Prep Course
Module I
Calculus Lecture 3

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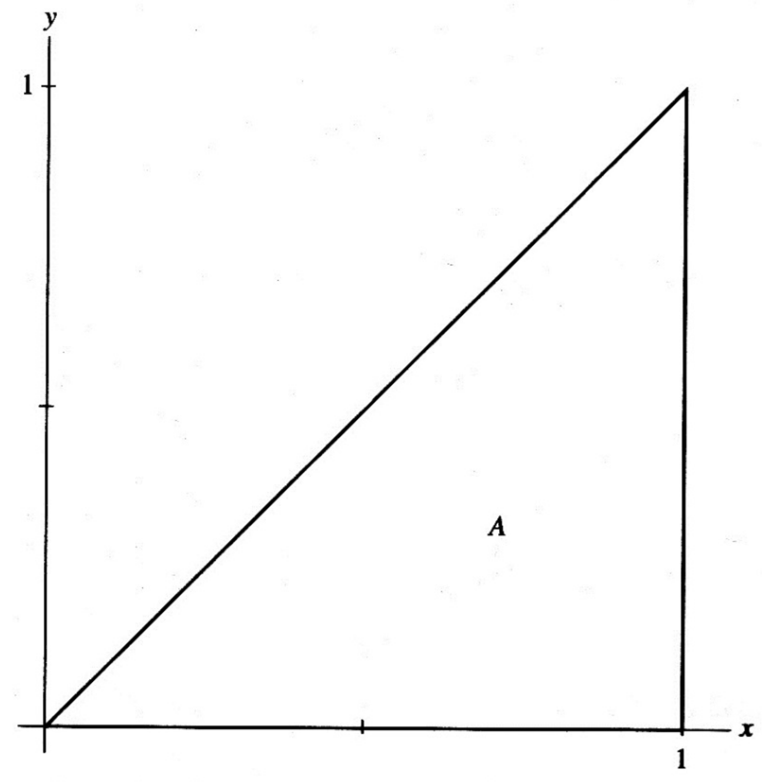
Double Integral

The idea of double integral

Given an area A on $\mathbb{R}^2$, the value of the double integral

$$\int_A f(x,y)dA = \iint\limits_A f(x,y)dxdy$$

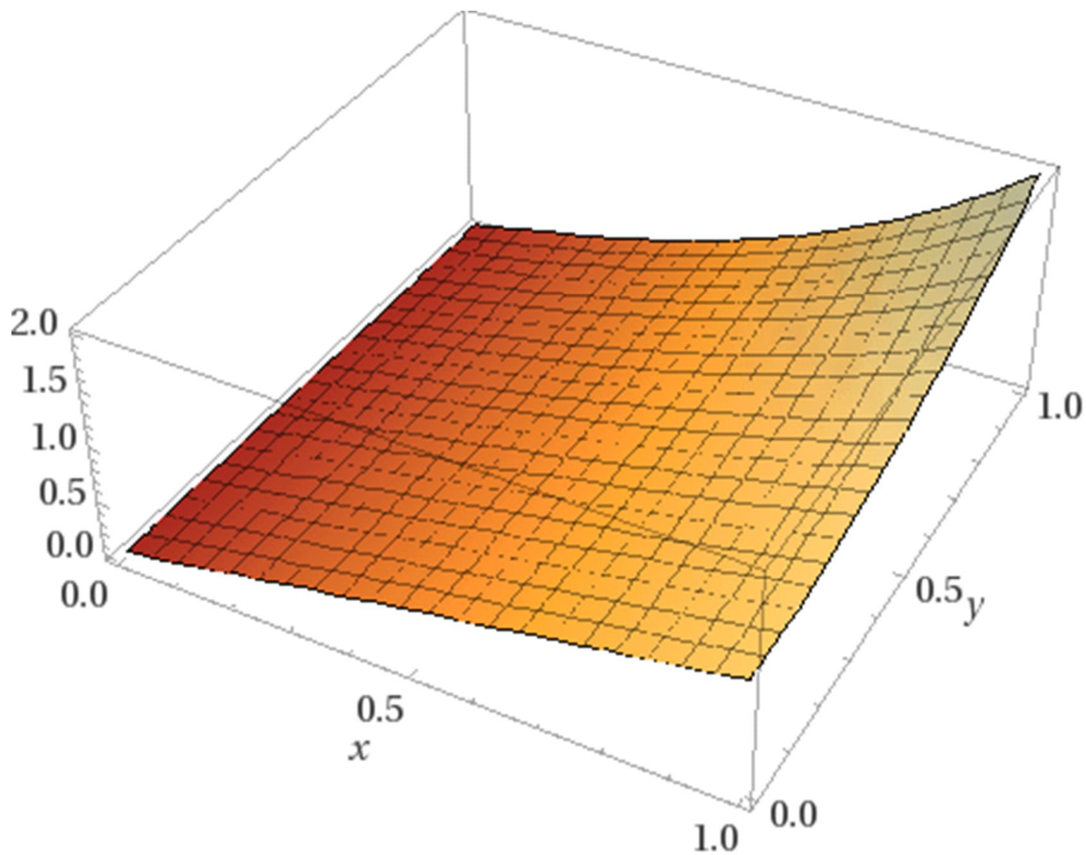
can usually be evaluated by an iterated process: that is, evaluating two successive integrals.



