Lab 2

1. In order to check whether a function f(t) is a solution of a differential equation, we have to substitute f(t), as we did in the seminar. This time we make the computations with Maple.

```
> diff(sin(t),t$4)-sin(t); diff(cos(t),t$4)-cos(t); diff(sinh(t),
  t$4)-sinh(t); diff(cosh(t),t$4)-cosh(t);
```

$$0$$
$$0$$
$$0$$
$$0 \quad (1)$$

```
>
```

We deduce that all functions are solutions.

2&3&4&5. We refer to section 2 of the tutorial. So, we use the command dsolve.

```
> dsolve(diff(x(t),t)+t*x(t)=0,x(t));
```

$$x(t) = \_C1\, e^{-\frac{1}{2} t^2} \quad (2)$$

```
> dsolve(diff(x(t),t$2)+x(t)=0,x(t));
```

$$x(t) = \_C1 \sin(t) + \_C2 \cos(t) \quad (3)$$

```
>
```

6.

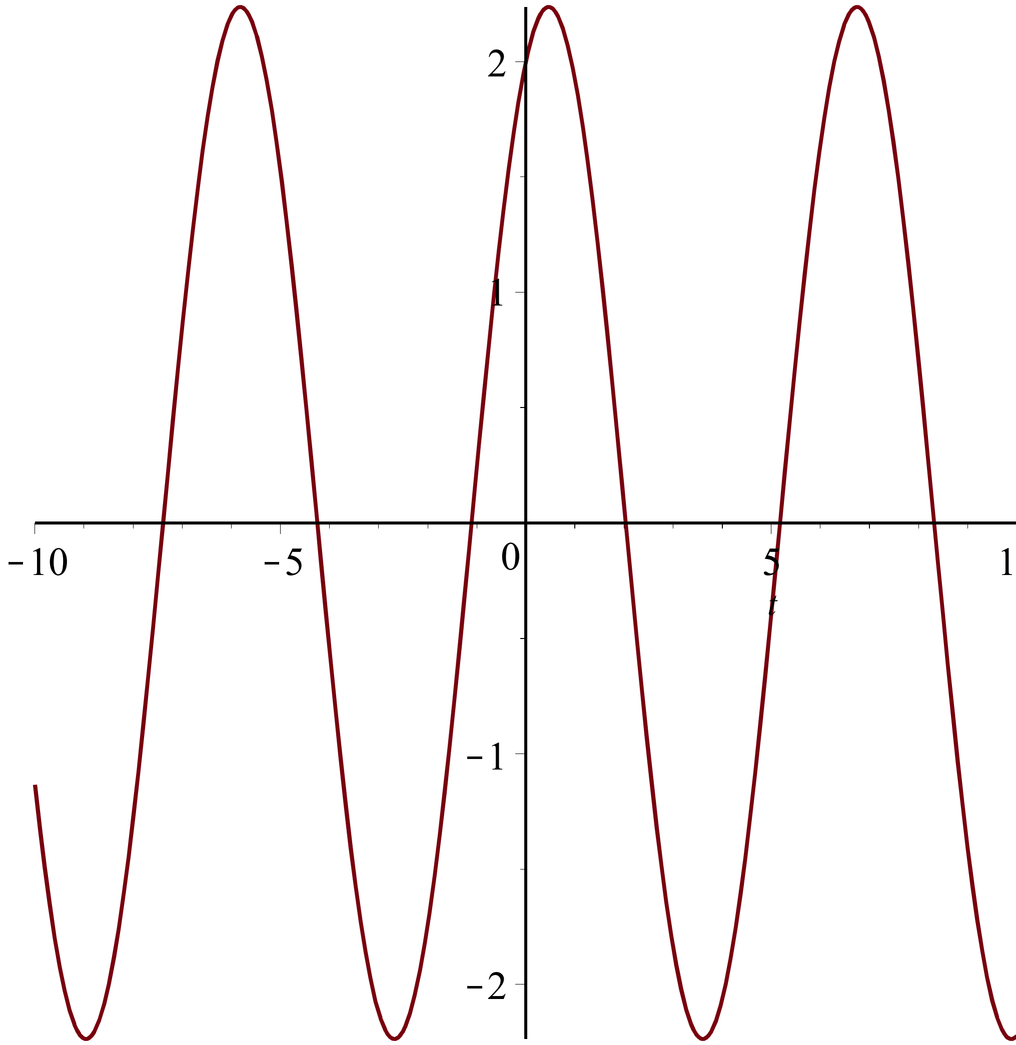
```
> sol:=dsolve({diff(x(t),t$2)+x(t)=0,x(Pi/2)=1,D(x)(Pi/2)=-2},x(t))
  ;
```

$$sol := x(t) = \sin(t) + 2 \cos(t) \quad (4)$$

```
> exprsol:=rhs(sol);
```

$$exprsol := \sin(t) + 2 \cos(t) \quad (5)$$

```
> plot(exprsol,t=-10..10);
```




```
> z:=solve(diff(exprsol,t),t);
```

$$z := \arctan\left(\frac{1}{2}\right) \quad \textbf{(6)}$$

```
> eval(exprsol,t=z);
```

$$\sqrt{5} \quad \textbf{(7)}$$

```
> expand(sin(t)+2*cos(t)-sqrt(5)*cos(t-arctan(1/2)));
```

$$0 \quad \textbf{(8)}$$

```
>
```

We have the formula: sin(t)+2cos(t)=sqrt(5)*cos(t-arctan(1/2)).
From here we deduce that the solution of this IVP is periodic with the main period 2*Pi and it oscillates around 0 with constant amplitude sqrt(5).

