Prep Course
Module I
Calculus Lecture 1

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General Assumptions and Reference Books

- The following assumptions apply throughout the entire Calculus course, unless otherwise indicated:
  - Real-valued functions only
    - Real or integral variables only
  - Both ∞ and $+\infty$ denote positive infinity. $-\infty$ denotes negative infinity.
  - $\mathbb{R}$ denotes the set of all real numbers.
  - $\mathbb{N}$ denotes the set of all positive integers.
  - $\mathbb{Z}$ denotes the set of all integers.
  - $\mathbb{Q}$ denotes the set of all rational numbers.

- Reference Books
  - Differential and Integral Calculus Volume I, 2nd Edition, by Richard Courant.
  - A First Course in Real Analysis, 2nd Edition, by M. H. Protter and C. B. Morrey.

Example: Let Volatility Go Crazy

Black-Scholes Option Pricing Formula

- If we were to price an European call C on stock S with spot price S_0, volatility σ, strike K, time to maturity T, and interest rate r, we have the following nice analytical formula for today's call price C_0:

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where $d_1 := \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$, $d_2 := \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$,

and $N(x)$ is the standard normal CDF.

Interview Question

- It's a well-known result that the higher the volatility is, the more expensive the call option. How much will the call option cost if the volatility tends to infinity?
- In other words, if we view the formula of C_0 as a function of σ with all other parameters fixed, we want to compute the following:

$$\lim_{\sigma \to \infty} C_0 = ?$$

- In order to compute the limit, we have to understand the concepts of limits and continuous functions in Calculus.

Limits

Definition of Limit

- Informal definition: if $f(x)$ can be made sufficiently close to a real number L by making x sufficiently close (but not equal to) to a, then we say:
  - there exists a limit L for $f(x)$, as x tends to a.

- Formal definition: if for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$, then we write:
  - $\lim_{x \to a} f(x) = L$ or;
  - $f(x) \to L \; as \; x \to a$

- Typically what we need to do when applying the definition is:
  - Show that, for any given $\epsilon > 0$, there exists a $\delta > 0$, which validates the definition above.
  - Keep in mind that if any such δ exists, any positive real smaller than δ can also play the role of δ.

Example of applying the definition of limit

Let $f(x) = x^2$. Show $\lim_{x \to 2} f(x) = 4$

Answer:

Our goal: For any given $\epsilon > 0$, we want to find a $\delta > 0$ such that $|f(x) - 4| < \epsilon$ whenever $0 < |x - 2| < \delta$.

Initially let $\delta_1 = 2$. Then for any x satisfying $0 < |x - 2| < \delta_1$, we have

$$|f(x) - 4| = |x^2 - 4| = |(x - 2)(x + 2)| = (x + 2)|x - 2| < 6|x - 2|$$

Let $\delta = \min(\frac{\epsilon}{6}, \delta_1)$. Then for any x satisfying $0 < |x - 2| < \delta$, we have $|f(x) - 4| < 6|x - 2| < \epsilon$. As such, $\lim_{x \to 2} f(x) = 4$.

Remark about the definition of limit

- Please note that the existence of the limit of function f at $x = a$ does NOT guarantee either of the following:
  - $\lim_{x \to a} f(x) = f(a)$ or;
  - $f(x)$ is defined at $x = a$.
- For example, let $f(x) = \frac{\sin x}{x}$, which is defined for $x \neq 0$ because anything divided by 0 is indeterminate. However, the following limit does exist.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

One-sided Limit

Definition: the function $f(x)$ tends to L as x tends to a from the right if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$. We can then write

