Fractals, at first glance, are intriguing geometric figures that hold the beauty of infinite, repetitive patterns. Zooming into a certain section of the figure, one will discover an exact same, reduced copy of the whole figure. In the 17^{th} century, people start to notice the recurring patterns in nature, the notions of recursion in fractals (they were not called fractals at the time). One of the earliest examples of a fractal curve is the geometric visualization of the Weierstrass function that is continuous but differentiable nowhere.[1] Fractals are extremely useful to define, model, and predict the nature, as it extends the concept of geometric patterns in the world such as clouds, lightning, leaves, shells, and even the surface of a mountain.

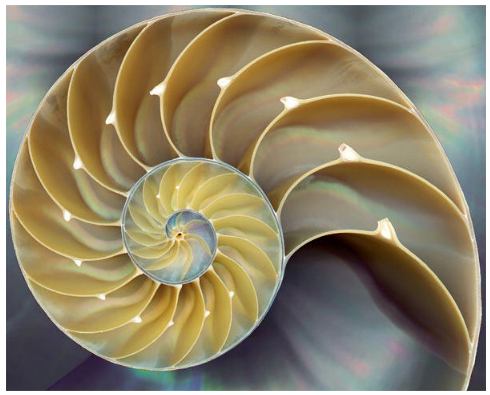


Figure 1: A Nautilus shell that demonstrates the logarithmic spiral fractal[2]

There was a long process to settle on the formal definition of a fractal. Benoit Mandelbrot, the mathematician who first used the term "fractal" to describe this kind of structures in 1975, simplified his original definition to be that "a fractal is a shape made of parts similar to the whole in some way".[3] Fractals are *self-similar* objects that has the property of being approximately the same to a part of itself. Recalling how normal geometric shapes are defined by a set of rules and an initial input, for example, a square consists of four straight lines connected at the right angles, fractals are also defined by a set of rules and an initial input. The input is either a domain of numbers for functions or a shape, and we recursively apply the rule to the previous output over and over again.[4] Reading over the definition might not tell us so much about fractals, so let's try to create one!

A classical example is the Koch snowflake, where the initial input is an equilateral triangle with a side length of 1 unit (it could be any length). The rule is as follows: we remove the middle one third of each side of the triangle, replace it with an equilateral triangle that has the same side length as the line segment being removed, and remove the side of the smaller triangle corresponding to the initial straight line.[4] Every implementation of the rule of removing the middle one third and

[1] What Are Fractals? When Do You Use Them In The Real World?, Teach-Nology, www.teach-nology.com/teachers/subject_matter/math/fractals/.

[2] Picture source

[3] "Fractal." Wikipedia, Wikimedia Foundation, 28 Jan. 2020, https://en.wikipedia.org/wiki/Fractal.

[4] Dallas, George. "What Are Fractals and Why Should I Care?", 2 May 2014, https://georgemdallas.wordpress.com/2014/05/02/what-are-fractals-and-why-should-i-care/.

replacing it with a new equilateral triangle to each side is called an iteration of the Koch snowflake.

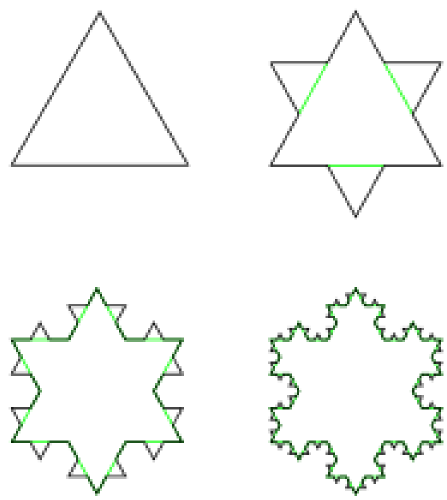


Figure 2: The first four iterations of the Koch snowflake[5]

If we take the iterations to infinity, the fractal shape of the Koch snowflake is defined. This probably sounds very bizarre as it is difficult to imagine repeating a process by infinite times. The idea behind it is if we keep zooming into any segment of the Koch snowflake, the visual pattern seems to be repeating itself, but in fact each edge and spike of the snowflake is smaller compared to the initial ones.

