15. $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \end{cases}$

a) Find the equilibria and study their stability.

b) Find $\varphi(t, 0, \frac{2}{3})$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, \frac{2}{3})$

c) Represent in the phase plane the orbits corresponding to the initial values $(0, \frac{2}{3})$, $(4, 0)$ and $(1, \frac{2}{3})$.

a) $-x + xy = 0$

$-2y + 3y^2 = 0 \quad \Rightarrow \quad y(-2+3y) = 0 \quad < \begin{matrix} y_1 = 0 \\ y_2 = \frac{2}{3} \end{matrix}$

$-x + 0 = 0 \quad \Rightarrow \quad x_1 = 0$

$-x + x \cdot \frac{2}{3} = 0 \quad \Rightarrow \quad -\frac{x}{3} = 0 \quad \Rightarrow \quad x_2 = 0$

$\Rightarrow A_1(0,0),\ A_2(0, \frac{2}{3})$ equilibrium points

b) • By definition $\varphi(t, 0, \frac{2}{3})$ is the unique solution of the IVP

IVP: $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 0 \\ y(0) = \frac{2}{3} \end{cases}$

Since $(0, \frac{2}{3})$ is an equilibrium point, we have that $\varphi(t, 0, \frac{2}{3}) = (0, \frac{2}{3})$, $\forall t \in \mathbb{R}$

• By def., $\varphi(t, 4, 0)$ is the unique sol. of the ivp

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 4 \\ y(0) = 0 \end{cases}$$

First we solve the ivp $\begin{cases} \dot{y} = -2y + 3y^2 \\ y(0) = 0 \end{cases} \Rightarrow y = 0$

ivp: $\begin{cases} \dot{x} = -x \\ x(0) = 4 \end{cases} \Rightarrow \dot{x} + x = 0$

$$r + 1 = 0$$
$$r = -1$$
$$\left.\begin{array}{l} x = e^{-t} \cdot c \quad c \in \mathbb{R} \\ x(0) = 4 = c \end{array}\right\} \Rightarrow x = 4e^{-t}$$

$\varphi(t, 4, 0) = \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}$, $\forall t \in \mathbb{R}$

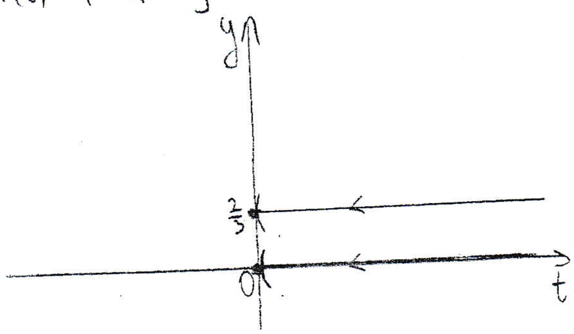
• $\varphi(t, 1, \frac{2}{3})$ is the unique sol. of the ivp $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 1 \\ y(0) = \frac{2}{3} \end{cases}$

We solve the ivp: $\begin{cases} \dot{y} = -2y + 3y^2 \\ y(0) = \frac{2}{3} \end{cases} \Rightarrow y = \frac{2}{3}$ unique sol.

$\begin{cases} \dot{x} = -x + x \cdot \frac{2}{3} \\ x(0) = 1 \end{cases} \Rightarrow \dot{x} = -\frac{x}{3} \mapsto r + \frac{1}{3} = 0 \Rightarrow r = -\frac{1}{3}$

$\left.\begin{array}{l} x = e^{-\frac{1}{3}t} \cdot c \\ x(0) = 1 = 1 \end{array}\right\} \Rightarrow x = e^{-\frac{1}{3}t} \Rightarrow \varphi(t, 1, \frac{2}{3}) = \begin{pmatrix} e^{-\frac{1}{3}t} \\ \frac{2}{3} \end{pmatrix}$, $\forall t \in \mathbb{R}$

c)



22. Consider the following planar systems

22. i) $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

22. iii) $\begin{cases} \dot{x} = -y + xy \\ \dot{y} = x - x^2 \end{cases}$

22. ii) $\begin{cases} \dot{x} = -y - x(x^2+y^2) \\ \dot{y} = x - y(x^2+y^2) \end{cases}$

22. iv) $\begin{cases} \dot{x} = -y + x(x^2+y^2) \\ \dot{y} = x + y(x^2+y^2) \end{cases}$

a) What is the type of the linear system i)?

b) Show that the equilibrium (0,0) of (ii), (iii), (iv) is not hyperbolic.

c) Passing to polar coordinates, represent the phase portraits of these systems.

d) Study the type and stability of (0,0), reading the phase portrait.

a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

We find the eigenvalues:

$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \iff \lambda^2 + 1 = 0 \iff \lambda^2 = -1 \; \begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

$\Rightarrow$ the system is a center

b) For system (ii):

Eq. point (0,0)

$f(x,y) = \begin{pmatrix} -y - x^3 - xy^2 \\ x - yx^2 - y^3 \end{pmatrix}$

$Jf(x,y) = \begin{pmatrix} -3x^2 - y^2 & -1 - 2xy \\ 1 - 2yx & -x^2 - 3y^2 \end{pmatrix}$

