Time Series and Sequence Learning

Lecture 8 – Parameter Estimation in Non-Linear/Non-Gaussian State Space Models

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Summary of Lecture 7

General State-Space Models

Def. A General State-Space model is given by:

$$\alpha_t \mid \alpha_{t-1} \sim q(\alpha_t \mid \alpha_{t-1})$$
$$y_t \mid \alpha_t \sim g(y_t \mid \alpha_t)$$

and initial distribution $\alpha_1 \sim q(\alpha_1)$.

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Thm. The joint-smoothing distribution is given by

$$p(\alpha_{1:n} \mid y_{1:n}) = \frac{q(\alpha_1)g(y_1 \mid \alpha_1)\prod_{i=2}^{n} q(\alpha_i \mid \alpha_{i-1})g(y_i \mid \alpha_i)}{L_n(y_{1:n})},$$

where $L_n(y_{1:n}) = \int q(\alpha_1)g(y_1 \mid \alpha_1)\prod_{i=2}^{n} q(\alpha_i \mid \alpha_{i-1})g(y_i \mid \alpha_i)\mathrm{d}\alpha_{1:n}$ is the likelihood.

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  - Draw $\alpha^i_{n+1} \sim f(\alpha_{n+1} \mid \alpha^i_{1:n})$ (propagating)
  - Set $\alpha^i_{1:n+1} = (\alpha^i_{1:n}, \alpha^i_{n+1})$
  - Set $\omega^i_{n+1} = \frac{q(\alpha^i_{n+1} \mid \alpha^i_n) g(y_{n+1} \mid \alpha^i_{n+1})}{f(\alpha^i_{n+1} \mid \alpha^i_{1:n})} \times \omega^i_n$. (weighting)

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- Simplest choice: $f(\alpha_{n+1} \mid \alpha^i_{1:n}) = q(\alpha_{n+1} \mid \alpha^i_n)$
- We estimate using:

$$\sum_{i=1}^{N} \frac{\omega^i_{n+1}}{\Omega_{n+1}} h(\alpha^i_{n+1}) \approx \mathbb{E}[h(\alpha_{n+1}) \mid y_{1:n+1}]$$

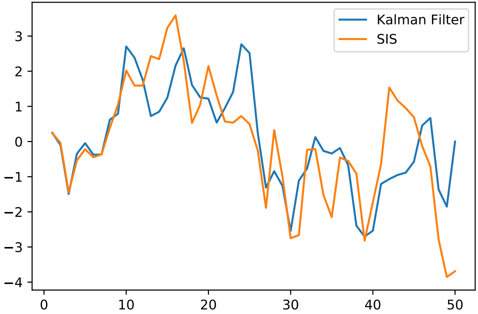
$$N^{-1}\Omega_{n+1} = \frac{1}{N} \sum_{i=1}^{N} \omega^i_{n+1} \approx L(y_{1:n+1})$$

Example: Linear Gaussian State Space Model

Def. A Linear Gaussian State-Space (LGSS) model is given by:

$$\alpha_t = T\alpha_{t-1} + R\eta_t, \qquad \eta_t \sim \mathcal{N}(0, Q),$$
$$y_t = Z\alpha_t + \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2),$$

and initial distribution $\alpha_1 \sim \mathcal{N}(a_1, P_1)$.



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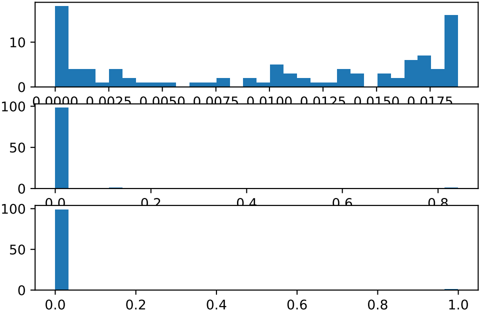

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- For $i = 1, 2, \ldots, N$ we let

$$\tilde{\alpha}^i_{1:n} = \alpha^j_{1:n} \quad \text{w. pr.} \quad \frac{\omega^j_n}{\Omega_n}$$

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$$\tilde{\alpha}^i_{1:n} = \alpha^j_{1:n} \quad \text{w. pr.} \quad \frac{\omega^j_n}{\Omega_n}$$

- Add this step before propagating the particles!

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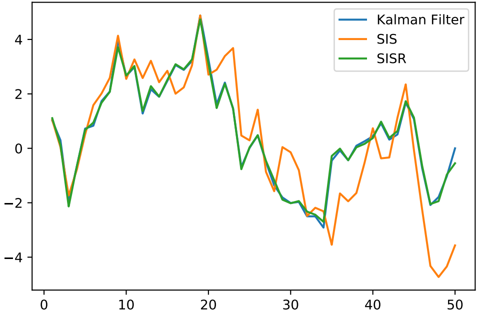


Algorithm: Particle Filter

Particle Filter:

Draw $\alpha_1^i \sim f(\alpha_1)$
Set $\omega_1^i = \frac{q(\alpha_1^i)g(y_1 \mid \alpha_1^i)}{f(\alpha_1^i)}$
Set $\Omega_1 = \sum_{i=1}^N \omega_1^i$
for $t = 2, 3, \ldots, n$ **do**
  Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$
  Draw $\alpha_t^i \sim f(\alpha_t \mid \alpha_{1:t-1}^{I^i})$
  Set $\omega_t^i = \frac{q(\alpha_t^i \mid \alpha_{t-1}^{I^i})g(y_t \mid \alpha_t^i)}{f(\alpha_t^i \mid \alpha_{1:t-1}^{I^i})}$
  Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$
  Set $\Omega_t = \sum_{i=1}^N \omega_t^i$
end for

Algorithm: Bootstrap Particle Filter

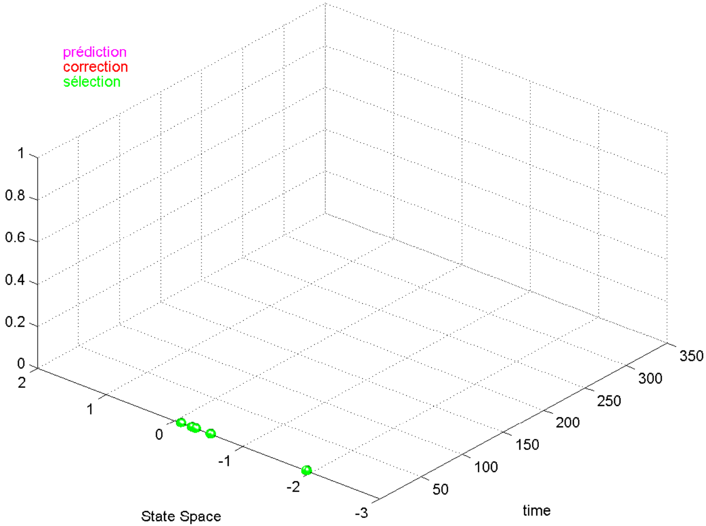
Bootstrap Particle Filter:

