Math & Music: A Topological Data Analysis of Harmonic Structure in Beethoven's Ninth (IV)

Quella Wang

December 9, 2025

Abstract

We study harmony and local voice leading in the fourth movement of Beethoven's Ninth by building precise mathematical objects on top of symbolic music data. Chords are encoded as pitch–class (PC) sets with auxiliary features (onset, duration, size), yielding a point cloud in a metric space. We summarize the large-scale organization of this cloud with the *Mapper* construction (a nerve of a cover pulled back by a lens), and we quantify multi-scale connectivity and cycles via persistent homology and Betti numbers. Simple, explicit examples introduce simplicial complexes, chain complexes over $\mathbb{Z}_2$, homology, filtrations, and the nerve theorem; then we detail our pipeline (encoding $\to$ lens/cover/cluster $\to$ Mapper; and Rips filtrations $\to$ barcodes and persistence diagrams). Interpreting loops and flares in the Mapper graph together with long-lived H_1 classes in barcodes reveals harmonic neighborhoods and recurring progressions in Beethoven 9/IV. We conclude with musical implications, learning outcomes, and methodological next steps.

1 Musical Problem and Project Overview

Motivating musical question

We ask: *how are the harmony and local voice leading in Beethoven 9/IV organized at scale?* Instead of analyzing a few measures, we want a geometric and topological picture of the entire movement's chordal language and recurring progressions.

Encoding musical events

Each musical *event* (e.g. a simultaneity in the MIDI) is encoded as a vector

$$x = \big(\text{PC-set, size, onset, duration, root PC}, \dots\big) \in \mathbb{R}^d,$$

where the PC-set is stored as a 12-dimensional $\{0, 1\}$ indicator and the remaining entries are real-valued contextual features. A distance (e.g. Hamming or Jaccard distance on PC indicators plus small penalties for temporal separation and voice-leading cost) turns the corpus into a finite metric space (X, ρ).

Topological pipeline

On top of this metric data we:

- build a 2D *lens* (UMAP or PCA) and a rectangular cover with overlap;

- cluster preimages of cover elements; take the nerve to obtain a Mapper graph;
- build Vietoris–Rips filtrations of (X, ρ) and compute persistent homology over $\mathbb{Z}_2$ to obtain Betti numbers, barcodes, and persistence diagrams.

This yields (i) a geometric "map" of harmonic neighborhoods and (ii) multi-scale topological summaries of connectivity (β_0) and cycles (β_1) that we can interpret musically.

2 Mathematical Background with Concrete Examples

Throughout this section all homology is taken with coefficients in $\mathbb{Z}_2 = \{0, 1\}$, so $1 + 1 = 0$ and orientations do not matter.

2.1 Metric Data, Covers, and Nerves

A *metric space* (M, ρ) is a set M equipped with a distance $\rho : M \times M \to \mathbb{R}_{\geq 0}$ satisfying non-negativity, symmetry, and the triangle inequality. In TDA we typically start from a finite subset $X = \{x_i\} \subset M$ with pairwise distances $\rho(x_i, x_j)$.

Definition 1 (Cover and Nerve)**.** *Let Y be a topological space. A* cover *of Y is a family of sets $\mathcal{U} = \{U_i\}_{i \in I}$ such that $Y = \bigcup_{i \in I} U_i$. The* nerve *$\mathcal{C}(\mathcal{U})$ is the abstract simplicial complex with*

- *one vertex for each U_i;*
- *a k-simplex $[U_{i_0}, \ldots, U_{i_k}]$ whenever $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$.*

Example 1 (Cover of a short chord progression)**.** *Let Y be four chord-events in B9/IV:*

$$V \to I \to vi \to IV.$$

Suppose we have three overlapping "windows in time":

$$U_1 = \{V, I\}, \quad U_2 = \{I, vi\}, \quad U_3 = \{vi, IV\}.$$

These three windows cover Y. Every adjacent pair intersects, but $U_1 \cap U_3 = \emptyset$. The nerve has three vertices (for U_1, U_2, U_3) and two edges

$$[U_1, U_2], \quad [U_2, U_3],$$

forming a short path. This path encodes the temporal ordering of the progression at a coarse level.

