# Graphics

Lectured by Bernhard Kainz and Abhijeet Ghosh

Typed by Aris Zhu Yi Qing

March 13, 2022

## Contents

# 1 Projections and Transformations

## 1.1 Parallel Projection

- For a vertex $\mathbf{V} = (V_x, V_y, V_z)^T$, the **projector** (projection line) is defined by the parametric line equation

$$\mathbf{P} = \mathbf{V} + \mu\mathbf{d}$$

- Assuming the projection plane is $z = 0$, we can establish

$$0 = P_z = V_z + \mu d_z$$

  to obtain $\mu$, thereby computing $P_x$ and $P_y$.

- **Orthographic projection** is a special type of parallel projection:
  - projection plane: $z = 0$
  - $\mathbf{d} = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^T$
  - $P_x = V_x$, $P_y = V_y$

## 1.2 Perspective Projections

- The **centre of projection** is the viewpoint, which all the projectors pass through, assumed to be at the origin.
- For a vertex $\mathbf{V} = (V_x, V_y, V_z)^T$, the projector $\mathbf{P}$ has the equation

$$\mathbf{P} = \mu\mathbf{V}$$

- Since the projection plane is at a constant $z$ value $f$, at the point of intersection we have

$$f = P_z = \mu V_z$$

  to obtain $\mu$, thereby computing $P_x$ and $P_y$.

## 1.3 Space Transformations

### 1.3.1 Homogeneous Coordinates

- A **homogeneous coordinate** is a three-dimensional coordinate with a fourth componenet called **ordinate** which acts as a scale factor.

- Assuming a point $\mathbf{P} = (p_x, p_y, p_z)$ in Cartesian coordinate, we introduce $s$ being the ordinate

$$\mathbf{P}' = (p_x, p_y, p_z, s)$$

to form a homogeneous coordinate.

- To convert $\mathbf{P}'$ back ot Cartesian, we will perform **perspective division**

$$\mathbf{P}'' = \left(\frac{p_x}{s}, \frac{p_y}{s}, \frac{p_z}{s}\right),$$

i.e. divide $x$, $y$ and $z$ values by the ordinate. Thus when $s = 1$, $\mathbf{P} = \mathbf{P}''$.

- If $s \neq 0$, we have a **position vector**. If $s = 0$, we have a **direction vector**.

### 1.3.2 Translation Matrix

To apply a translation vector $\mathbf{t} = (t_x, t_y, t_z)$ to a point $\mathbf{P} = (p_x, p_y, p_z)$, we do

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{pmatrix}$$

with the inverse of the translation matrix as

$$\begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

### 1.3.3 Scaling Matrix

To scale a point from the origin, we can do

$$\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x p_x \\ s_y p_y \\ s_z p_z \\ 1 \end{pmatrix}$$

with the inverse of the scaling matrix as

$$\begin{pmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

### 1.3.4 Rotation Matrix

To rotate <u>anti-clockwise</u> when looking <u>along the direction</u> of the axis with a <u>left-hand</u> axis system, we have

$$\mathcal{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{R}_x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{R}_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{R}_y^{-1} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{R}_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{R}_z^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

to rotate along $x$, $y$ and $z$ axis respectively. In other words, the right matrices are rotating clockwise.

### 1.3.5 Projection Matrix

For a perspective projection, placing the centre of projection at the origin and using $z = f$ as before, we can use

$$\mathcal{M}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix}$$

For an orthographic projection, with the projection plane at $z = 0$, we can use

$$\mathcal{M}_o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# 2 Clipping

- **Clipping** eliminates portions of objects outside the **viewing frustum**, which is the boundaries of the image plane projected in 3D with a near and far clipping plane.
- Why clipping?
  - avoid degeneracy: e.g. don't draw objects behind the camera
  - improve efficiency: e.g. do not process objects which are not visible.
- When to clip?
  - before perspective transform in 3D space:
    * 3D world space
    * use the equation of 6 planes
    * natural, not too degenerate
  - in homogeneous coordinates after perspective transform and before perspective division:
    * clip space
    * canonical, independent of camera
    * simplest to implement, since clipping plane can align with axis so that we can easily discard anything further than the far plane or closer than the near plane
  - in the transformed 3D screen space after perspective division:
    * Normalized Device Coordinates (NDC)
    * The regions extends from -1. to 1. in each axis. Anything outside from the volume is discarded.
    * problem — having negative orginates
- **Halfspace** We can define any plane as a test for a point $\mathbf{p}$:

