The direction field associated to a differential equation

(1) $y' = f(x,y)$ $\iff$ $y'(x) = f(x, y(x))$

- $y(x)$ is the notation of the unknown
- $f: \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1$

The solution curve in $\mathbb{R}^2$

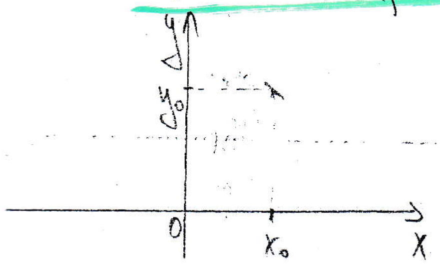
(2) $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is the unknown

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$, C^1

An orbit is represented in $\mathbb{R}^2$

(A) (1) $y' = f(x,y)$

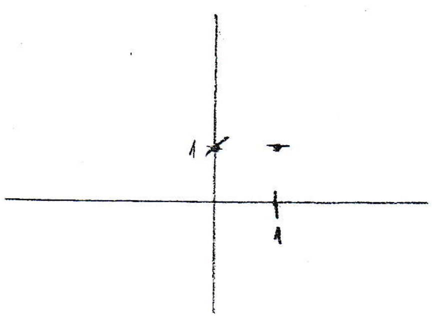
The direction field associated to (1) is a collection of vectors in $\mathbb{R}^2$.



The vector through the arbitrary point $(x_0, y_0) \in \mathbb{R}^2$ has the slope $m = f(x_0, y_0)$

Example: $y' = 1 - \dfrac{x}{y^2}$

Plot the vectors corresponding to $(1,1)$, $(0,1)$, $(1,0)$



$f(x,y) = 1 - \dfrac{x}{y^2}$

$f(1,1) = 0$

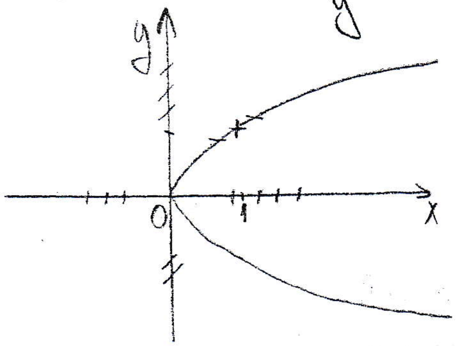
$f(0,1) = 1$

$f(1,0) = \infty$

Def. Let $m \in \mathbb{R} \cup \{\infty\}$. We define the m-izocline of the direction field of (1) as follows: $I_m = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = m\}$

Ex. Find the 0-izocline and 1-izocline of $y' = 1 - \frac{x}{y^2}$

- 0-izocline: $1 - \frac{x}{y^2} = 0 \iff y^2 = x$ (parabola)
- 1-izocline: $1 - \frac{x}{y^2} = 1 \iff x = 0$ (line) (same slope for all vectors $\iff m = 1$ on line $x = 0$)



Proposition: Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary, but fixed. The slope of the direction field of (1) in (x_0, y_0) is equal to the slope of the solution curve of (1), that passes through (x_0, y_0).

Proof. We consider the ivp $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$, which has a unique solution.

Denote it with $\varphi \Rightarrow \begin{cases} \varphi'(x) = f(x, \varphi(x)) \\ \varphi(x_0) = y_0 \end{cases} \Rightarrow \varphi$ is the unique solution curve of (1) that passes through (x_0, y_0)

Then $\varphi'(x_0)$ is the slope of the solution curve that passes through (x_0, y_0). So, we have to prove that $\varphi'(x_0) = f(x_0, y_0)$. But we have $\varphi'(x_0) = f(x_0, \varphi(x_0)) = f(x_0, y_0)$.

Ex. $\varphi' = -\frac{x}{y}$

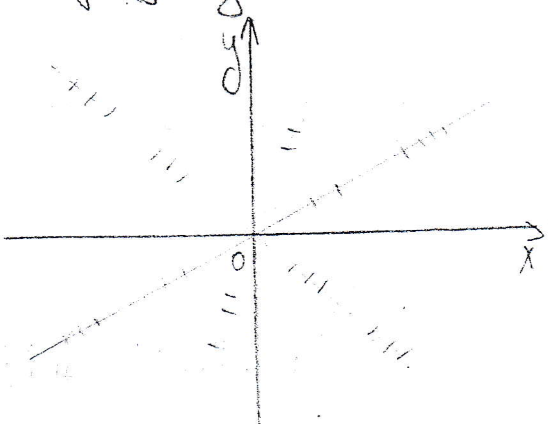
$f(x,y) = -\frac{x}{y}$

0-izocline $x = 0$

∞-izocline $y = 0$

m-izocline $-\frac{x}{y} = m \iff y = -\frac{x}{m}$ (a line)