Theorem 1 (Nerve Theorem (informal))**.** *If every set U_i in $\mathcal{U}$ and every finite intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is "simple" (e.g. convex or contractible), then the nerve $\mathcal{C}(\mathcal{U})$ has the same homotopy type as the union $\bigcup_i U_i$. In particular, they have isomorphic homology groups.*

Example 2 (Tiny circle cover)**.** *Let Y be a circle drawn with points, and let U_1, U_2, U_3 be three overlapping arcs that collectively cover it. Every pair intersects, and all three intersect, so the nerve is a filled triangle (a 2-simplex). Under the theorem's hypotheses, the filled triangle is homotopy equivalent to the union of the three arcs; they share the same "one-hole" topology as the circle's cover union.*

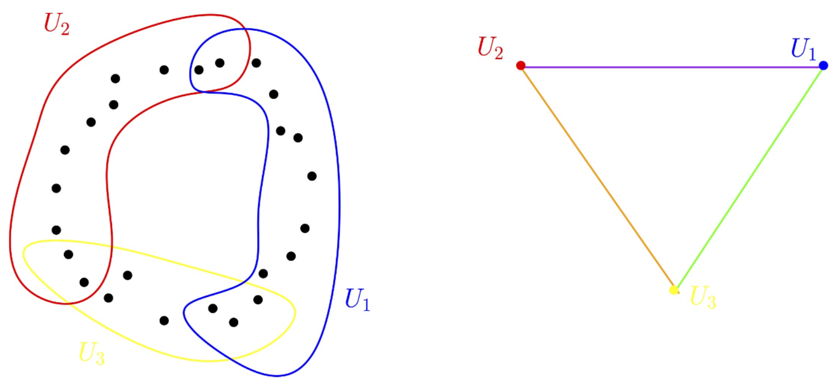

Figure 1: A cover of a point cloud by overlapping sets and its nerve. Think of the sets as chord neighborhoods and the nerve as the "skeleton" of their overlaps.

2.2 Abstract Simplicial Complexes and Simplex Examples

Definition 2 (Abstract Simplicial Complex)**.** *Let V be a set. An* abstract simplicial complex *K on V is a family of finite subsets of V (called* simplices*) such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. A k-simplex is a simplex with $k+1$ vertices.*

Example 3 (0-, 1-, and 2-simplices)**.**

- *A* 0*-simplex is a single vertex, e.g.* $\{a\}$.
- *A* 1*-simplex is an unordered edge, e.g.* $[a, b] = \{a, b\}$.
- *A* 2*-simplex is a filled triangle* $[a, b, c] = \{a, b, c\}$.

If K contains

$$\{a\}, \{b\}, \{c\}, \quad [a, b], [b, c], [a, c]$$

but not $\{a, b, c\}$*, then K is just the boundary of a triangle: one loop, no filled interior. If we* add *the* 2*-simplex $\{a, b, c\}$, we fill in the interior and the loop disappears.*

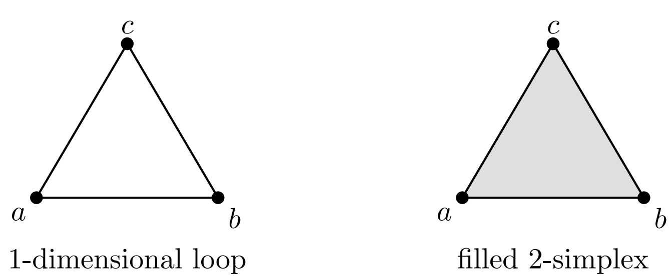

Figure 2: Left: three vertices and three edges form a 1-dimensional loop. Right: adding the 2-simplex fills the loop.

2.3 Rips and Čech Complexes from Metric Data

Given X in a metric space (M, ρ) and a scale $\alpha > 0$:

Definition 3 (Vietoris–Rips and Čech complexes)**.** *The* Vietoris–Rips *complex* $\mathrm{Rips}_\alpha(X)$ *has a k-simplex on $\{x_0, \ldots, x_k\}$ whenever all pairwise distances satisfy $\rho(x_i, x_j) \leq \alpha$.*

The Čech *complex* $\mathrm{Cech}_\alpha(X)$ *has a k-simplex when the closed balls $B(x_i, \alpha)$ have a nonempty common intersection.*

They satisfy

$$\mathrm{Rips}_\alpha(X) \subseteq \mathrm{Cech}_\alpha(X) \subseteq \mathrm{Rips}_{2\alpha}(X),$$

so Rips complexes are safe approximations for Čech complexes.