$$\lim_{x \to a^+} f(x) = L$$

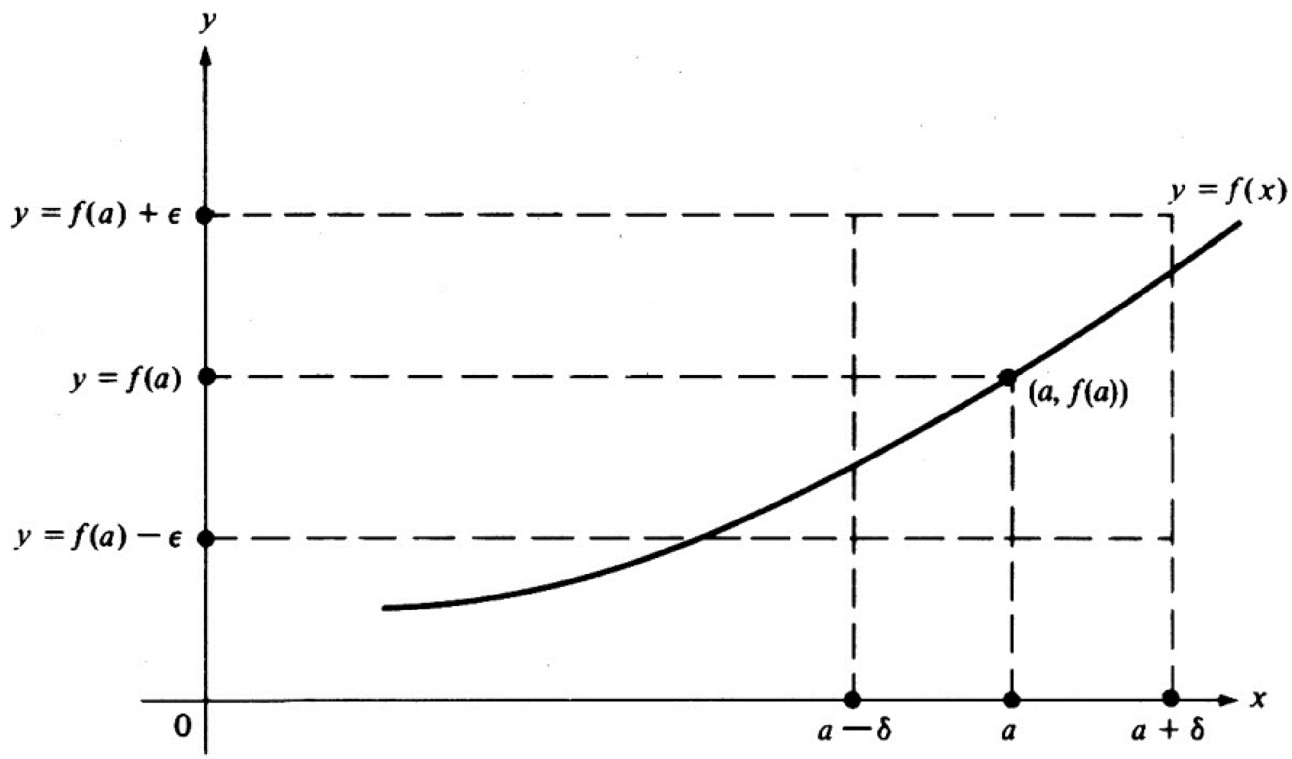
Definition: the function $f(x)$ tends to L as x tends to a from the left if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$. We can then write

$$\lim_{x \to a^-} f(x) = L$$

Theorem: $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Continuity of a function

Definition: if $\lim_{x \to a} f(x) = f(a)$, then we say f continuous at $x = a$. If f is continuous at each point of a set $S \subseteq \mathbb{R}$, then f is continuous on S.



The graph of f is in the rectangle for $a - \delta < x < a + \delta$.

Limit at Infinity

Definition: the function $f(x)$ tends to L as x tends to ∞ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > \delta$. We can then write

$$\lim_{x \to \infty} f(x) = L$$

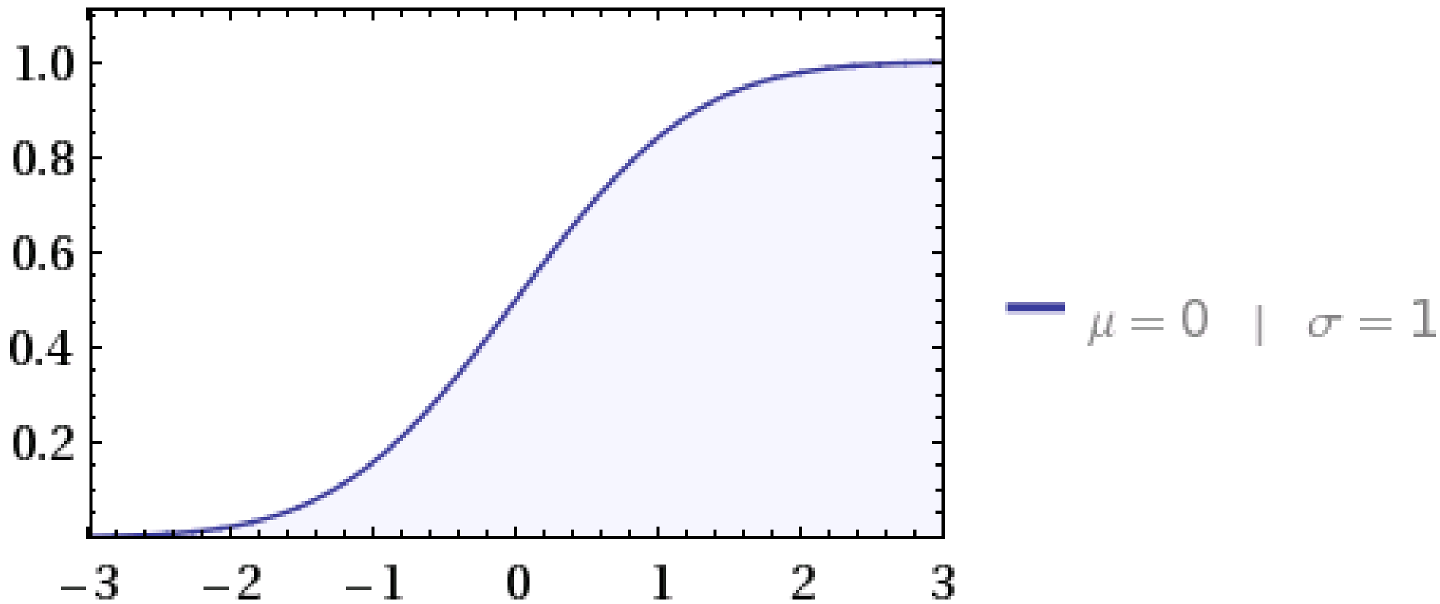
Definition: the function $f(x)$ tends to L as x tends to -∞ if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -\delta$. We can then write

