# Time Series and Sequence Learning

## Lecture 8 – Parameter Estimation in Non-Linear/Non-Gaussian State Space Models

Johan Alenlöv, Linköping University

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# Summary of Lecture 7

# General State-Space Models

**Def.** A **General State-Space** model is given by:

$$\alpha_t \,|\, \alpha_{t-1} \sim q(\alpha_t \,|\, \alpha_{t-1})$$
$$y_t \,|\, \alpha_t \sim g(y_t \,|\, \alpha_t)$$

and initial distribution $\alpha_1 \sim q(\alpha_1)$.

**Thm.** The **joint-smoothing distribution** is given by

$$p(\alpha_{1:n} \,|\, y_{1:n}) = \frac{q(\alpha_1) g(y_1 \,|\, \alpha_1) \prod_{i=2}^{n} q(\alpha_i \,|\, \alpha_{i-1}) g(y_i \,|\, \alpha_i)}{L_n(y_{1:n})},$$

where $L_n(y_{1:n}) = \int q(\alpha_1) g(y_1 \,|\, \alpha_1) \prod_{i=2}^{n} q(\alpha_i \,|\, \alpha_{i-1}) g(y_i \,|\, \alpha_i) \mathrm{d}\alpha_{1:n}$ is the **likelihood**.

# Sequential Importance Sampling

- A first idea was to use **importance sampling** targeting the **joint-smoothing distribution**.
- We go from time $n$ to $n+1$ in the following way:
  - Draw $\alpha^i_{n+1} \sim f(\alpha_{n+1} \mid \alpha^i_{1:n})$ (propagating)
  - Set $\alpha^i_{1:n+1} = (\alpha^i_{1:n}, \alpha^i_{n+1})$
  - Set $\omega^i_{n+1} = \frac{q(\alpha^i_{n+1} \mid \alpha^i_n) g(y_{n+1} \mid \alpha^i_{n+1})}{f(\alpha^i_{n+1} \mid \alpha^i_{1:n})} \times \omega^i_n$. (weighting)
- Simplest choice: $f(\alpha_{n+1} \mid \alpha^i_{1:n}) = q(\alpha_{n+1} \mid \alpha^i_n)$
- We estimate using:

$$\sum_{i=1}^{N} \frac{\omega^i_{n+1}}{\Omega_{n+1}} h(\alpha^i_{n+1}) \approx \mathbb{E}[h(\alpha_{n+1}) \mid y_{1:n+1}]$$

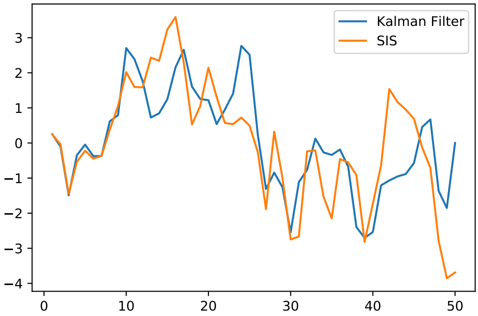
$$N^{-1}\Omega_{n+1} = \frac{1}{N} \sum_{i=1}^{N} \omega^i_{n+1} \approx L(y_{1:n+1})$$

# Example: Linear Gaussian State Space Model

**Def.** A **Linear Gaussian State-Space (LGSS)** model is given by:

$$\begin{aligned} \alpha_t &= T\alpha_{t-1} + R\eta_t, & \eta_t &\sim \mathcal{N}(0, Q), \\ y_t &= Z\alpha_t + \varepsilon_t & \varepsilon_t &\sim \mathcal{N}(0, \sigma_\epsilon^2), \end{aligned}$$

and initial distribution $\alpha_1 \sim \mathcal{N}(a_1, P_1)$.



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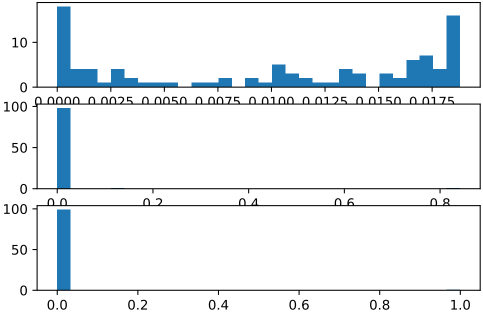

# Sequential Importance Sampling with Resampling

- We solve the problem of **weight degeneracy** by **resampling** the particles!
- The most natural selection is to draw new particles $(\tilde{\alpha}^i_{1:n})^N_{i=1}$ among the **SIS** produced $(\alpha^i_{1:n})^N_{i=1}$ with probabilities given by the **normalized importance weights**.
- For $i = 1, 2, \dots, N$ we let

$$\tilde{\alpha}^i_{1:n} = \alpha^j_{1:n} \quad \text{w. pr.} \quad \frac{\omega^j_n}{\Omega_n}$$

- Add this step **before** propagating the particles!

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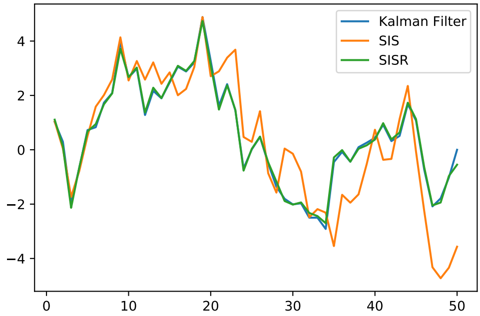


# Algorithm: Particle Filter

**Particle Filter:**

Draw $\alpha_1^i \sim f(\alpha_1)$

Set $\omega_1^i = \frac{q(\alpha_1^i) g(y_1 \mid \alpha_1^i)}{f(\alpha_1^i)}$

Set $\Omega_1 = \sum_{i=1}^N \omega_1^i$

**for** $t = 2, 3, \ldots, n$ **do**

  Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$

  Draw $\alpha_t^i \sim f(\alpha_t \,|\, \alpha_{1:t-1}^{I^i})$

  Set $\omega_t^i = \frac{q(\alpha_t^i \mid \alpha_{t-1}^{I^i}) g(y_t \mid \alpha_t^i)}{f(\alpha_t^i \mid \alpha_{1:t-1}^{I^i})}$

  Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$

  Set $\Omega_t = \sum_{i=1}^N \omega_t^i$

**end for**

# Algorithm: Bootstrap Particle Filter

**Bootstrap Particle Filter:**

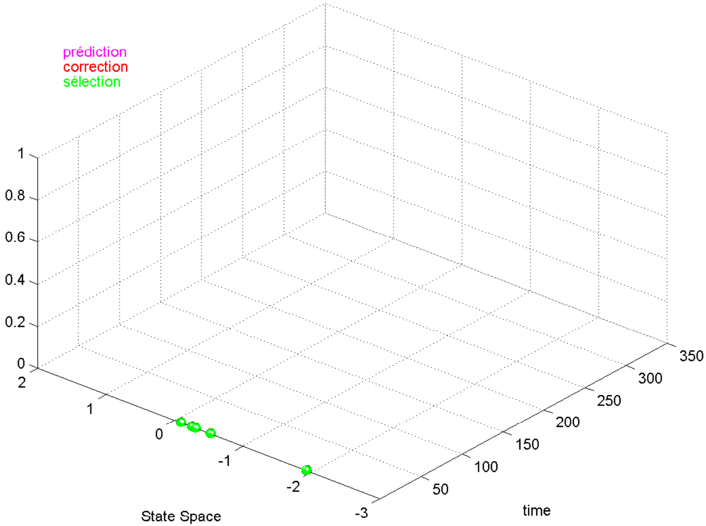
Draw $\alpha_1^i \sim q(\alpha_1)$
Set $\omega_1^i = g(y_1 \,|\, \alpha_1^i)$
Set $\Omega_1 = \sum_{i=1}^{N} \omega_1^i$
**for** $t = 2, 3, \ldots, n$ **do**
  Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$
  Draw $\alpha_t^i \sim q(\alpha_t \,|\, \alpha_{t-1}^{I^i})$
  Set $\omega_t^i = g(y_t \,|\, \alpha_t^i)$
  Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$
  Set $\Omega_t = \sum_{i=1}^{N} \omega_t^i$
**end for**

