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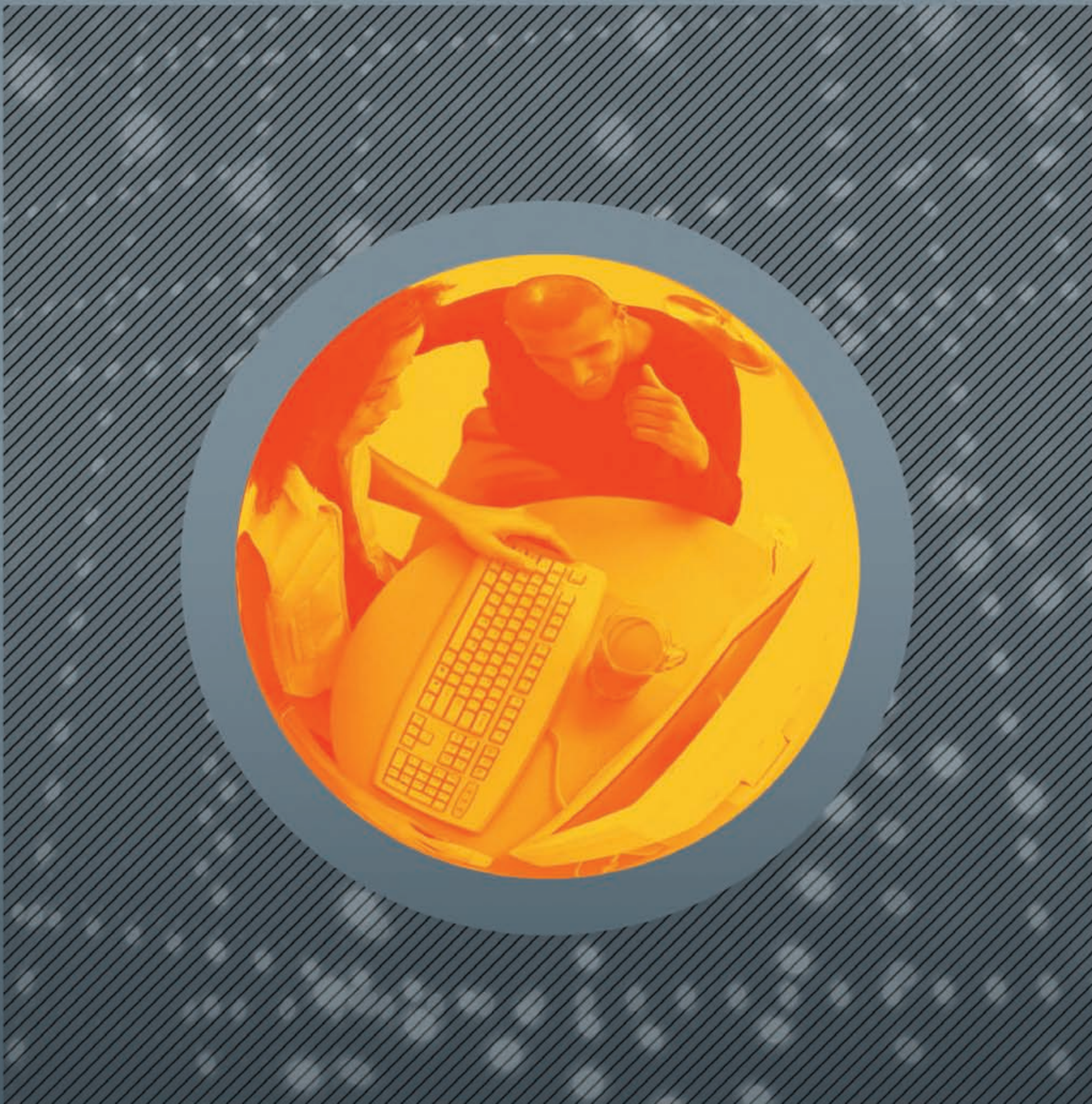

Discrete Mathematics

Richard Johnsonbaugh

Seventh Edition

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Sets and Logic

Logic, logic, logic. Logic is the beginning of wisdom, Valeris, not the end.

FROM *STAR TREK VI: THE UNDISCOVERED COUNTRY*

This chapter begins with sets. A **set** is a collection of objects; order is not taken into account. Discrete mathematics is concerned with objects such as graphs (sets of vertices and edges) and Boolean algebras (sets with certain operations defined on them). In this chapter, we introduce set terminology and notation. In Section 1, we provide a taste of the logic and proofs to come in the remainder of this chapter.

Logic is the study of reasoning; it is specifically concerned with whether reasoning is correct. Logic focuses on the relationship among statements as opposed to the content of any particular statement. Consider, for example, the following argument:

All mathematicians wear sandals.
Anyone who wears sandals is an algebraist.
Therefore, all mathematicians are algebraists.

Technically, logic is of no help in determining whether any of these statements is true; however, if the first two statements are true, logic assures us that the statement,

All mathematicians are algebraists,

is also true.

Logic is essential in reading and developing proofs. An understanding of logic can also be useful in clarifying ordinary writing. For example, at one time, the following ordinance was in effect in Naperville, Illinois: “It shall be unlawful for any person to keep more than three dogs and three cats upon his property within the city.” Was one of the citizens, who owned five dogs and no cats, in violation of the ordinance? Think about this question now, then analyze it (see Exercise 74, Section 2) after reading Section 2.

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1 → Sets

The concept of set is basic to all of mathematics and mathematical applications. A **set** is simply a collection of objects. The objects are sometimes referred to as elements or members. If a set is finite and not too large, we can describe it by listing the elements in it. For example, the equation

$$A = \{1, 2, 3, 4\} \tag{1.1}$$

describes a set A made up of the four elements 1, 2, 3, and 4. A set is determined by its elements and not by any particular order in which the elements might be listed. Thus the set A might just as well be specified as

$$A = \{1, 3, 4, 2\}.$$

The elements making up a set are assumed to be distinct, and although for some reason we may have duplicates in our list, only one occurrence of each element is in the set. For this reason we may also describe the set A defined in (1.1) as

$$A = \{1, 2, 2, 3, 4\}.$$

If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. For example, the equation

$$B = \{x \mid x \text{ is a positive, even integer}\} \tag{1.2}$$

describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. The vertical bar "|" is read "such that." Equation (1.2) would be read "B equals the set of all x such that x is a positive, even integer." Here the property necessary for membership is "is a positive, even integer." Note that the property appears after the vertical bar.

Some sets of numbers that occur frequently in mathematics generally, and in discrete mathematics in particular, are shown in Figure 1.1. The symbol $\mathbf{Z}$ comes from the German word, *Zahlen*, for *integer*. Rational numbers are quotients of integers, thus $\mathbf{Q}$ for *quotient*. The set of real numbers $\mathbf{R}$ can be depicted as consisting of all points on a straight line extending indefinitely in either direction (see Figure 1.2).†

To denote the negative numbers that belong to one of $\mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{R}$, we use the superscript minus. For example, $\mathbf{Z}^-$ denotes the set of negative integers, namely $-1, -2, -3, \ldots$. Similarly, to denote the positive numbers that belong to one of the three sets, we use the superscript plus. For example, $\mathbf{Q}^+$ denotes the set of positive rational numbers.

Symbol	*Set*	*Example of Members*
$\mathbf{Z}$	Integers	$-3, 0, 2, 145$
$\mathbf{Q}$	Rational numbers	$-1/3, 0, 24/15$
$\mathbf{R}$	Real numbers	$-3, -1.766, 0, 4/15, \sqrt{2}, 2.666\ldots, \pi$

Figure 1.1 Sets of numbers.

†The real numbers can be constructed by starting with a more primitive notion such as "set" or "integer," or they can be obtained by stating properties (axioms) they are assumed to obey. For our purposes, it suffices to think of the real numbers as points on a straight line.

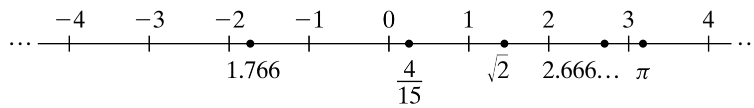


Figure 1.2 The real number line.

To denote the nonnegative numbers that belong to one of the three sets, we use the superscript *nonneg*. For example, $\mathbf{Z}^{nonneg}$ denotes the set of nonnegative integers, namely $0, 1, 2, 3, \ldots$.

If X is a finite set, we let

$$|X| = \text{ number of elements in } X.$$

We call $|X|$ the **cardinality** of X.

Example 1.1 ▶ For the set A in (1.1), we have $|A| = 4$, and the cardinality of A is 4. The cardinality of the set $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, namely the two sets $\mathbf{R}$ and $\mathbf{Z}$. ◀

Given a description of a set X such as (1.1) or (1.2) and an element x, we can determine whether or not x belongs to X. If the members of X are listed as in (1.1), we simply look to see whether or not x appears in the listing. In a description such as (1.2), we check to see whether the element x has the property listed. If x is in the set X, we write $x \in X$, and if x is not in X, we write $x \notin X$. For example, $3 \in \{1, 2, 3, 4\}$, but

$$3 \notin \{x \mid x \text{ is a positive, even integer}\}.$$

The set with no elements is called the **empty** (or **null** or **void**) **set** and is denoted $\varnothing$. Thus $\varnothing = \{\,\}$.

Two sets X and Y are **equal** and we write $X = Y$ if X and Y have the same elements. To put it another way, $X = Y$ if the following two conditions hold:

- For every x, if $x \in X$, then $x \in Y$,

and

- For every x, if $x \in Y$, then $x \in X$.

The first condition ensures that every element of X is an element of Y, and the second condition ensures that every element of Y is an element of X.

Example 1.2 ▶ If

$$A = \{1, 3, 2\} \qquad \text{and} \qquad B = \{2, 3, 2, 1\},$$

by inspection, A and B have the same elements. Therefore $A = B$. ◀

Example 1.3 ▶ Let us verify that if

$$A = \{x \mid x^2 + x - 6 = 0\} \qquad \text{and} \qquad B = \{2, -3\},$$

then $A = B$.

According to the criteria in the paragraph immediately preceding Example 1.2, we must show that for every x,

$$\text{if } x \in A, \text{ then } x \in B, \tag{1.3}$$

and for every x,

$$\text{if } x \in B, \text{ then } x \in A. \tag{1.4}$$

To verify equation (1.3), suppose that $x \in A$. Then

$$x^2 + x - 6 = 0.$$

Solving for x, we find that $x = 2$ or $x = -3$. In either case, $x \in B$. Therefore, equation (1.3) holds.

To verify equation (1.4), suppose that $x \in B$. Then $x = 2$ or $x = -3$. If $x = 2$, then

$$x^2 + x - 6 = 2^2 + 2 - 6 = 0.$$

Therefore, $x \in A$. If $x = -3$, then

$$x^2 + x - 6 = (-3)^2 + (-3) - 6 = 0.$$

Again, $x \in A$. Therefore, equation (1.4) holds. We conclude that $A = B$. ◀

For a set X to *not* be equal to a set Y (written $X \neq Y$), X and Y must *not* have the same elements: There must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Example 1.4 ▶ Let

$$A = \{1, 2, 3\} \qquad \text{and} \qquad B = \{2, 4\}.$$

Then $A \neq B$ since there is at least one element in A (1 for example) that is not in B. [Another way to see that $A \neq B$ is to note that there is at least one element in B (namely 4) that is not in A.] ◀

Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a **subset** of Y and write $X \subseteq Y$. In other words, X is a subset of Y if for every x, if $x \in X$, then $x \in Y$.

Example 1.5 ▶ If

$$C = \{1, 3\} \qquad \text{and} \qquad A = \{1, 2, 3, 4\},$$

by inspection, every element of C is an element of A. Therefore, C is a subset of A and we write $C \subseteq A$. ◀

Example 1.6 ▶ Let

$$X = \{x \mid x^2 + x - 2 = 0\}.$$

We show that $X \subseteq \mathbf{Z}$.

We must show that for every x, if $x \in X$, then $x \in \mathbf{Z}$. If $x \in X$, then

$$x^2 + x - 2 = 0.$$

Solving for x, we obtain $x = 1$ or $x = -2$. In either case, $x \in \mathbf{Z}$. Therefore, for every x, if $x \in X$, then $x \in \mathbf{Z}$. We conclude that X is a subset of $\mathbf{Z}$ and we write $X \subseteq \mathbf{Z}$. ◀

Example 1.7 ▶ The set of integers $\mathbf{Z}$ is a subset of the set of rational numbers $\mathbf{Q}$. If $n \in \mathbf{Z}$, n can be expressed as a quotient of integers, for example, $n = n/1$. Therefore $n \in \mathbf{Q}$ and $\mathbf{Z} \subseteq \mathbf{Q}$. ◀

Example 1.8 ▶ The set of rational numbers $\mathbf{Q}$ is a subset of the set of real numbers $\mathbf{R}$. If $x \in \mathbf{Q}$, x corresponds to a point on the number line (see Figure 1.2) so $x \in \mathbf{R}$. ◀

For X to *not* be a subset of Y, there must be at least one member of X that is not in Y.

Example 1.9 ▶ Let

$$X = \{x \mid 3x^2 - x - 2 = 0\}.$$

We show that X is not a subset of $\mathbf{Z}$.

If $x \in X$, then

$$3x^2 - x - 2 = 0.$$

Solving for x, we obtain $x = 1$ or $x = -2/3$. Taking $x = -2/3$, we have $x \in X$ but $x \notin \mathbf{Z}$. Therefore, X is not a subset of $\mathbf{Z}$. ◀

Any set X is a subset of itself, since any element in X is in X. Also, the empty set is a subset of every set. If $\varnothing$ is *not* a subset of some set Y, according to the discussion preceding Example 1.9, there would have to be at least one member of $\varnothing$ that is not in Y. But this cannot happen because the empty set, by definition, has no members.

If X is a subset of Y and X does not equal Y, we say that X is a **proper subset** of Y and write $X \subset Y$.

Example 1.10 ▶ Let

$$C = \{1, 3\} \qquad \text{and} \qquad A = \{1, 2, 3, 4\}.$$

Then C is a proper subset of A since C is a subset of A but C does not equal A. We write $C \subset A$. ◀

Example 1.11 ▶ Example 1.7 showed that $\mathbf{Z}$ is a subset of $\mathbf{Q}$. In fact, $\mathbf{Z}$ is a proper subset of $\mathbf{Q}$ because, for example, $1/2 \in \mathbf{Q}$, but $1/2 \notin \mathbf{Z}$. ◀

Example 1.12 ▶ Example 1.8 showed that $\mathbf{Q}$ is a subset of $\mathbf{R}$. In fact, $\mathbf{Q}$ is a proper subset of $\mathbf{R}$ because, for example, $\sqrt{2} \in \mathbf{R}$, but $\sqrt{2} \notin \mathbf{Q}$. ◀

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set** of X.

Example 1.13 ▶ If $A = \{a, b, c\}$, the members of $\mathcal{P}(A)$ are

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

All but $\{a, b, c\}$ are proper subsets of A. ◀

In Example 1.13,

$$|A| = 3 \qquad \text{and} \qquad |\mathcal{P}(A)| = 2^3 = 8.$$

It is worth noting in passing that this result holds in general; that is, the power set of a set with n elements has 2^n elements.

Given two sets X and Y, there are various set operations involving X and Y that can produce a new set. The set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

is called the **union** of X and Y. The union consists of all elements belonging to either X or Y (or both).

The set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

is called the **intersection** of X and Y. The intersection consists of all elements belonging to both X and Y.

The set

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

is called the **difference** (or **relative complement**). The difference $X - Y$ consists of all elements in X that are not in Y.

Example 1.14 ▶ If $A = \{1, 3, 5\}$ and $B = \{4, 5, 6\}$, then

$$\begin{aligned} A \cup B &= \{1, 3, 4, 5, 6\} \\ A \cap B &= \{5\} \\ A - B &= \{1, 3\} \\ B - A &= \{4, 6\}. \end{aligned}$$

Notice that, in general, $A - B \neq B - A$. ◀

Example 1.15 ▶ Since $\mathbf{Q} \subseteq \mathbf{R}$,

$$\begin{aligned} \mathbf{R} \cup \mathbf{Q} &= \mathbf{R} \\ \mathbf{R} \cap \mathbf{Q} &= \mathbf{Q} \\ \mathbf{Q} - \mathbf{R} &= \varnothing. \end{aligned}$$

The set $\mathbf{R} - \mathbf{Q}$, called the set of **irrational numbers,** consists of all real numbers that are not rational. ◀

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.

Example 1.16 ▶ The sets

$$\{1, 4, 5\} \qquad \text{and} \qquad \{2, 6\}$$

are disjoint. The collection of sets

$$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$$

is pairwise disjoint. ◀

Sometimes we are dealing with sets, all of which are subsets of a set U. This set U is called a **universal set** or a **universe.** The set U must be explicitly given or inferred from the context. Given a universal set U and a subset X of U, the set $U - X$ is called the **complement** of X and is written $\overline{X}$.

Example 1.17 ▸

Let $A = \{1, 3, 5\}$. If U, a universal set, is specified as $U = \{1, 2, 3, 4, 5\}$, then $\overline{A} = \{2, 4\}$. If, on the other hand, a universal set is specified as $U = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{7, 9\}$. The complement obviously depends on the universe in which we are working. ◂

Example 1.18 ▸

Let the universal set be $\mathbf{Z}$. Then $\overline{\mathbf{Z}^{-}}$, the complement of the set of negative integers, is $\mathbf{Z}^{nonneg}$, the set of nonnegative integers. ◂

Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set (see Figure 1.3). Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set. In Figure 1.3 we see two sets A and B within the universal set U. Region 1 represents $\overline{(A \cup B)}$, the elements in neither A nor B. Region 2 represents $A - B$, the elements in A but not in B. Region 3 represents $A \cap B$, the elements in both A and B. Region 4 represents $B - A$, the elements in B but not in A.

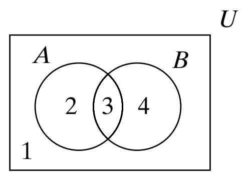


Figure 1.3 A Venn diagram.

Example 1.19 ▸

Particular regions in Venn diagrams are depicted by shading. The set $A \cup B$ is shown in Figure 1.4, and Figure 1.5 represents the set $A - B$. ◂

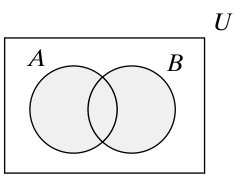


Figure 1.4 A Venn diagram of $A \cup B$.

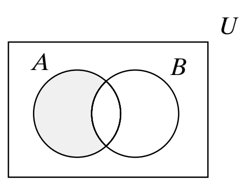


Figure 1.5 A Venn diagram of $A - B$.

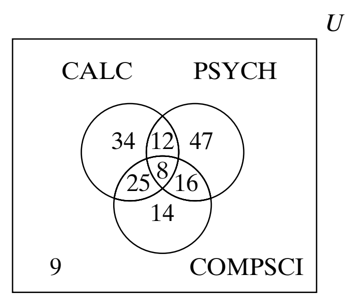


Figure 1.6 A Venn diagram of three sets CALC, PSYCH, and COMPSCI. The numbers show how many students belong to the particular region depicted.

To represent three sets, we use three overlapping circles (see Figure 1.6).

Example 1.20 ▸

Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology;

24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; and 63 are taking computer science. How many are taking none of the three subjects?

Let CALC, PSYCH, and COMPSCI denote the sets of students taking calculus, psychology, and computer science, respectively. Let U denote the set of all 165 students (see Figure 1.6). Since 8 students are taking calculus, psychology, and computer science, we write 8 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 33 students taking calculus and computer science, 8 are also taking psychology; thus 25 are taking calculus and computer science but not psychology. We write 25 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. Similarly, we write 12 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 16 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 79 students taking calculus, 45 have now been accounted for. This leaves 34 students taking only calculus. We write 34 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \overline{\text{COMPSCI}}$. Similarly, we write 47 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 14 in the region representing $\overline{\text{CALC}} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. At this point, 156 students have been accounted for. This leaves 9 students taking none of the three subjects. ◀

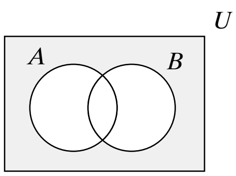


Figure 1.7 The shaded region depicts both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$; thus these sets are equal.

Venn diagrams can also be used to visualize certain properties of sets. For example, by sketching both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$ (see Figure 1.7), we see that these sets are equal. A formal proof would show that for every x, if $x \in \overline{(A \cup B)}$, then $x \in \overline{A} \cap \overline{B}$, and if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{(A \cup B)}$. We state many useful properties of sets as Theorem 1.21.

Theorem 1.21 *Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.*

(a) *Associative laws:*

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) *Commutative laws:*

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) *Distributive laws:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) *Identity laws:*

$$A \cup \varnothing = A, \quad A \cap U = A$$

(e) *Complement laws:*

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \varnothing$$

(f) *Idempotent laws:*

$$A \cup A = A, \quad A \cap A = A$$

(g) *Bound laws:*

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) *Absorption laws:*

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) *Involution law:*

$$\overline{\overline{A}} = A$$

(j) 0/1 laws:

$$\overline{\varnothing} = U, \quad \overline{U} = \varnothing$$

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

We define the union of an arbitrary family $\mathcal{S}$ of sets to be those elements x belonging to at least one set X in $\mathcal{S}$. Formally,

$$\cup\mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

Similarly, we define the intersection of an arbitrary family $\mathcal{S}$ of sets to be those elements x belonging to every set X in $\mathcal{S}$. Formally,

$$\cap\mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

If

$$\mathcal{S} = \{A_1, A_2, \ldots, A_n\},$$

we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{n} A_i,$$

and if

$$\mathcal{S} = \{A_1, A_2, \ldots\},$$

we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Example 1.22 ▶ For $i \geq 1$, define

$$A_i = \{i, i+1, \ldots\} \qquad \text{and} \qquad \mathcal{S} = \{A_1, A_2, \ldots\}.$$

Then

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i = \{1, 2, \ldots\}, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i = \varnothing.$$ ◀

A partition of a set X divides X into nonoverlapping subsets. More formally, a collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to exactly one member of $\mathcal{S}$. Notice that if $\mathcal{S}$ is a partition of X, $\mathcal{S}$ is pairwise disjoint and $\cup\mathcal{S} = X$.

Example 1.23 ▶ Since each element of

$$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

is in exactly one member of

$$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\},$$

$\mathcal{S}$ is a partition of X. ◀

At the beginning of this section, we pointed out that a set is an unordered collection of elements; that is, a set is determined by its elements and not by any particular order in which the elements are listed. Sometimes, however, we do want to take order into account. An **ordered pair** of elements, written (a, b), is considered distinct from the ordered pair (b, a), unless, of course, $a = b$. To put it another way, $(a, b) = (c, d)$ precisely when $a = c$ and $b = d$. If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the **Cartesian product** of X and Y.

Example 1.24 ▶ If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Y \times Y = \{(a, a), (a, b), (b, a), (b, b)\}.$$ ◀

Example 1.24 shows that, in general, $X \times Y \neq Y \times X$.

Notice that in Example 1.24, $|X \times Y| = |X| \cdot |Y|$ (both are equal to 6). The reason is that there are 3 ways to choose an element of X for the first member of the ordered pair, there are 2 ways to choose an element of Y for the second member of the ordered pair, and $3 \cdot 2 = 6$ (see Figure 1.8). The preceding argument holds for arbitrary finite sets X and Y; it is always true that $|X \times Y| = |X| \cdot |Y|$.

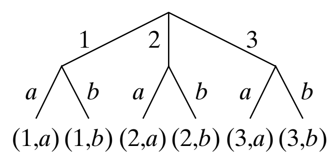


Figure 1.8 $|X \times Y| = |X| \cdot |Y|$, where $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. There are 3 ways to choose an element of X for the first member of the ordered pair (shown at the top of the diagram) and, for each of these choices, there are 2 ways to choose an element of Y for the second member of the ordered pair (shown at the bottom of the diagram). Since there are 3 groups of 2, there are $3 \cdot 2 = 6$ elements in $X \times Y$ (labeled at the bottom of the figure).

