People in Vancouver always bring an umbrella with them even if the forecast says that it will not rain. Why isn't there any trust in the weather forecast from the general public? How do we even predict the weather? To answer these questions, we will examine the mathematical tools that people have used to try to forecast the weather, how dynamical systems, chaos, and fractals play their role, and how the science behind them really explains why we cannot predict the future. In fact, we can't even predict the weather with complete certainty.

Mathematicians use dynamical systems to model things in nature, including the weather. We often encounter dynamical systems in daily life. Imagine how the moon orbits in outer space. Its position changes over the course of time. This is a simple example of a continuous dynamical system. A dynamical system is all about evolution of something over time. The evolution can occur smoothly, or in discrete time steps.[1] We can also change the orbit of the moon to a discrete dynamical system by taking a snapshot of its position once per day instead of recording in a continuous manner.[2] Its daily phases in a month are used to determine the Lunar Calendar. On the other hand, the dynamics of weather is very complex. Before we get to the business of modeling and predicting the weather, let's look into one of its simplest dynamical models.

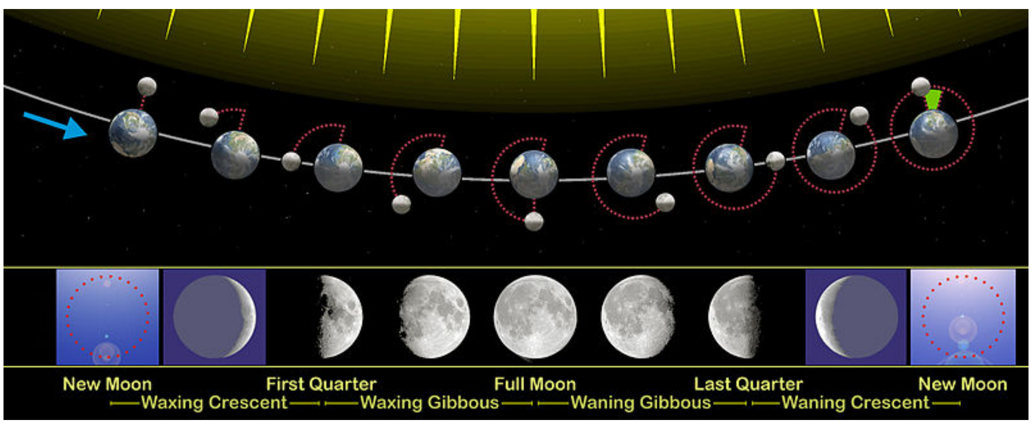


Figure 1: The phases of the Moon based on its orbit position around the Earth[3]

First, we need to understand what *Markov chains* are. Named after Andrey Markov, *Markov chains* are mathematical systems that hop from one "state" (a situation or set of values) to another.[4] This sounds familiar as it is very similar to a dynamical system that evolves over some time steps. For instance, we can make a Markov chain model of a typical UofT student's behavior during the daytime. We might include "studying", "procrastinating", "eating", "sleeping", and "crying" as states, in other words, the behaviours that the student exhibits. We could find more behaviors to form a *state space*, a list of all possible states. Moreover, a Markov chain tells us the probability

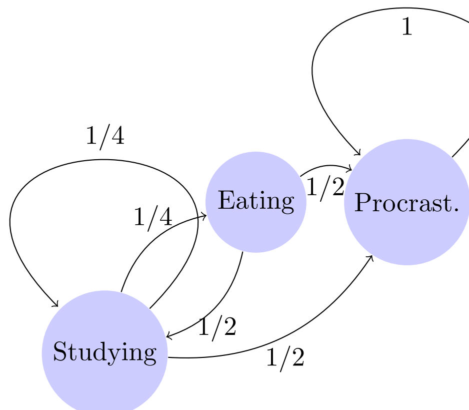
[1] Nykamp DQ, "An introduction to discrete dynamical systems." Math Insight, http://mathinsight.org/discrete_dynamical_system_introduction

[2] Nykamp DQ, "The idea of a dynamical system." Math Insight, http://mathinsight.org/dynamical_system_idea.