# Sequential Importance Sampling with Resampling

In Python:

```
part = np.zeros((n,N))
logwgt = np.zeros((n,N))
wgt = np.zeros((n,N))
ests = np.zeros(n)
part[0,:] = np.random.randn(N)
logwgt[0,:] = logwgtfun(xpart[0,:],y[0])
wgt[0,:] = np.exp(logwgt[0,:])
ests[0] = np.sum(wgt[0,:] * part[0,:])/np.sum(wgt[i+1,:])
for i in range(n-1):
    ind = np.random.choice(N, size=N, replace=True, p=wgt[i,:])
    part[i+1,:] = a*part[i,ind] + np.random.randn(N)
    logwgt[i+1,:] = logwgtfun(xpart[i+1,:],y[i+1])
    wgt[i+1,:] = np.exp(logwgt[i+1,:])
    ests[i+1] = np.sum(wgt[i+1,:] * part[i+1,:])/np.sum(wgt[i
        +1,:])
```

# Particle Filter Movie



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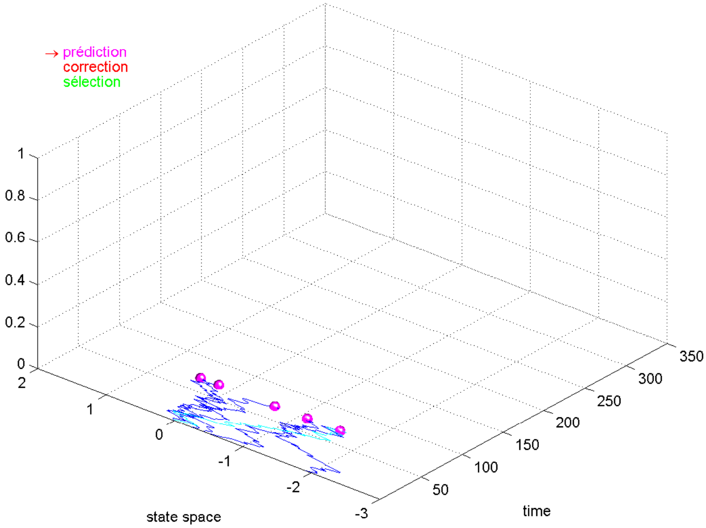

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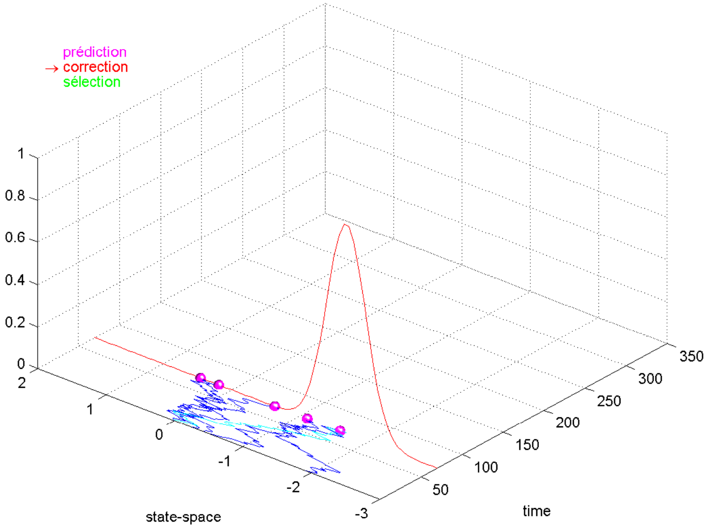

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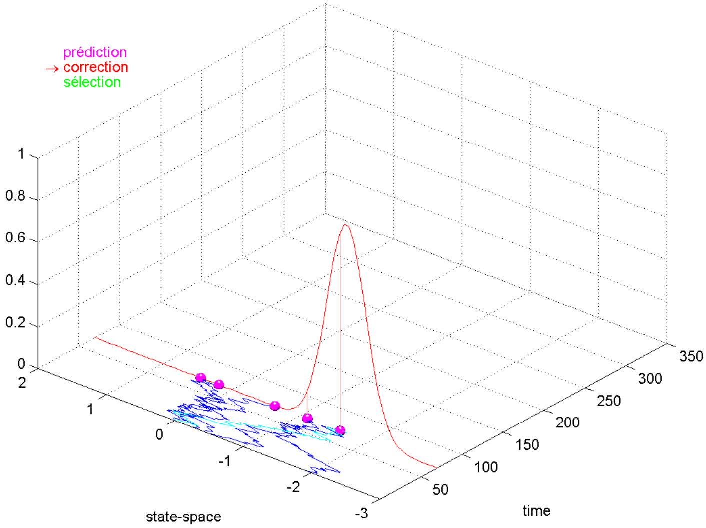

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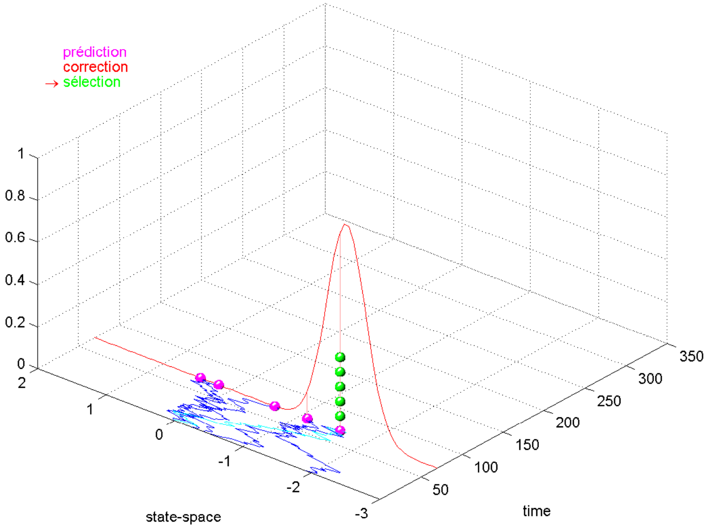

