Devoir3 IFT2125-A-H19

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Question 1

Provez que $f \in \Theta(g)$ si, et seulement si $g \in \Theta(f)$, si, et seulement si $\Theta(f) = \Theta(g)$.

Answer 1

1. Recall the justification of $f \in O(g) \Leftrightarrow g \in \Omega(f)$:

$$\begin{aligned} f \in O(g) &\Leftrightarrow \exists n_0 \geq 1, c_0 \geq 0, \forall n > n_0, f(n) \leq c_0 g(n) \\ &\Leftrightarrow \exists n_0 \geq 1, c_0 \geq 0, \forall n > n_0, g(n) \geq \frac{1}{c_0} f(n) \\ &\Leftrightarrow \exists n_0 \geq 1, c_1 = \frac{1}{c_0} \geq 0, \forall n > n_0, g(n) \geq c_1 f(n) \\ &\Leftrightarrow g \in \Omega(f) \end{aligned}$$

Recall the justification of $f \in \Theta(f)$:

$\exists n_0 = 1, c_0 = 1, c_0' = 1, \forall n \geq n_0, c_0 f(n) \leq f(n) \leq c_0' f(n)$. Obviously, it is true that:

$$f \in \Theta(f) \tag{1.0}$$

.

Part1. Prove that $g \in \Theta(f) \Leftrightarrow f \in \Theta(g)$.

$$\begin{aligned} g \in \Theta(f) &\Leftrightarrow g \in O(f) \wedge g \in \Omega(f) \\ &\Leftrightarrow f \in \Omega(g) \wedge f \in O(g) \\ &\Leftrightarrow f \in \Theta(g) \end{aligned}$$

Part2. Prove that $g \in \Theta(f) \Leftrightarrow \Theta(f) = \Theta(g)$.

First, we have:

$$\Theta(f) = \Theta(g) \Leftrightarrow (\forall h \in \Theta(f) \Rightarrow h \in \Theta(g)) \wedge (\forall h \in \Theta(g) \Rightarrow h \in \Theta(f))$$

and

$$\begin{aligned} f \in \Theta(g) &\Leftrightarrow f \in O(g) \wedge f \in \Omega(g) \\ &\Leftrightarrow \exists n_0 \geq 1, c_0 \geq 0, c_0' \geq 0, \forall n \geq n_0, c_0, c_0',\ c_0 g(n) \leq f(n) \leq c_0' g(n) \\ &\Leftrightarrow \exists n_0 \geq 1, c_0 \geq 0, c_0' \geq 0, \forall n \geq n_0, c_0, c_0',\ g(n) \leq \frac{1}{c_0} f(n) \wedge \frac{1}{c_0'} f(n) \leq g(n) \end{aligned} \tag{1.1}$$

Suppose any function $h : \mathbb{N} \to \mathbb{R}^{\geq 0} \in \Theta(g)$:

$$\begin{aligned} h \in \Theta(g) &\Leftrightarrow h \in O(g) \wedge h \in \Omega(g) \\ &\Leftrightarrow \exists n_1 \geq 1, c_1 \geq 0, c_1' \geq 0, \forall n \geq n_1, c_1 g(n) \leq h(n) \leq c_1' g(n) \end{aligned} \tag{1.2}$$

Based on formular (1.1),(1.2), we have:

$\exists n_2 = \max\{n_0, n_1\} \geq 1, c_0 \geq 0, c_0' \geq 0, c_1 \geq 0, \text{ and } c_1' \geq 0,$

$\forall n \geq n_2, \frac{c_1}{c_0'} f(n) \leq h(n) \leq \frac{c_1'}{c_0} f(n).$

That is,

$$h \in \Theta(g) \Rightarrow h \in \Theta(f) \tag{1.3}$$

Similarly, we also have:

$$h \in \Theta(f) \Rightarrow h \in \Theta(g) \tag{1.4}$$

Based on (1.1),(1.3),(1.4), we have,

$$f \in \Theta(g) \Rightarrow \Theta(f) = \Theta(g) \tag{1.5}$$

Based on (1.0), we have:

$$\begin{aligned} \Theta(f) = \Theta(g) &\Rightarrow f \in \Theta(f) \\ &\Rightarrow f \in \Theta(g) \end{aligned} \tag{1.6}$$

Finally, based on (1.5) and (1.6), we proved:

$$f \in \Theta(g) \Leftrightarrow \Theta(f) = \Theta(g)$$

Question2

Donnez un exemple de fonctions f, g telles que $f \in \Theta(g)$ et $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ n'existe pas. Prouvez vos dires. Les meilleurs reponses ont $g(n) > 0$.