The perimeter of the Koch snowflake is an interesting property that one should examine. Each iteration multiplies the number of sides in the Koch snowflake by four, and we start with three sides of the equilateral triangle, so the number of sides after n iterations, N_n is:

$$N_n = N_{n-1} \cdot 4 = 3 \cdot 4^n$$

If the original equilateral triangle has a side length l and the length of each edge of the snowflake is scaled by one third for each iteration, then the length of each edge after n iterations, L_n is:

$$L_n = \frac{L_{n-1}}{3} = \frac{l}{3^n}$$

Thus, we know the perimeter of the Koch snowflake at n^{th} iteration, P_n is:

$$P_n = N_n \cdot L_n = (3 \cdot 4^n) \cdot \left(\frac{l}{3^n}\right) = 3 \cdot l \cdot \left(\frac{4}{3}\right)^n$$

With the limit of n going to infinity, we finally get:

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 3 \cdot l \cdot \left(\frac{4}{3}\right)^n = \infty$$

since $\frac{4}{3} > 1$ and P_n gets greater for each larger n. This means that the perimeter of the Koch snowflake is infinite.[5]

On the other hand, the area of the Koch snowflake is finite. Clearly, we can subscribe the Koch snowflake in a circle, and a circle has a finite area.

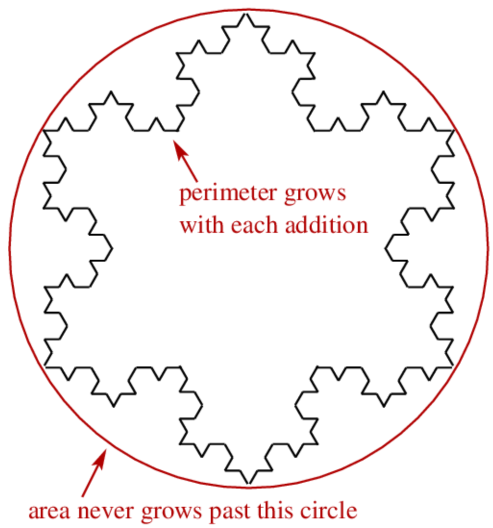


Figure 3: How Koch snowflake has a finite area but infinite perimeter[6]

One might question the dimension of the Koch snowflake, an object that has a finite area with an infinite perimeter. This is the more anti-intuitive part about fractals – they could be objects that exist between dimensions! In other words, they have a dimension that is not a whole number, not $2D$ or $3D$, but in between. The Koch snowflake seems to be a $2D$ object that appears on a plane (the surface of a paper), but is in fact $\frac{\log 4}{\log 3} \approx 1.26$.

We need to define a new tool to calculate the dimension of a fractal. The *Minkowski dimension*, also known as the *box-counting dimension*, measures how complexity of detail changes with the scale at which one views the fractal.[7] Let S be the set of points that build up the fractal. Imagine that there is an evenly spaced n by n grid which covers the fractal of interest, and each grid box has a side length of r_n. We keep counting the number of grid boxes that covers the fractal, name it C_n. Then, the box-counting dimension of the fractal, $\dim(S)$, is given by:

$$\dim(S) = \lim_{n\to\infty} \frac{\log(C_n)}{\log(\frac{1}{r_n})}$$

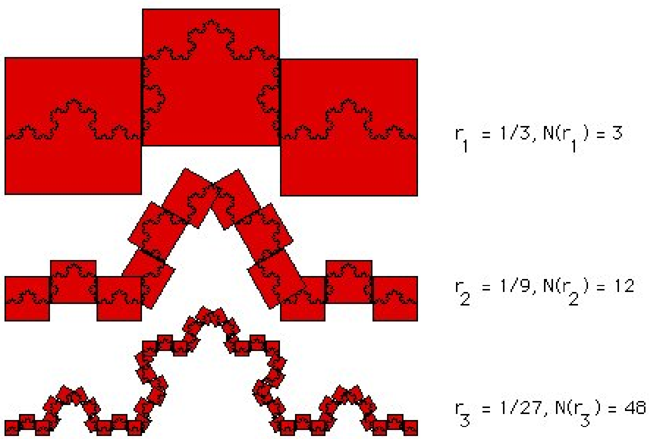
as we approximate the area of the fractal better with a finer grid when n goes to infinity.