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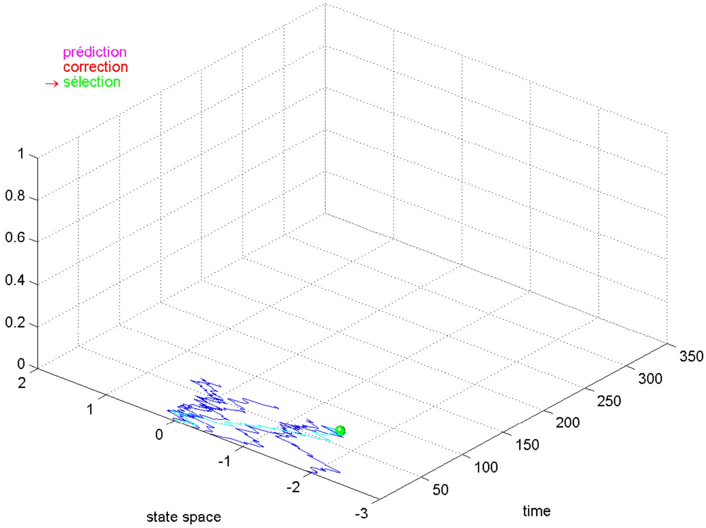

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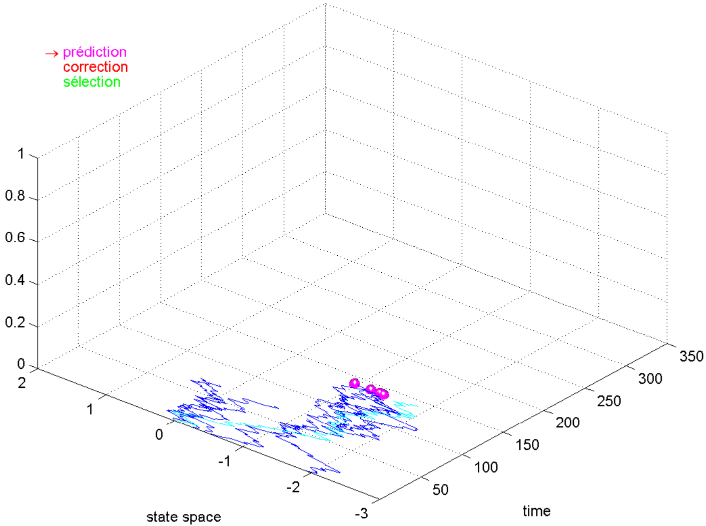

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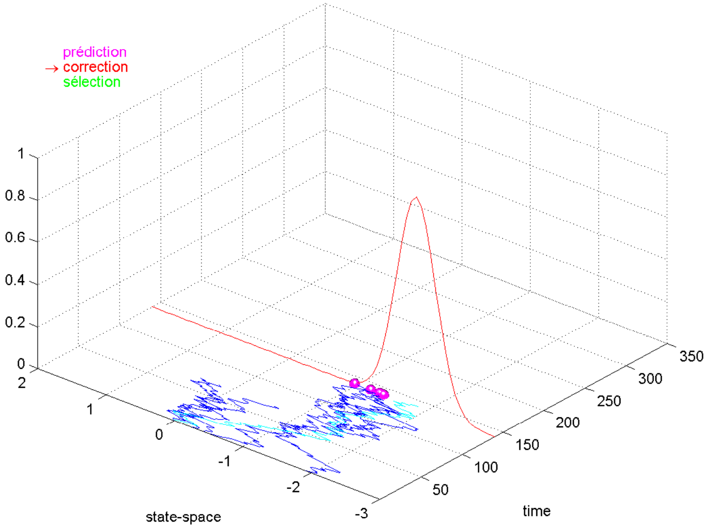

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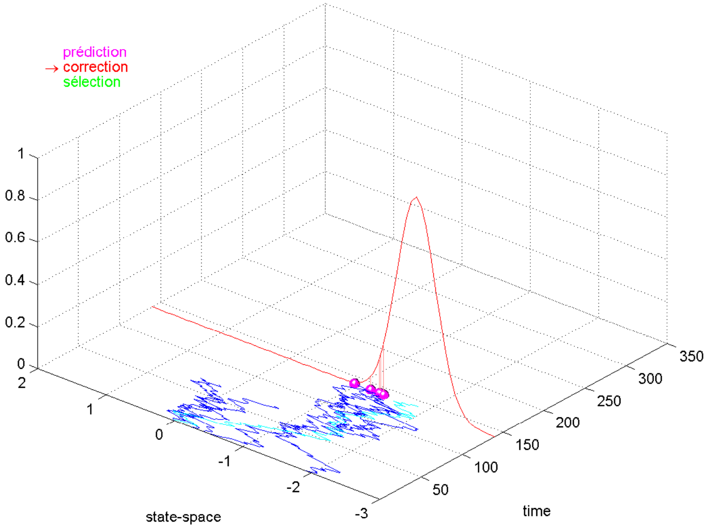

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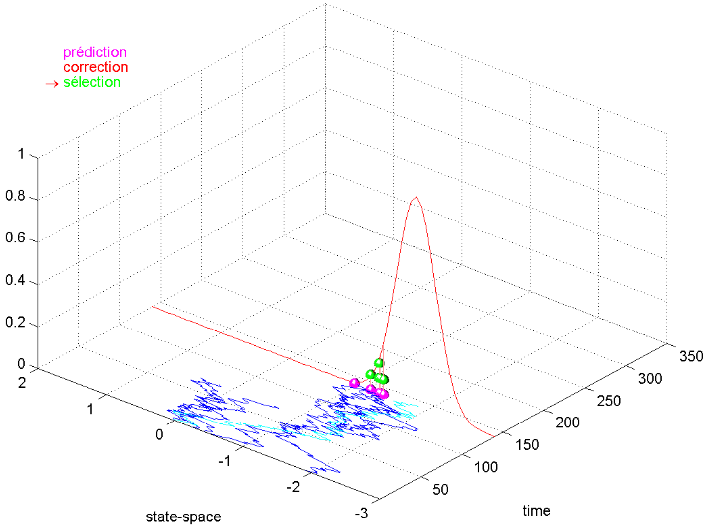

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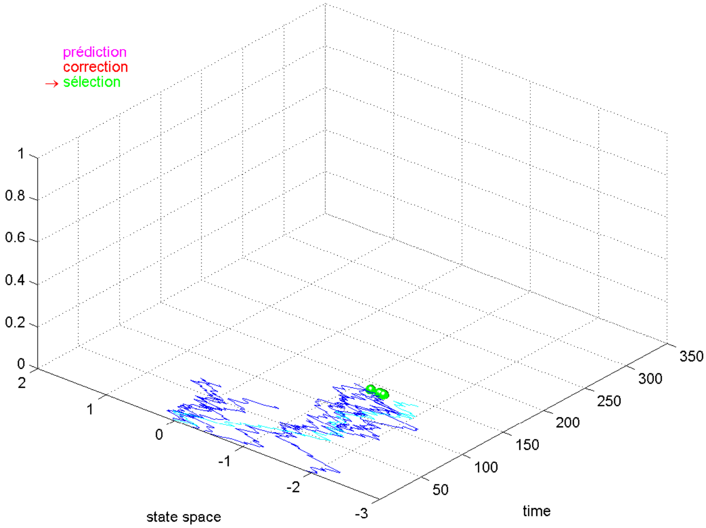

