Seminars 2 & 3

– Complete solutions of some problems –

1.4.1. a) $x' + 6x = 0.$

St1: we write the characteristic equation: $r + 6 = 0.$

St2: we find the roots in $\mathbb{C}$: $r = -6$

St3: $x = c \cdot e^{-6t}$

1.4.1. b) $x'' + 4x' + 4x = 0.$

St1: $r^2 + 4r + 4 = 0$

St2: $(r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2$ double root.

St3: The associated functions: e^{-2t}, $t \cdot e^{-2t}$

$\Rightarrow$ solution: $x(t) = c_1 \cdot e^{-2t} + c_2 \cdot t \cdot e^{-2t}$, $c_1, c_2 \in \mathbb{R}$

1.4.1. f) $x'' + x' + x = 0.$

St1: $r^2 + r + 1 = 0$

St2: $\Delta = 1 - 4 = -3 \Rightarrow r_{1,2} = \dfrac{-1 \pm \sqrt{3}\, i}{2} = \underbrace{-\frac{1}{2}}_{=\alpha} \pm \underbrace{\frac{\sqrt{3}}{2}}_{=\beta} i$

St3: The associated functions:

$e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t$; $e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t.$

$\Rightarrow$ the general solution: $x(t) = c_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$

$c_1, c_2 \in \mathbb{R}$

1.4.1. g) $x''' - 6x'' + 11x' - 6x = 0.$

St1: $r^3 - 6r^2 + 11r - 6 = 0.$

$r^3 - r^2 - 5r^2 + 5r + 6r - 6 = 0.$

$r^2(r-1) - 5r(r-1) + 6(r-1) = 0.$

$(r-1)(r^2 - 5r + 6) = 0.$

St2: $\Rightarrow \underline{r_1 = 1}$ $\quad r^2 - 5r + 6 = 0.$

$$\Delta = 25 - 24 = 1 \; ; \; r_{2,3} = \frac{5 \pm 1}{2}$$

$$\Rightarrow \underline{r_2 = 3}, \; \underline{r_3 = 2}$$

St3: The associated functions: e^{t}, e^{2t}, e^{3t}

$\Rightarrow$ general solution: $x(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t}$

$c_1, c_2, c_3 \in \mathbb{R}$

1.4.1.h. $x^{(4)} - x = 0.$

$r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0$

$\Rightarrow r_1 = 1, \; r_2 = -1, \; r_{3,4} = \pm i = \underbrace{0}_{\alpha} \pm \underbrace{1}_{\beta} \cdot i$

The associated functions: $e^{t}, e^{-t}, \sin t, \cos t.$

The general solution: $x(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \sin t + c_4 \cos t$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

1.4.1. i. $x' = kx$

The characteristic equation: $r = k$

The associated function: e^{kt}

The general solution: $x(t) = c \cdot e^{kt}, \; c \in \mathbb{R}.$

1.4.4. a) $\begin{cases} x'' + \pi^2 x = 0. \\ x(0) = 0 \ , x'(0) = \eta \end{cases}$

First we solve the diffential equation: $x'' + \pi^2 x = 0.$

$r^2 + \pi^2 = 0 \Rightarrow r^2 = -\pi^2 \Rightarrow r_{1,2} = \pm \pi \cdot i \Rightarrow \alpha = 0$
$\beta = \pi$

The associated functions: $\cos \pi t$, $\sin \pi t$

The solution of the equation: $x(t) = c_1 \cdot \cos \pi t + c_2 \sin \pi t$
$c_1, c_2 \in \mathbb{R}.$

Now we apply the conditions.