$$\lim_{x \to -\infty} f(x) = L$$

Example of Limit at Infinity

Definition: Let $N(x)$ denote the cumulative distribution function of the standard normal distribution. We know that for each x in $(-\infty, +\infty)$, $0 < N(x) < 1$. Moreover,

$$\lim_{x \to +\infty} N(x) = 1 \text{ and } \lim_{x \to -\infty} N(x) = 0$$



Infinite Limit

Definition: the function $f(x)$ tends to ∞ as x tends to $a \in \mathbb{R}$ if and only if for each $M > 0$, there is a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$. We can then write

$$\lim_{x \to a} f(x) = \infty$$

Definition: the function $f(x)$ tends to $-\infty$ as x tends to $a \in \mathbb{R}$ if and only if for each $M < 0$, there is a $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - a| < \delta$. We can then write

$$\lim_{x \to a} f(x) = -\infty$$

Infinite Limit at Infinity

Similar to all the definitions described so far, we can also define the following different types of infinite limit at infinity:

- $\lim_{x \to \infty} f(x) = \infty$
- $\lim_{x \to -\infty} f(x) = \infty$
- $\lim_{x \to \infty} f(x) = -\infty$
- $\lim_{x \to -\infty} f(x) = -\infty$

Basic Laws of Finite Limit

If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, where $L, M \in \mathbb{R}$ and a can be real or $\pm\infty$, then:

(1) $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x) = c \cdot L, where\ c\ is\ a\ constant.$

(2) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

(3) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right] = L \cdot M$

(4) $if\ M \neq 0, then\ \lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$

Please note that the laws above can apply to one-sided limits, i.e. a can be replaced with either a^+ or a^-.

When Infinite Limits are involved

If we allow the L or M at the previous slide to be infinity,

Given $c \in \mathbb{R}$ we have the following laws:

- $\infty + \infty = \infty$, $\infty \cdot \infty = \infty$, $\frac{c}{\infty} = 0$
- $c + \infty = \infty$, $c - \infty = -\infty$
- $c \cdot \infty = \infty$ for $c > 0$
- $c \cdot \infty = -\infty$ for $c < 0$

We do **NOT** have a law for the following expressions:

- $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, $\frac{0}{0}$
- $\infty - \infty$

Limit at infinity of a composite function

(Theorem) define the composite function $h(x) = f[g(x)]$. If $f(x)$ tends to $L \in \mathbb{R}$ as $x \to \infty$, and $g(x) \to \infty$ as $x \to \infty$, then

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} f[g(x)] = L$$

Answer:

For any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > \delta_1$.

There exists a $\delta > 0$ such that $g(x) > \delta_1$ whenever $x > \delta$.

As such, $\left|f\left(g(x)\right) - L\right| < \epsilon$ whenever $x > \delta$.

