Seminar 4 09-04-2019

Scalar dynamical systems

Let $f \in C^1(\mathbb{R})$ and consider the DE $\dot{x} = f(x)$ (1).
Here the unknown is denoted by $x(t)$ and $\dot{x}$ is the Newton's notation for x'. The ~~real~~ variable t has the significance of time.

The $\exists!$ Theorem ~~...~~ (presented in the last lecture) assures that $\forall \eta \in \mathbb{R}$ the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$ has a unique solution, denoted $\varphi(t, \eta)$, which is defined on an open interval $I_\eta = (\alpha_\eta, \beta_\eta)$.
The map $(t, \eta) \mapsto \varphi(t, \eta)$ is said to be the flow of (1) and $\mathbb{R}$ is said to be the state space of (1).

Definitions 1) $\eta^* \in \mathbb{R}$ is said to be an equilibrium point of (1) when $\varphi(t, \eta^*) = \eta^*$, $\forall t \in \mathbb{R}$.

2) For each initial state $\eta \in \mathbb{R}$ we define its orbit by
$$\gamma_\eta = \{ \varphi(t, \eta) : t \in (\alpha_\eta, \beta_\eta) \}$$
its positive orbit by $\gamma_\eta^+ = \{ \varphi(t, \eta) : t \in (0, \beta_\eta) \}$
and its negative orbit by $\gamma_\eta^- = \{ \varphi(t, \eta) : t \in (\alpha_\eta, 0) \}$.

Remarks 1) η^* is an equilibrium point if and only if $f(\eta^*) = 0 \iff \eta^*$ is a constant solution of $\dot{x} = f(x) \iff \gamma_{\eta^*} = \{\eta^*\}$.

2) γ_η is the image of the function $\varphi(\cdot, \eta)$.
γ_η^+ contains all the future states of the system when ... while γ_η^- contains all the past states.

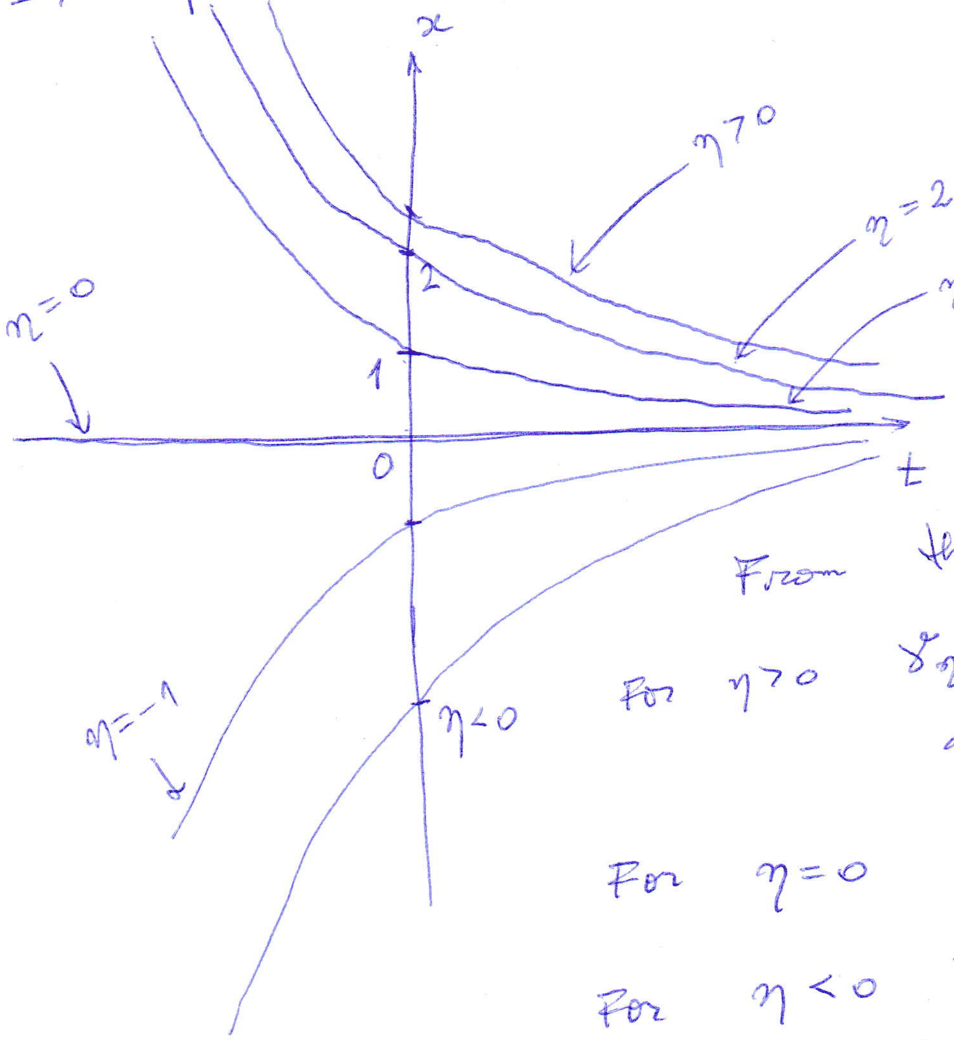
Example. Find the equilibrium points, the ~~orbits~~ flow and the orbits of the dynamical system $\dot{x} = -x$.