Draw $\alpha_1^i \sim q(\alpha_1)$
Set $\omega_1^i = g(y_1 \mid \alpha_1^i)$
Set $\Omega_1 = \sum_{i=1}^N \omega_1^i$
for $t = 2, 3, \ldots, n$ **do**
  Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$
  Draw $\alpha_t^i \sim q(\alpha_t \mid \alpha_{t-1}^{I^i})$
  Set $\omega_t^i = g(y_t \mid \alpha_t^i)$
  Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$
  Set $\Omega_t = \sum_{i=1}^N \omega_t^i$
end for

In Python:

```
part = np.zeros((n,N))
logwgt = np.zeros((n,N))
wgt = np.zeros((n,N))
ests = np.zeros(n)
part[0,:] = np.random.randn(N)
logwgt[0,:] = logwgtfun(xpart[0,:],y[0])
wgt[0,:] = np.exp(logwgt[0,:])
ests[0] = np.sum(wgt[0,:] * part[0,:])/np.sum(wgt[i+1,:])
for i in range(n-1):
    ind = np.random.choice(N, size=N, replace=True, p=wgt[i,:])
    part[i+1,:] = a*part[i,ind] + np.random.randn(N)
    logwgt[i+1,:] = logwgtfun(xpart[i+1,:],y[i+1])
    wgt[i+1,:] = np.exp(logwgt[i+1,:])
    ests[i+1] = np.sum(wgt[i+1,:] * part[i+1,:])/np.sum(wgt[i+1,:])
```

Particle Filter Movie



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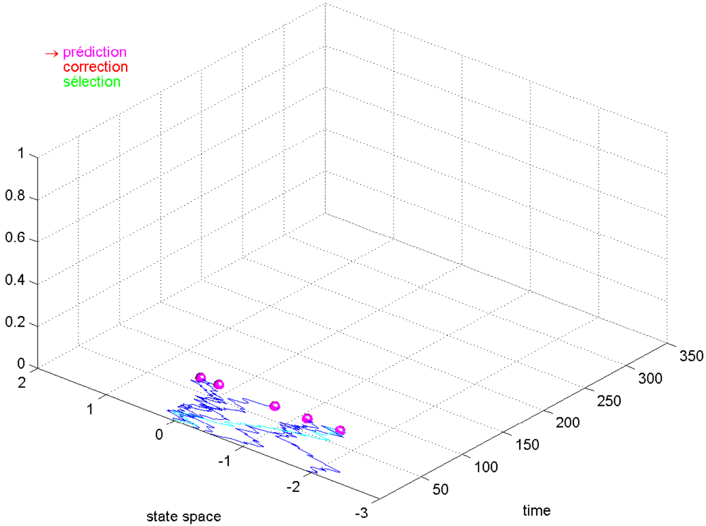

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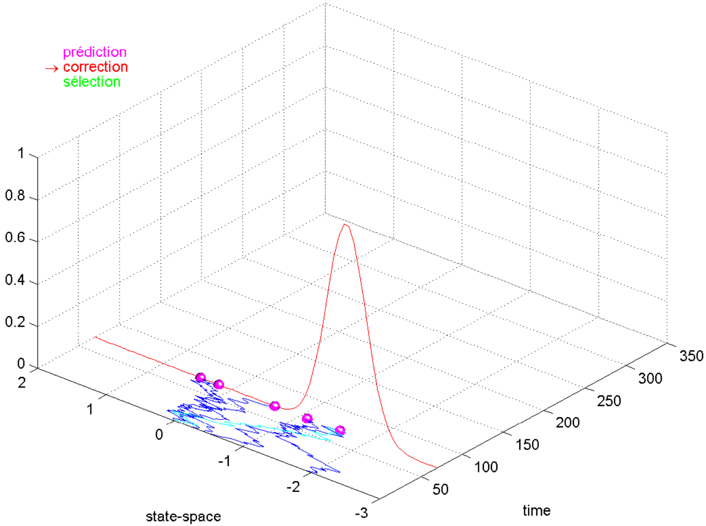

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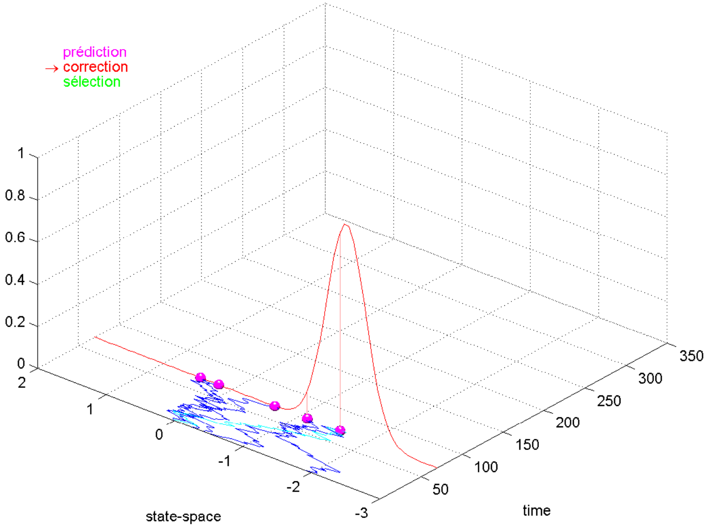

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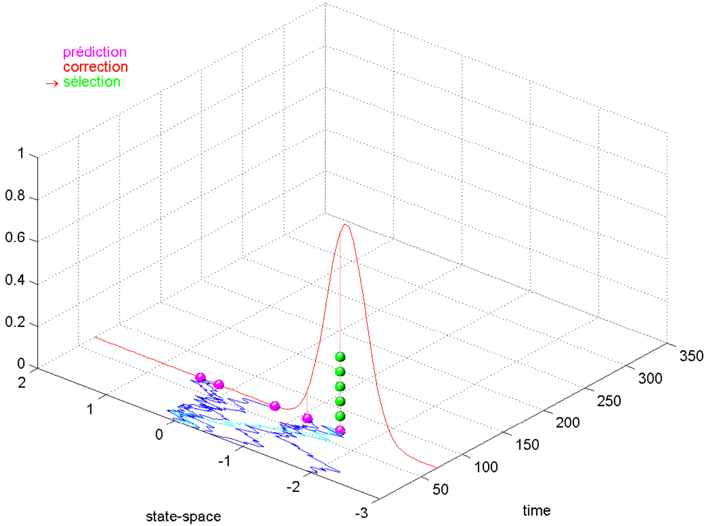

