Devoir 11 IFT2125-A-H19

Student: Qiang Ye (20139927)

Date: 5 Avril 2019

mailto: samuel.ducharme@umontreal.ca (mailto:samuel.ducharme@umontreal.ca)

Question

1. Soit G le graphe suivant donné par sa matrice d'adjacences pondérées.

$$\begin{bmatrix} 0 & \infty & 1 & 6 & 4 \\ 1 & 0 & \infty & \infty & 5 \\ \infty & \infty & 0 & 1 & 5 \\ \infty & 5 & \infty & 0 & 1 \\ \infty & 2 & \infty & 3 & 0 \end{bmatrix}$$

   Utilisez l'algorithme de Floyd pour remplir les tableaux en mettant dans la case (i, j) une paire de valeur (d, k), où d est la distance de i à j et k est le dernier sommet qui fait diminuer cette distance quand l'algorithme permet de passer par lui. Montrez toutes les itérations - c'est pourquoi vous avez cinq tableaux.

Answer

Itération1

$$\begin{bmatrix} [0,0] & [\infty,0] & [1,0] & [6,0] & [4,0] \\ [1,0] & [0,0] & [2,1] & [7,1] & [5,0] \\ [\infty,0] & [\infty,0] & [0,0] & [1,0] & [5,0] \\ [\infty,0] & [5,0] & [\infty,0] & [0,0] & [1,0] \\ [\infty,0] & [2,0] & [\infty,0] & [3,0] & [0,0] \end{bmatrix}$$

Itération2

$$\begin{bmatrix} [0,0] & [\infty,0] & [1,0] & [6,0] & [4,0] \\ [1,0] & [0,0] & [2,1] & [7,1] & [5,0] \\ [\infty,0] & [\infty,0] & [0,0] & [1,0] & [5,0] \\ [6,2] & [5,0] & [7,2] & [0,0] & [1,0] \\ [3,2] & [2,0] & [4,2] & [3,0] & [0,0] \end{bmatrix}$$

Itération3

$$\begin{bmatrix} [0,0] & [\infty,0] & [1,0] & [2,3] & [4,0] \\ [1,0] & [0,0] & [2,1] & [3,3] & [5,0] \\ [\infty,0] & [\infty,0] & [0,0] & [1,0] & [5,0] \\ [6,2] & [5,0] & [7,2] & [0,0] & [1,0] \\ [3,2] & [2,0] & [4,2] & [3,0] & [0,0] \end{bmatrix}$$

Itération4

$$\begin{bmatrix} [0,0] & [7,4] & [1,0] & [2,3] & [3,4] \\ [1,0] & [0,0] & [2,1] & [3,3] & [4,4] \\ [7,4] & [6,4] & [0,0] & [1,0] & [2,4] \\ [6,2] & [5,0] & [7,2] & [0,0] & [1,0] \\ [3,2] & [2,0] & [4,2] & [3,0] & [0,0] \end{bmatrix}$$

Itération5

$$\begin{bmatrix} [0,0] & [5,5] & [1,0] & [2,3] & [3,4] \\ [1,0] & [0,0] & [2,1] & [3,3] & [4,4] \\ [5,5] & [4,5] & [0,0] & [1,0] & [2,4] \\ [4,5] & [3,5] & [5,5] & [0,0] & [1,0] \\ [3,2] & [2,0] & [4,2] & [3,0] & [0,0] \end{bmatrix}$$

2. Quel est le chemin le plus court du sommet 4 au sommet 3?

   Answer

   According to the final matrix of previous question, we look at the element in 4_{th} row 3_{rd} column, which is the value paire of $(5, 5)$, indicating the shortest distance is 5, and the path goes through the 5_{th} node.

   Again in this matrix, we search the element in 4_{th} row 5_{th} column and the element in 5_{th} row 3_{rd} column, which are $(1, 0)$ and $(4, 2)$, respectively. k in first value pair is 0, meaning that the shortest path between them is the edge that connect them directly. k in second value pair is 2, meaning that 2_{nd} node is a node within the shortest path, we repeat the search for shortest sub-path again and again until all value paire has the $k = 0$.

   Finally, we got shortest path from node4 to node5:
   $$4 \rightarrow 5 \rightarrow 2 \rightarrow 1 \rightarrow 3$$
   ,and the shortest distance is 5.

3. Remplisser le tableau suivant afin que l'on connaisse l'ordre de multiplications des matrice qui minimuse le nombre de multiplication. Dans la case (i, j) mettez le nombre miniumum de multiplications $m_{i,j}$ nécessaire pour caculer le produit des matrices $M_i M_{i+1} \cdots M_j$ ainsi que l'endroit où on coupe pour obtenir ce minimum:$(m_{i,j}, k)$.

   Les dimensions de matrices sont $[3, 14, 2, 17, 6, 2]$

   Answer

$$\begin{bmatrix} 0,0 & 84,0 & 186,2 & 324,2 & 324,2 \\ & 0,0 & 476,0 & 372,2 & 284,1 \\ & & 0,0 & 204,0 & 228,4 \\ & & & 0,0 & 204,0 \\ & & & & 0,0 \end{bmatrix}$$

   Note:$k = 0$ means there is no division for the calculation.