$x(0) = 0 \Rightarrow x(0) = c_1 \cdot \cos 0 + c_2 \underbrace{\sin 0}_{=0} = 0 \Rightarrow \underline{c_1 = 0.}$ $\Big\} \Rightarrow$

$x'(t) = -c_1 \pi \sin \pi t + c_2 \pi \cos \pi t$

$\Rightarrow x'(t) = c_2 \pi \cos \pi t$

But $x'(0) = \eta \Rightarrow x'(0) = c_2 \cdot \pi \underbrace{\cos 0}_{=1} = \eta \Rightarrow c_2 = \frac{\eta}{\pi}$

$\Rightarrow$ The solution of the IVP : $x(t) = \frac{\eta}{\pi} \cdot \sin \pi t$

1.4.5. a) $\begin{cases} x'' + x = 0 \\ x(0) = x(\pi) = 0. \end{cases}$

First we solve the differential equation: $x'' + x = 0.$

$\Rightarrow r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

The associated functions: $\cos t$, $\sin t$

The solution of the equation: $x(t) = c_1 \cos t + c_2 \sin t$

The conditions: $x(0) = 0 \Rightarrow c_1 \cdot \cos 0 + c_2 \sin 0 = 0.$

$x(\pi) = 0 \Rightarrow c_1 \cdot \cos \pi + c_2 \sin \pi = 0.$

$$\Rightarrow \begin{cases} c_1 \cdot 1 + c_2 \cdot 0 = 0 \\ c_1 \cdot (-1) + c_2 \cdot 0 = 0 \end{cases} \Rightarrow c_1 = 0, \quad c_2 = c \in \mathbb{R}.$$

$\Rightarrow$ The solution of the BVP : $x(t) = c \cdot \sin t$, $c \in \mathbb{R}$.

1.5.1 b) $\begin{cases} x'' + 4x = 1 \\ x(0) = \frac{5}{4} \\ x'(0) = 0 \end{cases}$ $\qquad x(\pi) = 5/4$?

First we look for the general solution of the diff eq:

$x'' + 4x = 1$ (LNHDE with CC)

St1: we solve the homogenous equation: $x'' + 4x = 0$.

$r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm 2i$

The associated functions are: $\cos 2t$, $\sin 2t$.

The solution of the homogenous equation:

$x_0 = c_1 \cos 2t + c_2 \sin 2t$

St2: we look for a particular solution of the LNHDE : $x'' + 4x = 1$.

The right hand side of the equation is a constant, follows that we have to look for a constant solution x_p. Notice that $x_p = \frac{1}{4}$.

$$\left(\left(\tfrac{1}{4}\right)'' + 4 \cdot \tfrac{1}{4} = 1 \quad \checkmark \right)$$

St3: The general solution of the given equation:

$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}$

Now we put the conditions:

$$x(0) = \frac{5}{4} \Rightarrow x(0) = c_1 \cos 0 + c_2 \underbrace{\sin 0}_{=0} + \frac{1}{4} = \frac{5}{4}$$

$$\Rightarrow c_1 + \frac{1}{4} = \frac{5}{4} \Rightarrow \underline{c_1 = 1},$$

$$\left.\begin{aligned} &x'(0) = 0 \\ &x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t \end{aligned}\right\} \Rightarrow -2c_1 \underbrace{\sin 0}_{=0} + 2c_2 \cos 0 = 0$$

$$\Downarrow$$

$$c_2 = 0.$$

The unique solution of the IVP : $x(t) = \cos 2t + \frac{1}{4}$

Now we can see that :

$$x(\pi) = \cos 2\pi + \frac{1}{4} = \frac{5}{4}$$

1.5.4. (i) $\alpha > 0$, $\alpha \neq 2$, $x_p = a \cdot e^{\alpha t}$

The equation : $x'' - 4x = e^{\alpha t}$

$$x_p = a \cdot e^{\alpha t} \Rightarrow x_p' = \alpha \cdot a e^{\alpha t} \Rightarrow x_p'' = \alpha^2 \cdot a \cdot e^{\alpha t}$$