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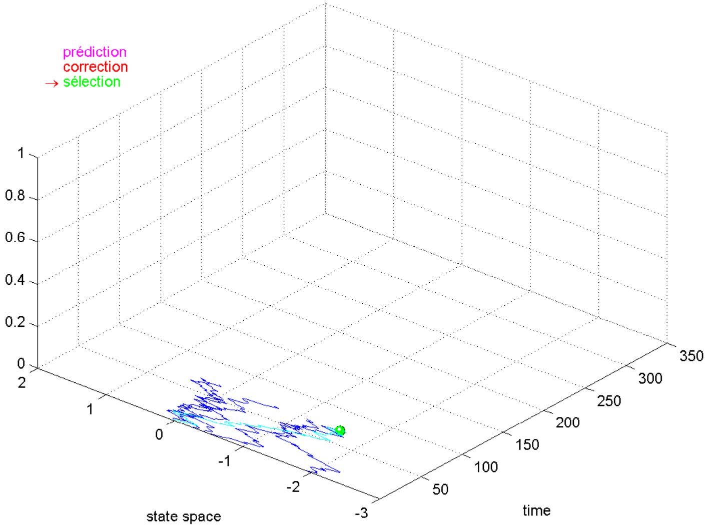

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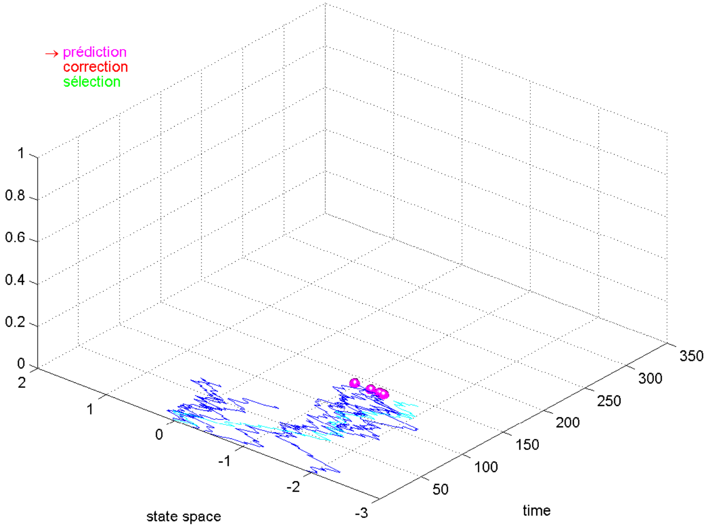

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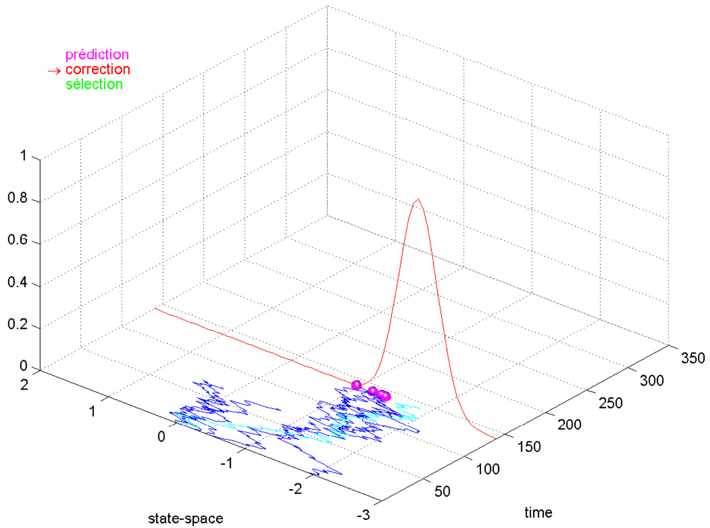

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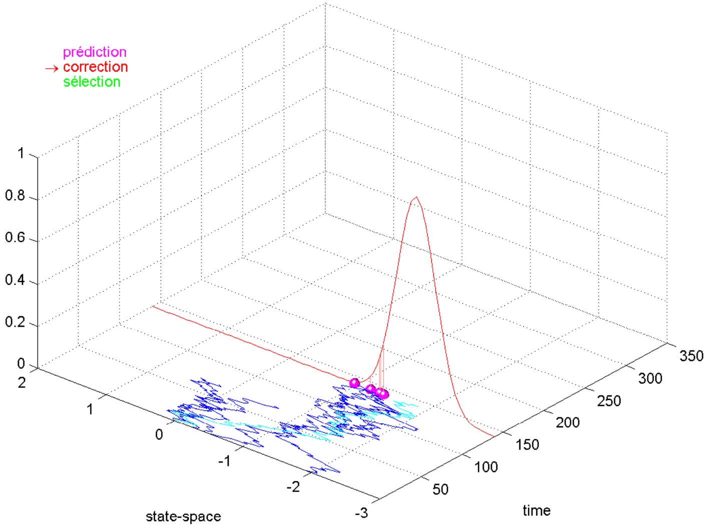

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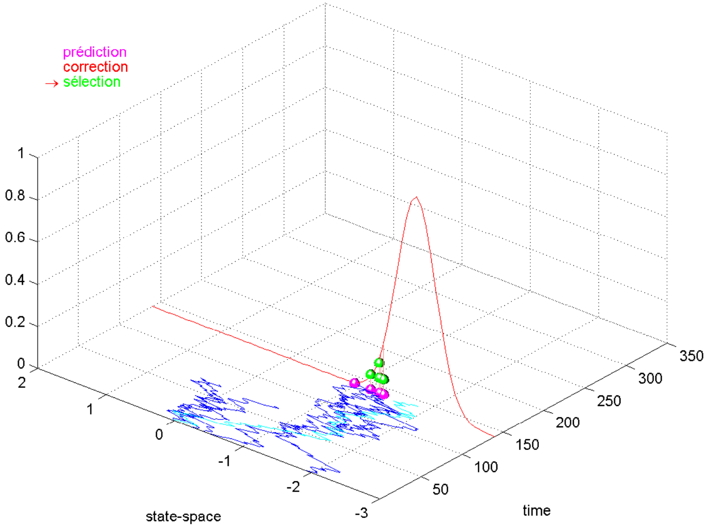

