Prep Course
Module I
Calculus Lecture 4

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Example: Distribution Approximation

Interview Question

Toss a fair coin 100 times. What’s the probability of getting exactly 50 heads? Don’t use a calculator. The final answer should be a decimal rather than a math expression. For example, the following

$$\frac{\binom{100}{50}}{2^{100}}$$

is correct, but it’s not what the interviewer is looking for.

Central Limit Theorem

Strong Law of Large Numbers

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables that are independently and identically distributed, and let $\mathbb{E}[X_i] = \mu$. Then, with probability 1,

$$\bar{X} := \frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu \quad as\ n \to \infty$$

This law is why Monte Carlo simulation works. However, what if we want to know that as $n \to \infty$:

- what does the distribution of $\bar{X}$ look like?
- what is the variance of $\bar{X}$?

The answer is the Central Limit Theorem.

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables that are independently and identically distributed, and let $\mathbb{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2$. Then, the distribution of

$$Y := \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

tends to the standard normal as $n \to \infty$. That is,

$$\mathbb{P}(Y \le a) \to N(a)$$

as $n \to \infty$. Note that this theorem holds for any distribution of the X_is; herein lies its power.

Example of approximation using CLT

If X is binomially distributed with parameters n and p (P1.44), then X has the same distribution as the sum of n independent Bernoulli random variables, each with parameter p. Hence, the distribution of

$$\frac{X - \mathbb{E}[X]}{\sqrt{Var[X]}} = \frac{X - np}{\sqrt{np(1-p)}}$$

approaches the standard normal distribution as n approaches ∞. The normal approximation (of a binomial distribution) will, in general, be quite good for values of n satisfying $np(1-p) \geq 10$. (p80, "Introduction to Probability Models," 10th Edition, by S. M. Ross)

Solution to the question of tossing 100 times (1/2)

Let X be the number of heads after tossing the coin 100 times.

$$X \coloneqq X_1 + X_2 + \cdots + X_{100}, where\ X_n = \begin{cases} 1, if\ toss\ n = head \\ 0, if\ toss\ n = tail \end{cases}$$

Obviously, X follows a binomial distribution with the following:

$$\mathbb{E}[X] = 100 \cdot \frac{1}{2} = 50$$

$$Var[X] = 100 \cdot Var[X_n] = 100 \cdot (\mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2)$$

$$= 100 \cdot \left(\frac{1}{2} - \frac{1}{4}\right) = 25$$

By Central Limit Theorem, X is approximately $Normal(50,25)$.

Solution to the question of tossing 100 times (2/2)

$$\mathbb{P}(X = 50) = \mathbb{P}(49.5 < X < 50.5)$$

$$= \mathbb{P}\left(\frac{49.5 - \mathbb{E}[X]}{\sqrt{Var[X]}} < \frac{X - \mathbb{E}[X]}{\sqrt{Var[X]}} < \frac{50.5 - \mathbb{E}[X]}{\sqrt{Var[X]}}\right)$$

$$= \mathbb{P}\left(-0.1 < \frac{X - \mathbb{E}[X]}{\sqrt{Var[X]}} < 0.1\right) \cong N(0.1) - N(-0.1)$$

$$= \int_{-0.1}^{0.1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-0.1}^{0.1} e^{-\frac{x^2}{2}} dx \cong 0.4 \times 0.2 = 0.08$$

The exact result is

$$\mathbb{P}(X = 50) = \frac{\binom{100}{50}}{2^{100}} \cong 0.0796$$

Taylor’s Theorem

Interview Question

No calculators allowed. Your interviewer says to you:

"Spot is 100. No dividends. What's the Black-Sholes price of a 1-year at-the-money-forward vanilla option with 20% volatility?"

At-the-money-forward (ATMF) means $K \coloneqq S_0 e^{rT}$.

Taylor's Theorem

Form 1 (without remainder): if $f(x)$ possesses derivatives of all orders on an interval $I = (a - \delta, a + \delta)$, then $\forall x \in I$,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Form 2 (with remainder): if $f(x)$ possesses derivatives of up to and including order $(n+1)$ on an interval $I = (a - \delta, a + \delta)$, then $\forall x \in I$, there exists c between x and a, such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Example 1

Expansion of e^x at $x = 0$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Expansion of e^x at $x = 1$,

$$e^x = e + \frac{e(x-1)}{1!} + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \cdots$$

Example 2

Expansion of $N(x)$ at $x = 0$,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$N''(x) = \frac{-x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$N'''(x) = \frac{(x^2 - 1)}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

$$N(x) = N(0) + \frac{N'(0)}{1!} x + \frac{N''(0)}{2!} x^2 + \frac{N'''(0)}{3!} x^3 + \cdots$$

$$= \frac{1}{2} + \frac{N'(0)}{1!} x + \frac{N'''(0)}{3!} x^3 + \frac{N^{(5)}(0)}{5!} x^5 + \cdots$$

$$= \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + \frac{N'''(c)}{3!} x^3, where\ c\ \ is\ between\ 0\ and\ x.$$

Solution to the interview question about ATMF (1/2)

Black Scholes call price is

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \text{ and } d_2 = \frac{\log \frac{S_0 e^{rT}}{K}}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$$

ATMF option price is

$$C_0 = S_0 N\left(\frac{1}{2}\sigma\sqrt{T}\right) - S_0 N\left(-\frac{1}{2}\sigma\sqrt{T}\right) = S_0\left(N\left(\frac{1}{2}\sigma\sqrt{T}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right)$$

Solution to the interview question about ATMF (2/2)

For small $|x|$, the Taylor approximation of $N(x)$ at 0 up to $N''(x)$:

$$N(x) \cong N(0) + N'(0)x + \frac{1}{2}N''(0)x^2 = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + 0$$

So the option price is approximately

$$S_0\left(N\left(\frac{1}{2}\sigma\sqrt{T}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right)$$

$$\cong S_0\left(\frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} - \frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right) = \frac{S_0\sigma\sqrt{T}}{\sqrt{2\pi}} \cong 0.4 \times S_0\sigma\sqrt{T}$$

$$= 0.4 \times 100 \times 0.2 = 8$$

(True answer: 7.97)