Computational Statistics Computer Lab 3 (Group 7)

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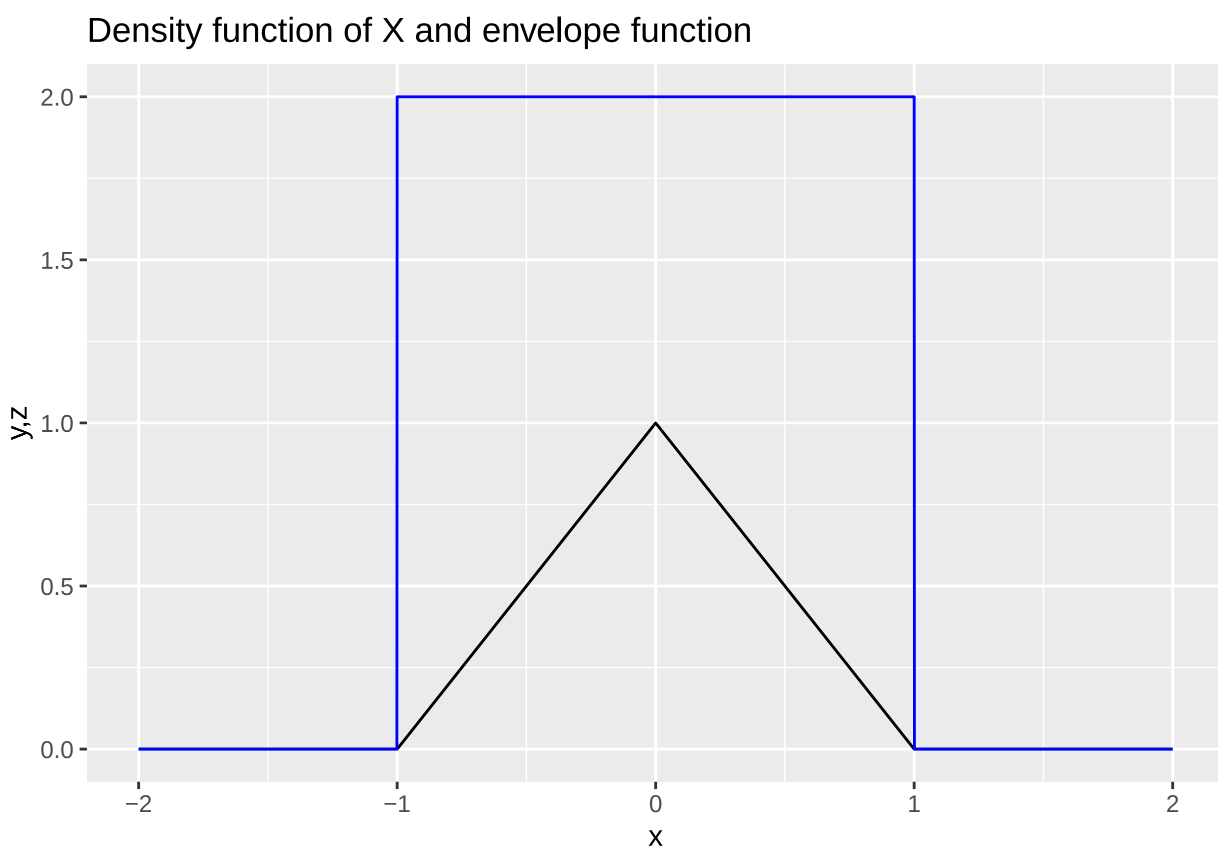
Question 1: Sampling algorithms for a triangle distribution (Solved by Qinqi Qi)

Answer:

(A)

Since we need to find a envelope function for the density function, so we need to plot the density function first. let $x \in [-3, 3]$, and we know it's
triangle sharped function. We also can plot it using the code(Check appendix). the target function $g(x) = 1$ in $[-1, 1]$, otherwise $g(x) = 0$ Since we set a = 0.5, so e(x) = g / a = 2 is our simple envelope function.