Example 1.25 ▶ A restaurant serves four appetizers,

$$r = \text{ribs}, \quad n = \text{nachos}, \quad s = \text{shrimp}, \quad f = \text{fried cheese},$$

and three entrees,

$$c = \text{chicken}, \quad b = \text{beef}, \quad t = \text{trout}.$$

If we let $A = \{r, n, s, f\}$ and $E = \{c, b, t\}$, the Cartesian product $A \times E$ lists the 12 possible dinners consisting of one appetizer and one entree. ◀

Ordered lists need not be restricted to two elements. An ***n*-tuple,** written $(a_1, a_2, \ldots, a_n)$, takes order into account; that is,

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

precisely when

$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n.$$

The Cartesian product of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$ for $i = 1, \ldots, n$; it is denoted $X_1 \times X_2 \times \cdots \times X_n$.

Example 1.26 ▶ If

$$X = \{1, 2\}, \qquad Y = \{a, b\}, \qquad Z = \{\alpha, \beta\},$$

then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}.$$ ◀

Notice that in Example 1.26, $|X \times Y \times Z| = |X| \cdot |Y| \cdot |Z|$. In general,

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdots |X_n|.$$

We leave the proof of this last statement as an exercise.

Example 1.27 ▶ If A is a set of appetizers, E is a set of entrees, and D is a set of desserts, the Cartesian product $A \times E \times D$ lists all possible dinners consisting of one appetizer, one entree, and one dessert. ◀

Problem-Solving Tips

To verify that two sets A and B are equal, written $A = B$, show that for every x, if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.

To verify that two sets A and B are *not* equal, written $A \neq B$, find at least one element that is in A but not in B, or find at least one element that is in B but not in A. One or the other conditions suffices; you need not (and may not be able to) show both conditions.

To verify that A is a subset of B, written $A \subseteq B$, show that for every x, if $x \in A$, then $x \in B$. Notice that if A is a subset of B, it is possible that $A = B$.

To verify that A is *not* a subset of B, find at least one element that is in A but not in B.

To verify that A is a proper subset of B, written $A \subset B$, verify that A is a subset of B as described previously, and that $A \neq B$, that is, that there is at least one element that is in B but not in A.

To visualize relationships among sets, use a Venn diagram. A Venn diagram can suggest whether a statement about sets is true or false.

A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n-tuples take order into account.

Section Review Exercises

[†]1. What is a set?

2. What is set notation?

3. Describe the sets **Z**, **Q**, **R**, $\mathbf{Z}^+$, $\mathbf{Q}^+$, $\mathbf{R}^+$, $\mathbf{Z}^-$, $\mathbf{Q}^-$, $\mathbf{R}^-$, $\mathbf{Z}^{nonneg}$, $\mathbf{Q}^{nonneg}$, and $\mathbf{R}^{nonneg}$, and give two examples of members of each set.

4. If X is a finite set, what is $|X|$?

5. How do we denote *x is an element of the set X*?

6. How do we denote *x is not an element of the set X*?

7. How do we denote the empty set?

[†]Exercise numbers in color indicate that a hint or solution appears at the end of this chapter.

8. Define *set X is equal to set Y*. How do we denote *X is equal to Y*?
9. Explain a method of verifying that sets X and Y are equal.
10. Explain a method of verifying that sets X and Y are *not* equal.
11. Define *X is a subset of Y*. How do we denote *X is a subset of Y*?
12. Explain a method of verifying that X is a subset of Y.
13. Explain a method of verifying that X is *not* a subset of Y.
14. Define *X is a proper subset of Y*. How do we denote *X is a proper subset of Y*?
15. Explain a method of verifying that X is a proper subset of Y.
16. What is the power set of X? How is it denoted?
17. Define *X union Y*. How is the union of X and Y denoted?
18. If $\mathcal{S}$ is a family of sets, how do we define the union of $\mathcal{S}$? How is the union denoted?
19. Define *X intersect Y*. How is the intersection of X and Y denoted?
20. If $\mathcal{S}$ is a family of sets, how do we define the intersection of $\mathcal{S}$? How is the intersection denoted?
21. Define *X and Y are disjoint sets*.
22. What is a pairwise disjoint family of sets?
23. Define the *difference* of sets X and Y. How is the difference denoted?
24. What is a universal set?
25. What is the complement of the set X? How is it denoted?
26. What is a Venn diagram?
27. Draw a Venn diagram of three sets and identify the set represented by each region.
28. State the associative laws for sets.
29. State the commutative laws for sets.
30. State the distributive laws for sets.
31. State the identity laws for sets.
32. State the complement laws for sets.
33. State the idempotent laws for sets.
34. State the bound laws for sets.
35. State the absorption laws for sets.
36. State the involution law for sets.
37. State the 0/1 laws for sets.
38. State De Morgan's laws for sets.
39. What is a partition of a set X?
40. Define the *Cartesian product* of sets X and Y. How is this Cartesian product denoted?
41. Define the *Cartesian product* of the sets $X_1, X_2, \ldots, X_n$. How is this Cartesian product denoted?

Exercises

In Exercises 1–16, let the universe be the set $U = \{1, 2, 3, \ldots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set.

1. $A \cup B$
2. $B \cap C$
3. $A - B$
4. $B - A$
5. $\overline{A}$
6. $U - C$
7. $\overline{U}$
8. $A \cup \varnothing$
9. $B \cap \varnothing$
10. $A \cup U$
11. $B \cap U$
12. $A \cap (B \cup C)$
13. $\overline{B} \cap (C - A)$
14. $(A \cap B) - C$
15. $\overline{A \cap B} \cup C$
16. $(A \cup B) - (C - B)$
17. What is the cardinality of $\varnothing$?
18. What is the cardinality of $\{\varnothing\}$?
19. What is the cardinality of $\{a, b, a, c\}$?
20. What is the cardinality of $\{\{a\}, \{a, b\}, \{a, c\}, a, b\}$?

In Exercises 21–24, show, as in Examples 1.2 and 1.3, that $A = B$.

21. $A = \{3, 2, 1\}$, $B = \{1, 2, 3\}$
22. $C = \{1, 2, 3\}$, $D = \{2, 3, 4\}$, $A = \{2, 3\}$, $B = C \cap D$
23. $A = \{1, 2, 3\}$, $B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 < 10\}$
24. $A = \{x \mid x^2 - 4x + 4 = 1\}$, $B = \{1, 3\}$

In Exercises 25–28, show, as in Example 1.4, that $A \neq B$.

25. $A = \{1, 2, 3\}$, $B = \varnothing$
26. $A = \{1, 2\}$, $B = \{x \mid x^3 - 2x^2 - x + 2 = 0\}$
27. $A = \{1, 3, 5\}$, $B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 - 1 \leq n\}$
28. $B = \{1, 2, 3, 4\}$, $C = \{2, 4, 6, 8\}$, $A = B \cap C$

In Exercises 29–32, determine whether each pair of sets is equal.

29. $\{1, 2, 2, 3\}$, $\{1, 2, 3\}$
30. $\{1, 1, 3\}$, $\{3, 3, 1\}$
31. $\{x \mid x^2 + x = 2\}$, $\{1, -1\}$
32. $\{x \mid x \in \mathbf{R} \text{ and } 0 < x \leq 2\}$, $\{1, 2\}$

In Exercises 33–36, show, as in Examples 1.5 and 1.6, that $A \subseteq B$.

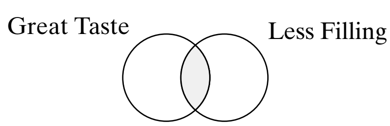
33. $A = \{1, 2\},\ B = \{3, 2, 1\}$
34. $A = \{1, 2\},\ B = \{x \mid x^3 - 6x^2 + 11x = 6\}$
35. $A = \{1\} \times \{1, 2\},\ B = \{1\} \times \{1, 2, 3\}$
36. $A = \{2n \mid n \in \mathbf{Z}^+\},\ B = \{n \mid n \in \mathbf{Z}^+\}$

In Exercises 37–40, show, as in Example 1.9, that A is not a subset of B.

37. $A = \{1, 2, 3\},\ B = \{1, 2\}$
38. $A = \{x \mid x^3 - 2x^2 - x + 2 = 0\},\ B = \{1, 2\}$
39. $A = \{1, 2, 3, 4\},\ C = \{5, 6, 7, 8\},\ B = \{n \mid n \in A$ and $n + m = 8$ for some $m \in C\}$
40. $A = \{1, 2, 3\},\ B = \varnothing$

In Exercises 41–48, draw a Venn diagram and shade the given set.

41. $A \cap \overline{B}$
42. $\overline{A} - B$
43. $B \cup (B - A)$
44. $(A \cup B) - B$
45. $B \cap \overline{(C \cup A)}$
46. $(\overline{A} \cup B) \cap (\overline{C} - A)$
47. $((C \cap A) - \overline{(B - A)}) \cap C$
48. $(B - \overline{C}) \cup ((B - \overline{A}) \cap (C \cup B))$
49. A television commercial for a popular beverage showed the following Venn diagram



What does the shaded area represent?

Exercises 50–54 refer to a group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

50. How many are taking French and music but not business?
51. How many are taking business and neither French nor music?
52. How many are taking French or business (or both)?
53. How many are taking music or French (or both) but not business?
54. How many are taking none of the three subjects?
55. A television poll of 151 persons found that 68 watched "Law and Disorder"; 61 watched "25"; 52 watched "The Tenors"; 16 watched both "Law and Disorder" and "25"; 25 watched both "Law and Disorder" and "The Tenors"; 19 watched both "25" and "The Tenors"; and 26 watched none of these shows. How many persons watched all three shows?
56. In a group of students, each student is taking a mathematics course or a computer science course or both. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eighth of those taking a computer science course are also taking a mathematics course. Are more than one-third of the students taking a mathematics course?

In Exercises 57–60, let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. List the elements in each set.

57. $X \times Y$
58. $Y \times X$
59. $X \times X$
60. $Y \times Y$

In Exercises 61–64, let $X = \{1, 2\}$, $Y = \{a\}$, and $Z = \{\alpha, \beta\}$. List the elements of each set.

61. $X \times Y \times Z$
62. $X \times Y \times Y$
63. $X \times X \times X$
64. $Y \times X \times Y \times Z$

In Exercises 65–72, give a geometric description of each set in words. (Consider the elements of the sets to be coordinates.)

65. $\mathbf{R} \times \mathbf{R}$
66. $\mathbf{Z} \times \mathbf{R}$
67. $\mathbf{R} \times \mathbf{Z}$
68. $\mathbf{R} \times \mathbf{Z}^{nonneg}$
69. $\mathbf{Z} \times \mathbf{Z}$
70. $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$
71. $\mathbf{R} \times \mathbf{R} \times \mathbf{Z}$
72. $\mathbf{R} \times \mathbf{Z} \times \mathbf{Z}$

In Exercises 73–76, list all partitions of the set.

73. $\{1\}$
74. $\{1, 2\}$
75. $\{a, b, c\}$
76. $\{a, b, c, d\}$

In Exercises 77–82, answer true or false.

77. $\{x\} \subseteq \{x\}$
78. $\{x\} \in \{x\}$
79. $\{x\} \in \{x, \{x\}\}$
80. $\{x\} \subseteq \{x, \{x\}\}$
81. $\{2\} \subseteq \mathcal{P}(\{1, 2\})$
82. $\{2\} \in \mathcal{P}(\{1, 2\})$
83. List the members of $\mathcal{P}(\{a, b\})$. Which are proper subsets of $\{a, b\}$?
84. List the members of $\mathcal{P}(\{a, b, c, d\})$. Which are proper subsets of $\{a, b, c, d\}$?
85. If X has 10 members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?
86. If X has n members, how many proper subsets does X have?

In Exercises 87–90, what relation must hold between sets A and B in order for the given condition to be true?

87. $A \cap B = A$
88. $A \cup B = A$
89. $\overline{A} \cap U = \varnothing$
90. $\overline{A \cap B} = \overline{B}$

The symmetric difference *of two sets A and B is the set*

$$A \triangle B = (A \cup B) - (A \cap B).$$

91. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, find $A \triangle B$.
92. Describe the symmetric difference of sets A and B in words.

93. Given a universe U, describe $A \triangle A$, $A \triangle \overline{A}$, $U \triangle A$, and $\varnothing \triangle A$.

94. Let C be a circle and let $\mathcal{D}$ be the set of all diameters of C. What is $\cap \mathcal{D}$? (Here, by "diameter" we mean a line segment through the center of the circle with its endpoints on the circumference of the circle.)

†★**95.** Let P denote the set of integers greater than 1. For $i \geq 2$, define

$$X_i = \{ik \mid k \in P\}.$$

Describe $P - \bigcup_{i=2}^{\infty} X_i$.

2 ➜ Propositions

Which of sentences (a)–(f) are either true or false (but not both)?

(a) The only positive integers that divide‡ 7 are 1 and 7 itself.
(b) Alfred Hitchcock won an Academy Award in 1940 for directing *Rebecca*.
(c) For every positive integer n, there is a prime number§ larger than n.
(d) Earth is the only planet in the universe that contains life.
(e) Buy two tickets to the "Unhinged Universe" rock concert for Friday.
(f) $x + 4 = 6$.

Sentence (a), which is another way to say that 7 is prime, is true.

Sentence (b) is false. Although *Rebecca* won the Academy Award for best picture in 1940, John Ford won the directing award for *The Grapes of Wrath*. It is a surprising fact that Alfred Hitchcock never won an Academy Award for directing.

Sentence (c), which is another way to say that the number of primes is infinite, is true.

Sentence (d) is either true or false (but not both), but no one knows which at this time.

Sentence (e) is neither true nor false [sentence (e) is a command].

The truth of equation (f) depends on the value of the variable x.

A sentence that is either true or false, but not both, is called a **proposition.** Sentences (a)–(d) are propositions, whereas sentences (e) and (f) are not propositions. A proposition is typically expressed as a declarative sentence (as opposed to a question, command, etc.). Propositions are the basic building blocks of any theory of logic.

We will use variables, such as p, q, and r, to represent propositions, much as we use letters in algebra to represent numbers. We will also use the notation

$$p\colon\ 1 + 1 = 3$$

to define p to be the proposition $1 + 1 = 3$.

In ordinary speech and writing, we combine propositions using connectives such as *and* and *or*. For example, the propositions "It is raining" and "It is cold" can be combined to form the single proposition "It is raining and it is cold." The formal definitions of *and* and *or* follow.

†A starred exercise indicates a problem of above-average difficulty.

‡"Divides" means "divides evenly." More formally, we say that a nonzero integer d *divides* an integer m if there is an integer q such that $m = dq$. We call q the *quotient*.

§An integer $n > 1$ is *prime* if the only positive integers that divide n are 1 and n itself. For example, 2, 3, and 11 are prime numbers.

Definition 2.1 ▶ Let p and q be propositions.

The *conjunction* of p and q, denoted $p \wedge q$, is the proposition

$$p \quad \text{and} \quad q.$$

The *disjunction* of p and q, denoted $p \vee q$, is the proposition

$$p \quad \text{or} \quad q.$$

Example 2.2 ▶ If

$$p: \text{ It is raining,}$$
$$q: \text{ It is cold,}$$

then the conjunction of p and q is

$$p \wedge q: \text{ It is raining and it is cold.}$$

The disjunction of p and q is

$$p \vee q: \text{ It is raining or it is cold.}$$

The truth value of the conjunction $p \wedge q$ is determined by the truth values of p and q, and the definition is based upon the usual interpretation of "and." Consider the proposition,

$$p \wedge q: \text{ It is raining and it is cold,}$$

of Example 2.2. If it is raining (i.e., p is true) and it is also cold (i.e., q is also true), then we would consider the proposition,

$$p \wedge q: \text{ It is raining and it is cold,}$$

to be true. However, if it is not raining (i.e., p is false) or it is not cold (i.e., q is false) or both, then we would consider the proposition,

$$p \wedge q: \text{ It is raining and it is cold,}$$

to be false.

The truth values of propositions such as conjunctions and disjunctions can be described by **truth tables.** The truth table of a proposition P made up of the individual propositions $p_1, \ldots, p_n$ lists all possible combinations of truth values for $p_1, \ldots, p_n$, T denoting true and F denoting false, and for each such combination lists the truth value of P. We use a truth table to formally define the truth value of $p \wedge q$.

Definition 2.3 ▶ The truth value of the proposition $p \wedge q$ is defined by the truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that, in the truth table in Definition 2.3, all four possible combinations of truth assignments for p and q are given.

Definition 2.3 states that the conjunction $p \wedge q$ is true provided that p and q are both true; $p \wedge q$ is false otherwise.

Example 2.4 ▶

If

$$p\colon \text{ A decade is 10 years,}$$
$$q\colon \text{ A millennium is 100 years,}$$

then p is true, q is false (a millennium is 1000 years), and the conjunction,

$$p \wedge q\colon \text{ A decade is 10 years and a millennium is 100 years,}$$

is false. ◀

Example 2.5 ▶

Most programming languages define "and" exactly as in Definition 2.3. For example, in the Java programming language, (logical) "and" is denoted `&&`, and the expression

```
x < 10 && y > 4
```

is true precisely when the value of the variable `x` is less than 10 (i.e., `x < 10` is true) and the value of the variable `y` is greater than 4 (i.e., `y > 4` is also true). ◀

The truth value of the disjunction $p \vee q$ is also determined by the truth values of p and q, and the definition is based upon the "inclusive" interpretation of "or." Consider the proposition,

$$p \vee q\colon \text{ It is raining or it is cold,}$$

of Example 2.2. If it is raining (i.e., p is true) or it is cold (i.e., q is also true) *or both,* then we would consider the proposition,

$$p \vee q\colon \text{ It is raining or it is cold,}$$

to be true (i.e., $p \vee q$ is true). If it is not raining (i.e., p is false) and it is not cold (i.e., q is also false), then we would consider the proposition,

$$p \vee q\colon \text{ It is raining or it is cold,}$$

to be false (i.e., $p \vee q$ is false). The **inclusive-or** of propositions p and q is true if p or q, *or both*, is true, and false otherwise. There is also an **exclusive-or** (see Exercise 66) that defines p *exor* q to be true if p or q, *but not both*, is true, and false otherwise.

Definition 2.6 ▶

The truth value of the proposition $p \vee q$, called the *inclusive-or* of p and q, is defined by the truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

◀

Example 2.7 ▶ If

$$p: \text{ A millennium is 100 years,}$$
$$q: \text{ A millennium is 1000 years,}$$

then p is false, q is true, and the disjunction,

$$p \vee q: \text{ A millennium is 100 years or a millennium is 1000 years,}$$

is true. ◀

Example 2.8 ▶ Most programming languages define (inclusive) "or" exactly as in Definition 2.6. For example, in the Java programming language, (logical) "or" is denoted `||`, and the expression

```
x < 10 || y > 4
```

is true precisely when the value of the variable `x` is less than 10 (i.e., `x < 10` is true) or the value of the variable `y` is greater than 4 (i.e., `y > 4` is true) or both. ◀

In ordinary language, propositions being combined (e.g., p and q combined to give the proposition $p \vee q$) are normally related; but in logic, these propositions are not required to refer to the same subject matter. For example, in logic, we permit propositions such as

$$3 < 5 \text{ or Paris is the capital of England.}$$

Logic is concerned with the form of propositions and the relation of propositions to each other and not with the subject matter itself. (The given proposition is true because $3 < 5$ is true.)

The final operator on a proposition p that we discuss in this section is the **negation** of p.

Definition 2.9 ▶ The *negation* of p, denoted $\neg p$, is the proposition

$$\text{not } p.$$

The truth value of the proposition $\neg p$ is defined by the truth table

p	$\neg p$
T	F
F	T

◀

In English, we sometime write $\neg p$ as "It is not the case that p." For example, if

$$p: \text{ Paris is the capital of England,}$$

the negation of p could be written

$$\neg p: \text{ It is not the case that Paris is the capital of England,}$$

or more simply as

$$\neg p: \text{ Paris is not the capital of England.}$$

Example 2.10 ▶ If

$$p: \pi \text{ was calculated to 1,000,000 decimal digits in 1954,}$$

the negation of p is the proposition

$$\neg p: \pi \text{ was not calculated to 1,000,000 decimal digits in 1954.}$$

It was not until 1973 that 1,000,000 decimal digits of π were computed; so, p is false. (Since then over one trillion decimal digits of π have been computed.) Since p is false, $\neg p$ is true. ◀

Example 2.11 ▶ Most programming languages define "not" exactly as in Definition 2.9. For example, in the Java programming language, "not" is denoted `!`, and the expression

```
!(x < 10)
```

is true precisely when the value of the variable `x` is not less than 10 (i.e., `x` is greater than or equal to 10). ◀

In expressions involving some or all of the operators $\neg$, $\wedge$, and $\vee$, in the absence of parentheses, we first evaluate $\neg$, then $\wedge$, and then $\vee$. We call such a convention **operator precedence.** In algebra, operator precedence tells us to evaluate $\cdot$ and $/$ before $+$ and $-$.

Example 2.12 ▶ Given that proposition p is false, proposition q is true, and proposition r is false, determine whether the proposition

$$\neg p \vee q \wedge r$$

is true or false.

We first evaluate $\neg p$, which is true. We next evaluate $q \wedge r$, which is false. Finally, we evaluate

$$\neg p \vee q \wedge r,$$

which is true. ◀

Example 2.13 ▶

Searching the Web

A variety of Web search engines are available (e.g., Google, Yahoo, AltaVista) that allow the user to enter keywords that the search engine then tries to match with Web pages. For example, entering *mathematics* produces a (huge!) list of pages that contain the word "mathematics." Some search engines allow the user to use *and*, *or*, and *not* operators to combine keywords (see Figure 2.1), thus allowing more complex searches. In the Google search engine, *and* is the default operator so that, for example, entering *discrete mathematics* produces a list of pages containing both of the words "discrete" *and* "mathematics." The *or* operator is OR, and the *not* operator is the minus sign $-$. Furthermore, enclosing a phrase, typically with embedded spaces, in double quotation marks causes the phrase to be treated as a single word. For example, to search for pages containing the keywords

"*Alfred Hitchcock*" and (*Herrmann* or *Waxman*) and (not *tv*),

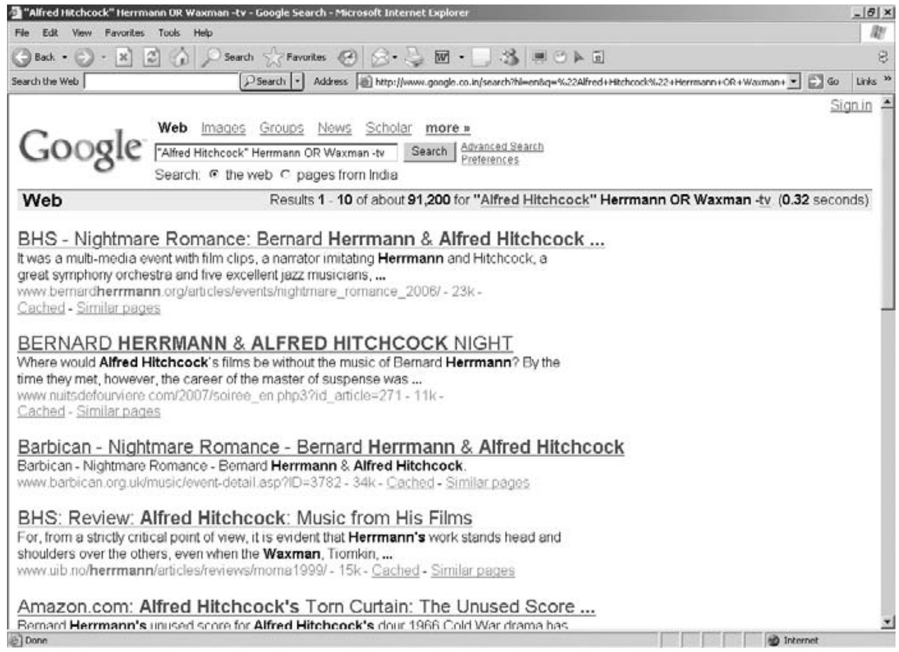


Figure 2.1 The Google search engine, which allows the user to use *and* (space), *or* (OR), and *not* (−) operators to combine keywords. As shown, Google found about 91,000 Web pages containing "*Alfred Hitchcock*" and (*Herrmann* or *Waxman*) and (not *tv*).

the user could enter

```
"Alfred Hitchcock" Herrmann OR Waxman -tv          (2.1)
```

[For those who have not studied discrete mathematics, clicking on *Advanced* on the Google home page yields a page in which the user can fill in boxes to achieve the same result as (2.1).] ◀

Problem-Solving Tips

Although there may be a shorter way to determine the truth values of a proposition P formed by combining propositions $p_1, \ldots, p_n$ using operators such as $\neg$ and $\vee$, a truth table will always supply all possible truth values of P for various truth values of the constituent propositions $p_1, \ldots, p_n$.

