Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, Semestru: Primăvara 2020

Seminars 2&3

Seminar 2: Linear homogeneous differential equations with constant coefficients

1.4.1. Apply the characteristic equation method.

1.4.2. Note that the possible solutions of a linear homogeneous differential equation with constant coefficients are of the form

$$t^k e^{\alpha t}, \quad t^k e^{\alpha t}\cos(\beta t), \quad t^k e^{\alpha t}\sin(\beta t),$$

or linear combinations of this type of functions. Here $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. The first function, $t^k e^{\alpha t}$ is a solution if and only if α is a real root of the characteristic equation of multiplicity *at least* $k+1$. One of the other two functions is a solution if and only if $\alpha \pm i\beta$ is a root of multiplicity *at least* $k+1$. The main idea to solve these type of problems is to recover the characteristic equation first, then the differential equation.

f) The function $(5-3t)e^{-3t}$ is a linear combination of e^{-3t} and te^{-3t}. Thus, it is a solution of a linear homogeneous differential equation with constant coefficients if and only if $r=-3$ is a root of the characteristic equation of multiplicity at least 2. The polynomial equation of minimal degree with this property is $(r+3)^2=0$, that is, $r^2+6r+9=0$. The DE having this ch eq is $x''+6x'+9x=0$. Its general solution is $x=c_1e^{-3t}+c_2te^{-3t}$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary.

o) The function $(t-1)^2$ is a linear combination of t^2, t and 1. Thus, it is a solution of a LHDE with CC if and only if $r=0$ is a root of the characteristic equation of multiplicity at least 3. The polynomial equation of minimal degree with this property is $r^3=0$. The DE having this ch eq is $x'''=0$. Its general solution is $x=c_1+c_2t+c_3t^2$, where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary.

p) The function $2\cos^2 t = 4\cos(2t)-2$ is a linear combination of $\cos(2t)$ and 1. Thus, it is a solution of a LHDE with CC if and only if $r=2i$ and $r=0$ are simple roots of the ch eq. The polynomial equation of minimal degree with this property is $r(r-2i)(r+2i)=0$, that is, $r^3+4r=0$. The DE having this ch eq is $x'''+4x'=0$. Its general solution is $x=c_1\cos(2t)+c_2\sin(2t)+c_3$, where $c_1,c_2,c_3\in\mathbb{R}$ are arbitrary.

1.4.3. Use, also, the ideas presented in 1.4.2.

1.4.4. Both DE are second order, LH with CC. First we have to find the general solution of the DE. It will depend on 2 arbitrary real constants, c_1 and c_2. We replace its expression in the other conditions and we obtain a linear algebraic system of 2 equations with 2 unknowns: c_1 and c_2. It will have a unique solution (c_1,c_2). We replace these values in the expression of the general solution to obtain the unique solution of the IVP.

Note that, for b) a discussion with respect to the parameter λ is needed.

1.4.5. The same ideas as the ones presented in 1.4.4., only that the existence and uniqueness of (c_1,c_2) is not guaranteed. There are BVP's without any solution, or with more solutions, see also Lab 2.

1.4.6. Hint: A LHDE with CC has periodic solutions if and only if its characteristic equation has pure imaginary roots. Conclusion: $\lambda>0$.

1.4.7. Hint: A LHDE with CC has periodic solutions if and only if its characteristic equation has pure imaginary roots. Conclusion: $\mu=0$. In this case, the general solution is $x=c_1\cos(\omega t)+c_2\sin(\omega t)$, $c_1,c_2\in\mathbb{R}$ arbitrary constants. The main period is $T=\frac{2\pi}{\omega}$.

1.4.8. As we already know, the possible solutions of a LHDE with CC are of the form

$$t^k e^{\alpha t},\quad t^k e^{\alpha t}\cos(\beta t),\quad t^k e^{\alpha t}\sin(\beta t),$$

or linear combinations of this type of functions. One can easily see that any solution of such equation goes to 0 as $t\to\infty$ if and only if the real part of any root of the ch eq is <0.