Solution.

$\eta^* = 0$ is the only eq. point

Let $\eta \in \mathbb{R}$ and consider the IVP $\dot{x} = -x$, $x(0) = \eta$.

$\Rightarrow \quad \varphi(t, \eta) = \eta e^{-t}, \quad \forall t \in \mathbb{R}, \ \forall \eta \in \mathbb{R}$.



we have

$\varphi(t, 1) = e^{t}$

$\varphi(t, 2) = 2e^{t}$

$\varphi(t, 0) = 0$

$\varphi(t, -1) = -e^{t}$

From ~~these~~ these graphs $\Rightarrow$

For $\eta > 0$ $\quad \gamma_\eta = (0, \infty)$, $\gamma_\eta^+ = (0, \eta)$ and $\gamma_\eta^- = (\eta, +\infty)$

For $\eta = 0$ $\quad \gamma_\eta = \gamma_\eta^+ = \gamma_\eta^- = \{0\}$

For $\eta < 0$ $\quad \gamma_\eta = (-\infty, 0)$, $\gamma_\eta^+ = (\eta, 0)$ and $\gamma_\eta^- = (-\infty, \eta)$.

Definition The phase portrait of $\dot{x} = f(x)$ is the representation on the real line ($\mathbb{R}$) of all its ~~or~~ orbits, together with an arrow on ~~its or~~ each orbit that indicates the future.

Ex: Represent the phase portrait of $\dot{x} = -x$.

the orbits are: $(-\infty, 0)$, $\{0\}$, $(0, \infty)$

Note also that for $\eta > 0$ $\varphi(\cdot, \eta)$ is strictly decreasing and this is reflected in the phase portrait by the fact that on the interval $(0, \infty)$ the arrow indicates to the left.
For $\eta < 0$, the sol. $\varphi(\cdot, \eta)$ is strictly increasing and this is reflected - - - - ,, — the arrow . . . . right.

An algorithm to represent the phase portrait of $\dot{x} = f(x)$.

<u>Step 1.</u> Find all the equilibrium points, that is, find $x \in \mathbb{R}$ s.t.

$$f(x) = 0.$$

<u>Step 2.</u> Find the sign of f on each interval delimited on $\mathbb{R}$ by the equilibrium points.

<u>Step 3.</u> Represent on the real line the orbits corresponding to the equilibrium points and the other orbits. The other orbits are precisely the intervals delimited by the eq. points. On each orbit insert an arrow following the rules:

- if $f > 0$ on that orbit, the arrow indicates to the right
- if $f < 0$ — ,, — ,, — — left.

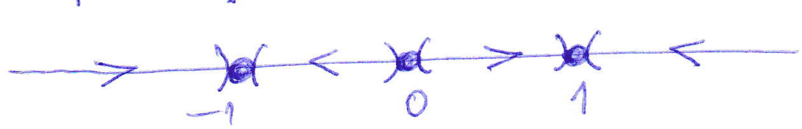
<u>Ex.</u> Represent the phase portrait of $\dot{x} = x - x^3$.

<u>Step 1.</u> $x - x^3 = 0$ The eq. points are $-1, 0, 1$.