Section Review Exercises

1. What is a proposition?
2. What is a truth table?
3. What is the conjunction of p and q? How is it denoted?
4. Give the truth table for the conjunction of p and q.
5. What is the disjunction of p and q? How is it denoted?
6. Give the truth table for the disjunction of p and q.
7. What is the negation of p? How is it denoted?
8. Give the truth table for the negation of p.

Exercises

Determine whether each sentence in Exercises 1–11 is a proposition. If the sentence is a proposition, write its negation. (You are not *being asked for the truth values of the sentences that are propositions.)*

1. $2 + 5 = 19$.
2. $6 + 9 = 15$.
3. $x + 9 = 15$.
4. Waiter, will you serve the nuts—I mean, would you serve the guests the nuts?
5. For some positive integer n, $19340 = n \cdot 17$.
6. Audrey Meadows was the original "Alice" in "The Honeymooners."
7. Peel me a grape.
8. The line "Play it again, Sam" occurs in the movie *Casablanca*.
9. Every even integer greater than 4 is the sum of two primes.
10. The difference of two primes.

★11. This statement is false.

Exercises 12–15 refer to a coin that is flipped 10 times. Write the negation of the proposition.

12. Ten heads were obtained.
13. Some heads were obtained.
14. Some heads and some tails were obtained.
15. At least one head was obtained.

Given that proposition p is false, proposition q is true, and proposition r is false, determine whether each proposition in Exercises 16–21 is true or false.

16. $p \vee q$
17. $\neg p \vee \neg q$
18. $\neg p \vee q$
19. $\neg p \vee \neg(q \wedge r)$
20. $\neg(p \vee q) \wedge (\neg p \vee r)$
21. $(p \vee \neg r) \wedge \neg((q \vee r) \vee \neg(r \vee p))$

Write the truth table of each proposition in Exercises 22–29.

22. $p \wedge \neg q$
23. $(\neg p \vee \neg q) \vee p$
24. $(p \vee q) \wedge \neg p$
25. $(p \wedge q) \wedge \neg p$
26. $(p \wedge q) \vee (\neg p \vee q)$
27. $\neg(p \wedge q) \vee (r \wedge \neg p)$
28. $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$
29. $\neg(p \wedge q) \vee (\neg q \vee r)$

In Exercises 30–32, represent the given proposition symbolically by letting

$$p:\ 5 < 9, \quad q:\ 9 < 7, \quad r:\ 5 < 7.$$

Determine whether each proposition is true or false.

30. $5 < 9$ and $9 < 7$.
31. It is not the case that ($5 < 9$ and $9 < 7$).
32. $5 < 9$ or it is not the case that ($9 < 7$ and $5 < 7$).

In Exercises 33–38, formulate the symbolic expression in words using

p : *Lee takes computer science.*
q : *Lee takes mathematics.*

33. $\neg p$
34. $p \wedge q$
35. $p \vee q$
36. $p \vee \neg q$
37. $p \wedge \neg q$
38. $\neg p \wedge \neg q$

In Exercises 39–43, formulate the symbolic expression in words using

p : *You play football.*
q : *You miss the midterm exam.*
r : *You pass the course.*

39. $p \wedge q$
40. $\neg q \wedge r$
41. $p \vee q \vee r$
42. $\neg(p \vee q) \vee r$
43. $(p \wedge q) \vee (\neg q \wedge r)$

In Exercises 44–48, formulate the symbolic expression in words using

p : *Today is Monday.*
q : *It is raining.*
r : *It is hot.*

44. $p \vee q$
45. $\neg p \wedge (q \vee r)$
46. $\neg(p \vee q) \wedge r$
47. $(p \wedge q) \wedge \neg(r \vee p)$
48. $(p \wedge (q \vee r)) \wedge (r \vee (q \vee p))$

In Exercises 49–54, represent the proposition symbolically by letting

p : *There is a hurricane.*
q : *It is raining.*

49. There is no hurricane.
50. There is a hurricane and it is raining.
51. There is a hurricane, but it is not raining.
52. There is no hurricane and it is not raining.
53. Either there is a hurricane or it is raining (or both).
54. Either there is a hurricane or it is raining, but there is no hurricane.

In Exercises 55–59, represent the proposition symbolically by letting

p : *You run 10 laps daily.*
q : *You are healthy.*
r : *You take multi-vitamins.*

55. You run 10 laps daily, but you are not healthy.
56. You run 10 laps daily, you take multi-vitamins, and you are healthy.
57. You run 10 laps daily or you take multi-vitamins, and you are healthy.

58. You do not run 10 laps daily, you do not take multi-vitamins, and you are not healthy.

59. Either you are healthy or you do not run 10 laps daily, and you do not take multi-vitamins.

In Exercises 60–65, represent the proposition symbolically by letting

p : *You heard the* "Flying Pigs" *rock concert.*
q : *You heard the* "Y2K" *rock concert.*
r : *You have sore eardrums.*

60. You heard the "Flying Pigs" rock concert, and you have sore eardrums.

61. You heard the "Flying Pigs" rock concert, but you do not have sore eardrums.

62. You heard the "Flying Pigs" rock concert, you heard the "Y2K" rock concert, and you have sore eardrums.

63. You heard either the "Flying Pigs" rock concert or the "Y2K" rock concert, but you do not have sore eardrums.

64. You did not hear the "Flying Pigs" rock concert and you did not hear the "Y2K" rock concert, but you have sore eardrums.

65. It is not the case that: You heard the "Flying Pigs" rock concert or you heard the "Y2K" rock concert or you do not have sore eardrums.

66. Give the truth table for the exclusive-or of p and q in which p *exor* q is true if either p or q, but not both, is true.

In Exercises 67–73, state the meaning of each sentence if "or" is interpreted as the inclusive-or; then, state the meaning of each sentence if "or" is interpreted as the exclusive-or (see Exercise 66). In each case, which meaning do you think is intended?

67. To enter Utopia, you must show a driver's license or a passport.

68. To enter Utopia, you must possess a driver's license or a passport.

69. The prerequisite to data structures is a course in Java or C++.

70. The car comes with a cupholder that heats or cools your drink.

71. We offer $1000 cash or 0 percent interest for two years.

72. Do you want fries or a salad with your burger?

73. The meeting will be canceled if fewer than 10 persons sign up or at least 3 inches of snow falls.

74. At one time, the following ordinance was in effect in Naperville, Illinois: "It shall be unlawful for any person to keep more than three [3] dogs and three [3] cats upon his property within the city." Was Charles Marko, who owned five dogs and no cats, in violation of the ordinance? Explain.

75. Write a command to search the Web for national parks in North or South Dakota.

76. Write a command to search the Web for information on lung disease other than cancer.

77. Write a command to search the Web for minor league baseball teams in Illinois that are not in the Midwest League.

3 ➜ Conditional Propositions and Logical Equivalence

The dean has announced that

$$\text{If the Mathematics Department gets an additional \$60,000,} \\ \text{then it will hire one new faculty member.} \tag{3.1}$$

Statement (3.1) states that on the condition that the Mathematics Department gets an additional $60,000, then the Mathematics Department will hire one new faculty member. A proposition such as (3.1) is called a **conditional proposition.**

Definition 3.1 ▶

If p and q are propositions, the proposition

$$\text{if } p \text{ then } q \tag{3.2}$$

is called a *conditional proposition* and is denoted

$$p \to q.$$

The proposition p is called the *hypothesis* (or *antecedent*) and the proposition q is called the *conclusion* (or *consequent*). ◀

Example 3.2 ▶

If we define

p: The Mathematics Department gets an additional $60,000,
q: The Mathematics Department will hire one new faculty member,

then proposition (3.1) assumes the form (3.2). The hypothesis is the statement "The Mathematics Department gets an additional \$60,000," and the conclusion is the statement "The Mathematics Department will hire one new faculty member." ◀

What is the truth value of the dean's statement (3.1)? First, suppose that the Mathematics Department gets an additional \$60,000. If the Mathematics Department does hire an additional faculty member, surely the dean's statement is true. (Using the notation of Example 3.2, if p and q are both true, then $p \to q$ is true.) On the other hand, if the Mathematics Department gets an additional \$60,000 and does *not* hire an additional faculty member, the dean is wrong—statement (3.1) is false. (If p is true and q is false, then $p \to q$ is false.) Now, suppose that the Mathematics Department does *not* get an additional \$60,000. In this case, the Mathematics Department might or might not hire an additional faculty member. (Perhaps a member of the department retires and someone is hired to replace the retiree. On the other hand, the department might not hire anyone.) Surely we would not consider the dean's statement to be false. Thus, if the Mathematics Department does *not* get an additional \$60,000, the dean's statement must be true, regardless of whether the department hires an additional faculty member or not. (If p is false, then $p \to q$ is true whether q is true or false.) This discussion motivates the following definition.

Definition 3.3 ▶ The truth value of the conditional proposition $p \to q$ is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

◀

For those who need additional evidence that we should define $p \to q$ to be true when p is false, we offer further justification. Most people would agree that the proposition,

$$\text{For all real numbers } x, \text{ if } x > 0, \text{ then } x^2 > 0, \tag{3.3}$$

is true. (In Section 5, we will discuss such "for all" statements formally and in detail.) In the following discussion, we let $P(x)$ denote $x > 0$ and $Q(x)$ denote $x^2 > 0$. That proposition (3.3) is true means that no matter which real number we replace x with, the proposition

$$\text{if } P(x) \text{ then } Q(x) \tag{3.4}$$

that results is true. For example, if $x = 3$, then $P(3)$ and $Q(3)$ are both true ($3 > 0$ and $3^2 > 0$ are both true), and, by Definition 3.3, (3.4) is true. Now let us consider the situation when $P(x)$ is false. If $x = -2$, then $P(-2)$ is false ($-2 > 0$ is false) and $Q(-2)$ is true [$(-2)^2 > 0$ is true]. In order for proposition (3.4) to be true in this case, we must define $p \to q$ to be true when p is false and q is true. This is exactly what occurs in the third line of the truth table of Definition 3.3. If $x = 0$, then $P(0)$ and $Q(0)$ are both false ($0 > 0$ and $0^2 > 0$ are both false). In order for proposition (3.4) to be true in this case, we must define $p \to q$ to be true when both p and q are false. This is exactly what occurs in the fourth line of the truth table of Definition 3.3. Even more motivation for defining $p \to q$ to be true when p is false is given in Exercises 74 and 75.

Example 3.4 ▶ Let

$$p\colon\ 1 > 2, \qquad q\colon\ 4 < 8.$$

Then p is false and q is true. Therefore,

$$p \to q \text{ is true}, \qquad q \to p \text{ is false}.$$ ◀

In expressions that involve the logical operators $\wedge$, $\vee$, $\neg$, and $\to$, the conditional operator $\to$ is evaluated last. For example,

$$p \vee q \to \neg r$$

is interpreted as

$$(p \vee q) \to (\neg r).$$

Example 3.5 ▶ Assuming that p is true, q is false, and r is true, find the truth value of each proposition.

(a) $p \wedge q \to r$
(b) $p \vee q \to \neg r$
(c) $p \wedge (q \to r)$
(d) $p \to (q \to r)$

(a) We first evaluate $p \wedge q$ because $\to$ is evaluated last. Since p is true and q is false, $p \wedge q$ is false. Therefore, $p \wedge q \to r$ is true (regardless of whether r is true or false).
(b) We first evaluate $\neg r$. Since r is true, $\neg r$ is false. We next evaluate $p \vee q$. Since p is true and q is false, $p \vee q$ is true. Therefore, $p \vee q \to \neg r$ is false.
(c) Since q is false, $q \to r$ is true (regardless of whether r is true or false). Since p is true, $p \wedge (q \to r)$ is true.
(d) Since q is false, $q \to r$ is true (regardless of whether r is true or false). Thus, $p \to (q \to r)$ is true (regardless of whether p is true or false). ◀

A conditional proposition that is true because the hypothesis is false is said to be **true by default** or **vacuously true.** For example, if the proposition,

> If the Mathematics Department gets an additional $60,000, then it will hire one new faculty member,

is true because the Mathematics Department did not get an additional $60,000, we would say that the proposition is true by default or that it is vacuously true.

Some statements not of the form (3.2) may be rephrased as conditional propositions, as the next example illustrates.

Example 3.6 ▶ Restate each proposition in the form (3.2) of a conditional proposition.

(a) Mary will be a good student if she studies hard.
(b) John takes calculus only if he has sophomore, junior, or senior standing.
(c) When you sing, my ears hurt.
(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.
(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

(a) The hypothesis is the clause following *if*; thus an equivalent formulation is

If Mary studies hard, then she will be a good student.

(b) The statement means that in order for John to take calculus, he must have sophomore, junior, or senior standing. In particular, if he is a freshman, he may *not* take calculus. Thus, we can conclude that if he takes calculus, then he has sophomore, junior, or senior standing. Therefore an equivalent formulation is

If John takes calculus, then he has sophomore, junior, or senior standing.

Notice that

If John has sophomore, junior, or senior standing, then he takes calculus,

is *not* an equivalent formulation. If John has sophomore, junior, or senior standing, he may or may *not* take calculus. (Although eligible to take calculus, he may have decided not to.)

The "if p then q" formulation emphasizes the hypothesis, whereas the "p only if q" formulation emphasizes the conclusion; the difference is only stylistic.

(c) *When* means the same as *if*; thus an equivalent formulation is

If you sing, then my ears hurt.

(d) A **necessary condition** is just that: a condition that is *necessary* for a particular outcome to be achieved. The condition does *not* guarantee the outcome; but, if the condition does not hold, the outcome will not be achieved. Here, the given statement means that if the Cubs win the World Series, we can be sure that they signed a right-handed relief pitcher since, without such a signing, they would not have won the World Series. Thus, an equivalent formulation of the given statement is

If the Cubs win the World Series, then they signed a right-handed relief pitcher.

The conclusion expresses a necessary condition.

Notice that

If the Cubs sign a right-handed relief pitcher, then they win the World Series,

is *not* an equivalent formulation. Signing a right-handed relief pitcher does not guarantee a World Series win. However, *not* signing a right-handed relief pitcher guarantees that they will not win the World Series.

(e) Similarly, a **sufficient condition** is a condition that *suffices* to guarantee a particular outcome. If the condition does not hold, the outcome might be achieved in other ways or it might not be achieved at all; but if the condition does hold, the outcome is guaranteed. Here, to be sure that Maria visits France, it suffices for her to go to the Eiffel Tower. (There are surely other ways to ensure that Maria visits France; for example, she could go to Lyon.) Thus, an equivalent formulation of the given statement is

If Maria goes to the Eiffel Tower, then she visits France.

The hypothesis expresses a sufficient condition.

Notice that

If Maria visits France, then she goes to the Eiffel Tower,

is *not* an equivalent formulation. As we have already noted, there are ways other than going to the Eiffel Tower to ensure that Maria visits France. ◀

Example 3.4 shows that the proposition $p \to q$ can be true while the proposition $q \to p$ is false. We call the proposition $q \to p$ the **converse** of the proposition $p \to q$. Thus a conditional proposition can be true while its converse is false.

Example 3.7 ▶ Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words. Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

Let

$$p\colon \text{ Jerry receives a scholarship,}$$
$$q\colon \text{ Jerry goes to college.}$$

The given proposition can be written symbolically as $p \to q$. Since the hypothesis p is false, the conditional proposition is true.

The converse of the proposition is

If Jerry goes to college, then he receives a scholarship.

The converse can be written symbolically as $q \to p$. Since the hypothesis q is true and the conclusion p is false, the converse is false. ◀

Another useful proposition is

$$p \text{ if and only if } q,$$

which is considered to be true precisely when p and q have the same truth values (i.e., p and q are both true or p and q are both false).

Definition 3.8 ▶ If p and q are propositions, the proposition

$$p \text{ if and only if } q$$

is called a *biconditional proposition* and is denoted

$$p \leftrightarrow q.$$

The truth value of the proposition $p \leftrightarrow q$ is defined by the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

◀

It is traditional in mathematical definitions to use "if" to mean "if and only if." Consider, for example, the definition of set equality: If sets X and Y have the same elements, then X and Y are equal. The meaning of this definition is that sets X and Y have the same elements *if and only if* X and Y are equal.

An alternative way to state "p if and only if q" is "p is a necessary and sufficient condition for q." The proposition "p if and only if q" is sometimes written "p iff q."

Example 3.9 ▶

The proposition

$$1 < 5 \text{ if and only if } 2 < 8 \tag{3.5}$$

can be written symbolically as

$$p \leftrightarrow q$$

if we define

$$p\colon 1 < 5, \qquad q\colon 2 < 8.$$

Since both p and q are true, the proposition $p \leftrightarrow q$ is true. ◀

An alternative way to state (3.5) is: A necessary and sufficient condition for $1 < 5$ is that $2 < 8$.

In some cases, two different propositions have the same truth values no matter what truth values their constituent propositions have. Such propositions are said to be **logically equivalent.**

Definition 3.10 ▶

Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$. We say that P and Q are *logically equivalent* and write

$$P \equiv Q,$$

provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false. ◀

Example 3.11 ▶

De Morgan's Laws for Logic

We will verify the first of **De Morgan's laws**

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q,$$

and leave the second as an exercise (see Exercise 76).

By writing the truth tables for $P = \neg(p \vee q)$ and $Q = \neg p \wedge \neg q$, we can verify that, given any truth values of p and q, either P and Q are both true or P and Q are both false:

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Thus P and Q are logically equivalent. ◀

Example 3.12 ▶

Show that, in Java, the expressions

```
x < 10 || x > 20
```

and

```
!(x >= 10 && x <= 20)
```

are equivalent. (In Java, >= means $\geq$, and <= means $\leq$.)

If we let p denote the expression `x >= 10` and q denote the expression `x <= 20`, the expression `!(x >= 10 && x <= 20)` becomes $\neg(p \wedge q)$. By De Morgan's second law, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Since $\neg p$ translates as `x < 10` and $\neg q$ translates as `x > 20`, $\neg p \vee \neg q$ translates as `x < 10 || x > 20`. Therefore, the expressions `x < 10 || x > 20` and `!(x >= 10 && x <= 20)` are equivalent. ◀

Our next example gives a logically equivalent form of the negation of $p \to q$.

Example 3.13 ▶

Show that the negation of $p \to q$ is logically equivalent to $p \wedge \neg q$.

We must show that

$$\neg(p \to q) \equiv p \wedge \neg q.$$

By writing the truth tables for $P = \neg(p \to q)$ and $Q = p \wedge \neg q$, we can verify that, given any truth values of p and q, either P and Q are both true or P and Q are both false:

p	q	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Thus P and Q are logically equivalent. ◀

Example 3.14 ▶

Use the logical equivalence of $\neg(p \to q)$ and $p \wedge \neg q$ (see Example 3.13) to write the negation of

If Jerry receives a scholarship, then he goes to college,

symbolically and in words.

We let

p: Jerry receives a scholarship,

q: Jerry goes to college.

The given proposition can be written symbolically as $p \to q$. Its negation is logically equivalent to $p \wedge \neg q$. In words, this last expression is

Jerry receives a scholarship and he does not go to college. ◀

We now show that, according to our definitions, $p \leftrightarrow q$ is logically equivalent to $p \to q$ *and* $q \to p$. In words,

p if and only if q

is logically equivalent to

if p then q and if q then p.

Example 3.15 ▶

The truth table shows that

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p).$$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Consider again the definition of set equality: If sets X and Y have the same elements, then X and Y are equal. We noted that the meaning of this definition is that sets X and Y have the same elements if and only if X and Y are equal. Example 3.15 shows that an equivalent formulation is: If sets X and Y have the same elements, then X and Y are equal, and if X and Y are equal, then X and Y have the same elements.

We conclude this section by defining the **contrapositive** of a conditional proposition. We will see (in Theorem 3.18) that the contrapositive is an alternative, logically equivalent form of the conditional proposition. Exercise 77 gives another logically equivalent form of the conditional proposition.

Definition 3.16 The *contrapositive* (or *transposition*) of the conditional proposition $p \to q$ is the proposition $\neg q \to \neg p$.

Notice the difference between the contrapositive and the converse. The converse of a conditional proposition merely reverses the roles of p and q, whereas the contrapositive reverses the roles of p and q *and* negates each of them.

Example 3.17 Write the conditional proposition,

If the network is down, then Dale cannot access the Internet,

symbolically. Write the contrapositive and the converse symbolically and in words. Also, assuming that the network is not down and Dale can access the Internet, find the truth value of the original proposition, its contrapositive, and its converse.

Let

p: The network is down,
q: Dale cannot access the Internet.

The given proposition can be written symbolically as $p \to q$. Since the hypothesis p is false, the conditional proposition is true.

The contrapositive can be written symbolically as $\neg q \to \neg p$ and, in words,

If Dale can access the Internet, then the network is not down.

Since the hypothesis $\neg q$ and conclusion $\neg p$ are both true, the contrapositive is true. (Theorem 3.18 will show that the conditional proposition and its contrapositive are logically equivalent, that is, that they always have the same truth value.)

The converse of the given proposition can be written symbolically as $q \to p$ and, in words,

If Dale cannot access the Internet, then the network is down.

Since the hypothesis q is false, the converse is true.

An important fact is that a conditional proposition and its contrapositive are logically equivalent.

Theorem 3.18 *The conditional proposition $p \to q$ and its contrapositive $\neg q \to \neg p$ are logically equivalent.*

Proof The truth table

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

shows that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

In ordinary language, "if" is often used to mean "if and only if." Consider the statement

If you fix my computer, then I'll pay you \$50.

The intended meaning is

If you fix my computer, then I'll pay you \$50, and
if you do not fix my computer, then I will not pay you \$50,

which is logically equivalent to (see Theorem 3.18)

If you fix my computer, then I'll pay you \$50, and
if I pay you \$50, then you fix my computer,

which, in turn, is logically equivalent to (see Example 3.15)

You fix my computer if and only if I pay you \$50.

In ordinary discourse, the intended meaning of statements involving logical operators can often (but, not always!) be inferred. However, in mathematics and science, precision is required. Only by carefully defining what we mean by terms such as "if" and "if and only if" can we obtain unambiguous and precise statements. In particular, logic carefully distinguishes among conditional, biconditional, converse, and contrapositive propositions.

Problem-Solving Tips

In formal logic, "if" and "if and only if" are quite different. The conditional proposition $p \to q$ (if p then q) is true except when p is true and q is false. On the other hand, the biconditional proposition $p \leftrightarrow q$ (p if and only if q) is true precisely when p and q are both true or both false.

To determine whether propositions P and Q, made up of the propositions $p_1, \ldots,$ p_n, are logically equivalent, write the truth tables for P and Q. If all of the entries for P and Q are always both true or both false, then P and Q *are* equivalent. If some entry is true for one of P or Q and false for the other, then P and Q are *not* equivalent.

