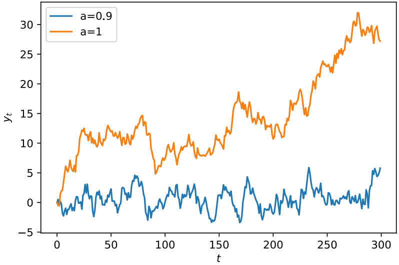
Time Series and Sequence Learning

Lecture 5e – Stability of LGSS

Johan Alenlöv, Linköping University

ex) Simulation of AR(1)

Simulation of $y_t = ay_{t-1} + \varepsilon_t$



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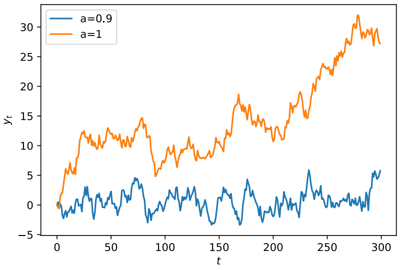

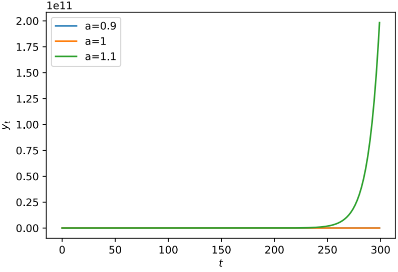

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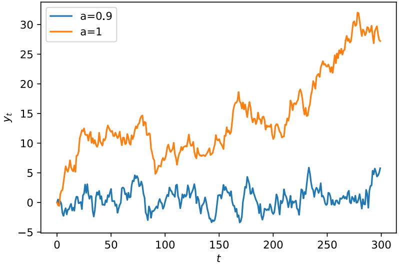
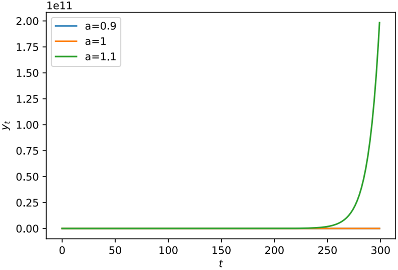




The AR(1) model is:

- Stable if $|a| < 1 \Rightarrow$ converges to stationary
- Marginally stable if $|a| = 1 \Rightarrow$ linear drift
- Unstable if $|a| > 1 \Rightarrow$ exponential explosion

Can this be generalized to an LGSS model?

AR(1)**:** $y_t = a y_{t-1} + \varepsilon_t$

LGSS: $\alpha_t = T\alpha_{t-1} + R\eta_t$

Stability of state space model

Can this be generalized to an LGSS model?

AR(1): $\quad y_t = a y_{t-1} + \varepsilon_t$

LGSS: $\quad \alpha_t = T \alpha_{t-1} + R \eta_t$

Intuitively, the state process is unstable if "size(T)" > 1.

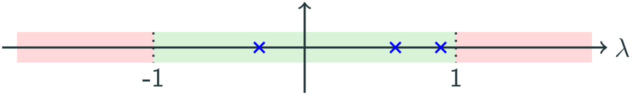
Stability of state space model

Thm. A state space model of dimension $d = \dim(\alpha_t)$ is:

- Stable iff $|\lambda_j| < 1, \quad j = 1, \dots, d,$
- Marginally stable iff $|\lambda_j| \leq 1, \quad j = 1, \dots, d,$
- Unstable iff $|\lambda_j| > 1$ for any $j = 1, \dots, d,$

where λ_j, $j = 1, \dots, d$ are the **eigenvalues of T.**

If all eigenvalues are real numbers,



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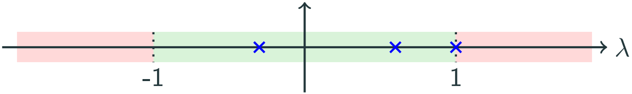

Marginally stable

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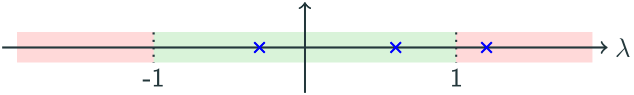


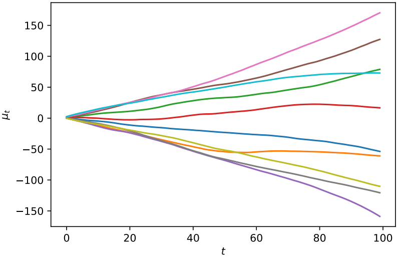
Unstable

ex) Eigenvalues of linear trend model

Linear trend model:

$$\alpha_t = \overbrace{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}^{F} \alpha_{t-1} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{R} \zeta_t$$

$$\mu_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{Z} \alpha_t$$



Check for stability by computing the eigenvalues

$$\mathrm{eig}(F) \Rightarrow \lambda_1 = \lambda_2 = 1.$$

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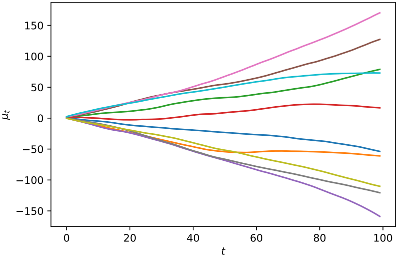

