Prep Course
Module I
Calculus Lecture 2

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Example: Delta Hedge for a Put

Black-Scholes Pricing Formula for Vanilla Puts

- If we hold one unit of a vanilla put P on a non-dividend-paying stock S with spot price S_0, volatility σ, strike K, time to maturity T, and interest rate r, we have the following nice analytical formula for today's put price P_0:

$$P_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

where $d_1 := \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$, $d_2 := \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$,
and $N(x)$ is the standard normal CDF.

Interview Question: What's the delta of a put?

- We view the formula of P_0 as a single-variable function of S_0 with all other parameters fixed.
- We want to know the following:
  - How much the option price P_0 moves, per unit move in S.
  - Number of shares of S needed to replicate one option.
- Mathematically we want to compute the following:

$$\frac{d}{dS_0} P_0 = ?$$

- In order to do this differentiation, we have to understand the concepts of differentiation in Calculus.

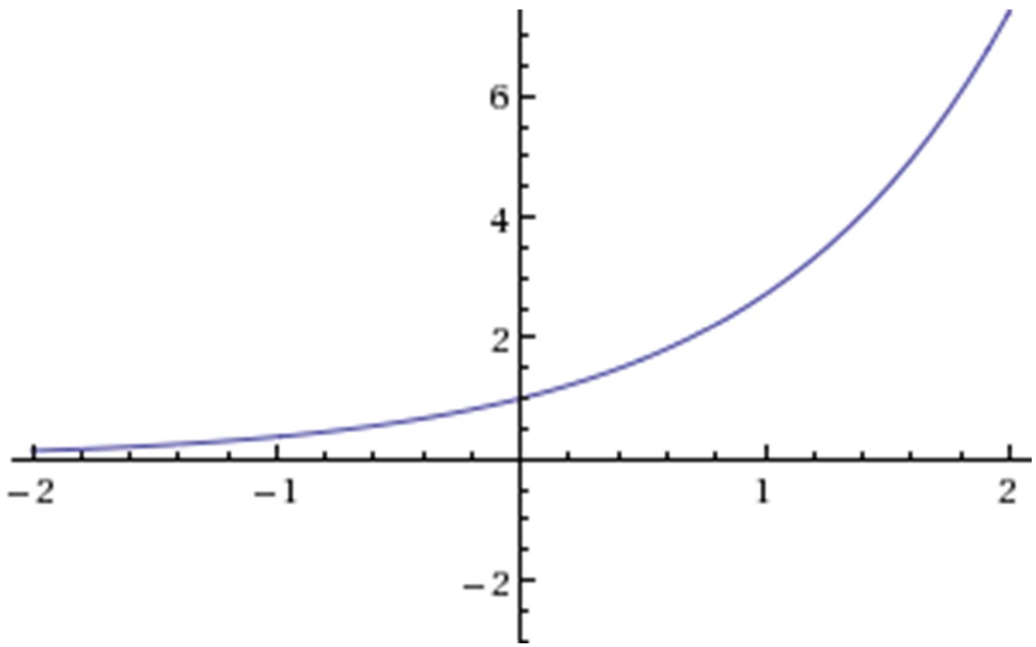
Exponential and Logarithm Functions

Exponential Function

Exponential Function $f(x) = e^x$, mapping x from $\mathbb{R}$ to $(0, \infty)$

Given $a, b, r \in \mathbb{R}$

- $e^{a+b} = e^a \cdot e^b$
- $e^{a-b} = \frac{e^a}{e^b}$
- $(e^a)^r = e^{ra}$
- $\lim_{x \to +\infty} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = 0$

