Module I Homework 4

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Prep24AutumnM1

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Problem 1.

Intuition:

For $0 \leq x < 1$, $x^2 \leq x$, so $\max(x, x^2) = x$, and for $1 \leq x \leq 2$, $x^2 \geq x$, so $\max(x, x^2) = x^2$

Therefore, we should compute:

$$\int_0^2 \max(x, x^2)\,dx = \int_0^1 x\,dx + \int_1^2 x^2\,dx$$

Solution. □

$$\begin{aligned}
&= \int_0^1 x\,dx + \int_1^2 x^2\,dx \\
&= \left.\frac{x^2}{2}\right|_0^1 + \left.\frac{x^3}{3}\right|_1^2 \\
&= \frac{1^2}{2} - \frac{0^2}{2} + \frac{2^3}{3} - \frac{1^3}{3} \\
&= \frac{1}{2} + \frac{7}{3} \\
&= \boxed{\frac{17}{6}}
\end{aligned}$$

Problem 2.

Intuition:

Given the complexity of the integrand, we will integrate by parts

Solution. □

Let $u = (\log x)^2$... $du = 2\log x \cdot \frac{1}{x}\,dx$
Let $dv = x\,dx$... $v = \frac{x^2}{2}$

Rewriting:

$$\int x(\log x)^2\,dx = \left(\frac{x^2}{2}\right)(\log x)^2 - \int \left(\frac{x^2}{2}\right)\cdot 2\log x \cdot \frac{1}{x}\,dx$$

$$= \frac{x^2(\log x)^2}{2} - \int x\log x\,dx$$

Given the complexity of the integrand, we will integrate by parts again

Let $u = \log x$... $du = \frac{1}{x}\,dx$
Let $dv = x\,dx$... $v = \frac{x^2}{2}$

Rewriting:

$$\int x\log x\,dx = \left(\frac{x^2}{2}\right)\log x - \int \left(\frac{x^2}{2}\right)\cdot \frac{1}{x}\,dx$$

$$= \frac{x^2\log x}{2} - \int \frac{x}{2}\,dx$$

$$= \frac{x^2\log x}{2} - \frac{x^2}{4}$$

Substituting back:

$$\int x(\log x)^2\,dx = \frac{x^2(\log x)^2}{2} - \left(\frac{x^2\log x}{2} - \frac{x^2}{4}\right)$$

$$= \frac{x^2(\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4}$$

Thus,

$$\int x(\log x)^2\,dx = \boxed{\frac{x^2(\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4} + C}$$

Problem 3.

Intuition: Given the complexity of the function, we will use logarithmic differentiation

Solution. □

Let $y = x^{x^x}$

Then,

$$\ln y = \ln\left(x^{x^x}\right)$$

$$= x^x \ln x$$

Differentiating w/r/t/ x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x^x \ln x)$$

Applying the chain rule on the LHS, product rule on the RHS:

$$\frac{1}{y}\frac{dy}{dx} = x^x \cdot \frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x^x)$$

Equating $x^x = e^{x \ln x}$ for ease of differentiation and applying the chain and product rules on the RHS:

$$\frac{1}{y}\frac{dy}{dx} = x^x \cdot \frac{1}{x} + \ln x \cdot e^{x \ln x}(x \cdot \frac{1}{x} + \ln x \cdot 1)$$

Multiplying both sides by y and simplifying the RHS:

$$\frac{dy}{dx} = y[x^{x-1} + \ln x(x^x(\ln x + 1)]$$

Thus,

$$\frac{d}{dx}x^{x^x} = \boxed{x^{x^x}[x^{x-1} + \ln x(x^x)(\ln x + 1)]}$$

Problem 4.

Intuition: Forget needing to utilize Bayes' Theorem and the Law of Total Probability! Given the choices in the presented answers, a), b), c), and d), we can deductively reason that the answer must be a), since the probability that the coin is normal given that it comes up heads must intuitively be less than one half. This is because any heads outcome is one of three satisfactory possible heads outcomes, of four total possible outcomes, between the normal and double-headed coins.

Solution. □

$$\boxed{\text{a) } \frac{1}{3}}$$

Problem 5.

Intuition: There are five choices for the first boy, four remaining for the second, three remaining for the third, *etc.*, representable as $5! = 120$

Solution. □

$$\boxed{\text{d) } 120}$$

Problem 6.

Solution. □

$$\frac{d}{d\sigma}C_0 = S_0 N'(d_1)\frac{d}{d\sigma}d_1 - Ke^{-rT}N'(d_2)\frac{d}{d\sigma}d_2$$

$$\boxed{\frac{d}{d\sigma}P_0 = -Ke^{-rT}N'(-d_2)\frac{d}{d\sigma}d_2 + S_0 N'(-d_1)\frac{d}{d\sigma}d_1 = \frac{d}{d\sigma}C_0}$$