Answer2

Let, $f(n) = 2n + n\sin(n),\ \text{and } g(n) = n$, where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f \in \Theta(g)$, but $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ doesn't exist. Here is the proof.

$$\begin{aligned} -1 \leq \sin(n) \leq 1 &\Rightarrow -n \leq n\sin(n) \leq n \\ &\Rightarrow 2n - n \leq 2n + n\sin(n) \leq 2n + n \\ &\Rightarrow n \leq 2n + n\sin(n) \leq 3n \end{aligned}$$

$n_0 = 1, c_0 = 1,\ \text{and } c_1 = 3, \forall n \geq n_0$:

$$c_0 g(n) \leq f(n) \leq c_1 g(n)$$

Therefore,

$$f \in \Theta(g)$$

$$\begin{aligned} \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \lim_{n\to\infty} \frac{2n + n\sin(n)}{n} \\ &= \lim_{n\to\infty} (2 + \sin(n)) \\ &= 2 + \lim_{n\to\infty} \sin(n) \end{aligned}$$

Since $\lim_{n\to\infty} \sin(n)$ doesn't exist, $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ doesn't exist either.

Question3

Prouvez que $f(n) = \lg n$, $p(n) = \sum_{i=0}^{k} a_i n^{k-i}, a_i \in \mathbb{R}, i, k \in \mathbb{N}$, sont lisses.

Answer3

part1 prove that $f(n) = \lg n$ is lisse.

First, $\exists n_0 = 1, \forall n \geq n_0$

$$\begin{aligned} f(n+1) - f(n) &= \lg(n+1) - \lg n \\ &= \lg \frac{n+1}{n} \\ &= \lg(1 + \frac{1}{n}) \\ &\geq 0 \end{aligned} \tag{3.1}$$

We can say, function $f(n) = \lg n$ is **e.n.d.** (éventuellement non-décroissante)

Let $b \in \mathbb{N}^{\geq 2}, c_b \in \mathbb{R}^{>1} \subset \mathbb{R}^{\geq 0}$,

$$\begin{aligned} f(bn) - c_b f(n) \leq 0 &\Leftrightarrow \lg(bn) - c_b \lg n \leq 0 \\ &\Leftrightarrow \lg n + \lg b - c_b \lg n \leq 0 \\ &\Leftrightarrow (1 - c_b) \lg n + \lg b \leq 0 \\ &\Leftrightarrow \lg n \geq \frac{\lg b}{c_b - 1} \\ &\Leftrightarrow n \geq 2^{\frac{\lg b}{c_b - 1}} \end{aligned}$$

Which means,

$$\exists c_b \in \mathbb{R}^{>1} \subset \mathbb{R}^{\geq 0}, n_b = \lceil 2^{\frac{\lg b}{c_b-1}} \rceil \in \mathbb{N}, , \forall n \geq n_b, \forall b \in \mathbb{N}^{\geq 2},$$
$$f(bn) \leq c_b f(n) \tag{3.2}$$

Based on (3.1) and (3.2) We can say: $\lg n$ is lisse.

part2 prove that $p(n) = \sum_{i=0}^{k} a_i n^{k-i}, a_i \in \mathbb{R}, i, k \in \mathbb{N}$ is lisse.

To prove $p(n) = \sum_{i=0}^{k} a_i n^{k-i}, a_i \in \mathbb{R}, i, k \in \mathbb{N}$ is lisse,

we **need more constrain for** a_i: $a_0 > 0$, which is a special case of more general constrains:

$$a_j > 0 \text{ and } \forall i < j, a_i = 0, \text{ where } j < k$$

Otherwise, $p(n)$ may not always **e.n.d.** or lisse.

We know that, for $x, r_i \in \mathbb{R}$, $a_0 \prod_{i=0}^{k} (x - r_i)$ can be expanded to the form of $\sum_{i=0}^{k} a_i x^{k-i}$, that is:

$$\prod_{i=0}^{k} (x - r_i) = \frac{1}{a_0} \sum_{i=0}^{k} a_i x^{k-i}$$

where $a_k = a_0 (-1)^k \prod_{i=0}^{k} |r_i|$.

So, $\exists C \in \mathbb{R}$, satisfies:

$$\sum_{i=0}^{k} a_i x^{k-i} = a_0 \prod_{i=0}^{k} (x - r_i) + C$$

As $n \in \mathbb{N} \subset \mathbb{R}$, it is true that:

$$p(n) = \sum_{i=0}^{k} a_i n^{k-i} = a_0 \prod_{i=0}^{k} (n - r_i) + C \tag{3.3}$$

$\exists n_0 = max\{r_i \mid i \in \mathbb{N}, i \leq k\}$, such that: $\forall n \geq n_0, n - r_i \geq 0$, and:

$$\begin{aligned} p(n+1) - p(n) &= a_0 \prod_{i=0}^{k} (n + 1 - r_i) + C - \left(a_0 \prod_{i=0}^{k} (n - r_i) + C \right) \\ &= a_0 \prod_{i=0}^{k} (n + 1 - r_i) - a_0 \prod_{i=0}^{k} (n - r_i) \\ &\geq a_0 \prod_{i=0}^{k} (n - r_i) - a_0 \prod_{i=0}^{k} (n - r_i) \\ &= 0 \end{aligned}$$

We can say, function $p(n)$ is **e.n.d.** (éventuellement non-décroissante)

Let

$$n_b = \lceil max\{r_i,\ r_i \left(1 + \frac{1}{b}\right) \mid i \in \mathbb{N}, i \leq k\} \rceil \tag{3.4}$$

such that $\forall n \geq n_b$, we have:

$$n - r_i \geq 0,\ bn - r_i \geq 0,\ p(bn) > 0,\ p(n) > 0 \tag{3.5}$$

, and,

$$r_i\left(1 + \frac{1}{b}\right) \leq n,\ \text{where } 0 \leq i \leq k \tag{3.6}$$

let $b \in \mathbb{N}^{\geq 2}, c_b = b^{2k}$, such that: $c_b \in \mathbb{R}^{>0}$.