   Here is how I got the result:

   Assume: the result matrix is m; the dimension list is d; the indices for row and column start from 1, and the index for dimensions starts from 0.

   From the length of dimensions, we know it's the multiplication of 5 matrices: $M_1 M_2 M_3 M_4 M_5$.

   step1: for $1 \leq i \leq 5$, set $m[i, i] = 0$

   step2: set $s = 1$, let $m[i, i + s] = d[i - 1] \times d[i] \times d[i + 1]$, we got:
$$\begin{aligned} m[1,2] &= d[0] \times d[1] \times d[2] = 3 \times 14 \times 2 &&= 84 \\ m[2,3] &= d[1] \times d[2] \times d[3] = 14 \times 2 \times 17 &&= 476 \\ m[3,4] &= d[2] \times d[3] \times d[4] = 2 \times 17 \times 6 &&= 204 \\ m[4,5] &= d[3] \times d[4] \times d[5] = 17 \times 6 \times 2 &&= 204 \end{aligned}$$

   step3: for $s = 2, 3, 4$, let
$$m[i, i + s] = \min_{i \leq k < i+s} (m[i, k] + m[k + 1, i + s] + d[i - 1] \times d[k] \times d[i + s])$$

   Therefore,

$$
\begin{aligned}
m[1,3] &= \min(m[1,1] + m[2,3] + d[0]d[1]d[3] = 0 + 476 + 714 = 1190\ (k=1) \\
&\qquad m[1,2] + m[3,3] + d[0]d[2]d[3] = 84 + 0 + 102 = 186\ (k=2)) \\
&= 186\ (k=2) \\
m[2,4] &= \min(m[2,2] + m[3,4] + d[1]d[2]d[4] = 0 + 204 + 168 = 372\ (k=2) \\
&\qquad m[2,3] + m[4,4] + d[1]d[3]d[4] = 476 + \cdots > 372\ (k=3)) \\
&= 372\ (k=2) \\
m[3,5] &= \min(m[3,3] + m[4,5] + d[2]d[3]d[5] = 0 + 204 + 68 = 272\ (k=3) \\
&\qquad m[3,4] + m[5,5] + d[2]d[4]d[5] = 204 + 0 + 24 = 228\ (k=4)) \\
&= 228\ (k=4) \\
m[1,4] &= \min(m[1,1] + m[2,4] + d[0]d[1]d[4] = 0 + 372 + 252 = 624\ (k=1) \\
&\qquad m[1,2] + m[3,4] + d[0]d[2]d[4] = 84 + 204 + 36 = 324\ (k=2) \\
&\qquad m[1,3] + m[4,4] + d[0]d[3]d[4] = 186 + 0 + 306 = 492\ (k=3)) \\
&= 324\ (k=2) \\
m[2,5] &= \min(m[2,2] + m[3,5] + d[1]d[2]d[5] = 0 + 228 + 56 = 284\ (k=1) \\
&\qquad m[2,3] + m[4,5] + d[1]d[3]d[5] = 476 + \cdots > 284\ (k=2) \\
&\qquad m[2,4] + m[5,5] + d[1]d[4]d[5] = 372 + \cdots > 284\ (k=3)) \\
&= 284\ (k=1) \\
m[1,5] &= \min(m[1,1] + m[2,5] + d[0]d[1]d[5] = 0 + 284 + 84 = 368\ (k=1) \\
&\qquad m[1,2] + m[3,5] + d[0]d[2]d[5] = 84 + 228 + 12 = 324\ (k=2) \\
&\qquad m[1,3] + m[4,5] + d[0]d[3]d[5] = 186 + 204 + 102 = 492\ (k=3) \\
&\qquad m[1,4] + m[5,5] + d[0]d[4]d[5] = 324 + 0 + 36 = 360\ (k=4)) \\
&= 324\ (k=2)
\end{aligned}
$$

4. Quel est l'ordre optimal des multiplications et quel est le nombre de multiplications minimum?

   Answer From result of previous question, the following division of the matrix multiplication is optimal:

$$
\begin{aligned}
M_1M_2M_3M_4M_5 &= (M_1M_2)(M_3M_4M_5) && (k = 2 \text{ for } m[1,5]) \\
&= (M_1M_2)(M_3M_4)(M_5) && (k = 4 \text{ for } m[3,5])
\end{aligned}
$$

   And, the minimal number of multiplication is $m[1,5] = 324$

In [1]:

```python
# codes for question 1-2
import numpy as np

def print_d_p(D, p):
    n = D.shape[0]
    for i in range(n):
        for j in range(n):
            d = "\infty" if D[i,j] == float("inf") else int(D[i,j])
            path = int(p[i,j])
            print("[{}, {}]".format(d, path),end = "& ")
        print("\\\\")

def Floyd(L):
    #D = copy.copy(L)
    D = np.array(L)
    p = np.zeros_like(D) # chemin
    n = D.shape[0]  # number of nodes
    for k in range(n):
        for i in range(n):
            for j in range(n):
                if D[i,k]+D[k,j]< D[i,j]:
                    p[i,j] = k+1
                D[i,j] = min(D[i,j], D[i,k] + D[k,j])

        print("iter{}".format(k+1))
        print_d_p(D, p)
    return D, p
```

