1. First order scalar linear differential equations

(1) $x' + \underbrace{a(t)}_{\uparrow \text{ the coefficient}} x = f(t)$, where $a, f \in C(I)$, $I \subset \mathbb{R}$ nonempty, open interval

$x' + a(t)x$ is the homogeneous part

$f(t)$ is the non-homogeneous part, or the force

Let $t_0 \in I$, $\eta \in \mathbb{R}$ an Initial Value Problem (IVP)

(2) $\begin{cases} x' + a(t)x = f(t) \\ x(t_0) = \eta \leftarrow \text{initial condition} \end{cases}$

Def 1. A function $\varphi : I \to \mathbb{R}$ is said to be a solution of (1) if $\varphi \in C^1(I)$ s.t. $\varphi'(t) + a(t)\varphi(t) = f(t)$, $\forall t \in I$.

Notations: $C(I) = \{\varphi : I \to \mathbb{R} \text{ continuous}\}$

$C^1(I) = \{\varphi : I \to \mathbb{R} \text{ s.t. } \exists \varphi' \text{ and both } \varphi \text{ and } \varphi' \text{ are continuous}\}$

We will multiply the DE (1) with a function $\mu(t)$ (called integrating factor) such that we will be able to integrate afterwards. This method is called the integrating factor method.

Notation: $A(t) = \int_{t_0}^{t} a(s)\,ds$ this is the primitive of a s.t. $A(t_0)=0$

$A'(t) = a(t)$

Proposition 1. $\mu(t) = e^{A(t)}$ is an integrating factor of (1)

$$x' + a(t)x = f(t) \quad \Big| \cdot e^{A(t)}$$

$$\underbrace{x' \cdot e^{A(t)} + x \cdot a(t) \cdot e^{A(t)}}_{[x \cdot e^{A(t)}]' = f(t) e^{A(t)}} = f(t)\, e^{A(t)}$$

$$[x \cdot e^{A(t)}]' = f(t)\, e^{A(t)} \quad \Big| \int_{t_0}^{t}$$

$$\int_{t_0}^{t} [x(s)\, e^{A(s)}]'\, ds = \int_{t_0}^{t} f(s)\, e^{A(s)}\, ds$$

$$x(s) \cdot e^{A(s)} \Big|_{t_0}^{t} = \int_{t_0}^{t} f(s)\, e^{A(s)}\, ds$$

$$x(t) \cdot e^{A(t)} - x(t_0)\, e^{A(t_0)} = \int_{t_0}^{t} f(s)\, e^{A(s)}\, ds$$

$$x(t) \cdot e^{A(t)} = x(t_0) + \int_{t_0}^{t} f(s) \cdot e^{A(s)}\, ds$$

$$x(t) = x(t_0)\, e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(s) \cdot e^{A(s)}\, ds$$

the general solution: $x = c \cdot e^{-A(t)} + e^{-A(t)} \cdot \int_{t_0}^{t} f(s)\, e^{A(s)}\, ds, \quad c \in \mathbb{R}$

Theorem 1. The IVP (2) has a unique solution, $\varphi(t) = \eta e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(s) e^{A(s)}\, ds$

1.1 First order linear (scalar) homogeneous d.e. (LHDEs)

(3) $x' + a(t)x = 0$

we know the general solution: $x = c \cdot e^{-A(t)}, \quad c \in \mathbb{R}$

Theorem 2. (i) Let x_1 be a solution of (3). Then either $x_1(t) = 0, \forall t \in I$ or $x_1(t) \neq 0 \quad \forall t \in I$.

(ii) Let x_1 be a non-null solution of (3). Then the general solution of (3) is $x = c \cdot x_1, \quad c \in \mathbb{R}$.

Proof. (i) x_1 sol. of (3) $\Rightarrow \exists c_1 \in \mathbb{R}$ s.t. $x_1 = c_1 \cdot e^{-A(t)}$

we have that either $c_1 = 0$ or $c_1 \neq 0$

Then it is easy to see the conclusion.

