Planar dynamical systems (continuation)

(1) $\dot{X} = f(X)$ where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

the flow $(t, \eta) \mapsto \varphi(t, \eta)$

the orbit for the initial state $\eta \in \mathbb{R}^2$ is $\gamma_\eta = \{\varphi(t, \eta) : t \in I_\eta\}$

Properties of the flow:

(i) $\varphi(0, \eta) = \eta, \ \forall \eta \in \mathbb{R}^2$

(ii) $\varphi(t, \varphi(s, \eta)) = \varphi(t+s, \eta), \ \forall \eta \in \mathbb{R}^2, \forall t, s$

(iii) the flow is a continuous function in (t, η)

An important property of the orbits:

Let $\eta, \tilde{\eta} \in \mathbb{R}^2$, $\eta \neq \tilde{\eta}$. Then either $\gamma_\eta = \gamma_{\tilde{\eta}}$ or $\gamma_\eta \cap \gamma_{\tilde{\eta}} = \emptyset$.

Remark. ~~Through any point in $\mathbb{R}^2$~~, There exists ~~at least an~~ a unique orbit passing. through any point in $\mathbb{R}^2$.

First integrals for planar systems

$$(1) \iff \begin{cases} \dot{x} = f_1(x, y) \\ \dot{y} = f_2(x, y) \end{cases}$$

Proposition. Let $U \in \mathbb{R}^2$ be open, connected, nonempty and $H \in C^1(U)$ be a non-locally constant function. We have that H is a first integral in U of (1) iff $\frac{\partial H}{\partial x}(x,y) \cdot f_1(x,y) + \frac{\partial H}{\partial y}(x,y) f_2(x,y) = 0, \ \forall (x,y) \in U$.

Proof. H is a first integral in U of (1) $\iff H(\varphi(t, \eta)) = H(\eta), \ \forall \eta \in U, \forall t$ s.t. $\varphi(t, \eta) \in U$

$$\iff \frac{d}{dt} H(\varphi(t, \eta)) = 0, \ \forall \eta \in U, \forall t \text{ s.t. } \varphi(t, \eta) \in U \iff$$

$$\iff \frac{\partial H}{\partial x}(\varphi(t, \eta)) \cdot \dot{\varphi}_1(t, \eta) + \frac{\partial H}{\partial y}(\varphi(t, \eta)) \cdot \dot{\varphi}_2(t, \eta) = 0, \ \forall \eta \in \mathbb{R}^2, \forall t \text{ s.t. } \varphi(t, \eta) \in U$$

$$\iff \frac{\partial H}{\partial x}(\varphi(t, \eta)) \cdot f_1(\varphi(t, \eta)) + \frac{\partial H}{\partial y}(\varphi(t, \eta)) \cdot f_2(\varphi(t, \eta)) = 0, \ \forall \eta \in \mathbb{R}, \forall t \text{ s.t. } \varphi(t, \eta) \in U$$

$$\iff \frac{\partial H}{\partial x} f_1 + \frac{\partial H}{\partial y} f_2 = 0 \text{ in } U$$

A method to find a first integral in U for some systems (1)

Step 1. Write $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$ (2)

Step 2. Integrate the above DE and put the general solution as $H(x,y) = c$, $c \in \mathbb{R}$.

Step 3. Find a domain U for the function H found above and check (using either the definition or the characterization) that, indeed, H is a first integral in U.

A method to integrate eq. (2) in the case that it is separable

i.e. it has the form $\frac{dy}{dx} = g_1(x)\, g_2(y)$

First we separate the variables: $\frac{dy}{g_2(y)} = g_1(x)\, dx$

Then we integrate $\int \frac{dy}{g_2(y)} = \int g_1(x)\, dx$ and obtain $G_2(y) = G_1(x) + c$, $c \in \mathbb{R}$

Now, if it is possible, we simplify the previous, if not we get $H(x,y) = G_2(y) - G_1(x)$.