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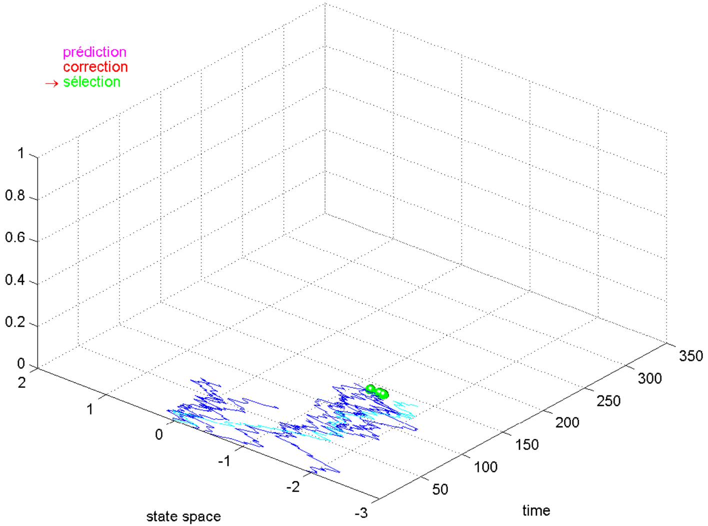

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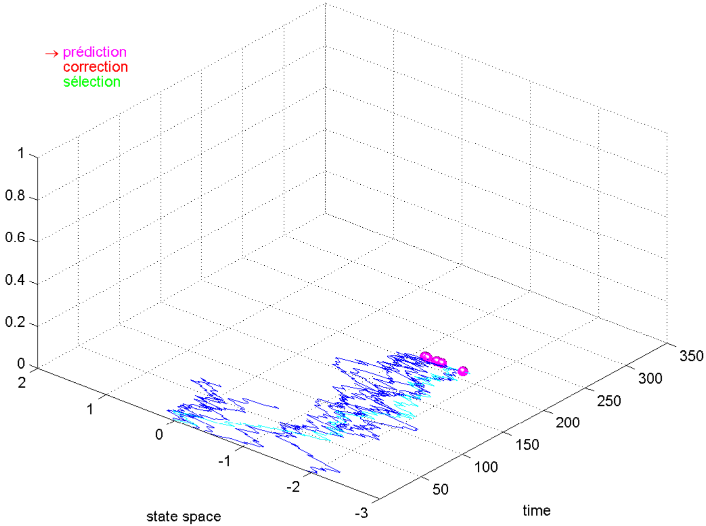

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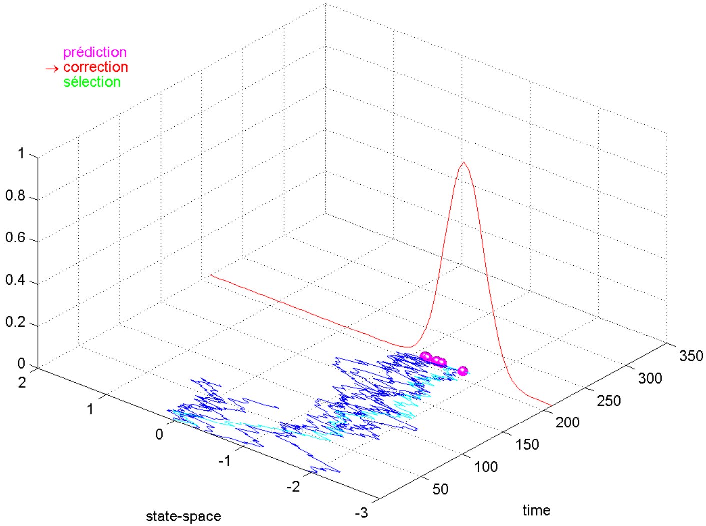

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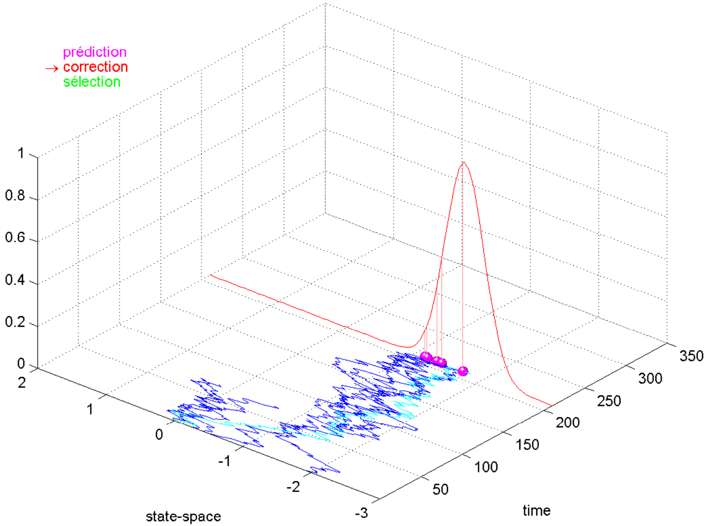

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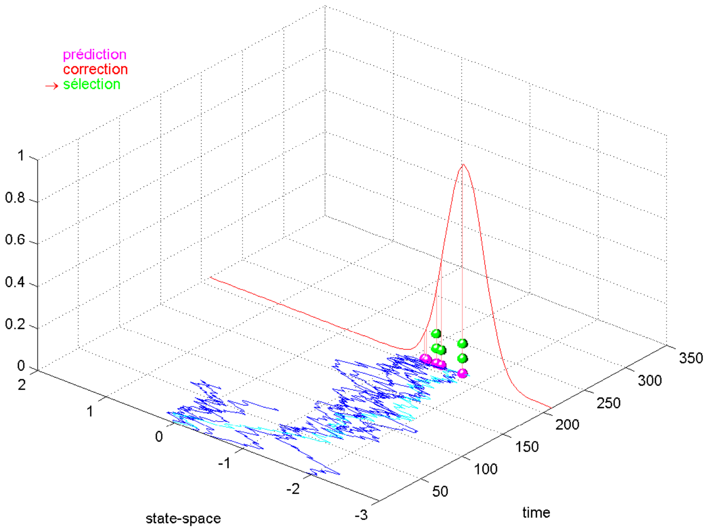

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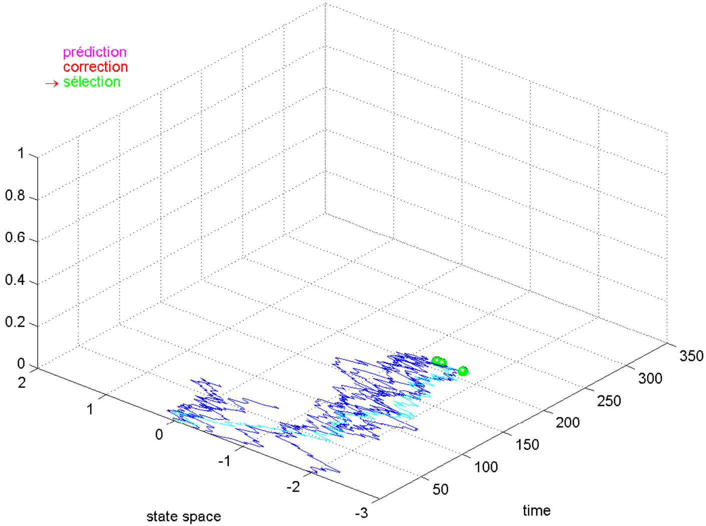