$$f(x, y, z) = \mathbf{H} \cdot \mathbf{p} = 0$$

  where $\mathbf{H} = (H_x, H_y, H_z, H_s)$ and $\mathbf{p} = (x, y, z, 1)$, such that

$$\begin{cases} \mathbf{H} \cdot \mathbf{p} > 0 & \text{in one halfspace (pass-through)} \\ \mathbf{H} \cdot \mathbf{p} < 0 & \text{in the other halfspace (clip/cull/reject)} \end{cases}$$

  - **Segment Clipping** Similarly we have

$$\begin{cases} \mathbf{H} \cdot \mathbf{p} > 0, \mathbf{H} \cdot \mathbf{q} < 0 & \text{clip } \mathbf{q} \text{ to plane} \\ \mathbf{H} \cdot \mathbf{p} < 0, \mathbf{H} \cdot \mathbf{q} > 0 & \text{clip } \mathbf{p} \text{ to plane} \\ \mathbf{H} \cdot \mathbf{p} > 0, \mathbf{H} \cdot \mathbf{q} > 0 & \text{pass through} \\ \mathbf{H} \cdot \mathbf{p} < 0, \mathbf{H} \cdot \mathbf{q} < 0 & \text{clipped out} \end{cases}$$

  - Test if an object is convex.
    1. For each face of the object, pick a random point.
    2. For this point, compare with points from other faces, check if

       ```
       sign(f(xj,yj,zj)) != sign(f(xi,yi,zi))
       ```

       then it is not convex.
  - Test if a point is contained in a concave object.
    * Cast a ray from the test point in any direction. If the number of intersections with the object is odd, then the test point is inside.

# 3 Graphics Pipeline

## 3.1 Application

- executed by the software on the main processor (CPU)
- typical tasks performed: collision detection, animation, morphing, perform spatial subdivision scheme (quadtree, octree).
- to reduce the amount of main memory required at a given time

## 3.2 Geometry

1. Modelling Transformations
2. Illumination (Shading)
3. Viewing Transformations (Perspective/Orthographic)
4. Clipping
5. Projection (to screen space — window-viewport transformation)

## 3.3 Rasterization

- **Rasterization** is the task of taking an image described in a vector graphics format (shapes) and converting it into a raster image (a series of pixels).
- During this process, fragments/raster points are created from continuous primitives. A **fragment** can be thought of as the data needed to shade the pixel (e.g. color, illumination, texture) and to test whether the fragment survives to become a pixel (depth, alpha, etc.)
- Eventually, one or more fragments are merged to become a **pixel**, which is the smallest addressable element in a raster image.
- To prevent from exposing the process of gradual screening of the primitives, double buffering is used so that the rasterization takes place in a special memory, and as soon as the image is completely rastered, it is copied into the visible area of the image memory (frame buffer).

## 3.4 Shading

**Shading** refers to the modification of individual vertices or fragments within the graphics pipeline. This is the *programmable* part of the graphics pipeline.

### 3.4.1 Vertex Shader

- executed once for each vertex
- only has access to the vertex and no neighbouring vertices, the topology, or similar

### 3.4.2 Tessellation Shader

- divides an area (triangle or square) into smaller areas
- advantage: allow detail to be dynamically added and subtracted from a 3D polygon mesh and its silhouette edges based on control parameters (e.g. camera distance)
- The *Tessellation Control Shader* (TCS) determines how much tessellation to do. It is optional; default tessellation values can be used.
- The *tessellation primitive generator* (not programmable) takes the input patch and subdivides it based on values computed by the TCS.
- The *Tessellation Evaluation Shader* (TES) takes the tessellated patch and computes the vertex values for each generated vertex.

### 3.4.3 Geometry Shader

- takes a single primitive as input and may output zero or more primitives of the same type
- has access to multiple vertices, if the primitive consists of multiple vertices
- A **primitive** can mean
  (a) the interpretation scheme to determine what a stream of vertices represents when being rendered, which can be arbitrarily long
  (b) the *result* of the interpretation of a vertex stream (also called the *base primitive*)
- Use case: in a particle system, the inputs are processed points, and geometry shader generates polygons/cubes/etc. to save computation in the previous pipelines.

