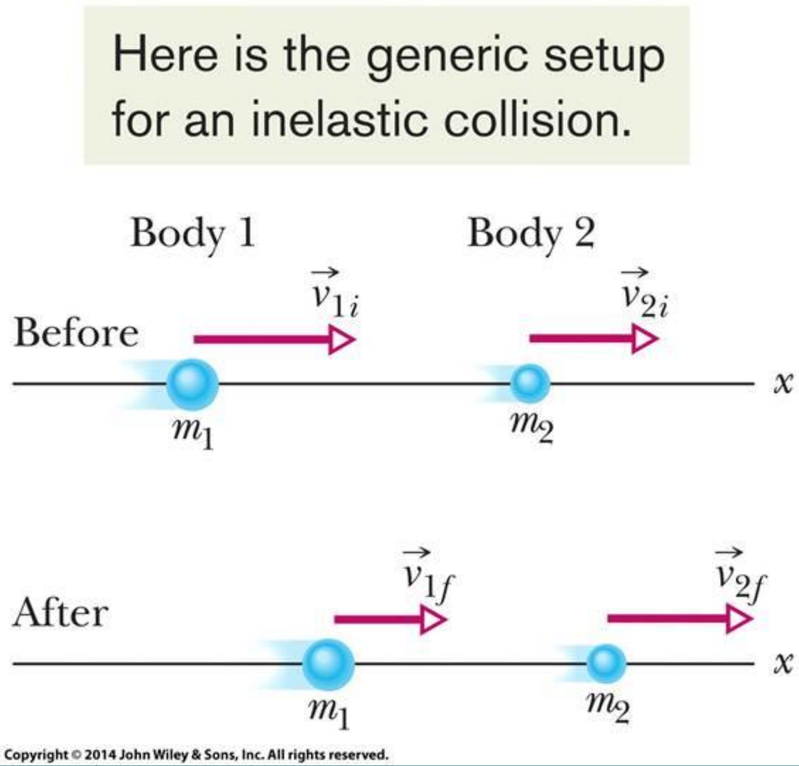


Figure 9-14

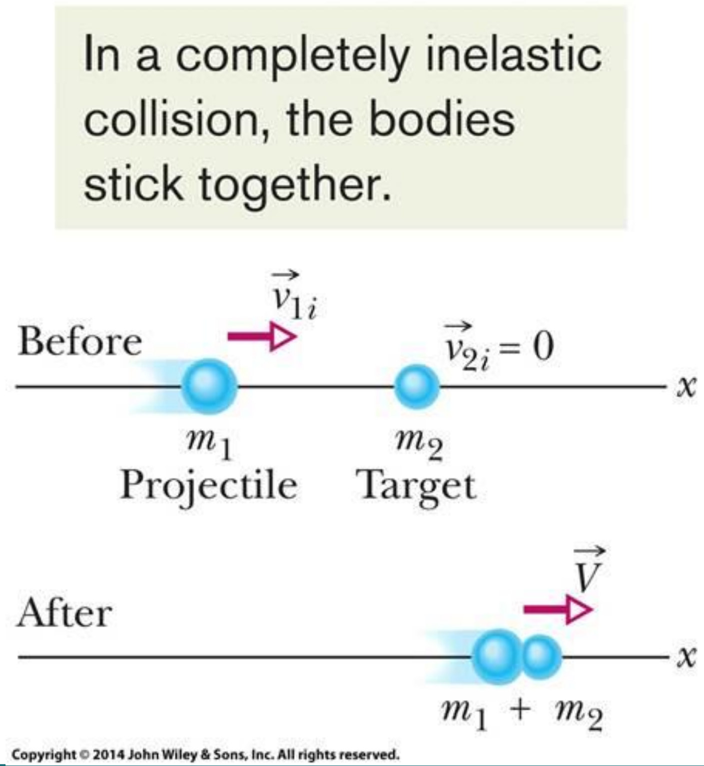


Figure 9-15

- The center of mass velocity remains unchanged:

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}.$$

Equation (9-56)

- Figure 9-16 shows freeze frames of a completely inelastic collision, showing center of mass velocity