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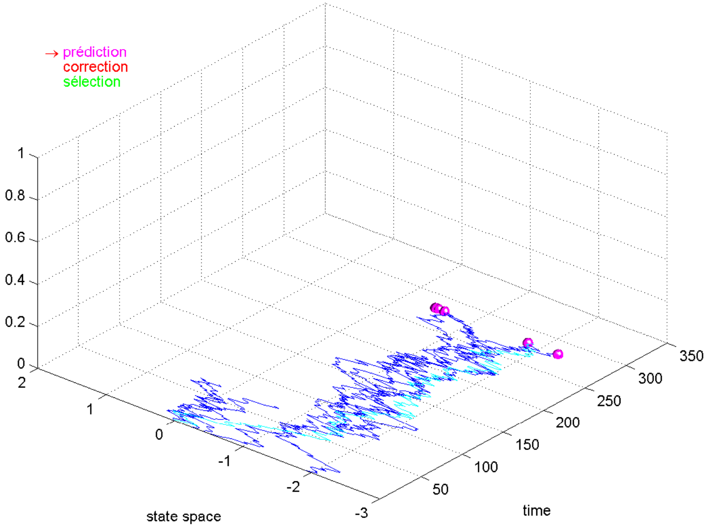

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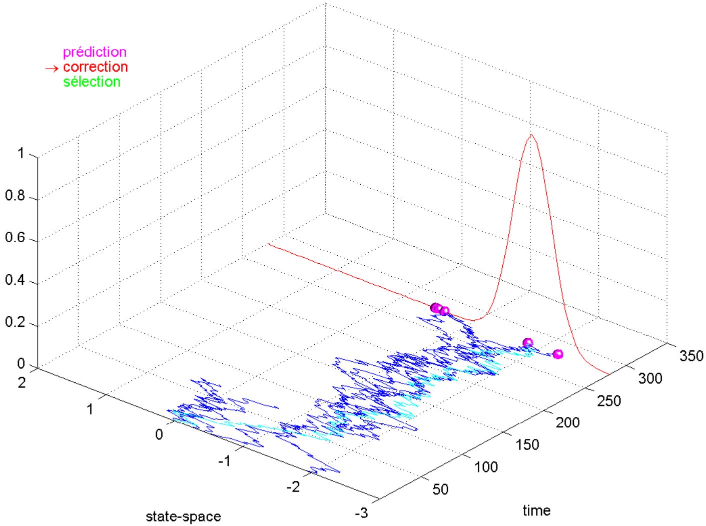

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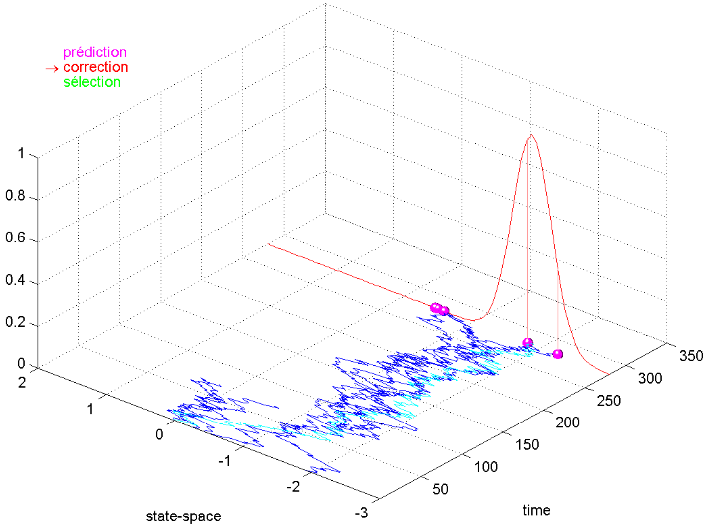

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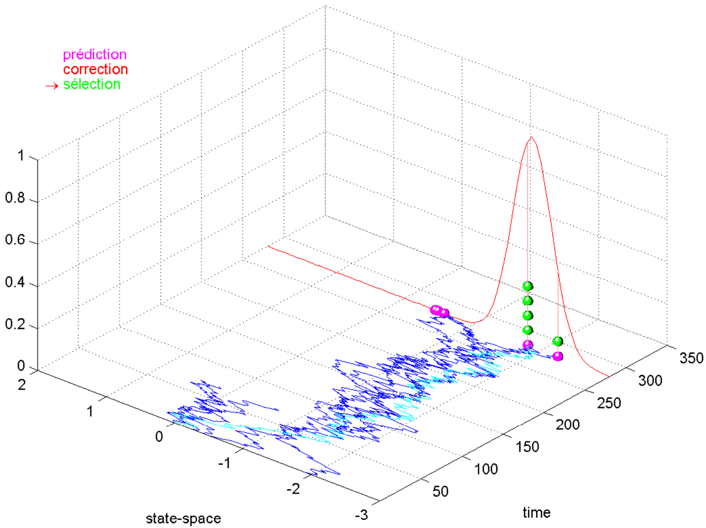

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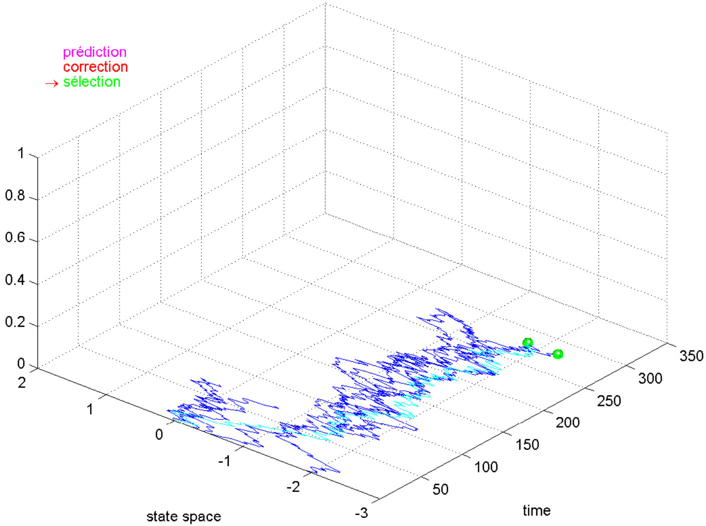