De Morgan's laws for logic

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

give formulas for negating "or" ($\vee$) and negating "and" ($\wedge$). Roughly speaking, negating "or" results in "and," and negating "and" results in "or."

Example 3.13 states a very important equivalence

$$\neg(p \to q) \equiv p \wedge \neg q.$$

This equivalence shows that the negation of the conditional proposition can be written using the "and" ($\wedge$) operator. Notice that there is no conditional operator on the right-hand side of the equation.

Section Review Exercises

1. What is a conditional proposition? How is it denoted?
2. Give the truth table for the conditional proposition.
3. In a conditional proposition, what is the hypothesis?
4. In a conditional proposition, what is the conclusion?
5. What is a necessary condition?
6. What is a sufficient condition?
7. What is the converse of $p \to q$?
8. What is a biconditional proposition? How is it denoted?
9. Give the truth table for the biconditional proposition.
10. What does it mean for P to be logically equivalent to Q?
11. State De Morgan's laws for logic.
12. What is the contrapositive of $p \to q$?

Exercises

In Exercises 1–10, restate each proposition in the form (3.2) of a conditional proposition.

1. Joey will pass the discrete mathematics exam if he studies hard.
2. Rosa may graduate if she has 160 quarter-hours of credits.
3. A necessary condition for Fernando to buy a computer is that he obtain \$2000.
4. A sufficient condition for Katrina to take the algorithms course is that she pass discrete mathematics.
5. Getting that job requires knowing someone who knows the boss.
6. You can go to the Super Bowl unless you can't afford the ticket.
7. You may inspect the aircraft only if you have the proper security clearance.
8. When better cars are built, Buick will build them.
9. The audience will go to sleep if the chairperson gives the lecture.
10. The program is readable only if it is well structured.
11. Write the converse of each proposition in Exercises 1–10.
12. Write the contrapositive of each proposition in Exercises 1–10.

Assuming that p and r are false and that q and s are true, find the truth value of each proposition in Exercises 13–20.

13. $p \to q$
14. $\neg p \to \neg q$
15. $\neg(p \to q)$
16. $(p \to q) \wedge (q \to r)$
17. $(p \to q) \to r$
18. $p \to (q \to r)$
19. $(s \to (p \wedge \neg r)) \wedge ((p \to (r \vee q)) \wedge s)$
20. $((p \wedge \neg q) \to (q \wedge r)) \to (s \vee \neg q)$

Exercises 21–30 refer to the propositions p, q, and r; p is true, q is false, and r's status is unknown at this time. Tell whether each proposition is true, is false, or has unknown status at this time.

21. $p \vee r$
22. $p \wedge r$
23. $p \to r$
24. $q \to r$
25. $r \to p$
26. $r \to q$
27. $(p \wedge r) \leftrightarrow r$
28. $(p \vee r) \leftrightarrow r$
29. $(q \wedge r) \leftrightarrow r$
30. $(q \vee r) \leftrightarrow r$

Determine the truth value of each proposition in Exercises 31–39.

31. If $3 + 5 < 2$, then $1 + 3 = 4$.
32. If $3 + 5 < 2$, then $1 + 3 \neq 4$.
33. If $3 + 5 > 2$, then $1 + 3 = 4$.
34. If $3 + 5 > 2$, then $1 + 3 \neq 4$.
35. $3 + 5 > 2$ if and only if $1 + 3 = 4$.
36. $3 + 5 < 2$ if and only if $1 + 3 = 4$.
37. $3 + 5 < 2$ if and only if $1 + 3 \neq 4$.
38. If the earth has six moons, then $1 < 3$.
39. If $1 < 3$, then the earth has six moons.

In Exercises 40–43, represent the given proposition symbolically by letting

$$p:\ 4 < 2, \quad q:\ 7 < 10, \quad r:\ 6 < 6.$$

40. If $4 < 2$, then $7 < 10$.
41. If ($4 < 2$ and $6 < 6$), then $7 < 10$.
42. If it is not the case that ($6 < 6$ and 7 is not less than 10), then $6 < 6$.
43. $7 < 10$ if and only if ($4 < 2$ and 6 is not less than 6).

In Exercises 44–49, represent the given proposition symbolically by letting

p : *You run 10 laps daily.*
q : *You are healthy.*
r : *You take multi-vitamins.*

44. If you run 10 laps daily, then you will be healthy.
45. If you do not run 10 laps daily or do not take multi-vitamins, then you will not be healthy.

46. Taking multi-vitamins is sufficient for being healthy.

47. You will be healthy if and only if you run 10 laps daily and take multi-vitamins.

48. If you are healthy, then you run 10 laps daily or you take multi-vitamins.

49. If you are healthy and run 10 laps daily, then you do not take multi-vitamins.

In Exercises 50–55, formulate the symbolic expression in words using

$$p: \text{ Today is Monday,}$$
$$q: \text{ It is raining,}$$
$$r: \text{ It is hot.}$$

50. $p \to q$

51. $\neg q \to (r \wedge p)$

52. $\neg p \to (q \vee r)$

53. $\neg(p \vee q) \leftrightarrow r$

54. $(p \wedge (q \vee r)) \to (r \vee (q \vee p))$

55. $(p \vee (\neg p \wedge \neg(q \vee r))) \to (p \vee \neg(r \vee q))$

In Exercises 56–59, write each conditional proposition symbolically. Write the converse and contrapositive of each proposition symbolically and in words. Also, find the truth value of each conditional proposition, its converse, and its contrapositive.

56. If $4 < 6$, then $9 > 12$.

57. If $4 > 6$, then $9 > 12$.

58. $|1| < 3$ if $-3 < 1 < 3$.

59. $|4| < 3$ if $-3 < 4 < 3$.

For each pair of propositions P and Q in Exercises 60–69, state whether or not $P \equiv Q$.

60. $P = p$, $Q = p \vee q$

61. $P = p \wedge q$, $Q = \neg p \vee \neg q$

62. $P = p \to q$, $Q = \neg p \vee q$

63. $P = p \wedge (\neg q \vee r)$, $Q = p \vee (q \wedge \neg r)$

64. $P = p \wedge (q \vee r)$, $Q = (p \vee q) \wedge (p \vee r)$

65. $P = p \to q$, $Q = \neg q \to \neg p$

66. $P = p \to q$, $Q = p \leftrightarrow q$

67. $P = (p \to q) \wedge (q \to r)$, $Q = p \to r$

68. $P = (p \to q) \to r$, $Q = p \to (q \to r)$

69. $P = (s \to (p \wedge \neg r)) \wedge ((p \to (r \vee q)) \wedge s)$, $Q = p \vee t$

Using De Morgan's laws for logic, write the negation of each proposition in Exercises 70–73.

70. Pat will use the treadmill or lift weights.

71. Dale is smart and funny.

72. Shirley will either take the bus or catch a ride to school.

73. Red pepper and onions are required to make chili.

Exercises 74 and 75 provide further motivation for defining $p \to q$ to be true when p is false. We consider changing the truth table for $p \to q$ when p is false. For the first change, we call the resulting operator imp1 (Exercise 74), and, for the second change, we call the resulting operator imp2 (Exercise 75). In both cases, we see that pathologies result.

74. Define the truth table for *imp1* by

p	q	p *imp1* q
T	T	T
T	F	F
F	T	F
F	F	T

Show that p *imp1* $q \equiv q$ *imp1* p.

75. Define the truth table for *imp2* by

p	q	p *imp2* q
T	T	T
T	F	F
F	T	T
F	F	F

(a) Show that

$$(p \textit{ imp2 } q) \wedge (q \textit{ imp2 } p) \not\equiv p \leftrightarrow q. \qquad \textbf{(3.6)}$$

(b) Show that (3.6) remains true if we change the third row of *imp2*'s truth table to F T F.

76. Verify the second of De Morgan's laws $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

77. Show that $(p \to q) \equiv (\neg p \vee q)$.

4 → Arguments and Rules of Inference

Consider the following sequence of propositions.

The bug is either in module 17 or in module 81.
The bug is a numerical error.
Module 81 has no numerical error. **(4.1)**

Assuming that these statements are true, it is reasonable to conclude

The bug is in module 17. **(4.2)**

This process of drawing a conclusion from a sequence of propositions is called **deductive reasoning.** The given propositions, such as (4.1), are called **hypotheses** or **premises,** and the proposition that follows from the hypotheses, such as (4.2), is called the **conclusion.** A **(deductive) argument** consists of hypotheses together with a conclusion. Many proofs in mathematics and computer science are deductive arguments.

Any argument has the form

$$\text{If } p_1 \text{ and } p_2 \text{ and } \cdots \text{ and } p_n, \text{ then } q. \tag{4.3}$$

Argument (4.3) is said to be **valid** if the conclusion follows from the hypotheses; that is, if p_1 and p_2 and $\cdots$ and p_n are true, then q must also be true. This discussion motivates the following definition.

Definition 4.1 ▶

An *argument* is a sequence of propositions written

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

or

$$p_1, p_2, \ldots, p_n / \therefore q.$$

The symbol $\therefore$ is read "therefore." The propositions $p_1, p_2, \ldots, p_n$ are called the *hypotheses* (or *premises*), and the proposition q is called the *conclusion.* The argument is *valid* provided that if p_1 and p_2 and $\cdots$ and p_n are all true, then q must also be true; otherwise, the argument is *invalid* (or a *fallacy*). ◀

In a valid argument, we sometimes say that the conclusion follows from the hypotheses. Notice that we are not saying that the conclusion is true; we are only saying that if you grant the hypotheses, you must also grant the conclusion. An argument is valid because of its form, not because of its content.

Each step of an extended argument involves drawing intermediate conclusions. For the argument as a whole to be valid, each step of the argument must result in a valid, intermediate conclusion. **Rules of inference,** brief, valid arguments, are used within a larger argument.

Example 4.2 ▶

Determine whether the argument

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

is valid.

[First solution] We construct a truth table for all the propositions involved:

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

We observe that whenever the hypotheses $p \to q$ and p are true, the conclusion q is also true; therefore, the argument is valid.

[Second solution] We can avoid writing the truth table by directly verifying that whenever the hypotheses are true, the conclusion is also true.

Suppose that $p \to q$ and p are true. Then q must be true, for otherwise $p \to q$ would be false. Therefore, the argument is valid. ◀

The argument in Example 4.2 is used extensively and is known as the **modus ponens rule of inference** or **law of detachment.** Several useful rules of inference for propositions, which may be verified using truth tables (see Exercises 24–29), are listed in Figure 4.1.

Rule of Inference	Name	Rule of Inference	Name
$\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$	Modus ponens	$\begin{array}{l} p \\ \underline{q \quad\quad} \\ \therefore p \wedge q \end{array}$	Conjunction
$\begin{array}{l} p \to q \\ \underline{\neg q \quad\quad} \\ \therefore \neg p \end{array}$	Modus tollens	$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$	Hypothetical syllogism
$\begin{array}{l} \underline{p \quad\quad} \\ \therefore p \vee q \end{array}$	Addition	$\begin{array}{l} p \vee q \\ \underline{\neg p \quad} \\ \therefore q \end{array}$	Disjunctive syllogism
$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array}$	Simplification		

Figure 4.1 Rules of inference for propositions.

Example 4.3 ▶

Which rule of inference is used in the following argument?

If the computer has one gigabyte of memory, then it can run "Blast 'em." If the computer can run "Blast 'em," then the sonics will be impressive. Therefore, if the computer has one gigabyte of memory, then the sonics will be impressive.

Let p denote the proposition "the computer has one gigabyte of memory," let q denote the proposition "the computer can run 'Blast 'em,'" and let r denote the proposition "the sonics will be impressive." The argument can be written symbolically as

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Therefore, the argument uses the hypothetical syllogism rule of inference. ◀

Example 4.4 ▶

Represent the argument

$$\begin{array}{l} \text{If } 2 = 3\text{, then I ate my hat.} \\ \underline{\text{I ate my hat.} \qquad\qquad\quad} \\ \therefore 2 = 3 \end{array}$$

symbolically and determine whether the argument is valid.

If we let

$$p\colon\ 2 = 3, \qquad q\colon\ \text{I ate my hat},$$

the argument may be written

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

If the argument is valid, then whenever $p \to q$ and q are both true, p must also be true. Suppose that $p \to q$ and q are true. This is possible if p is false and q is true. In this case, p is not true; thus the argument is invalid. This fallacy is known as the **fallacy of affirming the conclusion.** ◀

We can also determine whether the argument in Example 4.4 is valid or not by examining the truth table of Example 4.2. In the third row of the table, the hypotheses are true and the conclusion is false; thus the argument is invalid.

Example 4.5 ▶ Represent the argument

The bug is either in module 17 or in module 81.
The bug is a numerical error.
Module 81 has no numerical error.

∴ The bug is in module 17.

given at the beginning of this section symbolically and show that it is valid.

If we let

p : The bug is in module 17.
q : The bug is in module 81.
r : The bug is a numerical error.

the argument may be written

$$\begin{array}{l} p \vee q \\ r \\ r \to \neg q \\ \hline \therefore p \end{array}$$

From $r \to \neg q$ and r, we may use modus ponens to conclude $\neg q$. From $p \vee q$ and $\neg q$, we may use the disjunctive syllogism to conclude p. Thus the conclusion p follows from the hypotheses and the argument is valid. ◀

Example 4.6 ▶ We are given the following hypotheses: If the Chargers get a good linebacker, then the Chargers can beat the Broncos. If the Chargers can beat the Broncos, then the Chargers can beat the Jets. If the Chargers can beat the Broncos, then the Chargers can beat the Dolphins. The Chargers get a good linebacker. Show by using the rules of inference (see Figure 4.1) that the conclusion, the Chargers can beat the Jets and the Chargers can beat the Dolphins, follows from the hypotheses.

Let p denote the proposition "the Chargers get a good linebacker," let q denote the proposition "the Chargers can beat the Broncos," let r denote the proposition "the Chargers can beat the Jets," and let s denote the proposition "the Chargers can beat the Dolphins." Then the hypotheses are:

$$p \to q$$
$$q \to r$$
$$q \to s$$
$$p.$$

From $p \to q$ and $q \to r$, we may use the hypothetical syllogism to conclude $p \to r$. From $p \to r$ and p, we may use modus ponens to conclude r. From $p \to q$ and $q \to s$, we may use the hypothetical syllogism to conclude $p \to s$. From $p \to s$ and p, we may use modus ponens to conclude s. From r and s, we may use conjunction to conclude $r \wedge s$. Since $r \wedge s$ represents the proposition "the Chargers can beat the Jets and the Chargers can beat the Dolphins," we conclude that the conclusion does follow from the hypotheses. ◀

Problem-Solving Tips

The validity of a very short argument or proof might be verified using a truth table. In practice, arguments and proofs use rules of inference.

Section Review Exercises

1. What is deductive reasoning?
2. What is a hypothesis in an argument?
3. What is a premise in an argument?
4. What is a conclusion in an argument?
5. What is a valid argument?
6. What is an invalid argument?
7. State the modus ponens rule of inference.
8. State the modus tollens rule of inference.
9. State the addition rule of inference.
10. State the simplification rule of inference.
11. State the conjunction rule of inference.
12. State the hypothetical syllogism rule of inference.
13. State the disjunctive syllogism rule of inference.

Exercises

Formulate the arguments of Exercises 1–5 symbolically and determine whether each is valid. Let

p: *I study hard.* q: *I get A's.* r: *I get rich.*

1. If I study hard, then I get A's.
I study hard.
$\therefore$ I get A's.

2. If I study hard, then I get A's.
If I don't get rich, then I don't get A's.
$\therefore$ I get rich.

3. I study hard if and only if I get rich.
I get rich.
$\therefore$ I study hard.

4. If I study hard or I get rich, then I get A's.
I get A's.
$\therefore$ If I don't study hard, then I get rich.

5. If I study hard, then I get A's or I get rich.
I don't get A's and I don't get rich.
$\therefore$ I don't study hard.

In Exercises 6–10, write the given argument in words and determine whether each argument is valid. Let

p: *4 megabytes is better than no memory at all.*
q: *We will buy more memory.*
r: *We will buy a new computer.*

6. $$\begin{array}{l} p \to r \\ p \to q \\ \hline \therefore p \to (r \wedge q) \end{array}$$

7. $$\begin{array}{l} p \to (r \vee q) \\ r \to \neg q \\ \hline \therefore p \to r \end{array}$$

8. $$\begin{array}{l} p \to r \\ r \to q \\ \hline \therefore q \end{array}$$

9. $$\begin{array}{l} \neg r \to \neg p \\ r \\ \hline \therefore p \end{array}$$

10. $$\begin{array}{l} p \to r \\ r \to q \\ p \\ \hline \therefore q \end{array}$$

Determine whether each argument in Exercises 11–15 is valid.

11. $$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

12. $$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

13. $$\begin{array}{l} p \wedge \neg p \\ \hline \therefore q \end{array}$$

14. $$\begin{array}{l} p \to (q \to r) \\ q \to (p \to r) \\ \hline \therefore (p \vee q) \to r \end{array}$$

15. $$\begin{array}{l} (p \to q) \wedge (r \to s) \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

16. Show that if

$$p_1, p_2 / \therefore p \qquad \text{and} \qquad p, p_3, \ldots, p_n / \therefore c$$

are valid arguments, the argument

$$p_1, p_2, \ldots, p_n / \therefore c$$

is also valid.

17. Comment on the following argument:

$$\begin{array}{l} \text{Floppy disk storage is better than nothing.} \\ \text{Nothing is better than a hard disk drive.} \\ \hline \therefore\ \text{Floppy disk storage is better than a hard disk drive.} \end{array}$$

For each argument in Exercises 18–20, tell which rule of inference is used.

18. Fishing is a popular sport. Therefore, fishing is a popular sport or lacrosse is wildly popular in California.

19. If fishing is a popular sport, then lacrosse is wildly popular in California. Fishing is a popular sport. Therefore, lacrosse is wildly popular in California.

20. Fishing is a popular sport or lacrosse is wildly popular in California. Lacrosse is not wildly popular in California. Therefore, fishing is a popular sport.

In Exercises 21–23, give an argument using rules of inference to show that the conclusion follows from the hypotheses.

21. Hypotheses: If there is gas in the car, then I will go to the store. If I go to the store, then I will get a soda. There is gas in the car. Conclusion: I will get a soda.

22. Hypotheses: If there is gas in the car, then I will go to the store. If I go to the store, then I will get a soda. I do not get a soda. Conclusion: There is not gas in the car, or the car transmission is defective.

23. Hypotheses: If Jill can sing or Dweezle can play, then I'll buy the compact disc. Jill can sing. I'll buy the compact disc player. Conclusion: I'll buy the compact disc and the compact disc player.

24. Show that modus tollens (see Figure 4.1) is valid.

25. Show that addition (see Figure 4.1) is valid.

26. Show that simplification (see Figure 4.1) is valid.

27. Show that conjunction (see Figure 4.1) is valid.

28. Show that hypothetical syllogism (see Figure 4.1) is valid.

29. Show that disjunctive syllogism (see Figure 4.1) is valid.

5 → Quantifiers

The logic in Sections 2 and 3 that deals with propositions is incapable of describing most of the statements in mathematics and computer science. Consider, for example, the statement

$$p\colon\ n \text{ is an odd integer.}$$

A proposition is a statement that is either true or false. The statement p is not a proposition, because whether p is true or false depends on the value of n. For example, p is true if $n = 103$ and false if $n = 8$. Since most of the statements in mathematics and computer science use variables, we must extend the system of logic to include such statements.

Definition 5.1 ▶ Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a *propositional function* or *predicate* (with respect to D) if for each $x \in D$, $P(x)$ is a proposition. We call D the *domain of discourse* of P. ◀

In Definition 5.1, the domain of discourse specifies the allowable values for x.

Example 5.2 ▶ Let $P(n)$ be the statement

$$n \text{ is an odd integer.}$$

Then P is a propositional function with domain of discourse $\mathbf{Z}^+$ since for each $n \in \mathbf{Z}^+$, $P(n)$ is a proposition [i.e., for each $n \in \mathbf{Z}^+$, $P(n)$ is true or false but not both]. For example, if $n = 1$, we obtain the proposition

$$P(1): \text{ 1 is an odd integer}$$

(which is true). If $n = 2$, we obtain the proposition

$$P(2): \text{ 2 is an odd integer}$$

(which is false). ◀

A propositional function P, by itself, is neither true nor false. However, for each x in the domain of discourse, $P(x)$ is a proposition and is, therefore, either true or false. We can think of a propositional function as defining a class of propositions, one for each element in the domain of discourse. For example, if P is a propositional function with domain of discourse $\mathbf{Z}^+$, we obtain the class of propositions

$$P(1), P(2), \ldots .$$

Each of $P(1), P(2), \ldots$ is either true or false.

Example 5.3 ▶ The following are propositional functions.

(a) $n^2 + 2n$ is an odd integer (domain of discourse $= \mathbf{Z}^+$).

(b) $x^2 - x - 6 = 0$ (domain of discourse $= \mathbf{R}$).

(c) The baseball player hit over .300 in 2003 (domain of discourse = set of baseball players).

(d) The restaurant rated over two stars in *Chicago* magazine (domain of discourse = restaurants rated in *Chicago* magazine).

In statement (a), for each positive integer n, we obtain a proposition; therefore, statement (a) is a propositional function.

Similarly, in statement (b), for each real number x, we obtain a proposition; therefore, statement (b) is a propositional function.

We can regard the variable in statement (c) as "baseball player." Whenever we substitute a particular baseball player for the variable "baseball player," the statement is a proposition. For example, if we substitute "Barry Bonds" for "baseball player," statement (c) is

Barry Bonds hit over .300 in 2003,

which is true. If we substitute "Alex Rodriguez" for "baseball player," statement (c) is

Alex Rodriguez hit over .300 in 2003,

which is false. Thus statement (c) is a propositional function.

Statement (d) is similar in form to statement (c): Here the variable is "restaurant." Whenever we substitute a restaurant rated in *Chicago* magazine for the variable "restaurant," the statement is a proposition. For example, if we substitute "Yugo Inn" for "restaurant," statement (d) is

Yugo Inn rated over two stars in *Chicago* magazine,

which is false. If we substitute "Le Français" for "restaurant," statement (d) is

Le Français rated over two stars in *Chicago* magazine,

which is true. Thus statement (d) is a propositional function. ◀

Most of the statements in mathematics and computer science use terms such as "for every" and "for some." For example, in mathematics we have the following theorem:

For every triangle T, the sum of the angles of T is equal to $180°$.

In computer science, we have this theorem:

For some program P, the output of P is P itself.

We now extend the logical system of Sections 2 and 3 so that we can handle statements that include "for every" and "for some."

Definition 5.4 ▶ Let P be a propositional function with domain of discourse D. The statement

$$\text{for every } x,\ P(x)$$

is said to be a *universally quantified statement*. The symbol $\forall$ means "for every." Thus the statement

$$\text{for every } x,\ P(x)$$

may be written

$$\forall x\ P(x).$$

The symbol $\forall$ is called a *universal quantifier*.

The statement

$$\forall x\ P(x)$$

is true if $P(x)$ is true for every x in D. The statement

$$\forall x\ P(x)$$

is false if $P(x)$ is false for at least one x in D. ◀

Example 5.5 ▶ Consider the universally quantified statement

$$\forall x(x^2 \geq 0).$$

The domain of discourse is $\mathbf{R}$. The statement is true because, *for every* real number x, it is true that the square of x is positive or zero. ◀

According to Definition 5.4, the universally quantified statement

$$\forall x\ P(x)$$

is false if *for at least one* x in the domain of discourse, the proposition $P(x)$ is false. A value x in the domain of discourse that makes $P(x)$ false is called a **counterexample** to the statement

$$\forall x\ P(x).$$

Example 5.6 ▶ Consider the universally quantified statement

$$\forall x(x^2 - 1 > 0).$$

The domain of discourse is **R**. The statement is false since, if $x = 1$, the proposition

$$1^2 - 1 > 0$$

is false. The value 1 is a counterexample to the statement

$$\forall x(x^2 - 1 > 0).$$

Although there are values of x that make the propositional function true, the counterexample provided shows that the universally quantified statement is false. ◀

Example 5.7 ▶ Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \ldots, d_n\}$. The following pseudocode determines whether

$$\forall x\, P(x)$$

is true or false:

```
for i = 1 to n
    if (¬P(d_i))
        return false
return true
```

The for loop examines the members d_i of the domain of discourse one by one. If it finds a value d_i for which $P(d_i)$ is false, the condition $\neg P(d_i)$ in the if statement is true; so the code returns false [to indicate that $\forall x\, P(x)$ is false] and terminates. In this case, d_i is a counterexample. If $P(d_i)$ is true for every d_i, the condition $\neg P(d_i)$ in the if statement is always false. In this case, the for loop runs to completion, after which the code returns true [to indicate that $\forall x\, P(x)$ is true] and terminates.

