Prep Course
Module I
Calculus Lecture 5

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Moment Generating Function

Moment Generating Function (MGF)

The moment generating function $M(t)$ of a random variable X, if exists, is defined as follows:

$$M(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx, & \text{if } X \text{ is continuous} \end{cases}$$

Note that $\mathbb{E}[X^n]$, where $n \in \mathbb{N}$, is called the nth moment of X. We call $M(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M(t)$.

Moment Generating

Generating the first moment:

$$M'(t) = \frac{d}{dt}\mathbb{E}[e^{tX}] = \mathbb{E}\left[\frac{d}{dt}e^{tX}\right] = \mathbb{E}[Xe^{tX}]$$

$$\Rightarrow M'(0) = \mathbb{E}[X]$$

Generating the second moment:

$$M''(t) = \frac{d}{dt}M'(t) = \frac{d}{dt}\mathbb{E}[Xe^{tX}] = \mathbb{E}\left[\frac{d}{dt}(Xe^{tX})\right] = \mathbb{E}[X^2e^{tX}]$$

$$\Rightarrow M''(0) = \mathbb{E}[X^2]$$

As you can see, we get the expectation by taking the first moment. Moreover, we can also synthesize the variance with the first and second moments:

$$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = M''(0) - \left(M'(0)\right)^2$$

Important Properties of MGF

- If two random variables have the same MGF, then they follow the same distribution. As such, to specify a distribution:
  - Give its name and its parameters (if it has a name).
  - Or give its CDF.
  - Or give its probability density function or probability mass function.
  - Or give its MGF.
- The MGF of the sum of n independent random variables is equal to the product of their MGFs. Let $X_1, X_2, \dots, X_n$ be independent random variables and $Z \coloneqq \sum_{i=1}^{n} X_i$. Then

$$M_Z(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \cdots \cdot M_{X_n}(t)$$

Example: $X \sim Binomial(n, p)$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$

$$= (pe^t + 1 - p)^n$$

$$\Rightarrow M'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$\Rightarrow \mathbb{E}[X] = M'(0) = np$$

$$\Rightarrow M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

$$\Rightarrow \mathbb{E}[X^2] = M''(0) = n(n-1)p^2 + np$$

$$\Rightarrow Var[X] = M''(0) - \left(M'(0)\right)^2 = np(1-p)$$

Example: $X \sim N(\mu, \sigma^2)$ (1/3)

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} \frac{e^{tx}}{\sigma\sqrt{2\pi}} \exp\left(\frac{(x-\mu)^2}{-2\sigma^2}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{1}{-2\sigma^2}(x^2 - 2(\mu + \sigma^2 t)x + \mu^2)\right] dx$$

To evaluate this integral, we complete the square in the exponent

$$x^2 - 2(\mu + \sigma^2 t)x + \mu^2 = [x - (\mu + \sigma^2 t)]^2 - 2\mu\sigma^2 t - \sigma^4 t^2.$$

Thus

$$M(t)$$

$$= \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{[x - (\mu + \sigma^2 t)]^2}{-2\sigma^2}\right] dx$$

$$= \exp\left(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Example: $X \sim N(\mu, \sigma^2)$ (2/3)

The moments of X:

Order	Non-central moment	Central moment
1	μ	0
2	$\mu^2 + \sigma^2$	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$
5	$\mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	0
6	$\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$	$15\sigma^6$
7	$\mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$	0
8	$\mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\sigma^8$	$105\sigma^8$

Source: https://en.wikipedia.org/wiki/Normal_distribution#Moments

Example: $X \sim N(\mu, \sigma^2)$ (3/3)

Let $Y = e^X$. Then Y is lognormally distributed.

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = M(1) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

In the Black-Scholes world, share price S_t follows geometric Brownian motion. Then, under risk-neutral measure,

$$\log S_t \sim N\left(\log S_0 + \left(r - \frac{\sigma_S^2}{2}\right)t, \sigma_S^2 t\right)$$

