Prep Course

Module I Homework 3

Due 23:30, Saturday, July 27th, 2024 (Chicago time)

For this Homework, you should submit a single PDF named: **M1HW3_LASTNAME_Firstname.pdf**

Problem 1

Prove that $f(x) = x^3$ is continuous at $x = -2$.

Please prove it by imitating the delta-epsilon approach on slide 8 of Calculus lecture 1.

Problem 2

In Calculus lecture 1, we introduced the "sequence" version of the Sandwiching Theorem. In fact, we also have a "function" version of the same theorem: given functions $f(x)$, $g(x)$, and $h(x)$, if $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L \in \mathbb{R}$ and there exists a $\delta > 0$ such that $h(x) \leq f(x) \leq g(x)$ for any x satisfying $0 < |x-a| < \delta$, then $\lim_{x\to a} f(x) = L$.

If $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ x^4, & x \notin \mathbb{Q} \end{cases}$, please use the Sandwiching Theorem to show $\lim_{x\to 0} f(x) = 0$.

Problem 3

Given a sequence $a_n = \frac{2^n}{n!}$, please compute $\lim_{n\to\infty} a_n$.

Problem 4

<u>No justification or proof is required for this problem.</u>

Suppose that, over the course of the next year, a particular investment fund has a 40% probability of beating the market and a 60% probability of under-performing (perhaps they are poorly managed, or charge high fees). If the fund outperforms the market it will (certainly) continue operating for another year, but it is in danger if it under-performs: in that case there's a 50% probability that its investors will angrily withdraw their money, so that the fund simply ceases to exist.

If the fund still exists at the end of the year, which of the following options is the probability that it beat the market?

a) $\frac{1}{2}$
b) $\frac{3}{4}$
c) $\frac{4}{7}$

d) $\frac{4}{5}$

Problem 5

On slide 18 of Calculus lecture 1, we discussed a theorem about the limit of a composite function. In fact, there are many variants of the theorem. One of them is as follows: if $f(x)$ tends to ∞ as $x \to \infty$, and $g(x) \to \infty$ as $x \to L^+$, then

$$\lim_{x \to L^+} f[g(x)] = \infty \, .$$

Please prove it.

Problem 6

On slide 4 of Calculus lecture 1, we talked about the Black-Scholes call option pricing formula. Please compute the following:

$$\lim_{K \to 0^+} C_0$$

You may quote the theorem in Problem 5, if necessary. For the purpose of sanity check, I suggest that you think about whether your answer makes sense in the real world.