Notice that if $\forall x\, P(x)$ is true, the for loop necessarily runs to completion so that *every* member of the domain of discourse is checked to ensure that $P(x)$ is true for every x. If $\forall x\, P(x)$ is false, the for loop terminates as soon as *one* element x of the domain of discourse is found for which $P(x)$ is false. ◀

We call the variable x in the propositional function $P(x)$ a *free variable*. (The idea is that x is "free" to roam over the domain of discourse.) We call the variable x in the universally quantified statement

$$\forall x\, P(x) \tag{5.1}$$

a *bound variable*. (The idea is that x is "bound" by the quantifier $\forall$.)

We previously pointed out that a propositional function does not have a truth value. On the other hand, Definition 5.4 assigns a truth value to the quantified statement (5.1). In sum, a statement with free (unquantified) variables is not a proposition, and a statement with no free variables (no unquantified variables) is a proposition.

Alternative ways to write

$$\forall x\ P(x)$$

are

$$\text{for all } x,\ P(x)$$

and

$$\text{for any } x,\ P(x).$$

The symbol $\forall$ may be read "for every," "for all," or "for any."

To prove that

$$\forall x\ P(x)$$

is *true,* we must, in effect, examine *every* value of x in the domain of discourse and show that for every x, $P(x)$ is true. One technique for proving that

$$\forall x\ P(x)$$

is true is to let x denote an *arbitrary* element of the domain of discourse D. The argument then proceeds using the symbol x. Whatever is claimed about x must be true *no matter what value* x might have in D. The argument must conclude by proving that $P(x)$ is true.

Sometimes to specify the domain of discourse D, we write a universally quantified statement as

$$\text{for every } x \text{ in } D,\ P(x).$$

Example 5.8 ▶ The universally quantified statement

$$\text{for every real number } x, \text{ if } x > 1, \text{ then } x + 1 > 1$$

is true. This time we must verify that the statement

$$\text{if } x > 1, \text{ then } x + 1 > 1$$

is true *for every* real number x.

Let x be any real number whatsoever. It is true that for any real number x, either $x \leq 1$ or $x > 1$. If $x \leq 1$, the conditional proposition

$$\text{if } x > 1, \text{ then } x + 1 > 1$$

is vacuously true. (The proposition is true because the hypothesis $x > 1$ is false. Recall that when the hypothesis is false, the conditional proposition is true regardless of whether the conclusion is true or false.) In most arguments, the vacuous case is omitted.

Now suppose that $x > 1$. Regardless of the specific value of x, $x + 1 > x$. Since

$$x + 1 > x \qquad \text{and} \qquad x > 1,$$

we conclude that $x + 1 > 1$, so the conclusion is true. If $x > 1$, the hypothesis and conclusion are both true; hence the conditional proposition

$$\text{if } x > 1, \text{ then } x + 1 > 1$$

is true.

We have shown that for every real number x, the proposition

$$\text{if } x > 1, \text{ then } x + 1 > 1$$

is true. Therefore, the universally quantified statement

$$\text{for every real number } x,\ \text{ if } x > 1,\ \text{ then } x + 1 > 1$$

is true. ◀

The method of disproving the statement

$$\forall x\, P(x)$$

is quite different from the method used to prove that the statement is true. To show that the universally quantified statement

$$\forall x\, P(x)$$

is *false,* it is sufficient to find *one* value x in the domain of discourse for which the proposition $P(x)$ is false. Such a value, we recall, is called a counterexample to the universally quantified statement.

We turn next to existentially quantified statements.

Definition 5.9 ▶ Let P be a propositional function with domain of discourse D. The statement

$$\text{there exists } x,\ P(x)$$

is said to be an *existentially quantified statement*. The symbol $\exists$ means "there exists." Thus the statement

$$\text{there exists } x,\ P(x)$$

may be written

$$\exists x\, P(x).$$

The symbol $\exists$ is called an *existential quantifier*.

The statement

$$\exists x\, P(x)$$

is true if $P(x)$ is true for at least one x in D. The statement

$$\exists x\, P(x)$$

is false if $P(x)$ is false for every x in D. ◀

Example 5.10 ▶ Consider the existentially quantified statement

$$\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right).$$

The domain of discourse is $\mathbf{R}$. The statement is true because it is possible to find *at least one* real number x for which the proposition

$$\frac{x}{x^2+1} = \frac{2}{5}$$

is true. For example, if $x = 2$, we obtain the true proposition

$$\frac{2}{2^2 + 1} = \frac{2}{5}.$$

It is not the case that *every* value of x results in a true proposition. For example, if $x = 1$, the proposition

$$\frac{1}{1^2 + 1} = \frac{2}{5}$$

is false. ◀

According to Definition 5.9, the existentially quantified statement

$$\exists x\, P(x)$$

is false if for every x in the domain of discourse, the proposition $P(x)$ is false.

Example 5.11 ▶ To verify that the existentially quantified statement

$$\exists x \in \mathbf{R} \left(\frac{1}{x^2 + 1} > 1 \right)$$

is false, we must show that

$$\frac{1}{x^2 + 1} > 1$$

is false for every real number x. Now

$$\frac{1}{x^2 + 1} > 1$$

is false precisely when

$$\frac{1}{x^2 + 1} \leq 1$$

is true. Thus, we must show that

$$\frac{1}{x^2 + 1} \leq 1$$

is true for every real number x. To this end, let x be any real number whatsoever. Since $0 \leq x^2$, we may add 1 to both sides of this inequality to obtain $1 \leq x^2 + 1$. If we divide both sides of this last inequality by $x^2 + 1$, we obtain

$$\frac{1}{x^2 + 1} \leq 1.$$

Therefore, the statement

$$\frac{1}{x^2 + 1} \leq 1$$

is true for every real number x. Thus the statement

$$\frac{1}{x^2 + 1} > 1$$

is false for every real number x. We have shown that the existentially quantified statement

$$\exists x \left(\frac{1}{x^2 + 1} > 1 \right)$$

is false. ◀

Example 5.12 ▶ Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \ldots, d_n\}$. The following pseudocode determines whether

$$\exists x\, P(x)$$

is true or false:

```
for i = 1 to n
    if (P(d_i))
        return true
return false
```

The for loop examines the members d_i in the domain of discourse one by one. If it finds a value d_i for which $P(d_i)$ is true, the condition $P(d_i)$ in the if statement is true; so the code returns true [to indicate that $\exists x\, P(x)$ is true] and terminates. In this case, the code found a value in the domain of discourse, namely d_i, for which $P(d_i)$ is true. If $P(d_i)$ is false for every d_i, the condition $P(d_i)$ in the if statement is always false. In this case, the for loop runs to completion, after which the code returns false [to indicate that $\exists x\, P(x)$ is true] and terminates.

Notice that if $\exists x\, P(x)$ is true, the for loop terminates as soon as *one* element x in the domain of discourse is found for which $P(x)$ is true. If $\exists x\, P(x)$ is false, the for loop necessarily runs to completion so that *every* member in the domain of discourse is checked to ensure that $P(x)$ is false for every x. ◀

Alternative ways to write

$$\exists x\, P(x)$$

are

$$\text{there exists } x \text{ such that, } P(x)$$

and

$$\text{for some } x,\ P(x)$$

and

$$\text{for at least one } x,\ P(x).$$

The symbol $\exists$ may be read "there exists," "for some," or "for at least one."

Example 5.13 ▶ Consider the existentially quantified statement

for some n, if n is prime, then $n+1$, $n+2$, $n+3$, and $n+4$ are not prime.

The domain of discourse is $\mathbf{Z}^+$. This statement is true because we can find *at least one* positive integer n that makes the conditional proposition

if n is prime, then $n+1$, $n+2$, $n+3$, and $n+4$ are not prime

true. For example, if $n = 23$, we obtain the true proposition

if 23 is prime, then 24, 25, 26, and 27 are not prime.

(This conditional proposition is true because both the hypothesis "23 is prime" and the conclusion "24, 25, 26, and 27 are not prime" are true.) Some values of n make the conditional proposition true (e.g., $n = 23, n = 4, n = 47$), while others make it false (e.g., $n = 2, n = 101$). The point is that we found *one* value that makes the conditional proposition

if n is prime, then $n+1, n+2, n+3$, and $n+4$ are not prime

true. For this reason, the existentially quantified statement

for some n, if n is prime, then $n+1, n+2, n+3$, and $n+4$ are not prime

is true. ◀

In Example 5.11, we showed that an existentially quantified statement was false by proving that a related universally quantified statement was true. The following theorem makes this relationship precise. The theorem generalizes De Morgan's laws of logic (Example 3.11).

Theorem 5.14

Generalized De Morgan's Laws for Logic

If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e., either both are true or both are false).

(a) $\neg(\forall x\, P(x));\ \exists x\, \neg P(x)$

(b) $\neg(\exists x\, P(x));\ \forall x\, \neg P(x)$

Proof We prove only part (a) and leave the proof of part (b) to the reader (Exercise 68).

Suppose that the proposition $\neg(\forall x\, P(x))$ is true. Then the proposition $\forall x\, P(x)$ is false. By Definition 5.4, the proposition $\forall x\, P(x)$ is false precisely when $P(x)$ is false for at least one x in the domain of discourse. But if $P(x)$ is false for at least one x in the domain of discourse, $\neg P(x)$ is true for at least one x in the domain of discourse. By Definition 5.9, when $\neg P(x)$ is true for at least one x in the domain of discourse, the proposition $\exists x\, \neg P(x)$ is true. Thus, if the proposition $\neg(\forall x\, P(x))$ is true, the proposition $\exists x\, \neg P(x)$ is true. Similarly, if the proposition $\neg(\forall x\, P(x))$ is false, the proposition $\exists x\, \neg P(x)$ is false.

Therefore, the pair of propositions in part (a) always has the same truth values.

Example 5.15 ▶

Let $P(x)$ be the statement

$$\frac{1}{x^2+1} > 1.$$

In Example 5.11 we showed that

$$\exists x\, P(x)$$

is false by verifying that

$$\forall x\, \neg P(x) \tag{5.2}$$

is true.

The technique can be justified by appealing to Theorem 5.14. After we prove that proposition (5.2) is true, we may negate (5.2) and conclude that

$$\neg(\forall x\, \neg P(x))$$

is false. By Theorem 5.14, part (a),

$$\exists x \neg\neg P(x)$$

or, equivalently,

$$\exists x\, P(x)$$

is also false. ◀

Example 5.16 ▶ Write the statement

Every rock fan loves U2,

symbolically. Write its negation symbolically and in words.

Let $P(x)$ be the propositional function "x loves U2." The given statement can be written symbolically as

$$\forall x\, P(x).$$

The domain of discourse is the set of rock fans.

By Theorem 5.14, part (a), the negation of the preceding proposition $\neg(\forall x\, P(x))$ is equivalent to

$$\exists x \neg P(x).$$

In words, this last proposition can be stated as: There exists a rock fan who does not love U2. ◀

Example 5.17 ▶ Write the statement

Some birds cannot fly,

symbolically. Write its negation symbolically and in words.

Let $P(x)$ be the propositional function "x flies." The given statement can be written symbolically as

$$\exists x \neg P(x).$$

[The statement could also be written $\exists x\, Q(x)$, where $Q(x)$ is the propositional function "x cannot fly." As in algebra, there are many ways to represent text symbolically.] The domain of discourse is the set of birds.

By Theorem 5.14, part (b), the negation $\neg(\exists x \neg P(x))$ of the preceding proposition is equivalent to

$$\forall x \neg\neg P(x)$$

or, equivalently,

$$\forall x\, P(x).$$

In words, this last proposition can be stated as: Every bird can fly. ◀

A universally quantified proposition generalizes the proposition

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \tag{5.3}$$

in the sense that the individual propositions $P_1, P_2, \ldots, P_n$ are replaced by an arbitrary family $P(x)$, where x is in the domain of discourse, and (5.3) is replaced by

$$\forall x\, P(x). \tag{5.4}$$

The proposition (5.3) is true if and only if P_i is true for every $i = 1, \ldots, n$. The truth value of proposition (5.4) is defined similarly: (5.4) is true if and only if $P(x)$ is true for every x in the domain of discourse.

Example 5.18 ▶ Suppose that the domain of discourse of the propositional function P is $\{-1, 0, 1\}$. The propositional function $\forall x\, P(x)$ is equivalent to

$$P(-1) \wedge P(0) \wedge P(1).$$ ◀

Similarly, an existentially quantified proposition generalizes the proposition

$$P_1 \vee P_2 \vee \cdots \vee P_n \tag{5.5}$$

in the sense that the individual propositions $P_1, P_2, \ldots, P_n$ are replaced by an arbitrary family $P(x)$, where x is in the domain of discourse, and (5.5) is replaced by

$$\exists x\, P(x).$$

Example 5.19 ▶ Suppose that the domain of discourse of the propositional function P is $\{1, 2, 3, 4\}$. The propositional function $\exists x\, P(x)$ is equivalent to

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$ ◀

The preceding observations explain how Theorem 5.14 generalizes De Morgan's laws for logic (Example 3.11). Recall that the first of De Morgan's law for logic states that the propositions

$$\neg(P_1 \vee P_2 \vee \cdots \vee P_n) \qquad \text{and} \qquad \neg P_1 \wedge \neg P_2 \wedge \cdots \wedge \neg P_n$$

have the same truth values. In Theorem 5.14, part (b),

$$\neg P_1 \wedge \neg P_2 \wedge \cdots \wedge \neg P_n$$

is replaced by

$$\forall x\, \neg P(x)$$

and

$$\neg(P_1 \vee P_2 \vee \cdots \vee P_n)$$

is replaced by

$$\neg(\exists x\, P(x)).$$

Example 5.20 ▶ Statements in words often have more than one possible interpretation. Consider the well-known quotation from Shakespeare's "The Merchant of Venice":

> All that glitters is not gold.

One possible interpretation of this quotation is: Every object that glitters is not gold. However, this is surely not what Shakespeare intended. The correct interpretation is: Some object that glitters is not gold.

If we let $P(x)$ be the propositional function "x glitters" and $Q(x)$ be the propositional function "x is gold," the first interpretation becomes

$$\forall x(P(x) \to \neg Q(x)), \tag{5.6}$$

and the second interpretation becomes

$$\exists x(P(x) \wedge \neg Q(x)).$$

Using the result of Example 3.13, we see that the truth values of

$$\exists x(P(x) \wedge \neg Q(x))$$

and

$$\exists x \neg(P(x) \to Q(x))$$

are the same. By Theorem 5.14, the truth values of

$$\exists x \neg(P(x) \to Q(x))$$

and

$$\neg(\forall x\, P(x) \to Q(x))$$

are the same. Thus an equivalent way to represent the second interpretation is

$$\neg(\forall x\, P(x) \to Q(x)). \tag{5.7}$$

Comparing (5.6) and (5.7), we see that the ambiguity results from whether the negation applies to $Q(x)$ (the first interpretation) or to the entire statement

$$\forall x(P(x) \to Q(x))$$

(the second interpretation). The correct interpretation of the statement

All that glitters is not gold

results from negating the entire statement.

In positive statements, "any," "all," "each," and "every" have the same meaning. In negative statements, the situation changes:

Not all x satisfy $P(x)$.
Not each x satisfies $P(x)$.
Not every x satisfies $P(x)$.

are considered to have the same meaning as

For some x, $\neg P(x)$;

whereas

Not any x satisfies $P(x)$.
No x satisfies $P(x)$.

mean

For all x, $\neg P(x)$.

See Exercises 57–66 for other examples. ◀

Rules of Inference for Quantified Statements

We conclude this section by introducing some rules of inference for quantified statements and showing how they can be used with rules of inference for propositions (see Section 4).

Suppose that $\forall x\, P(x)$ is true. By Definition 5.4, $P(x)$ is true for every x in D, the domain of discourse. In particular, if d is in D, then $P(d)$ is true. We have shown that the argument

$$\frac{\forall x\, P(x)}{\therefore\ P(d) \text{ if } d \in D}$$

is valid. This rule of inference is called **universal instantiation.** Similar arguments (see Exercises 74–76) justify the other rules of inference listed in Figure 5.1.

Rule of Inference	Name
$\dfrac{\forall x\, P(x)}{\therefore\ P(d) \text{ if } d \in D}$	Universal instantiation
$\dfrac{P(d) \text{ for every } d \in D}{\therefore\ \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\ P(d) \text{ for some } d \in D}$	Existential instantiation
$\dfrac{P(d) \text{ for some } d \in D}{\therefore\ \exists x\, P(x)}$	Existential generalization

Figure 5.1 Rules of inference for quantified statements. The domain of discourse is D.

Example 5.21 ▶ Given that

$$\text{for every positive integer } n,\ n^2 \geq n$$

is true, we may use universal instantiation to conclude that $54^2 \geq 54$ since 54 is a positive integer (i.e., a member of the domain of discourse). ◀

Example 5.22 ▶ Let $P(x)$ denote the propositional function "x owns a laptop computer," where the domain of discourse is the set of students taking MATH 201 (discrete mathematics). Suppose that Taylor, who is taking MATH 201, owns a laptop computer; in symbols, $P(\text{Taylor})$ is true. We may then use existential generalization to conclude that $\exists x\, P(x)$ is true. ◀

Example 5.23 ▶ Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

For every real number x, if x is an integer, then x is a rational number. The number $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

If we let $P(x)$ denote the propositional function "x is an integer" and $Q(x)$ denote the propositional function "x is rational," the argument becomes

$$\begin{array}{l} \forall x \in \mathbf{R}\,(P(x) \to Q(x)) \\ \neg Q(\sqrt{2}) \\ \hline \therefore \neg P(\sqrt{2}) \end{array}$$

Since $\sqrt{2} \in \mathbf{R}$, we may use universal instantiation to conclude $P(\sqrt{2}) \to Q(\pi)$. Combining $P(\sqrt{2}) \to Q(\pi)$ and $\neg Q(\pi)$, we may use modus tollens (see Figure 4.1) to conclude $\neg P(\sqrt{2})$. Thus the argument is valid. ◀

The argument in Example 5.23 is called **universal modus tollens.**

Example 5.24 ▶ We are given these hypotheses: Everyone loves either Microsoft or Apple. Lynn does not love Microsoft. Show that the conclusion, Lynn loves Apple, follows from the hypotheses.

Let $P(x)$ denote the propositional function "x loves Microsoft," and let $Q(x)$ denote the propositional function "x loves Apple." The first hypothesis is $\forall x(P(x) \vee Q(x))$. By universal instantiation, we have $P(\text{Lynn}) \vee Q(\text{Lynn})$. The second hypothesis is $\neg P(\text{Lynn})$. The disjunctive syllogism rule of inference (see Figure 4.1) now gives $Q(\text{Lynn})$, which represents the proposition "Lynn loves Apple." We conclude that the conclusion does follow from the hypotheses. ◀

Problem-Solving Tips

To prove that the universally quantified statement

$$\forall x\, P(x)$$

is true, show that for *every* x in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for a *particular* value x does *not* prove that

$$\forall x\, P(x)$$

is true.

To prove that the existentially quantified statement

$$\exists x\, P(x)$$

is true, find *one* value of x in the domain of discourse for which the proposition $P(x)$ is true. *One* value suffices.

To prove that the universally quantified statement

$$\forall x\, P(x)$$

is false, find *one* value of x (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false.

To prove that the existentially quantified statement

$$\exists x\, P(x)$$

is false, show that for *every* x in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for a *particular* value x does *not* prove that

$$\exists x\, P(x)$$

is false.

Section Review Exercises

1. What is a propositional function?
2. What is a domain of discourse?
3. What is a universally quantified statement?
4. What is a counterexample?
5. What is an existentially quantified statement?
6. State the generalized De Morgan's laws for logic.
7. Explain how to prove that a universally quantified statement is true.
8. Explain how to prove that an existentially quantified statement is true.
9. Explain how to prove that a universally quantified statement is false.
10. Explain how to prove that an existentially quantified statement is false.
11. State the universal instantiation rule of inference.
12. State the universal generalization rule of inference.
13. State the existential instantiation rule of inference.
14. State the existential generalization rule of inference.

Exercises

In Exercises 1–6, tell whether the statement is a propositional function. For each statement that is a propositional function, give a domain of discourse.

1. $(2n + 1)^2$ is an odd integer.
2. Choose an integer between 1 and 10.
3. Let x be a real number.
4. The movie won the Academy Award as the best picture of 1955.
5. $1 + 3 = 4$.
6. There exists x such that $x < y$ (x, y real numbers).

Let $P(n)$ be the propositional function "n divides 77." Write each proposition in Exercises 7–11 in words and tell whether it is true or false. The domain of discourse is $\mathbf{Z}^+$.

7. $P(11)$
8. $P(1)$
9. $P(3)$
10. $\forall n\, P(n)$
11. $\exists n\, P(n)$

Let $P(x)$ be the propositional function "$x \geq x^2$." Tell whether each proposition in Exercises 12–20 is true or false. The domain of discourse is $\mathbf{R}$.

12. $P(1)$
13. $P(2)$
14. $P(1/2)$
15. $\forall x\, P(x)$
16. $\exists x\, P(x)$
17. $\neg(\forall x\, P(x))$
18. $\neg(\exists x\, P(x))$
19. $\forall x \neg P(x)$
20. $\exists x \neg P(x)$

Suppose that the domain of discourse of the propositional function P is $\{1, 2, 3, 4\}$. Rewrite each propositional function in Exercises 21–27 using only negation, disjunction, and conjunction.

21. $\forall x\, P(x)$
22. $\forall x \neg P(x)$
23. $\neg(\forall x\, P(x))$
24. $\exists x\, P(x)$
25. $\exists x \neg P(x)$
26. $\neg(\exists x\, P(x))$
27. $\forall x((x \neq 1) \to P(x))$

Let $P(x)$ denote the statement "x is taking a math course." The domain of discourse is the set of all students. Write each proposition in Exercises 28–33 in words.

28. $\forall x\, P(x)$
29. $\exists x\, P(x)$
30. $\forall x \neg P(x)$
31. $\exists x \neg P(x)$
32. $\neg(\forall x\, P(x))$
33. $\neg(\exists x\, P(x))$
34. Write the negation of each proposition in Exercises 28–33 symbolically and in words.

Let $P(x)$ denote the statement "x is a professional athlete," and let $Q(x)$ denote the statement "x plays soccer." The domain of discourse is the set of all people. Write each proposition in Exercises 35–42 in words. Determine the truth value of each statement.

35. $\forall x\,(P(x) \to Q(x))$
36. $\exists x\,(P(x) \to Q(x))$
37. $\forall x\,(Q(x) \to P(x))$
38. $\exists x\,(Q(x) \to P(x))$
39. $\forall x\,(P(x) \vee Q(x))$
40. $\exists x\,(P(x) \vee Q(x))$
41. $\forall x\,(P(x) \wedge Q(x))$
42. $\exists x\,(P(x) \wedge Q(x))$
43. Write the negation of each proposition in Exercises 35–42 symbolically and in words.

Let $P(x)$ denote the statement "x is an accountant," and let $Q(x)$ denote the statement "x owns a Porsche." Write each statement in Exercises 44–47 symbolically.