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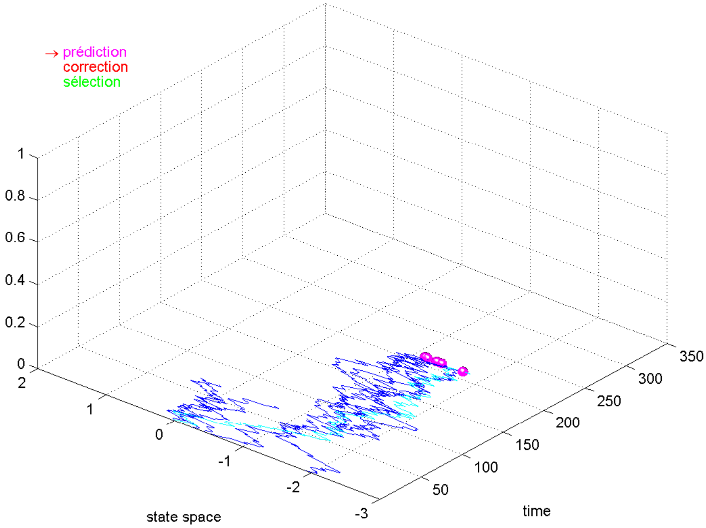

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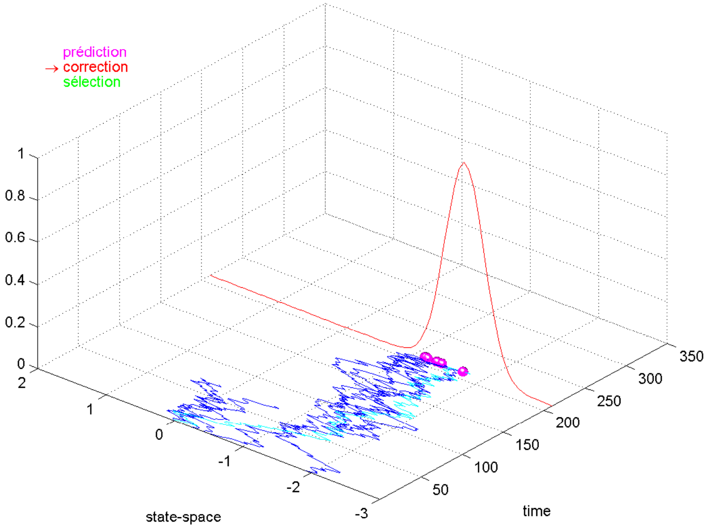

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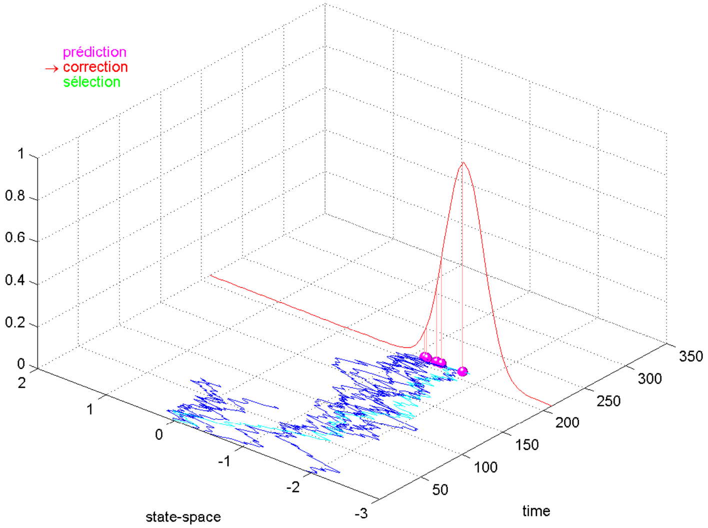

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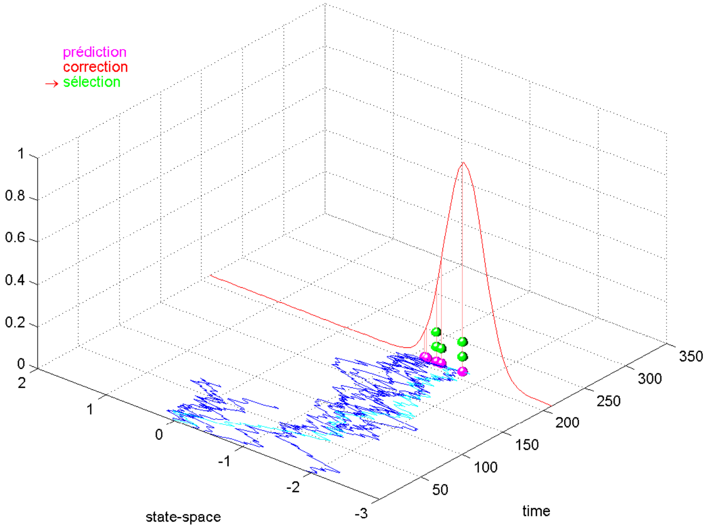

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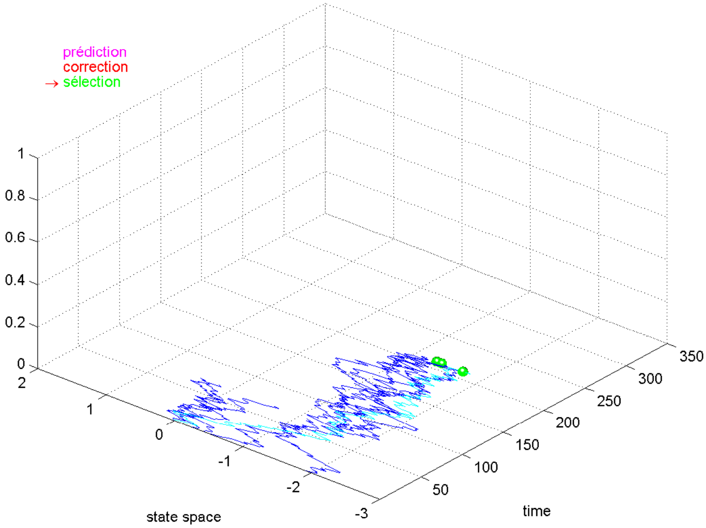

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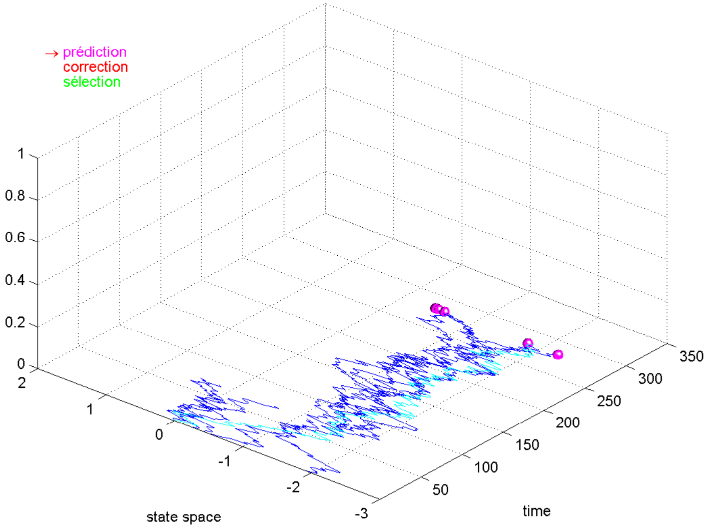