(ii) $x_1 = c_1 \cdot e^{-A(t)}, \quad c_1 \neq 0$

the general sol. is $x = k \cdot e^{-A(t)}$, $k \in \mathbb{R}$ arbitrary. ~~a solution of (1) if $\varphi \in C^1(I)$~~

$$x = k \cdot e^{-A(t)} = \frac{k}{c_1} \cdot x_1 = c \cdot x_1, \quad c \in \mathbb{R}$$

$\left\{ \frac{k}{c_1} : k \in \mathbb{R} \right\} = \mathbb{R}$

$c \stackrel{not}{=} \frac{k}{c_1}$

The separation of variables method to solve (3)

$x'(t) = -a(t)\, x(t)$ $\qquad$ $x = 0$ is a solution

now we want to find the non-null solutions

$$\frac{x'(t)}{x(t)} = -a(t) \quad \Big| \int_{t_0}^{t}$$

$$\int_{t_0}^{t} \frac{x'(s)}{x(s)}\, ds = -\int_{t_0}^{t} a(s)\, ds$$

$$\ln |x(s)| \Big|_{t_0}^{t} = -A(t)$$

$$\ln |x(t)| - \ln |x(t_0)| = -A(t)$$

$$\ln \left| \frac{x(t)}{x(t_0)} \right| = -A(t)$$

$$\ln \frac{x(t)}{x(t_0)} = -A(t) \quad \text{because } \frac{x(t)}{x(t_0)} > 0 \quad \forall t \in I$$

$$\frac{x(t)}{x(t_0)} = e^{-A(t)}$$

$$x(t) = \underbrace{x(t_0)}_{c} \cdot e^{-A(t)}$$

$$x = c \cdot e^{-A(t)}, \quad c \in \mathbb{R}$$

Short-cut separation of variables method

$x' = -a(t)\, x$ $\qquad$ $x' = \frac{dx}{dt}$

$x = 0$ solution

$$\frac{dx}{dt} = -a(t)\, x$$

$$\frac{dx}{x} = -a(t)\, dt \quad \Big| \int$$

$$\int \frac{dx}{x} = -\int a(t)\, dt$$

$$\ln |x| = -A(t) + c, \quad c \in \mathbb{R}$$

$$|x| = e^{-A(t) + c}$$

$$\begin{cases} x = \pm e^{c} \cdot e^{-A(t)}, & c \in \mathbb{R} \\ x = 0 & \text{solution} \end{cases}$$

$\{0, e^c, -e^{-c} : c \in \mathbb{R}\} = \mathbb{R}$

$x = k \cdot e^{-A(t)}, \quad k \in \mathbb{R}$

Conclusion:

4 methods for eq. (3):

1) "Guess" a non-null solution, x_1, and write the general solution as $x = c \cdot x_1, \quad c \in \mathbb{R}$

2) The separation of variables method

3) The integrating factor method

4) Memorize $x = c \cdot e^{-A(t)}$, where $A'(t) = a(t)$

Examples:

Find the general solution of:

a) $x' = \lambda x \qquad \lambda \in \mathbb{R}^*$ parameter

$x_1 = e^{\lambda t}$ a non-null sol. $\Rightarrow$ the gen. sol. is $x = c \cdot e^{\lambda t}, \ c \in \mathbb{R}$

b) $t \cdot x' + 2x = 0$

Method 1. The integrating factor method

$\mu(t) = e^{A(t)} \qquad A'(t) = a(t)$

$x' + \frac{2}{t} x = 0 \qquad a(t) = \frac{2}{t}, \quad t \neq 0 \qquad I$ can be either $(-\infty, 0)$ or $(0, \infty)$

$A(t) = 2\ln|t| = \ln t^2 \quad \Rightarrow \quad \mu(t) = e^{\ln t^2} = t^2$

$x' + \frac{2}{t} x = 0 \quad | \cdot t^2$

$t^2 x' + 2t x = 0$

$(t^2 x)' = 0$

$t^2 x = c \qquad x = \frac{c}{t^2}, \ c \in \mathbb{R}$

Method 2. Separation of variables

$x' = -\frac{2}{t} x \qquad x = 0$ sol.

