Module I Homework 8

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Prep24AutumnM1

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Problem 1.

Solution. □

First, simplifying the expression inside the limit:

$$\frac{x+a}{x-a} = \frac{x(1+\frac{a}{x})}{x(1-\frac{a}{x})} = \frac{1+\frac{a}{x}}{1-\frac{a}{x}}$$

N.b.: As $x \to \infty$, $\frac{a}{x} \to 0$ while it is not obvious how $\ln\left[\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x\right]$ behaves. To handle this complexity, we can approximate the fraction using a known Taylor Series expansion. Because for large x, the expression $\frac{1+\frac{a}{x}}{1-\frac{a}{x}}$ resembles the form $\frac{1}{1-z}$, where $z = -\frac{a}{x}$, we can apply the Taylor Series expansion of $\frac{1}{1-z}$, $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$, around $z = 0$. Accordingly, we substitute $z = \frac{2a}{x}$, which is small when x is large.

Thus, expanding $\frac{1+\frac{a}{x}}{1-\frac{a}{x}}$:

$$\begin{aligned}\frac{1+\frac{a}{x}}{1-\frac{a}{x}} &= \left(1+\frac{a}{x}\right)\left(1+\frac{a}{x}+\left(\frac{a}{x}\right)^2+\ldots\right)\\ &\approx 1+\frac{2a}{x}+\frac{2a^2}{x^2}+\ldots\\ &= 1+\frac{2a}{x}+O\left(\frac{1}{x^2}\right)\end{aligned}$$

N.b., for large x:

$$\frac{1+\frac{a}{x}}{1-\frac{a}{x}} \approx 1+\frac{2a}{x}$$

Next, proceeding by taking the natural logarithm of the original expression (to handle the power of x):

$$\ln\left[\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x\right] = x\ln\left(1+\frac{2a}{x}\right)$$

Applying the approximation $\ln(1+z) \approx z$ for small values of z:

$$x \ln\left(1 + \frac{2a}{x}\right) \approx x \cdot \frac{2a}{x} = 2a$$

Thus:

$$\lim_{x\to\infty} \ln\left[\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x\right] \approx \lim_{x\to\infty} 2a = 2a$$

Exponentiating:

$$\lim_{x\to\infty} \left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x = \boxed{e^{2a}}$$

Comment on solution methodology:

For this solution, we did not explicitly follow the traditional steps of computing a Taylor Series (*i.e.*, by finding derivatives of the function.) Instead, we recognized the expression inside the limit resembles a well-known function, $\frac{1}{1-z}$, which has a standard Taylor Series expansion around $z = 0$.
We leveraged this approximation because, for large x, the term $\frac{a}{x}$ becomes small. Rather than recalculating the derivatives of the expression, it is more efficient to use the known expansion of $\frac{1}{1-z}$ and substitute $z = \frac{2a}{x}$. This method is valid because Taylor Series expansions approximate functions well when the variable in question (here, z) is close to the point of expansion (in this case, 0).
The logarithmic transformation was applied to manage the x-th power in the original limit. After expanding the logarithm, we used the fact that $\ln(1+z) \approx z$ for small z, making the limit easier to compute.

Problem 2.

Intuition: When $r = 0$, m is today's call price and n is today's call payoff, meaning $m > n$ since both T and σ are positive

Solution. □

The defined term n simplifies to:

$$n = [S_0 - K]^+$$

This expression evaluates to $S_0 - K$ if $S_0 > K$ or to 0 if otherwise, and considers only the mean of the distribution

On the other hand, m is defined as the expected payoff of the option and considers all possible outcomes where $S_T > K$:

$$m = \mathbb{E}[(S_T - K)^+]$$

N.b., this incorporates the full probability distribution of S_T, and therefore is generally greater than or equal to n

Thus, given that m accounts for the complete distribution above the strike price K while n accounts only for the difference if the expected price exceeds K, typically:

$$\boxed{\text{(a) } m > n}$$

Alternative solution methodology:

Because the positive part function is a convex function, according to Jensen's inequality:

$$m = \mathbb{E}[(S_T - K)^+] > [\mathbb{E}(S_T - K)]^+ = (\mathbb{E}(S_T) - K)^+ = n$$

Problem 3.

Solution. □

$$\begin{aligned}\int_0^\infty e^{tx}\lambda e^{-\lambda x}dx &= \lambda \int_0^\infty e^{(t-\lambda)x}dx \\ &= \lambda \left[\frac{1}{t-\lambda}e^{(t-\lambda)x}\right]_0^\infty \\ &= \lambda \left[\frac{1}{t-\lambda}(0-1)\right] \\ &= \boxed{\frac{\lambda}{\lambda - t}}\end{aligned}$$

Problem 4.