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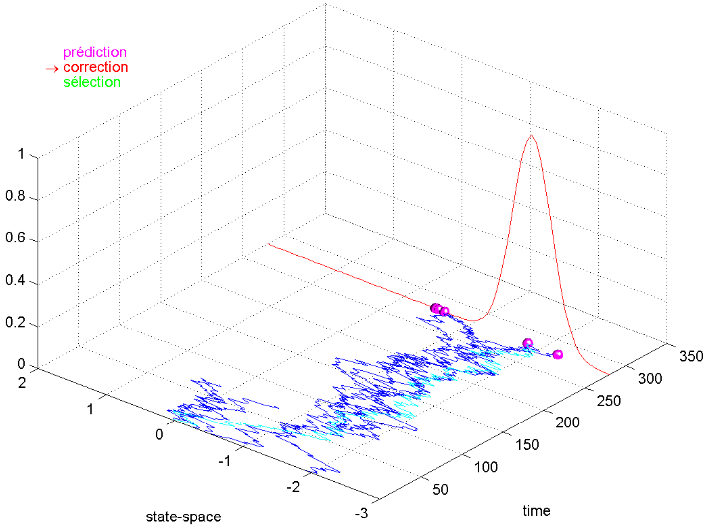

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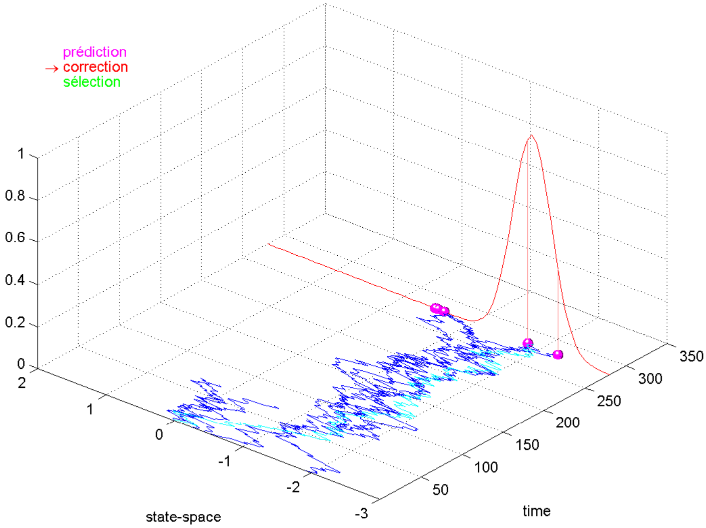

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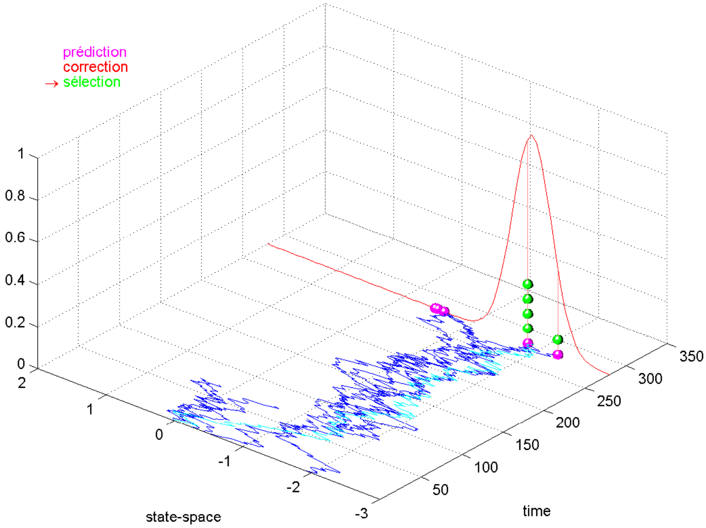

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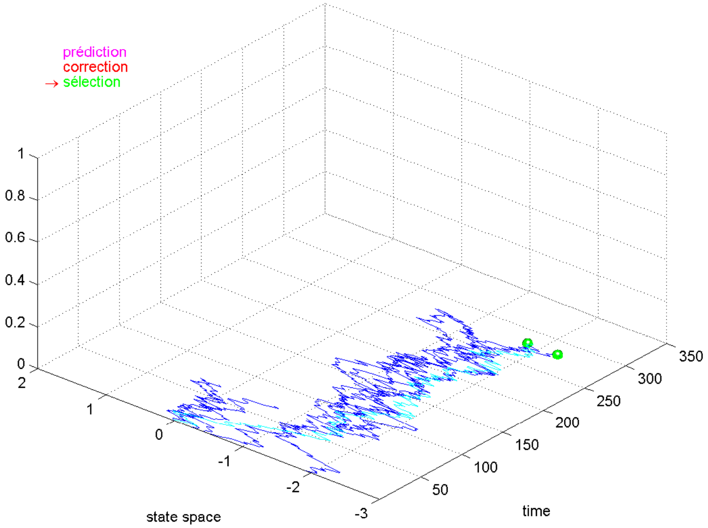

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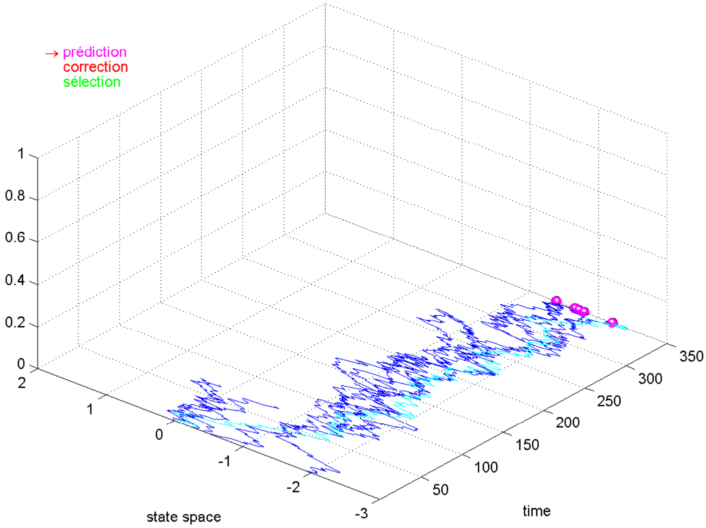

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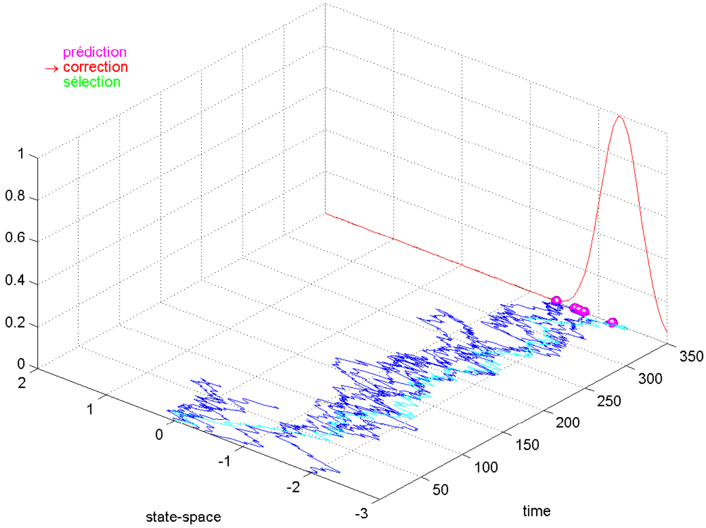