Based on (3.5), we have:

$$\begin{aligned} p(bn) - c_b p(n) <= 0 &\Leftrightarrow \frac{p(bn)}{c_b p(n)} \leq 1 \\ &\Leftrightarrow \frac{\prod_{i=0}^{k}(bn - r_i) + C}{c_b \prod_{i=0}^{k}(n - r_i) + C} \leq 1 \\ &\Leftrightarrow \frac{\prod_{i=0}^{k}(bn - r_i)}{b^{2k} \prod_{i=0}^{k}(n - r_i)} \leq 1 \\ &\Leftrightarrow \frac{\prod_{i=0}^{k}(bn - r_i)}{\prod_{i=0}^{k} b^2(n - r_i)} \leq 1 \end{aligned} \tag{3.7}$$

To prove (3.7), we only need to prove $\frac{bn-r_i}{b^2(n-r_i)} \leq 1,$ for all i where $0 \leq i \leq k$:

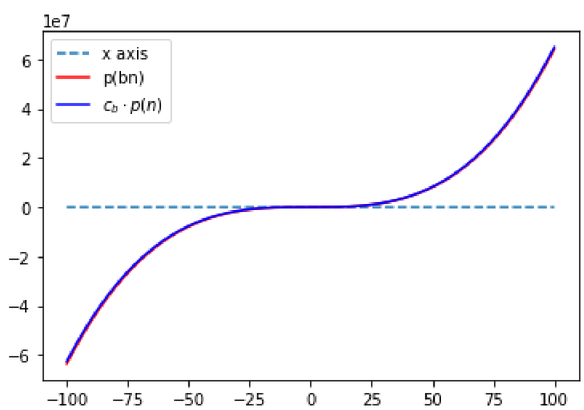
$$\begin{aligned} \frac{bn - r_i}{b^2(n - r_i)} \leq 1 &\Leftrightarrow bn - r_i \leq b^2(n - r_i) \\ &\Leftrightarrow bn - r_i \leq b^2 n - b^2 r_i \\ &\Leftrightarrow (b^2 - 1) r_i \leq (b^2 - b)n \\ &\Leftrightarrow (b+1)(b-1) r_i \leq b(b-1)n \\ &\Leftrightarrow \frac{b+1}{b} r_i \leq n \\ &\Leftrightarrow \left(1 + \frac{1}{b}\right) r_i \leq n \end{aligned} \tag{3.8}$$

(3.8) is same with (3.6),which is implied by (3.4), the choose of n_b.

Finally, we can say $p(n) = \sum_{i=0}^{k} a_i n^{k-i}, a_i \in \mathbb{R}, a_0 > 0, i, k \in \mathbb{N}$ is lisse.

In [2]:

```python
# tiny demostration of question2 or question3
import numpy as np
import matplotlib.pyplot as plt

def p_n(n, roots, const): # for question3
    k = len(roots)
    #roots = np.random.randint(-root_limit, root_limit, k)
    result = np.ones_like(n)
    for i in range(k):
        result *= (n-roots[i])
    return result + const

def n_plus_n_sin(n): # for question2
    return 2*n + n * np.sin(n)

def g_n(n): # for question2
    return n

C, b, k = 10, 4, 3
c_b = pow(b, k) # c_b should be no less than pow(b, k) to satisfy p(bn) <= c_b p(n)

roots = np.random.randint(-10, 10, k)
n = np.arange(-100, 100, 0.1)
p_bn = p_n(b*n, roots = roots, const = C)
cb_pn = c_b * p_n(n, roots = roots, const = C)

f_n = n_plus_n_sin(n)
z = n*0

# plot of second parth of question 3
plt.plot(n, z, '--', label = "x axis")
plt.plot(n, p_bn, 'r', label = 'p(bn)')
plt.plot(n, cb_pn, 'b', label = "$c_b \cdot p(n)$")
plt.legend()

# plot of question2
#plt.plot(n, f_n)
#plt.plot(n, 1*g_n(n))
#plt.plot(n, 3*g_n(n))
plt.show()

b = 3
n_b = int(np.ceil(np.max(np.concatenate((roots, (1+1/b)*roots)))))
print(n_b, p_n(n_b, roots, const = C) <= c_b*p_n(n_b, roots, const = C))
```

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6 True
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