Example of double integral

For illustration, say $A = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $f(x, y) = x + x^3y^2$

$$\iint_A (x + x^3y^2)\,dx\,dy$$



Example of double integral

Given $A = \{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$,

$$\iint_A (x + x^3y^2)dxdy = \int_0^1 \left[\int_0^1 (x + x^3y^2)dy\right] dx$$

$$= \int_0^1 \left[xy + \frac{x^3y^3}{3}\right]_0^1 dx = \int_0^1 \left(x + \frac{x^3}{3}\right) dx = \left[\frac{x^2}{2} + \frac{x^4}{3 \cdot 4}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$$

Note that:

- In the inner integral on y, x is treated as a constant.

Example of double integral, if A is not a square

Given $A = \{(x,y)|0 \le x \le 1, 0 \le y \le x\}$,

$$\iint_A (x + x^3y^2)dxdy = \int_0^1 \left[\int_0^x (x + x^3y^2)dy\right] dx$$

$$= \int_0^1 \left[xy + \frac{x^3y^3}{3}\right]_0^x dx = \int_0^1 \left(x^2 + \frac{x^6}{3}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^7}{3 \cdot 7}\right]_0^1 = \frac{1}{3} + \frac{1}{21} = \frac{8}{21}$$

Note that:

- For each fixed x between 0 and 1, y is restricted to the interval 0 and x.

Let's change the order of variables to integrate

$$\iint_A (x + x^3y^2)dxdy = \int_0^1 \left[\int_y^1 (x + x^3y^2)dx\right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + \frac{x^4y^2}{4}\right]_y^1 dy = \int_0^1 \left[\frac{1}{2} + \frac{y^2}{4} - \frac{y^2}{2} - \frac{y^6}{4}\right] dy$$

$$= \left[\frac{y}{2} - \frac{y^3}{3 \cdot 4} - \frac{y^7}{7 \cdot 4}\right]_0^1 = \frac{1}{2} - \frac{1}{12} - \frac{1}{28} = \frac{8}{21}$$

Note that in evaluating this double integral we could have restricted y to the interval 0 to 1, then x would be between y and 1. This is because $f(x, y)$ is continuous. (Fubini Theorem)

Increasing or Decreasing

QC (Quantitative Comparison) Question 1

We know $\pi > e$, but which of e^{π} and π^{e} is greater? Please prove your answer mathematically without using a calculator.