$Jf(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The eigenvalues are: $\lambda_1 = i$ and $\lambda_2 = -i$ $\Rightarrow$ $\left.\begin{matrix} Re(\lambda_1) = 0 \\ Re(\lambda_2) = 0 \end{matrix}\right\} \Rightarrow$ (0,0) is not hyperbolic

HW: Prove that the Jacobian matrix is the same for (iii) and (iv)

c) system i)

$(x,y) \to (\rho, \theta)$

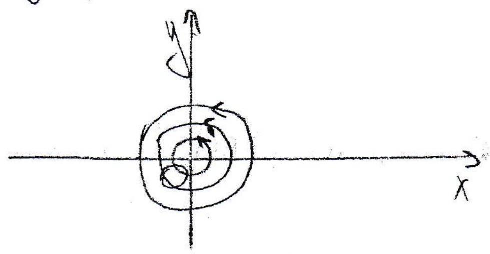
$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \leftrightarrow \begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$

$\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$

$$\begin{cases} \rho\dot\rho = x\cdot(-y) + y\cdot x \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2+y^2}{x^2} \end{cases} \iff \begin{cases} \rho\cdot\dot\rho = 0 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\rho^2}{x^2} = \dfrac{1}{\cos^2\theta} \end{cases}$$

$\dot\rho = 0 \Rightarrow \rho = c$

$\dot\theta = 1 \Rightarrow \theta$ increases with time



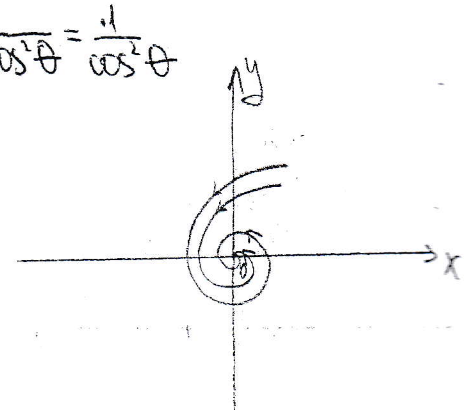
system ii):

$$\begin{cases} \rho\dot\rho = x\dot x + y\dot y \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot y x - \dot x y}{x^2} \end{cases} \iff \begin{cases} \rho\cdot\dot\rho = x[-y-x(x^2+y^2)] + y[x-y(x^2+y^2)] \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x[x-y(x^2+y^2)] - y[-y-x(x^2+y^2)]}{x^2} \end{cases}$$

$$\iff \begin{cases} \rho\cdot\dot\rho = -xy - x^2(x^2+y^2) + xy - y^2(x^2+y^2) = -\rho^4 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2 - xy(x^2+y^2) + y^2 + xy(x^2+y^2)}{x^2} = \dfrac{\rho^2}{\rho^2\cos^2\theta} = \dfrac{1}{\cos^2\theta} \end{cases}$$

$\dot\rho = -\rho^3 \Rightarrow \rho$ strictly decreases with time

$\dot\theta = 1 \Rightarrow \theta$ increases with time



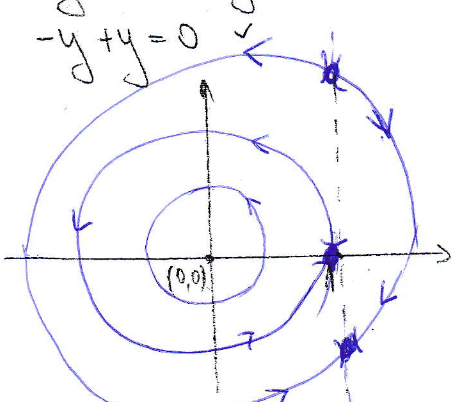
(iii) phase portrait: → all the equilibria
→ find a first integral

$$\begin{cases} \dot x = -y + xy \\ \dot y = x - x^2 \end{cases} \qquad \begin{cases} -y + xy = 0 \\ x - x^2 = 0 \end{cases} \qquad x - x^2 = 0, \quad x(1-x) = 0 \; \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$$

$-y = 0 \Rightarrow y_1 = 0$

$-y + y = 0$ ✓

$(0,0),\ (1,a),\ a \in \mathbb{R}$



$\dfrac{dy}{dx} = \dfrac{x - x^2}{-y + xy} \Rightarrow \dfrac{dy}{dx} = \dfrac{x(1-x)}{-y(1-x)} \Rightarrow \dfrac{dy}{dx}$

$-y\,dy = x\,dx \quad |\int$

$-\int y\,dy = \int x\,dx$

$-\dfrac{y^2}{2} = \dfrac{x^2}{2} + c \quad |\cdot 2 \Rightarrow x^2 + y^2 = c,\ c \in \mathbb{R}$

Let $H: \mathbb{R}^2 \to \mathbb{R},\ H(x,y) = x^2 + y^2$

the p.d.e. of f.i $(-y+xy)\dfrac{\partial H}{\partial x} + (x - x^2)\dfrac{\partial H}{\partial y} = 0$ $\Big\} \Rightarrow 0 = 0 \ \forall (x,y) \in \mathbb{R}^2$