### 3.4.4 Fragment Shader

- executed once for every fragment generated by the rasterization
- it takes in interpolated vertex attributes
- it calculates the color of the corresponding fragment

# 4 OpenGL

- The interface is platform independent, but the implementation is platform dependent.
- It defines an abstract rendering device and a set of functions to operate the device.
- It is a low-level "immediate mode" graphics API with drawing commands and no concept of permanent objects, operating as a state machine.
- To write an OpenGL programme, we need to
  1. create a render window via library such as glut, Qt, etc.
  2. setup viewport, model transformation and file I/O (shader, textures, etc.)
  3. implement frame-generataion (update/rendering) functions to define what happens in every frame
- Basic concepts:
  - **Context**
    - represents an instance of OpenGL
    - a process can have multiple contexts to share resources
    - one-to-one mapping between a context and a thread
  - **Resources**
    - act as sources of input and sinks for output
    - e.g. texture images(input), buffers(output)
  - **Object Model**
    - Object instances are identified by unique names (unsigned integer handle).
    - Commands work on targets, where each target is bounded by an object.
- **Buffer objects** are regular OpenGL objects that store an array of unformatted memory allocated by the OpenGL context (i.e. GPU).
- has primitive types such as `GL_POINTS`, `GL_LINES`, `GL_POLYGONS`, `GL_TRIANGLES`, etc.

# 5 Illumination and Shading

## 5.1 Physics of Shading

- object properties
  - the position of the object relative to the light sources
  - the surface normal vector
  - the albedo of the surface (ability to absorb light energy) and the reflectivity of the surface
- light source properties
  - intensity of the emitted light
  - distance to the point on the surface
- energy (Joule) of a photon is

$$e(\lambda) = \frac{hc}{\lambda}$$

  where $h \approx 6.63 \times 10^{-34} J \cdot s$ and $c \approx 3 \times 10^{8} m/s$.
- radiant energy (Joule) of $n$ photons is

$$Q = \sum_{i=1}^{n} e(\lambda_i)$$

- Radiation/radiant/electromagnetic flux (Watts) is

$$\Phi = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

- **Radiance** (Watt/(meter$^2$ · steradian)) is density of a incident flux falling onto a surface in a particular direction

$$L(\omega) = \frac{\mathrm{d}^2\Phi}{\cos\theta \, \mathrm{d}A \mathrm{d}\omega}$$

- **Irradiance** (Watt/meter$^2$) is density of the incident flux falling onto a surface

$$E = \frac{\mathrm{d}\Phi}{\mathrm{d}A}$$

- We define the **Bidirectional Reflectance Distribution Function** (BRDF) (1/steradian)

$$f_r(\theta_i, \phi_i, \theta_r, \phi_r) = f_r(\omega_i, \omega_r) = \frac{\mathrm{d}L_r(\omega_r)}{\mathrm{d}E_i(\omega_i)}$$

  by ignoring other physical phenomenon such as absorption, transmission, fluorescence, diffraction, etc.

  - *Isotropic BRDF* is such that rotation along surface normal does not change reflectance.
  - *Anisotropic BRDF* changes reflectance when rotating along surface normal, which happens on surfaces with strongly oriented microgeometry elements such as brushed metals, hair, cloth, etc.
  - non-negativity: $f_r(\omega_i, \omega_r) \geq 0$
  - energy conservation: $\forall \omega_i, \int_\Omega f_r(\omega_i, \omega_r) \cos\theta_r \mathrm{d}\omega_r \leq 1$
  - reciprocity: $f_r(\omega_i, \omega_r) = f_r(\omega_r, \omega_i)$

- To compute the reflected radiance discretely, with $n$ points light sources, we have

$$L_r(\omega_r) = \sum_{i=1}^{n} f_r(\omega_i, \omega_r) E_i = \sum_{i=1}^{n} f_r(\omega_i, \omega_r) \cos\theta_i \frac{\Phi_i}{4\pi d_i^2}$$

- Ideally, BRDF is constant, so with a single point light source

$$L(\omega_r) = k_d (n \cdot l) \frac{\Phi_s}{4\pi d^2}$$

  where $k_d$ is the diffuse reflection coefficient, $n$ is the (normalized) surface normal, and $l$ is the (normalized) light direction from surface.