We replace x_p'' and x_p in the equation:

$$\alpha^2 \cdot a \cdot e^{\alpha t} - 4 \cdot a e^{\alpha t} = e^{\alpha t} \quad \Big| : e^{\alpha t}$$

$$\alpha^2 \cdot a - 4a = 1 \quad \Rightarrow a(\alpha^2 - 4) = 1 \Rightarrow a = \frac{1}{\alpha^2 - 4}$$

$$\Rightarrow x_p = \frac{1}{\alpha^2 - 4} \cdot e^{\alpha t}.$$

(ii) $x_p = a \cdot t e^{2t}$ solution for $x'' - 4x = e^{2t}$

$$x_p' = a e^{2t} + 2at e^{2t}$$

$$x_p'' = 2ae^{2t} + 2ae^{2t} + 4ate^{2t} = 4ae^{2t} + 4ate^{2t}$$

$$\Rightarrow \underbrace{4ae^{2t} + 4ate^{2t}}_{x_p''} - 4\underbrace{ate^{2t}}_{x_p} = e^{2t}$$

$$\Rightarrow \quad 4ae^{2t} = e^{2t} \quad \Big|: e^{2t}$$

$$4a = 1 \Rightarrow a = \frac{1}{4} \quad \Rightarrow x_p = \frac{1}{4} t e^{2t}$$

(iii) $\alpha > 0$, $\begin{cases} x'' - 4x = e^{\alpha t} \\ x(0) = x'(0) = 0. \end{cases}$

First we solve the equation: $x'' - 4x = e^{\alpha t}$

St1: We solve the homogenous equation: $x'' - 4x = 0$.

$r^2 - 4 = 0 \Rightarrow r_1 = 2, \ r_2 = -2 \quad \Rightarrow x_0 = c_1 e^{2t} + c_2 e^{-2t}$

St2: We look for x_p. - particular solution.

We have here two cases:

- if $\alpha = 2 \Rightarrow x_p = \frac{1}{4} t e^{2t}$ (A)
- if $\alpha \neq 2 \Rightarrow x_p = \frac{1}{\alpha^2 - 4} \cdot e^{\alpha t}$ (B)

Thus we have:

(A) • if $\alpha = 2 \Rightarrow x = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4} t e^{2t}$

Here the conditions: $x(0) = 0 \Rightarrow c_1 + c_2 = 0. \Rightarrow c_1 = -c_2$

$x'(0) = 0$

$x'(t) = 2c_1 e^{2t} - 2c_2 e^{-2t} + \frac{1}{4} e^{2t} + \frac{1}{2} t e^{2t}$

$\Rightarrow x'(0) = 2c_1 - 2c_2 + \frac{1}{4} = 0.$

From $c_1 = -c_2$

$\Rightarrow -2c_2 - 2c_2 = -\frac{1}{4}$

$-4c_2 = -\frac{1}{4}$

$c_2 = \frac{1}{16}, \ c_1 = \frac{-1}{16}$

$$\Rightarrow \boxed{\varphi(t,2) = -\frac{1}{16} e^{2t} + \frac{1}{16} e^{-2t} + \frac{1}{4} t e^{2t}}$$

(B) • if $\alpha \neq 2 \Rightarrow x = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{\alpha^2-4} e^{\alpha t}$

Here the conditions: $x(0) = 0 \Rightarrow c_1 + c_2 + \frac{1}{\alpha^2-4} = 0.$

$$\left.\begin{aligned} &x'(0) = 0 \\ &x'(t) = 2c_1 e^{2t} - 2c_2 e^{-2t} + \frac{\alpha}{\alpha^2-4} e^{\alpha t} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} 2c_1 - 2c_2 + \frac{\alpha}{\alpha^2-4} = 0 \\ c_1 + c_2 + \frac{1}{\alpha^2-4} = 0 \quad \vert \cdot 2 \end{cases} \quad (+)$$

$$4c_1 + \frac{\alpha}{\alpha^2-4} + \frac{2}{\alpha^2-4} = 0.$$

$$4c_1 + \frac{\alpha+2}{(\alpha+2)(\alpha-2)} = 0 \Rightarrow 4c_1 = \frac{1}{2-\alpha} \Rightarrow c_1 = \frac{1}{4(2-\alpha)}$$