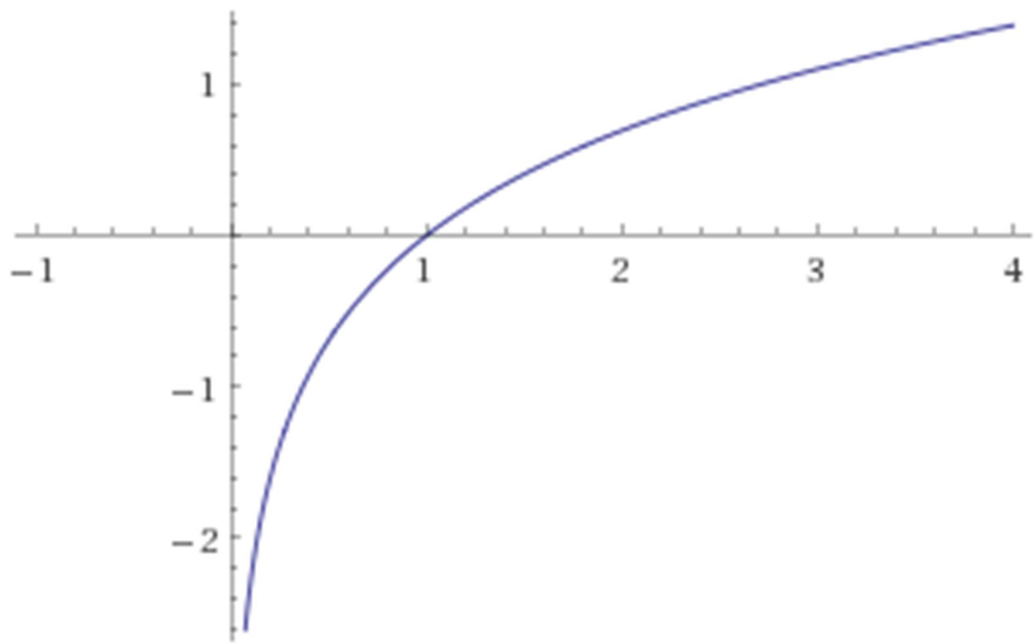


Logarithm Function

Logarithm Function $g(x) = \log x$, mapping x from $(0, \infty)$ to $\mathbb{R}$

Given $a, b > 0, r \in \mathbb{R}$

- $\log(ab) = \log a + \log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^r = r \cdot \log a$
- $\lim_{x \to +\infty} g(x) = \infty$
- $\lim_{x \to 0^+} g(x) = -\infty$
- $a = e^{\log a}$



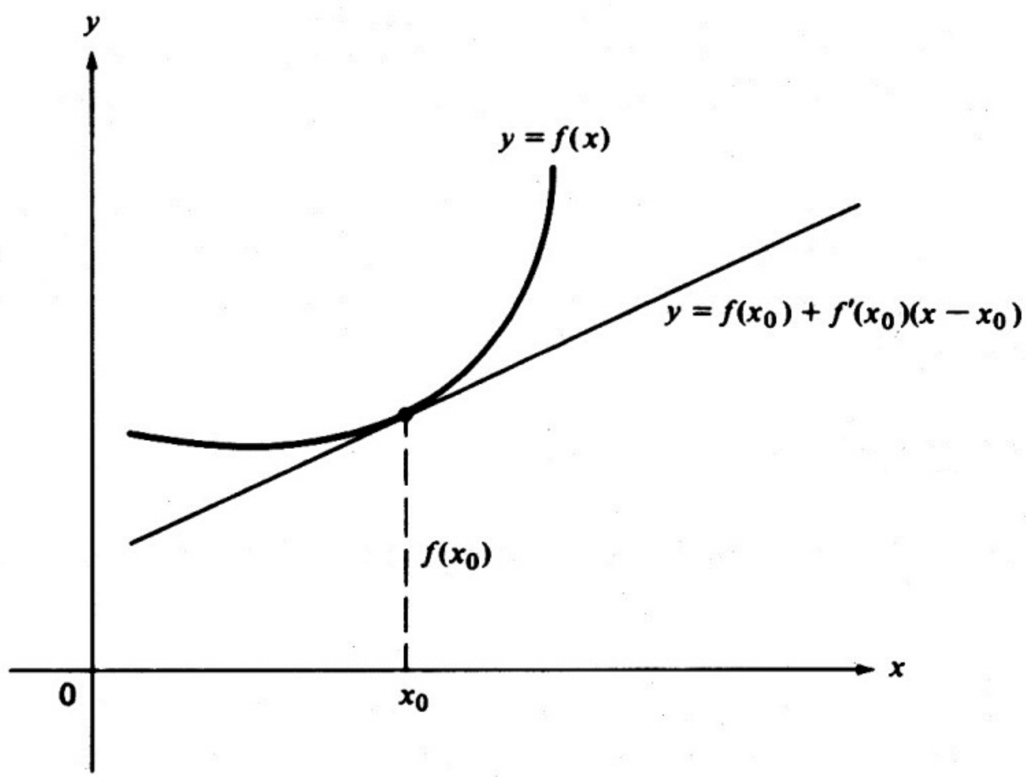
Differentiation

Definition of Derivative

Given a function $f(x)$, the derivativef' is defined by the following:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{x_1\to x}\frac{f(x_1)-f(x)}{x_1-x}$$

If the limit above exists, we say f is differentiable at point x. Please note that differentiability guarantees continuity, but continuity alone does not guarantee differentiability.



