1. For a dynamical system (T, X), a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.

   (a) Let $(T, [0,1))$ be the doubling map and let (σ, Ω) be the full two shift. i) A point that is eventually periodic for T is 0 and for σ is (0, 0, 0, ...) as $m = 2$, $m' = 1$ for both T and σ, and ii) a point that is eventually periodic but *not* periodic for T is $x = 1/2$ for $m = 2$, $m' = 1$ as $T^2(x) = T^1(x) = 0 \neq x$, and for σ the point $y = (1, 1, 1, 1, 0, 0, 0, ...)$ for $m = 5, m' = 4$, and it is clear that y is not periodic.

   (b) Let $\mathbb{C}$ be the coding function for the doubling map with the usual partition ($\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$).
   We want to show that $\mathbb{C}(x)$ is eventually periodic in (σ, Ω) if and only if x is eventually periodic in $(T, [0,1))$.
   Notice that applying T to any point $x \in [0,1)$ is equivalent as shifting one symbol in $\mathbb{C}(x)$, i.e., applying σ to $\mathbb{C}(x)$. This means that $\mathbb{C}(T(x)) = \sigma(\mathbb{C}(x))$. $(*)$
   $(\implies)$ Assume $\mathbb{C}(x)$ is eventually periodic in (σ, Ω), which means there exists $m > m'$ so that $\sigma^m(\mathbb{C}(x)) = \sigma^{m'}(\mathbb{C}(x))$.
   Then, $\sigma^m(\mathbb{C}(x)) = \mathbb{C}(T^m(x)) = \sigma^{m'}(\mathbb{C}(x)) = \mathbb{C}(T^{m'}x)$ by $(*)$.
   Since $\mathbb{C}$ is one-to-one (proved in class), we know $\mathbb{C}(T^m x) = \mathbb{C}(T^{m'}x) \implies T^m x = T^{m'} x$.
   Then, we know x is eventually periodic in $(T, [0,1))$ as well since there exists $n > n'$ so that $T^n x = T^{n'} x$ if we pick $n = m, n' = m'$.
   $(\impliedby)$ Similarly, assume x is eventually periodic in $(T, [0,1))$, i.e., there exists $m > m'$ s.t. $T^m x = T^{m'} x$, then $\mathbb{C}(T^m x) = \mathbb{C}(T^{m'} x)$.
   We know $\mathbb{C}(T^m x) = \sigma^m(\mathbb{C}(x)) = \mathbb{C}(T^{m'} x) = \sigma^{m'}(\mathbb{C}(x))$ by $(*)$.
   Thus, we know $\mathbb{C}(x)$ is eventually periodic in (σ, Ω), as there exists $n > n'$ so that $\sigma^n(\mathbb{C}(x)) = \sigma^{n'}(\mathbb{C}(x))$ if we pick $n = m, n' = m'$.
   This completes our proof.

   (c) We want to show that $\mathbb{C}(x)$ is eventually periodic if and only if x is a rational number.
   $(\implies)$ Assume $\mathbb{C}(x)$ is eventually periodic in (σ, Ω). From 1*b*), we know x is eventually periodic in $(T, [0,1))$, i.e., there exists $a > b$ so that $T^a x = T^b x$. Then, $2^a x = 2^b x \mod 1$. Then, $2^a x - 2^b x = x(2^a - 2^b) = 0 \mod 1$, and $x = \frac{p}{q}$ is rational if and only if $q \mid 2^a - 2^b$. Since $a > b$, we know $q \mid 2^a - 2^b = 2^b(2^a - 1) \geq 2^b$. Thus, x is rational.
   $(\impliedby)$ x is rational means the orbit of x, $O(x)$ under T the doubling map, is a finite set. Suppose $O(x)$ has m elements, then we know $\sigma^m(\mathbb{C}(x)) = \sigma^{m+m}(\mathbb{C}(x))$ as $\mathbb{C}(x)$ repeats every m digits by the property of coding function $\mathbb{C}$. Thus, there exists $m' > m$ such that $\sigma^m(\mathbb{C}(x)) = \sigma^{m'}(\mathbb{C}(x))$, so $\mathbb{C}(x)$ is eventually periodic in (σ, Ω).

   (d) We want to show that the binary expansion of x in $[0,1)$ is eventually periodic if and only if that x is rational.
   $(\implies)$ Assume the binary expansion of x in $[0,1)$, call it $C_x = (x_1, x_2, x_3 ...) \in \{0,1\}^{\mathbb{N}}$, is eventually periodic, i.e., there exists $m > m'$ s.t. $\sigma^m(C_x) = \sigma^{m'}(C_x)$ where σ is the full two-shift. If x can be generated by the coding function $\mathbb{C}$, i.e., $C_x = \mathbb{C}(x)$, then by 1*c*), x is rational.
   Suppose x cannot be generated by the coding function $\mathbb{C}$, then C_x has to be in the form of $C_x = (1, 1, 1, 1, ...)$ (discussed in class). Applying $\mathbb{E}$, the encoding function defined in the class notes, to C_x that transforms a binary expansion back to x, we get a convergent geometric series $\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} = x$, which is rational number, as needed.
   $(\impliedby)$ Assume x is rational. By 1*c*), we know that x is eventually periodic in $(T, [0,1))$. By 1*b*), we know $\mathbb{C}(x)$ is also eventually periodic in (σ, Ω). From the hint, we know that $\mathbb{C}(x)$ is always a binary expansion of x. Thus, the binary expansion of x, $\mathbb{C}(x)$ is eventually periodic.

   (e) We want to show that that the base n expansion of a number x in $[0,1)$ is eventually periodic if an only if that number x is rational. We generalize the $\mathbb{C}, \mathbb{E}$ into n-base encoding and decoding function, and everything else just follows from 1*d*).

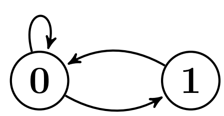
2. Let (σ, Ω) be the full two-shift. Let $G \subseteq \Omega$ bet the set of sequences without two ones in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* ones in a row.

   (a) We want to show that (σ, X) is a subshift.
   First, we know X is σ-invariant because $\sigma(X) = X$, as any sequence without three 1's in a row under σ still would not have three 1's in a row.
   Next, we want to show X is closed, i.e., it contains all its limit points.
   Suppose it is not, i.e., there exists a limit point, $y = (a_0, a_1, ..., a_n, 1, 1, 1, a_{n+4}, ...), n \in \mathbb{N}$ s.t. $y \notin X$.
   If y is a limit point, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ where each $x_n \in X$ and the sequence converges to y.
   This means that $d(x_n, y) \to 0$ as $n \to \infty$, but for all $x \in X$, $d(x, y) \geq \frac{1}{2^{n+3}} > 0$ (the closest point to y in X is $(a_0, a_1, ..., a_n, 1, 1, 0, a_{n+4}, ...)$, which is $\frac{1}{2^{n+3}}$ away from y).
   This is a contradiction.
   Thus, X is closed.

