Computational Finance

Lectured by Panos Parpas

Typed by Aris Zhu Yi Qing

November 28, 2022

Contents

1 Introduction

- **Forward contract**: a customized *binding* contract to buy/sell an asset at a specific price (**forward price**) on a *future* date.
- **American option**: gives the holder the *right* to exercise the option at any time before the date of expiration.
- **European option**: can only be exercised at the date of expiration.
- **call/put option**: gives the holder the *right* to buy/sell.
- **Derivative asset**: assets defined in terms of underlying financial asset.

2 The Binomial Model

2.1 The One Period Model

2.1.1 Definitions

- **Bond** price process is deterministic and given by

$$B_0 = 1,$$
$$B_1 = 1 + R.$$

  where constant R is the spot rate for the period.

- **Stock** price process is stochastic and given by

$$S_0 = s,$$
$$S_1 = s \cdot Z$$

  where Z is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u, \\ d, & \text{with probability } p_d. \end{cases}$$

  with $p_u + p_d = 1$ and the assumption that $d < u$.

- **Portfolio** on the (B, S) market is a vector

$$h = (x, y).$$

  with a deterministic market value at $t = 0$ and stochastic value at $t = 1$.

- **Value Process** of the portfolio h is defined as

$$V_t^h = xB_t + yS_t, \quad t = 0, 1,$$

  or, in more detail,

$$V_0^h = x + ys,$$
$$V_1^h = x(1 + R) + ysZ.$$

- **Arbitrage** portfolio is an h with the properties

$$V_0^h = 0,$$
$$V_1^h > 0, \quad \textit{with probability 1}$$

2.1.2 Contingent Claim Pricing

Proposition 1. The model h is free of arbitrage $\iff d \le (1+R) \le u$.

Comments The above proposition implies that $(1 + R)$ is a convex combination of u and d, i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where q_u and q_d can be interpreted as probabilities for a new probability measure Q with $P(Z = u) = q_u$ and $P(Z = d) = q_d$. Denoting expectation w.r.t. this measure by E^Q and with the following calculation

$$\frac{1}{1+R} E^Q[S_1] = \frac{1}{1+R}[q_u su + q_d sd] = \frac{1}{1+R} \cdot s(1 + R) = s,$$

we arrive at the following relation

$$s = \frac{1}{1+R} E^Q[S_1].$$

Definition 2. A probability measure Q is **martingale** if the following condition holds:

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Proposition 3. Arbitrage-free model $\iff$ $\exists$ martingale measure Q.

Proposition 4. For the binomial model above, the martingale probabilities are

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d}, \\ q_d = \dfrac{u-(1+R)}{u-d}. \end{cases}$$

Proof. Solve the system of equation

$$\begin{cases} 1+R = q_u \cdot u + q_d \cdot d \\ q_u + q_d = 1. \end{cases}$$

□

Definition 5. A **contingent claim** (financial derivative) is any stchastic variable X of the form $X = \Phi(Z)$, where Z is the stochastic variable driving the stock price process above, Φ is the **contract function**.

Example 6. European call option on the stock with strike price K. Assuming that $sd < K < su$, we have

$$X = \begin{cases} su - K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

so the contract function is given by

$$\Phi(u) = su - K,$$
$$\Phi(d) = 0.$$

Definition 7. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists h \text{ s.t. } V_1^h = X, \text{ with probability 1.}$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **complete**.

Proposition 8. If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is

$$\Pi_t[X] = V_t^h, \ t = 0, 1.$$

Here, "reasonable" means that $\Pi_0[X] \neq V_0^h \Rightarrow$ arbitrage possibility.

Proposition 9. The general binomial model is free of arbitrage $\Rightarrow$ it is complete.