[3] Picture Source

[4] Powell, Lehe. "Markov Chains Explained Visually." Explained Visually, 7 Nov. 2014, setosa.io/ev/markov-chains/.

of hopping, or "transitioning" from one state to any other state, the chance that the UofT student currently studying will fall asleep in the next thirty minutes without crying first.[4] Below is a Markov chain of my Friday afternoon labeled with probabilities.



We can clearly see that it is not a productive afternoon; if I start to procrastinate, I have a 100% chance of continuing to procrastinate.

Suppose we want to see if tomorrow will be sunny for a panic at the High Park to watch the cherry blossoms. Simplifying our universe with only two possible weathers, "Sunny" and "Rainy", we assume that the probability of it being sunny or rainy tomorrow does depend on whether it is sunny or rainy today in Toronto.[5] Then, we analyze Toronto's past data to calculate the probability of whether it is sunny or rainy given that the previous day was sunny or rainy.

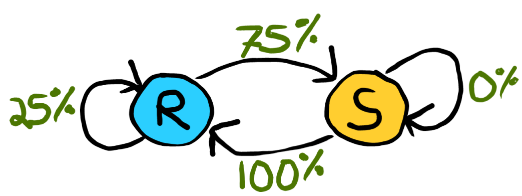


Figure 2: A naive weather Markov chain of Toronto (R = Rainy, S = Sunny)[5]

We have just created our first model to predict the weather! If it is sunny today, then the Markov chain in Figure 2 tells us that it will be 100% rainy tomorrow, so we should wait for one or more days to go picnic. If it is rainy today, there is a 75% chance to be sunny tomorrow, in which case, I'll give the picnic a try. Unfortunately, there are many limitations to this Markov chain model of weather. There are clearly more than two kinds of weather, and the probability of rainy or sunny does not just depend on the weather of the previous day.

Even though there are many interesting applications of Markov chains such as text generator and typing word prediction, using Markov chains is not an ideal tool to model the weather, as the weather is extremely sensitive to initial conditions. This sensibility is called the *butterfly effect*. In

[5] Chris. "Predicting the Weather with Markov Chains." Medium, Towards Data Science, 19 Mar. 2020, www.towardsdatascience.com/predicting-the-weather-with-markov-chains-a34735f0c4df.

order to model the weather adequately, we need to first understand some of its characteristics, such as the butterfly effect.

In pop culture, we often use the phrase "butterfly effect" to express how tiny, insignificant changes can later have huge consequences, making the future unpredictable.[6] A metaphor for the butterfly effect on weather is that a butterfly flapping its wings in China can cause a hurricane in Texas.[7] This describes how weather is extremely sensitive, as a small perturbations to the initial condition, such as a butterfly flapping its wings, will result in completely different outcomes. Sensitive dynamical systems like the system of weather are called chaotic.

The butterfly effect is an underlying principle of chaos.[6] Similar to the system of weather, a double pendulum also demonstrates the butterfly effect as well. A single pendulum has simple dynamics that moves in a elliptic trajectory, regardless where and what angle that one releases it. On the other hand, a double pendulum that is a pendulum attached with another pendulum to its end, presents a rich dynamic behavior with a strong sensitivity to initial conditions.[8] The figure below exhibits the drastic change of position after a minute of the same double pendulum with a similar releasing position.

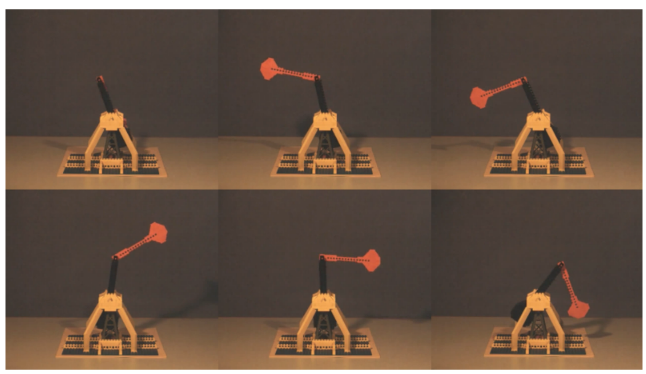


Figure 3: Double Pendulum: Drastic difference after a minute with a similar initial condition[9]

