Exercise

Let $k>0$ and consider the DE $\dot{x} = -k(x-21)$ (this describes the Newton's cooling process)

(i) Find the flow.

(ii) Let $x(t)$ denotes the temperature of a cup of tea at time t that obeys $\dot{x} = -k(x-21)$. An experiment revealed the following fact. If the cup of tea has initially $49°C$, then it has $37°C$ after 10 min. What is the initial temperature of another cup of tea if it has $37°C$ after 20 min.?

The dynamical system associated to a planar system

Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and we consider the planar autonomous system.

$$(1)\quad \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

For any (η with $(\eta_1, \eta_2) \in \mathbb{R}^2$) $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, the IVP

$$(2)\quad \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

has a unique solution

$$t \mapsto \varphi(t, \eta) \in \mathbb{R}^2, \ t \in J_\eta .$$

The function $(t,\eta) \mapsto \varphi(t,\eta)$ is said to be the flow of (1).

$\mathbb{R}^2$ is said to be the state space of (1).

We say that η^* is an equilibrium (or stationary state) of (1) if $\varphi(t, \eta^*) = \eta^*$, $\forall t \in \mathbb{R}$.

Remark: $\boxed{\eta \text{ is an equilibrium point} \iff f(\eta^*)=0.}$ So, in order to find the equilibrium of (1), we have to solve the system $\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$ for $(x,y)\in\mathbb{R}^2$.

For $\eta\in\mathbb{R}^2$ we define its orbit by $\gamma_\eta=\{\varphi(t,\eta): t\in I_\eta\}$.

Remark: η^* is an equilibrium point $\iff \gamma_{\eta^*}=\{\eta^*\}$.

Def. Let η^* be an equilibrium point. We say that η^* is an attractor if $\exists\ V\in\mathcal{V}(\eta^*)$ s.t. $\lim_{t\to\infty}\varphi(t,\eta)=\eta^*$, $\forall\,\eta\in V$.

If η^* is an attractor s.t. $\lim_{t\to\infty}(t,\eta)=\eta^*$, $\forall\eta\in\mathbb{R}^2$, we say that η^* is a global attractor.

If in the definition of an attractor we replace $+\infty$ with $-\infty$, we say that η^* is a repulsor (repellor).

Def. • We say that γ_η is a periodic orbit (closed orbit) when the solution $t\mapsto\varphi(t,\eta)$ is a non trivial periodic function.

Remark. A periodic orbit is a closed curve.

If an orbit is a closed curve, then it is a periodic orbit.

• Let $U\subset\mathbb{R}^2$ be open, nonempty and consider $H: U\to\mathbb{R}$ a C^1 function.

We say that H is a first integral in U of (1) if $H(\varphi(t,\eta))=H(\eta)$, $\forall t$ s.t. $\varphi(t,\eta)\in U$, $\forall\eta\in U$; and H is not locally constant.

A first integral in $\mathbb{R}^2$ of (1) is a global first integral.

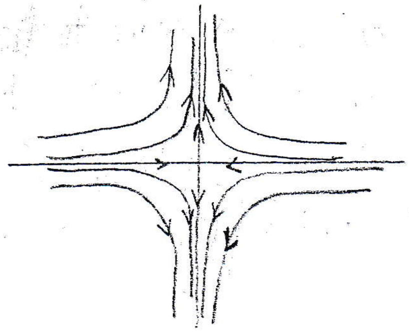
• Let $U\subset\mathbb{R}^2$ be nonempty. We say that U is an invariant for (1) if $\gamma_\eta\subset U$, $\forall\eta\in U$; ~~H not locally constant~~.

Def. • Let $H: U\to\mathbb{R}$ be a continuous function. Let a value $c\in\mathbb{R}$. The c-level curve of H is $\Gamma_c=\{(x,y)\in U: H(x,y)=c\}$.