<u>Step 2.</u>

$$+ \quad -1 \quad - \quad 0 \quad + \quad 1 \quad -$$

~~and~~ <u>Step 3.</u>



the orbits are: $(-\infty, -1), \{-1\}, \{0\}, (0, 1), \{1\}, (1, \infty)$

$(-1, 0)$

Ex. Reading the phase portrait of $\dot{x} = x - x^3$ describe the properties of ~~the solution of the IVP~~ $\varphi(t,1)$, $\varphi(t,2)$ and $\varphi(t,\frac{1}{2})$ and of the solution of ~~the IVP~~ each of the IVP's:

a) $\begin{cases} \dot{x} = x - x^3 \\ x(0) = 1 \end{cases}$, b) $\begin{cases} \dot{x} = x - x^3 \\ x(0) = 2 \end{cases}$, c) $\begin{cases} \dot{x} = x - x^3 \\ x(0) = \frac{1}{2} \end{cases}$.

Solution. First note that, by definition, $\varphi(t,1)$ is the unique sol. of the IVP a); $\varphi(t,2)$ is the unique solution of the IVP b) and $\varphi(t,\frac{1}{2})$ is the unique sol. of c).

- Since 1 is an eq. point we have $\varphi(t,1) = 1$, $\forall t \in \mathbb{R}$.
  So, $\varphi(t,1)$ is just a constant function.

- Since $2 \in (1,\infty)$ and $(1,\infty)$ is an orbit we have that $\gamma_2 = (1,\infty)$. Then the image of the function $\varphi(t,2)$ is $(1,\infty)$. Since the arrow on $(1,\infty)$ indicates to the left, we deduce that $\varphi(t,2)$ is strictly decreasing.
  Then $\boxed{\lim_{t\to\infty} \varphi(t,2) = 1}$.

- Since $\frac{1}{2} \in (0,1)$ and $(0,1)$ is an orbit we have that $\gamma_{\frac{1}{2}} = (0,1)$. Then the image of $\varphi(t,\frac{1}{2})$ is $(0,1)$, thus $\varphi(t,\frac{1}{2})$ is bounded. Since the arrow on $(0,1)$ indicates to the right, we deduce that $\varphi(t,\frac{1}{2})$ is strictly increasing. Then $\boxed{\lim_{t\to\infty} \varphi(t,\frac{1}{2}) = 1}$
  and $\boxed{\lim_{t\to-\infty} \varphi(t,\frac{1}{2}) = 0}$.

Definitions Let $\eta^* \in \mathbb{R}$ be an equilibrium point of $\dot{x} = f(x)$.
We say that η^* is an attractor when $\exists$ a neighborhood V of η^* s.t. $\lim_{t \to \infty} \varphi(t, \eta) = \eta^*$, $\forall \eta \in V$.

We say that η^* is a repulsor when $\exists$ a ~~set~~ neighborhood V of η^* s.t. $\lim_{t \to -\infty} \varphi(t, \eta) = \eta^*$, $\forall \eta \in V$.

If η^* is an attractor, we define its basin of attraction as

$$A_{\eta^*} = \{ \eta \in \mathbb{R} \text{ s.t. } \lim_{t \to \infty} \varphi(t, \eta) = \eta^* \}.$$

Example Reading the phase portrait of $\dot{x} = x - x^3$ we conclude that:

The eq. p. -1 is an attractor and $A_{-1} = (-\infty, 0)$

The eq. p. 0 is a repulsor.

The eq. p. 1 is an attractor and $A_1 = (0, \infty)$.

Theorem [The linearization method]

Let $\eta^* \in \mathbb{R}$ be an eq. point of $\dot{x} = f(x)$.

If $f'(\eta^*) < 0$ then η^* is an attractor.

If $f'(\eta^*) > 0$ then η^* is a repulsor.

Exercises 1) Let (i) $\dot{x} = x - x^3$, (ii) $\dot{x} = x^2 - 2x + 1$, (iii) $\dot{x} = \sin x$.
Use the linearization method to decide whether the equilibrium points are attractors or repulsors.
Represent the phase portraits of (ii) and (iii).

2) Let $d \in \mathbb{R}$ be a fixed parameter. Represent the phase portrait of $\dot{x} = d - x^2$.