Module I Homework 3

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Prep24AutumnM1

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Problem 1.

Intuition: $f(x) = x^3$ is continuous at $x = -2$ if for every $\epsilon > 0$, $\exists$ a real $\delta > 0$ such that $0 < |x - (-2)| = |x + 2| < \delta \implies |f(x) - f(-2)| = |x^3 + 8| < \epsilon$

Solution. $\square$

First, factoring the expression:

$$|x^3 + 8| = |(x + 2)(x^2 - 2x + 4)|$$

To bound this expression, we need to control $x+2$ as well as the quadratic term $x^2 - 2x + 4$

For x close to -2, the quadratic term $x^2 - 2x + 4$ does not vary too much...
Specifically, evaluating the term at the endpoints of a chosen interval $[-3, -1]$:

$$x^2 - 2x + 4 = 19 \quad \text{at } x = -3,$$

$$x^2 - 2x + 4 = 7 \quad \text{at } x = -1$$

Thus, for $x \in [-3, -1]$:

$$|x^2 - 2x + 4| \leq 19$$

Now, relating $|x^3 + 8|$ to $|x + 2|$ given the bound on the quadratic term:

$$|x^3 + 8| = |(x + 2)(x^2 - 2x + 4)| \leq 19|x + 2|$$

Note that to ensure that $|x^3 + 8| < \epsilon$, we need $19|x + 2| < \epsilon \implies |x + 2| < \frac{\epsilon}{19}$...

Now, choosing δ to satisfy both the requirement $|x + 2| < \frac{\epsilon}{19}$ and to ensure that x stays within the interval $[-3, -1]$ (*i.e.*, that $|x + 2| < 1$):

$$\delta = \min\left(1, \frac{\epsilon}{19}\right)$$

Thus, for any $\epsilon > 0$, we can choose $\delta = \min\left(1, \frac{\epsilon}{19}\right)$ such that:

$$0 < |x + 2| < \delta \implies |x^3 + 8| < \epsilon$$

Therefore, $f(x) = x^3$ is continuous at $x = -2$

Q.E.D.

Problem 2.

Solution. □

Let $h(x) = x^4$, $g(x) = x^2$

Then $h(x)$, $g(x)$ satisfy that $h(x) \leq f(x) \leq g(x)$ for any x satisfying $0 < |x - a| < \delta$ as well as that $\lim_{x \to 0} h(x) = \lim_{x \to 0} g(x) = 0$

Therefore, $x^4 \leq f(x) \leq x^2$ for all x around 0

Thus, by the Sandwiching Theorem: $\lim_{x \to 0} f(x) = 0$

Q.E.D.

Problem 3.

Intuition: The numerator scales at an exponential rate and the denominator scales at a factorial rate, and the numerator scales at a slower rate than the denominator does for $n > 2$, so the limit of the sequence as $n \to \infty$ is 0, and we can utilize the ratio test to rigorously compute the limit

Solution. □

Let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

For $a_n = \frac{2^n}{n!}$, we have:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1}$$

As such,

$$L = \lim_{n \to \infty} \left| \frac{2}{n+1} \right| = \lim_{n \to \infty} \frac{2}{n+1} = 0$$

Because $L < 1$, the sequence a_n converges to 0

Thus,

$$\boxed{\lim_{n \to \infty} \frac{2^n}{n!} = 0}$$

Problem 4.

Intuition: (Using Bayes Theorem) we can intuit that given the fund exists (either on the 40 per cent chance it beat the market or on the 30 per cent chance it underperformed but was not liquidated by its liquidity providers), there is a $\frac{4}{7}$ chance it beat the market.

Solution. $\square$

$\boxed{\text{c) } \frac{4}{7}}$

Problem 5.

Intuition: X

Solution. $\square$

Let $M > 0$ be given

We seek to show that for $M > 0$, $\exists\, \delta > 0$ such that $0 < x - L < \delta \implies f(g(x)) > M$, as this will demonstrate that $f(g(x))$ can get arbitrarily large as $x \to L$ from the right

Recalling and applying the definition of a limit...

- By the definition of $\lim_{x\to\infty} f(x) = \infty$, for every $M > 0$, $\exists$ a real $K > 0$ such that $x > K \implies f(x) > M$

- And by the definition of $\lim_{x\to L^+} g(x) = \infty$, for every $K > 0$, $\exists$ a real $\delta > 0$ such that $0 < x - L < \delta \implies g(x) > K$

Since $0 < x - L < \delta \implies g(x) > K$, and we know that $f(x) > M$ when $x > K$, it follows that for $0 < x - L < \delta$, $f(g(x)) > M$

Thus, for any $M > 0$, we can find a $\delta > 0$ such that whenever $0 < x - L < \delta$, $f(g(x)) > M$

Therefore, $\lim_{x\to L^+} f(g(x)) = \infty$

Q.E.D.

Problem 6.

From slide 4 of Calculus Lecture 1:

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$\text{where } d_1 = \frac{\log\left(\frac{S_0 e^{rT}}{K}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}, \quad d_2 = \frac{\log\left(\frac{S_0 e^{rT}}{K}\right)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2},$$

and $N(x)$ is the standard normal CDF

Intuition: Price today of European call C, C_0, approaches spot price of underlying stock S, S_0, as strike, K, approaches 0

Solution. □

It is clear to see $d_1, d_2 \to \infty$ as $K \to 0^+$ because of where the K term lies in d_1, d_2, and because $N(x)$ is the standard normal CDF, as $d_1, d_2 \to \infty$, $N(d_1), N(d_2) \to 1$

Thus:

$$\lim_{K \to 0^+} C_0 = S_0 - 0 = \boxed{S_0}$$