Devoir01 IFT2125-A-H19

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Question

Prouvez que la multiplication à la russe fonctionne en utilisant le fait que chaque entier peut être écrit comme la somme des puissances de 2.

Pour multiplier A par B, on fait un tableau T de taille n x 2 (n à détérminer dans le devoir). Dans la premiére rangée en met A et B dans les deux colonnes respectives, i.e. T(1,1)=A, T(1,2)=B. Quand on a T(i,1) et T(i,2), on obtient T(i+1,1)=[T(i,1)/2] et T(i+1,2)= 2T(i,2) (ici on revient a a vielle notation [x] de la partie entière du nombre x). On arrête quand T(n,1)=1. On obtient le résultat en faisant la somme des T(i,2) tels que T(i,1) est impair.

Answer

First, we rewrite the question in mathematical form:

Given the fact that "chaque entier peut être écrit comme la somme des puissances de 2", we can express any integer A by the formula:

$$A = \sum_{i=0}^{k} a_i 2^i \tag{1}$$

As to the process of "Russian multiplication", for two integers A and B, we build a table T with height of n, where:

$$\begin{aligned} T(1,1) &= A && (2)\\ T(1,2) &= B && (3)\\ T(i+1,1) &= [T(i,1)/2] && (4)\\ T(i+1,2) &= 2T(i,2) && (5)\\ T(n,1) &= 1 && (6) \end{aligned}$$

,where $[x]$ is the integer part of number x.

Given the Russian multiplication process ends with $T(n,1) = 1$, we are implied that an integer A should be positive; otherwise, it may end with the value -1 or 0. However, the proposition we need to justify does not restrict A always be positive. Therefore, we make a little modification to the end signal of the process:

$$T(n,1) = 0, \text{or} \pm 1 \tag{new 6}$$

as a replacement of formula (6).

It is a general situation where an integer A can be positive, negative, or 0, which matches better the proposition. We need to prove the multiplication of A and B can be expressed by:

$$AB = \sum_{i \in S} M(i)T(i,2), \text{ where } S = \{i | M(i) = \pm 1,\ 1 \leq i \leq n\} \tag{7}$$

,where

$$M(i) = T(i,1) \bmod 2 = \begin{cases} 0 & \text{if } T(i,1) \text{ is even} \\ 1 & \text{if } T(i,1) \text{ is positive odd} \\ -1 & \text{if } T(i,1) \text{ is negative odd} \end{cases} \tag{8}$$

Now, we give the justification:

Step1: we constrain the a_i in formula (1) with $a_i \in \{-1, 0, 1\}$ where $0 \leq i \leq k$.

An integer set ▯ $=$ ▯$^{\geq ▯}$ $\cup$ ▯$^{-}$

For any integers from set ▯$^{\geq ▯}$:

- each number can be transformed to a binary number,
- every bit of the corresponding binary number is only either 0 or 1,
- each bit of the binary number corresponds to a_i in the formula (1), where the right most bit equals to a_0 and the left most, a_k.

For any integers from set $\mathbb{Z}^-$, the situation is quite similar except:

- we use the binary number of the **absolute value** of the negative integer, and every bit of the binary number corresponds to $-a_i$ in the formula (1).

Based on the definition of M by fomula (8), it is **clear** that,

$$a_i = M(i+1) \tag{9}$$

For example,

The binary number of `23` is `10111`, which can be expressed by:

$$23 = 1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 = \sum_{i=0}^{4} a_i 2^i$$

where, $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 0,\ a_4 = 1$

while number `-23` can also be expressed by the expression with $a_0 = -1,\ a_1 = -1,\ a_2 = -1,\ a_3 = 0,\ a_4 = -1$

`0` can be expressed as 0×2^0.