Example 4 (Three harmonically close chords)**.** *Take three chord-events x_0, x_1, x_2 in B9/IV that are pairwise very similar (e.g. IV, ii^6, V^6 with strong pitch overlap).*

If the distance threshold α is larger than all three pairwise distances, then the triangle $[x_0, x_1, x_2]$ appears in $\mathrm{Rips}_\alpha(X)$*: these three events form a harmonic "face" of the space.*

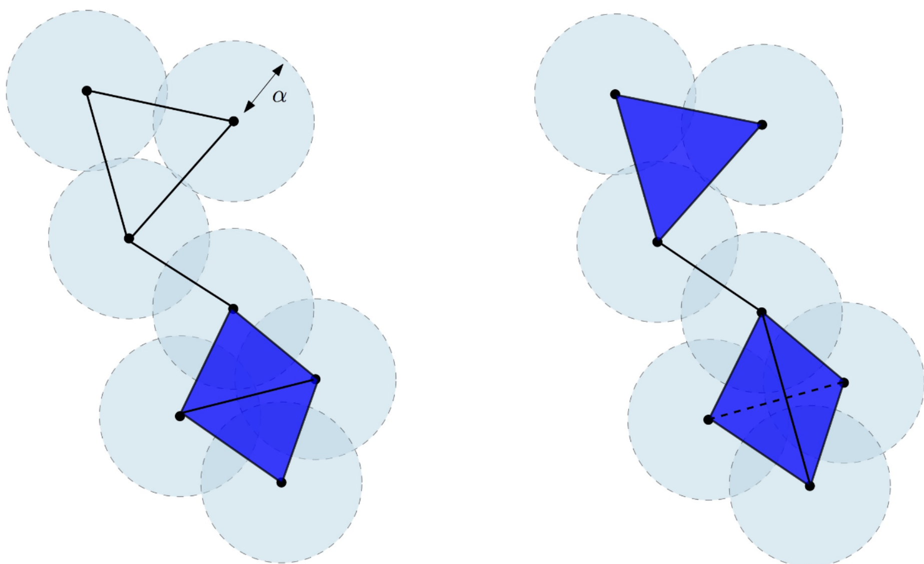

Figure 3: Left: a Čech complex built from balls of radius α. Right: the corresponding Rips complex at scale 2α.

2.4 Chains, Chain Complexes, Homology, and Betti Numbers

We now give a very concrete description of homology over $\mathbb{Z}_2$ using a single triangle.

Definition 4 (Chain groups over $\mathbb{Z}_2$)**.** *Let K be a simplicial complex. For each $k \geq 0$, the group of k*-chains *$C_k(K)$ is the $\mathbb{Z}_2$–vector space spanned by all k-simplices of K.*

Concretely, a k-chain is a finite sum

$$c = \sum_i \varepsilon_i \sigma_i, \qquad \varepsilon_i \in \mathbb{Z}_2,$$

where each σ_i is a k-simplex. Over $\mathbb{Z}_2$, adding chains corresponds to taking the symmetric difference of the underlying sets of simplices.

Definition 5 (Boundary maps and chain complex)**.** *The* boundary map $\partial_k : C_k(K) \to C_{k-1}(K)$ *is the linear map defined on a k-simplex* $\sigma = [v_0, \ldots, v_k]$ *by*

$$\partial_k(\sigma) = \sum_{i=0}^{k} [v_0, \ldots, \widehat{v_i}, \ldots, v_k],$$

where $\widehat{v_i}$ *means we omit* v_i*.*

The family

$$\cdots \xrightarrow{\partial_{k+1}} C_k(K) \xrightarrow{\partial_k} C_{k-1}(K) \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

is a chain complex*: it satisfies*

$$\partial_{k-1} \circ \partial_k = 0 \quad \textit{for all } k.$$

Example 5 (Explicit chain complex of one triangle)**.** *Let* K *be a single filled triangle with vertices* a, b, c*:*

$$\textit{vertices: } a, b, c; \quad \textit{edges: } [a, b], [b, c], [a, c]; \quad \textit{face: } [a, b, c].$$

Then over $\mathbb{Z}_2$*,*

$$C_2(K) = \langle [a, b, c] \rangle \cong \mathbb{Z}_2, \quad C_1(K) = \langle [a, b], [b, c], [a, c] \rangle \cong \mathbb{Z}_2^3, \quad C_0(K) = \langle a, b, c \rangle \cong \mathbb{Z}_2^3.$$