7.

```
> sol:=dsolve({4*diff(x(t),t$2)+8*diff(x(t),t)+5*x(t)=0,x(0)=0,D(x)
  (0)=0.5},x(t));
```

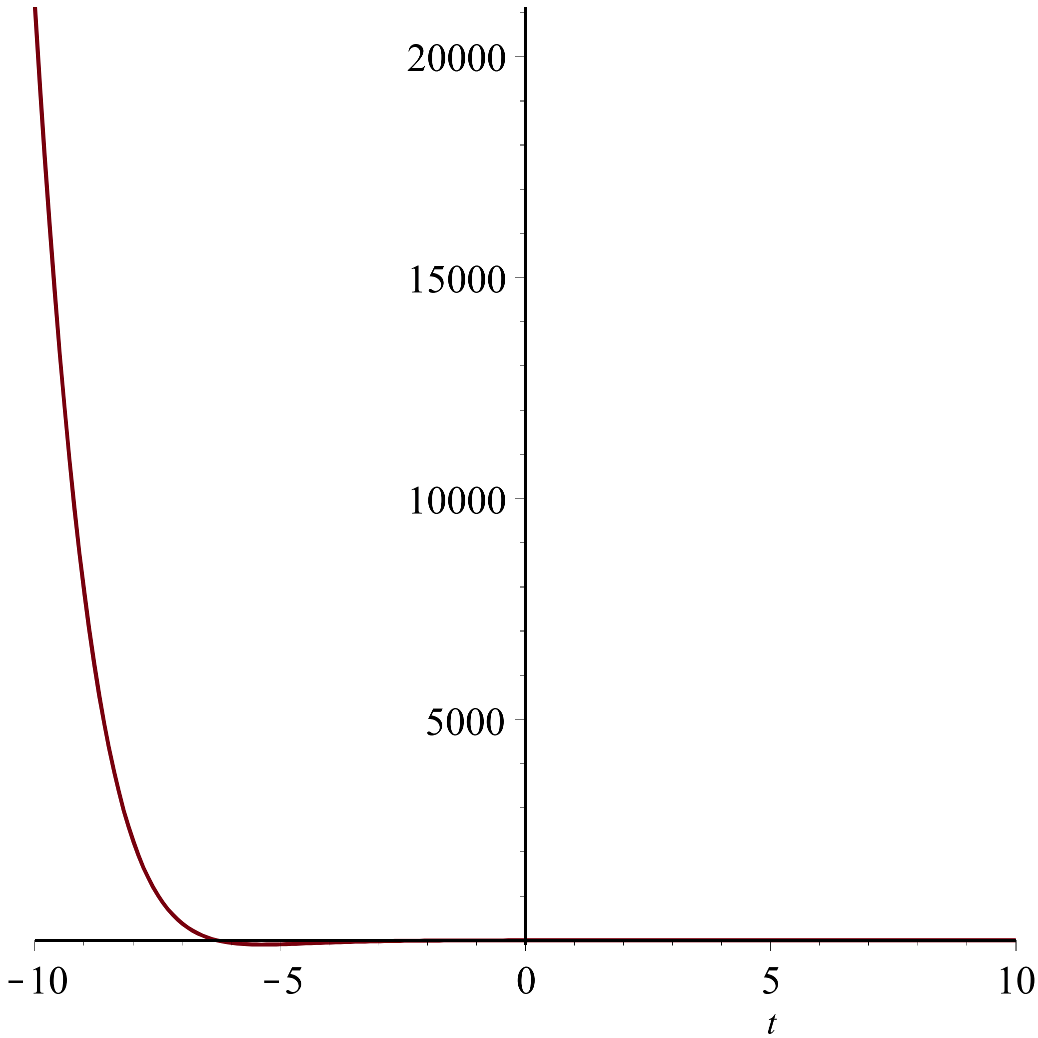
(9)