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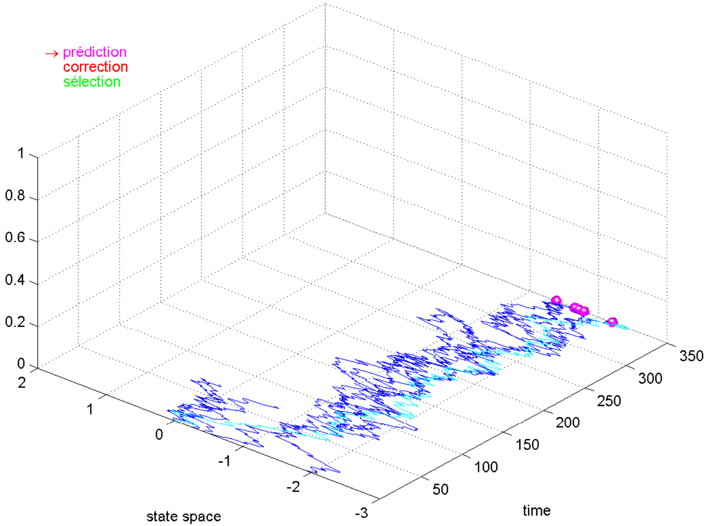

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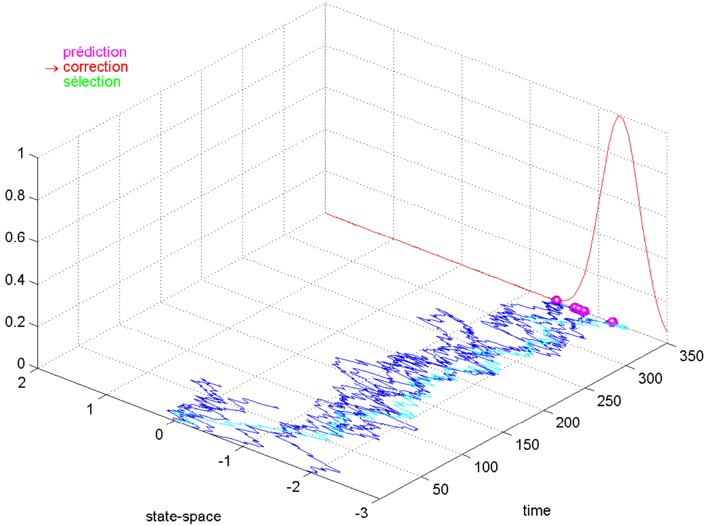

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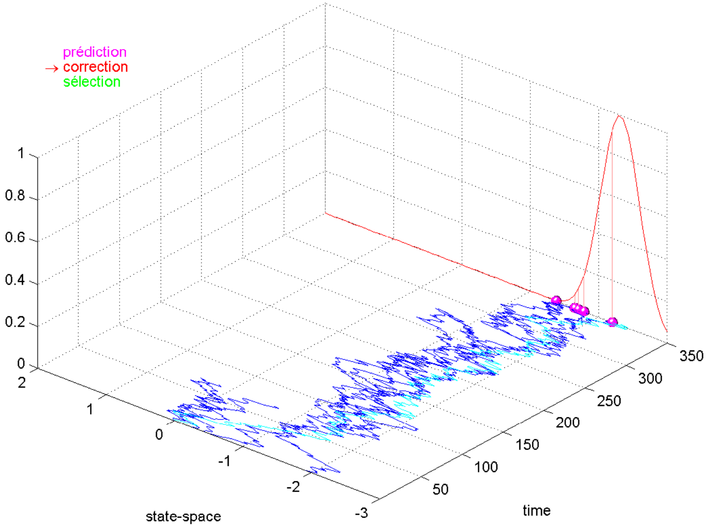

Parameter Estimation in General State-Space Models

Calculating the Log-Likelihood

- For the Sequential Importance Sampling algorithm we have that

$$\frac{1}{N}\sum_{i=1}^{N}\omega_n^i \approx L(y_{1:n})$$

- When adding resampling this is not a valid estimator for the likelihood.

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- Note: ω_t^i are the unnormalized weights!

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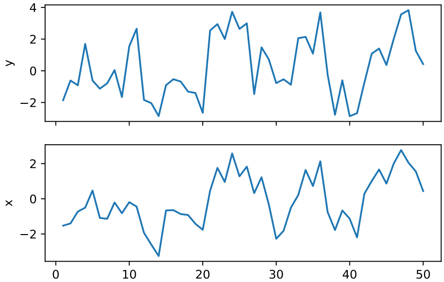
- Note: ω_t^i are the unnormalized weights!
- The is an unbiased estimator of the likelihood!

Example: Gaussian State Space Model

Linear Gaussian State Space Model We look at the model defined by the following equations:

$$\begin{cases} \alpha_{t+1} = a\alpha_t + \eta_{t+1}, & \eta_{t+1} \sim \mathcal{N}(0,1), \\ y_t = \alpha_t + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0,1), \end{cases}$$

with initial distribution $\alpha_1 \sim \mathcal{N}(0, 1/(1-a^2))$



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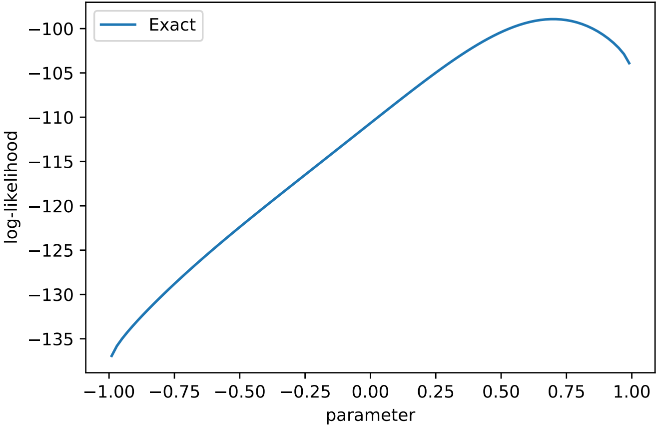

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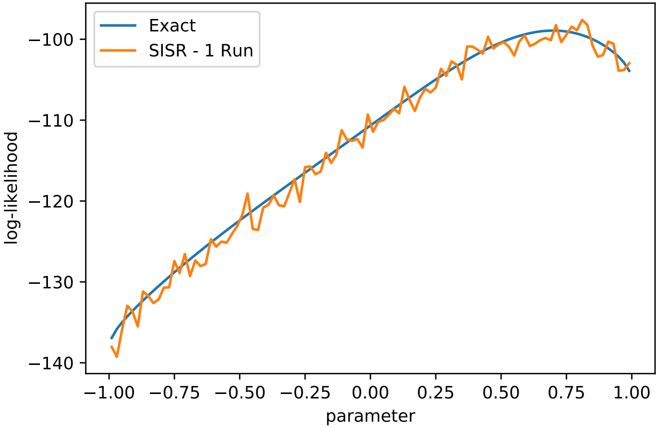

