ECS 230 - Homework 04

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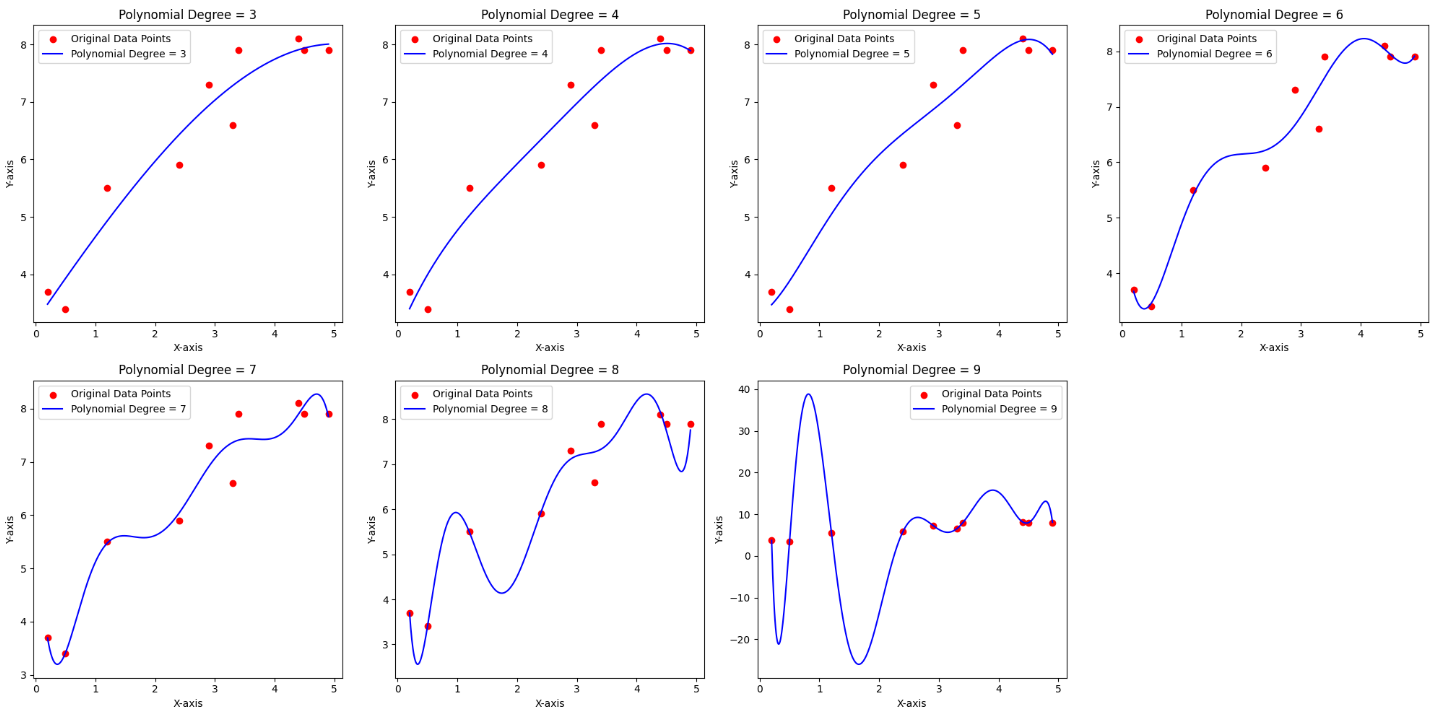
1. Polynomial Plots (degree from 3 to 9)



2. Program Outputs (degree 5 and 9)

2.1 Degree = 5

1. Initial Matrix X :

$$
\begin{pmatrix}
1.000000 & 0.500000 & 0.250000 & 0.125000 & 0.062500 & 0.031250 \\
1.000000 & 0.200000 & 0.040000 & 0.008000 & 0.001600 & 0.000320 \\
1.000000 & 1.200000 & 1.440000 & 1.728000 & 2.073600 & 2.488320 \\
1.000000 & 2.900000 & 8.410000 & 24.389000 & 70.728100 & 205.111490 \\
1.000000 & 3.400000 & 11.560000 & 39.304000 & 133.633600 & 454.354240 \\
1.000000 & 2.400000 & 5.760000 & 13.824000 & 33.177600 & 79.626240 \\
1.000000 & 4.400000 & 19.360000 & 85.184000 & 374.809600 & 1649.162240 \\
1.000000 & 4.500000 & 20.250000 & 91.125000 & 410.062500 & 1845.281250 \\
1.000000 & 4.900000 & 24.010000 & 117.649000 & 576.480100 & 2824.752490 \\
1.000000 & 3.300000 & 10.890000 & 35.937000 & 118.592100 & 391.353930
\end{pmatrix}
$$