$$sol := x(t) = e^{-t} \sin\left(\frac{1}{2}\, t\right) \tag{9}$$

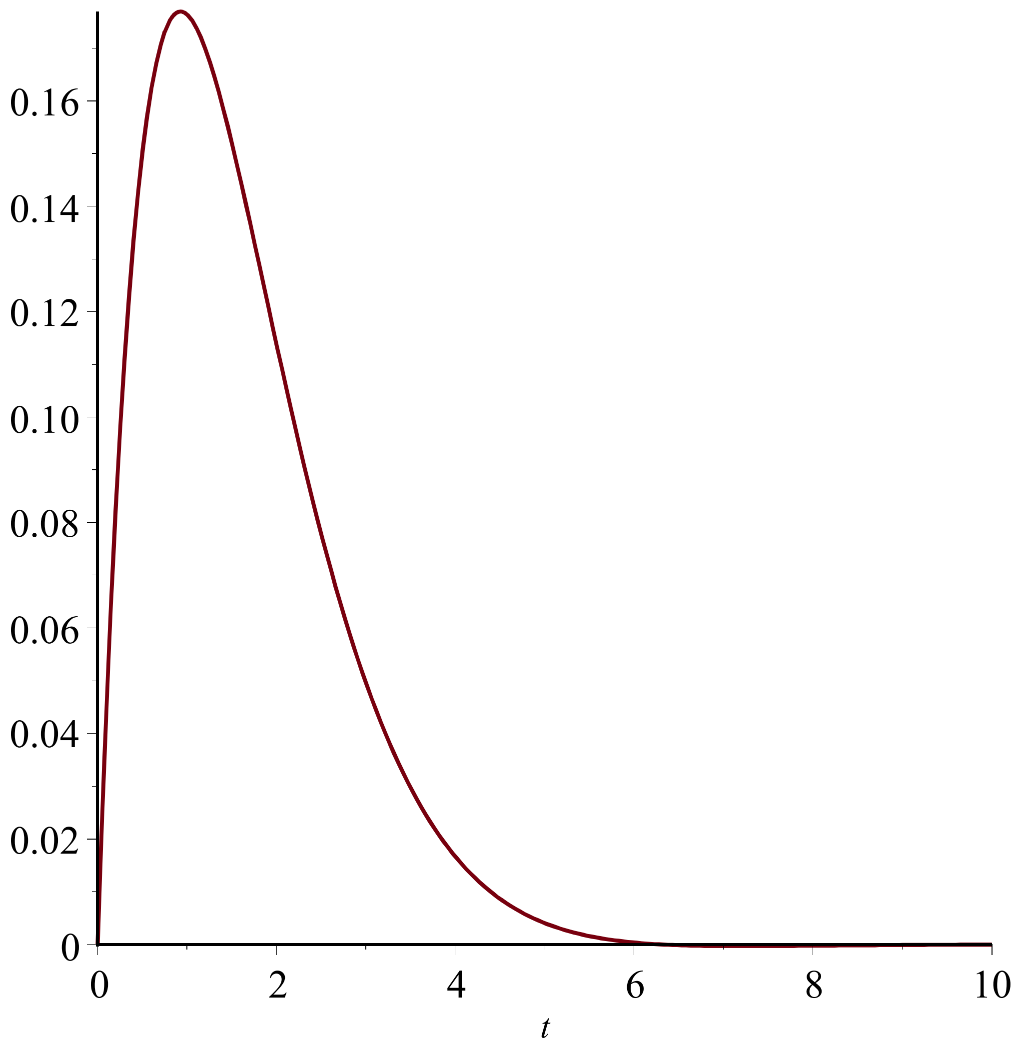
```
> exprsol:=rhs(sol);
```

$$exprsol := e^{-t} \sin\left(\frac{1}{2}\, t\right) \tag{10}$$

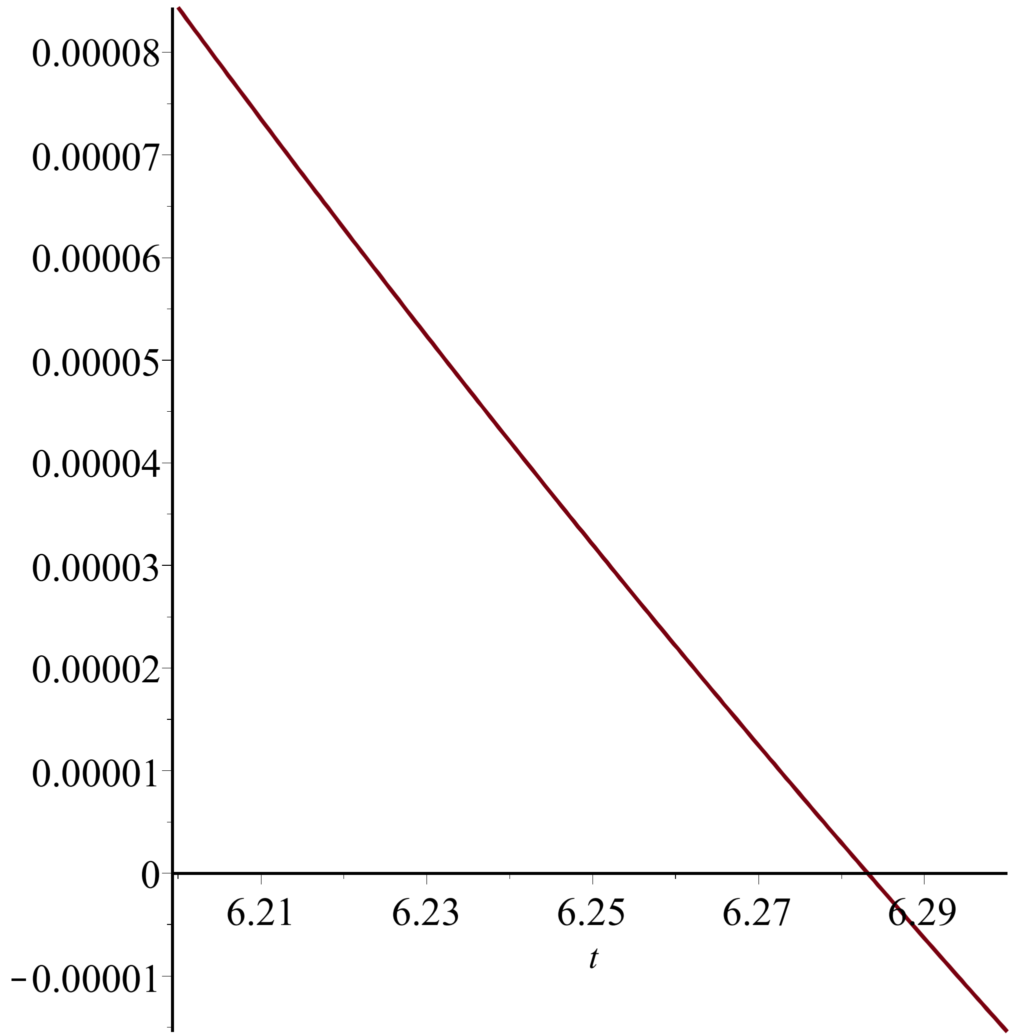
```
> plot(exprsol,t=-10..10);
```



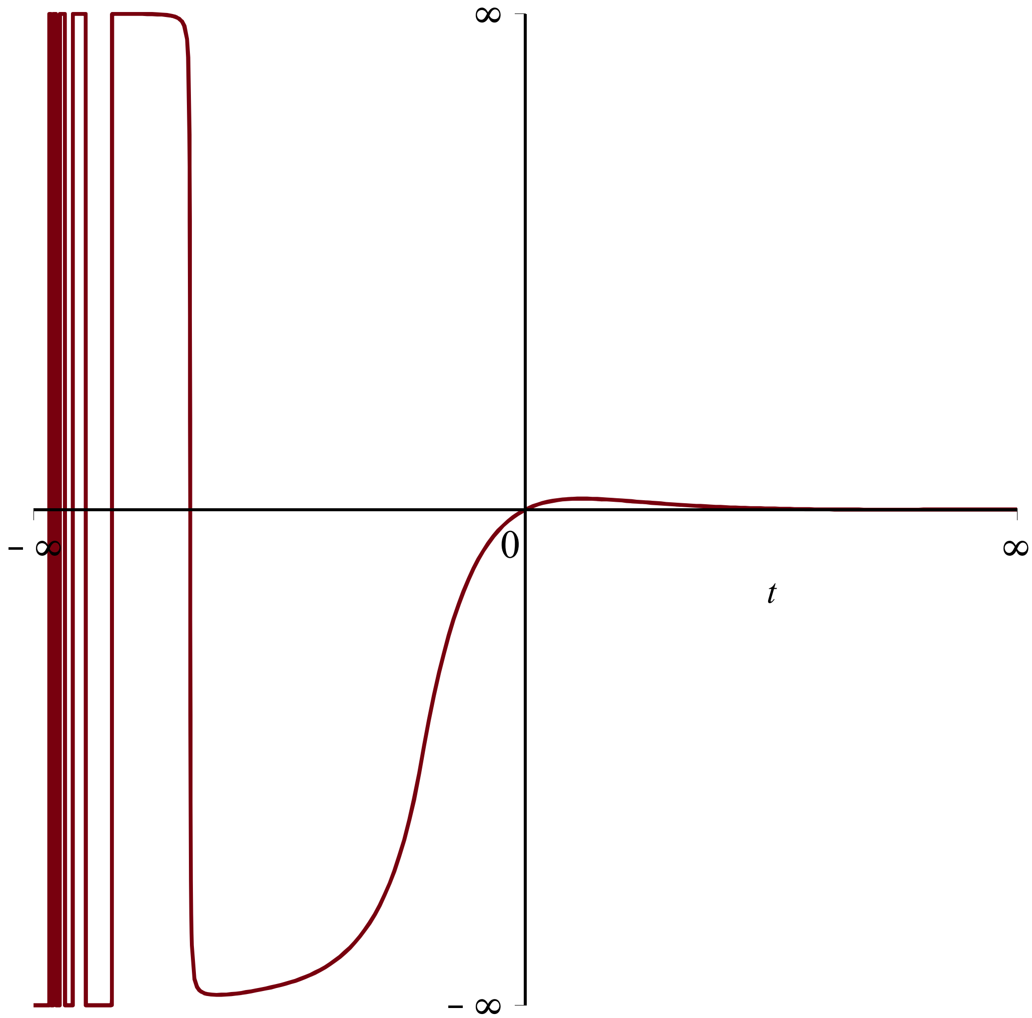

```
> plot(exprsol,t=0..10);
```




```
> plot(exprsol,t=6.2..6.3);
```




```
> plot(exprsol,t=-infinity..infinity);
```




```
> limit(exprsol,t=infinity);
```

$$0 \tag{11}$$

```
> solve(exprsol=0,t);
```

$$0 \tag{12}$$

```
>
```

It is difficult to have a nice global picture of this graph since, as we know, this function oscillates around 0 with exponentially decreasing amplitude. This means that, the