Before we dive into the math behind chaotic systems and the butterfly effect, we need to understand the term *deterministic*. Here is a famous thought experiment: imagine there is a demon named Laplace who knows everything that has happened. It knows what current GPA you have at the University, and the precise location and momentum of every atom in the universe. By some calculations using the laws of classical mechanics, Laplace should also know everything's future values for any given time, even your GPA after graduation.[10] Then, predicting the weather should not be problematic at all, since the weather would be determined completely by the past. However,

[6] "The Science Behind the Butterfly Effect." YouTube, 6 Dec. 2019, https://www.youtube.com/watch?v=fDek6cYijxI.

[7] "Chaos Theory." Wikipedia, Wikimedia Foundation, 17 Mar. 2020, https://en.wikipedia.org/wiki/Chaos_theory.

[8] "Double Pendulum." Wikipedia, Wikimedia Foundation, 4 Apr. 2020, https://en.wikipedia.org/wiki/Double_pendulum.

[9] Picture source

[10] "Laplace's Demon." Wikipedia, Wikimedia Foundation, 7 Mar. 2020, https://en.wikipedia.org/wiki/Laplace's_demon.

the assumptions of determinism were challenged when the mathematician, Edward Lorenz, had ran some simulations of weather of the Earth's atmosphere using initial condition data on a computer. He first submitted unrounded data and wanted to redo the run to double-check the result. To save time, he input rounded initial condition data for a second run. Lorenz was stunned by the result; it was not even close to the first simulation. Only few decimals difference that is less than one part in a thousand in the initial condition data simulates totally different weather! Lorenz re-ensured this phenomenon with a simpler model that had fewer equations, but the same thing happened again. A very small change in initial conditions creates a significantly different outcome. This is one of the fundamental ideas of chaos theory.[11]

If we plot out the system of equations, called the *Lorenz attractor* that Lorenz used for the weather simulation of atmospheric convection along with a wide range of initial conditions, we will get the following beautiful, butterfly-like graph. Notice that none of the paths cross each other or form a loop.

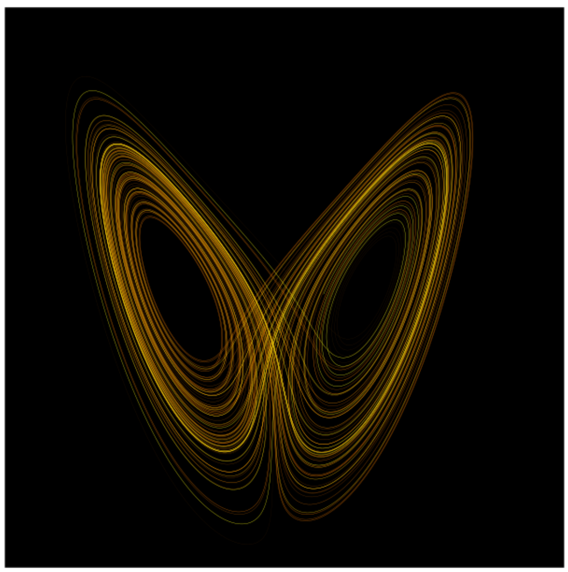


Figure 4: An Icon of Chaos Theory - The Lorenz Attractor[11]

The figure of the Lorenz attractor is an example of a graphical visualization of mathematical models for the nature having similar characteristics as fractals. To understand this model that uses the Lorenz equations better, we need to understand what fractals are, and the fractal dimension characteristic. This is because the Lorenz attractor shares some similar properties like fractals; it is an infinite complex of surfaces.

You might be wondering about what fractals really are. Have you ever looked closely at a snowflake that landed on your palm or the opening of a Nautilus shell? As examples of fractals in nature, they both have the feature of repeating patterns. Fractals are often viewed as intriguing figures that hold the beauty of infinite, repetitive patterns.