Examples of applying the definition of derivative

Example 1: Let $f(x) = x^2$, we can compute f' as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} 2x + h = 2x$$

Example 2: Let $\phi(x) = f(x)g(x)$, where f' and g' exist, then

$$\phi'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} = \lim_{h\to 0} \left[\left(f(x+h) \frac{g(x+h) - g(x)}{h} \right) + \left(g(x) \frac{f(x+h) - f(x)}{h} \right) \right] = f(x)g'(x) + g(x)f'(x)$$

Rules for Differentiation

Given $f(x)$ and $g(x)$ are differentiable:

Rule 1: if $c \in \mathbb{R}$ and $\phi(x) = c \cdot f(x)$, then $\phi'(x) = c \cdot f'(x)$.

Rule 2: if $\phi(x) = f(x) + g(x)$, then $\phi'(x) = f'(x) + g'(x)$.

Rule 3: if $\phi(x) = f(x)g(x)$, then $\phi'(x) = f(x)g'(x) + g(x)f'(x)$

Rule 4: if $\phi(x) = \frac{f(x)}{g(x)}$, then $\phi'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{\left(g(x)\right)^2}$.

Rule 5 (Chain Rule): if $\phi(x) = f\left(g(x)\right)$, then $\phi'(x) = f'\left(g(x)\right) \cdot g'(x)$

Derivative of elementary functions

- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \log x = \frac{1}{x}$
- $\frac{d}{dx} x^r = r \cdot x^{r-1}$, where $r \in \mathbb{R}$
- $\frac{d}{dx} a^x = a^x \cdot \log a$, where $a > 0$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$

Example of Differentiation

What is the derivative of x^x, where $x > 0$?

Answer:

Let

$$y = x^x$$

By taking logarithm of both sides,

$$\log y = x \cdot \log x$$

By taking derivative of both sides,

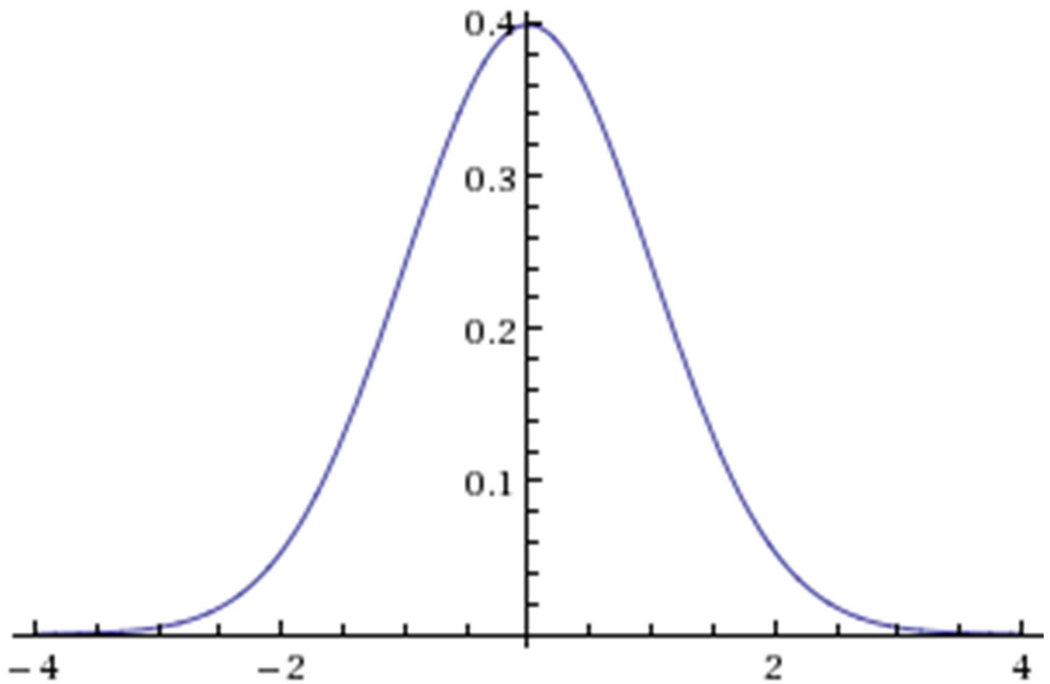
$$\frac{y'}{y} = \log x + 1 \Rightarrow y' = x^x(\log x + 1)$$