variation of the values of t, and the variantion of the values of the function are not comparable and they can not be seen on medium large intervals of time (like, for example, an interval that contains two zeros of the function). Note also that Maple returns only one value (in this case 0) for the zeros of a trigonometric function. But we know that the zeros of this function are 2k*Pi for any integer k.
Draw in your notebook the graph of this function such that to be able to see its main behaviour: that it oscillates around 0 with exponentially decreasing amplitude (even it is not scaled).

8.

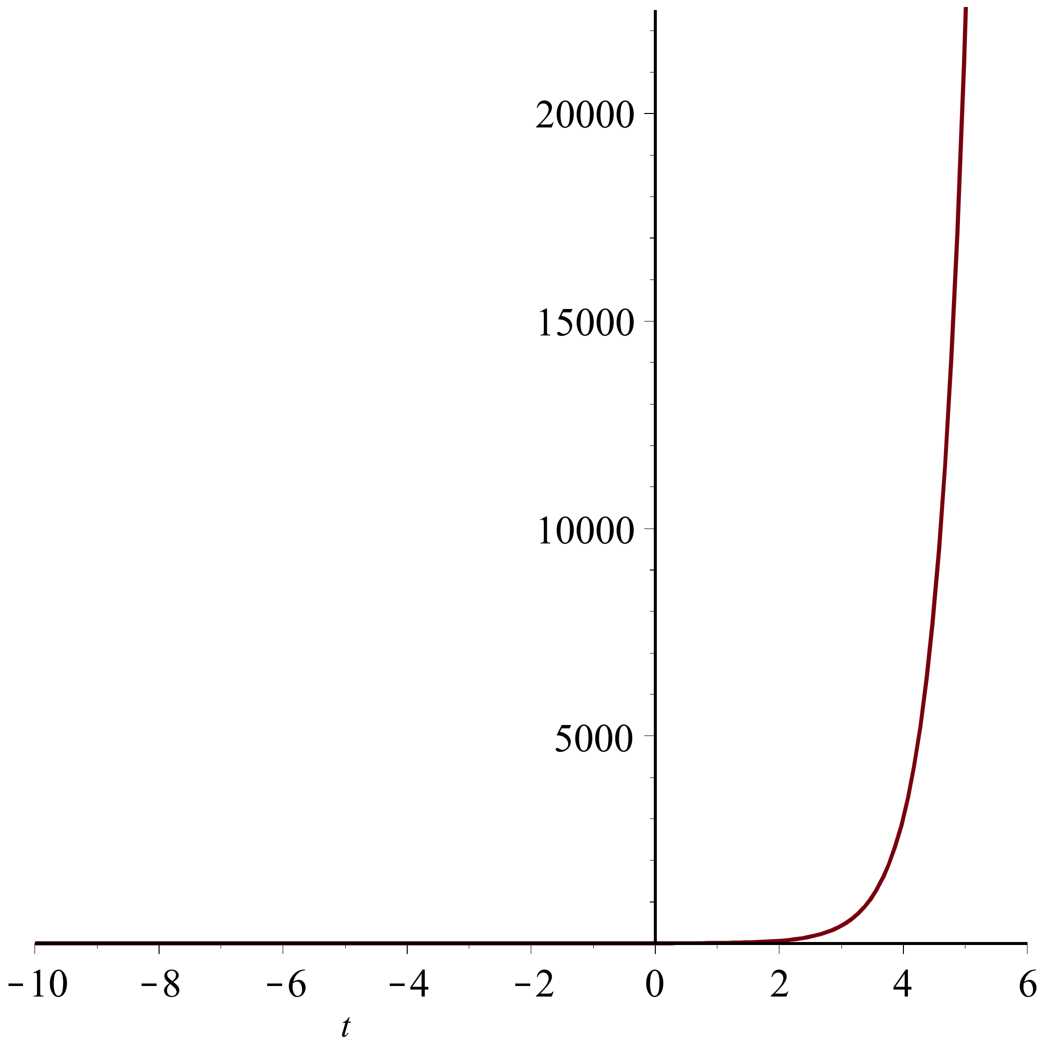
```
> sol:=dsolve({diff(x(t),t$2)-3*diff(x(t),t)+2*x(t)=0,x(0)=2,D(x)(0)=3},x(t));
```