Proof. Say a claim X has contract function Φ s.t.

$$V_1^h = \begin{cases} \Phi(u), & \text{if } Z = u \\ \Phi(d), & \text{if } Z = d, \end{cases}$$

we can obtain the following system of equations by expanding V_1^h

$$(1+R)x + xuy = \Phi(u),$$
$$(1+R)x + xdy = \Phi(d),$$

and solve it to find out the replicating portfolio as

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d},$$
$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}.$$

□

Proposition 10. If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X],$$

where the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Proof. Using the results derived previously, we can find out that

$$\begin{aligned}\Pi_0[X] &= x + sy \\ &= \frac{1}{1+R}\left[\frac{(1+R)-d}{u-d}\cdot\Phi(u) + \frac{u-(1+R)}{u-d}\cdot\Phi(d)\right] \\ &= \frac{1}{1+R}\left[\Phi(u)q_u + \Phi(d)q_d\right] \\ &= \frac{1}{1+R}E^Q[X].\end{aligned}$$

□

2.2 The Multiperiod Model

2.2.1 Definitions

- Model
  - The bond price dynamics is given by

    $$B_{n+1} = (1+R)B_n, \quad B_0 = 1.$$

  - The stock price dynamics is given by

    $$S_{n+1} = S_n \cdot Z_n, \quad S_0 = s.$$

    where $Z_0, Z_1, \ldots, Z_{T-1}$ are assumed to be i.i.d. stochastic variables, with only two variables u and d and probabilities

    $$P(Z_n = u) = p_u, \quad P(Z_n = d) = p_d.$$

  - Illustrating the stock dynamics by means of a tree, we can see that it is **recombining**.
- A **portfolio strategy** is a stochastic process

  $$\{h_t = (x_t, y_t);\ t = 1, \ldots, T\}$$

  s.t. h_t is a function of $S_0, S_1, \ldots, S_{t-1}$. By convention, $h_0 = h_1$.
  - Interpretation of h_t: at $t-1$, x_t and y_t of bonds and stocks are bought and held until time t.
- The **value process** corresponding to the portfolio h is defined by

  $$V_t^h = x_t(1+R) + y_t S_t.$$

- A portfolio strategy h_t is said to be **self-financing** if $\forall t = 0, \ldots, T-1$,

  $$x_t(1+R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

  which says that the market value of the "old" portfolio h_t equals to the purchase value of the "new" portfolio h_{t+1}.
- An **arbitrage** possibility is a self-financing portfolio h with the properties

  $$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

- The **martingale** probabilities q_u and q_d are defined as the probabilities for which the following relation holds

  $$s = \frac{1}{1+R}E^Q[S_{t+1}|S_t = s]$$

2.2.2 Binomial Model Pricing Algorithm

Proposition 11. The model is free of arbitrage $\Rightarrow d \leq (1+R) \leq u$.

Assumption 12. From now, assume that $d < u$ and $d \leq (1+R) \leq u$.

Definition 13. A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T),$$

where the **contract function** Φ is some given real valued function.

Definition 14. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists \text{ self-financing } h \text{ s.t. } V_T^h = X, \text{ with probability } 1.$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **(dynamically) complete**.

(Binomial Algorithm) Proposition 15. Consider a T-claim $X = \Phi(S_T)$, which could be replicated with a self-financing portfolio h. Let k be the number of up-moves occurred. So $V_t(k)$ can be computed recursively as

$$\begin{cases} V_t(k) &= \dfrac{1}{1+R}\left[q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\right], \\ V_T(k) &= \Phi(su^k d^{T-k}). \end{cases}$$

where the martingale probabilities q_u and q_d are

$$\begin{cases} q_u &= \dfrac{(1+R)-d}{u-d} \\ q_d &= \dfrac{u-(1+R)}{u-d}. \end{cases}$$

and the hedging portfolio is given by

$$\begin{cases} x_t(k) &= \dfrac{1}{1+R}\cdot\dfrac{uV_t(k)-dV_t(k+1)}{u-d}, \\ y_t(k) &= \dfrac{1}{S_{t-1}}\cdot\dfrac{V_t(k+1)-V_t(k)}{u-d}. \end{cases}$$

N.B. In Fig 2.9 in notes, (the figure might be confusing) $(-22.5, 5/8)$ is both h_0 and h_1, $(-42.5, 95/120)$ is $h_2(1)$, $(-2.5, 1/8)$ is $h_2(0)$, etc.