Useful facts about the standard normal CDF $N(x)$

- Derivative of CDF ➔ PDF

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \xrightarrow{\text{differentiation}} N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

- Symmetry of $N'(x)$: $\forall x \in \mathbb{R}, N'(x) = N'(-x) \Rightarrow N(x) + N(-x) = 1$



Solution to the delta of a put.

First, note that

$$\frac{d}{dS_0}N(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{d}{dS_0}d_1 = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

$$\frac{d}{dS_0}N(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \cdot \frac{d}{dS_0}d_2 = \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(d_1-\sigma\sqrt{T}\right)^2}{2}} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot e^{d_1\sigma\sqrt{T}} \cdot e^{-\frac{\sigma^2 T}{2}} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot e^{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T} \cdot e^{-\frac{\sigma^2 T}{2}} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{S_0}{K} \cdot e^{rT} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

Solution to the delta of a put.

Now take derivative with respect to S_0 of the pricing formula:

$$\frac{d}{dS_0}P_0 = Ke^{-rT}\frac{d}{dS_0}N(-d_2) - N(-d_1) - S_0\frac{d}{dS_0}N(-d_1)$$

Using the results at previous slide:

$$\frac{d}{dS_0}P_0$$

$$= Ke^{-rT}\frac{d}{dS_0}[1 - N(d_2)] - [1 - N(d_1)] - S_0\frac{d}{dS_0}[1 - N(d_1)]$$

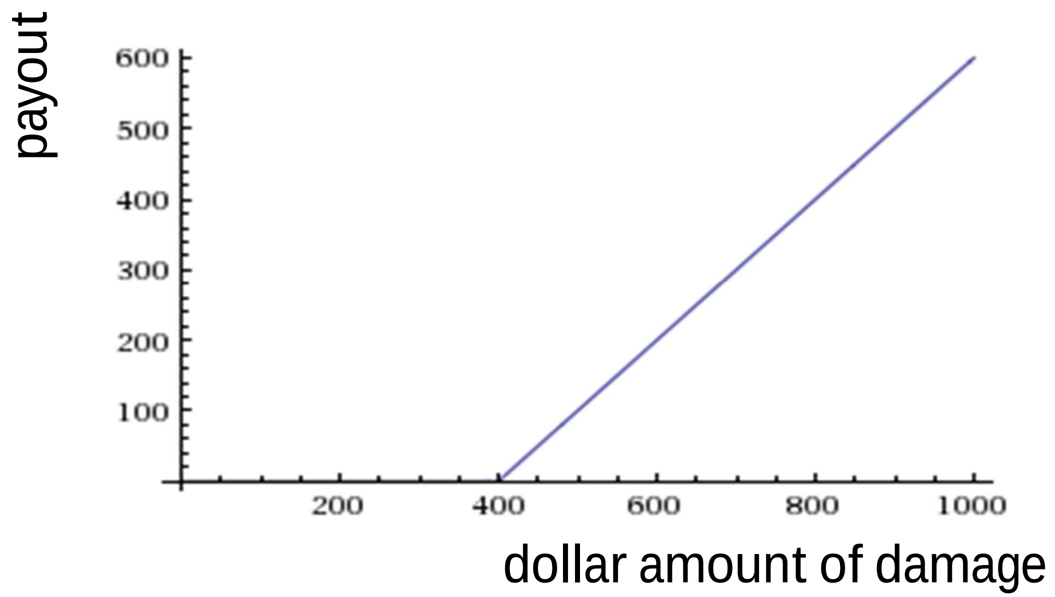
$$= -Ke^{-rT}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{S_0}{K} \cdot e^{rT} \cdot \frac{1}{S_0\sigma\sqrt{T}} - [1 - N(d_1)]$$

$$+ S_0\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \cdot \frac{1}{S_0\sigma\sqrt{T}} = N(d_1) - 1$$

Example: Automobile Accident Insurance

Automobile Accident Insurance

- The dollar amount of damage involved in an automobile accident is an exponential random variable with mean 1000. Of this, the insurance company only pays that amount exceeding (the deductible amount of) 400. Find the expected value of the amount the insurance company pays per accident. (Source: Introduction to Probability Models, 10th Edition, by Sheldon M. Ross)
- Payout = $(X - 400)^+$, where X is a random variable, representing the dollar amount of damage.



