Module I Homework 7

qquantt
Prep24AutumnM1

September 5, 2024

Problem 1.

Let i denote the ith ball in the queue

Defining an indicator random variable I_i for each ball i, where $I_i = 1$ if ball i has a neighbor of a different color, and $I_i = 0$ otherwise:

$$I_i = \begin{cases} 1, & \text{if the } i\text{-th ball has any neighbor of a different color} \\ 0, & \text{if the } i\text{-th ball has no neighbor of a different color} \end{cases}$$

The total expected number of balls that have a neighbor of a different color is given by summing the I_i expectations over all slots in the queue:

$$\mathbb{E}\left[\sum_{i=1}^{18} I_i\right] = \sum_{i=1}^{18} \mathbb{E}[I_i]$$

N.b., at this point, given the reasonably open interpretation of the interview question, let us clarify that $\mathbb{E}[1_2] = \mathbb{E}[1_3] = \mathbb{E}[1_4] = \cdots = \mathbb{E}[1_{17}]$, *i.e.*, because the balls are queued at once and not sequentially, we do not suppose any varying conditions at each slot (*i.e.*, that there would be, for a given ball, any consideration of the configuration of the balls placed before it)

Now, computing the $\mathbb{E}[I_i]s$:

For the first slot and the final slot, the balls each have only one neighbor:

$$\mathbb{E}[I_1] = \mathbb{E}[I_{17}] = \frac{10}{18} \times \frac{8}{17} + \frac{8}{18} \times \frac{10}{17} = \frac{160}{306} = \frac{80}{153}$$

For the second through the seventeenth slots:

$$\mathbb{E}[I_2|\text{First ball blue}] = \frac{8}{17} + \frac{9}{17} \times \frac{8}{16} = \frac{25}{34},$$
$$\mathbb{E}[I_2|\text{First ball red}] = \frac{10}{17} + \frac{7}{17} \times \frac{10}{16} = \frac{115}{136}$$

$$\mathbb{E}[I_2] = \frac{10}{18} \times \frac{25}{34} + \frac{8}{18} \times \frac{115}{136} = \frac{480}{612} = \frac{40}{51}$$

Thus,

$$\sum_{i=1}^{18} \mathbb{E}[I_i] = \mathbb{E}[1_1] + \mathbb{E}[1_2] + \cdots + \mathbb{E}[1_{17}] + \mathbb{E}[1_{18}]$$
$$= \frac{80}{153} \times 2 + \frac{40}{51} \times 16$$
$$\boxed{\approx 13.595}$$

Problem 2.

Solution. □

Preliminarily, defining the following states:

- E_0: the expected number of rolls needed to get two consecutive 6s, starting with no 6s

  ($N.b.$, $E_0 = \mathbb{E}[N]$)

- E_1: the expected number of rolls needed to get two consecutive 6s, starting with one 6 already rolled

- E_2: the expected number of rolls needed after getting two consecutive 6s

  ($N.b.$, $E_2 = 0$)

First, deriving the equation for E_0:

$$E_0 = 1 + \frac{1}{6}E_1 + \frac{5}{6}E_0$$

where...

- the term 1 accounts for the initial roll, which always happens;

- the term $\frac{1}{6}E_1$ corresponds to the case where the first roll is a 6, moving the state to E_1;

- the term $\frac{5}{6}E_0$ corresponds to the case where the first roll is not a 6.

Simplifying the equation:

$$E_0 = 6 + E_1$$

Next, deriving the equation for E_1:

$$E_1 = 1 + \frac{1}{6} \times 0 + \frac{5}{6} E_0$$

where...

- the term 1 accounts for the next roll, which always happens;
- the term $\frac{1}{6} \times 0$ corresponds to the case where the next roll is another 6, achieving the goal and thus requiring no additional rolls (*i.e.*, moving the state to E_2);
- the term $\frac{5}{6} E_0$ corresponds to the case where the next roll is not a 6, returning the state to E_0.

Simplifying the equation:

$$E_1 = 1 + \frac{5}{6} E_0$$

Solving the system of equations:

$$E_0 = 6 + 1 + \frac{5}{6} E_0$$

$$= \boxed{42}$$

Problem 3.

Solution. □

Let A, B, C, and D represent the corners of the square, with A and C being opposite corners and the bug starting from A

Preliminarily, defining the following states:

- $E(A)$: the expected number of steps to reach C from A

  ($N.b.$, $E(A) = \mathbb{E}[N]$)
- $E(B)$: the expected number of steps to reach C from B
- $E(D)$: the expected number of steps to reach C from D

  ($N.b.$, $E(B) = E(D)$)

- $E(C)$: the expected number of steps to reach C from C

  ($N.b.$, $E(C) = 0$)

First, deriving the equation for $E(A)$:

$$E(A) = 1 + \frac{1}{2}E(B) + \frac{1}{2}E(D)$$

Simplifying the equation:

$$E(A) = 1 + E(B)$$

Next, deriving the equation for $E(B)$:

$$E(B) = 1 + \frac{1}{2}E(A) + \frac{1}{2} \times 0$$

Simplifying the equation:

$$E(B) = 1 + \frac{1}{2}E(A)$$

Solving the system of equations:

$$E(A) = 1 + \left(1 + \frac{1}{2}E(A)\right)$$
$$= \boxed{4}$$

Problem 4.

Solution. □

Given:

- $E(A)$: X, Y are the times that the two people arrive, measured in hours after noon
- Since X and Y are uniformly distributed between 0 and 1, the joint density function $f(x, y)$ is given by $f(x, y) = 1$ for $0 \leq x, y \leq 1$

Intuitively, the waiting time $g(X, Y)$ is the absolute difference between the two arrival times:

$$g(x, y) = |x - y|$$

Computing the expected value of $g(X, Y)$:

$$\mathbb{E}[g(X,Y)] = \int_0^1 \int_0^1 |x - y| f(x, y)\, dx\, dy$$
$$= \int_0^1 \int_0^1 |x - y|\, dx\, dy$$

$$= \int_0^1 \int_0^x (x-y)\,dy\,dx + \int_0^1 \int_x^1 (y-x)\,dy\,dx$$

Integrating $\int_0^1 \int_0^x (x-y)\,dy\,dx$ first w/r/t/ y:

$$\int_0^x (x-y)\,dy = \left[xy - \frac{y^2}{2}\right]_0^x = x^2 - \frac{x^2}{2} = \frac{x^2}{2}$$

Then integrating w/r/t/ x:

$$\int_0^1 \frac{x^2}{2}\,dx = \frac{1}{2} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}$$

Next, integrating $\int_0^1 \int_x^1 (y-x)\,dy\,dx$ first w/r/t/ y:

$$\int_x^1 (y-x)\,dy = \left[\frac{y^2}{2} - xy\right]_x^1 = \frac{1}{2} - \frac{x^2}{2} - x + x^2 = \frac{1}{2} - x + \frac{x^2}{2}$$

Then integrating w/r/t/ x:

$$\int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx = \frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\Big|_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}$$

Thus, the expected waiting time for the first person to meet the second:

$$\mathbb{E}[g(X,Y)] = \frac{1}{6} + \frac{1}{6} = \boxed{\frac{1}{3} \text{ hours}}$$