Proposition 16. The arbitrage free price at $t = 0$ of a T-claim X is given by

$$\Pi_0[X] = \frac{1}{(1+R)^T}\cdot E^Q[X],$$

where Q denotes the martingale measure, or more explicitly,

$$\Pi_0[X] = \frac{1}{(1+R)^T}\cdot\sum_{k=0}^{T}\binom{T}{k} q_u^k q_d^{T-k}\Phi(su^k d^{T-k}).$$

Proposition 17. $d < (1+R) < u \iff$ free of arbitrage.

3 Brownian Motion

3.1 Definitions

- Let A, B be two random variables, we define **equal in distribution** as
$$A \overset{d}{=} B \iff P(A\in C) = P(B \in C)\ \forall \text{ possible sets } C.$$
- We say a random variable X has **stationary increments** if
$$X_t - X_s \overset{d}{=} X_{t+h} - X_{s+h} \quad \forall h > 0.$$
- A stochastic process B_t is called a **Brownian motion** if
  - $B_0 = 0$,
  - $B_t - B_s \sim N(0, t-s)\ \forall t \geq s \geq 0$, implying *stationary* increments,
  - it has *independent* increments.

  with
  - $\mathbb{E}[B_t] = 0$,
  - $\mathrm{Cov}(B_t, B_s) = \min\{t, s\}$. Show this using (Practice!)
    * $\mathrm{Cov}(B_t, B_s) = \mathbb{E}[(B_t - \mathbb{E}[B_t])(B_s - \mathbb{E}[B_s])]$
    * $\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_tB_s] - \mathbb{E}[B_t]\mathbb{E}[B_s]$.

3.2 Path Properties

- Paths are continuous (no jumps).
- Nowhere differentiable.
- Paths are **self-similar**, i.e.
$$(T^H B_{t_1}, \ldots, T^H B_{t_n}) \overset{d}{=} (B_{Tt_1}, \ldots, B_{Tt_n})$$
with $H = 0.5$ is the Hurst coefficient.

3.3 Brownian Motion with Drift

- A **Gaussian Process** is a stochastic process s.t. every finite collection of those random variables has a multivariate normal distribution.
- The process is
$$X_t = \mu t + \sigma B_t, \quad t \geq 0.$$
- It is a Gaussian process with the following properties:
  - $\mathbb{E}[X_t] = \mu t$,
  - $\mathrm{Cov}(X_t, X_s) = \sigma^2 \min\{t, s\}$. (Practice!)

3.4 Geometric Brownian Motion with Drift

- The process is
$$X_t = e^{\mu t + \sigma B_t}, \quad t \geq 0.$$
- It is *not* Gaussian, but log-normal instead, with the following properties:
  - $\mathbb{E}[X_t] = e^{\mu t + \frac{1}{2}\sigma^2 t}$, (Practice!)
  - $\mathrm{Cov}(X_t, X_s) = e^{\left(\mu + \frac{1}{2}\sigma^2\right)(t+s)} \left(e^{\sigma^2 s} - 1\right)$. (Practice!)

4 Stochastic Integration

4.1 Time Value of Money

- £x today is worth more than £x in the future.
- Compensation for postponed consumption
- Inflation: prices may rise so £x in the future will not have the same purchasing power
- Risk – money in the future may never be received.

4.2 Interest

4.2.1 Definitions

- $V(t)$: value of investment at time t
- r: interest rate $r \geq 0$
- t: time measured in years
- P: principal/initial investment (£)
- simple/annual compounding:
$$\begin{cases} V(0) &= P \\ V(t) &= (1 + tr)P \end{cases}$$

4.2.2 Compounding

- Periodic, e.g. monthly compounding:
$$V(t) = \left(1 + \frac{r}{12}\right)^t P$$
- Continuous:
$$\begin{aligned} V(t) = e^{tr} P \quad &\Longrightarrow \quad \frac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t) \\ &\Longrightarrow \quad V(t) = V(0) + \int_0^t rV(s)\mathrm{d}s. \end{aligned}$$