$$\Rightarrow c_2 = -\frac{1}{\alpha^2-4} - \frac{1}{4(2-\alpha)} = -\frac{{}^{4/}1}{\alpha^2-4} + \frac{{}^{\alpha+2/}1}{4(\alpha-2)}$$

$$\Rightarrow c_2 = \frac{-4+\alpha+2}{4(\alpha^2-4)} = \frac{\alpha-2}{4(\alpha-2)(\alpha+2)} = \frac{1}{4(\alpha+2)}$$

$$\Rightarrow \boxed{\varphi(t,\alpha) = -\frac{e^{2t}}{4(\alpha-2)} + \frac{e^{-2t}}{4(\alpha+2)} + \frac{1}{\alpha^2-4} e^{\alpha t}}$$

(iv) $\lim_{\alpha \to 2} \varphi(t,\alpha) = \varphi(t,2) \quad , \forall t \in \mathbb{R}.$

$$\lim_{\alpha \to 2} \varphi(t,\alpha) = \lim_{\alpha \to 2} \left[\frac{e^{-2t}}{4(\alpha+2)} + \frac{4e^{\alpha t} - (\alpha+2) e^{2t}}{4\underbrace{(\alpha^2-4)}_{=(\alpha-2)\cdot(\alpha+2)}} \right]$$

$$= \frac{1}{16} e^{-2t} + \frac{1}{16} \cdot \lim_{\alpha \to 2} \frac{4e^{\alpha t} - (\alpha+2) e^{2t}}{\alpha - 2} \overset{\frac{0}{0}}{=}$$

$$= \frac{1}{16} e^{-2t} + \frac{1}{16} \lim_{\alpha \to 2} \frac{4te^{\alpha t} - e^{2t}}{1} =$$

$$= \frac{1}{16} e^{-2t} + \frac{1}{16} \cdot 4t \cdot e^{2t} - \frac{1}{16} e^{2t} =$$

$$= \varphi(t,2).$$

1.3.2. c) $x' + \frac{2t}{1+t^2} x = 3$

We take the linear homogenous equation associated:

$x' + \frac{2t}{1+t^2} \cdot x = 0. \Rightarrow x' = -\frac{2t}{1+t^2} \cdot x$

We look for a non-null solution by separating variables.

$\Rightarrow \frac{dx}{x} = -\frac{2t}{1+t^2} dt$

We integrate and obtain:

$\ln|x| = -\int \frac{2t}{1+t^2} dt \rightarrow$

$u = 1+t^2$
$du = 2t\,dt$
$\int \frac{2t}{1+t^2} dt = \int \frac{du}{u} = \ln|u|$

$\ln|x| = -\ln|1+t^2| + \ln c$

$x_h = \frac{c}{1+t^2}, \quad c \in \mathbb{R}$

Now we apply the Lagrange method to find a particular solution: $x_p = \frac{\varphi(t)}{1+t^2}$

$\Rightarrow x_p' = \frac{\varphi'(t) \cdot (1+t^2) - \varphi(t) \cdot 2t}{(1+t^2)^2} = \frac{\varphi'(t)}{1+t^2} - \varphi(t) \cdot \frac{2t}{(1+t^2)^2}$

We replace in the equation: $x_p' + \frac{2t}{1+t^2} \cdot x_p = 3$

$$\Rightarrow \frac{\varphi'(t)}{1+t^2} \underbrace{- \varphi(t) \cdot \frac{2t}{(1+t^2)^2} + \frac{2t}{1+t^2} \cdot \frac{\varphi(t)}{1+t^2}}_{\text{cancel out}} = 3$$

$\frac{\varphi'(t)}{1+t^2} = 3 \Rightarrow \varphi'(t) = 3(1+t^2) \Rightarrow \varphi(t) = \int 3(1+t^2)\, dt$