## 5.2 The Phong Model

- light sources are assumed to be point-shaped, i.e. no spatial extent
- Reflected radiance calculation is

$$L(\omega_r) = k_s (v \cdot r)^q \frac{\Phi_s}{4\pi d^2} = k_s (v \cdot (2(n \cdot l)n - l))^q \frac{\Phi_s}{4\pi d^2}$$

  where $k_s$ is the specular reflection coefficient, $q$ is the specular reflection exponent, $v$ is the direction vector from surface to camera, and $r$ is the reflected ray.

- Blinn-Phong variation is that

$$L(\omega_r) = k_s (n \cdot h)^q \frac{\Phi_s}{4\pi d^2} \qquad \text{with} \qquad h = \frac{l + v}{\|l + v\|}$$

- The Phong model is the sum of three components: diffuse, specular and ambient, i.e.

$$L(\omega_r) = k_a + \left(k_d (n \cdot l) + k_s (v \cdot r)^q\right) \frac{\Phi_s}{4\pi d^2}$$

- Sometimes using $(d + s)$ instead of $d^2$ produces better result, where $s$ is a heuristic constant.

## 5.3 Shading

### 5.3.1 Flat Shading

- each polygon is shaded uniformly over its surface
- computed by taking a point in the center and at the surface normal
- normally only the diffuse and ambient components are used

### 5.3.2 Gouraud Shading

- interpolate color using shade value at each vertex
- can interpolate intensity at each vertex from all the polygons that meet at that vertex to create the impression of a smooth surface
- cannot accurately model specular components, since we don't have normal vector at each point on a polygon

### 5.3.3 Phong Shading

- interpolate normals across triangles at fragment stage
- more accurate modelling of specular components, but slower

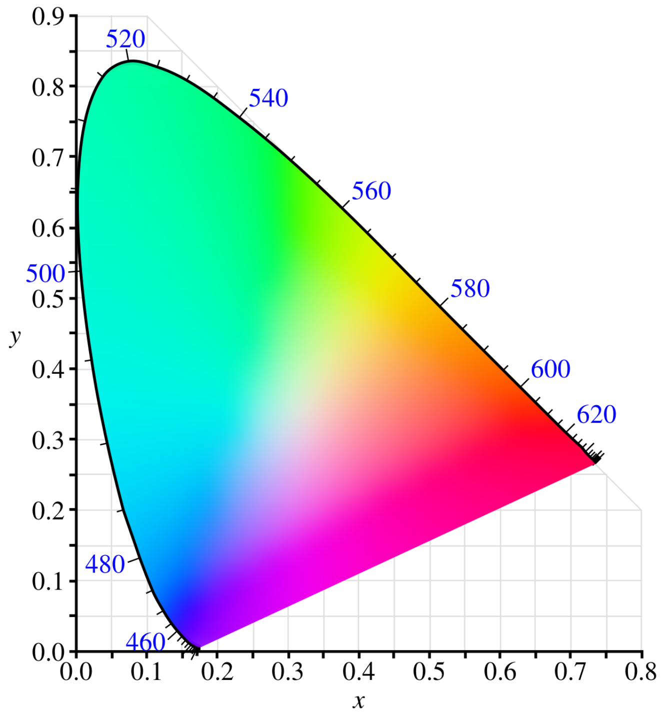



Figure 1: The CIE 1931 color space. Red is at $(0.628, 0.346, 0.026)$, Green is at $(0.268, 0.588, 0.144)$, Blue is at $(0.150, 0.07, 0.780)$

# 6 Color

## 6.1 RGB CIE Color Space

- a standard normalized representation of colors ranging from 0 to 1, with

$$x = \frac{r}{r+g+b}, y = \frac{g}{r+g+b}, z = \frac{b}{r+g+b} = 1 - x - y$$

- the actual visible colors are a subset of this as shown in Figure 1, done through manual testing