The boundary maps are:

$$\partial_2([a, b, c]) = [b, c] + [a, c] + [a, b],$$

$$\partial_1([a, b]) = b + a, \quad \partial_1([b, c]) = c + b, \quad \partial_1([a, c]) = c + a.$$

A direct check shows $\partial_1(\partial_2([a, b, c])) = 0$*, so indeed* $\partial_1 \circ \partial_2 = 0$*.*

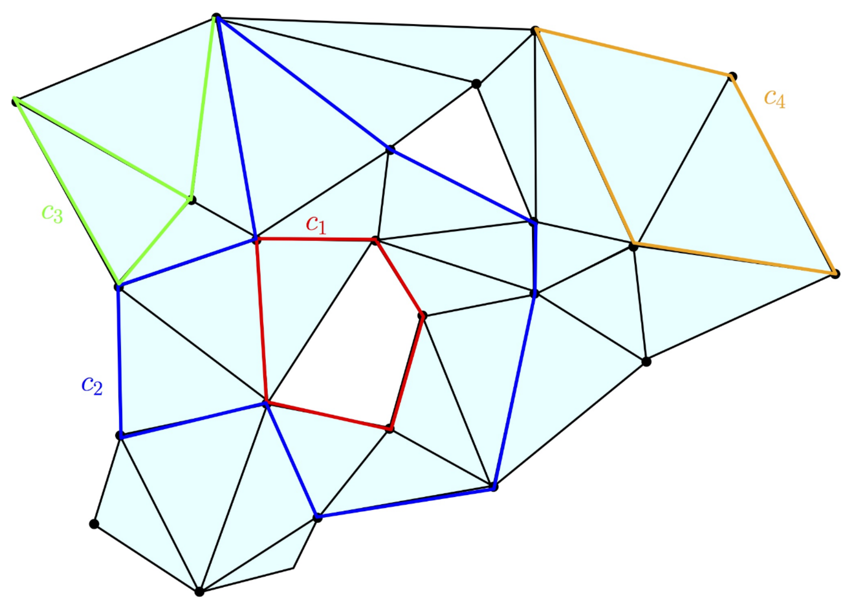

Figure 4: Examples of chains, cycles, and boundaries on a 2D complex. Over $\mathbb{Z}_2$, chains are sets of simplices; cycles are closed chains; boundaries are those that bound higher-dimensional chains.

Definition 6 (Cycles, boundaries, homology, Betti numbers)**.** *Given a chain complex* (C_k, ∂_k)*, define:*

$$Z_k(K) = \ker \partial_k \quad \textit{(space of k-cycles)}, \qquad B_k(K) = \operatorname{im} \partial_{k+1} \quad \textit{(space of k-boundaries).}$$

The kth homology group *of* K *is*

$$H_k(K) = Z_k(K)/B_k(K),$$

and its dimension

$$\beta_k(K) = \dim_{\mathbb{Z}_2} H_k(K)$$

is the kth Betti number*. Intuitively:*

$\beta_0 =$ *number of connected components,* $\quad \beta_1 =$ *number of independent loops,* $\quad \beta_2 =$ *number of voids,* $\ldots$

Example 6 (Homology of one triangle, worked out)**.** *For the single filled triangle* K *above:*

- $Z_2(K) = \ker \partial_2 = \langle [a,b,c] \rangle$ *(the whole 2-chain is a cycle), and* $B_2(K) = 0$*, so* $H_2(K) \cong \mathbb{Z}_2$ *and* $\beta_2 = 1$*.*
- $B_1(K) = \operatorname{im} \partial_2 = \langle [a,b] + [b,c] + [a,c] \rangle$*. One checks that any* 1*-cycle is a sum of edges whose endpoints appear an even number of times; for this complex all such cycles are boundaries, so* $H_1(K) = 0$ *and* $\beta_1 = 0$*.*
- $Z_0(K) = \ker \partial_0 = C_0(K)$*,* $B_0(K) = \operatorname{im} \partial_1$ *has rank* 2*, so* $H_0(K) \cong \mathbb{Z}_2$ *and* $\beta_0 = 1$*.*

Thus a single filled triangle has $(\beta_0, \beta_1, \beta_2) = (1, 0, 1)$*: one connected piece and one 2D "cap".*