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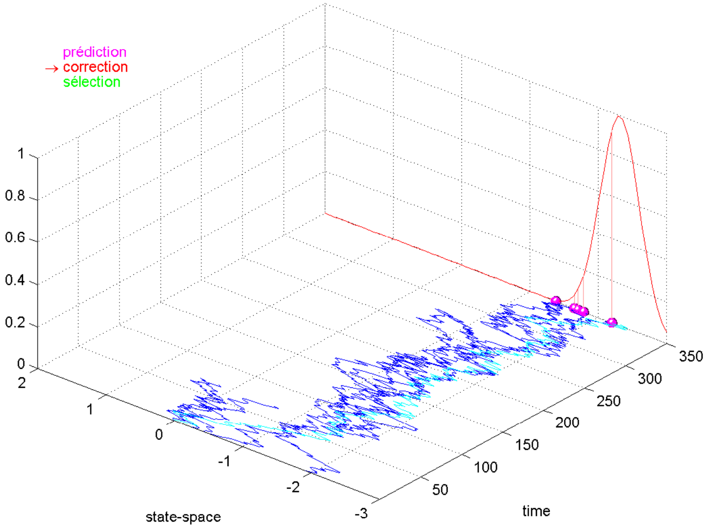


# Parameter Estimation in General State-Space Models

# Calculating the Log-Likelihood

- For the **Sequential Importance Sampling** algorithm we had that

$$\frac{1}{N}\sum_{i=1}^{N}\omega_n^i \approx L(y_{1:n})$$

- When adding resampling this is **not** a valid estimator for the likelihood.
- Instead we use

$$\prod_{t=1}^{n}\left(\frac{1}{N}\sum_{i=1}^{N}\omega_t^i\right) \approx L(y_{1:n})$$

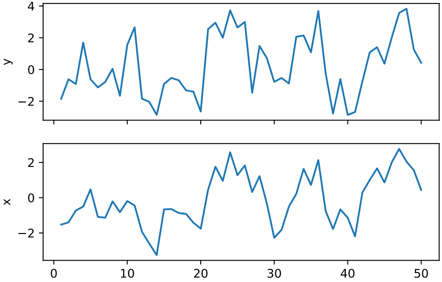
- **Note:** $\omega_t^i$ are the **unnormalized** weights!
- The is an **unbiased** estimator of the likelihood!

# Example: Gaussian State Space Model

**Linear Gaussian State Space Model** We look at the model defined by the following equations:

$$\begin{cases} \alpha_{t+1} = a\alpha_t + \eta_{t+1}, & \eta_{t+1} \sim \mathcal{N}(0,1), \\ y_t = \alpha_t + \varepsilon_t, & \varepsilon_t \sim \mathcal{N}(0,1), \end{cases}$$

with initial distribution $\alpha_1 \sim \mathcal{N}(0, 1/(1-a^2))$



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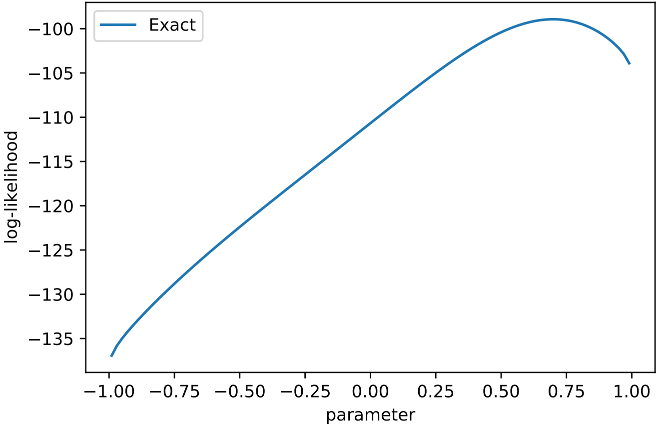

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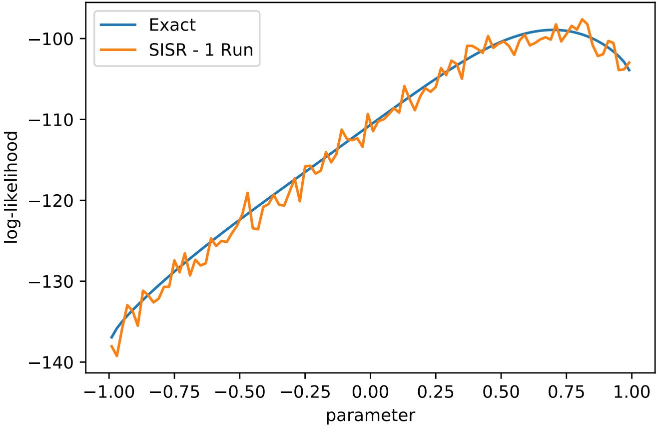

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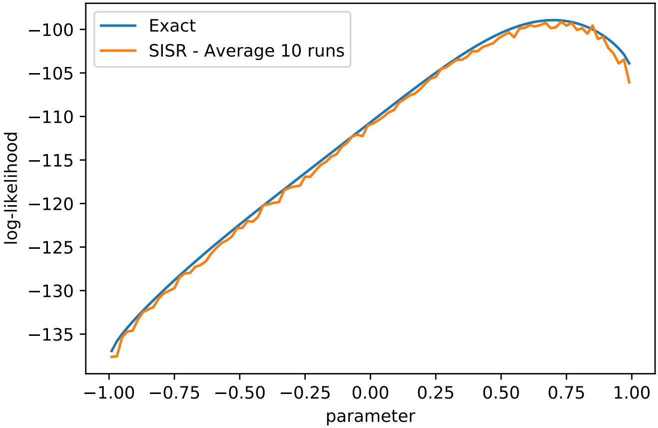

# Expectation-Maximization Algorithm

- As in the LGSS model we can use **expectation-maximization** algorithm for parameter estimation.
- The **E-step** is calculating:

$$\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) &= \mathbb{E}[\log p_{\theta}(\alpha_{1:n}, y_{1:n}) \mid y_{1:n}, \tilde{\theta}] \\
&= \mathbb{E}[\log q_{\theta}(\alpha_1) \mid y_{1:n}, \tilde{\theta}] + \sum_{t=2}^{n} \mathbb{E}[\log q_{\theta}(\alpha_t \mid \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] \\
&+ \sum_{t=1}^{n} \mathbb{E}[\log g_{\theta}(y_t \mid \alpha_t) \mid y_{1:n}, \tilde{\theta}]
\end{aligned}$$