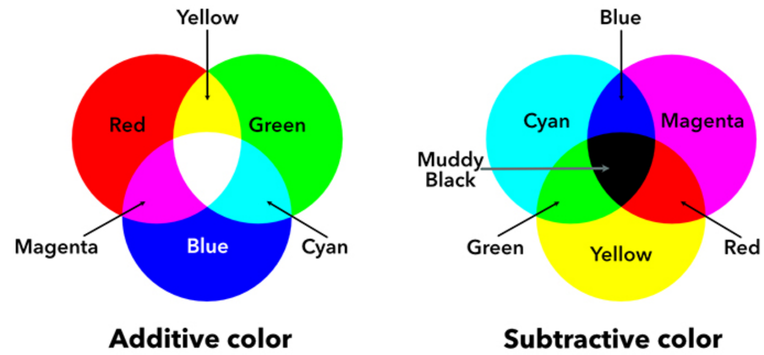
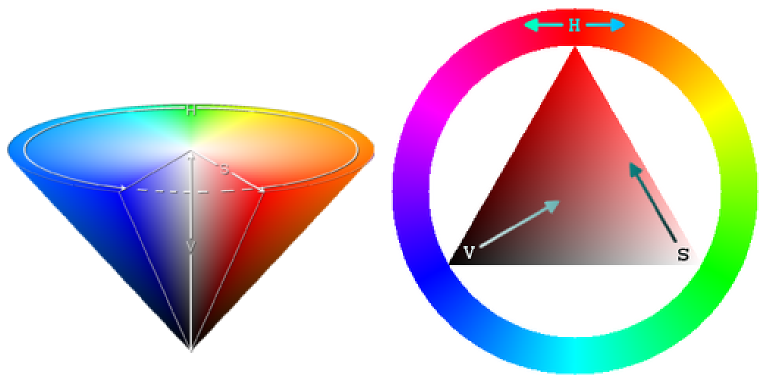



Figure 2: additive and subtractive primaries

- the shape must be convex, since any blend (interpolation) of pure colors should create a color in the visible region.
- the **pure colors** are around the edge of the diagram, also called **fully saturated**
- the line joining purple and red has no pure equivalent; the colours can only be created by blending
- **Saturation** of an arbitrary point is the ratio of its distance to the white point over the distance of the white point to the edge.
- white point: when $x = y = z = 0.\dot{3}$
- The **complement color** of a color is the point diametrically opposite through the white point. Computationally, if the color has value $(r, g, b)$, its complement color is $(255 - r, 255 - g, 255 - b)$.
- The **additive primaries** are RGB (Red, Green, Blue) and the **subtractive primaries** are CMY (Cyan, Magenta, Yellow). Red is the complement color of Cyan, and similarly for Green and Blue, as shown in Figure 2.
- RGB can be converted to CIE by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.628 & 0.268 & 0.15 \\ 0.346 & 0.588 & 0.07 \\ 0.026 & 0.144 & 0.78 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$




(a) The conical representation of the HSV model

(b) The HSV color wheel

Figure 3: HSV model

## 6.2 HSV Color Representation

- **Hue** corresponds notionally to pure color
- **Saturation** is the proportion of pure color
- **Value** is the brightness/intensity
- We can visualize the perceptual color space in HSV as in Figure 3.
- Conversion between RGB and HSV can be done as

$$V = \max(r, g, b)$$

$$S = \frac{\max(r, g, b) - \min(r, g, b)}{\max(r, g, b)}$$

$$H = \begin{cases} \text{undefined} & r = g = b \\ 120 \cdot \dfrac{g - b}{(r - b) + (g - b)} & (r > b) \wedge (g > b) \\ 120 + 120 \cdot \dfrac{b - r}{(g - r) + (b - r)} & (g > r) \wedge (b > r) \\ 240 + 120 \cdot \dfrac{r - g}{(r - g) + (b - g)} & (r > g) \wedge (b > g) \end{cases}$$

## 6.3 Transparency

We can model transparency with an $\alpha$ channel, with

- transparent: $\alpha = 0$
- semi-transparent: $0 < \alpha < 1$
- opaque: $\alpha = 1$

Suppose that we put $A$ over $B$ over background $G$,

- How much of $B$ is blocked by $A$? $\alpha_A$
- How much of $B$ shows through $A$? $(1 - \alpha_A)$
- How much of $G$ shows through both $A$ and $B$? $(1 - \alpha_A)(1 - \alpha_B)$
- How much does $G$ contribute to the overall color? $(1-\alpha_A)(1-\alpha_B)\alpha_G$

# 7 Texture

## 7.1 Definition

- **Texture (map)** is an image applied (mapped) to the surface of a shape or polygon.
- A texture can be 1D, 2D, or 3D, but 2D is the most common for visible surfaces.
- **Raster images** are 2D rectangular matrices or grid of square pixels, often used as textures.
- **Procedural texture** is a texture created using a mathematical description rather than directly stored data (e.g. raster image). Mathematically, it is a function $f$ defiend as

$$f : \mathbf{p} \mapsto \text{color},$$

where $\mathbf{p}$ is a coordinate.