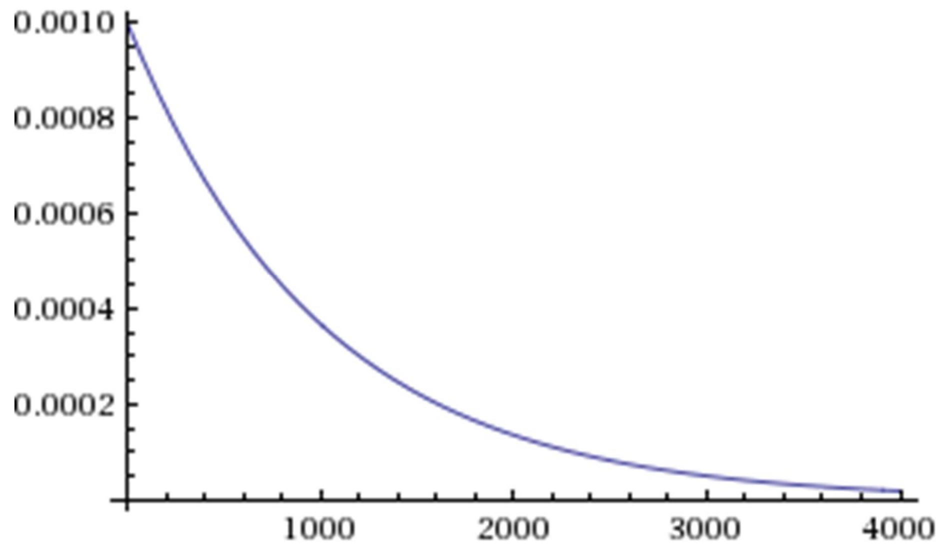
Exponential Distribution

X follows an exponential distribution, which has its PDF in the following form:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

Note that $\mathbb{E}[X] = \frac{1}{\lambda}$.

As such, $\lambda = \frac{1}{1000}$ in this case.



Basically we want to compute

$$\mathbb{E}[(X - 400)^+] = \int_0^\infty (x - 400)^+ f(x) dx = ?$$

Integration

Definition of Definite Integral

- If $f(x)$ is defined on a closed interval $I = \{x \in \mathbb{R}: a \leq x \leq b\}$, then we divide the interval I into N subintervals, $I_1, I_2, \ldots, I_N$ of length $\Delta = \frac{b-a}{N}$. For $n = 1,2, \ldots, N$, let $x_n = a + \left(n - \frac{1}{2}\right) \cdot \Delta$, i.e., x_n is the middle point of I_n. We say $f(x)$ is integrable on I if there exists a number $A \in \mathbb{R}$, such that

$$\lim_{N \to \infty} \sum_{n=1}^{N} f(x_n) \cdot \Delta = A = \int_a^b f(x) dx$$

- It is useful to make the following definitions:

$$\int_a^a f(x) dx = 0$$

$$\int_b^a f(x) dx = -\int_a^b f(x) dx$$

- If $f(x)$ is continuous on I, then $f(x)$ is integrable on I.

Properties of Definite Integral

- If $f_1(x) \le f_2(x)$ holds on a closed interval $I = \{x \in \mathbb{R}: a \le x \le b\}$, then
$$\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx$$
- If $f(x)$ is integrable on a closed interval $I = \{x \in \mathbb{R}: a \le x \le b\}$, then $\forall\ c \in (a, b)$
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
- Linearity of Definite Integral: given functions $f(x)$ and $g(x)$, both integrable on a closed interval $I = \{x \in \mathbb{R}: a \le x \le b\}$, and $\alpha, \beta \in \mathbb{R}$, we have
$$\int_a^b [\alpha \cdot f(x) \pm \beta \cdot g(x)]dx = \alpha \cdot \int_a^b f(x)dx \pm \beta \cdot \int_a^b g(x)dx$$

