Time Series and Sequence Learning

Lecture 5b – Structural Time-Series, Modelling the Seasonality

Johan Alenlöv, Linköping University

Seasonal component

Structural time series: $y_t = \mu_t + \gamma_t + \varepsilon_t$

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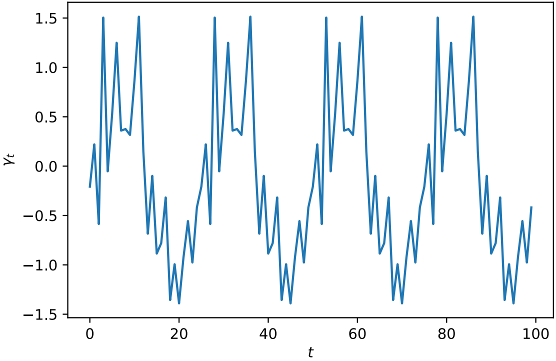


Seasonal component

For an exact periodic repition

with period s

$$\sum_{j=0}^{s-1} \gamma_{t-j} = \text{const} = 0$$

wlog

In practice we allow for small deviations or changes to the seasonal pattern

Model: $\sum_{j=0}^{s-1} \gamma_{t-j} = \omega_t$, $\omega_t \sim N(0, \sigma_\omega^2)$

$$\Rightarrow \gamma_t + \sum_{j=1}^{s-1} \gamma_{t-j} = \omega_t \Rightarrow \underbrace{\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j} + \omega_t}_{AR(s-1)}$$

Seasonal component

Seasonal component model:

$$\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j} + \omega_t, \qquad \omega_t \sim \mathcal{N}(0, \sigma_\omega^2).$$

Seasonal component

Seasonal component model:

$$\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j} + \omega_t, \qquad\qquad \omega_t \sim \mathcal{N}(0, \sigma_\omega^2).$$

Matrix form: By defining the **state vector**

$$\alpha_t = \begin{bmatrix} \gamma_t & \gamma_{t-1} & \cdots & \gamma_{t-s+2} \end{bmatrix}^\mathsf{T}$$

we can write this as

$$\alpha_t = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \omega_t$$

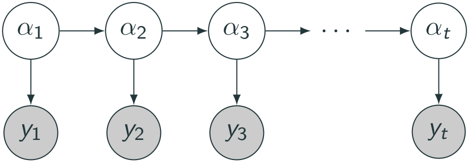
$$\gamma_t = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \end{bmatrix} \alpha_t,$$

A general state space model

Def. A **Linear Gaussian State-Space (LGSS)** model is given by:

$$\alpha_t = T\alpha_{t-1} + R\eta_t, \qquad \eta_t \sim \mathcal{N}(0, Q),$$
$$y_t = Z\alpha_t + \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2),$$

and initial distribution $\alpha_1 \sim \mathcal{N}(a_1, P_1)$.



Structural time series – block matrix model

A general structural time series model

$$y_t = \mu_t + \gamma_t + \varepsilon_t$$

can be written in state space form using **block matrices**.

State vector:

$$\alpha_t = \begin{bmatrix} \mu_t & \mu_{t-1} & \cdots & \mu_{t-k+1} & \gamma_t & \gamma_{t-1} & \cdots & \gamma_{t-s+2} \end{bmatrix}^\mathsf{T}$$

State space model:

$$\alpha_t = \begin{bmatrix} T_{[\mu]} & \\ & T_{[\gamma]} \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} R_{[\mu]} & \\ & R_{[\gamma]} \end{bmatrix} \eta_t, \quad \eta_t \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\zeta^2 & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix} \right),$$

$$y_t = \begin{bmatrix} Z_{[\mu]} & Z_{[\gamma]} \end{bmatrix} \alpha_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2).$$