[5] "Koch Snowflake." Wikipedia, Wikimedia Foundation, 15 Jan. 2020, https://en.wikipedia.org/wiki/Koch_snowflake.

[6] Francis, Matthew. "Fractals for Fun." Galileo's Pendulum, 31 Jan. 2012, galileospendulum.org/2012/01/31/fractals-for-fun/.

[7] "Fractal Dimension." Wikipedia, Wikimedia Foundation, 19 Dec. 2019, https://en.wikipedia.org/wiki/Fractal_dimension.

To simplify the case, we can assume that the Koch snowflake starts with an initial input of an equilateral triangle which has a side length of one unit, name it S. This assumption would not change the dimension of the Koch snowflake. The Koch snowflake has three sides, where each side is called the Koch curve. We name the top side S_1 such that $S = S_1 \cup S_2 \cup S_3$ where S_2 and S_3 are the two other sides of the Koch snowflake. Let N be a function that takes input r_n, the length of unit boxes on the n by n grid, and outputs the number of the unit boxes that cover the fractal, C_n such that $N(r_n) = C_n$. If we have a grid that has boxes with a side length of $r_1 = \frac{1}{3}$, then we need 3 copies of the grid boxes to cover up the Koch curve. If we have another grid that has boxes with a side length of $r_2 = \frac{1}{9}$, then we need $12 = 3 \cdot 4$ copies of the grid boxes to cover up the Koch curve. The pattern of how the number of sub-boxes increases with a finer and finer grid is demonstrated below:



N(1/3) = 3
N(1/9) = N($(1/3)^2$) = 12 = 3·4
N(1/27) = N($(1/3)^3$) = 48 = 3·4^2
and in general
N($(1/3)^n$) = 3·4^{n-1}

Figure 4: Box-counting dimension of the Koch curve[8]

Thus, the dimension of the top side, S_1, of the Koch snowflake is:[9]

$$\dim(S) = \lim_{n\to\infty} \frac{\log(C_n)}{\log(\frac{1}{r_n})} = \lim_{n\to\infty} \frac{\log(N(r_n))}{\log(\frac{1}{r_n})} = \lim_{n\to\infty} \frac{\log(\frac{3}{4^{n-1}})}{\log(\frac{1}{(\frac{1}{3})^n})}$$

$$= \lim_{n\to\infty} \frac{\log(3) + (n-1)\cdot\log(4)}{n\cdot\log(3)} = \lim_{n\to\infty} \frac{\log(3) - \log(4) + n\cdot\log(4)}{n\cdot\log(3)}$$

$$= \lim_{n\to\infty} \left(\frac{\log(3) - \log(4)}{n\cdot\log(3)} + \frac{n\cdot\log(4)}{n\cdot\log(3)} \right) = \frac{\log 4}{\log 3}$$

We know the other two sides of the Koch snowflake, S_2 and S_3 is simply a rotated copy of S_1, so their dimension is the same as S_1. The union of the three sides also maintains the same dimension, which means $\dim(S) = \dim(S_1) = \dim(S_2) = \dim(S_3) = \frac{\log 4}{\log 3} \approx 1.26$.

[8] "Box-Counting Dimension of the Koch Curve." Fractal Geometry, https://users.math.yale.edu/public_html/People/frame/Fractals/FracAndDim/BoxDim/KochBoxDim/KochBoxDim.html.

[9] "Box-Counting Dimension of the Koch Curve – Exact Value." Fractal Geometry, https://users.math.yale.edu/public_html/People/frame/Fractals/FracAndDim/BoxDim/KochBoxDim/KochExact.html.