Intuition: If drawing two marbles without replacement, then regardless of which color is picked (with probability $\frac{1}{2}$) for the first marble, the likelihood of drawing the same color for the second marble, assuming the marbles are distributed independently and at random in the bag, must intuitively be less than $\frac{1}{2}$, given the count of marbles of each color remaining after the first draw

Solution. □

Modeling the intuition...

The probability of drawing the first red (or black) marble:

$$\frac{n}{2n} = \frac{1}{2}$$

The probability of drawing the same color marble second:

$$\frac{n-1}{2n-1}$$

Thus, the probability of drawing two marbles of the same color:

$$P(\text{Red-Red}) = P(\text{Black-Black})$$

$$= \frac{n}{2n} \times \frac{n-1}{2n-1}$$
$$= \frac{n-1}{2(2n-1)}$$

Since these two events are mutually exclusive, the total probability p:

$$p = P(\text{Red-Red}) + P(\text{Black-Black}) = \frac{n-1}{2n-1}$$

$$\frac{n-1}{2n-1} < \frac{1}{2} \text{ for } n > 1$$

So the intuition is correct:

$$\boxed{\text{(a) } p < \frac{1}{2}}$$

Problem 5.

Solution. □

Rewriting:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln(x)}$$

N.b., in effect, we are considering, simply, $\lim_{x \to 0^+} x \ln(x)$

For $\lim_{x \to 0^+} x \ln(x)$, as $x \to 0^+$, $\ln(x) \to -\infty$ and $x \to 0$, so the product $x \ln(x)$ is of the indeterminate form $0 \cdot (-\infty)$

Resolving the $0 \cdot (-\infty)$ indeterminate form to a satisfactory one:

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now, $\frac{\ln(x)}{\frac{1}{x}} = x \ln(x)$ is an indeterminate form of type $\frac{-\infty}{\infty}$

Applying L'Hôpital's Rule:

$$\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx} \left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} -x$$
$$= 0$$

Therefore:

$$\lim_{x\to 0^+} x^x = e^0$$
$$= \boxed{1}$$

Problem 6.

Solution. □

For $\lim_{x\to 1} \frac{x^{\frac{3}{2}}-\frac{3}{2}x+\frac{1}{2}}{(x-1)^2}$, as $x \to 1$, both the numerator and the denominator approach 0, so the expression is of the $\frac{0}{0}$ indeterminate form

Letting the numerator $x^{\frac{3}{2}} - \frac{3}{2}x + \frac{1}{2} = f(x)$ and the denominator $(x-1)^2 = g(x)$ and applying L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{\frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}}{2(x-1)}$$

The expression $\frac{\frac{3}{2}x^{\frac{1}{2}}-\frac{3}{2}}{2(x-1)}$ is an indeterminate form of type $\frac{0}{0}$ as $x \to 1$, so applying L'Hôpital's Rule again:

$$\lim_{x\to 1} \frac{f''(x)}{g''(x)} = \lim_{x\to 1} \frac{\frac{3}{4}x^{-\frac{1}{2}}}{2}$$
$$= \boxed{\frac{3}{8}}$$

Problem 7a.

Solution. □

Recall the Taylor series expansion of a function $f(x)$ about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Calculating the function $\sin(x)$ and its derivatives:

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$
$$f^{(4)}(x) = \sin(x), \ \ etc.$$

Evaluating the function $\sin(x)$ and its derivatives at $x = 0$:

$$f(0) = \sin(0) = 0$$

$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1$$
$$f^{(4)}(0) = \sin(0) = 0, \ \ etc.$$

Thus, the Taylor expansion for $\sin(x)$ at $x = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}}$$

N.b., I did not derive the summation expression myself :P

Problem 7b.

Solution. □

Calculating the function $\log(x)$ and its derivatives and evaluating at $x = 1$:

$$f(x) = \log(x)$$
$$f(1) = \log(1) = 0$$

$$f'(x) = \frac{1}{x}$$
$$f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2}$$
$$f''(1) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3}$$
$$f'''(1) = \frac{2}{1^3} = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$f^{(4)}(1) = -\frac{6}{1^4} = -6$$

etc.

Thus, the Taylor expansion for $\log(x)$ at $x = 1$:

$$\log(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots$$

$$= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4 + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \text{ for } |x-1| < 1}$$