Example 7 (Catalogue of tiny complexes)**.**

Complex	$(\beta_0, \beta_1, \beta_2)$	*Interpretation*
Two isolated vertices	$(2, 0, 0)$	*two separate clusters*
Triangle boundary (3 edges)	$(1, 1, 0)$	*one loop present*
Filled triangle	$(1, 0, 1)$	*no 1-loop; one 2D cap*
Square cycle (4 edges)	$(1, 1, 0)$	*one loop again*
Two loops sharing a vertex	$(1, 2, 0)$	*two independent cycles*

2.5 Filtrations and Persistent Homology

A filtration *is a nested sequence of complexes* $(K_r)_{r \in \mathbb{R}}$ *with* $K_r \subseteq K_s$ *whenever* $r \leq s$*. In data analysis,* r *is usually a scale parameter (e.g. the radius of balls or a distance threshold).*

Definition 7 (Persistent homology, barcodes, and diagrams)**.** *Fix* k *and a filtration* $(K_r)_{r \in \mathbb{R}}$*. Each time we increase* r*:*

- *some new* k*-dimensional homology class may be* born*;*
- *some existing class may become a boundary and* die*.*

We record for each class its birth scale b *and death scale* d*, giving an interval* $[b, d)$*. The multiset of these intervals is the* kth persistence barcode*. Plotting each interval as a point* (b, d) *in the plane gives the* persistence diagram*.*

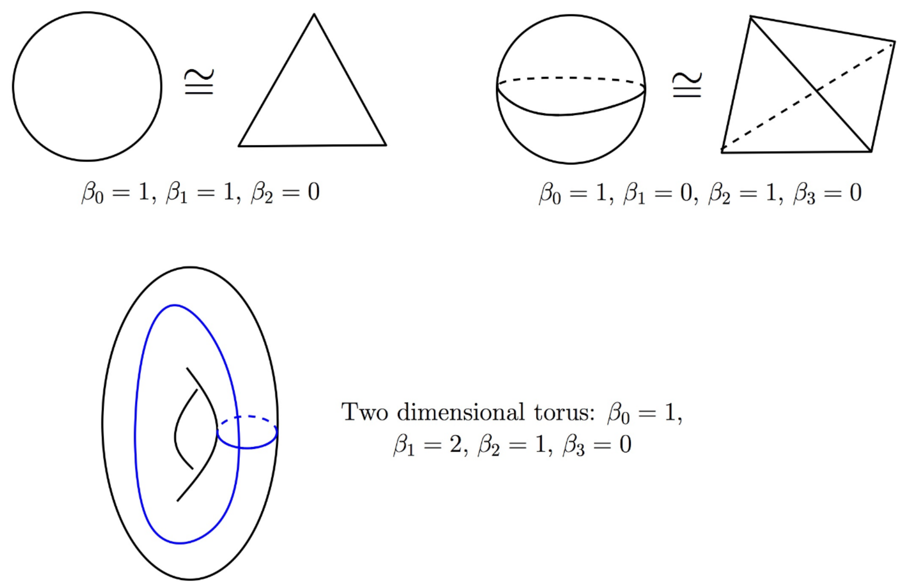

Figure 5: Betti numbers for classical spaces (circle, sphere, torus). Think of the torus as a "double loop" object with two independent 1-cycles.

Example 8 (Growing balls on a loop)**.** *Place points roughly on a circle and let K_r be the union of balls of radius r:*

- *For very small r, many components: large β_0, $\beta_1 = 0$.*
- *At a medium scale, the components merge and a single loop appears: $\beta_0 = 1$, $\beta_1 = 1$.*
- *At larger r, the loop fills in: $\beta_0 = 1$, $\beta_1 = 0$.*

The H_1 barcode shows one long bar representing the circle's main loop.