$$sol := x(t) = \mathrm{e}^{t} + \mathrm{e}^{2\,t} \tag{13}$$

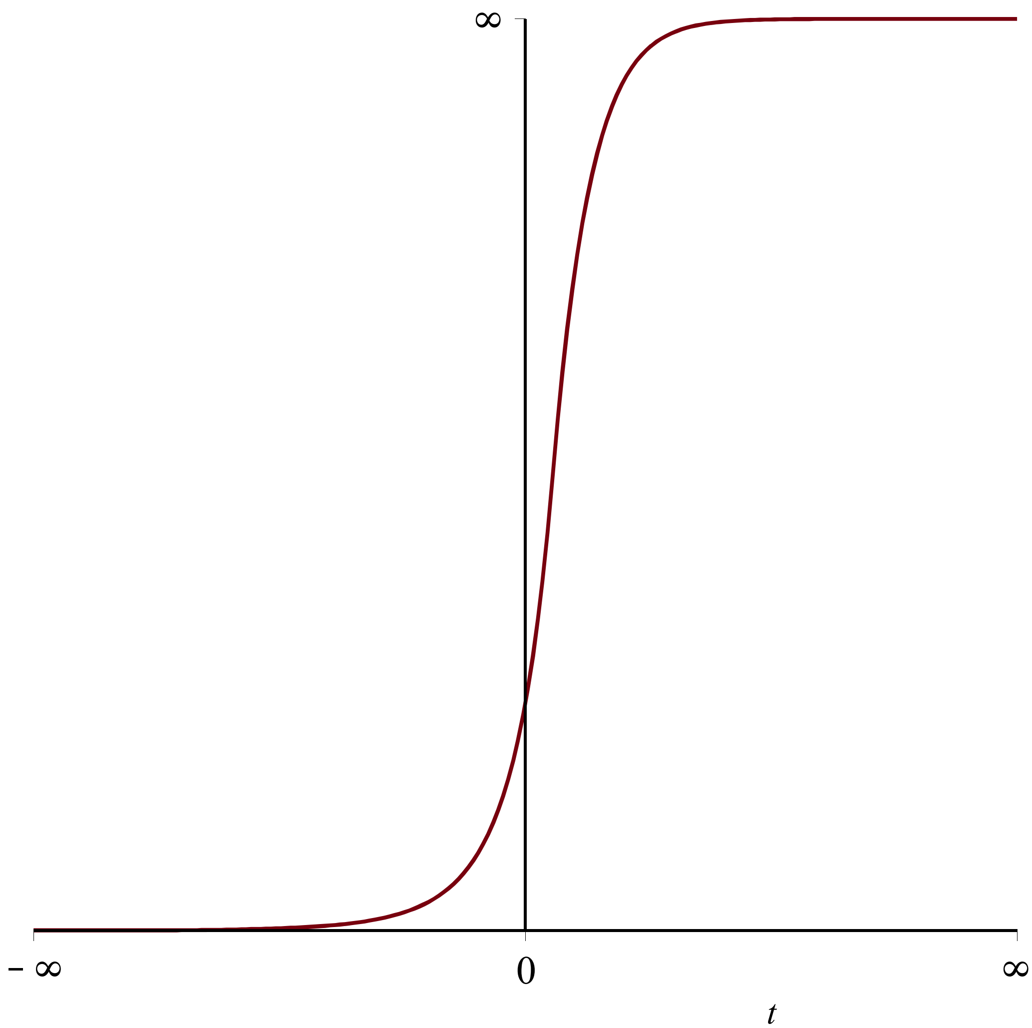
```
> exprsol:=rhs(sol);
```

$$exprsol := e^t + e^{2\,t} \tag{14}$$

```
> plot(exprsol,t=-10..10);
```




```
> plot(exprsol,t=-infinity..infinity);
```




```
> diff(exprsol,t$2);
```

$$e^{t} + 4\,e^{2\,t} \tag{15}$$

```
>
```

Note that, when plotting the graph from -infinity to infinity it looks like it is concave when it approaches infinity. Of course, this is not the case, since the second order derivative is positive for any real t. So, we must be careful when we interpret what a machine returns.

```
> limit(exprsol,t=-infinity); limit(exprsol,t=infinity);
```

$$0$$
$$\infty \tag{16}$$

```
>
```

In conclusion, this function is exponentially increasing, with positive values.