Definition of Improper Integral

- If $f(x)$ is integrable on interval $I_k = \{x \in \mathbb{R}: a \leq x \leq k\}$, $\forall\, k > a$, then we define
$$\int_a^\infty f(x)dx = \lim_{k\to\infty} \int_a^k f(x)dx$$
- If $f(x)$ is integrable on interval $I_h = \{x \in \mathbb{R}: h \leq x \leq b\}$, $\forall\, h < b$, then we define
$$\int_{-\infty}^b f(x)dx = \lim_{h\to-\infty} \int_h^b f(x)dx$$
- If $f(x)$ is defined on interval $I = \{x \in \mathbb{R} | -\infty < x < \infty\}$, and let $c \in \mathbb{R}$, then we define
$$\int_{-\infty}^\infty f(x)dx = \lim_{r\to\infty} \int_{-r}^c f(x)dx + \lim_{r\to\infty} \int_c^r f(x)dx$$

Fundamental theorem of Calculus

(First Form)

If $f(x)$ is continuous on a closed interval $I = \{x \in \mathbb{R}: a \leq x \leq b\}$ and we define $F(x)$ as follows

$$F(x) = \int_a^x f(t)dt$$

, then $F(x)$ is continuous on I and $F'(x) = f(x)\ \forall\ x \in (a, b)$.

(Second Form)

If $f(x)$ and $F(x)$ are continuous on a closed interval $I = \{x \in \mathbb{R}: a \leq x \leq b\}$ and $F'(x) = f(x)\ \forall\ x \in (a, b)$, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Definition of Indefinite Integral

- Given functions $f(x)$ and $g(x)$, if $g'(x) = f(x)$, then we say $g(x)$ is an indefinite integral of $f(x)$ and we write:

$$g(x) = \int f(x)dx$$

- If $g(x)$ is an indefinite integral of $f(x)$, then $\forall\, c \in \mathbb{R}$, $g(x) + c$ is also an indefinite integral of $f(x)$.

$$\int f(x)dx = g(x) + c$$

- Linearity of Indefinite Integral: given functions $f(x)$ and $g(x)$, and $\alpha, \beta \in \mathbb{R}$, we have

$$\int [\alpha \cdot f(x) \pm \beta \cdot g(x)]dx = \alpha \cdot \int f(x)dx \pm \beta \cdot \int g(x)dx$$

Integration of elementary functions

- $\int e^x dx = e^x$
- $\int \frac{1}{x} dx = \log x$
- $\int x^r dx = \frac{1}{r+1} \cdot x^{r+1}$, where $r \neq -1$
- $\int a^x dx = \frac{a^x}{\log a}$, where $a > 0$
- $\int \sin x \, dx = -\cos x$
- $\int \cos x \, dx = \sin x$

Integration by parts

- Given differentiable functions $f(x)$ and $g(x)$,

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- Example: $\int x \cdot 2^x dx$

$$\int x \cdot 2^x dx = \frac{x \cdot 2^x}{\log 2} - \int \frac{2^x}{\log 2} dx = \frac{x \cdot 2^x}{\log 2} - \frac{2^x}{(\log 2)^2} = \frac{2^x}{\log 2}\left(x - \frac{1}{\log 2}\right)$$

- Example: $\int \log x \, dx$

$$\int \log x \, dx = x \cdot \log x - \int dx = x(\log x - 1)$$

Solution to Automobile Accident Insurance

$$\mathbb{E}[(X-400)^+] = \int_0^\infty (x-400)^+ f(x)dx$$

$$= \int_0^{400} (x-400)^+ f(x)dx + \int_{400}^\infty (x-400)^+ f(x)dx$$

$$= 0 + \int_{400}^\infty (x-400) f(x)dx = \int_{400}^\infty x f(x)dx - 400\int_{400}^\infty f(x)dx$$

$$= \int_{400}^\infty \frac{x}{1000} e^{\frac{-x}{1000}} dx - 400\int_{400}^\infty \frac{1}{1000} e^{\frac{-x}{1000}} dx$$

$$= \left(\left[-x \cdot e^{\frac{-x}{1000}}\right]_{400}^\infty + \int_{400}^\infty e^{\frac{-x}{1000}} dx\right) - \frac{2}{5}\int_{400}^\infty e^{\frac{-x}{1000}} dx$$

$$= 0 + 400 \cdot e^{\frac{-400}{1000}} + \frac{3}{5} \cdot \left(-1000 \cdot \left[e^{\frac{-x}{1000}}\right]_{400}^\infty\right) = 1000 \cdot e^{\frac{-400}{1000}} \cong 670.32$$