Linear systems with constant coefficients

$\boxed{(1)\ X' = AX, \text{ where } A \in \mathcal{M}_n(\mathbb{R}),\ n \in \mathbb{N}^*}$

Recall from the last lecture:

- the F.T. If $X_1, \dots, X_n$ are n linearly independent solutions of (1), then the general solution of (1) is $X = c_1 X_1 + \dots + c_n X_n$, $c_1, \dots, c_n \in \mathbb{R}$
- we defined the matrix exponential and we showed that e^{At} is the principal matrix of (1) and that the general solution of (1) can be written as $X = e^{At} \cdot c$, $c \in \mathbb{R}^n$

Notation: $\boxed{\lambda_1, \dots, \lambda_n \in \mathbb{C}}$ the eigenvalues of A (counted with their multiplities, thus it is not necessary to be distinct)

$\boxed{u_1, \dots, u_m \in \mathbb{C}^n}$ a set of linearly independent eigenvectors $(m \le n)$

Similar matrices:

Def. Let $A, B \in \mathcal{M}_n(\mathbb{R})$. We say that A is similar to B if there exists $P \in \mathcal{M}_n(\mathbb{R})$ invertible s.t. $A = PBP^{-1}$.

Property. Let $\lambda \in \mathbb{C}$ and $u \in \mathbb{C}^n$ be such that u is an eigenvector of A corresponding to the eigenvalue λ. Then λ is an eigenvalue of B and $P^{-1}u$ is an eigenvector of B corresponding to λ.

The hypothesis is that A is similar to B.

Proof. $Au = \lambda u \quad u \neq 0$

$PBP^{-1}u = \lambda u \quad | \cdot P^{-1}$ to the left

$(P^{-1}P)BP^{-1}u = P^{-1}\underset{\substack{\downarrow \\ \text{scalar}}}{\lambda} u$

$\Rightarrow B \cdot \underbrace{[P^{-1}u]}_{\text{vector}} = \lambda \cdot \underbrace{[P^{-1}u]}_{\text{vector}}$

Since $u \neq 0$ we have $P^{-1}u \neq 0$

$\Rightarrow P^{-1}u$ is an eigenvector of B corresponding to the eigenvalue λ

Property. Let A be similar to B. Then:

(i) $A^k = PB^kP^{-1}$, $\forall k \ge 1$

(ii) $e^{At} = Pe^{Bt}P^{-1}$, $\forall t \in \mathbb{R}$

Proof. (i) $k=1 \quad A = PBP^{-1}$ (def)

$k=2 \quad A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB^2P^{-1}$

by induction assume that $A^k = PB^kP^{-1}$ prove that $A^{k+1} = PB^{k+1}P^{-1}$

indeed $A^{k+1} = A^k \cdot A = (PB^kP^{-1})\cdot(PBP^{-1}) = PB^k(P^{-1}P)BP^{-1} = PB^{k+1}P^{-1}$

(ii) $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}\cdot(At)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \overset{(i)}{=} \sum_{k=0}^{\infty} \frac{t^k}{k!} PB^kP^{-1} = P\cdot\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} B^k\right)P^{-1} =$

$= P\left(\sum_{k=0}^{\infty} \frac{1}{k!}(Bt)^k\right)P^{-1} \overset{def}{=} Pe^{Bt}P^{-1}$

Diagonalizable matrix

Def. We say that $A \in M_n(\mathbb{R})$ is diagonalizable (over $\mathbb{R}$) if there exists a diagonal matrix $\Delta \in M_n(\mathbb{R})$ s.t. A is similar to Δ.

A matrix that is not diagonalizable is said to be defective.

Property. We have that $A \in M_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ if and only if its eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and there exist n eigenvectors $u_1, \dots, u_n \in \mathbb{R}$

Remark. Denote by u_i an eigenvector corresp. to λ_i, $i = \overline{1,n}$.

! From now on we assume that A is diagonalizable.

Property. Let $\Delta = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and P a matrix whose i^{th} column is the eigenvector u_i. Then $A = P\Delta P^{-1}$.

Consequences:

$\Delta^k = \mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k) \Rightarrow A^k = P\,\mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k)P^{-1}$

$e^{\Delta t} \overset{\text{last lecture}}{=} \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \Rightarrow e^{At} = P\cdot\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}$

So, we gave a procedure to find the general sol. of $X' = AX$ in the case that A is diagonalizable.

Step 1. Find the eigenvalues and eigenvectors of A. If $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\exists\, n$ linearly independent eigenvectors, then we conclude that A is diagonalizable.