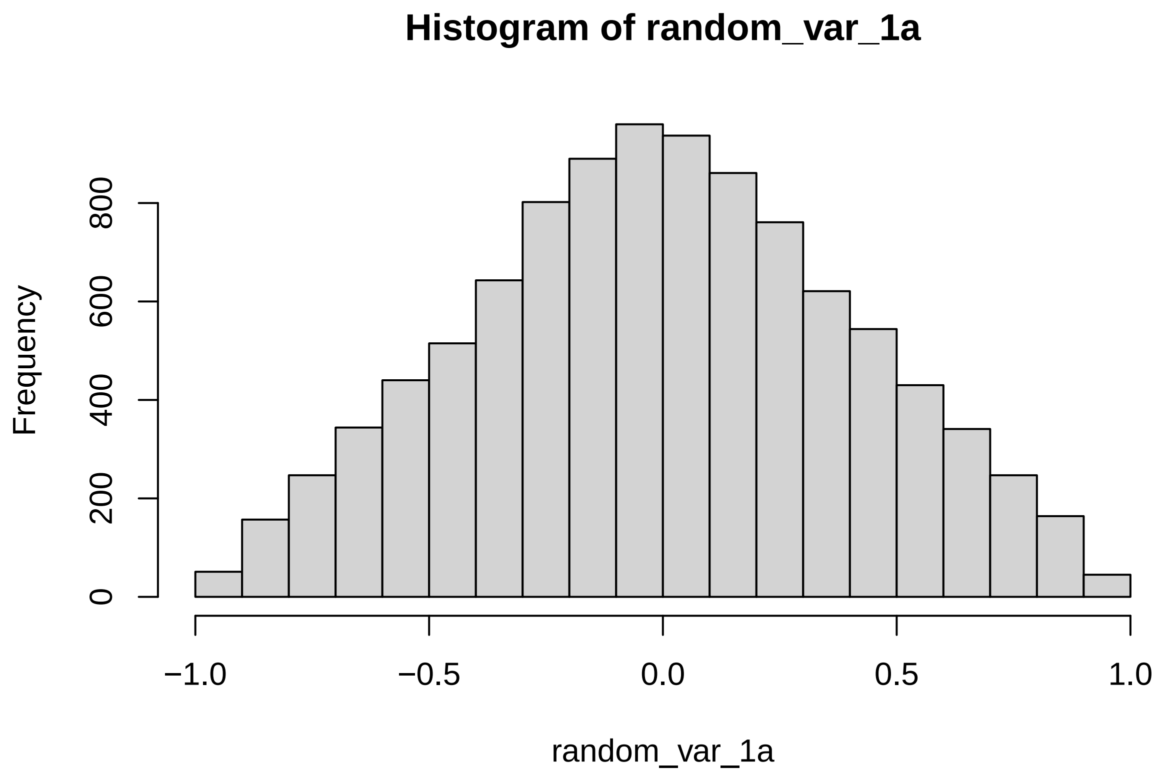


After we get the plot, we can see that the density function is a triangle we will use an envelope function to cover the density function, and we can use function $ g(x) = 1 $ to cover the density function in the [-1,1] range.

Now we will write a code for the rejection sampling algorithm our g function is $g(x) = 1$ in the range $[-1, 1]$, otherwise $g(x) = 0$ our e function is $e(x) = 1/a$ in the range $[-1, 1]$, otherwise $e(x) = 0$ and we will use

$a = 0.5$ here.

```r
########################## [ 1 a2 ] ##############################################

generate_random_var_1a <- function(n, a) {
  samples <- numeric(n)
  # Generate a sample from the proposal distribution in [-1,1]
  count <- 1
  e <- 1 / a

  while (count <= n) {

    x <- runif(1, -1, 1)
    # Calculate the acceptance probability
    u <- runif(1)

    # since g will always be 1 in the range [-1,1], so y = 1
    # f1(Y) = 1 - Y and e(Y) = 1 / a = 2
    # f2(Y) = Y + 1 and e(Y) = 1 / a = 2
    # so we have f(Y) / e(Y) = (1 - Y) / 2 or Y+1 / e

    # Calculate the acceptance probability

    if (x > 0 && u <= (1 - x) / e) {
      samples[count] <- x
      count <- count + 1
    }

    if (x <= 0 && u <= (x + 1) / e) {
      samples[count] <- x
      count <- count + 1
    }
  }

  return(samples)
}

a <- 0.5
random_var_1a <- generate_random_var_1a(10000, a)
hist(random_var_1a)
```




```r
var_of_random_var_1a <- var(random_var_1a)
cat("variance of 1st method is",var_of_random_var_1a)
```

```
## variance of 1st method is 0.1652277
```