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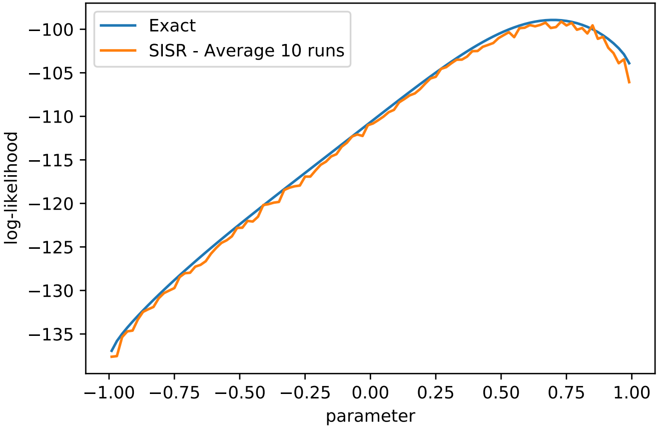

Expectation-Maximization

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- The E-step is calculating:

$$\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) &= \mathbb{E}[\log p_\theta(\alpha_{1:n}, y_{1:n}) \mid y_{1:n}, \tilde{\theta}] \\
&= \mathbb{E}[\log q_\theta(\alpha_1) \mid y_{1:n}, \tilde{\theta}] + \sum_{t=2}^{n} \mathbb{E}[\log q_\theta(\alpha_t \mid \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] \\
&+ \sum_{t=1}^{n} \mathbb{E}[\log g_\theta(y_t \mid \alpha_t) \mid y_{1:n}, \tilde{\theta}]
\end{aligned}$$

Expectation-Maximization Algorithm

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&+ \sum_{t=1}^{n} \mathbb{E}[\log g_\theta(y_t \mid \alpha_t) \mid y_{1:n}, \tilde{\theta}]
\end{aligned}
$$

- The M-step is to maximize $\mathcal{Q}(\tilde{\theta}, \theta)$:

$$\theta^* = \arg\max_\theta \mathcal{Q}(\tilde{\theta}, \theta)$$

Def: A distribution belongs to the exponential family if its density function can be written as,

$$p_\theta(x) = h(x) \exp\left(\mathsf{n}(\theta) \cdot \mathsf{T}(x) - A(\theta)\right),$$

where

- $\mathsf{T}(x)$ is a sufficient statistic
- $\mathsf{n}(\theta)$ is the natural parameter

Interlude I: Exponential Family

Def: A distribution belongs to the exponential family if its density function can be written as,

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where

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- $\mathsf{n}(\theta)$ is the natural parameter

Ex: A Gaussian distribution with mean μ and variance σ^2 belongs to the exponential family using

$$\underbrace{\frac{1}{\sqrt{2\pi}}}_{h(x)} \exp\left(\underbrace{\begin{pmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{pmatrix}}_{\mathsf{n}(\theta)} \cdot \underbrace{\begin{pmatrix} x \\ x^2 \end{pmatrix}}_{\mathsf{T}(x)} - \underbrace{\left(\frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log(\sigma^2) \right)}_{A(\theta)} \right)$$

Expectation-Maximization Algorithm II

- We assume that q_θ and g_θ belongs to the exponential family.

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- The E-step now reduces to,

$$
\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) = \mathbb{E}[\log h_q^1(\alpha_1) + \mathbf{n}_\mathbf{q}^1(\theta) \cdot \mathbf{T}_\mathbf{q}(\alpha_1) - A_q^1(\theta) \mid y_{1:n}, \tilde{\theta}] \\
+ \sum_{t=2}^{n} \mathbb{E}[\log h_q(\alpha_t, \alpha_{t-1}) + \mathbf{n}_\mathbf{q}(\theta) \cdot \mathbf{T}_\mathbf{q}(\alpha_t, \alpha_{t-1}) - A_q(\theta) \mid y_{1:n}\tilde{\theta}] \\
+ \sum_{t=1}^{n} \mathbb{E}[\log h_g(y_t, \alpha_t) + \mathbf{n}_\mathbf{g}(\theta) \cdot \mathbf{T}_\mathbf{g}(y_t, \alpha_t) - A_g(\theta) \mid y_{1:n}, \tilde{\theta}]
\end{aligned}
$$

Expectation-Maximization Algorithm II

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- The E-step now reduces to,

$$
\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) = \mathbb{E}[\log h_q^1(\alpha_1) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n}_\mathbf{q}^1(\theta) \cdot \mathbb{E}[\mathbf{T}_\mathbf{q}(\alpha_1) \mid y_{1:n}, \tilde{\theta}] - A_q^1(\theta) \\
+ \sum_{t=2}^{n} \mathbb{E}[\log h_q(\alpha_t, \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n}_\mathbf{q}(\theta) \cdot \mathbb{E}[\mathbf{T}_\mathbf{q}(\alpha_t, \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] \\
- A_q(\theta) \\
+ \sum_{t=1}^{n} \mathbb{E}[\log h_g(y_t, \alpha_t) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n}_\mathbf{g}(\theta) \cdot \mathbb{E}[\mathbf{T}_\mathbf{g}(y_t, \alpha_t) \mid y_{1:n}, \tilde{\theta}] \\
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Smoothing

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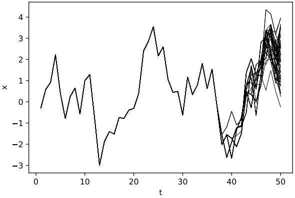
- Idea: Run the the particle filter, store the trajectories and approximate the smoothing distribution.
- Problem: Resampling collapses the trajectories.



Fixed-Lag Smoothing

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- Most of them involve intricate backward passes or Markov Chains to estimate the smoothing distribution.
- A simple smoothing algorithm is the fixed-lag smoothing.
- We approximate using

$$\mathbb{E}[h(\alpha_t) \mid y_{1:n}] \approx \mathbb{E}[h(\alpha_t) \mid y_{1:t+\ell}],$$

where ℓ is a fixed lag.

Example: Stochastic Volatility

Stochastic volatility model. The model is defined by the equations,

$$\begin{cases} \alpha_k = a\alpha_{k-1} + \sigma_\eta \eta_k, & \eta_k \sim \mathcal{N}(0, s^2) \\ y_k = b \exp(\alpha_k/2)\varepsilon_k, & \varepsilon_k \sim \mathcal{N}(0, 1) \end{cases}$$

For simplicity we assume that $\alpha_1 \sim \mathcal{N}(0, 1)$.

