Module I Homework 2

qquantt
Prep24AutumnM1

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Problem 1.

Intuition: Use Bayes Rule to evaluate $P(NewBabyWasBoy \mid PickedBabyWasBoy)$

Solution. □

$$\frac{\left(\frac{1}{2}\right)\left(\frac{3}{6}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{6}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{6}\right)}$$

$$=\boxed{\frac{3}{5}}$$

(...where the numerator is $P(NewBabyWasBoy)$ intersecting the set of events in which $P(PickedBabyWasBoy \mid NewBabyWasBoy)$ and the denominator is the universe Ω, which encapsulates $P(NewBabyWasBoy)$ intersecting the set of events in which $P(PickedBabyWasBoy \mid NewBabyWasBoy)$ OR $P(NewBabyWasNotBoy)$ intersecting the set of events in which $P(PickedBabyWasBoy \mid NewBabyWasNotBoy)$)

Problem 2.

Intuition: Identify the binomial probabilities for events in which $X = 0$ and $X = 1$ (since we are evaluating for X being at least 2 individuals in the random sample) and subtract those from the universe of total events to solve by complement

Solution. □

$$1 - P(X = 1) - P(X = 0)$$

$$= 1 - \binom{15}{1}(0.2)^1(0.8)^{14} - \binom{15}{0}(0.2)^0(0.8)^{15}$$

$$\approx \boxed{0.8329}$$

Problem 3.

Intuition: While intuitively it seems a four-engine plane would always be preferable to a two-engine plane, as more engines should provide more redundancy and increase the likelihood of a successful flight, to rigorously determine if a four-engine plane is always preferable, we need to compare the exact probabilities of successful flights for both planes under varying values of p

Solution. □

First expressing the probability of successful flight with a two-engine plane:

$$P(\text{success}_{\text{2eng}}) = 1 - P(\text{2EngFail})$$

$$= 1 - \binom{2}{0}(1-p)^2(1-(1-p))^0$$

$$= 1 - (1-p)^2$$

...and the probability with a four-engine plane:

$$P(\text{success}_{\text{4eng}}) = 1 - P(> \text{2EngFail})$$

$$= 1 - (P(\text{4EngFail}) + P(\text{3EngFail}))$$

$$= 1 - \binom{4}{0}(1-p)^4(1-(1-p))^0 - \binom{4}{1}(1-p)^3(1-(1-p))^1$$

$$= 1 - (1-p)^4 - 4p(1-p)^3$$

Then computing the values of p for which a four-engine plane is preferable to a two-engine plane:

$$1 - (1-p)^4 - 4p(1-p)^3 > 1 - (1-p)^2$$

$$(1-p)^2 > (1-p)^4 + 4p(1-p)^3$$

$$1 > (1-p)^2 + 4p(1-p)$$

$$1 > 1 - 2p + p^2 + 4p - 4p^2$$

$$3p^2 > 2p$$

$$3p > 2$$

$$\boxed{\frac{2}{3} < p \leq 1}$$

Problem 4.

Solution. $\square$

We know the following for player A:

$$P(A > B) = P(A < B)$$

We also know that the sample space, Ω, is partitioned by the three events $A > B,\ A = B,$ and $A < B$...

As such,

$$P(\Omega) = P(A > B) + P(A = B) + P(A < B)$$

$$= 2 \cdot P(A > B) + P(A = B)$$

$$\Rightarrow P(A > B) = \frac{1}{2}(1 - P(A = B))$$

$$\text{...where } P(A = B) = \frac{52 \times 3}{52 \times 51} = \frac{1}{17}$$

Thus,

$$P(A > B) = \frac{1}{2}\left(1 - \frac{1}{17}\right)$$

$$= \frac{8}{17}$$

$$\approx \boxed{0.4706}$$

Problem 5.

Solution. $\square$

$$P(\text{COVID}|\text{positive}) = \frac{P(\text{positive}|\text{COVID})P(\text{COVID})}{P(\text{positive}|\text{COVID})P(\text{COVID}) + P(\text{positive}|\text{no COVID})P(\text{no COVID})}$$

$$= \frac{0.9 \times P(\text{COVID})}{0.9 \times P(\text{COVID}) + 0.05 \times P(\text{no COVID})}$$

With 5 per-cent prevalence :

$$P(\text{COVID}|\text{positive}) = \frac{0.9 \times 0.05}{0.9 \times 0.05 + 0.05 \times 0.95}$$

$$\approx 48.6\% \approx 49\%$$

With 52 per-cent prevalence:

$$P(\text{COVID}|\text{positive}) = \frac{0.9 \times 0.52}{0.9 \times 0.52 + 0.05 \times 0.48}$$

$$\approx 95.1\% \approx 95\%$$