(B)

To generate a random variable from the density function using Inverse CDF, we will need to calculate the CDF function first.

CDF function as follows:

$$CDF = \begin{cases} 0, & \text{if } x \in (-\infty, -1) \\ \frac{x^2}{2} + x, & \text{if } x \in (-1, 0) \\ x - \frac{x^2}{2}, & \text{if } x \in (0, 1) \\ 1, & \text{if } x \in (1, \infty) \end{cases}$$

Since -Y has a triangle distribution on $[-1, 0]$, so we have

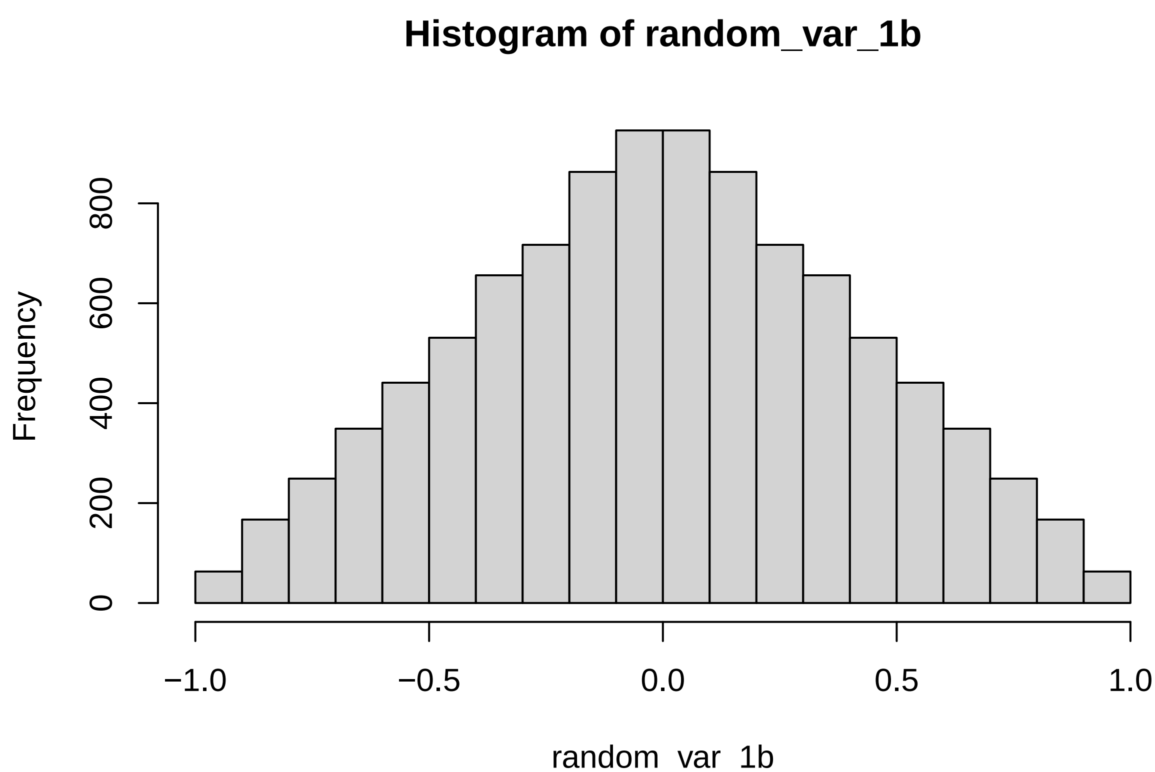
$$F^{-1} = \begin{cases} x = -1 + \sqrt{1 - 2y}, & \text{if } x \in [-1, 0) \\ x = 1 - \sqrt{1 - 2y}, & \text{if } x \in [0, 1] \\ 0, Otherwise \end{cases}$$

