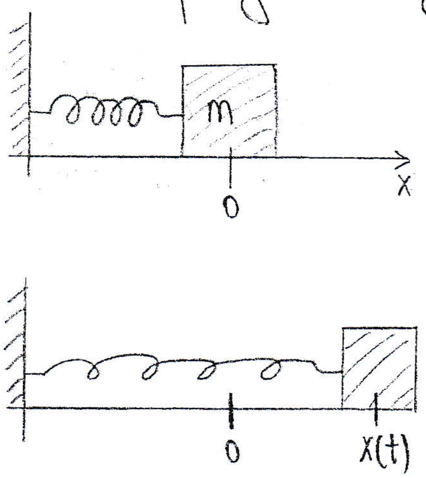
The spring-mass system



no motion → equilibrium state (or stationary state)

Second Newton's law: $F = ma$

$x(t)$ the displacement from the equilibrium pos.

$x'(t)$ the instant velocity

$x''(t)$ the acceleration

$F_r \approx k \cdot x(t)$ $\quad k>0$ the restoring force

$F_f \approx \gamma \cdot x(t)$ $\quad \gamma>0$ the friction (damping) force

$F_e = f(t)$ the external force

$F = -F_r - F_f + F_e = -kx - \gamma x' + f(t)$

$ma = F \iff mx'' = -kx - \gamma x' + f(t)$

$$\Rightarrow \boxed{x'' + \frac{\gamma}{m} \cdot x' + \frac{k}{m} \cdot x = f(t)} \quad \text{LNDE with cc}$$

Case I: No damping, no external force

$x'' + \frac{k}{m} \cdot x = 0$ $\quad W_0 = \sqrt{\frac{k}{m}}$ the internal frequancy of the spring

$X = C_1\cos(w_0 t) + C_2\sin(w_0 t)$ periodic, the main period $\frac{2\pi}{w_0}$

Case II: With damping, without F_e

$$x'' + \frac{\nu}{m}x' + \frac{k}{m}x = 0$$

$$\Delta = \frac{\nu^2}{m^2} - 4\cdot\frac{k}{m} = \frac{\nu^2 - 4km}{m^2}$$

II 1. $\underbrace{\nu > \sqrt{4km}}_{\text{overdamping}}$ $(\Leftrightarrow \Delta > 0)$ no oscillation

$$\left.\begin{array}{l} r_1 \cdot r_2 = \frac{k}{m} > 0 \\ r_1 + r_2 = -\frac{\nu}{m} \end{array}\right\} \Rightarrow \begin{cases} r_1 < 0 \\ r_2 < 0 \end{cases} \Rightarrow$$

$\Rightarrow$ any sol. goes to 0 as $t \to \infty$

II 2. $\nu = \sqrt{4km}$ $(\Leftrightarrow \Delta = 0)$
critically damped

II 3. $\nu < \sqrt{4km}$ (underdamping)

$\Delta < 0$, $r_{1,2} = \alpha \pm i\beta$ there are oscillations

$r_1 + r_2 = 2\alpha = -\frac{\nu}{m} \Rightarrow \alpha = -\frac{\nu}{2m} < 0 \Rightarrow$ any sol. goes to 0 as $t \to \infty$

Case III. No damping, with external force $F = A\cos(wt)$ where $A, w > 0$

$$x'' + w_0^2 x = A\cos(wt)$$

We assume that $\boxed{w = w_0}$ Resonance

$$x_p = \frac{1}{2w_0}t\sin(wt)$$

$$X = C_1\cos(w_0 t) + C_2\sin(w_0 t) + \frac{1}{2w_0}t\sin(wt)$$

Any sol. oscillates with unbounded amplitude