   (b) The set of possible transitions for $\mathcal{M}'$ is the same as $\mathcal{M}$. For $\mathcal{M}'$, the edge from state 00 to state 01 means the Markov chain steps from state 0 to state 0 then to state 1. This is equivalent as saying there are two edges, one from state 0 to state 0 and the other from state 0 to state 1 in $\mathcal{M}$. Similarly, the edge from state 10 to state 01 in $\mathcal{M}'$ means it steps from state 1 to state 0 to state 1, i.e., there are two edges from state 0 to state 1 and vice versa in $\mathcal{M}$. $\mathcal{M}'$ seems to have more edges as it considers the previous possible states. For example, for the edge that goes from state 0 to state 1, we could have it was previously on state 0 then to 0 and to 1, like $0 \to 0 \to 1$, or it was previously on state 1 then to 0 and to 1, like $1 \to 0 \to 1$, which are $00 \to 01$, $10 \to 01$ respectively in $\mathcal{M}'$. Thus, M' models M. In fact, M' can model more complicated situation, such as sequences that do not have three ones in a row.
   The graph for $\mathcal{M}'$ has no edge between 00 and 11 since you could not have previously stepped on state 0 then suddenly going from state 1 to state 1. If there is an edge between 00 and 11, this is saying that your path is $0 \to (0 \text{ and } 1) \to 1$. There is an ambiguity in the transition. You cannot be on both states in the same time.

   (c) Let $\mathcal{M}_G$ be a Markov chain whose set of realizations is G.

      i. We know that G is the set of sequences without two ones in a row, so there is no self-loop at state 1. Thus, the Markov chain graph $\mathcal{M}_G$ is

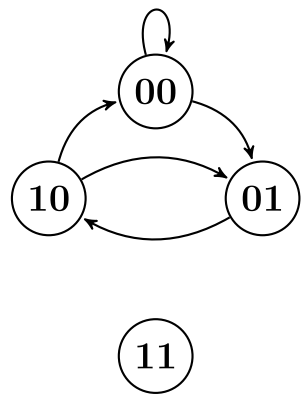
      

      And the transition matrix is
      $$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
      where the order of states of the transition matrix follows the order of $\{0, 1\}$ along the columns (left to right) and rows (going down).

      ii. We want to use the transition matrix A for $\mathcal{M}_G$ to compute the entropy of (σ, G). In class, we know A^{n+1}'s $(1, 1)$ entry is the number of words with length n in G, and the number of words with length n in Ω is 2^n.
      Moreover, the eigenvalues for A are $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$.
      A is diagonalizable so $A^{n+1} = PD^{n+1}P^{-1}$.
      Since $\lambda_1 > 1, |\lambda_2| < 1$, we get $\lim_{n\to\infty} D^{n+1} = \begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & 0 \end{bmatrix}$ for sufficiently large n.
      Then, the A^{n+1}'s $(1, 1)$ entry taking $n \to \infty$ is λ_1^{n+1} for sufficiently large n, which is also the number of words with length n in G, and the entropy is
      $$H(G) = \lim_{n\to\infty} \frac{\log(\# \text{ words of length } n \text{ in } G)}{\log(\# \text{ words of length } n \text{ in } \Omega)}$$

$$= \lim_{n\to\infty} \frac{\log(\lambda_1^{n+1})}{\log(2^n)} = \frac{\log(\lambda_1)}{\log(2)} = \frac{\log(1.618)}{\log(2)}.$$

iii. We want to model $\mathcal{M}_G$ as a two-step Markov chain $\mathcal{M}'_G$. We know there is no edges going in or out of state 11 as G contains sequences with no two one's in a row.



And its transition matrix is

$$A' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the order of states of the transition matrix follows the order of $\{00, 01, 10, 11\}$ along the columns (left to right) and rows (going down).

iv. We want to use the transition matrix for $\mathcal{M}'_G$ to compute the entropy of (σ, G). Similarly as $2c)ii)$, after computing several powers of A' using wolfram-alpha, we know A'^{n+1}'s $(1,1)$ entry is the number of words with length n in G, and the number of words with length n in Ω is 2^n.
Moreover, the eigenvalues for A are $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}, \lambda_3 = \lambda_4 = 0$, where λ_3 has a $2-$dimensional eigenspace.
A' is diagonalizable, so $A'^{n+1} = P'D'^{n+1}P'^{-1}$.
Since $\lambda_1 > 1, |\lambda_2| < 1, \lambda_3 = \lambda_4 = 0$, we get

$$\lim_{n\to\infty} D'^{n+1} = \lim_{n\to\infty} \begin{bmatrix} \lambda_1^{n+1} & 0 & 0 & 0 \\ 0 & \lambda_2^{n+1} & 0 & 0 \\ 0 & 0 & \lambda_3^{n+1} & 0 \\ 0 & 0 & 0 & \lambda_4^{n+1} \end{bmatrix} = \begin{bmatrix} \lambda_1^{n+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for sufficiently large n.
Then, the A'^{n+1}'s $(1,1)$ entry is λ_1^{n+1} for sufficiently large n, which is also the number of words with length n in G, and the entropy is

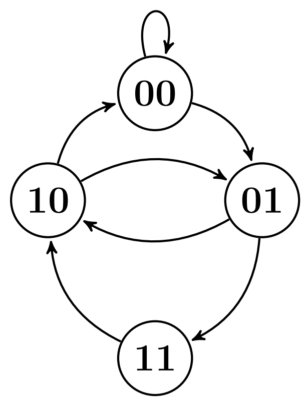
$$H(G) = \lim_{n\to\infty} \frac{\log(\#\text{ words of length } n \text{ in } G)}{\log(\#\text{ words of length } n \text{ in } \Omega)}$$

$$= \lim_{n\to\infty} \frac{\log(\lambda_1^{n+1})}{\log(2^n)} = \frac{\log(\lambda_1)}{\log(2)} = \frac{\log(1.618)}{\log(2)}$$

same as $2c)ii)$.

(d) X cannot be modeled by a one-step Markov chain because one-step Markov chain only tell us the transition to next step given the step before, but X requires to have no three 1's in a row, so we need to have restriction on the next step given the previous two stepped states.

(e) We want to find the entropy of (σ, X). We set up a two-step Markov chain and its transition matrix to compute it. We know there is no self loop for state (11) as that will produce three 1's in a row.