4.2.3 Stochastic Integral

Example:
$$I(T) = \int_0^T B(t)\mathrm{d}B(t)$$

Using left-hand point approximation, we can get
$$\begin{aligned} I(T) &= \lim_{L \to \infty} \sum_{i=0}^{L-1} B(t_i)\left[B(t_{i+1}) - B(t_i)\right] \\ &= \lim_{L \to \infty} \sum_{i=0}^{L-1} -\frac{1}{2}B(t_i)^2 - \frac{1}{2}[B(t_{i+1}) - B(t_i)]^2 + \frac{1}{2}B(t_{i+1})^2 \end{aligned}$$

$$= \frac{1}{2} \lim_{L \to \infty} \sum_{i=0}^{L-1} [B(t_{i+1})^2 - B(t_i)^2] - \delta B_i^2$$

where $\delta B_i = B(t_{i+1}) - B(t_i)$. Analyzing the above expression (exercise!), we can obtain

$$I(T) = \frac{1}{2} B(T)^2 - \frac{1}{2} T.$$

We can also show the following: (exercise!)

- $\mathbb{E}[\delta B_i] = 0$
- $\mathrm{Var}(\delta B_i) = \delta t$
- $\mathbb{E}[\delta B_i^2] = \delta t$
- $\mathrm{Var}(\delta B_i^2) = 2(\delta t)^2$ (hint: $\mathbb{E}[Z^4] = 3$ for $Z \sim N(0,1)$)

5 Stochastic Calculus

5.1 Ito's Formula

5.1.1 Ito's Multiplication Rules

$$\begin{aligned} (\mathrm{d}t)^2 &= 0, \\ (\mathrm{d}t)(\mathrm{d}B(t)) &= 0, \\ (\mathrm{d}B(t))^2 &= \mathrm{d}t. \end{aligned}$$

5.1.2 Ito's Lemma

Consider an **Ito's process**

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

where

- μ_t (or just μ): drift process,
- σ_t (or just σ): diffusion process,

both can be either stochastic or deterministic. Using the multiplication rules, we can derive that

$$(\mathrm{d}X_t)^2 = \mu^2 (\mathrm{d}t)^2 + \sigma^2 (\mathrm{d}B_t)^2 + 2\mu\sigma(\mathrm{d}t)(\mathrm{d}B_t) = \sigma^2 \mathrm{d}t.$$

Let $Z(t) = f(t, X_t)$, then

$$\begin{aligned} \mathrm{d}Z(t) &= \frac{\partial f}{\partial X_t} \mathrm{d}X_t + \frac{\partial f}{\partial t} \mathrm{d}t + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (\mathrm{d}X_t)^2}_{\text{Itô correction}} \\ &= \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial X_t^2} \right) \mathrm{d}t + \frac{\partial f}{\partial X_t} \sigma \mathrm{d}B(t). \end{aligned}$$

5.1.3 Examples

1. $\mathrm{d}X_t = \mathrm{d}W_t$, $f(t, x) = te^{\alpha x}$. Since

$$\frac{\partial f}{\partial t} = e^{\alpha x}, \quad \frac{\partial f}{\partial x} = \alpha t e^{\alpha x}, \quad \frac{\partial^2 f}{\partial x^2} = \alpha^2 t e^{\alpha x},$$

and in this case $\mu = 0$, $\sigma = 1$, we have

$$\mathrm{d}f(t, W_t) = \left(e^{\alpha x} + + \frac{1}{2} \alpha^2 t e^{\alpha W_t} \right) \mathrm{d}t + \alpha t e^{\alpha W_t} \mathrm{d}W_t$$

2. $\mathrm{d}X = \mathrm{d}W$, $f(x) = x^2$.

$$\begin{aligned} \mathrm{d}f(t, W_t) &= \frac{\partial f}{\partial t} \mathrm{d}t + \frac{\partial f}{\partial x} \mathrm{d}x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \mathrm{d}t \\ &= 0 + 2W \mathrm{d}W + \frac{1}{2} 2 \mathrm{d}t \\ &= \mathrm{d}t + 2W_t \mathrm{d}W_t \end{aligned}$$

