Time Series and Sequence Learning

Lecture 5a – Structural Time-Series, Modelling the Trend

Johan Alenlöv, Linköping University

Trend component

Structural time series: $y_t = \mu_t + \gamma_t + \varepsilon_t$

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Let's start by considering the trend component μ_t.

Exact linear trend $\mu_t = a \cdot t + b$

$$\Rightarrow \Delta \mu_t := \mu_t - \mu_{t-1} = a \cdot t + b - (a \cdot (t-1) + b) = \underline{\underline{a}} \text{ const.}$$

$$\Delta^2 \mu_t := \Delta \mu_t - \Delta \mu_{t-1} = a - a = 0$$

This can be generalized to a

k-th order polynomial using

$$\Delta^{k+1} \mu_t = 0$$

Trend component

In practice we don't assume exact linear/polynomial trend $\Delta^{k+1}\mu_t \approx 0$

Model: $\Delta^{k+1}\mu_t = \zeta_t \ , \ \zeta_t \sim N(0, \sigma_\zeta^2)$

Example: Linear trend

$$\Delta^2 \mu_t = \zeta_t$$

$$\Delta^2 \mu_t = \Delta\mu_t - \Delta\mu_{t-1} = \mu_t - \mu_{t-1} - (\mu_{t-1} - \mu_{t-2})$$

$$= \mu_t - 2\mu_{t-1} + \mu_{t-2} = \zeta_t$$

Trend component

Solve for μ_t gives

$$\underbrace{\mu_t = 2\mu_{t-1} - \mu_{t-2} + \zeta_t}_{\text{AR(2) model}}$$

Trend component

Introduce the state-vector

$$\alpha_t = \begin{bmatrix} \mu_t \\ \mu_{t-1} \end{bmatrix} \Rightarrow \alpha_{t-1} = \begin{bmatrix} \mu_{t-1} \\ \mu_{t-2} \end{bmatrix}$$

$$\therefore \begin{cases} \alpha_{t,1} = \mu_t = 2\mu_{t-1} - \mu_{t-2} + \xi_t = 2 \cdot \alpha_{t-1,1} - \alpha_{t-1,2} + \xi_t \\ \alpha_{t,2} = \mu_{t-1} = \alpha_{t-1,1} \end{cases}$$

On matrix form we get

$$\alpha_t = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi_t$$

$$\mu_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \alpha_t$$

Trend component

Similarly, a $k-1$th order polynomial trend model $\Delta^k \mu_t = \zeta_t$ can be written as

$$\alpha_t = \begin{bmatrix} c_1 & c_2 & \cdots & c_{k-1} & c_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \zeta_t,$$

$$\mu_t = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \end{bmatrix} \alpha_t,$$

where the **state vector** is

$$\alpha_t = \begin{bmatrix} \mu_t & \mu_{t-1} & \cdots & \mu_{t-k+1} \end{bmatrix}^\mathsf{T}$$

and $c_i = (-1)^{i+1} \binom{k}{i}$.