This seems to be a popular interview question, which I quoted from the book, “Quant Job Interview Questions and Answers,” by Mark Joshi, Nick Denson, and Andrew Downes.

Monotone functions

Let function f have an interval I of $\mathbb{R}$ as its domain and a set in $\mathbb{R}$ as its range. We say

- The function f is increasing on I if and only if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
- The function f is non-decreasing on I if and only if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.
- The function f is decreasing on I if and only if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.
- The function f is non-increasing on I if and only if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$.

A function which has any one of these four properties is called monotone on I.

First derivative v.s. Decreasing/Increasing

(Theorem) Suppose that function f is continuous on the closed interval I and has a derivative at each point of I_1, the interior of I.

- If f' is positive on I_1, then f is increasing on I.
- If f' is negative on I_1, then f is decreasing on I.

Solution to QC Question 1 (1/2)

If we could compare

$$\pi \cdot \log e \text{ and } e \cdot \log \pi$$

then we know which of e^{π} and π^{e} is greater, because $\log x$ is increasing.

If we could compare

$$\frac{\log e}{e} \text{ and } \frac{\log \pi}{\pi}$$

then we know which of $\pi \cdot \log e$ and $e \cdot \log \pi$ is greater, because both e and π are positive.

It's now a statement about the function

$$f(x) = \frac{\log x}{x}$$

Solution to QC Question 1 (2/2)

We check the first derivative of $f(x)$

$$f'(x) = \frac{1 - \log x}{x^2}.$$

This is equal to zero for $x = e$, and less than zero for $x > e$, so f is a decreasing function on $[e, \infty)$. As such, we know

$$f(e) = \frac{\log e}{e} > \frac{\log \pi}{\pi} = f(\pi)$$

$$\Rightarrow \pi \cdot \log e > e \cdot \log \pi$$

$$\Rightarrow e^{\pi} > \pi^{e}$$

Convex or Concave

QC (Quantitative Comparison) Question 2

Which of

$$e^{\frac{a+b}{2}} \quad \text{and} \quad \frac{e^a + e^b}{2}$$

is greater? Please prove your answer mathematically without using a calculator.

If you approach this question by taking log of both numbers, you will be probably getting nowhere.

Convex functions

A function f defined on an interval I is said to be convex on I if its graph lies on or below all of its chords. In other words, for all $x, y \in I$, all $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

Properties

- A twice differentiable function f is convex on I if and only if $f'' \geq 0$ on I.
- A convex function's graph lies on or above all of its tangents.
- Jensen's inequality: if f is convex on I and X is an integrable random variable taking values in I then
$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

Concave functions

A function f defined on an interval I is said to be concave on I if its graph lies above all of its chords. In other words, for all $x, y \in I$, all $\alpha \in [0,1]$,

$$f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y)$$

Properties

- A twice differentiable function f is concave on I if and only if $f'' < 0$ on I.
- A concave function's graph lies below all of its tangents.
- Jensen's inequality: if f is concave on I and X is an integrable random variable taking values in I then
$$\mathbb{E}[f(X)] < f(\mathbb{E}[X])$$

Solution to QC (Quantitative Comparison) Question 2

Let $f(x) = e^x$ and X be a random variable with $P(X = a) = P(X = b) = 0.5$. Then