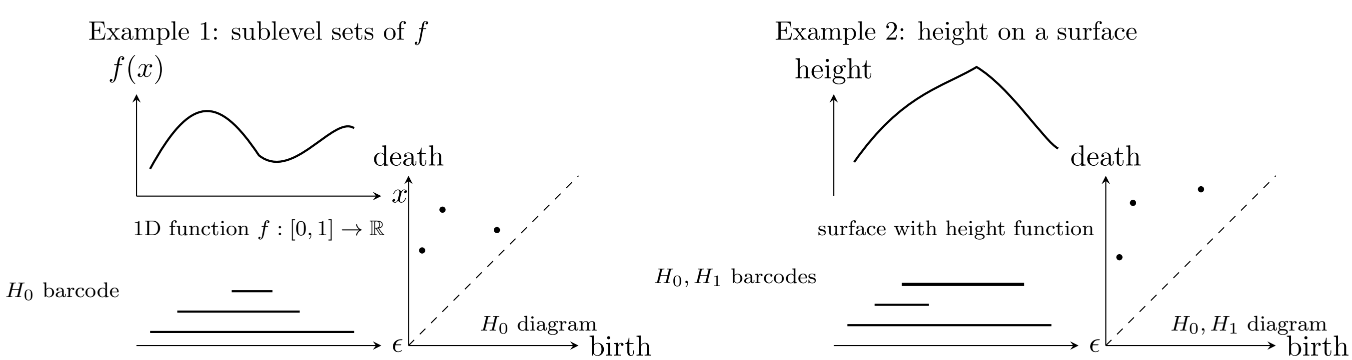

Figure 6: Example barcodes and diagrams for simple filtrations: a 1D function (left) and a height function on a surface (right).

2.6 The Mapper Construction and Its Music Interpretation

Given a function (lens) $f : X \to \mathbb{R}^d$ and a cover $\mathcal{U}$ of $f(X)$ by overlapping bins, the Mapper algorithm proceeds in four steps:

1. ***Lens:*** *compute a low-dimensional lens (UMAP or PCA) so the chord-events spread out in* $\mathbb{R}^2$ *by similarity.*
2. ***Cover:*** *lay an overlapping grid of windows (intervals or rectangles) on the lens image* $f(X)$*.*
3. ***Pullback and clustering:*** *for each window* $U \in \mathcal{U}$*, form the preimage* $f^{-1}(U)$ *and cluster the events inside it. Each cluster is a tight, "blob-like" set that we expect to be contractible.*
4. ***Nerve:*** *build the nerve of these clusters:*
   - *one node per cluster;*
   - *an edge between two nodes if the clusters share at least one data point.*

   This graph is the Mapper complex.

By the Nerve Theorem intuition, the connectivity pattern of this graph (chains, branches, loops) reflects the shape of harmonic space in B9/IV: branches correspond to harmonic alternatives; loops correspond to cyclical circuits of chord progressions.

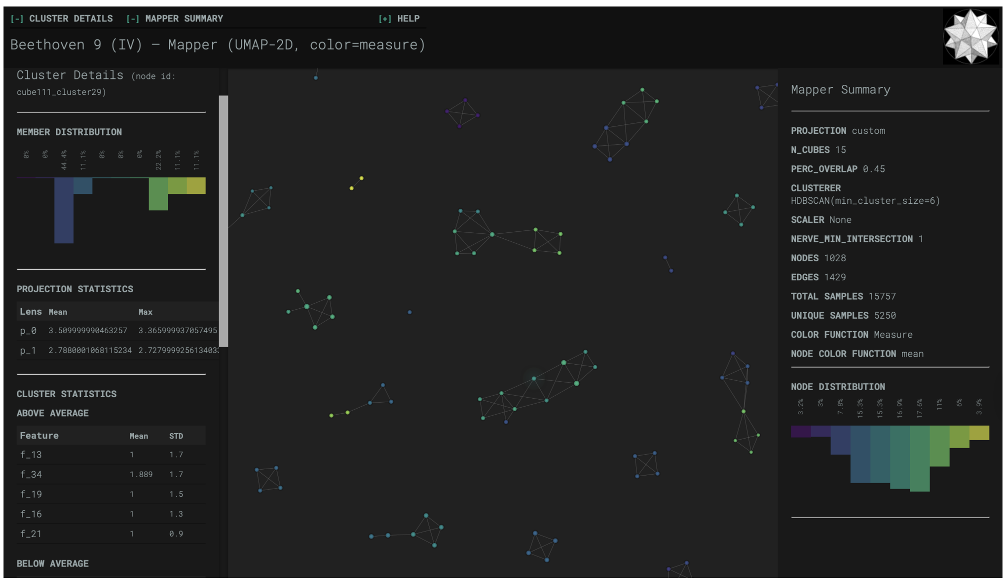

Figure 7: Mapper graph for B9/IV (schematic). Nodes are harmonic neighborhoods; edges indicate chord events shared between neighborhoods.