And its transition matrix is

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where the order of states of the transition matrix follows the order of $\{00, 01, 10, 11\}$ along the columns (left to right) and rows (going down).
We want to use the transition matrix for $\mathcal{M}'_G$ to compute the entropy of (σ, G). Similarly as $2c)iv)$, after computing several powers of B using wolfram-alpha, we know B^{n+2}'s $(1, 1)$ entry is the number of words with length n in X, and the number of words with length n in Ω is still 2^n, $n \in \mathbb{N}^+$.
Moreover, the eigenvalues for B are $\lambda_1 = 1.83929, \lambda_2 = -0.419643 - 0.606291i, \lambda_3 = -0.419643 + 0.606291i, \lambda_4 = 0$.
B is diagonalizable with complex eigenvalues, so $B^{n+2} = QE^{n+2}Q^{-1}$.
Since $\lambda_1 > 1, |\lambda_2| = |\lambda_3| = 0.737353 < 1, \lambda_4 = 0$, we get

$$\lim_{n\to\infty} E^{n+2} = \lim_{n\to\infty} \begin{bmatrix} \lambda_1^{n+2} & 0 & 0 & 0 \\ 0 & \lambda_2^{n+2} & 0 & 0 \\ 0 & 0 & \lambda_3^{n+2} & 0 \\ 0 & 0 & 0 & \lambda_4^{n+2} \end{bmatrix} = \begin{bmatrix} \lambda_1^{n+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for sufficiently large n.
Then, the B^{n+2}'s $(1, 1)$ entry is λ_1^{n+2} for sufficiently large n, which is also the number of words with length n in X, and the entropy becomes

$$H(X) = \lim_{n\to\infty} \frac{\log(\#\text{ words of length } n \text{ in } X)}{\log(\#\text{ words of length } n \text{ in } \Omega)}$$

$$= \lim_{n\to\infty} \frac{\log(\lambda_1^{n+2})}{\log(2^n)} = \lim_{n\to\infty} \frac{\log(\lambda_1^n) + log(\lambda_1^2)}{\log(2^n)}$$

$$= \lim_{n\to\infty} \frac{\log(\lambda_1^n)}{\log(2^n)} + \lim_{n\to\infty} \frac{log(\lambda_1^2)}{\log(2^n)} = \frac{\log(\lambda_1)}{\log(2)} = \frac{\log(1.83929)}{\log(2)}$$

3. Let (σ, Ω) be the full two shift.

   (a) We want to show that (σ, Ω) is expansive.
   Fix $x \in \Omega$. Pick $\epsilon = \frac{1}{4}$. Fix $y \neq x \in \Omega$.
   Since $y \neq x$, then they at least first differs at some k^{th} digit, $k \in \mathbb{N}^+$. This means that $y = (a_1, a_2, ..., a_{k-1}, a_k, a_{k+1}, ...), x = (a_1, a_2, ..., a_{k-1}, b_k, b_{k+1}, ...)$ and $a_k \neq b_k$.
   We want to show there exists a $n \geq 0$ s.t. $d(\sigma^n(x), \sigma^n(y)) > \epsilon$.
   Pick $n = k-1 \geq 0$ s.t. $d(\sigma^m(x), \sigma^m(y)) = 1 > \epsilon = \frac{1}{4}$, shifting both x, y $k-1$ digits so the first different digit is at the first position to get a unit distance away.
   Thus, (σ, Ω) is expansive.

   (b) We want to show that (σ, Ω) is transitive, i.e., we want to find a transitive point $x \in \Omega$ s.t. $\overline{\mathcal{O}(x)} = \Omega$.
   Consider $x = 0|1|00|01|10|11|000|100|010|001|110|101|011|111|0000... \in \Omega$, the concatenation of all the finite strings using 0's and 1's.
   We want to show that $\overline{\mathcal{O}(x)} = \Omega$, i.e., every element in Ω is a limit point of $\mathcal{O}(x)$.
   Fix $\alpha \in \Omega$, we know there exists a $n \in \mathbb{N}$ s.t. α has $n-$symbols in x and $2^{-n} < \epsilon$.
   Thus, $d(\alpha, x) = 2^{-n} < \epsilon$, so α is a limit point of $\mathcal{O}(x)$.
   Then, $\overline{\mathcal{O}(x)}$, the closure of the orbit of x, is Ω.

   (c) We want to show that (σ, Ω) is chaotic. From $3a$) and b), we know (σ, Ω) is both expansive and transitive. Thus, it is chaotic by definition.

   (d) Let $x, y \in \Omega$ be points that are at distance 2^{-k} of each other, i.e., the first digit that they differ at is the $k+1^{th}$ digit. We want to see how many times we can apply σ to x, y so their distance is at most $\frac{1}{4}$ to get the number of steps for x, y being $\frac{1}{4}-$correlated. We know the first different digit for x, y at $k+1^{th}$ position can be shifted at most to the 3^{rd} position so their distance is at most $\frac{1}{4}$, then the number of digits that we can shift is $k+1-3 = k-2$. Since applying each σ is shifting one symbol of x, y, we know the number of steps is $k-2$ s.t. $d(\sigma^n(x), \sigma^n(y)) \leq \frac{1}{4}$ for $n = 0, 1, 2, ..., k-2$.
   Thus, the number of steps is $k-2$ so that x, y are $\frac{1}{4}-$correlated.

   (e) Assume (T, X) is a chaotic dynamical system, so it is expansive and transitive. Then, we know there is a transitive point x in X s.t. $\overline{\mathcal{O}(x)} = X$, and there exists ϵ such that for all $y \neq x \in X$, $d(T^n(x), T^n(y)) > \epsilon$ for some $n \geq 0$. Since the transitive point eventually visits every point in X ($\overline{\mathcal{O}(x)} = X$), we get no two distinct points can be ϵ-correlated indefinitely. This also tells us that knowing two points are close at the start does not say anything about them after a certain time/few transformations of T. This explains the sensitivity of initial condition to chaotic system.

4. (a) Let (T, X) and (S, Y) be conjugate dynamical systems.

   i. Assume there exists a point y of period k in Y, i.e., $S^k(y) = y$. We want to show that there exists a point of period k in X.
   Claim: $\Phi(y) = x \in X$ is a point of period k, where $\Phi : Y \to X$ is a conjugacy for T and S.
   We first check $T^k(x) = \Phi(S^k(\Phi^{-1}(x))) = \Phi(S^k(y))) = \Phi(y) = x$.
   Next, we want to check there does not exist a $k' < k$ s.t. $T^{k'}(x) = x$.
   Suppose there is such k', then $T^{k'}(x) = x = \Phi(S^{k'}(\Phi^{-1}(x))) = \Phi(S^{k'}(y)))$.
   We know Φ is invertible, so Φ is bijective, especially one-to-one.
   Then, we get $\Phi(S^{k'}(y)) = x \implies S^{k'}(y) = \Phi^{-1}(x) = y$. This means y has a period $k' < k$. This is a contradiction.