Solution to the Interview Question with $\sigma \to \infty$ (1/2)

$$\lim_{\sigma\to\infty} \frac{\log\frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} = \frac{\log\frac{S_0 e^{rT}}{K}}{\sqrt{T}} \lim_{\sigma\to\infty} \frac{1}{\sigma} = 0$$

$$\lim_{\sigma\to\infty} \frac{\sigma\sqrt{T}}{2} = \frac{\sqrt{T}}{2} \lim_{\sigma\to\infty} \sigma = \infty \text{ (note that } \sqrt{T} > 0)$$

$$\lim_{\sigma\to\infty} d_1 = \lim_{\sigma\to\infty} \left[\frac{\log\frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right] = \lim_{\sigma\to\infty} \frac{\log\frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} + \lim_{\sigma\to\infty} \frac{\sigma\sqrt{T}}{2} = \infty$$

$$\lim_{\sigma\to\infty} d_2 = \lim_{\sigma\to\infty} \left[\frac{\log\frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right] = \lim_{\sigma\to\infty} \frac{\log\frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} - \lim_{\sigma\to\infty} \frac{\sigma\sqrt{T}}{2} = -\infty$$

Solution to the Interview Question with $\sigma \to \infty$ (2/2)

As N is continuous, we know that

$\lim_{\sigma\to\infty} N(d_1) = 1$ and $\lim_{\sigma\to\infty} N(d_2) = 0$

$$\lim_{\sigma\to\infty} S_0 N(d_1) = S_0 \lim_{\sigma\to\infty} N(d_1) = S_0$$

$$\lim_{\sigma\to\infty} Ke^{-rT} N(d_2) = Ke^{-rT} \lim_{\sigma\to\infty} N(d_2) = 0$$

$$\lim_{\sigma\to\infty} C_0 = \lim_{\sigma\to\infty} [S_0 N(d_1) - Ke^{-rT} N(d_2)]$$

$$= \lim_{\sigma\to\infty} S_0 N(d_1) - \lim_{\sigma\to\infty} Ke^{-rT} N(d_2) = S_0$$

Example: Continuous Compounding

How interest is compounded does matter!

- If we set up a 1-year time deposit of $100 with a bank, which pays interest at an annualized rate of 10%. How much will we get back in one year time?
- The question above cannot be answered unless we know what is compounding frequency of the deposit.
- When we compound m times per year at rate r, an amount A grows to $A \cdot \left(1+\frac{r}{n}\right)^n$ in one year

Compounding frequency	*Value of $100 in one year at 10%*	*Value Difference*
Annual (n=1)	110.00	Nil
Semi-annual (n=2)	110.25	0.25
Quarterly (n=4)	110.38	0.13
Monthly (n=12)	110.47	0.09
Weekly (n=52)	110.51	0.04
Daily (n=365)	110.52	0.01

Question

- If we take a close look at the compounding table on the previous slide, we can see that as we increase m, the final value increases whereas the value difference decreases.
- If we compound more and more times per year, what will the value of our deposit be in one year? Will it tend to infinity or converge to a certain number?
- Mathematically, we want to compute:

$$\lim_{n\to\infty} A \cdot \left(1 + \frac{r}{n}\right)^{n} = ?$$

Sequences and Series

Sequence and Series

- A sequence can be viewed as a function from $\mathbb{N}$ to $\mathbb{R}$.
  - Example: Let the sequence $a_n = \frac{1}{n}$, then $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, …, and so on.
- Given a sequence a_n, we can define $s_n = \sum_{j=1}^{n} a_j$.
  - Example: Let $s_n = \sum_{j=1}^{n} \frac{1}{j}$, then $s_1 = 1$, $s_2 = 1 + \frac{1}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3}$, $s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, …, and so on.
  - s_n is so called a series (with underlying sequence a_n). Obviously, a series itself is also a sequence.
- Taking the limit of a sequence or series can be used to define or generate a number, which cannot be stated in advance. In this section, our goal is to generate the number e, which we use excessively in financial mathematics.

Limit of a sequence

- Given a sequence a_n and a $L \in \mathbb{R}$, if for any $\epsilon > 0$, there exists a $M \in \mathbb{N}$, such that $|a_n - L| < \epsilon$ whenever $n > M$, then we say $a_n \to L$ as $n \to \infty$, or
$$\lim_{n\to\infty} a_n = L$$
- Given a sequence a_n, if for any $\epsilon > 0$, there exists a $M \in \mathbb{N}$, such that $a_n > \epsilon$ whenever $n > M$, then we say $a_n \to \infty$ as $n \to \infty$, or
$$\lim_{n\to\infty} a_n = \infty$$
- Similarly, we can define $\lim_{n\to\infty} a_n = -\infty$.
- If $\lim_{n\to\infty} a_n = L$, then we say a_n is convergent; otherwise it's divergent.