[11] "Butterfly Effect." Wikipedia, Wikimedia Foundation, 16 Mar. 2020, https://en.wikipedia.org/wiki/Butterfly_effect.

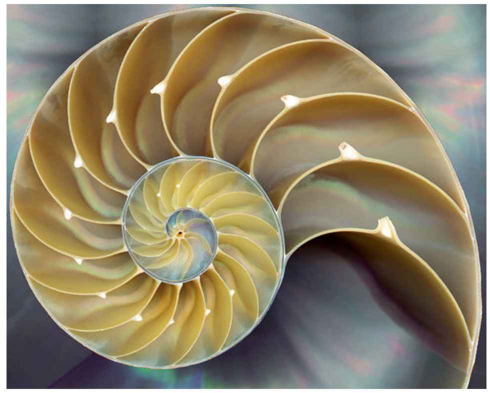


Figure 5: Cross-section of a Nautilus shell that demonstrates the logarithmic spiral fractal[12]

There was a long process to settle on the formal definition of a fractal. Benoit Mandelbrot, the mathematician who first used the term "fractal" to describe this kind of structures in 1975, simplified his original definition to be that "a fractal is a shape made of parts similar to the whole in some way".[13] Fractals are *self-similar* objects that has the property of being approximately the same to a part of itself. The definition might not tell us so much about fractals, so let's go back to the snowflake example!

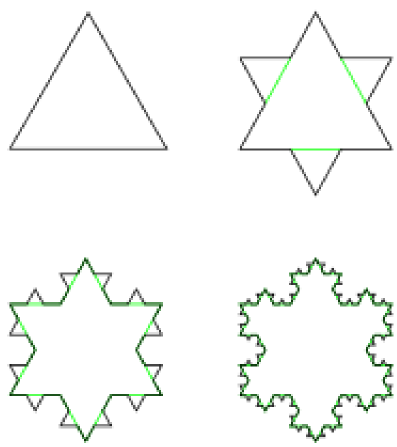


Figure 6: The first four iterations of the Koch snowflake[14]

The picture above is how the *Koch snowflake* looks like. Beginning with an equilateral triangle, we paste a smaller equilateral triangle onto the middle part of each side. If one repeats the pasting

[12] Picture source

[13] "Fractal." Wikipedia, Wikimedia Foundation, 28 Jan. 2020, https://en.wikipedia.org/wiki/Fractal.

[14] "Koch Snowflake." Wikipedia, Wikimedia Foundation, 15 Jan. 2020, https://en.wikipedia.org/wiki/Koch_snowflake.

business for infinite times, a fractal of the Koch snowflake is defined. This probably sounds very bizarre as it is difficult to imagine repeating a process by infinite times. The idea behind it is simple. If we keep zooming into any segment of the Koch snowflake, the visual pattern seems to be repeating itself, but in fact each edge and spike of the snowflake is smaller compared to the initial ones.

You might have noticed that the perimeter of the Koch snowflake is infinite as there are infinite number of repeating spikes. On the other hand, the area of the Koch snowflake is finite. Clearly, we can inscribe the Koch snowflake in a circle, and a circle has a finite area. This is a very counter-intuitive property of fractals.

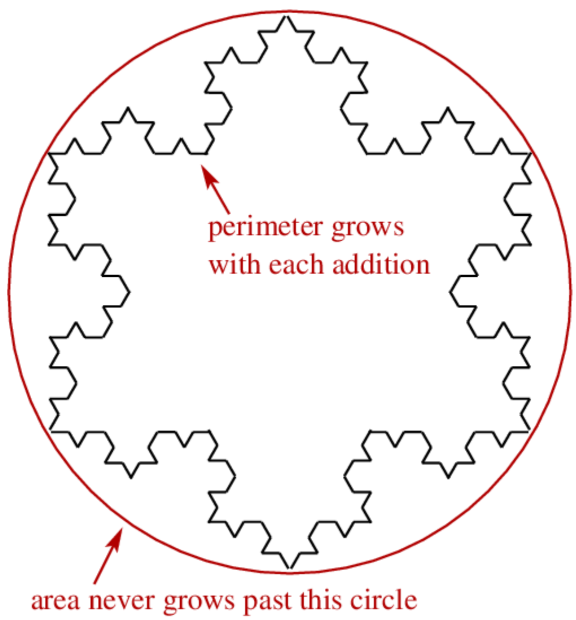


Figure 7: How Koch snowflake has a finite area but infinite perimeter[15]