9.

```
> infolevel[dsolve]:=3;
```

$$infolevel_{dsolve} := 3 \tag{17}$$

```
> dsolve(diff(x(t),t$2)+5*x(t),x(t));
```

```
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
```

$$x(t) = \_C1 \sin\left(\sqrt{5}\, t\right) + \_C2 \cos\left(\sqrt{5}\, t\right) \quad (18)$$

```
> dsolve(diff(x(t),t$2)+t*x(t),x(t));
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing 'y'
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful
```

$$x(t) = \_C1\, \mathrm{AiryAi}(-t) + \_C2\, \mathrm{AiryBi}(-t) \quad (19)$$

```
> dsolve(diff(x(t),t$2)+t^5*x(t),x(t));
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing 'y'
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful
```

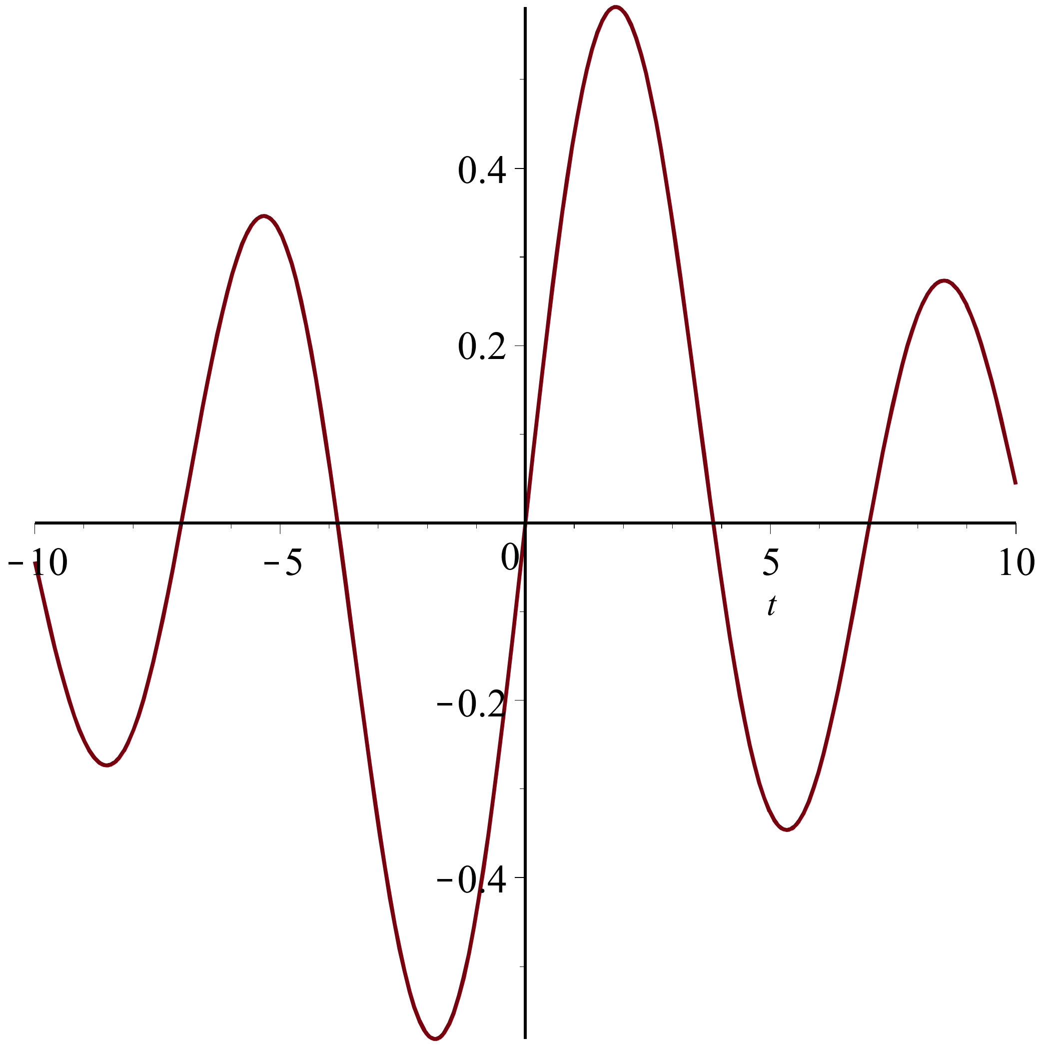
$$x(t) = \_C1 \sqrt{t}\, \mathrm{BesselJ}\left(\frac{1}{7}, \frac{2}{7}\, t^{7/2}\right) + \_C2 \sqrt{t}\, \mathrm{BesselY}\left(\frac{1}{7}, \frac{2}{7}\, t^{7/2}\right) \quad (20)$$

```
>
```