Geometric Series

- Given constants $a \neq 0$ and r, the geometric series is defined as follows:

$$S_n = \sum_{j=1}^{n} a \cdot r^{(j-1)} = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1}$$

$$= a \cdot (1 + r + r^2 + \cdots + r^{n-1})$$

$$= a \cdot \frac{(1-r) \cdot (1 + r + r^2 + \cdots + r^{n-1})}{1-r} = a \cdot \frac{1-r^n}{1-r}$$

- If we try to take the limit of S_n, we see that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[a \cdot \frac{1-r^n}{1-r}\right] = \frac{a}{1-r} \cdot \left[1 - \lim_{n\to\infty} r^n\right]$$

- Dependent on the value of r, we can have either of the following:
  - $|r| < 1 \Rightarrow \lim_{n\to\infty} S_n = \frac{a}{1-r}$, so S_n is convergent
  - $|r| \geq 1 \Rightarrow S_n$ is divergent.

Monotonic Sequences and Their Limit

- We say that a given sequence a_n is monotonic, if either of the following holds:
  - $a_j < a_{j+1}\ \forall\ j \Rightarrow a_n$ is a monotonic increasing sequence;
  - $a_j > a_{j+1}\ \forall\ j \Rightarrow a_n$ is a monotonic decreasing sequence.
- Monotonic Sequence Theorem:
  - If a_n is monotonic increasing and bounded above by a fixed number M, i.e., $a_j < M\ \forall\ j$, then a_n is convergent with $\lim_{n\to\infty} a_n < M$.
  - If a_n is monotonic decreasing and bounded below by a fixed number N, i.e., $a_j > N\ \forall\ j$, then a_n is convergent with $\lim_{n\to\infty} a_n > N$.
- Given the “bounded-ness” of a monotonic sequence, the theorem above tells us about the existence of its limit, but the theorem dose not tell us what is the exact value of the limit.

Sandwiching Theorem (for Sequences)

- (Sandwiching Theorem) given sequences a_n, b_n, and c_n, if a_n and c_n converge to the same limit L and there exists a $M \in \mathbb{N}$ such that $a_n \geq b_n \geq c_n \ \forall\, n > M$, then
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

- Example: $\lim_{n\to\infty} \frac{n!}{n^n} = ?$
$$\forall\, n \in \mathbb{N}, \qquad 0 < \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \dots \cdot \frac{n-1}{n} \cdot \frac{n}{n} \leq \frac{1}{n}$$
$$\lim_{n\to\infty} \frac{1}{n} = 0$$
$$By\ Sandwiching\ Theorem, \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Binomial Expansion

- (Binomial Theorem) Suppose that $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then
$$(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} x^{n-j} y^j \ .$$
- Example 1:
$$(x+y)^2 = \frac{2!}{0!\,2!} x^{2-0} y^0 + \frac{2!}{1!\,1!} x^{2-1} y^1 + \frac{2!}{2!\,0!} x^{2-2} y^2$$
$$= x^2 + 2xy + y^2$$
- Example 2:
$$\left(1+\frac{1}{n}\right)^n$$
$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2)\ldots 1}{n!}\frac{1}{n^n}$$

Existence of the number e

- We define a series S_n and a sequence T_n as follows

$$S_n := \sum_{j=0}^{n} \frac{1}{j!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$T_n := \left(1 + \frac{1}{n}\right)^n$$

- It turns out that both of S_n and T_n converge to the same limit. We denote their limit as e, so we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = e$$

- Now let's prove $\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n$.

Existence of the number e (proof, step 1)

- Firstly, let's show S_n is convergent. We know

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

- We observe that

$$S_n \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

$$= 1 + 2 \cdot \left(1 - \frac{1}{2^n}\right) < 3$$

- Since S_n is monotonic increasing and bounded above, S_n converges to a limit, which we denote by e:

$$\lim_{n \to \infty} S_n = e$$

Existence of the number e (proof, step 2)

- Secondly, let's show T_n is convergent. By Binomial Theorem,

$$T_n = \left(1 + \frac{1}{n}\right)^n$$
$$= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2)\ldots 1}{n!}\frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

- We can see that $T_n \leq S_n$ so T_n is bounded above. In addition, T_n is a monotonic increasing sequence. By Monotonic Sequence Theorem, we know that T_n converges to a limit, which we denote by T:

$$\lim_{n\to\infty} T_n = T$$

Existence of the number e (proof, step 3)

- Finally, let's show $T = e$.
- Provided that $m > n$, we observe that

$$T_m > 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\ldots\left(1 - \frac{n-1}{m}\right)$$

- If we now keep n fixed and let m tend to infinity, we obtain on the left the number T and on the right the series S_n. As such, we can see that

$$T \geq S_n \geq T_n \quad \forall\, n$$

- By Sandwiching Theorem,

$$\lim_{n \to \infty} S_n = T \Rightarrow T = e$$