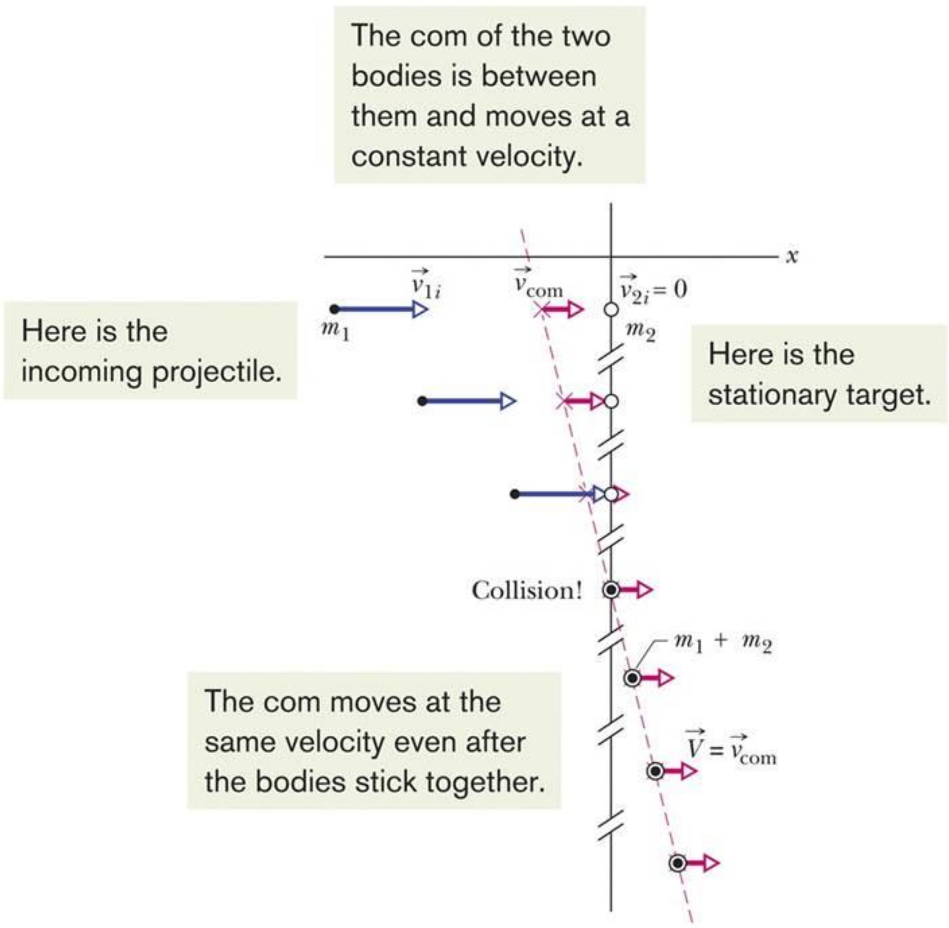
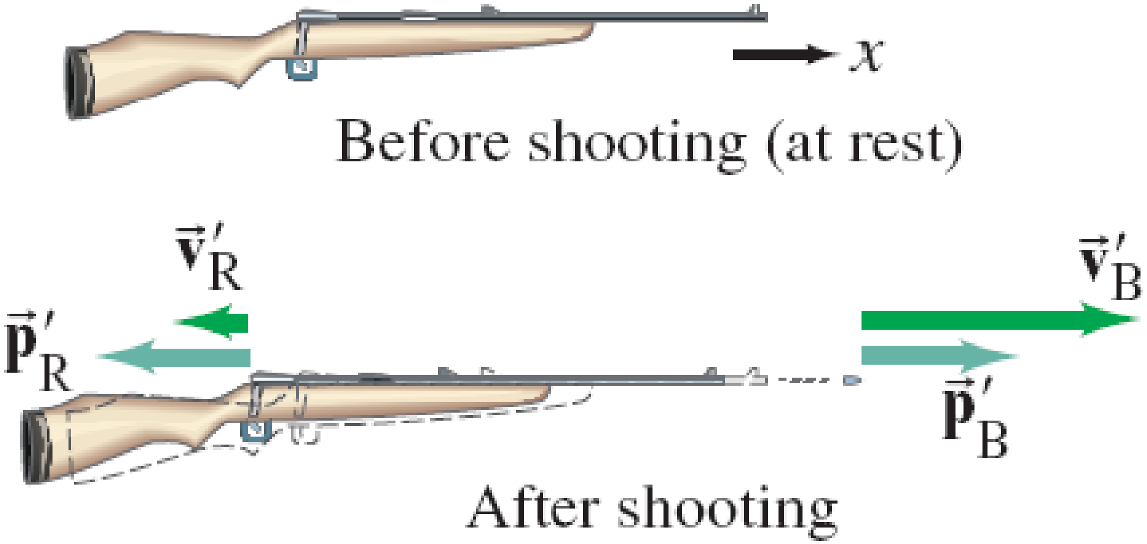


Figure 9-16

Example: Rifle recoil.