1-izocline $\Rightarrow y = -x$



Note that the solution curves are circles centered in the origin

$\left(\frac{dy}{dx} = -\frac{x}{y} \quad y\,dy = -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c \Rightarrow x^2 + y^2 = c,\ c \in \mathbb{R}\right)$

Def. The slope of the direction field associated to (2) (planar sys.) in the point $(x_0, y_0) \in \mathbb{R}^2$ is $m = \dfrac{f_2(x_0, y_0)}{f_1(x_0, y_0)}$

Proposition The slope of the direction field of (2) in $(x_0, y_0) \in \mathbb{R}^2$ = the slope of the orbit of (2) that passes through (x_0, y_0).

Proof. We consider the IVP $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$ with a unique solution,

denoted by $\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} \Rightarrow \begin{cases} \varphi_1'(t) = f_1(\varphi_1(t), \varphi_2(t)) \\ \varphi_2'(t) = f_2(\varphi_1(t), \varphi_2(t)) \\ \varphi_1(0) = x_0 \\ \varphi_2(0) = y_0 \end{cases} \Rightarrow \gamma_{(x_0,y_0)} = \{(\varphi_1(t), \varphi_2(t)) \in \mathbb{R}^2\}$

We know that the vector $(\varphi_1'(t), \varphi_2'(t))$ is tangent to $\gamma_{(x_0,y_0)}$ in the point $(\varphi_1(t), \varphi_2(t))$.

In particular $(\varphi_1'(0), \varphi_2'(0))$ is tangent to $\gamma_{(0,0)}$ in the point $(\varphi_1(0), \varphi_2(0)) = (x_0, y_0) \Rightarrow \dfrac{\varphi_2'(0)}{\varphi_1'(0)}$ = the slope of the orbit of (2) that passes through (x_0, y_0). Thus, we have to prove that $\dfrac{f_2(x_0, y_0)}{f_1(x_0, y_0)} = \dfrac{\varphi_2'(0)}{\varphi_1'(0)}$.

We have $\dfrac{\varphi_2'(0)}{\varphi_1'(0)} = \dfrac{f_2(\varphi_1(0), \varphi_2(0))}{f_1(\varphi_1(0), \varphi_2(0))} = \dfrac{\varphi_2(x_0, y_0)}{\varphi_1(x_0, y_0)}$ = the slope of the d.f. in (x_0, y_0)

Ex. HW. $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ Direction field + shape of orbits.

Numerical methods to find approximate solutions of DE's

(1) $y' = f(x,y)$

$f : \mathbb{R}^2 \to \mathbb{R}$, $f \in C^1$

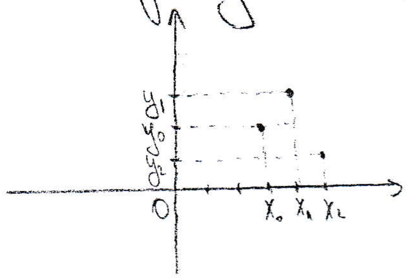
$(x_0, y_0) \in \mathbb{R}^2$ and the IVP $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ with the unique solution $\varphi : [x_0, x^*] \to \mathbb{R}$

We consider a partition of $[x_0, x^*]$: $x_0 < x_1 < x_2 < \dots < x_n = x^*$

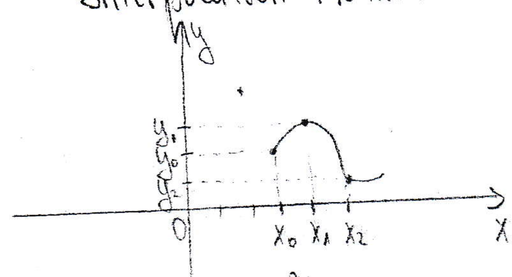
(We cover in n steps the interval $[x_0, x^*]$)

We want to find $y_1, y_2, \dots, y_n$ good approximations for $\varphi(x_1)$, $\varphi(x_2)$, $\dots$, $\varphi(x_n)$ $(y_i \simeq \varphi(x_i))$