   ii. Assume (S, Y) is transitive, i.e., it has a transitive point $y \in Y$ where $\overline{\mathcal{O}(y)} = Y$.
   We want to show that (T, X) is transitive.
   Claim: $\Phi(y) = x \in X$ is a transitive point of X, where $\Phi : Y \to X$ is a conjugacy.
   We know $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(\Phi(y))} = \overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}}$.
   From assumption, we know $\overline{\mathcal{O}(y)} = \overline{\{S^n(y) : n \in \mathbb{N}\}} = Y$.
   Since Φ is invertible, it is also bijective, so $\Phi(Y) = X$, then,
   $$\Phi(\mathcal{O}(y)) = \Phi(\overline{\{S^n(y) : n \in \mathbb{N}\}}) = \Phi(Y) = X$$
   and by $\Phi : Y \to X$ being a conjugacy for Y and X, we know $T = \Phi \circ S \circ \Phi^{-1} \implies T^n = \Phi \circ S^n \circ \Phi^{-1}$, so $T^n(\Phi(y)) = \Phi(S^n(y))$ for all $y \in Y$, then
   $$\Phi(\overline{\{S^n(y) : n \in \mathbb{N}\}}) = \overline{\{\Phi(S^n(y)) : n \in \mathbb{N}\}} = \overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}}$$
   Thus, we get
   $$\overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}} = \overline{\mathcal{O}(\Phi(y))} = X$$
   as needed.

   iii. No, consider the $(T, (0, \infty))$ the doubling map, $(S, (0, \infty))$ to be the halving map where $S(x) = \frac{1}{2}x$. Define $\Phi : (0, \infty) \to (0, \infty)$ s.t. $\Phi(x) = \frac{1}{x}$. Know Φ is a conjugacy where $\Phi^{-1}(S(\Phi(x))) = \Phi^{-1}(\frac{1}{2x}) = 2x = T(x)$ for all $x \in (0, \infty)$.
   We know the doubling map T is chaotic from $4e$), so it is expansive, but the halving map is not expansive.

   (b) Assume (T, X) and (S, Y) are conjugate dynamical systems, i.e., there exists a continuous, invertible function $\Phi : Y \to X$ so that $T = \Phi \circ S \circ \Phi^{-1}$.
   We want to show that they are also semi-conjugate dynamical systems, i.e., there exists continuous, onto function $\Phi' : Y \to X$ so that $T \circ \Phi' = \Phi' \circ S$.
   Pick $\Phi' = \Phi$. Φ that is continuous and invertible is also on-to, and $T = \Phi' \circ S \circ \Phi'^{-1} \implies T \circ \Phi' = \Phi' \circ S$, as needed.

   (c) Let (T, X) and (S, Y) be semi-conjugate dynamical systems.

   i. Assume there exists a point y of period k in Y, i.e., $S^k(y) = y$. We want to show that there exists a point of period $\leq k$ in X.
   Claim: $\Phi(y) = x \in X$ is a point of period $\leq k$, where $\Phi : Y \to X$ is a semi-conjugacy s.t. $T(\Phi(y))) = \Phi(S(y)))$
   We first check $T^k(x) = T^k(\Phi(y))) = \Phi(S^k(y))) = \Phi(y) = x$.
   Unlike $4a)i)$, we do not have ϕ being one-to-one, so we only know $\Phi(y) = x$ is a point of period $\leq k$.

   ii. Assume (S, Y) is transitive, i.e., it has a transitive point $y \in Y$ where $\overline{\mathcal{O}(y)} = Y$.
   We want to show that (T, X) is transitive.
   Claim: $\Phi(y) = x \in X$ is a transitive point of X, where $\Phi : Y \to X$ is a semi-conjugacy.
   We know $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(\Phi(y))} = \overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}}$.

From assumption, we know $\overline{\mathcal{O}(y)} = \overline{\{S^n(y) : n \in \mathbb{N}\}} = Y$.
Since Φ is on-to, $\Phi(Y) = X$, then,

$$\Phi(\mathcal{O}(y)) = \Phi(\overline{\{S^n(y) : n \in \mathbb{N}\}}) = \Phi(Y) = X$$

and by $\Phi : Y \to X$ being a semi-conjugacy for Y and X, we know $T \circ \Phi = \Phi \circ S$, so $T^n(\Phi(y)) = T^{n-1}(T(\Phi(y)) = T^{n-1}(\Phi S(y)) = ... = \Phi(S^n(y))$ for all $y \in Y$, then

$$\Phi(\overline{\{S^n(y) : n \in \mathbb{N}\}}) = \overline{\{\Phi(S^n(y)) : n \in \mathbb{N}\}} = \overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}}$$

Thus, we get

$$\overline{\{T^n(\Phi(y)) : n \in \mathbb{N}\}} = \overline{\mathcal{O}(\Phi(y))} = X$$

as needed.

iii. Similar as 4*a*)*iii*), consider the $(T, (0, \infty))$ the doubling map, $(S, (0, \infty))$ to be the halving map where $S(x) = \frac{1}{2}x$. Define $\Phi : (0, \infty) \to (0, \infty)$ s.t. $\Phi(x) = \frac{1}{x}$. Know Φ is a conjugacy where $\Phi^{-1}(S(\Phi(x))) = \Phi^{-1}(\frac{1}{2x}) = 2x = T(x)$ for all $x \in (0, \infty)$.
From 4*b*), we know they are also semi-conjugate.
We know the doubling map T is chaotic from 4*e*), so it is expansive, but the halving map is not expansive.

(d) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift.

i. We want to show that $(T, [0, 1))$ and (σ, Ω) are semi-conjugate, i.e., there exists a $\Phi : Y \to X$ that is a semi-conjugacy of Y and X and $T(\Phi(y))) = \Phi(\sigma(y)))$.
Consider the encoding function $\mathbb{E} : \Omega \to [0, 1)$, we know it is on-to from class. It is also continuous by sequence convergence.
Fix a convergent sequence $(x_n) \to x$ in Ω. We want to show that $\mathbb{E}(x_n) \to \mathbb{E}(x)$.
Pick $\Phi = \mathbb{E}$. Check if $T(\Phi(y))) = \Phi(\sigma(y)))$ for all $y \in \Omega$.
Fix $y = (a_1, a_2, a_3, ...) \in \Omega$.

$$T(\Phi(y)) = T(\mathbb{E}((a_1, a_2, a_3, ...))) = T(\sum_{i=1}^{\infty} \frac{a_i}{2^i}) = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$$

$$= \mathbb{E}((a_2, a_3, ...)) = \mathbb{E}(\sigma(y)) = \Phi(\sigma(y))$$

as needed.

ii. We want to show that $(T, [0, 1))$ is chaotic.
By 4*d*)*i*), we know $(T, [0, 1))$ and (σ, Ω) are semi-conjugate. By 3*b*) and 4*c*)*ii*), we know $(T, [0, 1))$ is transitive.
We want to show that the doubling map T is also expansive.
Fix $x \in [0, 1)$. Pick $\epsilon = \frac{1}{4}$. Fix $y \neq x \in [0, 1)$.
Observe that if $d(x, y) < \frac{1}{4}$, then $d(T(x), T(y)) \geq 2d(x, y)$. $(*)$
If $d(x, y) \geq \frac{1}{4}$, then we are done as $d(T^n(x), T^n(y)) \geq 2^n d(x, y) \geq \frac{1}{4}$ for $n = 0$.
If $d(T(x), (y)) \geq \frac{1}{4}$, then we are also done as $d(T^n(x), T^n(y)) = d(T(x), T(y)) \geq 2^n d(x, y) \geq \frac{1}{4}$ for $n = 1$.
Otherwise for $d(x, y) < \frac{1}{4}$, we can generalize $(*)$ such that:
If for all $0 \leq k \leq n-1, n \in \mathbb{N}^+$, $d(T^k(x), T^k(y)) < \frac{1}{4}$, then $d(T^n(x), T^n(y)) \geq 2^n d(x, y)$.
We know there exists $n \geq 0$ s.t. $2^n d(x, y) \geq \epsilon = \frac{1}{4}$ since $x \neq y$.
Thus, $d(T^n(x), T^n(y)) \geq 2^n d(x, y) \geq \epsilon = \frac{1}{4}$, as needed.