44. All accountants own Porsches.
45. Some accountant owns a Porsche.
46. All owners of Porsches are accountants.
47. Someone who owns a Porsche is an accountant.
48. Write the negation of each proposition in Exercises 44–47 symbolically and in words.

Determine the truth value of each statement in Exercises 49–54. The domain of discourse is $\mathbf{R}$. *Justify your answers.*

49. $\forall x(x^2 > x)$
50. $\exists x(x^2 > x)$
51. $\forall x(x > 1 \to x^2 > x)$
52. $\exists x(x > 1 \to x^2 > x)$
53. $\forall x(x > 1 \to x/(x^2 + 1) < 1/3)$
54. $\exists x(x > 1 \to x/(x^2 + 1) < 1/3)$

55. Write the negation of each proposition in Exercises 49–54 symbolically and in words.

56. Could the pseudocode of Example 5.7 be written as follows?

```
for i = 1 to n
  if (¬P(d_i))
    return false
  else
    return true
```

What is the literal meaning of each statement in Exercises 57–66? What is the intended meaning? Clarify each statement by rephrasing it and writing it symbolically.

57. From *Dear Abby:* All men do not cheat on their wives.

58. From the *San Antonio Express-News:* All old things don't covet twenty-somethings.

59. All 74 hospitals did not report every month.

60. Economist Robert J. Samuelson: Every environmental problem is not a tragedy.

61. Comment from a Door County alderman: This is still Door County and we all don't have a degree.

62. Headline over a Martha Stewart column: All lampshades can't be cleaned.

63. Headline in the *New York Times:* A World Where All Is Not Sweetness and Light.

64. Headline over a story about subsidized housing: Everyone can't afford home.

65. George W. Bush: I understand everybody in this country doesn't agree with the decisions I've made.

66. From *Newsweek:* Formal investigations are a sound practice in the right circumstances, but every circumstance is not right.

67. (a) Use a truth table to prove that if p and q are propositions, at least one of $p \to q$ or $q \to p$ is true.

(b) Let $I(x)$ be the propositional function "x is an integer" and let $P(x)$ be the propositional function "x is a positive number." The domain of discourse is $\mathbf{R}$. Determine whether or not the following proof that all integers are positive or all positive real numbers are integers is correct.

By part (a),

$$\forall x\,((I(x) \to P(x)) \lor (P(x) \to I(x)))$$

is true. In words: For all x, if x is an integer, then x is positive; or if x is positive, then x is an integer. Therefore, all integers are positive or all positive real numbers are integers.

68. Prove Theorem 5.14, part (b).

69. Analyze the following comments by film critic Roger Ebert: No good movie is too long. No bad movie is short enough. *Love Actually* is good, but it is too long.

70. Which rule of inference is used in the following argument? Every rational number is of the form p/q, where p and q are integers. Therefore, 9.345 is of the form p/q.

In Exercises 71–73, give an argument using rules of inference to show that the conclusion follows from the hypotheses.

71. Hypotheses: Everyone in the class has a graphing calculator. Everyone who has a graphing calculator understands the trigonometric functions. Conclusion: Ralphie, who is in the class, understands the trigonometric functions.

72. Hypotheses: Ken, a member of the Titans, can hit the ball a long way. Everyone who can hit the ball a long way can make a lot of money. Conclusion: Some member of the Titans can make a lot of money.

73. Hypotheses: Everyone in the discrete mathematics class loves proofs. Someone in the discrete mathematics class has never taken calculus. Conclusion: Someone who loves proofs has never taken calculus.

74. Show that universal generalization (see Figure 5.1) is valid.

75. Show that existential instantiation (see Figure 5.1) is valid.

76. Show that existential generalization (see Figure 5.1) is valid.

6 → Nested Quantifiers

Consider writing the statement

The sum of any two positive real numbers is positive,

symbolically. We first note that since two numbers are involved, we will need two variables, say x and y. The assertion can be restated as: If $x > 0$ and $y > 0$, then $x + y > 0$. The given statement says that the sum of *any* two positive real numbers is positive, so we need two universal quantifiers. If we let $P(x, y)$ denote the expression $(x > 0) \land (y > 0) \to (x + y > 0)$, the given statement can be written symbolically as

$$\forall x \forall y\, P(x, y).$$

In words, for every x and for every y, if $x > 0$ and $y > 0$, then $x + y > 0$. The domain of discourse of the two-variable propositional function P is $\mathbf{R} \times \mathbf{R}$, which means that each variable x and y must belong to the set of real numbers. Multiple quantifiers such as $\forall x \forall y$ are said to be **nested quantifiers.** In this section we explore nested quantifiers in detail.

Example 6.1 ▶ Restate

$$\forall m \exists n (m < n)$$

in words. The domain of discourse is the set $\mathbf{Z} \times \mathbf{Z}$.

We may first rephrase this statement as: For every m, there exists n such that $m < n$. Less formally, this means that if you take any integer m whatsoever, there is an integer n greater than m. Another restatement is then: There is no greatest integer. ◀

Example 6.2 ▶ Write the assertion

Everybody loves somebody,

symbolically, letting $L(x, y)$ be the statement "x loves y."

"Everybody" requires universal quantification and "somebody" requires existential quantification. Thus, the given statement may be written symbolically as

$$\forall x \exists y\, L(x, y).$$

In words, for every person x, there exists a person y such that x loves y.

Notice that

$$\exists x \forall y\, L(x, y)$$

is *not* a correct interpretation of the original statement. This latter statement is: There exists a person x such that for all y, x loves y. Less formally, someone loves everyone. The order of quantifiers is important; changing the order can change the meaning. ◀

By definition, the statement

$$\forall x \forall y\, P(x, y),$$

with domain of discourse $X \times Y$, is true if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is true. The statement

$$\forall x \forall y\, P(x, y)$$

is false if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 6.3 ▶ Consider the statement

$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x and for every real number y, the conditional proposition

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

is true. In words, for every real number x and for every real number y, if x and y are positive, their sum is positive. ◀

Example 6.4 ▶ Consider the statement

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is false because if $x = 1$ and $y = -1$, the conditional proposition

$$(x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)$$

is false. We say that the pair $x = 1$ and $y = -1$ is a counterexample. ◀

Example 6.5 ▶ Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. The following pseudocode determines whether

$$\forall x \forall y\, P(x, y)$$

is true or false:

```
for i = 1 to n
   for j = 1 to n
      if (¬P(d_i, d_j))
         return false
return true
```

The for loops examine members of the domain of discourse. If they find a pair d_i, d_j for which $P(d_i, d_j)$ is false, the condition $\neg P(d_i, d_j)$ in the if statement is true; so the code returns false [to indicate that $\forall x \forall y\, P(x, y)$ is false] and terminates. In this case, the pair d_i, d_j is a counterexample. If $P(d_i, d_j)$ is true for every pair d_i, d_j, the condition $\neg P(d_i, d_j)$ in the if statement is always false. In this case, the for loops run to completion, after which the code returns true [to indicate that $\forall x \forall y\, P(x, y)$ is true] and terminates. ◀

By definition, the statement

$$\forall x \exists y\, P(x, y),$$

with domain of discourse $X \times Y$, is true if, for *every* $x \in X$, there is *at least one* $y \in Y$ for which $P(x, y)$ is true. The statement

$$\forall x \exists y\, P(x, y)$$

is false if there is *at least one* $x \in X$ such that $P(x, y)$ is false for *every* $y \in Y$.

Example 6.6 ▶ Consider the statement

$$\forall x \exists y (x + y = 0).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x, there is at least one y (namely $y = -x$) for which $x + y = 0$ is true. In words, for every real number x, there is a number that when added to x makes the sum zero. ◀

Example 6.7 ▶ Consider the statement

$$\forall x \exists y (x > y).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because there is at least one x, namely $x = 1$, such that $x > y$ is false for every positive integer y. ◀

Example 6.8 ▶

Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. The following pseudocode determines whether

$$\forall x \exists y\, P(x, y)$$

is true or false:

```
for i = 1 to n
   if (¬ exists_dj(i))
      return false
return true
exists_dj(i) {
   for j = 1 to n
      if (P(d_i, d_j))
         return true
   return false
}
```

If for each d_i, there exists d_j such that $P(d_i, d_j)$ is true, then for each i, $P(d_i, d_j)$ is true for some j. Thus, $exists\_dj(i)$ returns true for every i. Since $\neg\, exists\_dj(i)$ is always false, the first for loop eventually terminates and true is returned to indicate that $\forall x \exists y\, P(x, y)$ is true.

If for some d_i, $P(d_i, d_j)$ is false for every j, then, for this i, $P(d_i, d_j)$ is false for every j. In this case, the for loop in $exists\_dj(i)$ runs to termination and false is returned. Since $\neg\, exists\_dj(i)$ is true, false is returned to indicate that $\forall x \exists y\, P(x, y)$ is false. ◀

By definition, the statement

$$\exists x \forall y\, P(x, y),$$

with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ such that $P(x, y)$ is true for *every* $y \in Y$. The statement

$$\exists x \forall y\, P(x, y)$$

is false if, for *every* $x \in X$, there is *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 6.9 ▶

Consider the statement

$$\exists x \forall y (x \leq y).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one positive integer x (namely $x = 1$) for which $x \leq y$ is true for every positive integer y. In words, there is a smallest positive integer (namely 1). ◀

Example 6.10 ▶

Consider the statement

$$\exists x \forall y (x \geq y).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because, for every positive integer x, there is at least one positive integer y, namely $y = x + 1$, such that $x \geq y$ is false. In words, there is no greatest positive integer. ◀

By definition, the statement

$$\exists x \exists y\, P(x, y),$$

with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is true. The statement

$$\exists x \exists y\, P(x, y)$$

is false if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is false.

Example 6.11 ▶ Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 6)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one integer $x > 1$ (namely $x = 2$) and at least one integer $y > 1$ (namely $y = 3$) such that $xy = 6$. In words, 6 is composite (i.e., not prime). ◀

Example 6.12 ▶ Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 7)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because for every positive integer x and for every positive integer y,

$$(x > 1) \wedge (y > 1) \wedge (xy = 7)$$

is false. In words, 7 is prime. ◀

The generalized De Morgan's laws for logic (Theorem 5.14) can be used to negate a proposition containing nested quantifiers.

Example 6.13 ▶ Using the generalized De Morgan's laws for logic, we find that the negation of

$$\forall x \exists y\, P(x, y)$$

is

$$\neg(\forall x \exists y\, P(x, y)) \equiv \exists x \neg(\exists y\, P(x, y)) \equiv \exists x \forall y\, \neg P(x, y).$$

Notice how in the negation, $\forall$ and $\exists$ are interchanged. ◀

Example 6.14 ▶ Write the negation of $\exists x \forall y (xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

Using the generalized De Morgan's laws for logic, we find that the negation is

$$\neg(\exists x \forall y (xy < 1)) \equiv \forall x \neg(\forall y (xy < 1)) \equiv \forall x \exists y \neg(xy < 1) \equiv \forall x \exists y (xy \geq 1).$$

The given statement $\exists x \forall y (xy < 1)$ is true because there is at least one x (namely $x = 0$) such that $xy < 1$ for every y. Since the given statement is true, its negation is false. ◀

We conclude with a logic game, which presents an alternative way to determine whether a quantified propositional function is true or false. André Berthiaume contributed this example.

Example 6.15 ▶ **The Logic Game**

Given a quantified propositional function such as

$$\forall x \exists y\, P(x, y),$$

you and your opponent, whom we call Farley, play a logic game. Your goal is to try to make $P(x, y)$ true, and Farley's goal is to try to make $P(x, y)$ false. The game begins with the first (left) quantifier. If the quantifier is $\forall$, Farley chooses a value for that variable; if the quantifier is $\exists$, you choose a value for that variable. The game continues with the second quantifier. After values are chosen for all the variables, if $P(x, y)$ is true, you win; if $P(x, y)$ is false, Farley wins. We will show that if you can always win regardless of how Farley chooses values for the variables, the quantified statement is true, but if Farley can choose values for the variables so that you cannot win, the quantified statement is false.

Consider the statement

$$\forall x \exists y (x + y = 0).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\forall$, Farley goes first and chooses a value for x. Since the second quantifier is $\exists$, you go second. Regardless of what value Farley chose, you can choose $y = -x$, which makes the statement $x + y = 0$ true. You can always win the game, so the statement

$$\forall x \exists y (x + y = 0)$$

is true.

Next, consider the statement

$$\exists x \forall y (x + y = 0).$$

Again, the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\exists$, you go first and choose a value for x. Since the second quantifier is $\forall$, Farley goes second. Regardless of what value you chose, Farley can always choose a value for y, which makes the statement $x + y = 0$ false. (If you choose $x = 0$, Farley can choose $y = 1$. If you choose $x \neq 0$, Farley can choose $y = 0$.) Farley can always win the game, so the statement

$$\exists x \forall y (x + y = 0)$$

is false.

We discuss why the game correctly determines the truth value of a quantified propositional function. Consider

$$\forall x \forall y\, P(x, y).$$

If Farley can always win the game, this means that Farley can find values for x and y that make $P(x, y)$ false. In this case, the propositional function is false; the values Farley found provide a counterexample. If Farley cannot win the game, no counterexample exists; in this case, the propositional function is true.

Consider

$$\forall x \exists y\, P(x, y).$$

Farley goes first and chooses a value for x. You choose second. If, no matter what value Farley chose, you can choose a value for y that makes $P(x, y)$ true, you can always win the game and the propositional function is true. However, if Farley can choose a value for x so that every value you choose for y makes $P(x, y)$ false, then you will always lose the game and the propositional function is false.

An analysis of the other cases also shows that if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false.

The logic game extends to propositional functions of more than two variables. The rules are the same and, again, if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false. ◀

Problem-Solving Tips

To prove that

$$\forall x \forall y\, P(x, y)$$

is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols x and y to stand for *arbitrary* elements in X and Y.

To prove that

$$\forall x \forall y\, P(x, y)$$

is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ false.

To prove that

$$\forall x \exists y\, P(x, y)$$

is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.

To prove that

$$\forall x \exists y\, P(x, y)$$

is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always false.

To prove that

$$\exists x \forall y\, P(x, y)$$

is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always true.

To prove that

$$\exists x \forall y\, P(x, y)$$

is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.

To prove that

$$\exists x \exists y\, P(x, y)$$

is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ true.

To prove that

$$\exists x \exists y\, P(x, y)$$

is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols x and y to stand for *arbitrary* elements in X and Y.

To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \to q$ is equivalent to $p \wedge \neg q$.

Section Review Exercises

1. What is the interpretation of $\forall x \forall y P(x, y)$? When is this quantified expression true? When is it false?
2. What is the interpretation of $\forall x \exists y P(x, y)$? When is this quantified expression true? When is it false?
3. What is the interpretation of $\exists x \forall y P(x, y)$? When is this quantified expression true? When is it false?
4. What is the interpretation of $\exists x \exists y P(x, y)$? When is this quantified expression true? When is it false?
5. Give an example to show that, in general, $\forall x \exists y P(x, y)$ and $\exists x \forall y P(x, y)$ have different meanings.
6. Write the negation of $\forall x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.
7. Write the negation of $\forall x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.
8. Write the negation of $\exists x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.
9. Write the negation of $\exists x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.
10. Explain the rules for playing the logic game. How can the logic game be used to determine the truth value of a quantified expression?

Exercises

In Exercises 1–27, the set D_1 consists of three students: Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_2 consists of four students: Dale, who is 6 feet tall; Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_3 consists of one student: Dale, who is 6 feet tall. The set D_4 consists of three students: Pat, Sandy, and Gale, each of whom is 5 feet 11 inches tall.

In Exercises 1–9, $T_1(x, y)$ is the propositional function "x is taller than y." Write each proposition in Exercises 1–4 in words.

1. $\forall x \forall y\, T_1(x, y)$
2. $\forall x \exists y\, T_1(x, y)$
3. $\exists x \forall y\, T_1(x, y)$
4. $\exists x \exists y\, T_1(x, y)$
5. Write the negation of each proposition in Exercises 1–4 in words and symbolically.
6. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_1 \times D_1$.
7. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_2 \times D_2$.
8. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_3 \times D_3$.
9. Tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_4 \times D_4$.

In Exercises 10–18, $T_2(x, y)$ is the propositional function "x is taller than or the same height as y." Write each proposition in Exercises 10–13 in words.

10. $\forall x \forall y\, T_2(x, y)$
11. $\forall x \exists y\, T_2(x, y)$
12. $\exists x \forall y\, T_2(x, y)$
13. $\exists x \exists y\, T_2(x, y)$
14. Write the negation of each proposition in Exercises 10–13 in words and symbolically.

15. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is $D_1 \times D_1$. (The set D_1 is defined before Exercise 1.)

16. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is $D_2 \times D_2$. (The set D_2 is defined before Exercise 1.)

17. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is $D_3 \times D_3$. (The set D_3 is defined before Exercise 1.)

18. Tell whether each proposition in Exercises 10–13 is true or false if the domain of discourse is $D_4 \times D_4$. (The set D_4 is defined before Exercise 1.)

In Exercises 19–27, $T_3(x, y)$ is the propositional function "if x and y are distinct persons, then x is taller than y." Write each proposition in Exercises 19–22 in words.

19. $\forall x \forall y\ T_3(x, y)$

20. $\forall x \exists y\ T_3(x, y)$

21. $\exists x \forall y\ T_3(x, y)$

22. $\exists x \exists y\ T_3(x, y)$

23. Write the negation of each proposition in Exercises 19–22 in words and symbolically.

24. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is $D_1 \times D_1$. (The set D_1 is defined before Exercise 1.)

25. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is $D_2 \times D_2$. (The set D_2 is defined before Exercise 1.)

26. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is $D_3 \times D_3$. (The set D_3 is defined before Exercise 1.)

27. Tell whether each proposition in Exercises 19–22 is true or false if the domain of discourse is $D_4 \times D_4$. (The set D_4 is defined before Exercise 1.)

Let $L(x, y)$ be the propositional function "x loves y." The domain of discourse is the Cartesian product of the set of all living people with itself (i.e., both x and y take on values in the set of all living people). Write each proposition in Exercises 28–31 symbolically. Which do you think are true?

28. Someone loves everybody.

29. Everybody loves everybody.

30. Somebody loves somebody.

31. Everybody loves somebody.

32. Write the negation of each proposition in Exercises 28–31 in words and symbolically.

Let $A(x, y)$ be the propositional function "x attended y's office hours" and let $E(x)$ be the propositional function "x is enrolled in a discrete math class." Let S be the set of students and let T denote the set of teachers—all at Hudson University. The domain of discourse of A is $S \times T$ and the domain of discourse of E is S. Write each proposition in Exercises 33–36 symbolically.

33. Brit attended someone's office hours.

34. No one attended Professor Sandwich's office hours.

35. Every discrete math student attended someone's office hours.

36. All teachers had at least one student attend their office hours.

Let $P(x, y)$ be the propositional function $x \geq y$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. Tell whether each proposition in Exercises 37–40 is true or false.

37. $\forall x \forall y\ P(x, y)$

38. $\forall x \exists y\ P(x, y)$

39. $\exists x \forall y\ P(x, y)$

40. $\exists x \exists y\ P(x, y)$

41. Write the negation of each proposition in Exercises 37–40.

Determine the truth value of each statement in Exercises 42–59. The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Justify your answers.

42. $\forall x \forall y (x^2 < y + 1)$

43. $\forall x \exists y (x^2 < y + 1)$

44. $\exists x \forall y (x^2 < y + 1)$

45. $\exists x \exists y (x^2 < y + 1)$

46. $\exists y \forall x (x^2 < y + 1)$

47. $\forall y \exists x (x^2 < y + 1)$

48. $\forall x \forall y (x^2 + y^2 = 9)$

49. $\forall x \exists y (x^2 + y^2 = 9)$

50. $\exists x \forall y (x^2 + y^2 = 9)$

51. $\exists x \exists y (x^2 + y^2 = 9)$

52. $\forall x \forall y (x^2 + y^2 \geq 0)$

53. $\forall x \exists y (x^2 + y^2 \geq 0)$

54. $\exists x \forall y (x^2 + y^2 \geq 0)$

55. $\exists x \exists y (x^2 + y^2 \geq 0)$

56. $\forall x \forall y ((x < y) \to (x^2 < y^2))$

57. $\forall x \exists y ((x < y) \to (x^2 < y^2))$

58. $\exists x \forall y ((x < y) \to (x^2 < y^2))$

59. $\exists x \exists y ((x < y) \to (x^2 < y^2))$

60. Write the negation of each proposition in Exercises 42–59.

61. Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. Write pseudocode that determines whether

$$\exists x \forall y\ P(x, y)$$

is true or false.

62. Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. Write pseudocode that determines whether

$$\exists x \exists y\ P(x, y)$$

is true or false.

63. Explain how the logic game (Example 6.15) determines whether each proposition in Exercises 42–59 is true or false.

64. Use the logic game (Example 6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z > x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

65. Use the logic game (Example 6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z < x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

66. Use the logic game (Example 6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((x < y) \to ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

67. Use the logic game (Example 6.15) to determine whether the proposition

$$\forall x \forall y \exists z((x < y) \to ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

Assume that $\forall x \forall y\ P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 68–70 must also be true? Prove your answer.

68. $\forall x \exists y\ P(x, y)$ **69.** $\exists x \forall y\ P(x, y)$ **70.** $\exists x \exists y\ P(x, y)$

Assume that $\exists x \forall y\ P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 71–73 must also be true? Prove your answer.

71. $\forall x \forall y\ P(x, y)$ **72.** $\forall x \exists y\ P(x, y)$ **73.** $\exists x \exists y\ P(x, y)$

Assume that $\exists x \exists y\ P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 74–76 must also be true? Prove your answer.

74. $\forall x \forall y\ P(x, y)$ **75.** $\forall x \exists y\ P(x, y)$ **76.** $\exists x \forall y\ P(x, y)$

Assume that $\forall x \forall y\ P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 77–79 must also be false? Prove your answer.

77. $\forall x \exists y\ P(x, y)$ **78.** $\exists x \forall y\ P(x, y)$ **79.** $\exists x \exists y\ P(x, y)$

Assume that $\forall x \exists y\ P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 80–82 must also be false? Prove your answer.

80. $\forall x \forall y\ P(x, y)$

81. $\exists x \forall y\ P(x, y)$

82. $\exists x \exists y\ P(x, y)$

Assume that $\exists x \forall y\ P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 83–85 must also be false? Prove your answer.

83. $\forall x \forall y\ P(x, y)$ **84.** $\forall x \exists y\ P(x, y)$ **85.** $\exists x \exists y\ P(x, y)$

Assume that $\exists x \exists y\ P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 86–88 must also be false? Prove your answer.

86. $\forall x \forall y\ P(x, y)$ **87.** $\forall x \exists y\ P(x, y)$ **88.** $\exists x \forall y\ P(x, y)$

Which of Exercises 89–92 is logically equivalent to $\neg(\forall x \exists y\ P(x, y))$? Explain.

89. $\exists x \neg(\forall y\ P(x, y))$ **90.** $\forall x \neg(\exists y\ P(x, y))$

91. $\exists x \forall y \neg P(x, y)$ 92. $\exists x \exists y \neg P(x, y)$

93. [Requires calculus] The definition of

$$\lim_{x \to a} f(x) = L$$

is: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Write this definition symbolically using $\forall$ and $\exists$.

94. [Requires calculus] Write the negation of the definition of limit (see Exercise 93) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

★95. [Requires calculus] Write the definition of "$\lim_{x \to a} f(x)$ does not exist" (see Exercise 93) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

96. Consider the headline: Every school may not be right for every child. What is the literal meaning? What is the intended meaning? Clarify the headline by rephrasing it and writing it symbolically.

Problem-Solving Corner Quantifiers

Problem

Assume that $\forall x \exists y\ P(x, y)$ is true and that the domain of discourse is nonempty. Which of the following must also be true? If the statement is true, explain; otherwise, give a counterexample.