  + Very small memory footprint before the texture map is generated. The ultimate way in image compression.

+ No texture memory is really needed since generated "on the fly" in a fragment shader, resulting in the exactly right level of detail for each pixel on the screen.

- Hard to get a formula to get the exact/natural look.

- On-the-fly generation can take a lot of shader program instructions, almost always slower than just loading/looking up one.

## 7.2 Photo Textures Mapping

### 7.2.1 Mechanism

- Define a 2D coordinate system on an image mapped onto a 3D object.
- For each fragment on an object's surface, work out what coordinate needs to be sampled in the image's 2D space to get the right color.
- Conventionally, texture coordinates are denoted with $(s, t)$ (*texture space*). Canonically it goes from (0,0) to (0,1). The object surface is denoted with $(u, v)$ (*object space*) and the pixel on the screen is denoted with $(x, y)$ (*screen space*). We need to define

$$\underline{\textbf{Parameterization}} : (s, t) \mapsto (u, v)$$

which is the process of finding parametric equations of textures and objects so that texture can be mapped onto object surface, and

$$\underline{\textbf{Rendering}} : (u, v) \mapsto (x, y)$$

which is the process of generating an image from a model.

### 7.2.2 Parameterization

- Planar mapping: ignore one of the coordinates
- Cylindrical/Spherical mapping: compare to cylindrical/spherical coordinate systems
- Box mapping: 6 planar mapping
- **Unwrapping**: the process of creating manual mapping

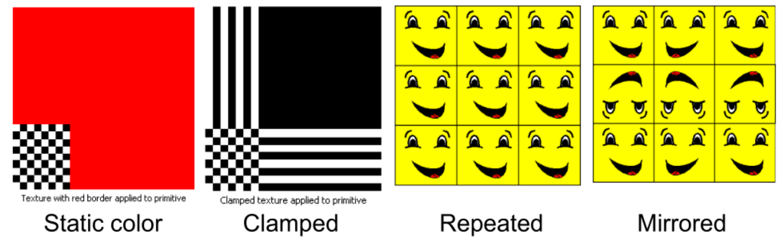



Figure 4: Different texture addressing

### 7.2.3 Texture Addressing (Mode)

What happens outside $[0, 1]$? Following the order in Figure 4,

- static color: we use a static color (red in this case)
- clamped: we use the last color in the range
- repeated: wrap back to the first coordinate, repeating the texture
- mirrored: similar to repeated, but go backwards insteads after ending the texture

## 7.3 Perspective Correct Interpolation

- cannot simply perform linear interpolation as in Gouraud shading on texture coordinates
- The problem:
  - perspective projection **does not preserve** linear combinations of points
  - e.g. equal distances in 3D do not map to equal distances in screen space, as shown in figure 5.
- The solution:
  - Assume image plane is at $z = f = 1$.
  - Let $t$ controls linear blend of texture coordinates of $\mathbf{p}$ and $\mathbf{r}$ and let

$$t_p = 0, \; t_r = 1.$$

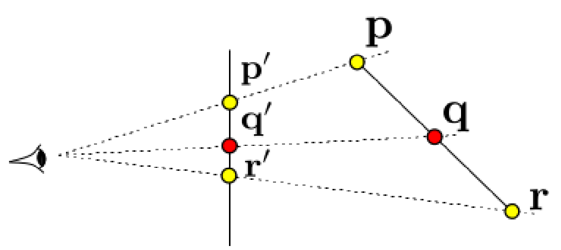



Figure 5: perspective interpolation error

- Let $\text{lerp}(x, y)$ be the linear interpolation function, and as such

$$t_q = t_{q'} z_q = \frac{\text{lerp}(t_{p'}, t_{r'})}{\frac{1}{z_q}} = \frac{\text{lerp}(\frac{t_p}{z_p}, \frac{t_r}{z_r})}{\text{lerp}(\frac{1}{z_p}, \frac{1}{z_r})}$$

- The algorithm: given texture parameter $t$ at vertices,
  1. compute $\frac{1}{z}$ for each vertex
  2. linearly interpolate $\frac{1}{z}$ across the triangle
  3. linearly interpolate $\frac{t}{z}$ across the triangle
  4. perform perspective division via dividing $\frac{t}{z}$ by $\frac{1}{z}$ to obtain $t$

## 7.4 Texture Mapping and Illumination

- For texture-mapped object, changing the lighting will not show the unevenness of the object's surface.
- **Bump mapping**: textures to alter the surface normal of an object. Shading on the object is changed, but its silhouette is not.
- **Displacement mapping**: textures to change the shading of both the object and its silhouette. It actually moves the surface point — geometry is displaced before determining visibility.
- **Environment mapping**: we can simulate reflections by using the direction of the reflected ray to index a spherical texture map at "infinity".