3. Compute $\mathbb{E}\left[B_t^4\right]$. This could be transformed to the question

$$\mathrm{d}X_t = \mathrm{d}B_t, \quad Z_t = X_t^4.$$

Thus,

$$\mathrm{d}Z_t = 6B_t^2 \mathrm{d}t + 4B_t^3 \mathrm{d}B_t,$$

i.e.

$$Z_t = Z_0 + 6 \int_0^t B_s^2 \mathrm{d}s + 4 \int_0^t B_s^3 \mathrm{d}B_s$$

so

$$\begin{aligned}
\mathbb{E}[B_t^4] = \mathbb{E}Z_t &= 0 + 6\int_0^t \mathbb{E}[B_s^2]\mathrm{d}s + 4\int_0^t \mathbb{E}[B_s^3]\mathrm{d}B_s \\
&= 6\int_0^t s\mathrm{d}s + 0 \\
&= 3t^2
\end{aligned}$$

5.2 Stochastic Differential Equation

Example Given that

$$\mathrm{d}S(t) = \mathrm{d}B(t), \quad u(t,x) = y_0 e^{\mu t + \sigma x},$$

and

$$Y(t) = u(t,S), \quad Y(0) = y_0,$$

we have

$$\begin{aligned}
\mathrm{d}Y_t &= \mu u(t,S)\mathrm{d}t + \sigma u(t,s)\mathrm{d}B + \frac{1}{2}\sigma^2 u(t,s)\mathrm{d}t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) Y(t)\mathrm{d}t + \sigma Y(t)\mathrm{d}B_t.
\end{aligned}$$

This gives hint on how to solve

$$\mathrm{d}Y(t) = \mu Y(t)\mathrm{d}t + \frac{1}{2}\sigma Y(t)\mathrm{d}B_t,$$

whose solution is

$$Y(t) = u(t,B_t) = y(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}.$$

This can be solved with a change of variable

$$Z(t) = \ln Y(t).$$

Applying Ito's lemma, we can derive that

$$\begin{aligned}
\mathrm{d}Z(t) &= \frac{1}{Y(t)}\mathrm{d}Y(t) - \frac{1}{2}\frac{1}{Y(t)^2}\sigma^2 Y(t)^2\mathrm{d}t \\
&= \mu\mathrm{d}t + \sigma\mathrm{d}B(t) - \frac{1}{2}\sigma^2\mathrm{d}t \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}B_t
\end{aligned}$$

Integrating both sides from 0 to t, we get

$$Z(t) - Z(0) = \ln\frac{Y(t)}{Y(0)} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t$$

leading to the solution.

5.3 SDE & PDE

Consider the SDE

$$\begin{aligned}
\mathrm{d}X_s &= \mu(s,X_s)\mathrm{d}s + \sigma(s,B_s)\mathrm{d}B_s, \\
X_t &= x,
\end{aligned}$$

which starts at x at time t and evolves in the interval $[t,T]$. Now the parabolic PDE

$$\begin{aligned}
\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} &= \frac{\partial F}{\partial t} + \mathcal{A}F = 0 \\
F(T,x) &= \phi(x),
\end{aligned}$$

where

$$\mathcal{A} = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}$$

is called the *infinitesimal operator*. Applying Ito's Lemma to $F(s,X(s))$, we have

$$\mathrm{d}F = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial X}\mathrm{d}X + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\mathrm{d}t.$$