$$e^{\frac{a+b}{2}} = e^{\mathbb{E}[X]} = f(\mathbb{E}[X])$$

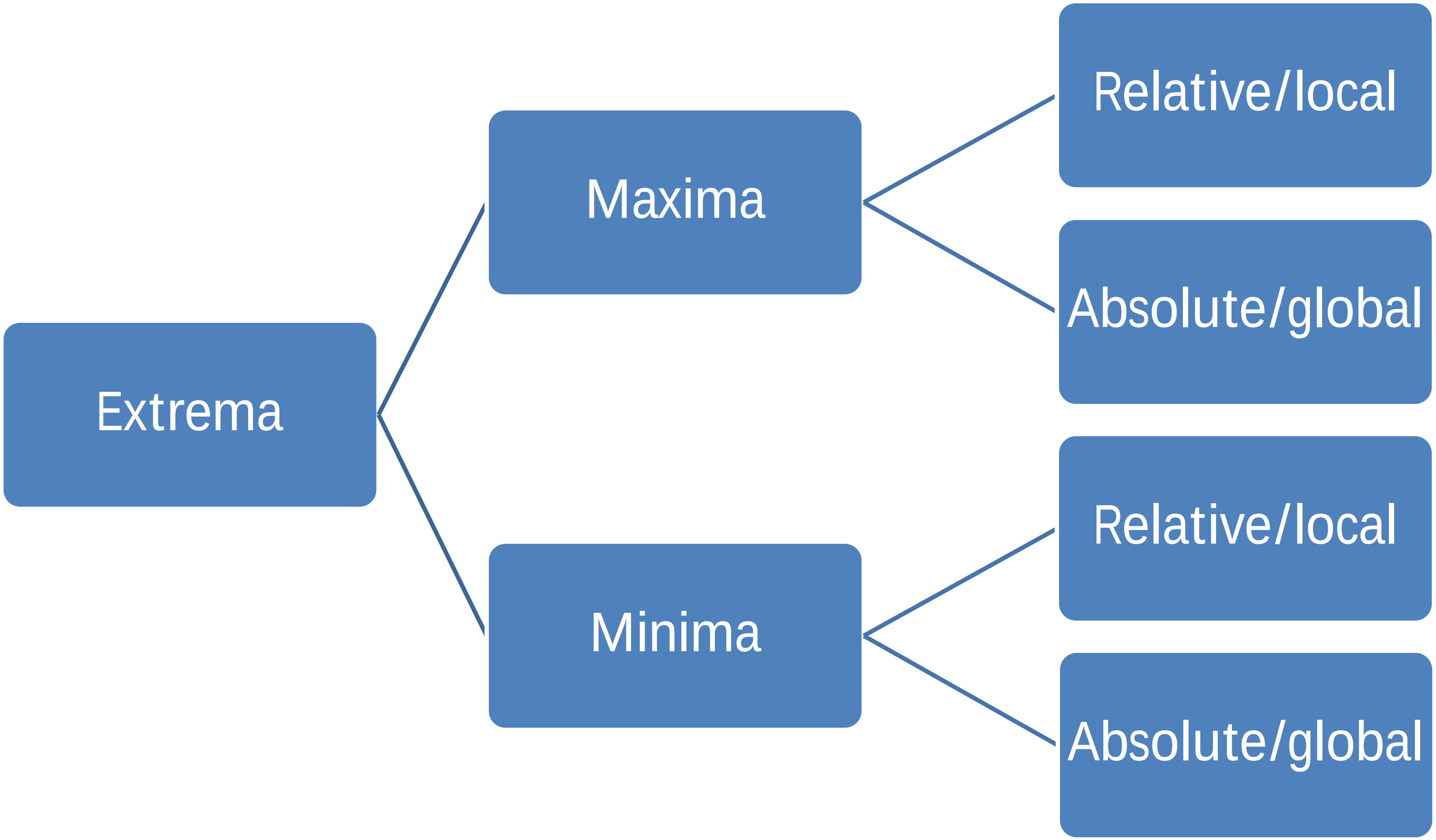
$$\frac{e^a + e^b}{2} = \frac{f(a) + f(b)}{2} = \mathbb{E}[f(X)]$$

We know that f is convex because $f'' > 0$. By Jensen's inequality:

$$\frac{e^a + e^b}{2} > e^{\frac{a+b}{2}}$$

Maxima and Minima

Types of Extreme Values



Definitions of Extreme Values

Relative extrema (or local extrema): let D_f denote the domain of function f and $x_0 \in D_f$, if there exists $\delta > 0$ such that $\forall x \in D_f$ and $|x - x_0| < \delta$,

- if $f(x) < f(x_0)$ always holds, then $f(x_0)$ is a local maxima.
- if $f(x) > f(x_0)$ always holds, then $f(x_0)$ is a local minima.