(e) Let $(T, [0, 1))$ be the doubling map and let $(L, [0, 1)$ be the *logistic map* defined by $L(x) = rx(1-x)$ for $r = 4$.

i. Define $f : [0, 1) \to [0, 1)$ by $f(x) = \sin^2(2\pi x)$. We want to show that that f is a semi-conjugacy between $(L, [0, 1))$ and $(T, [0, 1))$, i.e., $L \circ f = f \circ T$.
We know, using trigonometric identities,

$$L(f(x)) = 4\sin^2(2\pi x)(1 - \sin^2(2\pi x)) = 4\sin^2(2\pi x)\cos^2(2\pi x)$$

$$= (2\sin(2\pi x)\cos(2\pi x))^2 = (\sin(4\pi x))^2 = \sin^2(4\pi x)$$

and

$$f(T(x)) = \sin^2(2\pi(2x)) = \sin^2(4\pi x)$$

for all $x \in [0, 1)$.
Thus, $L \circ f = f \circ T$, as needed.

ii. We want to show that $(L, [0, 1))$ contains points of every period.
First, observe that $f(x) = \sin^2(2\pi x)$ and

$$L(f(x)) = 4\sin^2(2\pi x)\cos^2(2\pi x) = \sin^2(4\pi x) = f(2x)$$

So if a point x has even period k in T, then the point $f(x)$ has $k/2$ period in L. $(*)$
Claim: If $2 \nmid q$, then $\mathcal{O}(\frac{\infty}{\infty})$, the orbit of $\frac{1}{q}$ in T, is symmetric about $\frac{1}{2}$ but not $\frac{1}{4}$.
Proof: First, notice that q is odd and greater than one. We know $\mathcal{O}(\frac{1}{q}) = \{\frac{1}{q}, \frac{2}{q}, ..., \frac{q-1}{q}\}$, which we divides the interval $[0, 1)$ into $q - 1$, even number of sub-intervals.
Since q is odd, the largest point in the orbit of $\frac{1}{q}$ that is less than $\frac{1}{2}$ is $\frac{1}{2q}$ distance away from $\frac{1}{2}$, and the smallest point in the orbit that is greater than $\frac{1}{2}$ is $\frac{1}{2q}$ distance away from $\frac{1}{2}$ as well. However, the largest point in the orbit of $\frac{1}{q}$ that is less than $\frac{1}{4}$ is $\frac{3}{4q}$ distance away from $\frac{1}{4}$, and the smallest point in the orbit that is greater than $\frac{1}{4}$ is $\frac{1}{4q}$ distance away from $\frac{1}{4}$.
Combined with $(*)$, we know if $2 \nmid q$, then $\frac{1}{q} \in [0, 1)$ has $q - 1$ period in T, so $f(\frac{1}{q})$ has period $\frac{q-1}{2}$ in L.
Thus, we get L contains points of every period as T contains points of every even period (using points $\frac{1}{q} \in [0, 1), q \in \mathbb{N}$ and $2 \nmid q$).

iii. We want to show that $(L, [0, 1))$ is chaotic.
By 4*e*)*i*), we know $(L, [0, 1))$ and $(T, [0, 1))$ are semi-conjugate. By 3*b*) and 4*c*)*ii*), we know $(L, [0, 1))$ is also transitive.
We want to show that $(L, [0, 1))$ is expansive.
Fix $x \in [0, 1)$. Pick $\epsilon =$. Fix $y \neq x \in [0, 1)$.
I am not sure... :(

Homework 4

Do the programing part of Homework 4 in this notebook. Predefined are function *stubs*. That is, the name of the function and a basic body is predefined. You need to modify the code to fulfil the requirements of the homework.

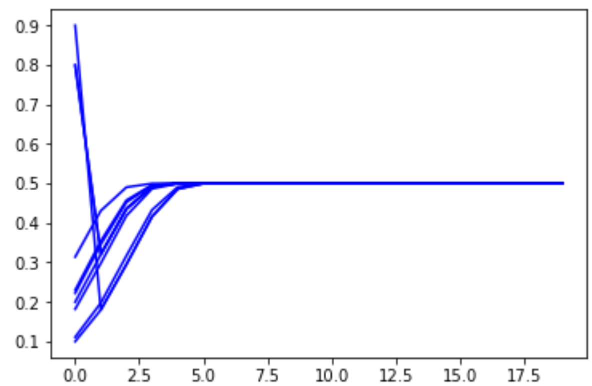
In [1]:
```
# import numpy and matplotlib
import numpy as np
import matplotlib.pylab as plt
from collections import Counter
# We give the matplotlib instruction twice, because firefox sometimes gets upset if we don't.
# note these `%`-commands are not actually Python commands. They are Jupyter-notebook-specific
 commands.
%matplotlib notebook
%matplotlib notebook
```

In [2]:
```
# The logistic map
def f(x, r):
    return r*x*(1-x)
# find the oribit of a point under the logistic map with parameter `r`.
def orbit_f(x_0, r, n=1000):
    ret = [x_0]
    curr = x_0
    for i in range(n-1):
        curr = f(curr, r)
        ret.append(curr)
    return ret

print(orbit_f(0.2, 1, 10))
```

```
[0.2, 0.16000000000000003, 0.13440000000000002, 0.11633664, 0.1028024261935104, 0.09223408736
223825, 0.08372696049069325, 0.07671675657768315, 0.07083129583788365, 0.06581422336780986]
```