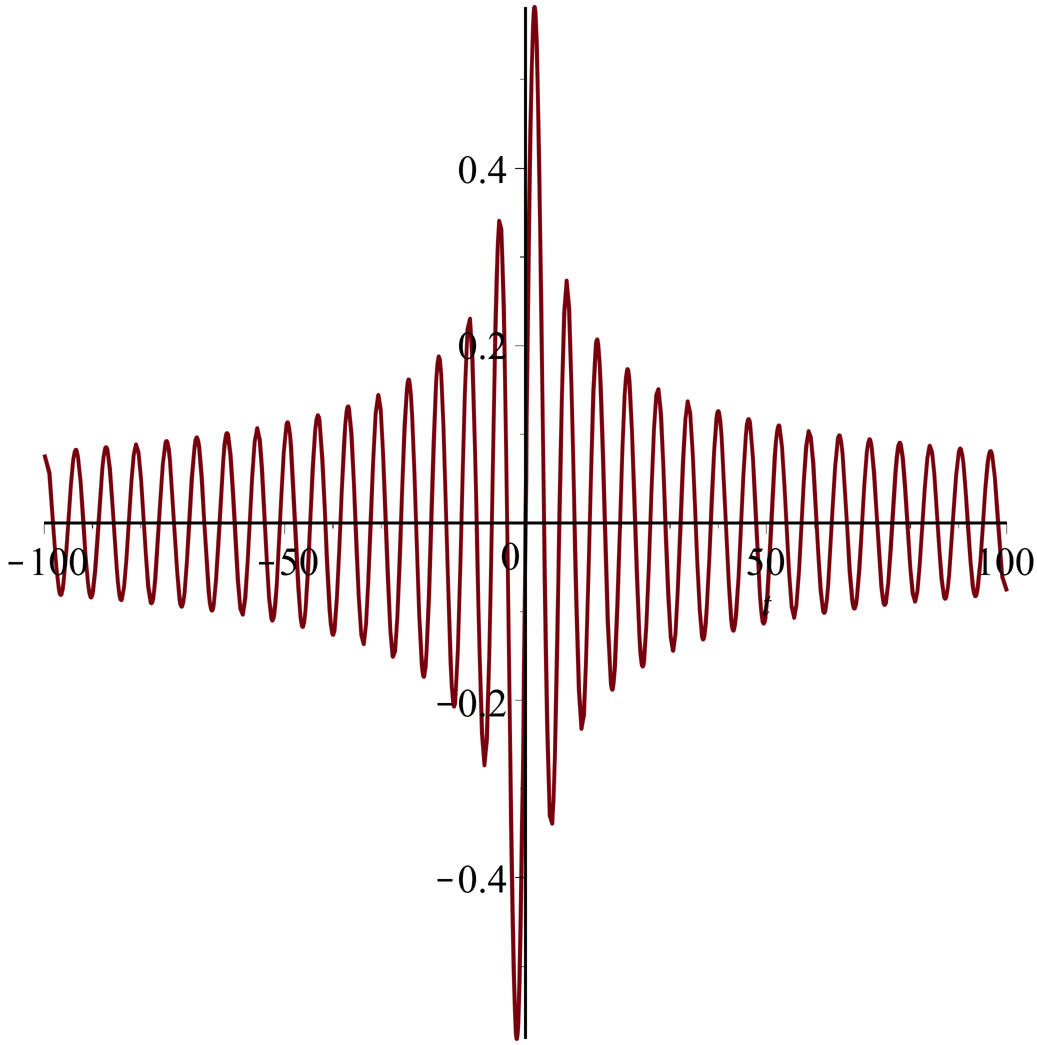
Note that the solutions of the last two equations (with variable coefficients) are given in terms of the special functions Airy and Bessel. Their expression can not be written as a finite combination of elementary functions. They are defined as the solutions of some given second order linear homogeneous equation with variable coefficients. There are many more special functions similar to these. They important in physics, and, of course, in mathematics. That is why one can find whole books dedicated to their study

.

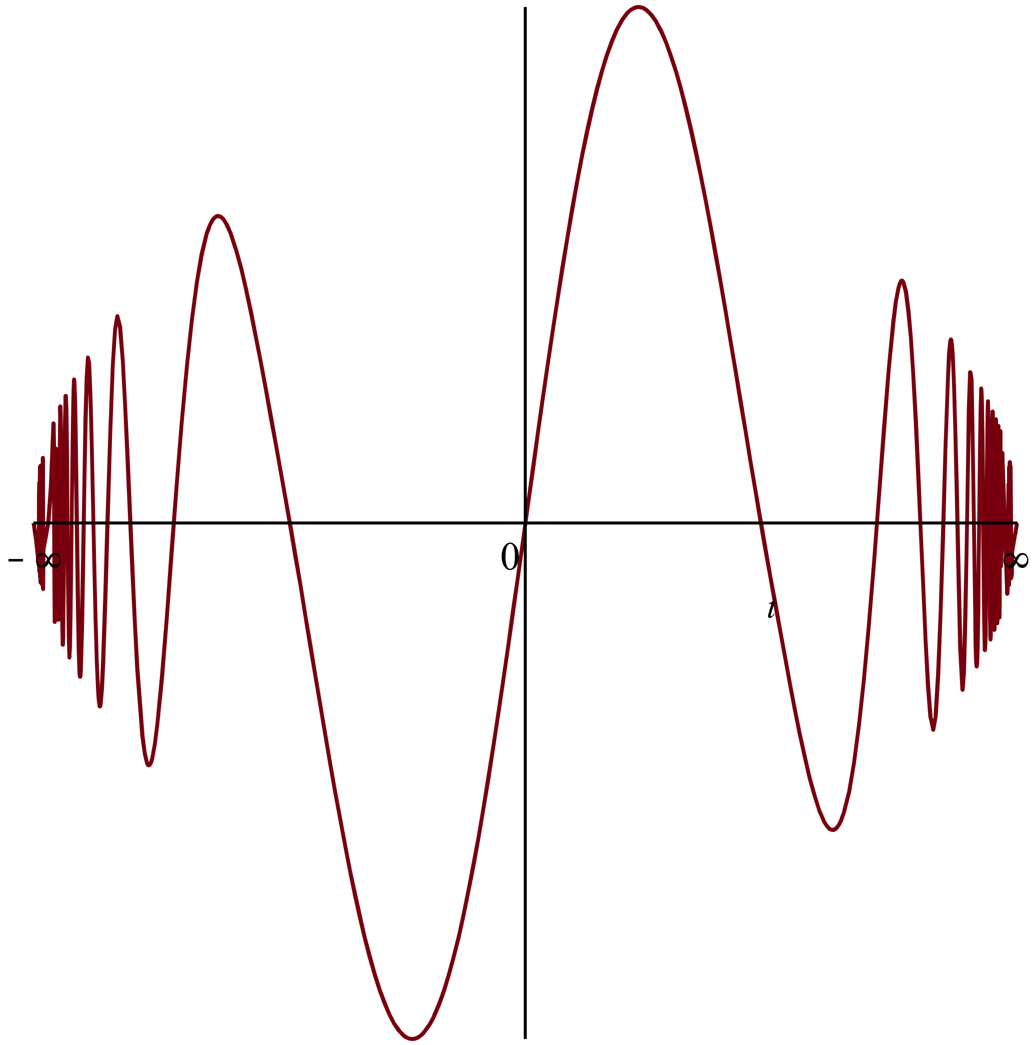
```
> plot(BesselJ(1,t),t=-10..10);
```




```
> plot(BesselJ(1,t),t=-100..100);
```




```
> plot(BesselJ(1,t),t=-infinity..infinity);
```