The following is the function to generate a random variable from it.

```r
########################## [ 1 b ] ##############################################
generate_random_var_1b <- function(n){
  u <- runif(n)
  x1 <-  - 1 + sqrt(1 - 2 * u)
  x2 <-  1 - sqrt(1 - 2 * u)
  return(c(x1, x2))
}

random_var_1b <- generate_random_var_1b(10000)
```

```
## Warning in sqrt(1 - 2 * u): NaNs produced

## Warning in sqrt(1 - 2 * u): NaNs produced
```

```r
hist(random_var_1b)
```




```r
var_of_random_var_1b <- var(random_var_1b)
cat("variance of 2nd method is",var_of_random_var_1b)
```

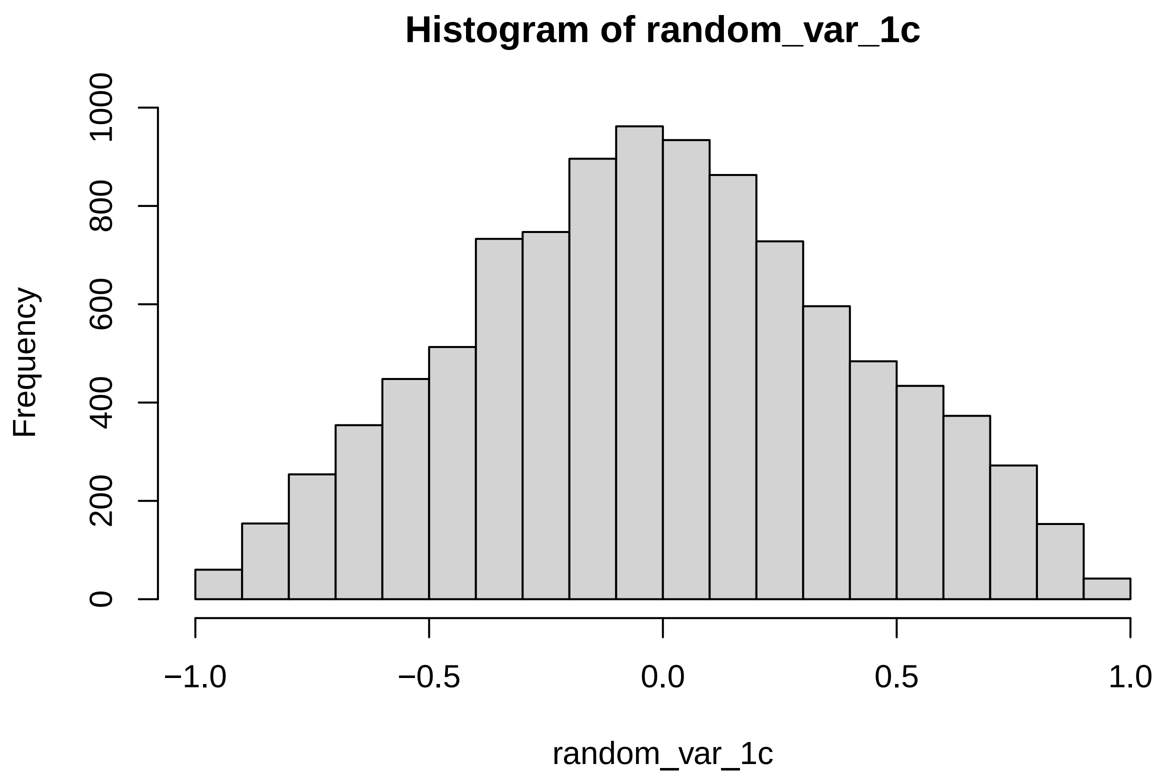
```
## variance of 2nd method is NA
```

(C)