In [3]:
```
#
# Plot n-orbits of x from various x's using a paramter of r=2
#
r = 2
many_xs = np.linspace(0, 19, 20) #time steps
many_ys_x0 = orbit_f(0.1, r, 20) #x0 = 0.1
many_ys_x1 = orbit_f(0.2, r, 20) #x1 = 0
many_ys_x2 = orbit_f(0.9, r, 20) #x2 = 0.9
many_ys_x3 = orbit_f(0.314, r, 20) #x3 = 0.314
many_ys_x4 = orbit_f(0.22222, r, 20) #x4 = 0.22222
many_ys_x5 = orbit_f(0.798322, r, 20) #x5 = 0.798322
many_ys_x6 = orbit_f(0.11111111, r, 20) #x6 = 0.11111111
many_ys_x7 = orbit_f(0.8, r, 20) #x7 = 0.8
many_ys_x8 = orbit_f(0.23, r, 20) #x8 = 0.23
many_ys_x9 = orbit_f(0.182, r, 20) #x9 = 0.182

# create a new figure
fig = plt.figure()
# create an "axis" inside the figure
ax = fig.add_subplot(1, 1, 1)

#plot the 20-orbit for different x's
ax.plot(many_xs, many_ys_x0, color="blue")
ax.plot(many_xs, many_ys_x1, color="blue")
ax.plot(many_xs, many_ys_x2, color="blue")
ax.plot(many_xs, many_ys_x3, color="blue")
ax.plot(many_xs, many_ys_x4, color="blue")
ax.plot(many_xs, many_ys_x5, color="blue")
ax.plot(many_xs, many_ys_x6, color="blue")
ax.plot(many_xs, many_ys_x7, color="blue")
ax.plot(many_xs, many_ys_x8, color="blue")
ax.plot(many_xs, many_ys_x9, color="blue")
```

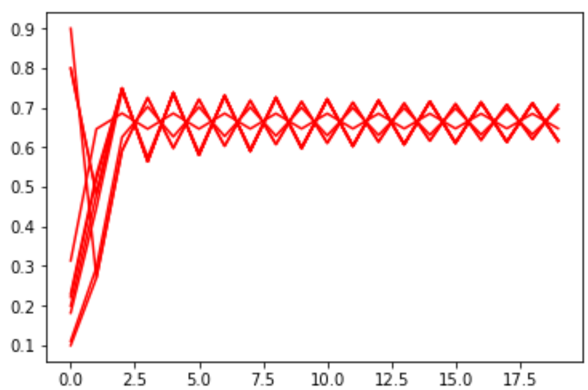
Out[3]: [<matplotlib.lines.Line2D at 0x7fec95414f60>]



I notice that for different x's, the orbit elements converge to the value around 0.5 for $r = 2$. This means, the population stablizes over time. The logistic map with $r = 2$ does have a basin of attraction A_x where x is the value around 0.5, and at least the 10 different x's except $x = 0$ are in A_x.

In [4]:
```
#
# Plot n-orbits of x from various x's using a paramter of r=3
#
r_2 = 3
many_xs = np.linspace(0, 19, 20) #time steps
many_ys_x0_r2 = orbit_f(0.1, r_2, 20) #x0 = 0.1
many_ys_x1_r2 = orbit_f(0.2, r_2, 20) #x1 = 0
many_ys_x2_r2 = orbit_f(0.9, r_2, 20) #x2 = 0.9
many_ys_x3_r2 = orbit_f(0.314, r_2, 20) #x3 = 0.314
many_ys_x4_r2 = orbit_f(0.22222, r_2, 20) #x4 = 0.22222
many_ys_x5_r2 = orbit_f(0.798322, r_2, 20) #x5 = 0.798322
many_ys_x6_r2 = orbit_f(0.11111111, r_2, 20) #x6 = 0.11111111
many_ys_x7_r2 = orbit_f(0.8, r_2, 20) #x7 = 0.8
many_ys_x8_r2 = orbit_f(0.23, r_2, 20) #x8 = 0.23
many_ys_x9_r2 = orbit_f(0.182, r_2, 20) #x9 = 0.182

# create a new figure
fig = plt.figure()
# create an "axis" inside the figure
ax = fig.add_subplot(1, 1, 1)

#plot the 20-orbit for different x's
ax.plot(many_xs, many_ys_x0_r2, color="red")
ax.plot(many_xs, many_ys_x1_r2, color="red")
ax.plot(many_xs, many_ys_x2_r2, color="red")
ax.plot(many_xs, many_ys_x3_r2, color="red")
ax.plot(many_xs, many_ys_x4_r2, color="red")
ax.plot(many_xs, many_ys_x5_r2, color="red")
ax.plot(many_xs, many_ys_x6_r2, color="red")
ax.plot(many_xs, many_ys_x7_r2, color="red")
ax.plot(many_xs, many_ys_x8_r2, color="red")
ax.plot(many_xs, many_ys_x9_r2, color="red")
```

Out[4]: [<matplotlib.lines.Line2D at 0x7fec953572e8>]

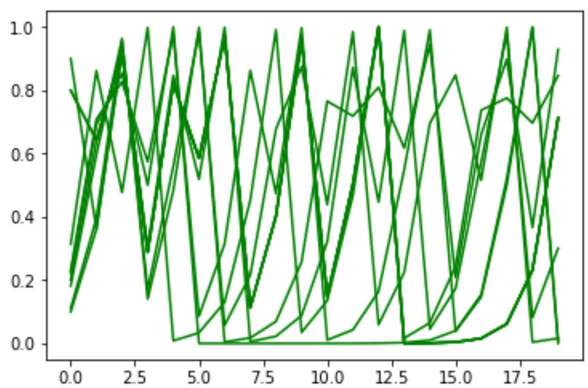


I notice that for different x's (except 0), the orbit elements alternate between two limit points for each x for $r = 3$. The logistic map with $r = 3$ does not have a basin of attraction as the elements alter.

In [5]:

```
#
# Plot n-orbits of x from various x's using a paramter of r=4
#
r_3 = 4
many_xs = np.linspace(0, 19, 20) #time steps
many_ys_x0_r3 = orbit_f(0.1, r_3, 20) #x0 = 0.1
many_ys_x1_r3 = orbit_f(0.2, r_3, 20) #x1 = 0
many_ys_x2_r3 = orbit_f(0.9, r_3, 20) #x2 = 0.9
many_ys_x3_r3 = orbit_f(0.314, r_3, 20) #x3 = 0.314
many_ys_x4_r3 = orbit_f(0.22222, r_3, 20) #x4 = 0.22222
many_ys_x5_r3 = orbit_f(0.798322, r_3, 20) #x5 = 0.798322
many_ys_x6_r3 = orbit_f(0.11111111, r_3, 20) #x6 = 0.11111111
many_ys_x7_r3 = orbit_f(0.8, r_3, 20) #x7 = 0.8
many_ys_x8_r3 = orbit_f(0.23, r_3, 20) #x8 = 0.23
many_ys_x9_r3 = orbit_f(0.182, r_3, 20) #x9 = 0.182

# create a new figure
fig = plt.figure()
# create an "axis" inside the figure
ax = fig.add_subplot(1, 1, 1)

#plot the 20-orbit for different x's
ax.plot(many_xs, many_ys_x0_r3, color="green")
ax.plot(many_xs, many_ys_x1_r3, color="green")
ax.plot(many_xs, many_ys_x2_r3, color="green")
ax.plot(many_xs, many_ys_x3_r3, color="green")
ax.plot(many_xs, many_ys_x4_r3, color="green")
ax.plot(many_xs, many_ys_x5_r3, color="green")
ax.plot(many_xs, many_ys_x6_r3, color="green")
ax.plot(many_xs, many_ys_x7_r3, color="green")
ax.plot(many_xs, many_ys_x8_r3, color="green")
ax.plot(many_xs, many_ys_x9_r3, color="green")
```