It seems that BesselJ(1,t) oscillates around 0 with decresing amplitude as t goes to infinity, as well as t goes to -infinity.

```
> limit(BesselJ(1,t),t=-infinity); limit(BesselJ(1,t),t=infinity);
```

$$0$$

$$0 \tag{21}$$

```
> infolevel[dsolve]:=1;
```

$$infolevel_{dsolve} := 1 \tag{22}$$

```
> dsolve({diff(x(t),t$2)+5*x(t)=0,x(0)=0,D(x)(0)=0},x(t));
```

$$x(t) = 0 \tag{23}$$

```
> dsolve({diff(x(t),t$2)+t*x(t)=0,x(0)=0,D(x)(0)=0},x(t));
<- No Liouvillian solutions exists
```

$$x(t) = 0 \tag{24}$$

```
> dsolve({diff(x(t),t$2)+t^5*x(t)=0,x(0)=0,D(x)(0)=0},x(t));
<- No Liouvillian solutions exists
```

(25)

$$x(t) = \begin{cases} \_C1\,\sqrt{t}\,\mathrm{BesselJ}\left(\frac{1}{7}, \frac{2}{7}\,t^{7/2}\right) & t < 0 \\ 0 & t = 0 \\ \_C1\,\sqrt{t}\,\mathrm{BesselJ}\left(\frac{1}{7}, \frac{2}{7}\,t^{7/2}\right) & 0 < t \end{cases} \quad \textbf{(25)}$$

The result returned by Maple in the first two cases is correct. For the last IVP it is not correct, since we know that the unique solution is x=0. In addition, note that the expression above contain sqrt(t) for t<0. A possible explanation could be that the implementation of the Bessel function does not permit the proper computation of its value at t=0, neither of the value of its derivative at t=0.

10&11&12.

```
> dsolve({diff(x(t),t$2)+x(t)=0,x(0)=0,x(Pi)=0},x(t));
```

$$x(t) = \_C1\,\sin(t) \quad \textbf{(26)}$$

```
> dsolve({diff(x(t),t$2)+x(t)=0,x(0)=0,x(1)=0},x(t));
```

$$x(t) = 0 \quad \textbf{(27)}$$

```
> dsolve({diff(x(t),t$2)+x(t)=1,x(0)=0,x(Pi)=0},x(t));
>
```

We have here 3 BVP's, apparently very similar, with very few differences. But they have 3 different behaviours. The first one has as many solution as real numbers are. The second one has a unique solution x=0. The last one has no solution. Check by hand that Maple returned the correct answer.

13&14&15.

```
> dsolve(diff(x(t),t)+x(t)=15);
```

$$x(t) = 15 + \mathrm{e}^{-t}\,\_C1 \quad \textbf{(28)}$$

The right-hand side is a constant, a particular solution is also constant.

```
> dsolve(diff(x(t),t)+x(t)=2*exp(t)-7*exp(-3*t));
```

$$x(t) = \mathrm{e}^{t} + \frac{7}{2}\,\mathrm{e}^{-3t} + \mathrm{e}^{-t}\,\_C1 \quad \textbf{(29)}$$

```
>
```

The right-hand side is a linear combination of exp(t) si exp(-3t), the same for a particular solution.

```
> dsolve(diff(x(t),t)+x(t)=-t^2+3*t-7);
```

$$x(t) = -t^2 + 5\,t - 12 + \mathrm{e}^{-t}\,\_C1 \quad \textbf{(30)}$$

```
>
```

The right-hand side is a second degree polynomial, the same for a particular solution.

19.

```
> dsolve(diff(x(t),t)+x(t)=2/sqrt(Pi)*exp(-t^2-t)); int(exp(t^2),t)
  ; int(2/sqrt(Pi)*exp(-t^2),t);
```

$$x(t) = (\mathrm{erf}(t) + \_C1)\,\mathrm{e}^{-t}$$

$$\frac{1}{2}\,\sqrt{\pi}\,\mathrm{erfi}(t)$$

$$\mathrm{erf}(t) \quad (31)$$

erf(t) is called the error function and it is defined as the primitive of exp(-t^2) multiplied with the constant 2/sqrt(Pi). As we know, this can not be written as a finite combination of elementary functions.

20.

```
> sol:=dsolve(diff(x(t),t$2)+3*diff(x(t),t)+x(t)=1);
```

$$sol := x(t) = e^{\frac{1}{2}\left(\sqrt{5}-3\right)t}\_C2 + e^{-\frac{1}{2}\left(\sqrt{5}+3\right)t}\_C1 + 1 \quad (32)$$

```
> exprgensol:=rhs(sol);
```

$$exprgensol := e^{\frac{1}{2}\left(\sqrt{5}-3\right)t}\_C2 + e^{-\frac{1}{2}\left(\sqrt{5}+3\right)t}\_C1 + 1 \quad (33)$$

```
> limit(exprgensol,t=infinity);
```

$$1 \quad (34)$$

```
>
```

The statement is true.

22.

```
> dsolve(diff(x(t),t)=3*x(t)+t^3);
```

$$x(t) = -\frac{1}{3}t^2 - \frac{1}{3}t^3 - \frac{2}{9}t - \frac{2}{27} + e^{3t}\_C1 \quad (35)$$

```
>
```

Since this is the general solution, _C1 is an arbitrary real constant. It is easy to see that, for _C1=0, the function is a third degree polynomial.