The code to generate a random variable following a triangle distribution as follows.

```r
########################## [ 1 c ] ##############################################
generate_random_var_1c <- function(n){
  u1_1c <- runif(n)
```

```
  u2_1c <- runif(n)
  return(u1_1c - u2_1c)
}

random_var_1c <- generate_random_var_1c(10000)
hist(random_var_1c)
```




```
var_of_random_var_1c <- var(random_var_1c)
cat("variance of 3rd method is",var_of_random_var_1c)
```

```
## variance of 3rd method is 0.1673967
```

(D)

Since we already plot the data, we will not plot them here.

The variance of the three method is: 0.1691043, NA and 0.1699383.

	variance
1a	0.1691043
1b	NA
1c	0.1699383

The 2nd method generate lots of NA value , so the variance is NA.

According to the result , the 1st method, which is rejection sampling have the smallest variance.

since the 1c is the simplest way to generate the random variable, but since it can not adapt to some specific distribution, for 1b , since we need to calculate CDF and inverse function, in some cases, it's hard to do that, so we will use 1a to generate the random variable(also because of the smallest variance).

Question 2: Laplace distribution (Solved by Satya Sai Naga Jaya Koushik Pilla)

Answer:

(a) We have the function

$$DE(\mu, \lambda) = \frac{\lambda}{2} exp(-\lambda|x - \mu|)$$

Calculate the integral of the function on x to get CDF since $\mu = 0$ and $\lambda = 1$, we have $\frac{1}{2}exp(-1|x|)$. Also because we have a absolute value, we need to split the integral into two parts.

$$f(x) = \begin{cases} \frac{1}{2} * exp(-x) \\ \frac{1}{2} * exp(x) \end{cases}$$

We calculate the CDF

$$CDF = \begin{cases} -\frac{1}{2}exp(-x), forx >= 0 \\ \frac{1}{2}exp(x), forx < 0 \end{cases}$$

Then we calculate the inverse function of CDF

$$F^{-1}(y) = \begin{cases} log(2y), forx < 0 \\ -log(-2y), forx >= 0 \end{cases}$$

According to the plot, we know that most of the data are located in [-3,3] area.

```
########################## [ 2 a ] ######################################

generate_random_var_2a <- function(n){
  u1 <- runif(n,0,1)
  x1 <-  log(2 * u1)
  u2 <- runif(n,-1,0)
  x2 <-  - log(-2  * u2)
  return(c(x1, x2))
}

random_var_2a <- generate_random_var_2a(10000)
hist(random_var_2a)
```

(b) Now we will write code for the rejection sampling algorithm. since $\mu = 0$ and $\lambda = 1$, we have

$$g(x) = \begin{cases} 1/2 * exp(-x), for x >= 0 \\ 1/2 * exp(x), for x < 0 \end{cases}$$

Our e function is $e(x) = g(x)/a$, which means

$$e(x) = \begin{cases} 1/(2a) * exp(-x) \text{ for } x >= 0 \\ 1/(2a) * exp(x) for x < 0 \end{cases}$$