Calculate the recoil velocity of a 5.0-kg rifle that shoots a 0.020-kg bullet at a speed of 620 m/s.



Elastic Collisions in One Dimension

- Total kinetic energy is conserved in elastic collisions

  In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

- For *a stationary target in 1D*, conservation laws give:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{(linear momentum).} \qquad \textbf{Equation (9-63)}$$

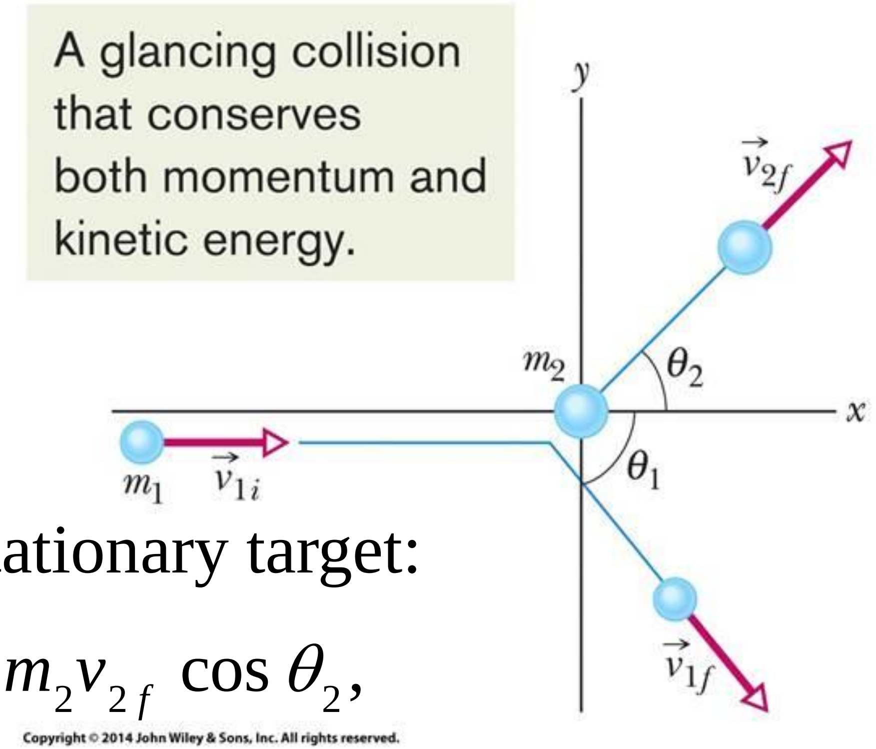
$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \text{(kinetic energy).} \qquad \textbf{Equation (9-64)}$$

- With some algebra we get:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

Collisions in Two Dimensions

- *Apply the conservation of momentum along each axis*
- Apply conservation of energy for elastic collisions



Example For Figure 9-21 for a stationary target:

- Along x: $m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2,$
- Along y: $0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2.$
- Energy: $\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$

Figure 9-21

- These 3 equations for a stationary target have 7 unknowns (since $v_{2i} = 0$) : if we know 4 of them we can solve for the remaining ones.

Checkpoint 9

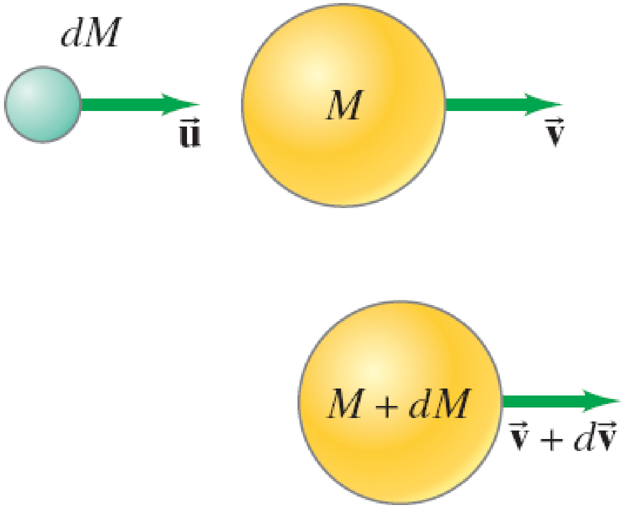
In Fig. 9-21, suppose that the projectile has an initial momentum of $6\ \text{kg}\cdot\text{m/s}$, a final *x* component of momentum of $4\ \text{kg}\cdot\text{m/s}$, and a final *y* component of momentum of $-3\ \text{kg}\cdot\text{m/s}$. For the target, what then are (a) the final *x* component of momentum and (b) the final *y* component of momentum?