The dimension of a fractal helps us to study more about its appearance in nature. Have you ever wondered what is the longest river on Earth? If we want to measure the length of a river, we have to account that the length depends on the measuring unit which one uses. Suppose we start with 100km as the unit so that we can fit 8 unit segments into the winding river somewhat off of its path, and conclude it is 800 km long. But if we start with 50km as the unit, we might be able to fit 19 of them and conclude it is 950 km long. There is a 150 km difference if we start measuring with a different unit length![10]

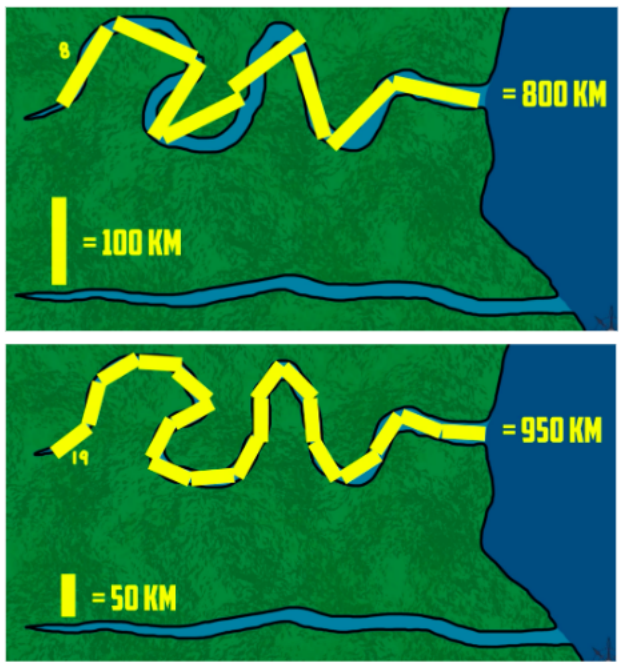


Figure 5: The coastline paradox: What is the longest river on Earth?[10]

It is impossible to measure the length of the river precisely since its value changes relatively to the unit length of measurement. This is called the *coastline paradox*.[11] Nevertheless, fractal dimension allows us to measure the complexity of an object by evaluating how fast the measurements increase as the scale becomes finer,[12] which resolves the problem. It measures the ratio that is independent of the unit length. Fractals provide us a new scientific way of viewing natural phenomena. They form a new appropriate tool to study irregular objects, while the classical Euclidean geometry characterizes only regular objects such as triangles, squares, tetrahedrons, etc. Unfortunately, most of the objects in reality are not regular, but fractals allow us to approximate the real world around us better.

Fractals have been used to create the *JPEG* algorithm for sending pictures between computers.

[10] Atlas Pro. What's the Longest River on Earth? YouTube, 8 Feb. 2018, www.youtube.com/watch?v=g3Y3vCgVeM0&feature=share.

[11] "Coastline Paradox." Wikipedia, Wikimedia Foundation, 25 Jan. 2020, https://en.wikipedia.org/wiki/Coastline_paradox.

[12] Green, Eric. "Fractal Dimension." Measuring Fractal Dimension, http://pages.cs.wisc.edu/~ergreen/honors_thesis/dimension.html.

The algorithm compresses pictures if it identifies self-similar patterns in the pictures. Modern cell phones use a fractal-looking antenna that has different shapes nested within one another. Each shape corresponds to a radio signal so that the antenna can work at different frequencies at the same time with a higher efficiency than regularly shaped antennas.[13] Mathematically, fractals interact closely with dynamical systems and provide geometric interpretations of chaos. One of the dynamical systems is the Newton's method applied to a function in order to locate its roots. Graphing the boundary of the outputs from the Newton's method on a certain function actually would create a fractal. Using Mandelbrot's words to summarize: "beautiful, damn hard, increasingly useful. That's fractals".[3]

[13] Hein, Simeon. "What Are Fractals? The Complex History of Fractals." What Are Fractals?, Gaia, 12 Dec. 2016, www.gaia.com/article/what-are-fractals.