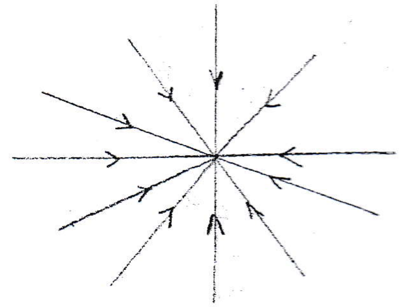
Remark. Let H be a first integral in U of (1) and U be an invariant set of (1). Then, $\forall\eta\in U$ we have that the $\gamma_\eta\subset\Gamma_{H(\eta)}$.

• The phase portrait of (1) is the representation in the state space $\mathbb{R}^2$ of some representative orbits together with an arrow on each orbit that indicates the future.

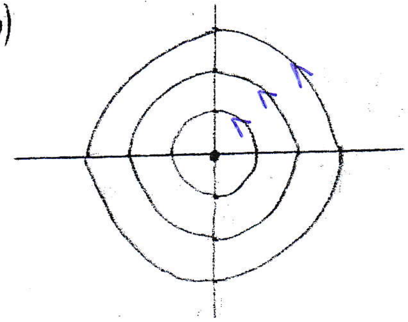
1)



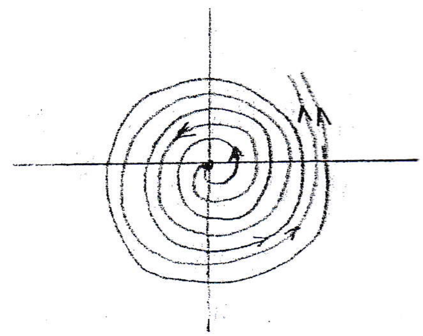
2)



3)



4)



Remark. Systems a) - d) are L.S. $X' = AX$, $X \in \mathcal{M}_2(\mathbb{R})$

The system $X' = AX$ has a unique equilibrium point.

$\eta^* = 0 \iff \det A \neq 0$

Exercise a) Find the equilibrium point, the flow. Prove that it has a global first integral. Represent the phase portrait.

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

$\det A = 1 \neq 0 \Rightarrow (0,0)$ is an equilibrium point

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>OR</u>. solve $\begin{cases} -y = 0 \\ x = 0 \end{cases} \Rightarrow (0,0)$ equilibrium point

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, the IVP $\begin{cases} \dot{x} = -y \\ \dot{y} = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

$\ddot{x} = -\dot{y} = -x \Rightarrow \ddot{x} + x = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \mapsto \cos t, \sin t$

$$\Rightarrow \begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = -\dot{x} = c_1 \sin t - c_2 \cos t \end{cases} \qquad c_1, c_2 \in \mathbb{R}$$

$\Rightarrow \begin{cases} x(0)=c_1 \\ y(0)=c_2 \end{cases} \Rightarrow \varphi(t,\eta_1,\eta_2) = \begin{pmatrix} \eta_1 \cos t - \eta_2 \sin t \\ \eta_1 \sin t + \eta_2 \cos t \end{pmatrix}, \ \forall t \in \mathbb{R}, \ \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$

We check if $H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2+y^2$ is a first integral.

$$H(\varphi(t,\eta)) = (\eta_1 \cos t - \eta_2 \sin t)^2 + (\eta_1 \sin t + \eta_2 \cos t)^2$$
$$= \eta_1^2 \cos^2 t - \cancel{2\eta_1\eta_2 \cos t \sin t} + \eta_2^2 \sin^2 t + \eta_1^2 \sin^2 t + \cancel{2\eta_1\eta_2 \sin t \cos t} + \eta_2^2 \cos^2 t$$
$$= (\eta_1^2+\eta_2^2)\cos^2 t + (\eta_1^2+\eta_2^2)\sin^2 t = \eta_1^2+\eta_2^2 \quad \forall t \in \mathbb{R}.$$

$\Rightarrow H$ is a global first integral

The level curves of H are $x^2+y^2=c$, $c \in \mathbb{R}$; thus they are circles centered in the origin with arbitrary radius. Then the phase portrait is 3).

b) Find the equilibrium point, first integral and prove that the system does not have a global first integral. Represent the phase portrait.