Examples: Find a global first integral and represent the phase portrait of the following systems.

a) $\begin{cases} \dot{x} = -2y \\ \dot{y} = 3x \end{cases}$ b) $\begin{cases} \dot{x} = x \\ \dot{y} = -3y \end{cases}$

a) $\frac{dy}{dx} = \frac{3x}{-2y}$ this is separable $\Rightarrow$ $2y\, dy = -3x\, dx$

we integrate $\int 2y\, dy = -3 \int x\, dx$ $\Rightarrow$ $y^2 = -\frac{3}{2}x^2 + c$, $c \in \mathbb{R}$

$\Rightarrow H(x,y) = \frac{3}{2}x^2 + y^2$ $U = \mathbb{R}^2$

, we write the pde for first integrals of (1)

, $(-2y) \cdot \frac{\partial H}{\partial x}(x,y) + 3x \cdot \frac{\partial H}{\partial y}(x,y) = 0$

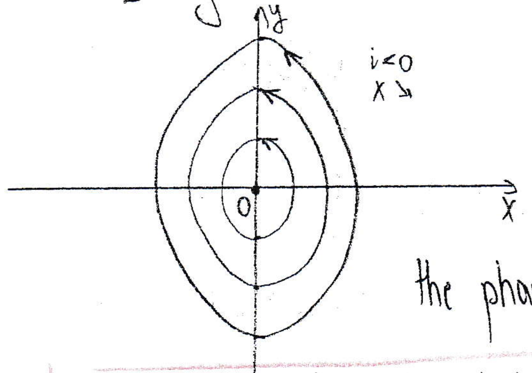
we check $\frac{\partial H}{\partial x} = 3x$ $\frac{\partial H}{\partial y} = 2y$

replace: "$(-2y) \cdot 3x + 3x \cdot 2y = 0$, $\forall (x,y) \in \mathbb{R}^2$"

TRUE $\Rightarrow$ the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = \frac{3}{2}x^2 + y^2$ is a global first integral

Represent the level curves of H: $\frac{3}{2}x^2+y^2=c$, $c\in\mathbb{R}$

$c=1$ $\frac{3}{2}x+y^2=1$ the level curves of H are ellipses



the phase portrait of a)

Since the nontrivial orbits are closed, we deduce that any solution of a) is periodic in time.

b) $\begin{cases} \dot{x}=x \\ \dot{y}=-3y \end{cases}$ the pde for f.i.: $x\frac{\partial H}{\partial x}-3y\frac{\partial H}{\partial y}=0$

$\frac{dy}{dx}=\frac{-3y}{x}$ is separable $\Rightarrow$ $\frac{dy}{y}=-\frac{3\,dx}{x}$

$\int\frac{dy}{y}=-3\int\frac{dx}{x}$

$\ln|y|=-3\ln|x|+c$

$\ln|yx^3|=c$ $\quad yx^3=k$

Take $H(x,y)=yx^3$, $U\in\mathbb{R}^2$

we check "$x\cdot 3x^2y-3y\,x^3=0,\ \forall(x,y)\in\mathbb{R}^2$" TRUE

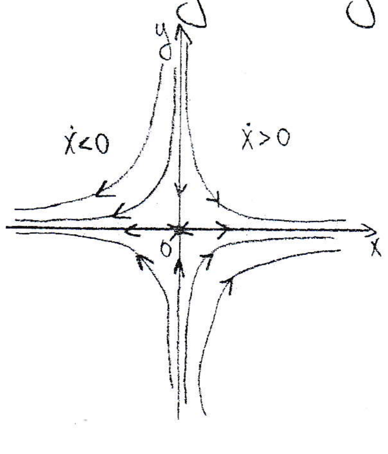
Then H is a global f.i.

the level curves of H

$yx^3=c$

$c=1$ $\quad yx^3=1$ $\quad y=\frac{1}{x^3}$ $\qquad\qquad c=0$ $\quad yx^3=0$

$y=0$ or $x=0$



$f(x)=\frac{1}{x^3}$

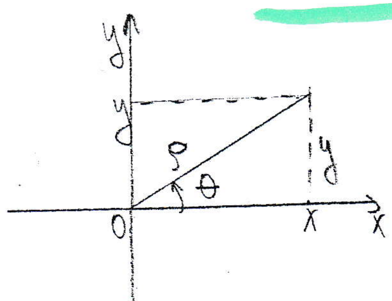
$\lim_{x\to\infty}\frac{1}{x^3}=0$

$\lim_{x\to 0_+}\frac{1}{x^3}=+\infty$

$f'(x)=-3\frac{1}{x^4}<0$

$\begin{cases} \dot{x}=x \\ \dot{y}=-3y \end{cases}$ the only equilibrium is $(0,0)$

Polar coordinates in the plane



Property. For any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ there exists a unique pair $(\rho, \theta) \in (0,\infty) \times [0, 2\pi)$ such that (1) $\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$ $\Leftrightarrow z = x + iy = \rho e^{i\theta}$