2. Matrix $X^T X$:

$$\begin{pmatrix} 10.000000 & 27.700000 & 101.970000 & 409.273000 & 1719.621300 & 7452.161770 \\ 27.700000 & 101.970000 & 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 \\ 101.970000 & 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 \\ 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 \\ 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 \\ 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 \end{pmatrix}$$

3. Vector $X^T y$:

$$\begin{pmatrix} 64.200000 \\ 202.910000 \\ 773.963000 \\ 3156.550700 \\ 13391.743790 \\ 58404.717011 \end{pmatrix}$$

4. Lower Triangular Matrix L:

$$\begin{pmatrix} 3.162278 & 0 & 0 & 0 & 0 & 0 \\ 8.759509 & 5.024042 & 0 & 0 & 0 & 0 \\ 32.245745 & 25.241846 & 6.533178 & 0 & 0 & 0 \\ 129.423486 & 116.626228 & 51.268111 & 6.784688 & 0 & 0 \\ 543.792002 & 535.188731 & 303.441456 & 68.546818 & 8.957628 & 0 \\ 2356.580469 & 2464.963161 & 1623.792326 & 481.578786 & 117.436679 & 10.613064 \end{pmatrix}$$

5. Coefficients of the polynomial fit b :

$$\begin{pmatrix} 3.302285 \\ 0.548540 \\ 1.698495 \\ -1.062627 \\ 0.254667 \\ -0.021502 \end{pmatrix}$$

2.2 Degree = 9

1. Initial Matrix X:

$$\begin{pmatrix} 1.000000 & 0.500000 & 0.250000 & 0.125000 & 0.062500 & 0.031250 & 0.015625 & 0.007812 & 0.003906 & 0.001953 \\ 1.000000 & 0.200000 & 0.040000 & 0.008000 & 0.001600 & 0.000320 & 0.000064 & 0.000013 & 0.000003 & 0.000001 \\ 1.000000 & 1.200000 & 1.440000 & 1.728000 & 2.073600 & 2.488320 & 2.985984 & 3.583181 & 4.299817 & 5.159780 \\ 1.000000 & 2.900000 & 8.410000 & 24.389000 & 70.728100 & 205.111490 & 594.823321 & 1724.987631 & 5002.464130 & 14507.145976 \\ 1.000000 & 3.400000 & 11.560000 & 39.304000 & 133.633600 & 454.354240 & 1544.804416 & 5252.335014 & 17857.939049 & 60716.992766 \\ 1.000000 & 2.400000 & 5.760000 & 13.824000 & 33.177600 & 79.626240 & 191.102976 & 458.647142 & 1100.753142 & 2641.807540 \\ 1.000000 & 4.400000 & 19.360000 & 85.184000 & 374.809600 & 1649.162240 & 7256.313856 & 31927.780966 & 140482.236252 & 618121.839510 \\ 1.000000 & 4.500000 & 20.250000 & 91.125000 & 410.062500 & 1845.281250 & 8303.765625 & 37366.945312 & 168151.253906 & 756680.642578 \\ 1.000000 & 4.900000 & 24.010000 & 117.649000 & 576.480100 & 2824.752490 & 13841.287201 & 67822.307285 & 332329.305696 & 1628413.597910 \\ 1.000000 & 3.300000 & 10.890000 & 35.937000 & 118.592100 & 391.353930 & 1291.467969 & 4261.844298 & 14064.086182 & 46411.484402 \end{pmatrix}$$