Absolute extrema (or global extrema): let D_f denote the domain of function f and $x_0 \in D_f$, $\forall x \in D_f$

- if $f(x) < f(x_0)$ always holds, then $f(x_0)$ is the global maxima.
- if $f(x) > f(x_0)$ always holds, then $f(x_0)$ is the global minima.

Where an extreme value could occur

- (Theorem) If $f(x_0)$ is an extreme value of f and $f'(x_0)$ exists, then $f'(x_0) = 0$ and x_0 is called a stationary point of f.

- A point x_0 is called a critical point, if it satisfies one of the following:
  - $f'(x_0) = 0$, i.e., x_0 is a stationary point;
  - $f'(x_0)$ does not exist, i.e., x_0 is a singular point;
  - x_0 is an end point.

- When looking for an extreme values, one has to consider all the 3 types of critical points.

Whether an extreme value is a maxima or minima

- By first derivative (if f' exists),
  - $f(x_0)$ is a maxima, if there exists $\delta > 0$ such that $f'(x) > 0$ when $0 < x_0 - x < \delta$ and $f'(x) < 0$ when $0 < x - x_0 < \delta$.
  - $f(x_0)$ is a minima, if there exists $\delta > 0$ such that $f'(x) < 0$ when $0 < x_0 - x < \delta$ and $f'(x) > 0$ when $0 < x - x_0 < \delta$
- By second derivative (if f'' exists),
  - $f(x_0)$ is a maxima, if $f'(x_0) = 0$ and $f''(x_0) < 0$.
  - $f(x_0)$ is a minima, if $f'(x_0) = 0$ and $f''(x_0) > 0$.

Where a convex function starts going concave

Intuitively, an inflection point is the point x_0 where f will change from being convex to being concave, and vice versa. If f'' exists, x_0 is the place where f''will change its sign, so you can imagine that it can be determined as follows:

x_0 is called an inflection point when the 2 conditions below hold

- $f''(x_0) = 0$
- There exists $\delta > 0$ such that either one of the following holds:
  - $f''(a) > 0$ and $f''(b) < 0$, $\forall a \in (x_0 - \delta, x_0)$ and $\forall b \in (x_0, x_0 + \delta)$
  - $f''(a) < 0$ and $f''(b) > 0$, $\forall a \in (x_0 - \delta, x_0)$ and $\forall b \in (x_0, x_0 + \delta)$

If f''' exists, then the second condition above can be replaced with:

- $f'''(x_0) \neq 0$

Note that $f''(x_0) = 0$ alone does not warrant an inflection point. For example, 0 is not an inflection point of $f(x) = x^4$.

Example of finding maxima, minima, and inflection

Given $f(x) = x^4 - 4x^3 - 2x^2 + 12x + 7$, find the maxima, minima, and inflection point of $f(x)$

Answers:

$$f'(x) = 4x^3 - 12x^2 - 4x + 12$$
$$\Rightarrow f'(x) = 0, when\ x = \pm 1\ or\ 3$$

$$f''(x) = 12x^2 - 24x - 4$$
$$\Rightarrow f''(x) = 0, when\ x = 1 \pm \frac{2}{3}\sqrt{3}$$

$$f'''(x) = 24x - 24$$

$$f(-1) = f(3) = -2 \text{ are the minima as } f''(-1) > 0\ and\ f''(3) > 0$$
$$f(1) = 14 \text{ is the maxima as } f''(1) < 0$$
$$x = 1 \pm \frac{2}{3}\sqrt{3} \text{ are inflection points as } f'''\left(1 \pm \frac{2}{3}\sqrt{3}\right) \neq 0$$