# 8 Rasterization, Visibility, and Anti-aliasing

## 8.1 Background

- **Rasterization** determines which pixels are drawn into the framebuffer.
- Pixels have unique framebuffer location, but multiple fragments can be at the same address.
- **Alias effects**: an unreal visual artefacts caused by undersampling e.g. straight lines look jagged.

## 8.2 Barycentric Coordinates

- let vertices of a triangle be $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, then any point $\mathbf{p}$ can be specified in the plane as

$$\begin{aligned}\mathbf{p} &= \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}) = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \\ &= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},\end{aligned}$$

  and $(\alpha, \beta, \gamma)$ is called (absolute) barycentric coordinates.
- If $\mathbf{p}$ is inside the triangle $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, we have $0 < \alpha, \beta, \gamma < 1$; if $\mathbf{p}$ is on an edge, one coefficient is 0; if $\mathbf{p}$ is on a vertex, one coefficient is 1.
- Since implicit equation in 2D is defined as

$$f(x, y) = Ax + By + c = 0$$

  and an implicit line through $(x_a, y_a)$ and $(x_b, y_b)$ is

$$f_{ab}(x, y) = (y_a - y_b)x + (x_b - x_a)y + x_a y_b - x_b y_a = 0,$$

  note that a barycentric coordinate such as $\beta$ is a **signed distance** from a line (in this case, the line through $\mathbf{a}$ and $\mathbf{c}$), and we can use implicit line equations to evaluate the signed distance, we have

$$\frac{f_{ac}(x_b, y_b)}{f_{ac}(x, y)} = \frac{1}{\beta} \Longrightarrow \beta = \frac{f_{ac}(x, y)}{f_{ac}(x_b, y_b)}$$

  and we can obtain similar results for $\alpha$ and $\gamma$, thereby computing the barycentric coordinates with the given cartesian coordinates.

- In general, the barycentric coordinates for $\mathbf{p}$ are the solution of the linear system
$$\begin{pmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Let $a, b, c$ be the side lengths of the triangle, from barycentric to trilinear coordinates:
$$(\alpha, \beta, \gamma) \mapsto (\frac{\alpha}{a}, \frac{\beta}{b}, \frac{\gamma}{c}),$$
from trilinear to barycentric coordinates:
$$(t_1, t_2, t_3) \mapsto (t_1 a, t_2 b, t_3 c)$$

- **Triangle Rasterization**: we can check if $(0 < \alpha, \beta, \gamma < 1)$ to determine if a fragment of a triangle should be generated.

## 8.3 Visibility

- **Painter's algorithm**: sort the triangles using the $z$ values in camera space and draw them from back to front.
  - not efficient due to costly sorting
  - suffer from correctness issues such as intersections, cycles, etc.
- **depth buffer (z-buffer)**: perform hidden surface removal per-fragment by saving the $z$ value for each fragment and keep only the fragment with the smallest $z$ value at each pixel. Use *z-buffer (2D buffer)* of the same size as image to save $z$ values.
  - \+ facilitates hardware implementation
  - \+ handles intersections and cycles
  - \+ draw opaque polygons in any order

## 8.4 Anti-aliasing

- apply a degree of blurring to the boundary such that the aliasing effect is reduced.
- **Supersampling**
  1. Compute the picture at a high resolution to that of the display area.
  2. Supersamples are averaged to find the pixel value.
  3. This blurs the boundaries and leaves the coherent areas of color unchanged.
- **Convolution filtering**: use a filter that takes a (weighted) average over a small region around the pixel to blur the image, such as
$$\frac{1}{36}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$
  - \+ very fast, can be done in hardware
  - \+ generally applicable
  - \- degrade the image while enhancing its visual appearance