Module I Homework 5

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Prep24AutumnM1

September 3, 2024

Problem 1.

Intuition: $\lambda = 1$ means that on average, there is one typographical error per page, informing how to approach calculating the probability that there is at least one typographical error on a randomly chosen page

Solution. $\square$

First, note that $P(X \geq 1) = 1 - P(X = 0)$

Then, computing $P(X = 0)$:

$$P(X = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$

Thus, for $\lambda = 1$:

$$P(X \geq 1) = 1 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

$$\approx \boxed{0.6321}$$

Problem 2.

Solution. $\square$

CDF of M

First, note that the CDF $F_M(m)$ is given by:

$$F_M(m) = P(M \leq m)$$

Since M is the minimum of the X_is, $M \le m$ iff at least one $X_i \le m$:

$$F_M(m) = P(\min(X_1, X_2, \dots, X_n) \le m)$$

$$= 1 - P(\min(X_1, X_2, \dots, X_n) > m)$$

$$= 1 - P(X_1 > m) \times P(X_2 > m) \times \dots \times P(X_n > m)$$

Since $X_i \sim \text{Uniform}[0, 1]$:

$$P(X_i > m) = 1 - P(X_i \le m) = 1 - m$$

As such, the CDF of M:

$$\boxed{F_M(m) = 1 - (1 - m)^n}$$

PDF of M

Note, the PDF $f_M(m)$ can be expressed as:

$$f_M(m) = \frac{d}{dm} F_M(m)$$

Computing:

$$\frac{d}{dm} F_M(m) = \frac{d}{dm} \left(1 - (1 - m)^n\right)$$

$$= n(1 - m)^{n-1}$$

Thus, the PDF of M:

$$\boxed{f_M(m) = \begin{cases} n(1 - m)^{n-1} & \text{if } 0 \le m \le 1, \\ 0 & \text{otherwise} \end{cases}}$$

Problem 3.

Solution. □

First, computing the CDF:

$$F_Z(z) = P(Z \le z)$$

$$= P(3X^2 + 1 \leq z) = P\left(-\sqrt{\frac{z-1}{3}} \leq X \leq \sqrt{\frac{z-1}{3}}\right)$$

$$= P\left(-\sqrt{\frac{z-1}{3}} \leq X \leq \sqrt{\frac{z-1}{3}}\right)$$

$$= F_X\left(\sqrt{\frac{z-1}{3}}\right) - F_X\left(-\sqrt{\frac{z-1}{3}}\right)$$

... where $z \geq 1$

Then taking the derivative of the CDF $F_Z(z)$ to compute the PDF $f_Z(z)$:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{1}{2\sqrt{3}\sqrt{z-1}} f_X\left(\sqrt{\frac{z-1}{3}}\right) - \left(\frac{1}{2\sqrt{3}\sqrt{z-1}}\right) f_X\left(-\sqrt{\frac{z-1}{3}}\right)$$

$$= \frac{1}{2\sqrt{3}\sqrt{z-1}\sqrt{2\pi}} e^{-\frac{z-1}{6}} + \frac{1}{2\sqrt{3}\sqrt{z-1}\sqrt{2\pi}} e^{-\frac{z-1}{6}}$$

$$= \boxed{\frac{1}{\sqrt{3}\sqrt{z-1}\sqrt{2\pi}} e^{-\frac{z-1}{6}}}$$

where $z \geq 1$

(Note the utilization of the chain rule and the fact that because X follows a standard normal distribution, its PDF is given by $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.)

Problem 4.

Intuition: Apply Bayes' Theorem

Solution. □

$$P(\text{disease} \mid \text{positive})$$

$$= \frac{P(\text{disease}) \cdot P(\text{positive} \mid \text{disease})}{P(\text{disease}) \cdot P(\text{positive} \mid \text{disease}) + P(\text{no disease}) \cdot P(\text{positive} \mid \text{no disease})}$$

$$= \frac{150 \cdot 0.99}{150 \cdot 0.99 + (100000 - 150) \cdot 0.01}$$

$$\approx \boxed{0.1295}$$

Problem 5.

First, identifying the limits of integration and rewriting the integral:

$$x \text{ ranges from } 0 \text{ to } \sqrt{y}\ ,$$
$$y \text{ ranges from } 0 \text{ to } 4$$

Rewriting the integral:

$$\iint_R xe^{y^2}\,dx\,dy = \int_0^4 \int_0^{\sqrt{y}} xe^{y^2}\,dx\,dy$$

Integrating first w/r/t/ x:

$$\int_0^{\sqrt{y}} xe^{y^2}\,dx = e^{y^2}\int_0^{\sqrt{y}} x\,dx = e^{y^2}\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} = \frac{ye^{y^2}}{2}$$

Integrating next w/r/t/ y:

$$\int_0^4 \frac{ye^{y^2}}{2}\,dy = ...$$

Given the complexity of the integrand, we will integrate via substitution...

Let $u = y^2$... $du = 2y\,dy$

As such:

$$\int_0^4 \frac{ye^{y^2}}{2}\,dy = \frac{1}{4}\int_0^{16} e^u\,du$$
$$= \frac{1}{4}\left(e^{16}-1\right)$$
$$\approx \boxed{2221527.6}$$

Problem 6.

Intuition: Apply Bayes' Theorem

Solution. □

First, let's define the following events:

- A: the man reports 5
- B_i: the outcome of rolling the die is i
  (Thus, $\{B_1, \ldots, B_6\}$ is the partition of the sample space Ω.)

Then, computing $P(B_5 \mid A)$:

$$P(B_5 \mid A)$$

$$= \frac{P(B_5)P(A \mid B_5)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + ... + P(B_6)P(A \mid B_6)}$$

$$= \frac{\frac{1}{6}P(A \mid B_5)}{\frac{1}{6}P(A \mid B_1) + \frac{1}{6}P(A \mid B_2) + \frac{1}{6}P(A \mid B_3) + \frac{1}{6}P(A \mid B_4) + \frac{1}{6}P(A \mid B_5) + \frac{1}{6}P(A \mid B_6)}$$

$$= \frac{\frac{1}{6} \cdot \frac{4}{5}}{\frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{4}{5} + \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{5}}$$

$$= \frac{\frac{4}{5}}{\frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{4}{5} + \frac{1}{5} \cdot \frac{1}{5}}$$

$$= \boxed{\frac{4}{9}}$$