Out[5]: [<matplotlib.lines.Line2D at 0x7fecca74dba8>]



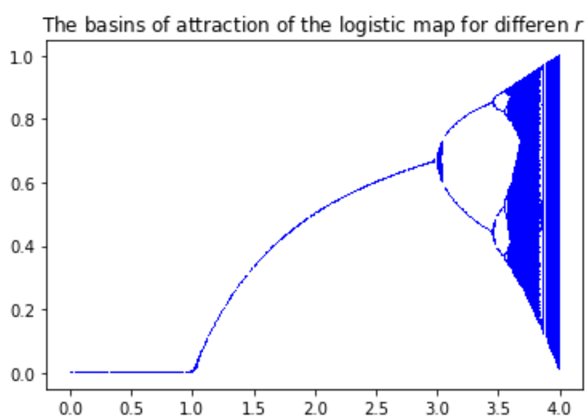
I notice that for different x's, the orbit elements alter insanely. The logistic map with $r = 4$ does not have a basin of attraction A_x for any x.

In [29]:
```
import random

def approximate_basin(r):
    list_pts = []
    for i in range(100): #try 100 x's
        value = random.uniform(0, 1)
        orbit = orbit_f(value, r, 1000)
        last_100 = orbit[-900:]
        rounded_last_100 = np.around(last_100, decimals = 4)
#         rounded_last_100 = []
#         for num in last_100:
#             rounded_last_100.append(round(num, 3))
        list_pts.extend(rounded_last_100)
    return np.unique(list_pts)

print("The basin of attraction for r=2 consists (approximately) of the points", approximate_bas
in(2))
```

```
The basin of attraction for r=2 consists (approximately) of the points [0.5]
```

In [37]:
```
#
# Plot the basins of attraction vs r
#

xs = [] # r's
ys = [] # basins of attractions of the logisics map

fig, ax = plt.subplots()

rs= [random.uniform(0, 4) for i in range(1000)] # get xs for 1000 r's

for r in rs:
    ys_r = approximate_basin(r)
    for num in ys_r:
        xs.append(r)
        ys.append(num)

ax.plot(xs, ys, ',', color="blue")

ax.set_title("The basins of attraction of the logistic map for differen $r$")
```

Out[37]:
```
Text(0.5, 1.0, 'The basins of attraction of the logistic map for differen $r$')
```



The plot of basin of attractions vs r looks very pretty. I did not expect that it would form a fractal (I have seen it before and it is still suprising).