For the given DE, the ch eq is $r^2 + \mu r + \omega^2 = 0$. Of course, its roots can be real or not.

In the case that they are real, we look for conditions to be both < 0. Using the Viéte relations, their sum is $-\mu$ and their product is ω^2. Then both roots have the same sign, opposed to the sign of μ. Conclusion: $\mu > 0$.

In the case that they are not real, denote them by $\alpha \pm i\beta$, where $\beta \neq 0$ and α is the real part of both. Their sum is $2\alpha = -\mu$. We look for conditions such that $\alpha < 0$. Conclusion: $\mu > 0$.

Final conclusion: $\mu > 0$.

Seminar 3: Linear nonhomogeneous equations with constant coefficients. First order equations.

1.5.1. a) Our first aim is to find the general solution of the DE $x'' + 3x' + x = 1$. Note that it is a second order LNHDE with CC, the term 1 being the nonhomogeneous part.

Step 1. Write the LHDE associated, $x'' + 3x' + x = 0$. Its characteristic equation is $r^2 + 3r + 1 = 0$, with the discriminant $\Delta = 5$. Then it has two real, distinct roots, denoted α_1 and α_2. Note that $\alpha_1 + \alpha_2 = -3 < 0$ and $\alpha_1\alpha_2 = 1 > 0$. Then $\alpha_1 < 0$ and $\alpha_2 < 0$. Moreover, we have $x_h = c_1e^{\alpha_1} + c_2e^{\alpha_2}$, $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

Step 2. We need to find a particular solution of $x'' + 3x' + x = 1$. Since the right-hand side is a constant, we look for a constant solution. Indeed, we immediately notice that $x_p = 1$ is a solution.

Step 3. The general solution of the given equation is $x(t) = c_1e^{\alpha_1} + c_2e^{\alpha_2} + 1$, $c_1, c_2 \in \mathbb{R}$ arbitrary constants. We saw at Step 1 that $\alpha_1 < 0$ and $\alpha_2 < 0$. Then, it is easy to see that $\lim_{t\to\infty} x(t) = 1$.

1.5.2. This is a second order LNHDE with CC. The LHDE associated is $x'' - x = 0$. Its characteristic equation is $r^2 - 1 = 0$, whose roots are -1 and 1. Then its general solution $x_h = c_1e^{-t} + c_2e^t$, $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

We look, first, for a particular solution of the form $x_p = ae^{\lambda t}$, where we want to determine that coefficient $a \in \mathbb{R}$. After we substitute in $x'' - x = e^{\lambda t}$, we obtain $a(\lambda^2 - 1) = 1$, thus $a = \frac{1}{\lambda^2-1}$ which is well-defined only when $\lambda \in \mathbb{R} \setminus \{-1, 1\}$.

Using again the hint, when $\lambda \in \{-1, 1\}$, we look for a particular solution of the form $x_p = ate^{\lambda t}$. After we substitute in $x'' - x = e^{\lambda t}$, we obtain $a = \frac{1}{2\lambda}$.

In conclusion, the general solution of $x'' - x = e^{\lambda t}$ is $x = c_1 e^{-t} + c_2 e^t + \frac{1}{\lambda^2 - 1} e^{\lambda t}$ when $\lambda \in \mathbb{R} \setminus \{-1, 1\}$, respectively $x = c_1 e^{-t} + c_2 e^t + \frac{1}{2\lambda} e^{\lambda t}$ when $\lambda \in \{-1, 1\}$.

1.5.3. is partially solved in Lecture 4.

1.5.4. is similar to 1.5.3.

1.2.5. a) $x_{p1} = -e^t$. b) $x_{p2} = -\frac{1}{3} e^{-t}$. c) $x_p = 5(-e^t) - 3(-\frac{1}{3} e^{-t}) = -5e^t + e^{-t}$. d) $x = ce^{2t} - 5e^t + e^{-t}$, $c \in \mathbb{R}$.

For 1.3.1. - 1.3.5. there are models in Lecture 3.

1.3.6. is optional.

Solve also 1.1.14. and 1.1.15.