Stanford University Mathematics Camp

SUMaC 2026 Admission Exam

For use by SUMaC 2026 applicants only. Not for distribution.

- Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to the questions on the application form are all used to evaluate your SUMaC application. Although SUMaC is very selective with a competitive applicant pool, correct answers on every problem are not required for admission.
- There is no time limit for this exam other than the application deadline.
- *Be sure to include clear, detailed explanations for all of your solutions, and describe the key ideas that led you to a solution. Numerical answers or formulas with no explanation are not useful for evaluating your application. We are interested in learning your understanding and insights on each problem, and in addition to correctness, solutions will be evaluated on your demonstration of sound logical reasoning and your ability to clearly communicate mathematical ideas.*
- In the event you are unable to solve a problem completely, you are encouraged to write up any partial progress that you feel captures your work toward a solution.
- You will need to create a separate document with your solutions and explanations. This document may be typed or handwritten, as long as the final document you upload is legible for our review.
- None of these problems require a calculator or computer, and they are all designed so that they can be done without computational tools.
- You are expected to do your own work without the use of any outside source (books, teacher or parent help, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this in your write up.
- If you do not clearly understand what the problem is asking, indicate this in your solution explanation, and solve the problem according to your best interpretation of what is being asked.
- **Do not share these problems or your solutions with anyone.**

1. Solve for x:
$$x^2 - 2026 = \sqrt{x + 2026}.$$

   Please take note of the following extracted from the instructions above, here as well as with the rest of the exam: Be sure to include clear, detailed explanations for all of your solutions, and describe the key ideas that led you to a solution. Numerical answers or formulas with no explanation are not useful for evaluating your application. In the case of this problem, do not use quartic formula, and do not simply “guess and check” to find your solution.

2. You are locked in a room and have to solve a puzzle to get out. Here is the description: there is a small cylindrical symmetric table with four large identical cupholders, each wide enough to fit your hand inside. Each cupholder contains a coin which you can easily feel with your hands, and you can detect whether the coin is *heads up* or *heads down*. This puzzle is happening in a dark room where you cannot see inside the cups, but only feel with your hands. If all four coins have the same orientation at any point – all flipped heads up or all flipped heads down - then you have solved the puzzle and you may leave the room. The rules of the puzzle are as follows:

   A move in solving the puzzle consists of putting each of your two hands simultaneously into two of the cupholders (either adjacent to each other, or along a diagonal), checking the orientation of the coins in the cupholders, and choosing to flip either 0, 1, or 2 of the coins in their respective cupholders, and immediately removing both hands. After removing your hands, the table starts spinning around its axis. It spins such that the coins cannot reverse orientation by themselves – they can only be reversed during one of your moves. When it stops, you cannot know what coins were accessed in the previous move because it’s too dark to see and you don’t know how many rotations, including fractional rotations, the table made.

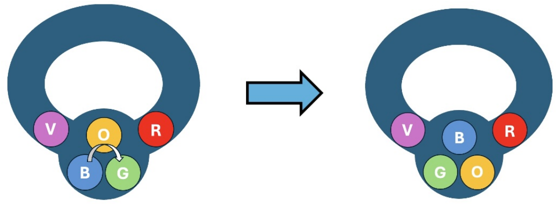
   (a) What is the minimum number of moves to guarantee your escape?

   (b) Let’s replace the number of cupholders and hands to eight large identical cupholders set up analogously, and with a friend, you are allowed to each simultaneously put two hands into a total of four cups. You are allowed to communicate the orientations of the coins in your cupholders, and change 0, 1, 2, 3, or 4 of the coins’ orientations. What is the minimum number of moves to guarantee your escape in this new situation?

3. Suppose you have a table with five balls colored violet, orange, red, blue, and green, arranged on a table as shown in this diagram:

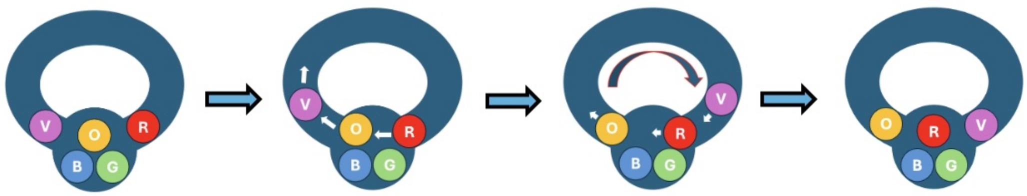
Starting Position

   The red and violet balls are on a track just wide enough for one ball, and the track leads to a circular section that fits exactly three balls at a time (orange, blue, and green shown above). You can move the balls around the track, and within the circular section. When there are three balls in the circular section, you can rotate them as follows.



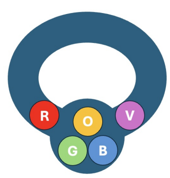
Rotating around the circular section

Also, you may move the ball on the left of the circle around the track, while moving one ball out of the circle to a position on the left of the circle, and moving the ball on the right of the circle into the circle in place of the ball that was moved out, as shown:



Rotating around the track

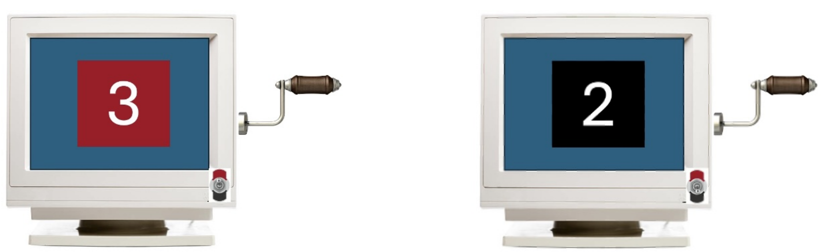
For each configuration below determine whether it is possible to obtain the given configuration from the **Starting Position** (above) using a sequence of the two allowable moves described previously. Be sure to explain your conclusions.



(a)

(b)

4. Suppose p is a prime number such that $p > 2$. A p machine is a very basic deterministic machine consisting of a hand crank, a toggle switch, and a display. The display shows one number from the set $\{1, 2, 3, \ldots, p\}$ inside of a square that is either black or red. For example, here are two possible states of a p-machine:



The p machine operates on a computer program consisting of a single positive integer k where $k \leq p$. In particular, with program k, flipping the toggle switch and turning the crank has a *deterministic* outcome in the sense that starting with any color and number on the display, the same combination of crank turns and flips of the toggle switch, in the same order, will always yield the same results, only depending on k and the starting state of the display. Furthermore, the toggle switch and crank operate according to the following rules and features:

i. Flipping the toggle switch up or down changes the square from red to black, or black to red, leaving the number unchanged.

ii. Turning the crank one full turn does not change the color, but it changes the number as follows.

   - if the square is red and the displayed number is less than p, then the number is increased by one, and if the square is red and the displayed number is p, then the number changes to 1.
   - if the square is black and the number is less than or equal to $p-k$, then the number increases by k, and if the number is greater than $p-k$, then it decreases by $p-k$.

iii. A program k is *successful* for the p machine if whenever i and j are distinct positive integers less than or equal to k, and i is displayed (on either a red or black square) then there is a sequence of crank turns such that j is displayed.

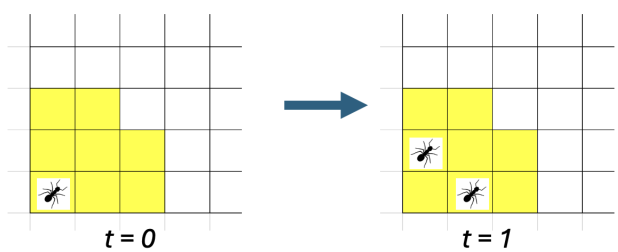
iv. A program k is *special* for the p machine if it has the property that toggling the switch followed by a single turn of the crank has the same outcome as first turning the crank k full turns and then flipping the switch.

Suppose p is a prime and $p > 2$, and suppose k is a positive integer such that $k \leq p$. Please answer the following:

(a) Find necessary and sufficient conditions on k for program k to be successful for the p machine. Be sure to show that your conditions are both necessary and sufficient.

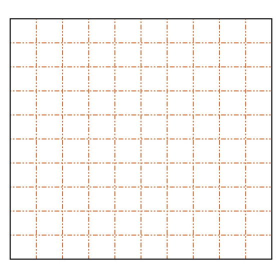
(b) Find necessary and sufficient conditions on k for program k to be both successful and special for the p machine. Be sure to show that your conditions are both necessary and sufficient.

5. At time $t = 0$ an ant is on the lower left corner of a grid that goes infinitely far to the right and infinitely far up. Every second, one ant on the grid can duplicate itself and move one copy of itself up and one copy of itself to the right, *assuming both of the cells on the grid the ant copies move into are vacant.* For example, the following shows going from $t = 0$ to $t = 1$

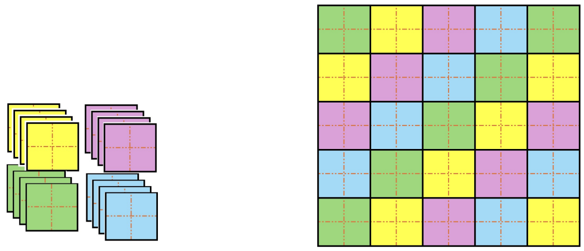


Is there a time t such that the ants have completely vacated the yellow region?

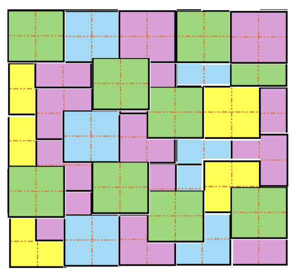
6. Suppose we have a 10×10 grid like this:



and we have a deck of 2×2 cards with squares that match the size of the squares of the grid. Clearly, 25 of these cards can be used to cover the grid as follows



For this problem, we have an unlimited number of cards available to us, and the card colors are not relevant. If we use more cards, and we allow some of the cards to overlap others, there are many ways to cover the entire grid. For example, in the following, the cards overlap.



i. We say that an arrangement of cards is a *topper* if all of the cards in the arrangement are lined up with the squares on the grid, and the cards are completely on the grid, possibly overlapping, and every square of the grid has at least one card on top of it.

ii. We call a topper *silly* if one of the cards can be removed and the arrangement of cards remaining is still a topper.

iii. A topper is *A+* if it is no longer a topper when any one card is removed from the arrangement.

Clearly, the smallest A+ topper has 25 cards. The aim of this problem is to find upper and lower bounds on the number of cards in the *largest possible A+ topper*.

(a) Show that there is an A+ topper with 35 cards.

(b) Show that every topper with 55 cards is silly.

(c) Can you improve these bounds?