We finally obtain (x_k, y_k), $k = \overline{1,n}$

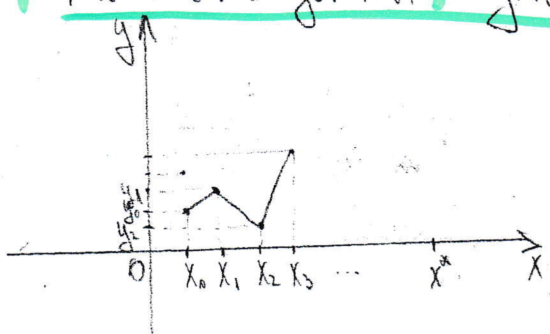


Interpolation method



a planar curve

The Euler's formula: $\boxed{y_{k+1} = y_k + (x_{k+1} - x_k)\, f(x_k, y_k) \qquad k = \overline{0, n-1}}$



$(x_0, y_0) \longrightarrow m_0 = f(x_0, y_0)$

$y - y_0 = m_0 (x - x_0)$

$y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$

if $x_{k+1} - x_k = h$, $\forall k \Rightarrow h =$ stepsize

Example: We consider the IVP $\begin{cases} y' = 2xy \\ y(0) = 1 \end{cases}$ $x \in [0,1]$

a) Write the Euler's numerical formula for this ~~numerical~~ IVP with constant stepsize $h = 0.1$.

b) Compute y_1, y_2

a) $f(x,y) = 2xy$

$x_0 = 0 \Rightarrow x^* = 1$, $y_0 = 1$, $h = 0{,}1$

$\begin{cases} y_{k+1} = y_k + 0{,}1 \cdot 2 \cdot x_k \cdot y_k & \forall k = \overline{0,9} \\ x_{k+1} = x_k + h \end{cases}$ $\Longleftrightarrow$

$$\iff \begin{cases} y_{k+1} = y_k + 0{,}2 \cdot x_k \cdot y_k, & \forall k = \overline{0,9} \\ x_{k+1} = x_k + h \end{cases} \Rightarrow x_k = \frac{k}{10}, \quad k = \overline{1,10}$$

b) $y_1 = y_0 + 0{,}2 \cdot x_0 \cdot y_0 = 1 + 0{,}2 \cdot 0 \cdot 1 = 1$

$y_2 = y_1 + 0{,}2 \cdot x_1 \cdot y_1 = 1 + 0{,}2 \cdot \frac{1}{10} \cdot 1 = 1 + 0{,}02 = 1{,}02$

Improved Euler's Formula:

$$y_{k+1} = y_k + \frac{h}{2} f(x_k, y_k) + \frac{h}{2}\left(x_k + h, y_k + h \cdot f(x_k, y_k)\right)$$

$$y_{k+1} = y_k + \frac{h}{2} f(x_k, y_k) + \frac{h}{2} f\left(x_k + h, y_k + h f(x_k, y_k)\right)$$

Exercise: We consider the IVP $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$. Write the Euler's formula with stepsize $h_n = \frac{1}{n}$, $(n \geq 1)$. ~~A~~ obtain approximations for e

$e : \mathbb{R} \to \mathbb{R}$, $e(x) = e^x$ is the unique sol of this IVP. Note that $e(1) = e$.

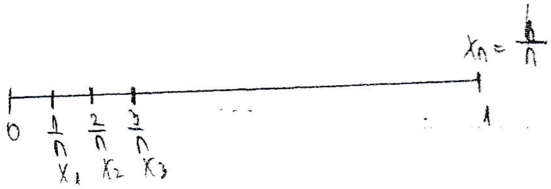
$y_{k+1} = y_k + \frac{1}{n} f(x_k, y_k)$

$y_{k+1} = y_k + \frac{1}{n} y_k, \ \forall k \geq 0 \iff \begin{cases} y_{k+1} = \left(1 + \frac{1}{n}\right) y_k, & \forall k \geq 0 \\ y_0 = 1 \end{cases}$

Note that $y_k = \left(1 + \frac{1}{n}\right)^k, \ \forall k \geq 0$

We work on the interval $[0,1]$ with a partition with constant stepsize $h_n = \frac{1}{n}$.

So, $x_n = 1$



$y_n \approx e(x_n) = e(1) = e \Rightarrow y_n \approx e$

$y_n = \left(1 + \frac{1}{n}\right)^n \approx e$

$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$