Integrate from t to T, we get

$$\begin{aligned}
&F(T,X_T) - F(t,X_t) \\
&= \underbrace{\int_t^T \left[\frac{\partial F(s,X_s)}{\partial t} + \frac{\partial F(s,X_s)}{\partial X}\mu(s,X_s) + \frac{1}{2}\sigma^2\frac{\partial^2 F(s,X_s)}{\partial X^2}\right]\mathrm{d}s}_{\text{exactly the PDE, } = 0} \\
&\quad + \int_t^T \sigma(s,X_s)\frac{\partial F(s,X_s)}{\partial X}\mathrm{d}W_s \\
&= \int_t^T \left[\frac{\partial F(s,X_s)}{\partial t} + \mathcal{A}F(s,X_s)\right]\mathrm{d}s + \int_t^T \sigma(s,X_s)\frac{\partial F(s,X_s)}{\partial X}\mathrm{d}W_s
\end{aligned}$$

$$= \int_t^T \sigma(s, X_s) \frac{\partial F(s, X_s)}{\partial X} \mathrm{d}W_s,$$

$$\implies \quad \phi(X_T) - F(t, X) = \int_t^T \sigma(s, X_s) \frac{\partial F(s, X_s)}{\partial X} \mathrm{d}W_s$$

Taking expectation on both sides (conditional on t),

$$\mathbb{E}\phi(X_T) - \mathbb{E}[F(t, X(t))|X(t) = x] = \mathbb{E}\left[\int_t^T \sigma(s, X_s) \frac{\partial F(s, X_s)}{\partial X} \mathrm{d}W_s | X(t) = x\right].$$

Since stochastic integral has 0 expectation, i.e.

$$F(t, X) = \mathbb{E}\left[\phi(X_T)|X(t) = x\right]$$

The above conclusion is the **Feynman Kac Theorem**

Proposition 18. If F is a solution to

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0, \quad F(T, X) = \phi(X),$$

where $\mathcal{A}$ is the infinitesimal operator associated with the SDE

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}s + \sigma(s, X_s)\mathrm{d}W_s, \quad X_t = x$$

then

$$F(t, x) = \mathbb{E}\left[\phi(X_T)|X_t = x\right]$$

Note By "associated" means that the quantity μ and σ are the same in both SDE and PDE.

Example 19. Solve the PDE

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F(t, x)}{\partial x^2} = 0, \quad F(T, x) = x^2$$

using the Feynman Kac formula. Let

$$\mathrm{d}X_s = \sigma \mathrm{d}W_s, \quad X_t = x$$

The solution to the SDE is

$$X_t = x + \sigma\left[W_T - W_t\right]$$

implying that $X_T \sim N(x, \sigma^2(T - t))$. So

$$F(t, x) = \mathbb{E}X_T^2 = \sigma^2(T - t) + x^2.$$

6 Arbitrage Pricing

6.1 Definitions

- **out-of-the-money (OTM)** for a call means $S(t) - K < 0$.
- **in-the-money** for a call means $S(t) - K > 0$.
- **at-the-money** for a call means $S(t) = K$.
- **Exotic options**: more complicated products that have "exotic" features, e.g. early exercise, multiple strikes, etc.

6.2 The Black & Scholes Model and Arbitrage

6.2.1 Model Introduction

The model is based on:

1. risk-free bank (letter B stands for bank):

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t \quad \Rightarrow \quad B(t) = B(0)e^{rt}$$

2. stock:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \quad S(0) = s(0)$$
$$\Rightarrow \quad S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

3. option, that depends on stock price and the current time:

$$V(t, S).$$

6.2.2 The PDE approach of pricing

The model:

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad \mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

Applying Ito's lemma on $V(t, S)$, we have

$$\begin{aligned} \mathrm{d}V &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\mathrm{d}S)^2 \\ &= \left[\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\mathrm{d}t + \sigma S \frac{\partial V}{\partial S}\mathrm{d}W_t. \end{aligned}$$

Consider a portfolio $P(t)$ with Δ stocks and one short option (**Delta hedging**). The value of the portfolio is

$$P(t) = \underbrace{\Delta S(t)}_{\text{long the stock}} - \underbrace{V(t)}_{\text{short the option}}$$

The portfolio evolves according to

$$\begin{aligned} \mathrm{d}P(t) &= \Delta \mathrm{d}S(t) - \mathrm{d}V \\ &= \left[\Delta\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\mathrm{d}t + \left[\Delta\sigma S - \sigma S\frac{\partial V}{\partial S}\right]\mathrm{d}W_t. \end{aligned}$$