- The **M-step** is to maximize $\mathcal{Q}(\tilde{\theta}, \theta)$:

$$\theta^* = \arg\max_{\theta} \mathcal{Q}(\tilde{\theta}, \theta)$$

# Interlude I: Exponential Family

**Def:** A distribution belongs to the **exponential family** if its density function can be written as,

$$p_\theta(x) = h(x) \exp\left(\mathbf{n}(\theta) \cdot \mathbf{T}(x) - A(\theta)\right),$$

where

- $\mathbf{T}(x)$ is a **sufficient statistic**
- $\mathbf{n}(\theta)$ is the **natural parameter**

**Ex:** A **Gaussian** distribution with mean $\mu$ and variance $\sigma^2$ belongs to the **exponential family** using

$$\underbrace{\frac{1}{\sqrt{2\pi}}}_{h(x)} \exp\left( \underbrace{\begin{pmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{pmatrix}}_{\mathbf{n}(\theta)} \cdot \underbrace{\begin{pmatrix} x \\ x^2 \end{pmatrix}}_{\mathbf{T}(x)} - \underbrace{\left( \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log(\sigma^2 \right)}_{A(\theta)} \right)$$

# Expectation-Maximization Algorithm II

- We assume that $q_\theta$ and $g_\theta$ belongs to the **exponential family**.
- The **E-step** now reduces to,

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- The **E-step** now reduces to,

$$
\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) = \mathbb{E}[\log h_q^1(\alpha_1) + \mathbf{n}_\mathbf{q}^\mathbf{1}(\theta) \cdot \mathbf{T}_\mathbf{q}(\alpha_1) - A_q^1(\theta) \mid y_{1:n}, \tilde{\theta}] \\
+ \sum_{t=2}^{n} \mathbb{E}[\log h_q(\alpha_t, \alpha_{t-1}) + \mathbf{n}_\mathbf{q}(\theta) \cdot \mathbf{T}_\mathbf{q}(\alpha_t, \alpha_{t-1}) - A_q(\theta) \mid y_{1:n}\tilde{\theta}] \\
+ \sum_{t=1}^{n} \mathbb{E}[\log h_g(y_t, \alpha_t) + \mathbf{n}_\mathbf{g}(\theta) \cdot \mathbf{T}_\mathbf{g}(y_t, \alpha_t) - A_g(\theta) \mid y_{1:n}, \tilde{\theta}]
\end{aligned}
$$

# Expectation-Maximization Algorithm II

- We assume that $q_\theta$ and $g_\theta$ belongs to the **exponential family**.
- The **E-step** now reduces to,

$$
\begin{aligned}
\mathcal{Q}(\tilde{\theta}, \theta) = {} & \mathbb{E}[\log h_q^1(\alpha_1) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n_q^1}(\theta) \cdot \mathbb{E}[\mathbf{T_q}(\alpha_1) \mid y_{1:n}, \tilde{\theta}] - A_q^1(\theta) \\
& + \sum_{t=2}^{n} \mathbb{E}[\log h_q(\alpha_t, \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n_q}(\theta) \cdot \mathbb{E}[\mathbf{T_q}(\alpha_t, \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}] \\
& \qquad - A_q(\theta) \\
& + \sum_{t=1}^{n} \mathbb{E}[\log h_g(y_t, \alpha_t) \mid y_{1:n}, \tilde{\theta}] + \mathbf{n_g}(\theta) \cdot \mathbb{E}[\mathbf{T_g}(y_t, \alpha_t) \mid y_{1:n}, \tilde{\theta}] \\
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\end{aligned}
$$

# Expectation-Maximization Algorithm II

- We assume that $q_\theta$ and $g_\theta$ belongs to the **exponential family**.
- The **E-step** now reduces to,
  Calculate the **smoothed sufficient statistics**,

$$
\begin{aligned}
\mathbf{T_1} &= \mathbb{E}[\mathbf{T_q}(\alpha_1) \,|\, y_{1:n}, \tilde{\theta}], \\
\mathbf{T_2} &= \sum_{t=2}^{n} \mathbb{E}[\mathbf{T_q}(\alpha_t, \alpha_{t-1}) \,|\, y_{1:n}, \tilde{\theta}], \\
\mathbf{T_3} &= \sum_{t=1}^{n} \mathbb{E}[\mathbf{T_g}(y_t, \alpha_t) \,|\, y_{1:n}, \tilde{\theta}].
\end{aligned}
$$

# Expectation-Maximization Algorithm II

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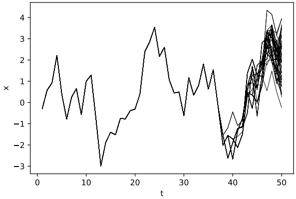
$$\mathbf{T_1} = \mathbb{E}[\mathbf{T_q}(\alpha_1) \mid y_{1:n}, \tilde{\theta}],$$
$$\mathbf{T_2} = \sum_{t=2}^{n} \mathbb{E}[\mathbf{T_q}(\alpha_t, \alpha_{t-1}) \mid y_{1:n}, \tilde{\theta}],$$
$$\mathbf{T_3} = \sum_{t=1}^{n} \mathbb{E}[\mathbf{T_g}(y_t, \alpha_t) \mid y_{1:n}, \tilde{\theta}].$$

- The **M-step** becomes maximize,

$$\mathbf{n_q^1}(\theta) \cdot \mathbf{T_1} - A_q^1(\theta) + \mathbf{n_q}(\theta) \cdot \mathbf{T_2} - A_q(\theta) + \mathbf{n_g}(\theta) \cdot \mathbf{T_3} - A_g(\theta)$$

# Joint-Smoothing Distribution

- As previously for the EM-algorithm we need the **smoothing distribution**.
- Good news! When we derived the particle filter we did it for the **joint-smoothing distribution**.
- The **particle trajectories** approximate the joint-smoothing distribution.
- **Idea:** Run the the particle filter, store the trajectories and approximate the smoothing distribution.
- **Problem:** Resampling collapses the trajectories.



# Fixed-Lag Smoothing

- There are many algorithms to perform **smoothing** in general State Space models.
- Most of them involve intricate backward passes or Markov Chains to estimate the smoothing distribution.
- A **simple** smoothing algorithm is the **fixed-lag smoothing**.
- We approximate using

$$\mathbb{E}[h(\alpha_t) \mid y_{1:n}] \approx \mathbb{E}[h(\alpha_t) \mid y_{1:t+\ell}],$$

where $\ell$ is a **fixed lag**.

# Example: Stochastic Volatility

**Stochastic volatility model**. The model is defined by the equations,

$$\begin{cases} \alpha_k = a\alpha_{k-1} + \sigma_\eta \eta_k, & \eta_k \sim \mathcal{N}(0, s^2) \\ y_k = b\exp(\alpha_k/2)\varepsilon_k, & \varepsilon_k \sim \mathcal{N}(0,1) \end{cases}$$

For simplicity we assume that $\alpha_1 \sim \mathcal{N}(0,1)$.