In [2]:

```python
# matrix of the exercise
L = np.array([
     [0, float("inf"), 1, 6, 4],
     [1, 0, float("inf"), float("inf"), 5],
     [float("inf"), float("inf"), 0, 1, 5],
     [float("inf"), 5, float("inf"), 0, 1],
     [float("inf"), 2, float("inf"), 3, 0]
    ])

# matrix from the book
L2 = np.array([
    [0, 5, float('inf'), float('inf')],
    [50, 0, 15, 5],
    [30, float('inf'), 0, 15],
    [15, float('inf'), 5, 0]
])
```

In [3]:
```
d, p = Floyd(L)
```

```
iter1
[0, 0]& [\infty, 0]& [1, 0]& [6, 0]& [4, 0]& \\
[1, 0]& [0, 0]& [2, 1]& [7, 1]& [5, 0]& \\
[\infty, 0]& [\infty, 0]& [0, 0]& [1, 0]& [5, 0]& \\
[\infty, 0]& [5, 0]& [\infty, 0]& [0, 0]& [1, 0]& \\
[\infty, 0]& [2, 0]& [\infty, 0]& [3, 0]& [0, 0]& \\
iter2
[0, 0]& [\infty, 0]& [1, 0]& [6, 0]& [4, 0]& \\
[1, 0]& [0, 0]& [2, 1]& [7, 1]& [5, 0]& \\
[\infty, 0]& [\infty, 0]& [0, 0]& [1, 0]& [5, 0]& \\
[6, 2]& [5, 0]& [7, 2]& [0, 0]& [1, 0]& \\
[3, 2]& [2, 0]& [4, 2]& [3, 0]& [0, 0]& \\
iter3
[0, 0]& [\infty, 0]& [1, 0]& [2, 3]& [4, 0]& \\
[1, 0]& [0, 0]& [2, 1]& [3, 3]& [5, 0]& \\
[\infty, 0]& [\infty, 0]& [0, 0]& [1, 0]& [5, 0]& \\
[6, 2]& [5, 0]& [7, 2]& [0, 0]& [1, 0]& \\
[3, 2]& [2, 0]& [4, 2]& [3, 0]& [0, 0]& \\
iter4
[0, 0]& [7, 4]& [1, 0]& [2, 3]& [3, 4]& \\
[1, 0]& [0, 0]& [2, 1]& [3, 3]& [4, 4]& \\
[7, 4]& [6, 4]& [0, 0]& [1, 0]& [2, 4]& \\
[6, 2]& [5, 0]& [7, 2]& [0, 0]& [1, 0]& \\
[3, 2]& [2, 0]& [4, 2]& [3, 0]& [0, 0]& \\
iter5
[0, 0]& [5, 5]& [1, 0]& [2, 3]& [3, 4]& \\
[1, 0]& [0, 0]& [2, 1]& [3, 3]& [4, 4]& \\
[5, 5]& [4, 5]& [0, 0]& [1, 0]& [2, 4]& \\
[4, 5]& [3, 5]& [5, 5]& [0, 0]& [1, 0]& \\
[3, 2]& [2, 0]& [4, 2]& [3, 0]& [0, 0]& \\
```

In [4]:
```
# codes for question 3-4
d = [3, 14, 2, 17, 6, 2] # dimensions
n = len(d) - 1 # number of matrices

m = np.zeros((n,n))
for i in range(n):
    m[i,i] = 0
    s = 1
    if i+s< n:
        m[i,i+s] = d[i]*d[i+1]*d[i+2]

print(m)
```

```
[[  0.  84.   0.   0.   0.]
 [  0.   0. 476.   0.   0.]
 [  0.   0.   0. 204.   0.]
 [  0.   0.   0.   0. 204.]
 [  0.   0.   0.   0.   0.]]
```

In [5]:

```
for s in range(2, n):
    for i in range(n):
        if i + s < n:
            temps = []
            for k in range(i, i+s):
                cals = m[i,k] + m[k+1, i+s] + d[i]*d[k+1]*d[i+s+1]
                temps.append(cals)
            m[i,i+s] = min(temps)
    print(m)
    print()
```

```
[[  0.  84. 186.   0.   0.]
 [  0.   0. 476. 372.   0.]
 [  0.   0.   0. 204. 228.]
 [  0.   0.   0.   0. 204.]
 [  0.   0.   0.   0.   0.]]

[[  0.  84. 186. 324.   0.]
 [  0.   0. 476. 372. 284.]
 [  0.   0.   0. 204. 228.]
 [  0.   0.   0.   0. 204.]
 [  0.   0.   0.   0.   0.]]

[[  0.  84. 186. 324. 324.]
 [  0.   0. 476. 372. 284.]
 [  0.   0.   0. 204. 228.]
 [  0.   0.   0.   0. 204.]
 [  0.   0.   0.   0.   0.]]
```