Answer:

(a) 2 kg m/s   (b) 3 kg m/s

Systems of Variable Mass; Rocket Propulsion

Applying Newton's second law to the system shown gives:



$$d\vec{\mathbf{P}}/dt = \Sigma \vec{\mathbf{F}}_{\text{ext}}.$$

Therefore,

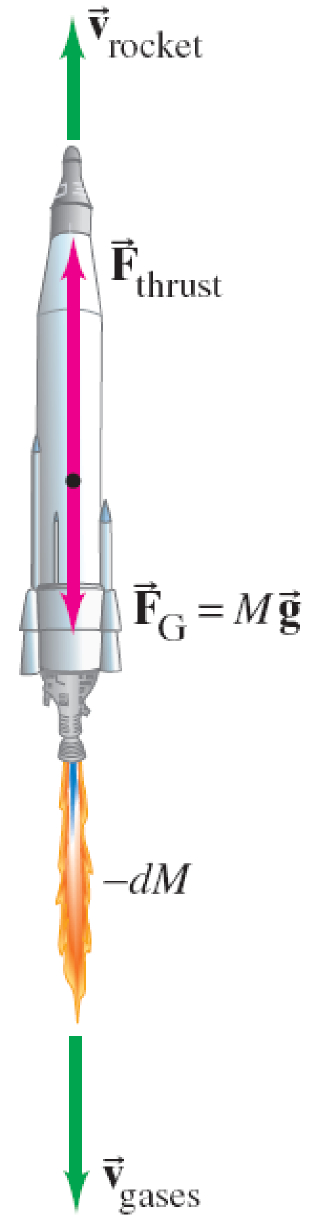
$$\Sigma \vec{\mathbf{F}}_{\text{ext}} = M\frac{d\vec{\mathbf{v}}}{dt} - (\vec{\mathbf{u}} - \vec{\mathbf{v}})\frac{dM}{dt},$$

or

$$M\frac{d\vec{\mathbf{v}}}{dt} = \Sigma \vec{\mathbf{F}}_{\text{ext}} + \vec{\mathbf{v}}_{\text{rel}}\frac{dM}{dt}.$$

Example: Rocket propulsion.

A fully fueled rocket has a mass of 21,000 kg, of which 15,000 kg is fuel. The burned fuel is spewed out the rear at a rate of 190 kg/s with a speed of 2800 m/s relative to the rocket. If the rocket is fired vertically upward calculate: (a) the thrust of the rocket; (b) the net force on the rocket at blastoff, and just before burnout (when all the fuel has been used up); (c) the rocket's velocity as a function of time, and (d) its final velocity at burnout. Ignore air resistance and assume the acceleration due to gravity is constant at g = 9.80 m/s^2.



Learning Outcomes

- ✓ Calculate the center of mass of an object/a system (of objects)
- ✓ Analyze the motion of the CoM.
- ✓ Find the linear momentum/impulse of a particle/an object.
- ✓ Apply conservation of linear momentum in 1- and 2-D.
- ✓ Solve problems of variable mass system.

CHAPTER REVIEW AND EXAMPLES, DEMOS

https://openstax.org/books/university-physics-volume-1/pages/9-summary

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. *The mass of the system must remain constant during the interaction.*
   As the objects interact (apply forces on each other), they may *transfer* mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:

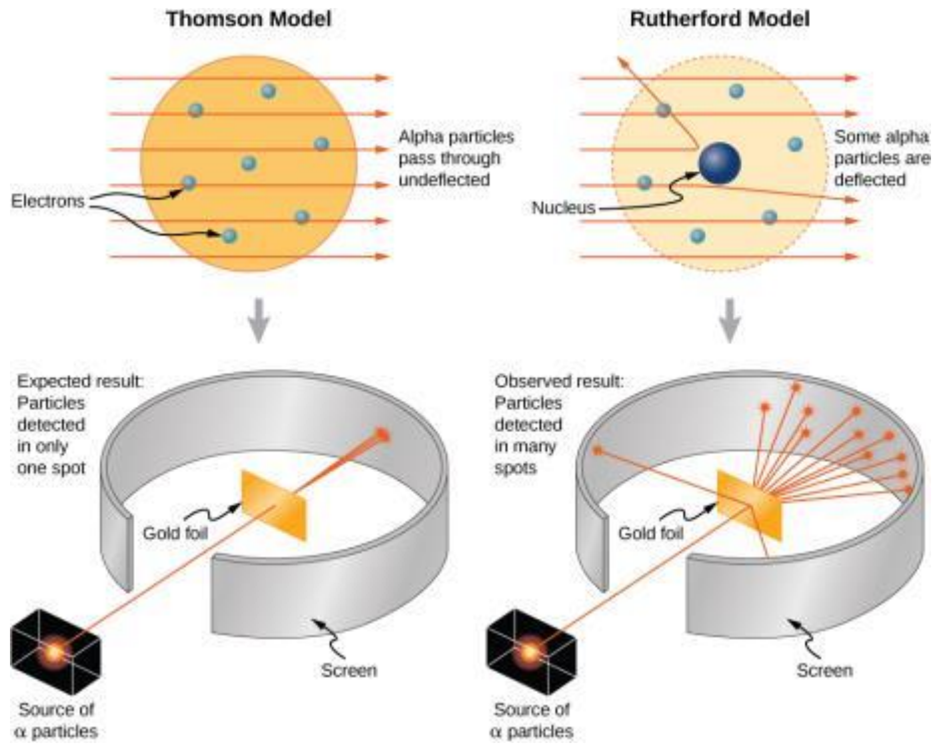
$$\left[\frac{dm}{dt}\right]_{\text{system}} = 0.$$