(a) $\forall x \forall y\ P(x, y)$

(b) $\exists x \forall y\ P(x, y)$

(c) $\exists x \exists y\ P(x, y)$

Attacking the Problem

Let's begin with part (a). We are given that $\forall x \exists y\ P(x, y)$ is true, which says, in words, for every x, there exists at least one y for which $P(x, y)$ is true. If (a) is also true, then, in words, for every x, *for every* y, $P(x, y)$ is true. Let the words sink in. If for every x, $P(x, y)$ is true for *at least one* y, doesn't it seem unlikely that it would follow that $P(x, y)$ is true for *every* y? We suspect that (a) could be false. We'll need to come up with a counterexample.

Contrasting statement (b) with the given statement, we see that the quantifiers $\forall$ and $\exists$ have been swapped. There is a difference. In the given true statement $\forall x \exists y\ P(x, y)$, given *any* x, it's possible to find a y, which may depend on x, that makes $P(x, y)$ true. For statement (b), $\exists x \forall y\ P(x, y)$, to be true, for some x, $P(x, y)$ would need to be true for every y. Again, let the words sink in. These two statements seem quite different. We suspect that (b) also could be false. Again, we'll need to come up with a counterexample.

Now let's turn to part (c). We are given that $\forall x \exists y\ P(x, y)$ is true, which says, in words, for every x, there exists at least one y for which $P(x, y)$ is true.

For statement (c), $\exists x \exists y\, P(x, y)$, to be true, for some x and for some y, $P(x, y)$ must be true. But the given statement says that for *every* x, there exists at least one y for which $P(x, y)$ is true. So if we pick one x (and we know we can since the domain of discourse is nonempty), the given statement assures us that there exists at least one y for which $P(x, y)$ is true. Thus part (c) must be true. In fact, we have just given an explanation!

Finding a Solution

As noted, we have already solved part (c). We need counterexamples for parts (a) and (b).

For part (a), we need the given statement, $\forall x \exists y\, P(x, y)$, to be true and $\forall x \forall y\, P(x, y)$ to be false. In order for the given statement to be true, we must find a propositional function $P(x, y)$ satisfying

for every x, there exists y such that $P(x, y)$ is true. **(1)**

In order for (a) to be false, we must have

at least one value of x and at least one value of y such that $P(x, y)$ is false. **(2)**

We can arrange for (1) and (2) to hold simultaneously if we choose $P(x, y)$ so that for every x, $P(x, y)$ is true for some y, *but* for at least one x, $P(x, y)$ is also false for some other value of y. Upon reflection, many mathematical statements have this property. For example, $x > y$, $x, y \in \mathbf{R}$, suffices. For every x, there exists y such that $x > y$ is true. Furthermore, for *every* x (and, in particular, for at least one value of x), there exists y such that $x > y$ is false.

For part (b), we again need the given statement, $\forall x \exists y\, P(x, y)$, to be true and $\exists x \forall y\, P(x, y)$ to be false. In order for the given statement to be true, we must find a propositional function $P(x, y)$ satisfying (1). In order for (b) to be false, we must have

for every x, there exists at least one value of y such that $P(x, y)$ is false. **(3)**

We can arrange for (1) and (3) to hold simultaneously if we choose $P(x, y)$ so that for every x, $P(x, y)$ is true for some y *and* false for some other value of y. We noted in the preceding paragraph that $x > y$, $x, y \in \mathbf{R}$, has this property.

Formal Solution

(a) We give an example to show that statement (a) can be false while the given statement is true. Let $P(x, y)$ be the propositional function $x > y$ with domain of discourse $\mathbf{R} \times \mathbf{R}$. Then

$$\forall x \exists y\, P(x, y)$$

is true since for any x, we may choose $y = x - 1$ to make $P(x, y)$ true. At the same time,

$$\forall x \forall y\, P(x, y)$$

is false. A counterexample is $x = 0$, $y = 1$.

(b) We give an example to show that statement (b) can be false while the given statement is true. Let $P(x, y)$ be the propositional function $x > y$ with domain of discourse $\mathbf{R} \times \mathbf{R}$. As we showed in part (a),

$$\forall x \exists y\, P(x, y)$$

is true. Now we show that

$$\exists x \forall y\, P(x, y)$$

is false. Let x be an arbitrary element in $\mathbf{R}$. We may choose $y = x + 1$ to make $x > y$ false. Thus for every x, there exists y such that $P(x, y)$ is false. Therefore statement (b) is false.

(c) We show that if the given statement is true, statement (c) is necessarily true.

We are given that for every x, there exists y such that $P(x, y)$ is true. We must show that there exist x and y such that $P(x, y)$ is true. Since the domain of discourse is nonempty, we may choose a value for x. For this chosen x, there exists y such that $P(x, y)$ is true. We have found at least one value for x and at least one value for y that make $P(x, y)$ true. Therefore

$$\exists x \exists y\, P(x, y)$$

is true.

Summary of Problem-Solving Techniques

- When dealing with quantified statements, it is sometimes useful to write out the statements in words. For example, in this problem, it helped to write out exactly what $\forall x \exists y\, P(x, y)$ means. Take time to let the words sink in.
- If you have trouble finding examples, look at existing examples (e.g., examples in this book). To solve problems (a) and (b), we could have used the statement in Example 6.6. Sometimes, an existing example can be modified to solve a given problem.

Exercises

1. Show that the statement in Example 6.6 solves problems (a) and (b) in this Problem-Solving Corner.

2. Could examples in Section 6 other than Example 6.6 have been used to solve problems (a) and (b) in this Problem-Solving Corner?

Notes

General references on discrete mathematics are [Graham, 1994; Liu, 1985; Tucker]. [Knuth, 1997, 1998a, 1998b] is the classic reference for much of this material.

[Halmos; Lipschutz; and Stoll] are recommended to the reader wanting to study set theory in more detail.

[Barker; Copi; Edgar] are introductory logic textbooks. A more advanced treatment is found in [Davis]. The first chapter of the geometry book by [Jacobs] is devoted to basic logic. For a history of logic, see [Kline]. The role of logic in reasoning about computer programs is discussed by [Gries].

Chapter Review

Section 1

1. Set: any collection of objects
2. Notation for sets: $\{x \mid x \text{ has property } P\}$
3. $|X|$, the cardinality of X: the number of elements in the set X
4. $x \in X$: x is an element of the set X
5. $x \notin X$: x is not an element of the set X
6. Empty set: $\varnothing$ or $\{\ \}$
7. $X = Y$, where X and Y are sets: X and Y have the same elements
8. $X \subseteq Y$, X is a subset of Y: every element in X is also in Y
9. $X \subset Y$, X is a proper subset of Y: $X \subseteq Y$ and $X \neq Y$
10. $\mathcal{P}(X)$, the power set of X: set of all subsets of X
11. $|\mathcal{P}(X)| = 2^{|X|}$
12. $X \cup Y$, X union Y: set of elements in X *or* Y or both
13. Union of a family $\mathcal{S}$ of sets: $\cup \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}$
14. $X \cap Y$, X intersect Y: set of elements in X *and* Y
15. Intersection of a family $\mathcal{S}$ of sets: $\cap \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}$
16. Disjoint sets X and Y: $X \cap Y = \varnothing$
17. Pairwise disjoint family of sets
18. $X - Y$, difference of X and Y, relative complement: set of elements in X but not in Y
19. Universal set, universe
20. $\overline{X}$, complement of X: $U - X$, where U is a universal set
21. Venn diagram
22. Properties of sets (see Theorem 1.21)
23. De Morgan's laws for sets: $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$, $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
24. Partition of X: a collection $\mathcal{S}$ of nonempty subsets of X such that every element in X belongs to exactly one member of $\mathcal{S}$
25. Ordered pair: (x, y)
26. Cartesian product of X and Y: $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
27. Cartesian product of $X_1, X_2, \ldots, X_n$:

$$X_1 \times X_2 \times \cdots \times X_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in X_i\}$$

Section 2

28. Logic
29. Proposition
30. Conjunction: p and q, $p \wedge q$
31. Disjunction: p or q, $p \vee q$
32. Negation: not p, $\neg p$
33. Truth table
34. Exclusive-or of propositions p, q: p or q, but not both

Section 3

35. Conditional proposition: if p, then q; $p \to q$
36. Hypothesis
37. Conclusion
38. Necessary condition
39. Sufficient condition
40. Converse of $p \to q$: $q \to p$
41. Biconditional proposition: p if and only if q, $p \leftrightarrow q$
42. Logical equivalence: $P \equiv Q$
43. De Morgan's laws for logic: $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$
44. Contrapositive of $p \to q$: $\neg q \to \neg p$

Section 4

45. Deductive reasoning
46. Hypothesis
47. Premises
48. Conclusion
49. Argument
50. Valid argument
51. Invalid argument
52. Rules of inference for propositions: modus ponens, modus tollens, addition, simplification, conjunction, hypothetical syllogism, disjunctive syllogism

Section 5

53. Propositional function
54. Domain of discourse

55. Universal quantifier

56. Universally quantified statement

57. Counterexample

58. Existential quantifier

59. Existentially quantified statement

60. Generalized De Morgan's laws for logic:

$$\neg(\forall x P(x)) \text{ and } \exists x \neg P(x) \text{ have the same truth values.}$$
$$\neg(\exists x P(x)) \text{ and } \forall x \neg P(x) \text{ have the same truth values.}$$

61. To prove that the universally quantified statement

$$\forall x\, P(x)$$

is true, show that for every x in the domain of discourse, the proposition $P(x)$ is true.

62. To prove that the existentially quantified statement

$$\exists x\, P(x)$$

is true, find one value of x in the domain of discourse for which $P(x)$ is true.

63. To prove that the universally quantified statement

$$\forall x\, P(x)$$

is false, find one value of x (a counterexample) in the domain of discourse for which $P(x)$ is false.

64. To prove that the existentially quantified statement

$$\exists x\, P(x)$$

is false, show that for every x in the domain of discourse, the proposition $P(x)$ is false.

65. Rules of inference for quantified statements: universal instantiation, universal generalization, existential instantiation, existential generalization

Section 6

66. To prove that

$$\forall x \forall y\, P(x, y)$$

is true, show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$, where the domain of discourse is $X \times Y$.

67. To prove that

$$\forall x \exists y\, P(x, y)$$

is true, show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true, where the domain of discourse is $X \times Y$.

68. To prove that

$$\exists x \forall y\, P(x, y)$$

is true, show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$, where the domain of discourse is $X \times Y$.

69. To prove that

$$\exists x \exists y\, P(x, y)$$

is true, find one value of $x \in X$ and one value of $y \in Y$ that make $P(x, y)$ true, where the domain of discourse is $X \times Y$.

70. To prove that

$$\forall x \forall y\, P(x, y)$$

is false, find one value of $x \in X$ and one value of $y \in Y$ that make $P(x, y)$ false, where the domain of discourse is $X \times Y$.

71. To prove that

$$\forall x \exists y\, P(x, y)$$

is false, show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$, where the domain of discourse is $X \times Y$.

72. To prove that

$$\exists x \forall y\, P(x, y)$$

is false, show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false, where the domain of discourse is $X \times Y$.

73. To prove that

$$\exists x \exists y\, P(x, y)$$

is false, show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$, where the domain of discourse is $X \times Y$.

74. To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic.

75. The logic game

Chapter Self-Test

Section 1

1. If $A = \{1, 3, 4, 5, 6, 7\}$, $B = \{x \mid x \text{ is an even integer}\}$, $C = \{2, 3, 4, 5, 6\}$, find $(A \cap B) - C$.
2. If $A \cup B = B$, what relation must hold between A and B?
3. Are the sets

$$\{3, 2, 2\}, \qquad \{x \mid x \text{ is an integer and } 1 < x \leq 3\}$$

equal? Explain.

4. If $A = \{a, b, c\}$, how many elements are in $\mathcal{P}(A) \times A$?

Section 2

5. If p, q, and r are true, find the truth value of the proposition $(p \vee q) \wedge \neg((\neg p \wedge r) \vee q)$.
6. Write the truth table of the proposition $\neg(p \wedge q) \vee (p \vee \neg r)$.
7. Formulate the proposition $p \wedge (\neg q \vee r)$ in words using

p: I take hotel management.
q: I take recreation supervision.
r: I take popular culture.

8. Assume that $a, b,$ and c are real numbers. Represent the statement

$$a < b \text{ or } (b < c \text{ and } a \geq c)$$

symbolically, letting

$$p\colon\ a < b, \qquad q\colon\ b < c, \qquad r\colon\ a < c.$$

Section 3

9. Restate the proposition "A necessary condition for Leah to get an A in discrete mathematics is to study hard" in the form of a conditional proposition.
10. Write the converse and contrapositive of the proposition of Exercise 9.
11. If p is true and q and r are false, find the truth value of the proposition

$$(p \vee q) \rightarrow \neg r.$$

12. Represent the statement

$$\text{If } (a \geq c \text{ or } b < c), \text{ then } b \geq c$$

symbolically using the definitions of Exercise 8.

Section 4

13. Which rule of inference is used in the following argument? If the Skyscrapers win, I'll eat my hat. If I eat my hat, I'll be quite full. Therefore, if the Skyscrapers win, I'll be quite full.
14. Write the following argument symbolically and determine whether it is valid. If the Skyscrapers win, I'll eat my hat. If I eat my hat, I'll be quite full. Therefore, if I'm quite full, the Skyscrapers won.
15. Determine whether the following argument is valid.

$$\begin{array}{l} p \rightarrow q \vee r \\ p \vee \neg q \\ \underline{r \vee q} \\ \therefore q \end{array}$$

16. Give an argument using rules of inference to show that the conclusion follows from the hypotheses.

Hypotheses: If the Council approves the funds, then New Atlantic will get the Olympic Games. If New Atlantic gets the Olympic Games, then New Atlantic will build a new stadium. New Atlantic does not build a new stadium. Conclusion: The Council does not approve the funds, or the Olympic Games are canceled.

Section 5

17. Is the statement

The team won the 2006 National Basketball Association championship

a proposition? Explain.

18. Is the statement of Exercise 17 a propositional function? Explain.

Let $P(n)$ be the statement

$$n \text{ and } n+2 \text{ are prime.}$$

In Exercises 19 and 20, write the statement in words and tell whether it is true or false.

19. $\forall n\ P(n)$
20. $\exists n\ P(n)$

Section 6

21. Let $K(x, y)$ be the propositional function "x knows y." The domain of discourse is the Cartesian product of the set of students taking discrete math with itself (i.e., both x and y take on values in the set of students taking discrete math). Represent the assertion "someone does not know anyone" symbolically.
22. Write the negation of the assertion of Exercise 21 symbolically and in words.
23. Determine whether the statement

$$\forall x \exists y (x = y^3)$$

is true or false. The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Explain your answer. Explain, in words, the meaning of the statement.

24. Use the generalized De Morgan's laws for logic to write the negation of

$$\forall x \exists y \forall z\ P(x, y, z).$$

Computer Exercises

In Exercises 1–6, assume that a set X of n elements is represented as an array A of size at least $n+1$. The elements of X are listed consecutively in A starting in the first position and terminating with 0. Assume further that no set contains 0.

1. Write a program to represent the sets $X \cup Y$, $X \cap Y$, $X - Y$, and $X \,\Delta\, Y$, given the arrays representing X and Y. (The symmetric difference is denoted Δ.)
2. Write a program to determine whether $X \subseteq Y$, given arrays representing X and Y.
3. Write a program to determine whether $X = Y$, given arrays representing X and Y.
4. Assuming a universe represented as an array, write a program to represent the set $\overline{X}$, given the array representing X.

5. Given an element E and the array A that represents X, write a program that determines whether $E \in X$.
6. Given the array representing X, write a program that lists all subsets of X.
7. Write a program that reads a logical expression in p and q and prints the truth table of the expression.
8. Write a program that reads a logical expression in p, q, and r and prints the truth table of the expression.
9. Write a program that tests whether two logical expressions in p and q are logically equivalent.
10. Write a program that tests whether two logical expressions in p, q, and r are logically equivalent.

Hints/Solutions to Selected Exercises

Section 1 Review

1. A set is a collection of objects.
2. A set may be defined by listing the elements in it. For example, $\{1, 2, 3, 4\}$ is the set consisting of the integers 1, 2, 3, 4. A set may also be defined by listing a property necessary for membership. For example,

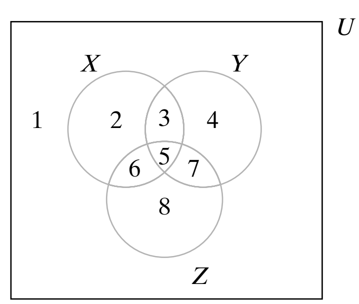
$$\{x \mid x \text{ is a positive, real number}\}$$

defines the set consisting of the positive, real numbers.

3.

Set	*Description*	*Examples of Members*
Z	Integers	$-3, 2$
Q	Rational numbers	$-3/4$, 2.13074
R	Real numbers	-2.13074, $\sqrt{2}$
$\mathbf{Z}^+$	Positive integers	2, 10
$\mathbf{Q}^+$	Positive rational numbers	$3/4$, 2.13074
$\mathbf{R}^+$	Positive real numbers	2.13074, $\sqrt{2}$
$\mathbf{Z}^-$	Negative integers	$-12, -10$
$\mathbf{Q}^-$	Negative rational numbers	$-3/8, -2.13074$
$\mathbf{R}^-$	Negative real numbers	$-2.13074, -\sqrt{2}$
$\mathbf{Z}^{nonneg}$	Nonnegative integers	0, 3
$\mathbf{Q}^{nonneg}$	Nonnegative rational numbers	0, 3.13074
$\mathbf{R}^{nonneg}$	Nonnegative real numbers	0, $\sqrt{3}$

4. The cardinality of X (i.e., the number of elements in X)
5. $x \in X$
6. $x \notin X$
7. $\varnothing$
8. Sets X and Y are equal if they have the same elements. Set equality is denoted $X = Y$.
9. Prove that for every x, if x is in X, then x is in Y, and if x is in Y, then x is in X.
10. Prove one of the following: (a) There exists x such that $x \in X$ and $x \notin Y$. (b) There exists x such that $x \notin X$ and $x \in Y$.
11. X is a subset of Y if every element of X is an element of Y. X is a subset of Y is denoted $X \subseteq Y$.
12. To prove that X is a subset of Y, let x be an arbitrary element of X and prove that x is in Y.
13. Find x such that x is in X, but x is not in Y.
14. X is a proper subset of Y if $X \subseteq Y$ and $X \neq Y$. X is a proper subset of Y is denoted $X \subset Y$.
15. To prove that X is a proper subset of Y, prove that X is a subset of Y and find x in Y such that x is not in X.
16. The power set of X is the collection of all subsets of X. It is denoted $\mathcal{P}(X)$.
17. *X union Y* is the set of elements that belong to either X or Y or both. It is denoted $X \cup Y$.
18. The union of $\mathcal{S}$ is the set of elements that belong to at least one set in $\mathcal{S}$. It is denoted $\cup\mathcal{S}$.
19. *X intersect Y* is the set of elements that belong to both X and Y. It is denoted $X \cap Y$.
20. The intersection of $\mathcal{S}$ is the set of elements that belong to every set in $\mathcal{S}$. It is denoted $\cap\mathcal{S}$.
21. $X \cap Y = \varnothing$
22. A collection of sets $\mathcal{S}$ is pairwise disjoint if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.
23. The difference of X and Y is the set of elements that are in X but not in Y. It is denoted $X - Y$.
24. A universal set is a set that contains all of the sets under discussion.
25. The complement of X is $U - X$, where U is a given universal set. The complement of X is denoted $\overline{X}$.
26. A Venn diagram provides a pictorial view of sets. In a Venn diagram, a rectangle depicts a universal set, and subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set.
27.



Region 1 represents elements in none of X, Y, or Z. Region 2 represents elements in X, but in neither Y nor Z. Region 3 represents elements in X and Y, but not in Z. Region 4 represents elements in Y, but in neither X nor Z. Region 5 represents elements in X, Y, and Z. Region 6 represents elements in X and Z, but not in Y. Region 7 represents elements in Y and Z, but not in X. Region 8 represents elements in Z, but in neither X nor Y.

28. $(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$

29. $A \cup B = B \cup A,\ A \cap B = B \cap A$

30. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

31. $A \cup \varnothing = A,\ A \cap U = A$

32. $A \cup \overline{A} = U,\ A \cap \overline{A} = \varnothing$

33. $A \cup A = A,\ A \cap A = A$

34. $A \cup U = U,\ A \cap \varnothing = \varnothing$

35. $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$

36. $\overline{\overline{A}} = A$ **37.** $\overline{\varnothing} = U,\ \overline{U} = \varnothing$

38. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \overline{(A \cap B)} = \overline{A} \cup \overline{B}$

39. A collection $\mathcal{S}$ of nonempty subsets of X is a partition of X if every element in X belongs to exactly one member of $\mathcal{S}$.

40. The Cartesian product of X and Y is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. It is denoted $X \times Y$.

41. The Cartesian product of $X_1, X_2, \ldots, X_n$ is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$ for $i = 1, \ldots, n$. It is denoted $X_1 \times X_2 \times \cdots \times X_n$.

Section 1

1. $\{1, 2, 3, 4, 5, 7, 10\}$ **4.** $\{2, 3, 5\}$

7. $\varnothing$ **10.** U

13. $\{6, 8\}$

16. $\{1, 2, 3, 4, 5, 7, 10\}$

17. 0 **20.** 5

21. If $x \in A$, then x is one of 3, 2, 1. Thus $x \in B$. If $x \in B$, then x is one of 1, 2, 3. Thus $x \in A$. Therefore, $A = B$.

24. If $x \in A$, then x satisfies $x^2 - 4x + 4 = 1$. Factoring $x^2 - 4x + 4$, we find that $(x-2)^2 = 1$. Thus $(x-2) = \pm 1$. If $(x-2) = 1$, then $x = 3$. If $(x-2) = -1$, then $x = 1$. Since $x = 3$ or $x = 1$, $x \in B$. Therefore $A \subseteq B$.

If $x \in B$, then $x = 1$ or $x = 3$. If $x = 1$, then

$$x^2 - 4x + 4 = 1^2 - 4 \cdot 1 + 4 = 1$$

and thus $x \in A$. If $x = 3$, then

$$x^2 - 4x + 4 = 3^2 - 4 \cdot 3 + 4 = 1$$

and again $x \in A$. Therefore $B \subseteq A$. We conclude that $A = B$.

25. Since $1 \in A$, but $1 \notin B$, $A \neq B$.

28. Note that $A = B \cap C = \{2, 4\}$. Since $1 \in B$, but $1 \notin A$, $A \neq B$.

29. Equal **32.** Not equal

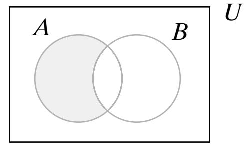
33. Let $x \in A$. Then $x = 1$ or $x = 2$. In either case, $x \in B$. Therefore $A \subseteq B$.

36. First note that $B = \mathbf{Z}^+$. Now let $x \in A$. Then $x = 2n$ for some $n \in \mathbf{Z}^+$. Since $2 \in \mathbf{Z}^+$, $2n \in \mathbf{Z}^+ = B$. Therefore $A \subseteq B$.

37. Since $3 \in A$, but $3 \notin B$, A is not a subset of B.

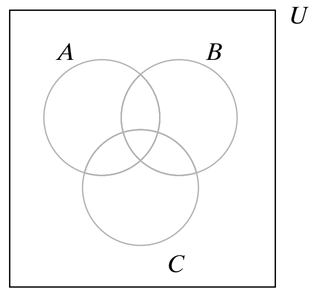
40. Since $3 \in A$, but $3 \notin B$, A is not a subset of B.

41.



44. Same as Exercise 41

47.