$x \neq 0$

$\frac{dx}{dt} = -\frac{2}{t} x$

$\frac{dx}{x} = -\frac{2}{t}\,dt$

$\ln|x| = -2\ln|t| + c$

$\ln|x| = \ln\frac{1}{t^2} + c$

$|x| = e^{\ln\frac{1}{t^2}+c}$

$x = \pm e^{c}\cdot e^{\ln\frac{1}{t^2}}$

$$\begin{cases} x = \pm e^{c}\cdot\frac{1}{t^2}, & c\in\mathbb{R} \\ x = 0 \end{cases}$$

$x = c\cdot\frac{1}{t^2}, \quad c\in\mathbb{R}$

Method 3

(i) Check that $x_1 = \frac{1}{t^2}$ is a sol

(ii) Find the general sol $\quad x = c\cdot\frac{1}{t^2}, \quad c\in\mathbb{R}$

1.2. First order scalar linear non-homogeneous DE (LNDE)

(1) $x' + a(t)x = f(t)$

Theorem. Let x_h denote the general solution of the LHDE associated, $x' + a(t)x = 0$.

Let x_p denote a particular solution of (1). Then the general solution of (1) is: $\boxed{x = x_h + x_p}$

Proof. $x_p' + a(t)x_p = f(t), \ \forall t\in I \quad (*)$

φ is an arbitrary sol. of (1) $\Leftrightarrow \varphi' + a(t)\varphi = f(t) \overset{(*)}{\Longleftrightarrow} \varphi' + a(t)\varphi - \underbrace{[x_p' + a(t)x_p]}_{=0}$

$\Leftrightarrow (\varphi - x_p)' + a(t)[\varphi - x_p] = 0 \Leftrightarrow \varphi - x_p$ is a sol. of $x' + a(t)x = 0$

So, to solve (1):

Step 1. Write the LHDE associated $x' + a(t)x = 0$

Step 2. Find its general sol, denoted x_h.

Step 3. Find x_p.

Step 4. The gen. sol. of (1) $\quad x = x_h + x_p$

The Lagrange method (the variation of constant) to find x_p.

Step 2 $\quad x_h = c \cdot e^{-A(t)}, \quad c \in \mathbb{R}$

Step 3 $\quad$ Find x_p

$x_p = \varphi(t) \cdot e^{-A(t)} \qquad \varphi = ?$

$\varphi' e^{-A(t)} + \cancel{\varphi[-a(t)] \cdot e^{-A(t)}} + \cancel{a(t) \cdot \varphi \cdot e^{-A(t)}} = f(t) \quad | \cdot e^{A(t)}$

$\varphi' = f(t) \cdot e^{A(t)}$

$\varphi(t) = \int_{t_0}^{t} f(s) \cdot e^{A(s)} \, ds \quad \Rightarrow \quad x_p = e^{-A(t)} \cdot \int_{t_0}^{t} f(s) \cdot e^{A(s)} \, ds$

Rules to find x_p in some particular situations

1) The eq $\; x' - \lambda x = \alpha, \; \lambda \in \mathbb{R}^*, \; \alpha \in \mathbb{R}$ $\;$ has a constant solution

2) The eq $\; x' - \lambda x = \alpha \cdot e^{bt}, \; \lambda \in \mathbb{R}^*, \; b \neq \lambda$ $\;$ has a sol. of the form $x_p = a \cdot e^{bt}, \; a = ?$

3) The eq. $\; x' - \lambda x = \alpha \cdot e^{\lambda t}$ has a solution of the form $x_p = a t e^{\lambda t}$ $\; a = ?$
has a sol. of the form $x_p = a t e^{\lambda t} \quad a = ?$

4) The eq. $\; x' - \lambda x = \alpha_1 \cdot t + \alpha_2 \quad \lambda \in \mathbb{R}^*$ $\;$ has a sol. of the form $x_p = a_1 t + a_2 \qquad a_1, a_2 = ?$

Scalar linear differential equations

Let $n \geq 1$ be an integer.
the unknown $\quad t \in \mathbb{R} \longmapsto x(t) \in \mathbb{R}$

(1) $\; x^{(n)} + a_1(t) x^{(n-1)} + \ldots + a_{n-1}(t) x' + a_n(t) x = f(t)$

where $a_1, \ldots, a_n, f \in C(I)$, $I \subset \mathbb{R}$ open, nonempty interval
$a_1, \ldots, a_n$ are the coefficients, f is the force (or the non-homogeneous part)
When $a_1, \ldots, a_n$ are constant functions, we say that equation (1) has constant coefficients.