Step2: we determine the size n of the table T:

Based on formula (2) and (4), we get,

$$(A-1)/2 \leq T(2,1) \leq A/2$$

and

$$(T[i,1]-1)/2 \leq T(i+1,1) \leq T(i,1)/2$$

therefore,

$$A \geq 2\,T(2,1) \geq 2^2\,T(3,1) \geq \cdots \geq 2^{n-1}\,T(n,1)$$
$$A \leq 1 + 2\,T(2,1) \leq 1 + 2(1 + 2\,T(3,1)) = (2^2-1) + 2^2\,T(3,1) \leq \cdots \leq (2^{n-1}-1) + 2^{n-1}\,T(n,1)$$

Based on formula (2), we come to,

$$2^{n-1} \leq A \leq 2^n - 1$$
$$log_2(A+1) \leq n \leq 1 + log_2 A$$

Therefore,

$$n = 1 + \lfloor log_2 A \rfloor \tag{10}$$

As row index of table T starts at 1 and ends with n, we can build the relationship between n in formula (10) and k occured in formula (1):

$$n = k + 1 \tag{11}$$

Step3: For two integers A and B, their multiplication,

$$\begin{aligned} AB &= \sum_{i=0}^{k} a_i 2^i \times B \\ &= (a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k) \times B \\ &= a_0 \times 2^0 B + a_1 \times 2^1 B + \cdots + a_k \times 2^k B \\ &= a_0 T(1,2) + a_1 T(2,2) + \cdots + a_k T(k+1,2) \end{aligned}$$

Based on formula (8) and (11), we arrive at:

$$\begin{aligned} AB &= M(1)T(1,2) + M(2)T(2,2) + \cdots + M(n)T(n,2) \\ &= \sum_{i \in S} M(i)T(i,2),\ \text{where } S = \{i \mid M(i) = \pm 1,\ 1 \leq i \leq n\} \end{aligned}$$

End of the justification

Appendix1: an example explaining the notation and process of Russian multiplication

Let $A = 23$ and $B = 19$, the binary number of 23 is `10111`. Therefore, $k = 4, n = 5$, and $S = \{1, 2, 3, 5\}$.

The table of the process of Russian multiplication will be as follow:

i	a_{i-1}	T(i,1)	T(i,2)	M(i)
1	1	23	19	1
2	1	11	38	1
3	1	5	76	1
4	0	2	152	0
5	1	1	304	1

And,

$$AB = 23 \times 19 = 19 + 38 + 76 + 304 = 437$$

In [1]:
```python
import math
A, B = 19, 23
```

In [2]:
```python
bin(A)
```

Out[2]: '0b10011'

In [3]:
```python
n = 1 + math.floor(math.log2(A))
print(n)
```

5

In [4]:
```python
19 + 38 + 76 + 304 == 23 * 19
```

Out[4]: True

Appendix2: Another justification for step3 without using $A = \sum_{i=0}^{k} a_i 2^i$

Based on formula 2,3,4,5,6, and 8, we can infer,

$$\begin{aligned} T(i,1)T(i,2) &= T(i,1)/2 \times 2T(i,2) \\ &= \big([T(i,1)/2] \times 2 + M(i)\big)/2 \times 2T(i,2) \\ &= \big(T(i+1,1) + M(i)/2\big) \times T(i+1,2) \\ &= T(i+1,1)T(i+1,2) + M(i)/2 \times T(i+1,2) \\ &= T(i+1,1)T(i+1,2) + M(i)T(i,2) \end{aligned}$$

Therefore,

$$\begin{aligned} A \times B &= T(1,1)T(1,2) \\ &= T(2,1)T(2,2) + M(1)T(1,2) \\ &= T(3,1)T(3,2) + M(2)T(2,2) + M(1)T(1,2) \\ &\dots \\ &= T(n,1)T(n,2) + M(n-1)T(n-1,2) + M(n-2)T(n-2,2) + \dots + M(1)T(1,2) \\ &= M(n)T(n,2) + M(n-1)T(n-1,2) + \dots + M(1)T(1,2) \\ &= \Sigma_{i \in S} M(i)T(i,2), \text{ where } S = \{i | M(i) = \pm 1,\ 1 \le i \le n\} \end{aligned}$$

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