$\Rightarrow \varphi(t) = 3t + t^3$

$$\Rightarrow x_p = \frac{3t+t^3}{1+t^2} = \text{particular solution}$$

Thus, the general solution is: $x = x_h + x_p$

$$\Rightarrow x = \frac{c}{1+t^2} + \frac{3t+t^3}{1+t^2}, \quad c \in \mathbb{R}$$

1.3.2. d) $\quad x' - \frac{2}{t}\cdot x = t^2 \sin(2t) - 4t^3, \; t \in (0, \infty)$

First we solve the linear homogenous equation associated here: $x' - \frac{2}{t}x = 0 \Rightarrow x' = \frac{2}{t}\cdot x$.

We look for non-null solution. We separate the variables

$$\Rightarrow \frac{dx}{x} = \frac{2}{t}\, dt$$

We integrate $\Rightarrow \int \frac{dx}{x} = 2\int \frac{1}{t}\, dt$

$$\ln|x| = 2\ln|t| + \ln c$$

$$\ln|x| = \ln(t^2 \cdot c)$$

$$x = c \cdot t^2$$

$$\Rightarrow x_h = c\cdot t^2 = \text{general solution of the homogenous eq.}$$

Now we apply the Lagrange method to find a particular solution of the given equation: $x' - \frac{2}{t}x = t^2 \sin(2t) - 4t^3$

Here x_p has the formula: $x_p = \varphi(t)\cdot t^2$

$$\Rightarrow x_p' = \varphi'(t)\cdot t^2 + \varphi(t)\cdot 2t$$

We replace in the nonhomogenous equation:

$$\underbrace{\varphi'(t)\cdot t^2 + 2t\cdot\varphi(t)}_{x_p'} - \frac{2}{t}\cdot\underbrace{\varphi(t)\cdot t^2}_{x_p} = t^2 \sin(2t) - 4t^3$$

$$\varphi'(t)\cdot t^2 + \cancel{2t\cdot\varphi(t)} - \cancel{2t\cdot\varphi(t)} = t^2 \sin(2t) - 4t^3$$

$$\varphi'(t) = \sin(2t) - 4t$$

$$\varphi(t) = \int(\sin 2t)\, dt - \int 4t\, dt$$

$$\varphi(t) = -\frac{\cos 2t}{2} - 2t^2$$

We replace $\varphi(t)$ in the particular solution:

$$x_p = -\frac{t^2 \cdot \cos 2t}{2} - 2t^4$$

We deduce now that the general solution is:

$$x = C \cdot t^2 - \frac{t^2 \cdot \cos 2t}{2} - 2t^4, \quad C \in \mathbb{R}.$$

1.3.5. a) $x''' - x'' = 0.$

We denote $x'' = z \Rightarrow x''' = z'$ and replace in eq:

$\Rightarrow z' - z = 0$ (linear homogenous equation)

$z' = z$

We find a non-null solution by separating the variables: $\frac{dz}{z} = dt$

We integrate: $\ln|z| = t + \ln C$

$\ln|z| = \ln(e^t \cdot C)$

$z = C \cdot e^t$

Follows that: $x'' = C \cdot e^t \quad \Big| \int dt$

We integrate here two times: $x' = C_1 e^t + C_2$

$x = C_1 e^t + C_2 t + C_3$

– the general solution. $(C_1, C_2, C_3 \in \mathbb{R})$.

1.3.5. b) $x'' = \frac{2}{t} x'$

We denote here: $x' = z \Rightarrow x'' = z'$

We replace in the equation: $z' = \frac{2}{t} \cdot z$ (linear hom. eq.)

We separate the variables: $\frac{dz}{z} = \frac{2}{t} \cdot dt$

We integrate: $\ln|z| = 2\ln|t| + \ln C \Rightarrow z = C \cdot t^2$

We obtain: $x' = C \cdot t^2$

We integrate: $x = c_1 \frac{t^3}{3} + c_2$, $c_1, c_2 \in \mathbb{R}$