Let's go back to the Lorenz attractor example. It also has a infinite perimeter and finite area that inscribed into a butterfly shape, just like a fractal. If we can input the exact same initial condition to the system of Lorenz equations each time, it will return the same result; in other words, the same loop of the path will be traced on the graph. This might seem contradicting as this system, the computer simulation of weather, is both deterministic and unpredictable! The following figure demonstrates the sensitivity of Lorenz's model of the weather. The yellow line and blue line represent the path inputting by different values that vary by a hundred thousandth digit. Initially, at $t = 1$, the two trajectories seem coincident as only the yellow one can be seen, drawing over the blue one. However, after some time ($t = 3$), one can see there are not overlapping anymore; the divergence is obvious.[16]

[15] Francis, Matthew. "Fractals for Fun." Galileo's Pendulum, 31 Jan. 2012, www.galileospendulum.org/2012/01/31/fractals-for-fun/.

[16] "Lorenz System." Wikipedia, Wikimedia Foundation, 28 Mar. 2020, https://en.wikipedia.org/wiki/Lorenz_system.

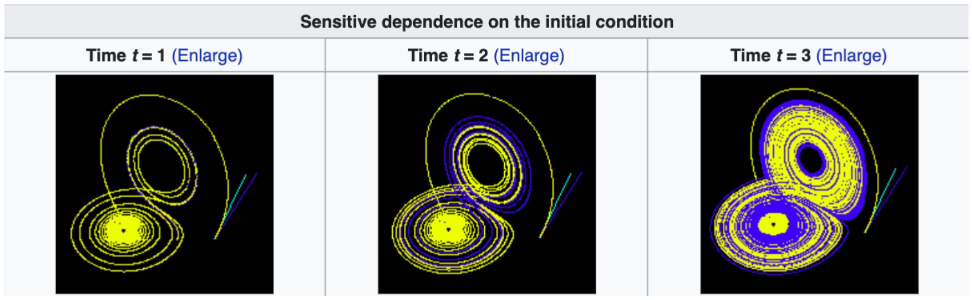


Figure 8: The sensitivity on initial conditions of the system of Lorenz equations[16]

In practice, we can never get the perfect precision of any initial condition that has infinite decimal places.[6] A 32 bits computer can store up to 4294967295 digits that is sufficient for approximation to some extent, but it is never prefect. Since it is impossible to input the exact same initial condition every time, it makes the system of weather chaotic. This is why even with current technologies that can create advanced artificial intelligence like Alpha Go, we still cannot forecast the weather a week advance. Studies have shown that by the eighth day, the prediction of weather is less accurate than by taking the average of the historical temperature.

People have many mathematical tools to understand this world, through using dynamical systems for modeling. We study properties of a Markov Chain for practical usage through dynamics. Scientific or Economic modelling involves building dynamical systems — to simplify the complicated reality into a set of equations and use computers to study the behaviour of the simulated system.[17] However, the weather example is like the red sky at morning, a warning bell that starts to ring. Making predictions is less easy than what determinists would expect, at least when it comes to chaotic systems that model the weather. There are many more chaotic systems which have behaviours like fractals in the world that hinder us from predicting the future. People have suggested to use fractals to predict the weather, but the rounding error is still a big challenge that we have to face. Maybe someone will be able to develop an accurate and efficient way to store data, including initial conditions perfectly, and then we might predict the future. But life might be boring if we were not living in a fog, finding our path in this world with non-predetermined events.

[17] "Calculus, Analysis, and Dynamical Systems." Maths Careers, www.mathscareers.org.uk/article/calculus-analysis-dynamical-systems/