3 From Score to Mathematics: Our Construction

3.1 Encoding of chord-events and voice leading

Each event is represented by:

$$\textit{PC incidence vector} \in \{0,1\}^{12}, \quad \textit{size}, \quad \textit{onset}, \quad \textit{duration}, \quad \textit{root PC}, \quad \textit{optional voice-leading cost.}$$

Distances combine PC overlap and timing:

$$\rho(x,y) = \lambda\,\mathrm{Ham}\big(\mathrm{PC}(x),\mathrm{PC}(y)\big) + \mu\,|\textit{onset}(x) - \textit{onset}(y)| + \nu\,\mathrm{VL}(x,y),$$

where VL *is a simple voice-leading distance counting semitone moves between voice-leading realizations.*

Example 9 (Voice-leading toy example)**.** *The move C–major* $\{0, 4, 7\}$ *to A–minor* $\{9, 0, 4\}$ *can be realized as* $G \mapsto A$ *(a +2 semitone step), with C and E fixed. We assign* $\mathrm{VL} = 2$ *to this transition, so harmonically close, parsimonious moves are close in* ρ*.*

3.2 Persistent homology on B9/IV and musical reading

We build the Vietoris–Rips filtration $\mathrm{Rips}_\varepsilon(X)$ *over scales* ε *and compute* H_0 *and* H_1 *over* $\mathbb{Z}_2$*.*

- H_0 ***barcodes:*** *long-living* H_0 *intervals correspond to harmonic regions that remain separated until large* ε*—for example, distinct key areas or contrasting episodes in B9/IV.*
- H_1 ***barcodes:*** *long* H_1 *bars correspond to robust loops in harmonic space: families of progressions (and their variants) that "wrap around" the PC/voice-leading geometry and recur throughout the movement.*

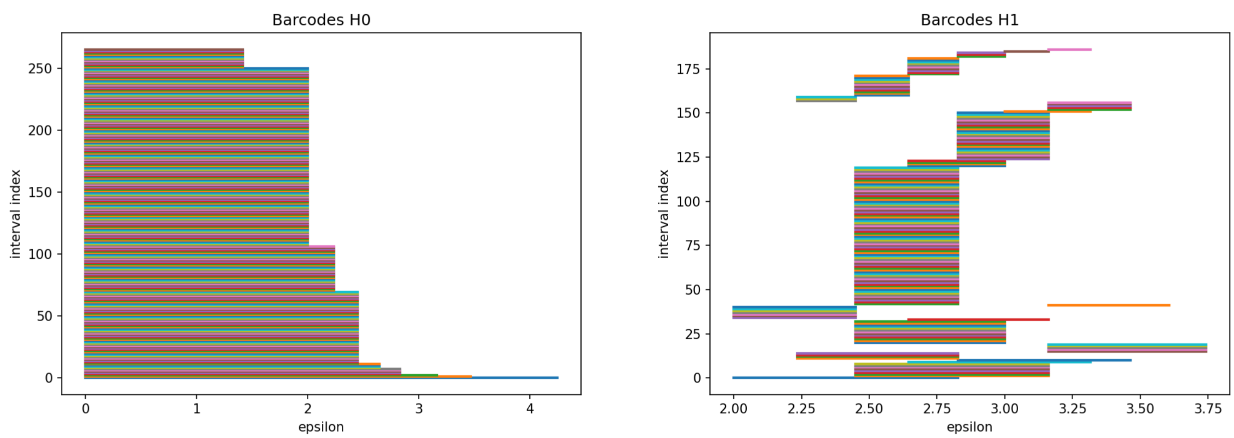

Figure 8: Barcodes for H_0 (left) and H_1 (right) for B9/IV. Each horizontal bar is a homology class persistent across ε.

How to read these plots for B9/IV.

