# Module I Homework 6

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Prep24AutumnM1

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**Problem 1.**

*Solution.* □

**(a)**

Calculating $P(XY > z)$:

$$P(XY > z) = \iint_C f(x,y)\,dx\,dy$$

where $C$ is the region of the unit square where $xy > z$, *i.e.*, $C = \{(x,y) : 0 \leq x \leq 1,\ \frac{z}{x} \leq y \leq 1\}$
*N.b.*: $x$ must start from $z$ to possibly satisfy $xy > z$, given $0 \leq y \leq 1$
*N.b.*: We derive $\frac{z}{x} \leq y \leq 1$ from the desired condition that $z < xy$

Accordingly:

$$\iint_C f(x,y)\,dx\,dy = \int_{x=z}^{1} \int_{y=z/x}^{1} 2x\,dy\,dx$$

Integrating first w/r/t/ $y$:

$$\int_{y=z/x}^{1} 2x\,dy$$
$$= 2x\left(1 - \frac{z}{x}\right)$$
$$= 2x - 2z$$

Integrating next w/r/t/ $x$:

$$P(XY > z) = \int_{z}^{1} (2x - 2z)\,dx$$

$$= \left[x^2 - 2xz\right]_z^1$$
$$= \boxed{1 - 2z + z^2}$$

**(b)**

Deriving the CDF of $Z$, $F_Z(z)$:

$$F_Z(z) = P(XY \leq z)$$
$$= 1 - P(XY > z)$$
$$= 1 - \left(1 - 2z + z^2\right)$$
$$= \boxed{2z - z^2} \text{ for } 0 \leq z \leq 1$$

**(c)**

First, computing the PDF of $Z$, $f_Z(z)$:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$
$$= \frac{d}{dz}\left(2z - z^2\right)$$
$$= 2 - 2z \text{ for } 0 \leq z \leq 1$$

Then, computing the Expected Value of $Z$, $\mathbb{E}[Z]$:

$$\mathbb{E}[Z] = \int_0^1 z f_Z(z)\, dz$$
$$= 2\int_0^1 \left(z - z^2\right)\, dz$$
$$= 2\left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^1$$
$$= 2\left(\frac{1^2}{2} - \frac{1^3}{3}\right)$$
$$= \boxed{\frac{1}{3}}$$

**Problem 2.**

Intuition: Maxima exist where $\frac{d}{dx}f(x) = 0$, *i.e.*, where the first derivative changes sign, and inflection points exist where $\frac{d}{dx}f'(x) = 0$, *i.e.*, where the second derivative changes sign

*Solution.* □

First, calculating $\frac{d}{dx}f(x)$:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot (-x)$$

$$= -\frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

Then, deriving the maxima:

$$0 = -\frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$f'(x) = 0 \text{ is satisfied for } x = 0$$

Thus, evaluating $f(0)$, the maximum of $f(x)$:

$$\boxed{\left(0, \frac{1}{\sqrt{2\pi}}\right)}$$

Next, calculating $\frac{d}{dx}f'(x)$:

$$\frac{d}{dx}f'(x) = \frac{d}{dx}\left(-\frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}}(-1)\,e^{-\frac{x^2}{2}} + \frac{1}{\sqrt{2\pi}}\left(x^2\right)e^{-\frac{x^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\left(x^2 - 1\right)e^{-\frac{x^2}{2}}$$

Then, deriving the inflection point:

$$0 = \frac{1}{\sqrt{2\pi}}\left(x^2 - 1\right) e^{-\frac{x^2}{2}}$$
$$f''(x) = 0 \text{ is satisfied for } x = 1 \text{ and } x = -1$$

Thus, evaluating $f'(x) = 1$, $f'(x) = -1$, the inflection points of $f(x)$:

$$\boxed{\left(1, \frac{1}{e^2\sqrt{2\pi}}\right), \left(-1, \frac{1}{e^2\sqrt{2\pi}}\right)}$$

**Problem 3.**

*Solution.* □

$$\begin{aligned}
F_U(u) &= P(U \leq u) \\
&= P(40(1 - X) \leq u) \\
&= P\left(X \geq 1 - \frac{u}{40}\right) \\
&= 1 - P\left(X \leq 1 - \frac{u}{40}\right) \\
&= \int_{1-\frac{u}{40}}^{1} 3x^2 \, dx \\
&= \left[x^3\right]_{1-\frac{u}{40}}^{1} \\
&= \boxed{(a)\ 1 - \left(1 - \frac{u}{40}\right)^3}
\end{aligned}$$

**Problem 4.**

Intuition:

The PDF of the continuous random variable $U$ can be expressed as the first-order derivative of the CDF of $U$

*Solution.* □

$$\begin{aligned}
\frac{d}{du}F_U(u) &= \frac{d}{du}\left(1 - \left(1 - \frac{u}{40}\right)^3\right) \\
&= -(3)\left(1 - \frac{u}{40}\right)^2\left(-\frac{1}{40}\right) \\
&= \boxed{(a)\ \frac{3}{40}\left(1 - \frac{u}{40}\right)^2}
\end{aligned}$$