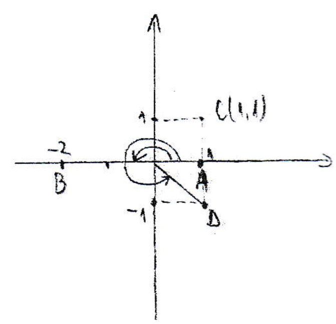
Def. For a point of cartesian coordinates $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we say that (ρ, θ) given by (1) are its polar coordinates.

$$(1) \Longleftrightarrow \begin{cases} \rho = \sqrt{x^2+y^2} \\ \cos\theta = \frac{x}{\rho} \\ \sin\theta = \frac{y}{\rho} \end{cases}$$

Examples. For the following points of given cartesian coordinates, find the polar coordinates:

a) $\begin{cases} x = 1 \\ y = 0 \end{cases}$ $\quad \begin{matrix} \rho = 1 \\ \theta = 0 \end{matrix}$

b) $\begin{cases} x = -2 \\ y = 0 \end{cases}$ $\quad \begin{matrix} \rho = 2 \\ \theta = \pi \end{matrix}$



c) $\begin{cases} x = 1 \\ y = 1 \end{cases}$ $\quad \begin{matrix} \rho = \sqrt{2} \\ \theta = \frac{\pi}{4} \end{matrix}$

d) $\begin{cases} x = 1 \\ y = 1 \end{cases}$ $\quad \begin{matrix} \rho = \sqrt{2} \\ \theta = \frac{7\pi}{4} \end{matrix}$

e) $\begin{cases} x = \eta_1 \cos t + \eta_2 \sin t \\ y = \eta_1 \sin t - \eta_2 \cos t \end{cases}$, $t \in \mathbb{R}$

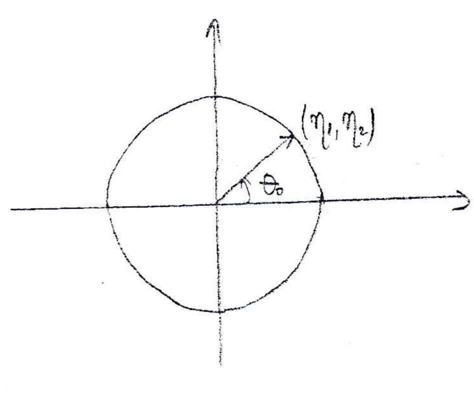
$\rho = \sqrt{\eta_1^2 + \eta_2^2}$

Denote by $\rho_0 = \sqrt{\eta_1^2 + \eta_2^2}$ and $\theta_0 \in [0, 2\pi)$ s.t. $\begin{cases} \eta_1 = \rho_0 \cos\theta_0 \\ \eta_2 = \rho_0 \sin\theta_0 \end{cases}$

$\Rightarrow x = \rho_0 \cos\theta_0 \cos t + \rho_0 \sin\theta_0 \sin t$

$\quad y = \rho_0 \cos\theta_0 \sin t - \rho_0 \sin\theta_0 \cos t$

$\Rightarrow \begin{cases} x = \rho_0 \cos(t - \theta_0) \\ y = \rho_0 \sin(t - \theta_0) \end{cases}$



the polar coordinates are:

$\begin{cases} \rho(t) = \rho_0 \\ \theta(t) = t - \theta_0 \end{cases}$, $\forall t \in \mathbb{R}$

To transform a planar system $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ in polar coordinates means to consider new unknowns $\rho(t)$ and $\theta(t)$ related by $\begin{cases} x(t) = \rho(t)\cos\theta(t) \\ y(t) = \rho(t)\sin\theta(t) \end{cases}$ and find a system in ρ and θ.

Practically, we have to use $\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$

$\begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$ derivating w.r.t. t

$$\Rightarrow \begin{cases} \dot{\rho} = \cos\theta \cdot f_1(\rho\cos\theta, \rho\sin\theta) + \sin\theta \cdot f_2(\rho\cos\theta, \rho\sin\theta) \\ \dot{\theta} = \frac{1}{\rho}\cos\theta \cdot f_2(\rho\cos\theta, \rho\sin\theta) - \frac{1}{\rho}\sin\theta \cdot f_1(\rho\cos\theta, \rho\sin\theta) \end{cases}$$