Thus, the forward price is

$$\mathbb{E}[S_t] = \mathbb{E}\left[e^{\log S_t}\right] = \exp\left(\log S_0 + \left(r - \frac{\sigma_S^2}{2}\right)t + \frac{\sigma_S^2 t}{2}\right)$$

$$= \exp(\log S_0 + rt) = S_0 e^{rt}$$

L’Hospital’s Rule

L'Hospital's Rule

Given differentiable functions $f(x)$ and $g(x)$, if

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = L$$

and either of the following 2 forms holds

$$\left(\frac{0}{0}\text{ form}\right): \lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0$$

$$\left(\frac{\infty}{\infty}\text{ form}\right): \lim_{x \to p} f(x) = \lim_{x \to p} g(x) = \infty$$

then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = L$$

where

p can be a real c, c^+, c^-, ∞, or $-\infty$

and L can be a real, ∞, or $-\infty$

Example of applying L'Hospital's rule

At the last slide of lecture 2, we left out the detail of the following limit, which is of the indeterminate form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{-x}{e^{\frac{x}{1000}}} = -1\cdot\lim_{x\to\infty}\frac{x}{e^{\frac{x}{1000}}} = -1\cdot\lim_{x\to\infty}\frac{1}{\frac{1}{1000}e^{\frac{x}{1000}}} = -1\cdot 0$$

$$= 0$$

Example of recursively applying L'Hospital's rule

Evalute

$$\lim_{x\to 0} \frac{1 + x - e^x}{2x^2}$$

Answer:

Set $f(x) = 1 + x - e^x$ and $g(x) = 2x^2$. Then $\lim_{x\to 0} \frac{f(x)}{g(x)}$ yields the indeterminate form $\frac{0}{0}$. If we apply L'Hospital's rule, we get

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{1 - e^x}{4x}$$

which still give the $\frac{0}{0}$ form. If we apply the rule one more time,

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{-e^x}{4} = -\frac{1}{4}$$

Example of L'Hospital's rule messing things up (1/2)

Evalute

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{2^x}{e^{x^2}}$$

Wrong Approach:

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{2^x \log 2}{2x \cdot e^{x^2}}$$

$$\lim_{x\to\infty}\frac{f''(x)}{g''(x)} = \lim_{x\to\infty}\frac{2^x (\log 2)^2}{(4x^2 + 2) \cdot e^{x^2}}$$

The more differentiation we do, the messier the expression is.

Example of L'Hospital's rule messing things up (2/2)

Evalute

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{2^x}{e^{x^2}}$$

Answer:

For any $x > 2$, we have

$$e^x > e^2 > 2^2 = 4$$

$$\Rightarrow 0 < \frac{2}{e^x} < \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow 0 < \left(\frac{2}{e^x}\right)^x = \frac{2^x}{e^{x^2}} < \left(\frac{1}{2}\right)^x$$

We know $\lim_{x\to\infty}\left(\frac{1}{2}\right)^x = 0$. Then, by Sandwiching theorem,

$$\lim_{x\to\infty}\frac{2^x}{e^{x^2}} = 0$$

Example of indeterminate form: $\infty - \infty$

Evaluate

$$\lim_{x\to 0^+}\left(\frac{1}{\log(1+x)}-\frac{1}{x}\right)$$

Answer: We need to convert it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x\to 0^+}\left(\frac{1}{\log(1+x)}-\frac{1}{x}\right)=\lim_{x\to 0^+}\frac{x-\log(1+x)}{x\cdot\log(1+x)}$$

$$=\lim_{x\to 0^+}\frac{1-\frac{1}{1+x}}{\log(1+x)+\frac{x}{1+x}}=\lim_{x\to 0^+}\frac{\frac{1}{(1+x)^2}}{\frac{1}{1+x}+\frac{1+x-x}{(1+x)^2}}=\frac{1}{2}$$

Example of indeterminate form: $0 \cdot \infty$

Evaluate

$$\lim_{x \to -\infty} x^2 e^x$$

Answer: We need to convert it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

Example of indeterminate form: $1^{\infty}, 0^{0}, \infty^{0}$

Evaluate

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{xT}, where\ r > 0\ and\ T > 0$$

Answer: solution pattern $f(x) = g(x)^{h(x)} = e^{h(x)\cdot\log g(x)}$

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{xT} = \lim_{x\to\infty} e^{xT\cdot\log\left(1+\frac{r}{x}\right)} = e^{T\cdot\lim_{x\to\infty}\left(x\cdot\log\left(1+\frac{r}{x}\right)\right)}$$

$$= e^{T\cdot\lim_{x\to\infty}\left(\frac{\log\left(1+\frac{r}{x}\right)}{\frac{1}{x}}\right)} = e^{T\cdot\lim_{x\to\infty}\left(\frac{\frac{1}{\left(1+\frac{r}{x}\right)}\cdot\left(-rx^{-2}\right)}{-x^{-2}}\right)} = e^{T\cdot\lim_{x\to\infty}\left(\frac{r}{\left(1+\frac{r}{x}\right)}\right)} = e^{rT}$$