EM-algorithm

$$q(\alpha_t|\alpha_{t-1}) = \frac{1}{\sqrt{2\pi s^2}} \exp\left\{-\frac{1}{2}\frac{(\alpha_t - a\alpha_{t-1})^2}{s^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\alpha_t^2 - 2a\overset{\alpha_t}{\alpha_{t-1}} + a^2\alpha_{t-1}^2}{s^2} - \frac{1}{2}\log s^2\right\}$$

We notice that α_t^2, $\alpha_t\alpha_{t-1}$, α_{t-1}^2 are the data and all multiplied with parameters so those are candidates for sufficient statistics

$$= \underbrace{\frac{1}{\sqrt{2\pi}}}_{h_q(\alpha_t,\alpha_{t-1})} \exp\left\{\underbrace{\begin{bmatrix} -\frac{1}{2s^2} \\ \frac{a}{s^2} \\ -\frac{a^2}{2s^2}\end{bmatrix}}_{\eta_q(\theta)} \cdot \underbrace{\begin{bmatrix} \alpha_t^2 \\ \alpha_t\alpha_{t-1} \\ \alpha_{t-1}^2 \end{bmatrix}}_{T_q(\alpha_t,\alpha_{t-1})} - \underbrace{\frac{1}{2}\log(s^2)}_{A_q(\theta)}\right\}$$

$$g(y_t|\alpha_t) = \frac{1}{\sqrt{2\pi b^2\exp\{-\alpha_t\}}} \exp\left\{-\frac{1}{2}\frac{y_t^2}{b^2\exp\{-\alpha_t\}}\right\}$$

$$= \frac{1}{\sqrt{2\pi\exp\{-\alpha_t\}}} \exp\left\{-\frac{1}{2}\frac{y_t^2\exp\{\alpha_t\}}{b^2} - \frac{1}{2}\log b^2\right\}$$

Same reasoning

$$= \underbrace{\frac{1}{\sqrt{2\pi\exp\{-\alpha_t\}}}}_{h_g(\alpha_t)} \exp\left\{\underbrace{\left[-\frac{1}{2b^2}\right]}_{\eta_g(\theta)} \cdot \underbrace{\left[y_t^2\exp\{\alpha_t\}\right]}_{T_g(\alpha_t)} - \underbrace{\frac{1}{2}\log b^2}_{A_g(\theta)}\right\}$$

Now we look at $Q(\theta, \tilde{\theta})$ skipping initial distribution

$$Q(\theta, \tilde{\theta}) = \sum_{t=2}^{n} \mathbb{E}[\log q(\alpha_t | \alpha_{t-1}) | y_{1:n}]$$

$$+ \sum_{t=1}^{n} \mathbb{E}[\log g(y_t | \alpha_t) | y_{1:n}]$$

$$= \ldots = \text{const} + \eta_q(\theta)\Big(\sum_{t=2}^{n} \mathbb{E}[T_q(\alpha_t, \alpha_{t-1}) | y_{1:n}\Big) - \frac{(n-1)}{2}\log s^2$$

$$+ \eta_g(\theta)\Big(\sum_{t=1}^{n} \mathbb{E}[T_g(\alpha_t) | y_{1:n}]\Big) - \frac{n}{2}\log b^2$$

let
$$t_1 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t^2 | y_{1:n}]$$

$$t_2 = \sum_{t=2}^{n} \mathbb{E}[\alpha_t \alpha_{t-1} | y_{1:n}]$$

$$t_3 = \sum_{t=2}^{n} \mathbb{E}[\alpha_{t-1}^2 | y_{1:n}]$$

$$t_4 = \sum_{t=1}^{n} \mathbb{E}[y_t^2 \exp\{\alpha_t\} | y_{1:n}]$$

then

$$Q(\theta, \tilde{\theta}) = \text{const} - \frac{1}{2s^2} t_1 + \frac{a}{s^2} t_2 - \frac{a^2}{2s^2} t_3 - \frac{1}{2b^2} t_4 - \frac{(n-1)}{2}\log s^2 - \frac{n}{2}\log b^2$$

$$\frac{\partial Q}{\partial a} = \frac{1}{s^2} t_2 - \frac{a}{s^2} t_3 = 0 \Rightarrow a = \frac{t_2}{t_3}$$

$$\frac{\partial Q}{\partial s^2} = \frac{1}{2s^4} t_1 - \frac{a}{s^4} t_2 + \frac{a^2}{2s^4} t_3 - \frac{(n-1)}{2} \cdot \frac{1}{s^2} = 0 \Rightarrow s^2 = \frac{1}{n-1}\Big(t_1 - \frac{t_2^2}{t_3}\Big)$$

$$\frac{\partial Q}{\partial b^2} = \frac{1}{2b^4} t_4 - \frac{n}{2} \cdot \frac{1}{b^2} = 0 \Rightarrow b^2 = \frac{t_4}{n}$$