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$

equilibrium points: $\begin{cases} -x=0 \\ -y=0 \end{cases} \Rightarrow (0,0)$ is an equilibrium point

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, the ivp $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \Rightarrow \varphi(t,\eta_1,\eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{-t} \end{pmatrix} \quad \forall \eta \in \mathbb{R}^2$

$H(x,y) = \dfrac{x}{y}$ $\quad U_1 = \mathbb{R} \times (0,\infty)$, $\quad U_2 = \mathbb{R} \times (-\infty, 0)$

$H(\varphi(t,\eta)) = \dfrac{\eta_1 e^{-t}}{\eta_2 e^{-t}} = \dfrac{\eta_1}{\eta_2}$

$\Rightarrow H$ is first integral in U_1, U_2, but it is not a global first integral

Note that $\lim\limits_{t\to\infty} \varphi(t,\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\forall \eta \in \mathbb{R}^2$ $\left(\sim \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \text{global attractor}\right)$

Assume by contradiction that $\exists \bar{H}: \mathbb{R}^2 \to \mathbb{R}$ a global first integral

$\Rightarrow \bar{H}(\varphi(t,\eta)) = \bar{H}(\eta)$, $\forall t \in \mathbb{R}$, $\forall \eta \in \mathbb{R}^2$

$\Rightarrow \lim\limits_{t\to\infty} \bar{H}(\varphi(t,\eta)) = \bar{H}(\eta)$, $\forall \eta \in \mathbb{R}^2$

$\xrightarrow{\bar{H} \text{ is cont.}} \bar{H}(0,0) = \bar{H}(\eta)$, $\forall \eta \in \mathbb{R}^2 \Rightarrow \bar{H}$ is constant in $\mathbb{R}^2$. this contradicts the definition of ~~$\Rightarrow \bar{H}$ is not~~ a first integral.

Phase portrait: $H=\frac{x}{y}$ is first integral $\Rightarrow$ the orbits lie on $\frac{x}{y}=c \iff y=\frac{1}{c}x$
there are lines through the origin $\Rightarrow$ The phase portrait is (2).

c) Find the equilibrium point, the flow. Show that it has a global first integral. Represent the phase portrait.

$$\begin{cases} \dot{x}=-x \\ \dot{y}=y \end{cases}$$

equilibrium points: $\begin{cases} -x=0 \\ y=0 \end{cases} \Rightarrow (0,0)$ is the only equilibrium point

the flow: Let $\eta=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}$ and the ivp $\begin{cases} \dot{x}=-x \\ \dot{y}=+y \\ x(0)=\eta_1 \\ y(0)=\eta_2 \end{cases}$

$\Rightarrow \varphi(t,\eta_1,\eta_2)=\begin{pmatrix}\eta_1 e^{-t}\\ \eta_2 e^{-t}\end{pmatrix}$, $\forall\, t\in\mathbb{R}, \forall\,(\eta_1,\eta_2)\in\mathbb{R}^2$.

Consider $H(x,y)=x\cdot y$ defined on $\mathbb{R}^2$

$H(\varphi(t,\eta))=\eta_1 e^{-t}\cdot\eta_2 e^{t}=\eta_1\eta_2=H(\eta), \forall t\in\mathbb{R}, \eta\in\mathbb{R}^2$

$\Rightarrow H$ is a global first integral

Phase portraits:

The level curves of H: $x\cdot y=c \Rightarrow y=\frac{c}{x}$

Consider $c=1 \Rightarrow y=\frac{1}{x}$

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	$-$	$-$	
$f(x)$		0	∞	0	

When $x>0 \Rightarrow \dot{x}<0 \Rightarrow$ arrow points to the left

When $x<0 \Rightarrow \dot{x}>0 \Rightarrow$ arrow points to the right

The phase portrait is 1).