- *The* H_0 *barcode starts with as many bars as chord-events; bars merge quickly as* ε *increases. The longest surviving* H_0 *bars represent harmonically distinct regions (e.g. tonic vs. "Amen" coda textures) that only connect under a very generous harmonic/timing tolerance.*
- *The* H_1 *barcode and persistence diagram show a cluster of* 1*-dimensional intervals born at intermediate* ε *and dying later. Together with the Mapper loops, these suggest stable recurrence of certain progressions and voice-leading patterns—musical "circuits" rather than isolated cadences.*
- *The Betti curves clearly show* β_0 *decreasing (components merging) and* β_1 *rising then falling (loops appear and are eventually filled). The* ε*-range where* β_1 *is largest indicates the scale at which the harmonic space of B9/IV is most "loop-like".*

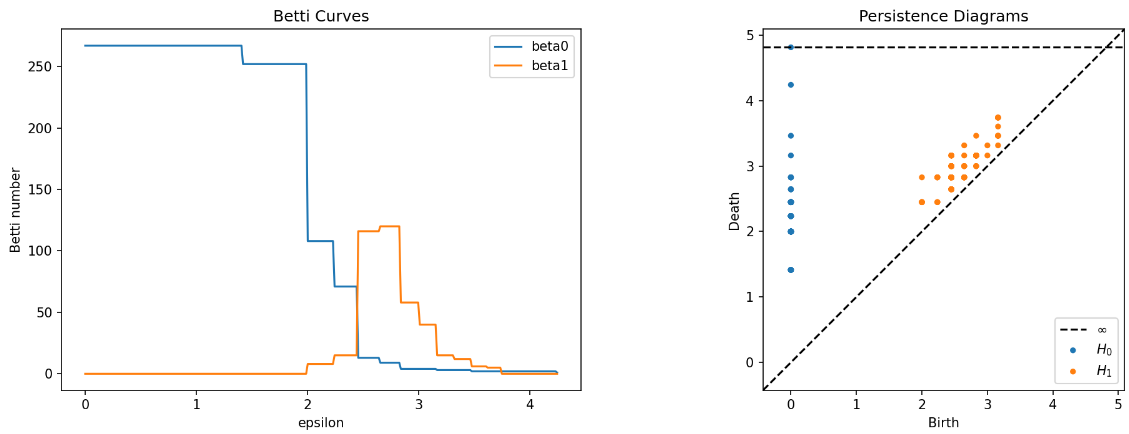

Figure 9: Left: Betti curves $\beta_0(\varepsilon)$ and $\beta_1(\varepsilon)$. Right: persistence diagrams for H_0 and H_1.

4 Learning Outcomes and Musical Conclusions

This project has two intertwined conclusions:

Mathematical learning

- *Built chain complexes over* $\mathbb{Z}_2$ *explicitly (simplices, chains, boundary maps* ∂_k*) and computed homology and Betti numbers in concrete examples.*
- *Implemented Vietoris–Rips filtrations and interpreted persistent homology via barcodes, Betti curves, and persistence diagrams.*
- *Applied the Nerve Theorem and Mapper construction to real symbolic data, seeing how covers, nerves, and simplicial complexes appear in practice.*

Musical learning

- *Saw that Beethoven's harmonic language in B9/IV can be treated as a point cloud in a geometric space: pitch-class sets and voice leading are not just metaphors but coordinates.*
- *Identified harmonic neighborhoods (clusters) and recurrent progressions (loops) that persist across scales, suggesting that certain voice-leading "routes" form a backbone of the movement.*
- *Experienced first-hand how topology and geometry give new, global perspectives on a familiar piece:* music and mathematics genuinely support one another*, with homology groups and Betti numbers providing rigorous language for intuitive notions like "region", "bridge", and "cycle" in harmony.*

Acknowledgments

Foundational definitions, examples, and the TDA pipeline (metric spaces, covers, simplicial complexes, Mapper, filtrations, persistent homology) closely follow the survey by Chazal and Michel on Topological Data Analysis for data scientists.

References

[1] *A. Alcalá-Álvarez and P. Padilla-Longoria. A Framework for Topological Music Analysis (TMA).* arXiv:2204.09744*, 2022.*

[2] *D. Tymoczko.* A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Oxford University Press, 2011.*

[3] *D. Tymoczko. Scale Networks in Debussy.* Journal of Music Theory*, 48(2):215–292, 2004.*

[4] *D. Tymoczko. The Geometry of Musical Chords.* Science*, 313:72–74, 2006.*

[5] *D. Tymoczko. Scale Theory, Serial Theory, and Voice Leading.* Music Analysis*, 27(1):1–49, 2008.*

[6] *D. Tymoczko. Three Conceptions of Musical Distance. (manuscript / article).*

[7] *F. Chazal and B. Michel. Topological Data Analysis: fundamental and practical aspects for data scientists. (survey/tutorial).*