Check for stability by computing the eigenvalues

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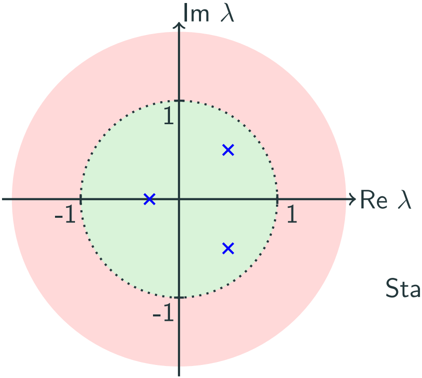
The trend model is **marginally stable!**

Complex eigenvalues

Note. The eigenvalues can be complex numbers in general. Thus, the stability condition reads

all eigenvalues of T are within the unit circle in the complex plane.



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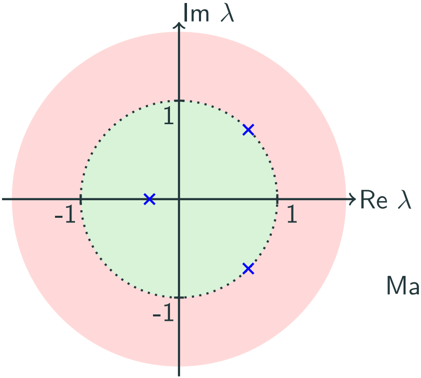

Marginally stable

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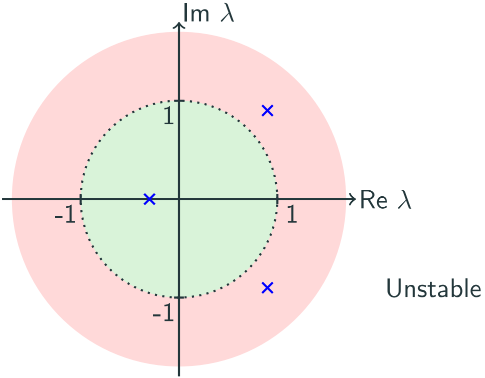
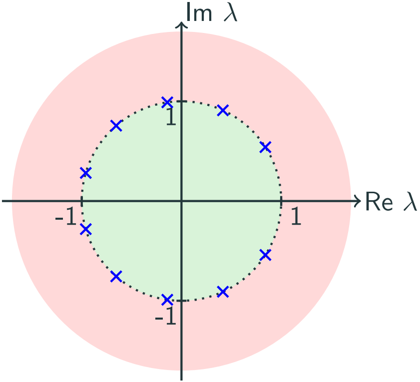


ex) Eigenvalues of seasonal model

Seasonal model:

$$T = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



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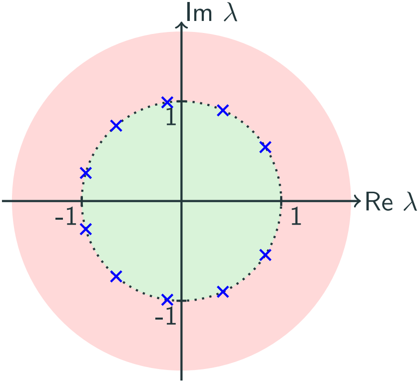
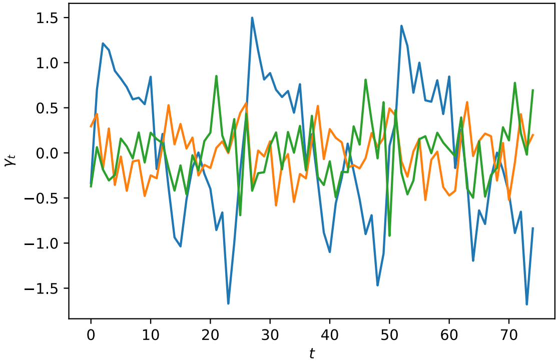


The seasonal model is **marginally stable!**

ex) Sample trajectories for seasonal model



Stability of structural time series

The structural time series models that we have proposed are

designed to be marginally stable!

Marginal stability results in desirable properties:

- Real eigenvalues $\lambda_j = 1 \Rightarrow$ polynomial drift/trend.
- Complex eigenvalues with $|\lambda_j| = 1 \Rightarrow$ periodicity/seasonality.