49. The shaded area represents the beverage, which has great taste *and* is less filling.

50. 10 **53.** 64 **55.** 4

57. $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

60. $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

61. $\{(1, a, \alpha), (1, a, \beta), (2, a, \alpha), (2, a, \beta)\}$

64. $\{(a, 1, a, \alpha), (a, 2, a, \alpha), (a, 1, a, \beta), (a, 2, a, \beta)\}$

65. The entire xy-plane

68. Parallel horizontal lines spaced one unit apart. There is a lowest line [passing through $(0, 0)$] but the lines continue indefinitely above the lowest line.

71. Parallel planes stacked one above another one unit apart. The planes continue indefinitely in both directions [above and below the origin $(0, 0, 0)$].

73. $\{\{1\}\}$

76. $\{\{a, b, c, d\}\}, \{\{a, b, c\}, \{d\}\},$
$\{\{a, b, d\}, \{c\}\}, \{\{a, c, d\}, \{b\}\}, \{\{b, c, d\}, \{a\}\},$
$\{\{a, b\}, \{c\}, \{d\}\}, \{\{a, c\}, \{b\}, \{d\}\},$
$\{\{a, d\}, \{b\}, \{c\}\},$
$\{\{b, c\}, \{a\}, \{d\}\}, \{\{b, d\}, \{a\}, \{c\}\}, \{\{c, d\}, \{a\}, \{b\}\},$
$\{\{a, b\}, \{c, d\}\}, \{\{a, c\}, \{b, d\}\}, \{\{a, d\}, \{b, c\}\},$
$\{\{a\}, \{b\}, \{c\}, \{d\}\}$

77. True

80. True

83. $\varnothing, \{a\}, \{b\}, \{a, b\}$. All but $\{a, b\}$ are proper subsets.

86. $2^n - 1$ **87.** $A \subseteq B$

90. $B \subseteq A$ **91.** $\{1, 4, 5\}$

94. The center of the circle

Section 2 Review

1. A proposition is a sentence that is either true or false, but not both.

2. The truth table of a proposition P made up of the individual propositions $p_1, \ldots, p_n$ lists all possible combinations of truth values for $p_1, \ldots, p_n$, T denoting true and F denoting false, and for each such combination lists the truth value of P.

3. The conjunction of propositions p and q is the proposition p *and* q. It is denoted $p \wedge q$.

4.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

5. The disjunction of propositions p and q is the proposition p *or* q. It is denoted $p \vee q$.

6.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

7. The negation of proposition p is the proposition *not* p. It is denoted $\neg p$.

8.

p	$\neg p$
T	F
F	T

Section 2

1. Is a proposition. Negation: $2 + 5 \neq 19$

4. Not a proposition; it is a question.

7. Not a proposition; it is a command.

10. Not a proposition; it is a description of a mathematical expression (i.e., $p - q$, where p and q are primes).

12. Ten heads were not obtained. (Alternative: At least one tail was obtained.)

15. No heads were obtained. (Alternative: Ten tails were obtained.)

16. True

19. True

22.

p	q	$p \wedge \neg q$
T	T	F
T	F	T
F	T	F
F	F	F

25.

p	q	$(p \wedge q) \wedge \neg q$
T	T	F
T	F	F
F	T	F
F	F	F

28.

p	q	$(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$
T	T	F
T	F	F
F	T	F
F	F	F

30. $p \wedge q$; false

33. Lee does not take computer science.

36. Lee takes computer science or Lee does not take mathematics.

39. You play football and you miss the midterm exam.

42. It is not the case that you play football or you miss the midterm exam, or you pass the course.

44. Today is Monday or it is raining.

47. (Today is Monday and it is raining) and it is not the case that (it is hot or today is Monday).

49. $\neg p$

52. $\neg p \wedge \neg q$

55. $p \wedge \neg q$

58. $\neg p \wedge \neg r \wedge \neg q$

60. $p \wedge r$

63. $(p \vee q) \wedge \neg r$

67. Inclusive-or: To enter Utopia, you must show a driver's license or a passport or both. Exclusive-or: To enter Utopia, you must show a driver's license or a passport but not both. Exclusive-or is the intended meaning.

70. Inclusive-or: The car comes with a cupholder that heats or cools your drink or both. Exclusive-or: The car comes with a cupholder that heats or cools your drink but not both. Exclusive-or is the intended meaning.

73. Inclusive-or: The meeting will be canceled if fewer than 10 persons sign up or at least 3 inches of snow falls or both. Exclusive-or: The meeting will be canceled if fewer than 10 persons sign up or at least 3 inches of snow falls but not both. Inclusive-or is the intended meaning.

74. No, assuming the interpretation: It shall be unlawful for any person to keep more than three [3] dogs and more than three [3] cats upon his property within the city. A judge ruled that the ordinance was "vague." Presumably, the intended meaning was: "It shall be unlawful for any person to keep more than three [3] dogs *or* more than three [3] cats upon his property within the city."

75. `"national park" "north dakota" OR "south dakota"`

Section 3 Review

1. If p and q are propositions, the conditional proposition is the proposition if p then q. It is denoted $p \rightarrow q$.

2.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

3. In the conditional proposition $p \to q$, p is the hypothesis.

4. In the conditional proposition $p \to q$, q is the conclusion.

5. In the conditional proposition $p \to q$, q is a necessary condition.

6. In the conditional proposition $p \to q$, p is a sufficient condition.

7. The converse of $p \to q$ is $q \to p$.

8. If p and q are propositions, the biconditional proposition is the proposition p if and only if q. It is denoted $p \leftrightarrow q$.

9.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

10. If the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$, P and Q are logically equivalent provided that given any truth values of $p_1, \ldots, p_n$, either P and Q are both true or P and Q are both false.

11. $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$

12. The contrapositive of $p \to q$ is $\neg q \to \neg p$.

Section 3

1. If Joey studies hard, then he will pass the discrete mathematics exam.

4. If Katrina passes discrete mathematics, then she will take the algorithms course.

7. If you inspect the aircraft, then you have the proper security clearance.

10. If the program is readable, then it is well structured.

11. (For Exercise 1) If Joey passes the discrete mathematics exam, then he studied hard.

13. True **16.** False **19.** False

21. True **24.** True **27.** True

30. True **31.** True **34.** False

37. True **40.** $p \to q$ **43.** $q \leftrightarrow (p \wedge \neg r)$

44. $p \to q$

47. $q \leftrightarrow (p \wedge r)$

50. If today is Monday, then it is raining.

53. It is not the case that today is Monday or it is raining if and only if it is hot.

56. Let p: $4 < 6$ and q: $9 > 12$.
Given statement: $p \to q$; false.
Converse: $q \to p$; if $9 > 12$, then $4 < 6$; true.
Contrapositive: $\neg q \to \neg p$; if $9 \leq 12$, then $4 \geq 6$; false.

59. Let p: $|4| < 3$ and q: $-3 < 4 < 3$.
Given statement: $q \to p$; true.
Converse: $p \to q$; if $|4| < 3$, then $-3 < 4 < 3$; true.
Contrapositive:
$\neg p \to \neg q$, if $|4| \geq 3$, then $-3 \geq 4$ or $4 \geq 3$; true.

60. $P \not\equiv Q$ **63.** $P \not\equiv Q$

66. $P \not\equiv Q$ **69.** $P \not\equiv Q$

70. Pat will not use the treadmill and will not lift weights.

73. To make chili, you do not need red pepper or you do not need onions.

74.

p	q	p *imp1* q	q *imp1* p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Since p *imp1* q is true precisely when q *imp1* p is true, p *imp1* $q \equiv q$ *imp1* p.

77.

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Since $p \to q$ is true precisely when $\neg p \vee q$ is true, $p \to q \equiv \neg p \vee q$.

Section 4 Review

1. Deductive reasoning refers to the process of drawing a conclusion from a sequence of propositions.

2. In the argument $p_1, p_2, \ldots, p_n / \therefore q$, the hypotheses are $p_1, p_2, \ldots, p_n$.

3. "Premise" in another name for hypothesis.

4. In the argument $p_1, p_2, \ldots, p_n / \therefore q$, the conclusion is q.

5. The argument $p_1, p_2, \ldots, p_n / \therefore q$ is valid provided that if p_1 and p_2 and ... and p_n are all true, then q must also be true.

6. An invalid argument is an argument that is not valid.

7. $$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

8. $$\begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore \neg p \end{array}$$

9. $$\begin{array}{l} \underline{p \qquad\quad} \\ \therefore p \vee q \end{array}$$

10. $$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array}$$

11. $$\begin{array}{l} p \\ \underline{q \qquad\quad} \\ \therefore p \wedge q \end{array}$$

12. $$\begin{array}{l} p \to q \\ \underline{q \to r \quad} \\ \therefore p \to r \end{array}$$

13. $\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$

Section 4

1. Valid $\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$ **4.** Invalid $\begin{array}{l} (p \vee r) \to q \\ \underline{q \quad\quad\quad\quad} \\ \therefore \neg p \to r \end{array}$

6. Valid. If 4 megabytes is better than no memory at all, then we will buy a new computer. If 4 megabytes is better than no memory at all, then we will buy more memory. Therefore, if 4 megabytes is better than no memory at all, then we will buy a new computer and we will buy more memory.

9. Invalid. If we will not buy a new computer, then 4 megabytes is not better than no memory at all. We will buy a new computer. Therefore, 4 megabytes is better than no memory at all.

11. Invalid **14.** Invalid

17. An analysis of the argument must take into account the fact that "nothing" is being used in two very different ways.

18. Addition

21. Let p denote the proposition "there is gas in the car," let q denote the proposition "I go to the store," and let r denote the proposition "I get a soda." Then the hypotheses are as follows:

$$p \to q$$
$$q \to r$$
$$p$$

From $p \to q$ and $q \to r$, we may use the hypothetical syllogism to conclude $p \to r$. From $p \to r$ and p, we may use modus ponens to conclude r. Since r represents the proposition "I get a soda," we conclude that the conclusion does follow from the hypotheses.

24. We construct a truth table for all the propositions involved:

p	q	$p \to q$	$\neg q$	$\neg p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

We observe that whenever the hypotheses $p \to q$ and $\neg q$ are true, the conclusion $\neg p$ is also true; therefore, the argument is valid.

27. We construct a truth table for all the propositions involved:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

We observe that whenever the hypotheses p and q are true, the conclusion $p \wedge q$ is also true; therefore, the argument is valid.

Section 5 Review

1. If $P(x)$ is a statement involving the variable x, we call P a propositional function if for each x in the domain of discourse, $P(x)$ is a proposition.

2. A domain of discourse for a propositional function P is a set D such that $P(x)$ is defined for every x in D.

3. A universally quantified statement is a statement of the form for all x in the domain of discourse, $P(x)$.

4. A counterexample to the statement $\forall x\, P(x)$ is a value of x for which $P(x)$ is false.

5. An existentially quantified statement is a statement of the form for some x in the domain of discourse, $P(x)$.

6. $\neg(\forall x\, P(x))$ and $\exists x \neg P(x)$ have the same truth values. $\neg(\exists x\, P(x))$ and $\forall x \neg P(x)$ have the same truth values.

7. To prove that the universally quantified statement $\forall x\, P(x)$ is true, show that for every x in the domain of discourse, the proposition $P(x)$ is true.

8. To prove that the existentially quantified statement $\exists x\, P(x)$ is true, find one value of x in the domain of discourse for which the proposition $P(x)$ is true.

9. To prove that the universally quantified statement $\forall x\, P(x)$ is false, find one value of x in the domain of discourse for which the proposition $P(x)$ is false.

10. To prove that the existentially quantified statement $\exists x\, P(x)$ is false, show that for every x in the domain of discourse, the proposition $P(x)$ is false.

11. $\dfrac{\forall x\, P(x)}{\therefore P(d) \text{ if } d \in D}$ **12.** $\dfrac{P(d) \text{ for every } d \in D}{\therefore \forall x\, P(x)}$

13. $\dfrac{\exists x\, P(x)}{\therefore P(d) \text{ for some } d \in D}$ **14.** $\dfrac{P(d) \text{ for some } d \in D}{\therefore \exists x\, P(x)}$

Section 5

1. Is a propositional function. The domain of discourse could be taken to be all integers.

4. Is a propositional function. The domain of discourse is the set of all movies.

7. 11 divides 77. True.

10. For every positive integer n, n divides 77. False.

12. True **15.** False **18.** False

21. $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

24. $P(1) \vee P(2) \vee P(3) \vee P(4)$

27. $P(2) \wedge P(3) \wedge P(4)$

28. Every student is taking a math course.

31. Some student is not taking a math course.

34. (For Exercise 28) $\exists x \neg P(x)$. Some student is not taking a math course.

35. Every professional athlete plays soccer. False.

38. Either someone does not play soccer or some soccer player is a professional athlete. True.

41. Everyone is a professional athlete and plays soccer. False.

43. (For Exercise 35) $\exists x(P(x) \wedge \neg Q(x))$. Someone is a professional athlete and does not play soccer.

44. $\forall x(P(x) \rightarrow Q(x))$

47. $\exists x(P(x) \wedge Q(x))$

48. (For Exercise 44) $\exists x(P(x) \wedge \neg Q(x))$. Some accountant does not own a Porsche.

49. False. A counterexample is $x = 0$.

52. True. The value $x = 2$ makes $(x > 1) \rightarrow (x^2 > x)$ true.

55. (For Exercise 49) $\exists x(x^2 \leq x)$. There exists x such that $x^2 \leq x$.

57. The literal meaning is: No man cheats on his wife. The intended meaning is: Some man does not cheat on his wife. Let $P(x)$ denote the statement "x is a man," and $Q(x)$ denote the statement "x cheats on his wife." Symbolically, the clarified statement is $\exists x(P(x) \wedge \neg Q(x))$.

60. The literal meaning is: No environmental problem is a tragedy. The intended meaning is: Some environmental problem is not a tragedy. Let $P(x)$ denote the statement "x is an environmental problem," and $Q(x)$ denote the statement "x is a tragedy." Symbolically, the clarified statement is $\exists x(P(x) \wedge \neg Q(x))$.

63. The literal meaning is: Everything is not sweetness and light. The intended meaning is: Not everything is sweetness and light. Let $P(x)$ denote the statement "x is sweetness and light." Symbolically, the clarified statement is $\exists x \neg P(x)$.

66. The literal meaning is: No circumstance is right for a formal investigation. The intended meaning is: Some circumstance is not right for a formal investigation. Let $P(x)$ denote the statement "x is a circumstance," and $Q(x)$ denote the statement "x is right for a formal investigation." Symbolically, the clarified statement is $\exists x(P(x) \wedge \neg Q(x))$.

67. (a)

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

One of $p \rightarrow q$ or $q \rightarrow p$ is true since in each row, one of the last two entries is true.

(b) The statement, "All integers are positive or all positive numbers are integers," which is false, in symbols is

$$(\forall x(I(x) \rightarrow P(x))) \vee (\forall x(P(x) \rightarrow I(x))).$$

This is *not* the same as the given statement

$$\forall x((I(x) \rightarrow P(x)) \vee (P(x) \rightarrow I(x))),$$

which is true. The ambiguity results from attempting to distribute $\forall$ across the *or*.

70. Universal instantiation

71. Let $P(x)$ denote the propositional function "x has a graphing calculator," and let $Q(x)$ denote the propositional function "x understands the trigonometric functions." The hypotheses are $\forall x\, P(x)$ and $\forall x(P(x) \rightarrow Q(x))$. By universal instantiation, we have $P(\text{Ralphie})$ and $P(\text{Ralphie}) \rightarrow Q(\text{Ralphie})$. The modus ponens rule of inference now gives $Q(\text{Ralphie})$, which represents the proposition "Ralphie understands the trigonometric functions." We conclude that the conclusion does follow from the hypotheses.

74. By definition, the proposition $\forall x\, P(x)$ is true when $P(x)$ is true for all x in the domain of discourse. We are given that $P(d)$ is true for any d in the domain of discourse D. Therefore, $\forall x\, P(x)$ is true.

Section 6 Review

1. For every x and for every y, $P(x, y)$. Let the domain of discourse be $X \times Y$. The statement is true if, for every $x \in X$ and for every $y \in Y$, $P(x, y)$ is true. The statement is false if there is at least one $x \in X$ and at least one $y \in Y$ such that $P(x, y)$ is false.

2. For every x, there exists y such that $P(x, y)$. Let the domain of discourse be $X \times Y$. The statement is true if, for every $x \in X$, there is at least one $y \in Y$ for which $P(x, y)$ is true. The statement is false if there is at least one $x \in X$ such that $P(x, y)$ is false for every $y \in Y$.

3. There exists x such that for every y, $P(x, y)$. Let the domain of discourse be $X \times Y$. The statement is true if there is at least one $x \in X$ such that $P(x, y)$ is true for every $y \in Y$. The statement is false if, for every $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false.

4. There exists x and there exists y such that $P(x, y)$. Let the domain of discourse be $X \times Y$. The statement is true if there is at least one $x \in X$ and at least one $y \in Y$ such that $P(x, y)$ is true. The statement is false if, for every $x \in X$ and for every $y \in Y$, $P(x, y)$ is false.

5. Let $P(x, y)$ be the propositional function "$x \leq y$" with domain of discourse $\mathbf{Z} \times \mathbf{Z}$. Then $\forall x \exists y P(x, y)$ is true since, for every integer x, there exists an integer y (e.g., $y = x$) such that $x \leq y$ is true. On the other hand, $\exists x \forall y P(x, y)$ is false. For every integer x, there exists an integer y (e.g., $y = x - 1$) such that $x \leq y$ is false.

6. $\exists x \exists y \neg P(x, y)$

7. $\exists x \forall y \neg P(x, y)$

8. $\forall x \exists y \neg P(x, y)$

9. $\forall x \forall y \neg P(x, y)$

10. Given a quantified propositional function, you and your opponent, whom we call Farley, play a logic game. Your goal is to try to make the propositional function true, and Farley's goal is to try to make it false. The game begins with the first (left) quantifier. If the quantifier is $\forall$, Farley chooses a value for that variable; if the quantifier is $\exists$, you choose a value for that variable. The game continues with the second quantifier. After values are chosen for all the variables, if the propositional function is true, you win; if it is false, Farley wins. If you can always win regardless of how Farley chooses values for the variables, the quantified propositional function is true, but if Farley can choose values for the variables so that you cannot win, the quantified propositional function is false.

Section 6

1. Everyone is taller than everyone.

4. Someone is taller than someone.

5. (For Exercise 1) In symbols: $\exists x \exists y \neg T_1(x, y)$. In words: Someone is not taller than someone.

6. (For Exercise 1) False; Garth is not taller than Garth.

9. (For Exercise 1) False; Pat is not taller than Pat.

10. Everyone is taller than or the same height as everyone.

13. Someone is taller than or the same height as someone.

14. (For Exercise 10) In symbols: $\exists x \exists y \neg T_2(x, y)$. In words: Someone is shorter than someone.

15. (For Exercise 10) False; Erin is not taller than or the same height as Garth.

18. (For Exercise 10) True

19. For any two people, if they are distinct, the first is taller than the second.

22. There are two people and, if they are distinct, the first is taller than the second.

23. (For Exercise 19) In symbols: $\exists x \exists y \neg T_3(x, y)$. In words: There are two distinct people and the first is shorter than or the same height as the second.

24. (For Exercise 19) False; Erin and Garth are distinct persons, but Erin is not taller than Garth.

27. (For Exercise 19) False; Pat and Sandy are distinct persons, but Pat is not taller than Sandy.

28. $\exists x \forall y L(x, y)$. True (think of a saint).

31. $\forall x \exists y L(x, y)$. True (according to Dean Martin's song, "Everybody Loves Somebody Sometime").

32. (For Exercise 28) Everyone does not love someone. $\forall x \exists y \neg L(x, y)$

33. $\exists y\, A(\text{Brit}, y)$

36. $\forall y \exists x\, A(x, y)$

37. False **40.** True

41. (For Exercise 37) $\exists x \exists y \neg P(x, y)$ or $\exists x \exists y (x < y)$

42. False. A counterexample is $x = 2$, $y = 0$.

45. True. Take $x = y = 0$.

48. False. A counterexample is $x = y = 2$.

51. True. Take $x = 1$, $y = \sqrt{8}$.

54. True. Take $x = 0$. Then for all y, $x^2 + y^2 \geq 0$.

57. True. For any x, if we set $y = x - 1$, the conditional proposition, if $x < y$, then $x^2 < y^2$, is true because the hypothesis is false.

60. (For Exercise 42) $\exists x \exists y (x^2 \geq y + 1)$

63. (For Exercise 42) Since both quantifiers are $\forall$, Farley chooses values for both x and y. Since Farley can choose values that make $x^2 < y + 1$ false (e.g., $x = 2$, $y = 0$), Farley can win the game. Therefore, the proposition is false.

66. Since the first two quantifiers are $\forall$, Farley chooses values for both x and y. The last quantifier is $\exists$, so you choose a value for z. Farley can choose values (e.g., $x = 1$, $y = 2$) so that no matter which value you choose for z, the expression

$$(x < y) \rightarrow ((z > x) \wedge (z < y))$$

is false. Since Farley can choose values for the variables so that you cannot win, the quantified statement is false.

68. $\forall x \exists y\, P(x, y)$ must be true. Since $\forall x \forall y\, P(x, y)$ is true, regardless of which value of x is selected, $P(x, y)$ is true *for all* y. Thus for any x, $P(x, y)$ is true *for any* particular y.

71. $\forall x \forall y\, P(x, y)$ might be false. Let $P(x, y)$ denote the expression $x \leq y$. If the domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$, $\exists x \forall y\, P(x, y)$ is true; however, $\forall x \forall y\, P(x, y)$ is false.

74. $\forall x \forall y\, P(x, y)$ might be false. Let $P(x, y)$ denote the expression $x \leq y$. If the domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$, $\exists x \exists y\, P(x, y)$ is true; however, $\forall x \forall y\, P(x, y)$ is false.

77. $\forall x \exists y\, P(x, y)$ might be true. Let $P(x, y)$ denote the expression $x \leq y$. If the domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$, $\forall x \exists y\, P(x, y)$ is true; however, $\forall x \forall y\, P(x, y)$ is false.

80. $\forall x \forall y\, P(x, y)$ must be false. Since $\forall x \exists y\, P(x, y)$ is false, there exists x, say $x = x'$, such that for all y, $P(x, y)$ is false. Choose $y = y'$ in the domain of discourse. Then $P(x', y')$ is false. Therefore $\forall x \forall y\, P(x, y)$ is false.

83. $\forall x \forall y\, P(x, y)$ must be false. Since $\exists x \forall y\, P(x, y)$ is false, for every x there exists y such that $P(x, y)$ is false. Choose $x = x'$ in the domain of discourse. For this choice of x, there exists $y = y'$ such that $P(x', y')$ is false. Therefore $\forall x \forall y\, P(x, y)$ is false.

86. $\forall x \forall y\, P(x, y)$ must be false. Since $\exists x \exists y\, P(x, y)$ is false, for every x and for every y, $P(x, y)$ is false. Choose $x = x'$ and $y = y'$ in the domain of discourse. For these choices of x and y, $P(x', y')$ is false. Therefore $\forall x \forall y\, P(x, y)$ is false.

89. $\exists x \neg(\forall y\, P(x, y))$ is not logically equivalent to $\neg(\forall x \exists y\, P(x, y))$. Let $P(x, y)$ denote the expression $x < y$. If the domain of discourse is $\mathbf{Z} \times \mathbf{Z}$, $\exists x \neg(\forall y\, P(x, y))$ is true; however, $\neg(\forall x \exists y\, P(x, y))$ is false.

92. $\exists x \exists y \neg P(x, y)$ is not logically equivalent to $\neg(\forall x \exists y\, P(x, y))$. Let $P(x, y)$ denote the expression $x < y$. If the domain of discourse is $\mathbf{Z} \times \mathbf{Z}$, $\exists x \exists y \neg P(x, y)$ is true; however, $\neg(\forall x \exists y\, P(x, y))$ is false.

93. $\forall \varepsilon > 0\, \exists \delta > 0\, \forall x\, ((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \varepsilon))$

Chapter Self-Test

1. $\varnothing$

2. $A \subseteq B$

3. Yes