Step 2. We find Δ and P

Step 3. We find e^{At} using (1)

Step 4. $X = e^{At}C$, $C \in \mathbb{R}^n$

Proposition. Let $A \in M_n(\mathbb{R})$, $\lambda \in \mathbb{R}$ be an eigenvalue of A, $u \in \mathbb{R}^n$ be an eigenvector of A corresponding to λ. Then the function $\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{\lambda t} u$ is a solution of $X' = AX$.

Proof. We have to prove that $\varphi'(t) = A\varphi(t)$, $\forall t \in \mathbb{R}$.

We compute $\varphi'(t) = \underbrace{\lambda e^{\lambda t}}_{\text{scalars}} u = \underbrace{e^{\lambda t}}_{\text{scalar}} (Au) = A(e^{\lambda t} u) = A\varphi(t)$

Hyp $\Rightarrow Au = \lambda u$

The characteristic equation method to find the general solution of $X' = AX$ in the case that A is diagonalizable

Step 1. We find $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ the eigenvalues of A as roots of $\det(A - \lambda I_n) = 0$ (this is called the characteristic equation). We also find the n linearly indep. eigenvectors $u_1, \dots, u_n \in \mathbb{R}^n$

Step 2. We associate n functions: $e^{\lambda_i t} u_i$, $i = \overline{1,n}$

Theorem The n functions are lin. indep. solutions of $X' = AX$.

Step 3. The general sol: $X = c_1 e^{\lambda_1 t} u_1 + \dots + c_n e^{\lambda_n t} u_n$, $c_1, \dots, c_n \in \mathbb{R}$

Example

(i) Prove that the matrix $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable.

(ii) Using the char. eq. method find the general sol. of $X' = AX$.

Solution (i) $\det(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)(-1-\lambda) - 3 = 0$

$\Leftrightarrow \lambda^2 - 1 - 3 = 0 \Leftrightarrow \lambda^2 - 4 = 0 \Leftrightarrow \lambda_1 = -2$ and $\lambda_2 = 2$

Find an eigenvector corresp. to $\lambda_1 = -2$: $Au = -2u$, $u = ?$ $u \in \mathbb{R}^2$

$\Leftrightarrow (A + 2I_2)u = 0 \Leftrightarrow \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow a + b = 0 \overset{\text{choose}}{\Longrightarrow} \begin{matrix} a = 1 \\ b = -1 \end{matrix}$

$u = \begin{pmatrix} a \\ b \end{pmatrix}$ Then $u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ an eigenvector corresp. to $\lambda_1 = -2$

Find an eigenvector corresp to $\lambda_2 = 2$

$Au = 2u \Leftrightarrow (A - 2I_2)u = 0 \Leftrightarrow \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -a + 3b = 0$

We choose $b = 1 \Rightarrow a = 3$

Then $u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector corresp. to $\lambda_2 = 2$

$\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1+3 = 4 \neq 0 \Rightarrow u_1, u_2$ are lin. indep.

We proved that A is diagonalizable

(ii) the general sol. of $\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = x_1 - x_2 \end{cases}$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $c_1, c_2 \in \mathbb{R}$

$\Rightarrow \begin{cases} x_1 = c_1 e^{-2t} + 3c_2 e^{2t} \\ x_2 = -c_1 e^{-2t} + c_2 e^{2t} \end{cases}$ $c_1, c_2 \in \mathbb{R}$

Henri Poincaré

The dynamical system associated to a differential equation

We consider the autonomous system in $\mathbb{R}^n$.

(1) $\dot{x} = f(x)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1- functions.

The unknown is $x(t)$ and $\dot{x}$ is just the notation of Newton for x' used when t is the time.

Keywords: flow, initial state, equilibrium point, orbit, attractor, phase portrait

Theorem (an existence and uniqueness theorem)

Let $\eta \in \mathbb{R}^n$. We have that the IVP $\begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$ has a unique solution denoted by $\varphi(t, \eta)$ which is defined on a minimal interval $I_\eta = (\alpha_\eta, \beta_\eta) \subset \mathbb{R}$

if $\varphi(\cdot, \eta)$ is bounded on $[0, \beta_\eta)$ then $\beta_\eta = +\infty$

if $\varphi(\cdot, \eta)$ is bounded on $(\alpha_\eta, 0]$ then $\alpha_\eta = -\infty$

if $\varphi(\cdot, \eta)$ is bounded on I_η then $I_\eta = \mathbb{R} = (-\infty, +\infty)$

Def. The map $\varphi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, \underset{\text{is the initial state}}{\eta}) \mapsto \underbrace{\varphi(t, \eta)}_{\text{the state of the system at time } t \text{ when it initiated at } \eta}$ is called the flow of (1)

The space $\mathbb{R}^n$ is called the state space.