2. Matrix $X^T X$:

$$
\begin{pmatrix}
10.000000 & 27.700000 & 101.970000 & 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 \\
27.700000 & 101.970000 & 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 \\
101.970000 & 409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 \\
409.273000 & 1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 \\
1719.621300 & 7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 \\
7452.161770 & 33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 & 7064473457.161751 \\
33026.567037 & 148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 & 7064473457.161751 & 33470793953.852802 \\
148818.438655 & 678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 & 7064473457.161750 & 33470793953.852802 & 158995811147.602295 \\
678992.342083 & 3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 & 7064473457.161751 & 33470793953.852802 & 158995811147.602295 & 757053992476.260010 \\
3127498.672417 & 14512038.543277 & 67732371.152119 & 317632499.957908 & 1495443977.308277 & 7064473457.161751 & 33470793953.852802 & 158995811147.602295 & 757053992476.260010 & 3612429064743.778320
\end{pmatrix}
$$

3. Vector $X^T y$:

$$
\begin{pmatrix}
64.200000 \\
202.910000 \\
773.963000 \\
3156.550700 \\
13391.743790 \\
58404.717011 \\
259935.897116 \\
1174549.902276 \\
5368639.315758 \\
24756528.655340
\end{pmatrix}
$$

4. Lower Triangular Matrix L :

$$
\begin{pmatrix}
3.162278 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8.759509 & 5.024042 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32.245745 & 25.241846 & 6.533178 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
129.423486 & 116.626228 & 51.268111 & 6.784688 & 0 & 0 & 0 & 0 & 0 & 0 \\
543.792002 & 535.188731 & 303.441456 & 68.546818 & 8.957628 & 0 & 0 & 0 & 0 & 0 \\
2356.580469 & 2464.963161 & 1623.792326 & 481.578786 & 117.436679 & 10.613064 & 0 & 0 & 0 & 0 \\
10443.917513 & 11412.095224 & 8289.700126 & 2927.706354 & 1001.530335 & 165.427909 & 12.143240 & 0 & 0 & 0 \\
47060.522398 & 53097.736152 & 41283.456271 & 16535.495790 & 7073.046800 & 1610.271754 & 209.050765 & 7.294521 & 0 & 0 \\
214716.231479 & 248144.787781 & 202768.251954 & 89512.168065 & 45008.406820 & 12607.957139 & 2187.198085 & 148.401049 & 5.064513 & 0 \\
989001.918399 & 1164175.576363 & 988083.660915 & 471920.750926 & 268462.145231 & 87058.156491 & 18076.109899 & 1781.693719 & 123.327046 & 0.982265
\end{pmatrix}
$$

5. Coefficients of the polynomial fit b :

$$
\begin{pmatrix}
280.120205 \\
-2632.708972 \\
8366.981270 \\
-12398.140518 \\
10053.695004 \\
-4830.562494 \\
1414.511540 \\
-248.059639 \\
23.953893 \\
-0.979794
\end{pmatrix}
$$

3. Discussion on the quality of the fit for different choices of the degree

3.1 Degree from 3 to 9

- As can be seen in the polynomial plots above, the quality of the fit consistently increases as the chosen degree increases from 3 to 9. This is intuitive, as there are 10 data points in the provided `data.dat` file. According to the basic property of polynomials, `(d+1)` data points (x,y) with different x values uniquely determine a polynomial of degree `d`. For `d=9`, it can be found that the polynomial perfectly fits all the 10 data points.

3.2 Degree >= 10

- Next let's analyze why this fit does not work for polynomials with degree >= 10. The reason is also straightforward. Since we only have 10 data points, if the choice of degree is 10, then matrix X will be of shape 10*11, thus the coefficient matrix in the normal equation $X^T X$ will be of shape 11*11 and will definitely be rank-deficient as its maximum possible rank cannot exceed 10. In this way, the normal equation $X^T Xb = X^T y$ will have infinite number of solutions b instead of a unique one. Thus we cannot determine the best fit by simply solving the normal equation. Actually, since $X^T X$ is rank-deficient, it is sometimes impossible to even calculate the Cholesky Decomposition of it.
- Typically in such rank-deficient cases of Least Square, we can add an additional regularization term into our original optimization objective, and convert the basic Least Square problem into a Ridge/Lasso Regression problem.