In [45]:
```
# print out basin of attractions for r=4, r=3.82, and r=3.83
def approximate_basin_3_dec(r): #round to 3 decimals
    list_pts = []
    for i in range(100): #try 100 starting point x's
        value = random.uniform(0, 1)
        orbit = orbit_f(value, r, 1000)
        last_100 = orbit[-900:]
        rounded_last_100 = np.around(last_100, decimals = 3)
        list_pts.extend(rounded_last_100)
    return np.unique(list_pts)

print("The basin of attraction for r=4 consists (approximately) of the points", approximate_bas
in_3_dec(4))
print("The basin of attraction for r=3.82 consists (approximately) of the points", approximate_
basin_3_dec(3.82))
print("The basin of attraction for r=3.83 consists (approximately) of the points", approximate_
basin_3_dec(3.83))
print("The number of points in the basin of attraction for r=4 is",
      len(approximate_basin_3_dec(4)))
print("The number of points in the basin of attractions for r=3.82 is", len(approximate_basin_3
_dec(3.82)),
      ", which is greater than", len(approximate_basin_3_dec(3.83)),
      ", the number of points in the basin of attractions for r=3.83.")
```

```
The basin of attraction for r=4 consists (approximately) of the points [0.    0.001 0.002 ...
0.998 0.999 1.   ]
The basin of attraction for r=3.82 consists (approximately) of the points [0.164 0.165 0.166
0.167 0.168 0.169 0.17  0.171 0.172 0.173 0.174 0.175
 0.176 0.177 0.178 0.179 0.18  0.181 0.182 0.183 0.184 0.185 0.186 0.187
 0.188 0.189 0.19  0.191 0.192 0.193 0.194 0.195 0.196 0.197 0.198 0.199
 0.2   0.201 0.202 0.203 0.204 0.205 0.206 0.207 0.208 0.209 0.21  0.211
 0.212 0.213 0.214 0.215 0.216 0.217 0.218 0.219 0.22  0.221 0.222 0.223
 0.224 0.225 0.226 0.227 0.228 0.229 0.23  0.231 0.232 0.233 0.234 0.235
 0.236 0.237 0.238 0.239 0.24  0.241 0.242 0.243 0.244 0.245 0.246 0.247
 0.248 0.249 0.25  0.251 0.252 0.253 0.254 0.255 0.256 0.257 0.258 0.259
 0.26  0.261 0.262 0.263 0.264 0.265 0.266 0.267 0.268 0.269 0.27  0.271
 0.272 0.273 0.274 0.275 0.276 0.277 0.278 0.279 0.28  0.281 0.282 0.283
 0.284 0.285 0.286 0.287 0.288 0.289 0.29  0.291 0.292 0.293 0.294 0.295
 0.296 0.297 0.298 0.299 0.3   0.301 0.302 0.303 0.304 0.305 0.306 0.307
 0.308 0.309 0.31  0.311 0.312 0.313 0.314 0.315 0.316 0.317 0.318 0.319
 0.32  0.321 0.322 0.323 0.324 0.325 0.326 0.327 0.328 0.329 0.33  0.331
 0.332 0.333 0.334 0.335 0.336 0.337 0.338 0.339 0.34  0.341 0.342 0.343
 0.344 0.345 0.346 0.347 0.348 0.349 0.35  0.351 0.352 0.353 0.354 0.355
 0.356 0.357 0.358 0.359 0.36  0.361 0.362 0.363 0.364 0.365 0.366 0.367
 0.368 0.369 0.37  0.371 0.372 0.373 0.374 0.375 0.376 0.377 0.378 0.379
 0.38  0.381 0.382 0.383 0.384 0.385 0.386 0.387 0.388 0.389 0.39  0.391
 0.392 0.393 0.394 0.395 0.396 0.397 0.398 0.399 0.4   0.401 0.402 0.403
 0.404 0.405 0.406 0.407 0.408 0.409 0.41  0.411 0.412 0.413 0.414 0.415
 0.416 0.417 0.418 0.419 0.42  0.421 0.422 0.423 0.424 0.425 0.426 0.427
 0.428 0.429 0.43  0.431 0.432 0.433 0.434 0.435 0.436 0.437 0.438 0.439
 0.44  0.441 0.442 0.443 0.444 0.445 0.446 0.447 0.448 0.449 0.45  0.451
 0.452 0.453 0.454 0.455 0.456 0.457 0.458 0.459 0.46  0.461 0.462 0.463
 0.464 0.465 0.466 0.467 0.468 0.469 0.47  0.471 0.472 0.473 0.474 0.475
 0.476 0.477 0.478 0.479 0.48  0.481 0.482 0.483 0.484 0.485 0.486 0.487
 0.488 0.489 0.49  0.491 0.492 0.493 0.494 0.495 0.496 0.497 0.498 0.499
 0.5   0.501 0.502 0.503 0.504 0.505 0.506 0.507 0.508 0.509 0.51  0.511
 0.512 0.513 0.514 0.515 0.516 0.517 0.518 0.519 0.52  0.521 0.522 0.523
 0.524 0.525 0.526 0.527 0.528 0.529 0.53  0.531 0.532 0.533 0.534 0.535
 0.536 0.537 0.538 0.539 0.54  0.541 0.542 0.543 0.544 0.545 0.546 0.547
 0.548 0.549 0.55  0.551 0.552 0.553 0.554 0.555 0.556 0.557 0.558 0.559
 0.56  0.561 0.562 0.563 0.564 0.565 0.566 0.567 0.568 0.569 0.57  0.571
 0.572 0.573 0.574 0.575 0.576 0.577 0.578 0.579 0.58  0.581 0.582 0.583
 0.584 0.585 0.586 0.587 0.588 0.589 0.59  0.591 0.592 0.593 0.594 0.595
 0.596 0.597 0.598 0.599 0.6   0.601 0.602 0.603 0.604 0.605 0.606 0.607
 0.608 0.609 0.61  0.611 0.612 0.613 0.614 0.615 0.616 0.617 0.618 0.619
 0.62  0.621 0.622 0.623 0.624 0.625 0.626 0.627 0.628 0.629 0.63  0.631
 0.632 0.633 0.634 0.635 0.636 0.637 0.638 0.639 0.64  0.641 0.642 0.643
 0.644 0.645 0.646 0.647 0.648 0.649 0.65  0.651 0.652 0.653 0.654 0.655
 0.656 0.657 0.658 0.659 0.66  0.661 0.662 0.663 0.664 0.665 0.666 0.667
 0.668 0.669 0.67  0.671 0.672 0.673 0.674 0.675 0.676 0.677 0.678 0.679
 0.68  0.681 0.682 0.683 0.684 0.685 0.686 0.687 0.688 0.689 0.69  0.691
 0.692 0.693 0.694 0.695 0.696 0.697 0.698 0.699 0.7   0.701 0.702 0.703
 0.704 0.705 0.706 0.707 0.708 0.709 0.71  0.711 0.712 0.713 0.714 0.715
 0.716 0.717 0.718 0.719 0.72  0.721 0.722 0.723 0.724 0.725 0.726 0.727
 0.728 0.729 0.73  0.731 0.732 0.733 0.734 0.735 0.736 0.737 0.738 0.739
 0.74  0.741 0.742 0.743 0.744 0.745 0.746 0.747 0.748 0.749 0.75  0.751
 0.752 0.753 0.754 0.755 0.756 0.757 0.758 0.759 0.76  0.761 0.762 0.763
 0.764 0.765 0.766 0.767 0.768 0.769 0.77  0.771 0.772 0.773 0.774 0.775
 0.776 0.777 0.778 0.779 0.78  0.781 0.782 0.783 0.784 0.785 0.786 0.787
 0.788 0.789 0.79  0.791 0.792 0.793 0.794 0.795 0.796 0.797 0.798 0.799
 0.8   0.801 0.802 0.803 0.804 0.805 0.806 0.807 0.808 0.809 0.81  0.811
 0.812 0.813 0.814 0.815 0.816 0.817 0.818 0.819 0.82  0.821 0.822 0.823
 0.824 0.825 0.826 0.827 0.828 0.829 0.83  0.831 0.832 0.833 0.834 0.835
 0.836 0.837 0.838 0.839 0.84  0.841 0.842 0.843 0.844 0.845 0.846 0.847
 0.848 0.849 0.85  0.851 0.852 0.853 0.854 0.855 0.856 0.857 0.858 0.859
 0.86  0.861 0.862 0.863 0.864 0.865 0.866 0.867 0.868 0.869 0.87  0.871
 0.872 0.873 0.874 0.875 0.876 0.877 0.878 0.879 0.88  0.881 0.882 0.883
 0.884 0.885 0.886 0.887 0.888 0.889 0.89  0.891 0.892 0.893 0.894 0.895
 0.896 0.897 0.898 0.899 0.9   0.901 0.902 0.903 0.904 0.905 0.906 0.907
 0.908 0.909 0.91  0.911 0.912 0.913 0.914 0.915 0.916 0.917 0.918 0.919
 0.92  0.921 0.922 0.923 0.924 0.925 0.926 0.927 0.928 0.929 0.93  0.931
 0.932 0.933 0.934 0.935 0.936 0.937 0.938 0.939 0.94  0.941 0.942 0.943
 0.944 0.945 0.946 0.947 0.948 0.949 0.95  0.951 0.952 0.953 0.954 0.955]
The basin of attraction for r=3.83 consists (approximately) of the points [0.156 0.157 0.165
0.166 0.167 0.17  0.171 0.177 0.179 0.18  0.189 0.201
```

```
 0.202 0.208 0.21  0.218 0.252 0.263 0.32  0.322 0.33  0.346 0.367 0.37
 0.377 0.386 0.409 0.431 0.441 0.442 0.496 0.497 0.504 0.505 0.506 0.507
 0.508 0.526 0.527 0.529 0.532 0.533 0.54  0.542 0.559 0.563 0.564 0.586
 0.614 0.617 0.631 0.636 0.644 0.654 0.672 0.683 0.72  0.723 0.733 0.742
 0.749 0.768 0.773 0.829 0.833 0.836 0.844 0.847 0.867 0.878 0.886 0.889
 0.892 0.893 0.899 0.905 0.907 0.908 0.926 0.929 0.939 0.942 0.944 0.945
 0.951 0.953 0.954 0.955 0.957]
The number of points in the basin of attraction for r=4 is 1001
The number of points in the basin of attractions for r=3.82 is 792 , which is greater than 20
6 , the number of points in the basin of attractions for r=3.83.
```

In homework, we proved that the logistics map is chaotic when r = 4. Looking at the basin of attractions (the veritical line) for $r = 4$ of the x-axis, we notice that almost every points in $[0, 1]$ are a limit point of the basin of attractions, which agrees since chaotic system travels every point evetually (transitivity).

If a population is well-modeled by the logistic map with a parameter of $r = 3.82$, we have more basin of attraction limit points (792). If it is modeled by a logistic map with parameter $r = 3.83$, the population is more predictable as there are less limit points (approximately) of the basin of attractions, (206).

In [ ]: