HPS390: The notion of a proof on the Pythagorean Theorem - Ancient Greek vs. Ancient Chinese

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The Pythagorean Theorem, one of the most influential mathematical formulas in history and the field of geometry, has numerous explanations from different mathematicians throughout the past centuries. However, what distinguishes all these explanations is whether it holds the idea of proofs — the construction of a set of statements that are built from fundamental assumptions or axioms, with the accepted rule of inferences. In this essay, we will examine Euclid's axiomatic proof presented in his famous publication, *Elements*, and the ancient Chinese visual argument of the formulation of the Pythagorean Theorem from *Zhoubi Suanjing*. We will look at how the latter also holds a similar notion of a proof as Euclid's proof of the Pythagorean Theorem according to the criteria of acquiring a deductive structure and the usage of fundamental, intuitive ideas.

Euclid's *Elements* was published in 300 BC. It is a collection of 13 books completed by Euclid, the ancient Greek mathematician. The *Elements* is a mathematical treatise that consisting of axioms, propositions, postulates and their proofs that build up the foundation of Euclidean geometry. The rigorousness of math logic presented in the proofs from these works was not surpassed until the nineteenth century. On the other hand, *Zhoubi Suanjing*, or *Chou-pei Suan-ching* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven) was originally named *Zhoubi*, without the word "Suanjing", which means computational canons. There was not a certain publication date for this book, but it was thought to be approximately around 206 BC – 220 AD (Joyce, n.d.). It recorded various steps in the development of mathematics and astronomy in China, especially in mathematics. *Zhoubi Suanjing* was recognized as a classical work in mathematics and gained its new title that is relevant to computations (Gong S. & Liu D, 2006). It is one of the works in the *Ten Computational Canons*, the collection of ten influential Chinese mathematical works that reflected all the Chinese mathematics developments from Qing to Tang Dynasty (Ten Computational Canons, 2017).

Inevitably, both works hold their own essential position in the development of western and eastern mathematics. What made the *Elements* so influential in its time was its presentation of proofs and the level of abstraction contained in it. There was no clear evidence of notion of a proof before his publication. For instance, the proof of the Pythagorean Theorem, or proposition 47, that was stated in Euclid's first book, *Elements I*, was built up from the axioms, propositions, and postulates established beforehand. In modern mathematics, these proofs are classified as axiomatic, synthetic, or direct proofs, by which one deduces a result first by constructing the result geometrically, as indicated in the reference diagram of the proof in Figure 1 (Joyce, n.d.), and then proves and verifies the construction using the previously verified propositions, postulates, and axioms or common notions that were assumed to be true.

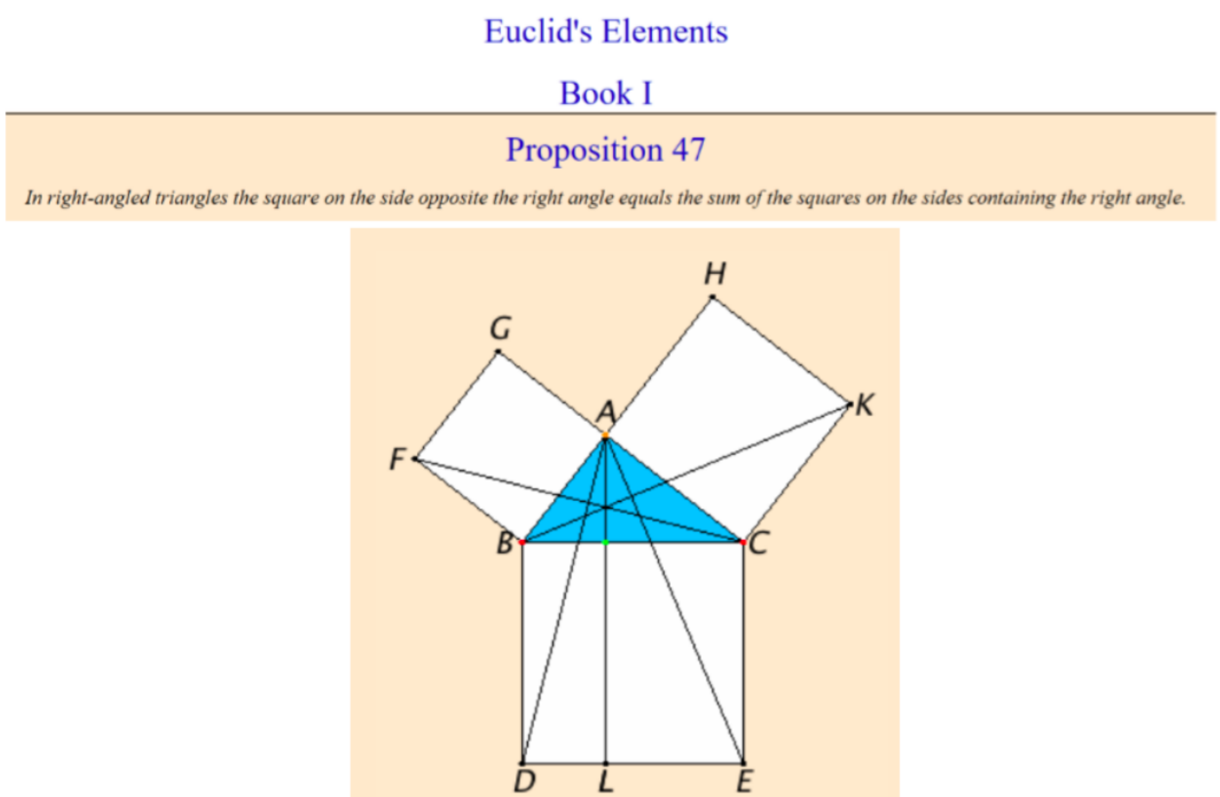

Figure 1: The proposition of the Pythagorean theorem, and its diagram of Euclid's proof in *Elements I*. Adapted from "Propsition 47," by D. E. Joyce, n.d., Euclid's Elements. Retrieved September 15, 2019, from https://mathcs.clarku.edu/~djoyce/java/elements/bookI/propI47.html.

Euclid's proof started as follows: "describe the square $BDEC$ on BC, and the squares GB and HC on BA and AC" by proposition 46 "to describe a square on a given straight line", then "draw AL through A parallel to either BD or CE, and join AD and FC" by proposition 31 "to draw a straight line through a given point parallel to a given straight line" and the first postulate "to draw a straight line from any point to any point", where the 2 propositions and 1 postulate were proven earlier in the book (Joyce, n.d.). Other common notions and postulates that he had used in this proof were common notion 1 "things equal to the same thing are also equal to one another other" to compare two lines, postulate 4 "that all right-angles are equal to one another" for comparing two right angles, and the parallel postulate or postulate 5 to guarantee the possibility of constructing a square (Fitzpatrick, 2008, p.7). These examples of usage of the previously proven facts in the new proof indicate how systemic and rigorous Euclid's proof of the Pythagorean Theorem was. He also wrote most of his proofs in this style in his other 12 books of the *Elements*.

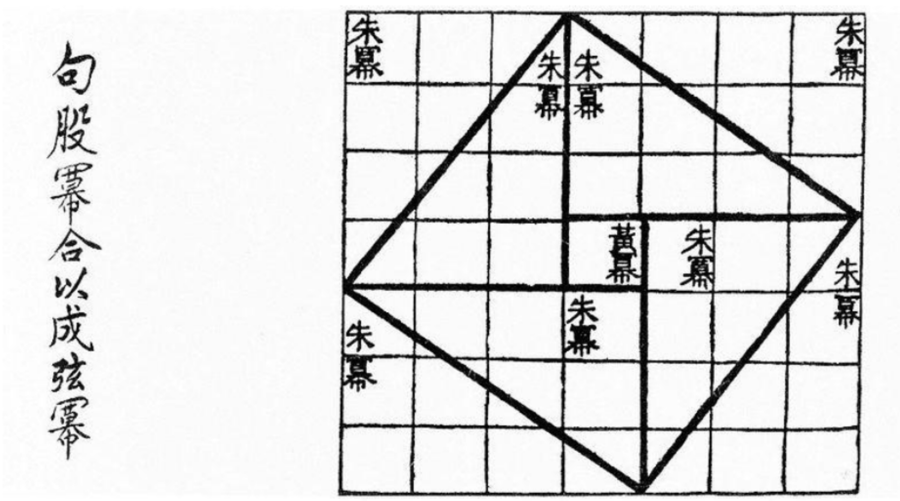

Figure 2: The original diagram of the Pythagorean Theorem's proof on *Zhoubi Suanjing*. Adapted from "Zhoubi Suanjing," 2019, Wikipedia. Retrieved September 15, 2019, from https://en.wikipedia.org/wiki/Zhoubi_Suanjing.

On the contrary, one might question whether the ancient Chinese explanation of the Pythagorean Theorem also shared the rigor shown in Euclid's proof to some extent. The short answer to this question is yes. Many explanations to the statements that were established in *Zhoubi Suanjing* (Figure 2) were well-explained by an ancient Chinese mathematician Zhao Shuang (around 300 AD). His commentaries of *Zhoubi Suanjing* included the Pythagorean Theorem's argument. The argument went as follows: first, label the four identical right-angled triangles Red and place them circularly, creating a square $ACHK$ with the side length of the hypotenuse c of the Red triangles by adding the missing part in the centre — the $Yellow$ square, as indicated in Figure 3.

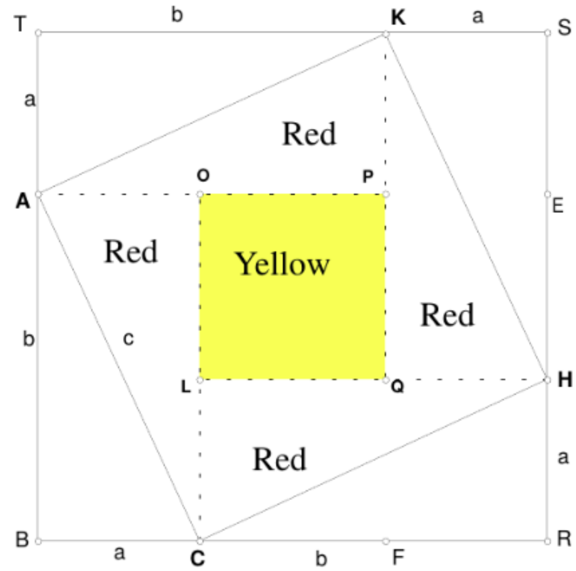

Figure 3: The translated version of the proof diagram based on the commentaries of Zhao Shuang. Adapted from "The Ancient Chinese Proofs on Pythagoras' Theorem," by n.d., Exemp21.Pdf. [PDF file]. Retrieved from `https://cd1.edb.hkedcity.net/cd/maths/en/ref_res/material/MSS_e/Exemp21.pdf`.

Then, Zhao Shuang outlined an outer square $TBRS$ with the length of the sum of opposite side a and adjacent side b of a Red right-angled triangle, which inscribed the square created by the Red triangles. The method he used to conclude $a^2+b^2=c^2$, the Pythagorean theorem, is comparing the resulted values from two different methods of finding the same area of the square $ACHK$ ("The Ancient Chinese Proofs on Pythagoras' Theorem," n.d., p.13). It is clear to see that the area of the square $ACHK$ is c^2 as its side length is the hypotenuse c. Another way to compute the area of the square $ACHK$ is by calculating the sum of the areas of the four Red triangles, which are $2ab$, and the area of the small $Yellow$ square, which is $(b-a)^2$. Hence, their sum is $2ab+(b-a)^2$. After some algebraic simplifications, it becomes a^2+b^2. By comparing the two answers of the same area of the square $ACHK$, one can conclude that $a^2+b^2=c^2$ ("The Ancient Chinese Proofs on Pythagoras' Theorem," n.d., p.13).

This proof used multiple mathematical techniques such as pasting and sticking shapes, rotating, combining, translating, and algebraic simplifications to justify its correctness ("The Ancient

Chinese Proofs on Pythagoras' Theorem," n.d., p.14). In a nutshell, the ancient Chinese proof of the Pythagorean Theorem presented a sequence of operations to simplify and to argue what one wants to prove in concise wording, which the proof commentators would elaborate and rephrase so that the citizens without advanced mathematical knowledge could also understand. Karine Chemla (2010) stated that "the commentator thus in some sense reads a proof in the structure of the text, writes down a text for an algorithm and at the same time, step by step, sub-procedure by sub-procedure, [and] provides an interpretation for each partial result" (p.257). What others might be skeptical about is whether the interpretations are not as accurate as the original proof. According to the criteria of a proof stated earlier, as long as the interpretations stand true and form a deduction, it does not really matter if it is completely what the original proof intended, as the concept of a proof would not be negated.

Examining Zhao Shuang's commentaries on the proof of the Pythagorean Theorem, we see that it first assumed what Euclid called the common notion 1 "things equal to the same thing are also equal to one another other", so that one can ensure the resulted values from the two ways of computing the same area of the square $ACHK$ have to be equal (Fitzpatrick, 2008, p.7). Even though it was not explicitly stated or explained, this assumption is acceptable and does not make the argument less rigorous since it is very trivial; Euclid did not prove axioms either. Other implicit assumptions, using Euclid's terminologies, were postulate 5 to guarantee the possibility of constructing the outer square $TBRS$, postulate 4 "that all right-angles are equal to one another" in order to conclude that $ACHK$ is a square whose corners have the same right-angles from the Red triangles (Fitzpatrick, 2008, p.7), etc. Judging this argument from the modern perspective of mathematics, one can argue that the ancient Chinese argument of the Pythagorean Theorem is less explicit and well-written than Euclid's proof, but it certainly holds the concept of a proof as one statement leads to another smoothly. The algebraic simplification from $2ab + (b-a)^2$ to $a^2 + b^2$ is also convincing, where the equation $2ab+(b-a)^2 = 2ab+(b^2-2ab+a^2) = 2ab-2ab+b^2+a^2 = b^2 + a^2 = a^2 + b^2$ is true. The ancient Chinese proof of the Pythagorean Theorem is a set of statements that holds a deductive structure with fundamental, intuitive geometric and algebraic ideas.

Nevertheless, some historians of mathematics affirmed that ancient Chinese mathematics did not hold the notion of a proof because of its societal structure — social hierarchy. Geoffrey E. R. Lloyd (1996) stated that “it may seem that a major contrast can be suggested between ancient Greek and ancient Chinese intellectuals, the former constantly criticising one another, the latter seeking the common ground, often under the banner of the revival of past wisdom; however, it is easy to see that the thesis suffers from serious limitations and weaknesses” (p.26). Ancient Greek society had slavery, where manual labour was completed by slaves, and the free citizens were able to preserve abstract and theoretical pursuits, such as Euclidean geometry. The Greek legal system, based on justification and argument rather than authority, fostered a deductive proof-centred approach to mathematics (Farringston, 1953). On the other hand, ancient Chinese society had an authorized emperor that ruled over almost everything, which might limit the scientific development under superior pressure from the imperial family if it was affecting their control over the society and citizens. Although ancient Chinese mathematicians tried to seek common ground on theoretical and philosophical subjects in favour of the emperor, they were “sometimes very far from irenic, ... each claiming that they were right” (Lloyd, 1996, p.27). Liu Hui, a well-known Chinese mathematician “[pointed] out the inaccuracies in the *Jiuzhang Suanshu* (‘Nine Chapters of the Mathematical Art’)”, another significant mathematics work in China (Lloyd, 1996, p.28). Lloyd (1996) also mentioned how “Chinese mathematicians frequently criticise the work of their predecessors” (p.28). One of the converse stands for these point of views is that preserving different, sometimes opposite, opinions on another colleague’s work do not cover up the notion of a proof in the somewhat flawed proofs. In fact, criticizing and pointing out inaccuracies in a mathematics work helps the authors to rethink and correct the flaws in proofs; it is a positive competition through which they can learn and avoid making the same mistakes for future publications. This is similar to what ancient Greeks did, constantly criticising one another due to the influence from their legal system. It is true that the Chinese emperor had absolute power, but the publications of many outstanding mathematics works on different subject areas, even astronomy throughout the dynasties, including *Zhoubi Suanjing*, demonstrate ancient China’s freedom on knowledge related to mathematics in its time.

Although Euclid presented the first axiomatic proof of the Pythagorean Theorem, not long after *Zhoubi Suanjing* also presented a different geometric and algebraic argument of the Pythagorean Theorem that shared a similar notion of a proof as Euclid's, in the sense of logical validity and generality. The concept of a proof is a revolutionary turning point in the history of mathematics, which standardizes the development of mathematics under strict criteria of maintaining logical consistency and rigorousness while achieving one's scope of interest in the subject. Moreover, proofs are not limited to mathematics. They have practical benefits such as the logical reasoning needed for problem-solving, and learning proofs is the greatest experience that one could ever have in mathematics.

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