Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, An: Primăvara 2020

**Laboratory 4. First Order Nonlinear Differential Equations**

1. **Experiment vs. mathematical theory.** We consider $x' = 1 - x^2$.

a) Find its constant solutions.

b) Find the solution $\phi(t, \eta)$ of this differential equation that satisfies $x(0) = \eta$, where $\eta \in \mathbb{R}$ is fixed. Notice that $\phi(t, 1) = 1$ and $\phi(t, -1) = -1$ but the formula returned by Maple for $\phi(t, \eta)$ is not defined neither for $\eta = 1$ nor for $\eta = -1$.

c) In your *notebooks represent simultaneously* various integral curves of this differential equation.

This means that you have to represent the graph of many solutions, as, for example: $\phi(t, -0.7)$, $\phi(t, -0.5)$, $\phi(t, 0.4)$, $\phi(t, 0.9)$, $\phi(t, 1.1)$, $\phi(t, 1.2)$, $\phi(t, 1.3)$, $\phi(t, -1.3)$, $\phi(t, -1.2)$, $\phi(t, -1.1)$. Of course you can use Maple. When using Maple, for better results, represent them separately and on small intervals, like $(0, 2)$ or $(-2, 0)$. Use Maple to understand better the shape of the graphs on large intervals. When $\eta \in (1, \infty)$ compute the limit of each solution as $t \to \infty$. When $\eta \in (-\infty, -1)$ compute the limit of each solution as $t \to -\infty$.

d) Study the validity of the proposition "Each nonconstant solution is strictly monotone."

e) Finally, in your notebooks represent the phase portrait of $x' = 1 - x^2$ and confirm the properties you found.

2. **It exists, but we can not see it!**

Find the general solution of $x' = 1 - tx^3$. Find the solution of the IVP $x' = 1 - tx^3$, $x(0) = 0$. Set **infolevel[dsolve]** to 3 to see what is going on. So, there is no algorithm available to Maple to find its solution. What if I tell you that nobody (machine or human) can not write down this solution! And for most of the nonlinear differential equations happens this! But, the mathematicians proved that the solution of the IVP $x' = 1 - tx^3$, $x(0) = 0$ exists! So, this function exists but we can not write its expression! Why?! Because we are able to write expressions using a **finite** combination of *elementary or special* functions. The mathematicians proved that this solution can be written as an **infinite** sum of polynomial functions.

Using **dfieldplot** represent the corresponding direction field. Using **DEplot** represent the numerical solution that satisfies, for example $x(0) = 0$. Add other initial conditions to represent more numerical solutions simultaneously.

3. Using **dfieldplot** represent the direction field in a box that contains the origin, for each of the differential equations. We know that these directions are tangent to the integral curves. What kind of curves could be the integral curves of each differential equation?

Find the general solution of each differential equation and see if your intuition is confirmed. Remember that $y = mx$ is the equation of a line, $xy = a$ is the equation of a hyperbola, $x^2 + y^2 = r^2$ is the equation of a circle, while $a^2x^2 + b^2y^2 = 1$ is the equation of an ellipse.

a) $y'(x) = \frac{y(x)}{x}$; b) $y'(x) = -\frac{y(x)}{x}$; c) $y'(x) = \frac{-x}{y(x)}$; d) $y'(x) = \frac{-2x}{y(x)}$.

4. Using **contourplot** find the level curves in a box that contains the origin of the scalar functions of two variables $H_1(x, y) = xy$, $H_2(x, y) = x^2 + y^2$ and, respectively, $H_3(x, y) = 2x^2 + y^2$. Relate these with the previous exercise. Notice that the differential equation found performing the calculations in $\frac{d}{dx}H(x, y(x)) = 0$, is the equation whose integral curves are the level curves of $H$. Find in this way the corresponding differential equation for each function $H_1$, $H_2$, $H_3$ written above.

5. **An inverse problem.** Find the differential equation whose integral curves are the level curves of $H(x, y) = x^2 + 4y^2$.

6. **A first integral of the ideal pendulum equation.**

Represent the level curves of $H(x, y) = y^2 - \cos x$ in a sufficiently small box that contains the origin. Notice that the level curves are closed. We will use this property during the Lecture.