**EM-algorithm**, in this model we find four **sufficient statistics**:

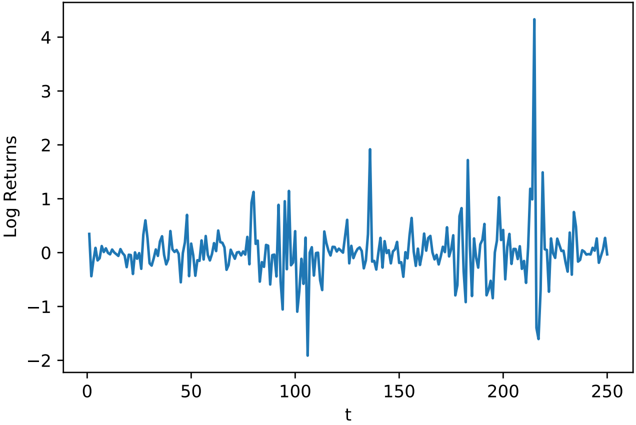
$$t_1 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t^2 \,|\, y_{1:n}] \qquad t_2 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t \alpha_{t-1} \,|\, y_{1:n}]$$

$$t_3 = \sum_{t=2}^{n} \mathbb{E}[\alpha_{t-1}^2 \,|\, y_{1:n}] \qquad t_4 = \sum_{t=1}^{n} \mathbb{E}[y_t^2 \exp(-\alpha_t) \,|\, y_{1:n}]$$

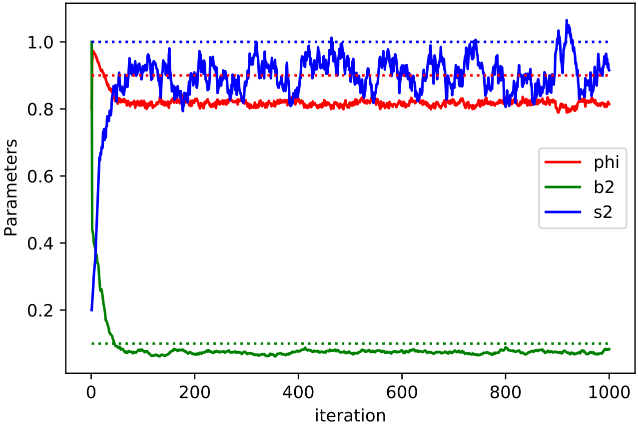
The parameters are updated as,

$$a = \tfrac{t_2}{t_3} \quad s^2 = \tfrac{1}{n-1}\left(t_1 - \tfrac{t_2^2}{t_3}\right) \quad b^2 = \tfrac{t_4}{n}$$

# Example: Stochastic Volatility



# Example: Stochastic Volatility



# Extensions to Particle Filters

# Adaptive Resampling

- So far we have done **resampling every iteration**.
- This is **unnecessary**, we only need to do it if the weights have degenerated.
- Use **effective sample size** $(\mathrm{ESS})$ as a measure for degeneracy,

$$\mathrm{ESS}_t = \frac{(\sum_{i=1}^{N} \omega_t^i)^2}{\sum_{i=1}^{N} (\omega_t^i)^2}$$

- If all weights are **equal** then $\mathrm{ESS}_t = N$.
- If all weights except one is zero then $\mathrm{ESS}_t = 1$.
- Choose a **threshold** $N_{\mathrm{ESS}}$, if $\mathrm{ESS}_t < N_{\mathrm{ESS}}$ then resample, otherwise no resampling happens.
- If no resampling happens the weights should be updated as in the SIS algorithm.

# Algorithm: Particle Filter with Adaptive Resampling

**Particle Filter:**

Draw $\alpha_1^i \sim f(\alpha_1)$

Set $\omega_1^i = \frac{q(\alpha_1^i)g(y_1 \mid \alpha_1^i)}{f(\alpha_1^i)}$

Set $\Omega_1 = \sum_{i=1}^N \omega_1^i$

**for** $t = 2, 3, \ldots, n$ **do**

  Calculate $\mathrm{ESS}_{t-1} = \frac{(\sum_{i=1}^N \omega_t^i)^2}{\sum_{i=1}^N (\omega_t^i)^2}$

  **if** $\mathrm{ESS}_{t-1} < N_{\mathrm{ESS}}$ **then**

    Draw $I^i = j$ w. pr. $\frac{\omega_{t-1}^j}{\Omega_{t-1}}$

    Set $\alpha_{1:t}^i = (\alpha_{1:t-1}^{I^i}, \alpha_t^i)$

    Set $\omega_{t-1}^i = 1/N$

  **end if**

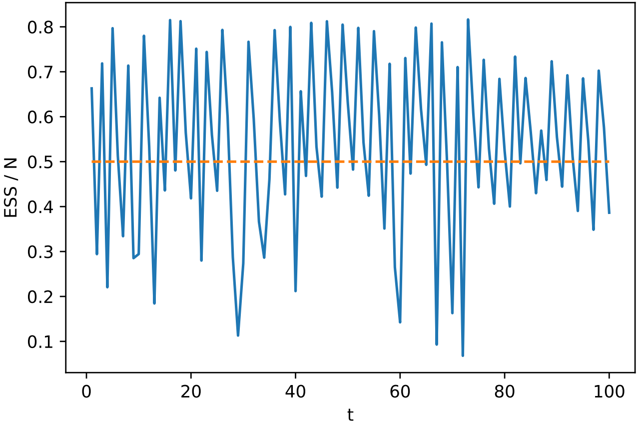
  Draw $\alpha_t^i \sim f(\alpha_t \mid \alpha_{1:t-1}^i)$

  Set $\omega_t^i = \frac{q(\alpha_t^i \mid \alpha_{t-1}^i)g(y_t \mid \alpha_t^i)}{f(\alpha_t^i \mid \alpha_{1:t-1}^i)} \times \omega_{t-1}^i$

  Set $\Omega_t = \sum_{i=1}^N \omega_t^i$

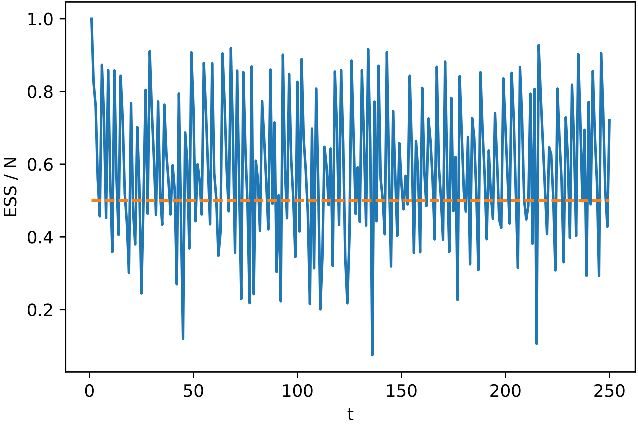
**end for**

# Example: Adaptive resampling




Resampling: 43%

# Example: Adaptive resampling




Resampling: 36%

# Different Resampling Schemes

- We have so far only considered **multinomial resampling**.
- There are many different schemes,
  - **Residual resampling**
  - **Stratified resampling**
  - **Systematic resampling**

# Summary

- We looked at how to calculate the **likelihood** using a **particle filter**.
  - The randomness of the algorithm gives us **noisy** likelihood estimate.
  - **Solve** by performing many runs and average.
- We looked at the **EM-algorithm**.
  - Easier if the model belongs to the **exponential family**.
  - Requires, again, the **smoothing distribution**.
  - Looked at the **fixed-lag smoother**.
- We looked at some extensions regarding the **resampling**.
  - By calculating the $\mathrm{ESS}$ we performed **adaptive resampling** by only resampling when needed.
  - We looked at some different resampling schemes.