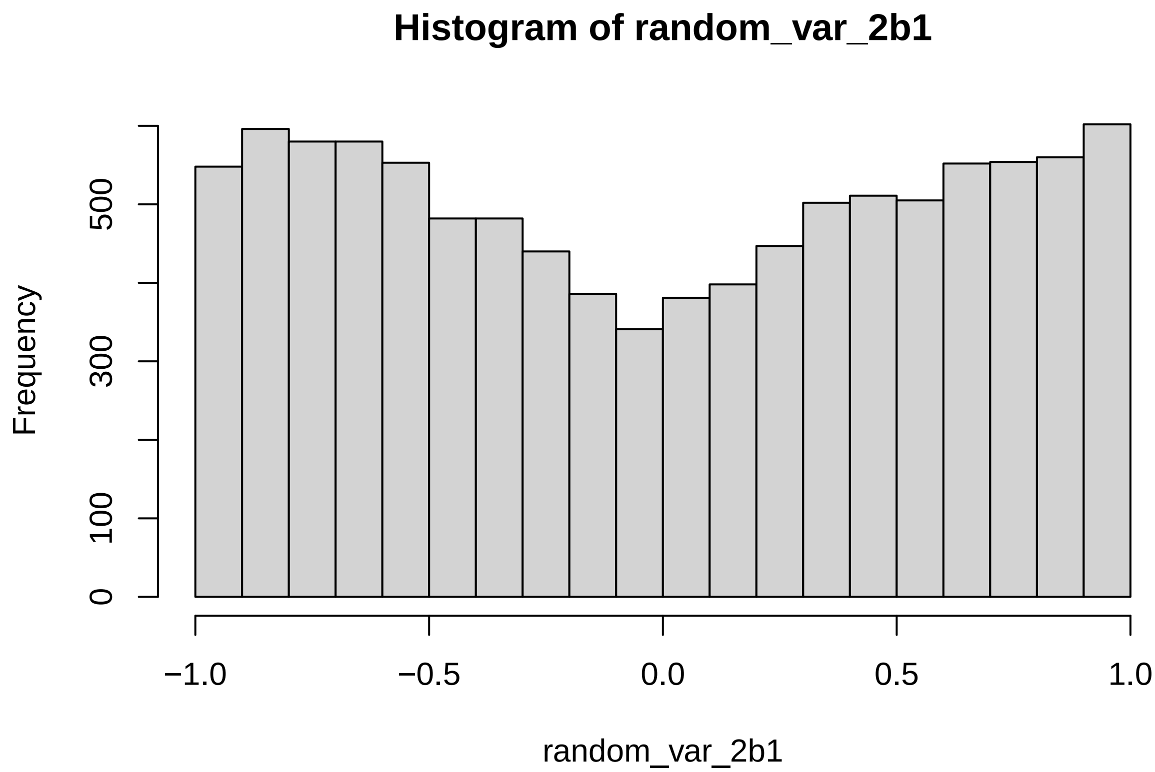
we will use $a = 0.5$ here. According to the execute result , we know that the average rejection rate is 0.4430274.

```
########################## [ 2 b ] #############################################

# generate a function to show normal distribution

normal_distribution_function <- function(x){
  return(1 / sqrt(2 * pi) * exp(-x^2 / 2))
}

generate_random_var_2b <- function(n, a) {
  samples <- numeric(n)
  # Generate a sample from the proposal distribution in [-1,1]
  count <- 1
```

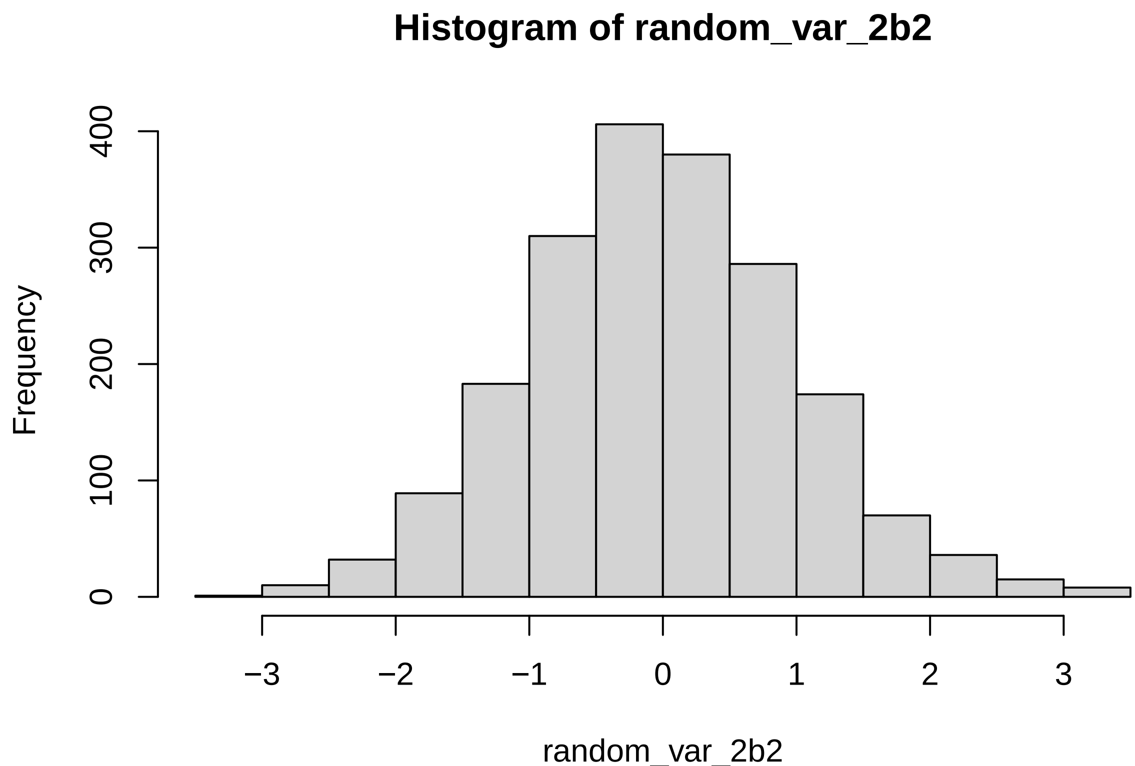
```r
  execute_count <- 1

  while (count <= n) {

    x <- runif(1, -1, 1)
    # Calculate the acceptance probability
    u <- runif(1)

    # since g will always be 1 in the range [-1,1], so y = 1
    # f(Y) = \frac{1}{\sqrt{2*pi}} * e^{\frac{1}{2} x^2 }
    # so we have f(Y) / e(Y) have 2 choices

    # Calculate the acceptance probability
    f_val <- normal_distribution_function(x)

    if (x > 0){
      e <- exp(-x)
      if (u <= (f_val / e)) {
        samples[count] <- x
        count <- count + 1
      }
    } else{
      e <- exp(x)
      if (u <= (f_val / e)) {
        samples[count] <- x
        count <- count + 1
      }
    }
    execute_count <- execute_count + 1
  }
  cat("Average rejection rate: ", (execute_count - count) / execute_count, "\n")
  return(samples)
}

a <- 0.5
random_var_2b1 <- generate_random_var_2b(10000, a)
```

```
## Average rejection rate:  0.4351951
```

```r
hist(random_var_2b1)
```




```r
# Generate 2000 random variables from the distribution using rnorm
random_var_2b2 <- rnorm(2000, 0, 1)
hist(random_var_2b2)
```


Histogram of random_var_2b2
Frequency
0
100
200
300
400
−3
−2
−1
0
1
2
3
random_var_2b2

Appendix: Code for this report

```
########################## Init code for question 1 ############################
rm(list = ls())
########################## [ 1 a1 ] ##############################################

x <- seq(-2, 2, by = 0.001)
y <- rep(0, length(x))
z <- rep(0, length(x))

x1_index <- which(x > 1 | x < -1)
x2_index <- which(x >= -1 & x <= 0)
x3_index <- which(x > 0 & x <= 1)

y[x1_index] <- 0
y[x2_index] <- x[x2_index] + 1
y[x3_index] <- 1 - x[x3_index]

z0_index <- which(x > 1 | x < -1)
z1_index <- which(x >= -1 & x <= 1)

z[z1_index] <- 2
z[-z1_index] <- 0

data <- data.frame(x, y, z)

graph <- ggplot2::ggplot(data) +
  ggplot2::geom_line(mapping=ggplot2::aes(x=x,y=y)) +
  ggplot2::geom_line(mapping=ggplot2::aes(x=x,y=z),color="blue") +
  ggplot2::ggtitle("Density function of X and envelope function") +
  ggplot2::xlab("x") +
  ggplot2::ylab("y,z")

graph
########################## [ 1 a2 ] ##############################################

generate_random_var_1a <- function(n, a) {
  samples <- numeric(n)
  # Generate a sample from the proposal distribution in [-1,1]
  count <- 1
  e <- 1 / a

  while (count <= n) {

    x <- runif(1, -1, 1)
    # Calculate the acceptance probability
    u <- runif(1)

    # since g will always be 1 in the range [-1,1], so y = 1
    # f1(Y) = 1 - Y and e(Y) = 1 / a = 2
    # f2(Y) = Y + 1 and e(Y) = 1 / a = 2
    # so we have f(Y) / e(Y) = (1 - Y) / 2 or Y+1 / e

    # Calculate the acceptance probability
```

```r
    if (x > 0 && u <= (1 - x) / e) {
      samples[count] <- x
      count <- count + 1
    }

    if (x <= 0 && u <= (x + 1) / e) {
      samples[count] <- x
      count <- count + 1
    }
  }

  return(samples)
}

a <- 0.5
random_var_1a <- generate_random_var_1a(10000, a)
hist(random_var_1a)
var_of_random_var_1a <- var(random_var_1a)
cat("variance of 1st method is",var_of_random_var_1a)
########################## [ 1 b ] #############################################
generate_random_var_1b <- function(n){
  u <- runif(n)
  x1 <-  - 1 + sqrt(1 - 2 * u)
  x2 <-  1 - sqrt(1 - 2 * u)
  return(c(x1, x2))
}

random_var_1b <- generate_random_var_1b(10000)
hist(random_var_1b)

var_of_random_var_1b <- var(random_var_1b)
cat("variance of 2nd method is",var_of_random_var_1b)
########################## [ 1 c ] #############################################
generate_random_var_1c <- function(n){
  u1_1c <- runif(n)
  u2_1c <- runif(n)
  return(u1_1c - u2_1c)
}

random_var_1c <- generate_random_var_1c(10000)
hist(random_var_1c)
var_of_random_var_1c <- var(random_var_1c)
cat("variance of 3rd method is",var_of_random_var_1c)
########################## Init code for question 2############################
knitr::opts_chunk$set(echo = TRUE)
rm(list = ls())
########################## [ 2 a ] ######################################

generate_random_var_2a <- function(n){
  u1 <- runif(n,0,1)
  x1 <-  log(2 * u1)
  u2 <- runif(n,-1,0)
  x2 <-  - log(-2  * u2)
```

```r
  return(c(x1, x2))
}

random_var_2a <- generate_random_var_2a(10000)
hist(random_var_2a)
########################## [ 2 b ] #############################################

# generate a function to show normal distribution

normal_distribution_function <- function(x){
  return(1 / sqrt(2 * pi) * exp(-x^2 / 2))
}

generate_random_var_2b <- function(n, a) {
  samples <- numeric(n)
  # Generate a sample from the proposal distribution in [-1,1]
  count <- 1
  execute_count <- 1

  while (count <= n) {

    x <- runif(1, -1, 1)
    # Calculate the acceptance probability
    u <- runif(1)

    # since g will always be 1 in the range [-1,1], so y = 1
    # f(Y) = \frac{1}{\sqrt{2*pi}} * e^{\frac{1}{2} x^2 }
    # so we have f(Y) / e(Y) have 2 choices

    # Calculate the acceptance probability
    f_val <- normal_distribution_function(x)

    if (x > 0){
      e <- exp(-x)
      if (u <= (f_val / e)) {
        samples[count] <- x
        count <- count + 1
      }
    } else{
      e <- exp(x)
      if (u <= (f_val / e)) {
        samples[count] <- x
        count <- count + 1
      }
    }
    execute_count <- execute_count + 1
  }
  cat("Average rejection rate: ", (execute_count - count) / execute_count, "\n")
  return(samples)
}

a <- 0.5
random_var_2b1 <- generate_random_var_2b(10000, a)
```

```r
hist(random_var_2b1)


# Generate 2000 random variables from the distribution using rnorm
random_var_2b2 <- rnorm(2000, 0, 1)
hist(random_var_2b2)
```