Time Series and Sequence Learning

Lecture 5d – AR and ARMA Models

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Auto-regressive models in state space form

Recall, seasonal component model:

$$\gamma_t = -\sum_{j=1}^{s-1} \gamma_{t-j} + \omega_t,$$

On matrix form:

$$\alpha_t = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \omega_t,$$

$$\gamma_t = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \alpha_t$$

Auto-regressive model in state space form

State space formulation of AR model: The AR(p) model,

$$y_t = \sum_{j=1}^{p} a_j y_{t-j} + \eta_t,$$

can equivalently be expressed in **state space form** as

$$\alpha_t = \begin{bmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_t,$$

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A scalar AR(p) model can be written as a vector-valued AR(1) model!

Auto-regressive moving average model in state space form

State space formulation of ARMA: Consider the ARMA(p, q) model,

$$y_t = \sum_{j=1}^{p} a_j y_{t-j} + \sum_{j=1}^{q} b_j \eta_{t-j} + \eta_t.$$

Let $d = \max(p, q+1)$ and define $a_j = 0$ for $j > p$ and $b_j = 0$ for $j > q$. Then, an equivalent **state space form** is given by

$$\alpha_t = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_t,$$

$$y_t = \begin{bmatrix} 1 & b_1 & \cdots & b_{d-2} & b_{d-1} \end{bmatrix} \alpha_t$$

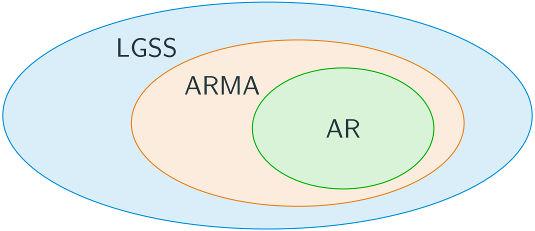
A general state space model

Both AR and ARMA models are special cases of the general linear Gaussian state space model,

Recall, the LGSS model is defined by

$$\alpha_t = T\alpha_{t-1} + R\eta_t, \qquad \eta_t \sim \mathcal{N}(0, Q),$$
$$y_t = Z\alpha_t + \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2),$$

and initial distribution $\alpha_1 \sim \mathcal{N}(a_1, P_1)$.



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Even more specifically:

Conditionally on α_t, any future state or observation variable is **conditionally independent** of past states and observations,

$$p(\alpha_\tau \mid \alpha_{1:t}, y_{1:t}) = p(\alpha_\tau \mid \alpha_t) \qquad \text{for any } \tau > t.$$