2. *The net external force on the system must be zero.*
   As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{\mathbf{F}}_{\text{ext}} = \vec{\mathbf{0}}.$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Thus, the more compact way to express this is shown below.

FIGURE 9.21

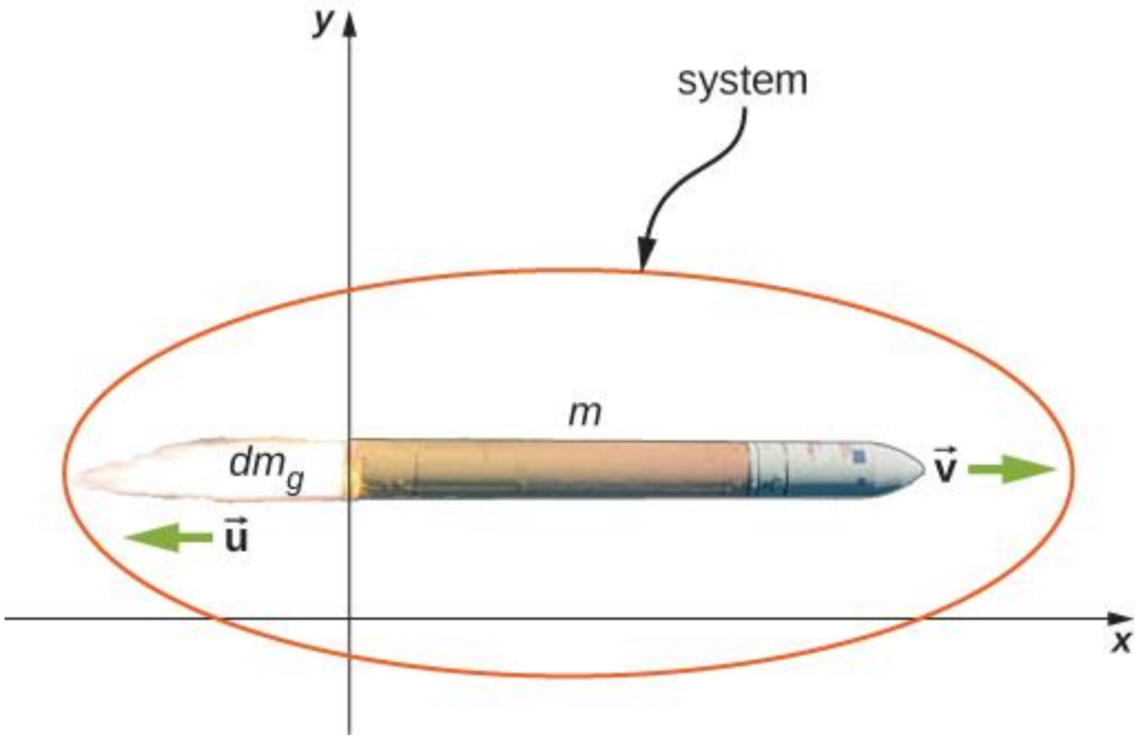


The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.

FIGURE 9.31

These exploding fireworks are a vivid example of conservation of momentum and the motion of the center of mass.

FIGURE 9.33



The rocket accelerates to the right due to the expulsion of some of its fuel mass to the left. Conservation of momentum enables us to determine the resulting change of velocity. The mass *m* is the instantaneous total mass of the rocket (i.e., mass of rocket body plus mass of fuel at that point in time). (credit: modification of work by NASA/Bill Ingalls)

Example about the demo, and the Rocket in the recorded video.

Questions