Example: Stochastic Volatility

EM-algorithm, in this model we find four sufficient statistics:

$$t_1 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t^2 \mid y_{1:n}] \qquad t_2 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t \alpha_{t-1} \mid y_{1:n}]$$

$$t_3 = \sum_{t=2}^{n} \mathbb{E}[\alpha_{t-1}^2 \mid y_{1:n}] \qquad t_4 = \sum_{t=1}^{n} \mathbb{E}[y_t^2 \exp(-\alpha_t) \mid y_{1:n}]$$

The parameters are updated as,

$$a = \frac{t_2}{t_3} \quad s^2 = \frac{1}{n-1}\left(t_1 - \frac{t_2^2}{t_3}\right) \quad b^2 = \frac{t_4}{n}$$

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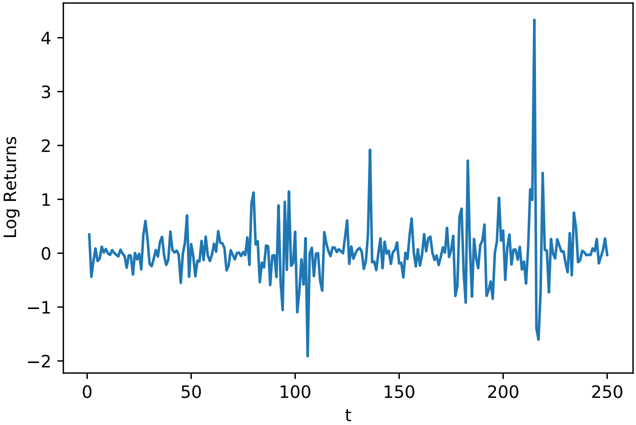

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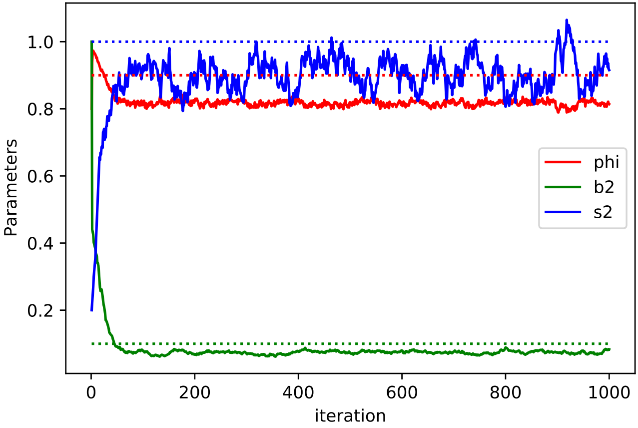

Extensions to Particle Filters

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- If all weights are equal then $\mathrm{ESS}_t = N$.
- If all weights except one is zero then $\mathrm{ESS}_t = 1$.
- Choose a threshold N_{ESS}, if $\mathrm{ESS}_t < N_{\mathrm{ESS}}$ then resample, otherwise no resampling happens.
- If no resampling happens the weights should be updated as in the SIS algorithm.

Algorithm: Particle Filter with Adaptive Resampling

Particle Filter:

Draw $\alpha_1^i \sim f(\alpha_1)$

Set $\omega_1^i = \frac{q(\alpha_1^i)g(y_1 \mid \alpha_1^i)}{f(\alpha_1^i)}$

Set $\Omega_1 = \sum_{i=1}^N \omega_1^i$

for $t = 2, 3, \ldots, n$ **do**

  Calculate $\mathrm{ESS}_{t-1} = \frac{(\sum_{i=1}^N \omega_t^i)^2}{\sum_{i=1}^N (\omega_t^i)^2}$

  if $\mathrm{ESS}_{t-1} < N_{\mathrm{ESS}}$ **then**

    Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$

    Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$

    Set $\omega_{t-1}^i = 1/N$

  end if

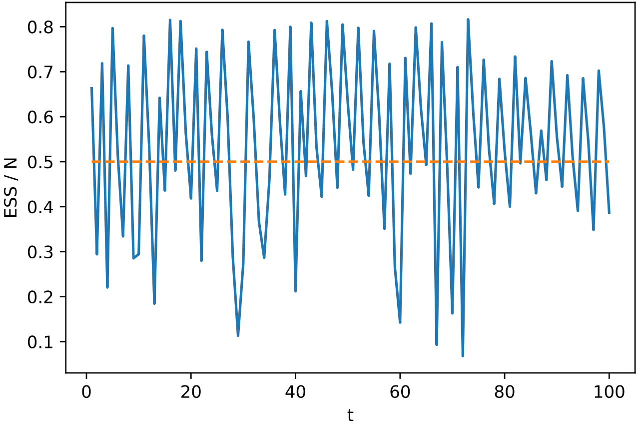
  Draw $\alpha_t^i \sim f(\alpha_t \mid \alpha_{1:t-1}^i)$

  Set $\omega_t^i = \frac{q(\alpha_t^i \mid \alpha_{t-1}^i)g(y_t \mid \alpha_t^i)}{f(\alpha_t^i \mid \alpha_{1:t-1}^i)} \times \omega_{t-1}^i$

  Set $\Omega_t = \sum_{i=1}^N \omega_t^i$

end for

Example: Adaptive resampling



Resampling: 43%

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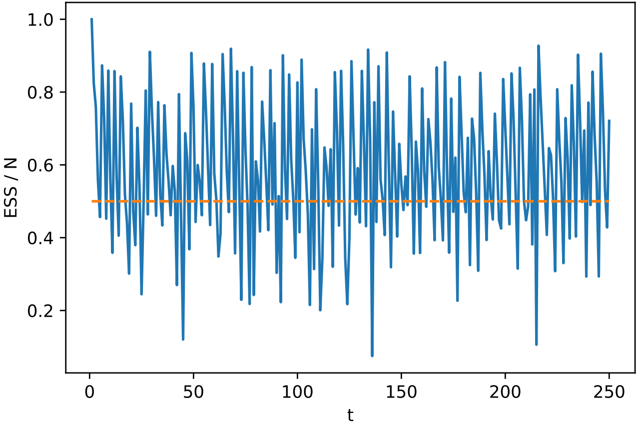

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Different Resampling Schemes

- We have so far only considered multinomial resampling.
- There are many different schemes,
    - Residual resampling

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    - Easier if the model belongs to the exponential family.
    - Requires, again, the smoothing distribution.
    - Looked at the fixed-lag smoother.
- We looked at some extensions regarding the resampling.
    - By calculating the ESS we performed adaptive resampling by only resampling when needed.
    - We looked at some different resampling schemes.