If we let $\Delta = \frac{\partial V}{\partial S}$, because if the market has no arbitrage then its return must equal $rP\mathrm{d}t$, we are left with

$$\mathrm{d}P = \left[\Delta\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\mathrm{d}t = rP\mathrm{d}t$$

$$\Rightarrow \quad \Delta\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rP$$

Since $P = \frac{\partial V}{\partial S}S - V, \Delta = \frac{\partial V}{\partial S}$, after these substitution, we obtain the following second-order parabolic PDE called the **Black & Scholes Equation (BSE)**:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV, \quad V(T,S) = \max(S-K,0),$$

where the second equation is the boundary condition for European call option.

Hedge Parameters The following parameters (appearing in the PDE) are the sensitivities of the option value w.r.t. small changes in the problem.

- $$\Delta = \frac{\partial V}{\partial S}$$
  It tells the trader how to balance the portfolio so that it is always equal to the option.

- $$\Gamma = \frac{\partial^2 V}{\partial S^2}$$
  It gives an indication of how stable the hedging portfolio is. If Γ is large, the trader needs to rebalance more often.

- $$\theta = \frac{\partial V}{\partial t}$$
  If S stays constant then the value of the option will change by θ

- $$\zeta = \frac{\partial V}{\partial \sigma}$$
  It measures the change in price w.r.t. volatility

- $$\rho = \frac{\partial V}{\partial r}$$
  It measures sensitivity w.r.t. interest rate.

6.2.3 The Martingale Method of Pricing

Option valuation could be reduced to the calculation of the following:

$$V(S_0) = \mathbb{E}^Q\left[e^{-rT}V(S_T)|S_0\right].$$

In the case of the Black-Scholes model, Q is the probability distribution of the following SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

or

$$S(t) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}$$

where

- r: risk-free rate
- σ: volatility
- w: standard brownian motion

This says that the assumption of the Black Scholes model is that the underlying asset follows a geometric Brownian motion i.e. lognormal random walk.

7 Monte-Carlo methods

7.1 Strong Law of Large Numbers

Let $\xi^{(i)}$, $i = 1, 2, \dots, N$ be i.i.d. random variables with values in $\mathbb{R}^d$, and mean $\mathbb{E}|\xi| < \infty$. Let $\hat{S}_N$ denote the empirical mean $\hat{S}_N = \frac{1}{N}\sum_{i=1}^{N} \xi^{(i)}$. Then the SLLN holds true:

$$\lim_{N\to\infty} \hat{S}_N = \mathbb{E}(\xi)$$

7.2 Central Limit Theorem

Let $\xi^{(i)}$, $i = 1, 2, \dots, N$ be i.i.d. random variables with mean $\mathbb{E}\xi$ and $\operatorname{Var}(\xi) < \infty$. Then

$$Y_N = \frac{\sqrt{N}}{\sigma}\left(\frac{1}{N}\sum_{i=1}^{N} \xi^i - \mathbb{E}\xi\right) \longrightarrow_d Z \sim N(0,1)$$

as $N \to \infty$. In practice, we use empirical variance to compute σ (unbiased estimate):

$$\sigma_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\xi^i - \overline{\xi}\right)^2.$$

In other words,

$$\mathbb{P}\left(\left|\frac{\sqrt{N}}{\sigma}\left(\frac{1}{N}\sum_{i=1}^{N}\xi^i - \mathbb{E}\xi\right)\right| < R\right) \longrightarrow \int_{-R}^{R} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x$$

7.3 Monte Carlo for Option Valuation

Since

$$S(T) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z}, \quad Z \sim N(0,1),$$

if we can generate i.i.d. normal random variables $\{Z^i\}$, $i = 1, 2, \dots, N$, we have

$$V(S_0) \approx \frac{1}{N}\sum_{i=1}^{N} V(S_T^i)e^{-rT} = V_N(S_0),$$

where

$$S_T^i = e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sqrt{T}Z^i}$$

7.4 Variance Reduction methods for MC

7.4.1 Antithetic Variables

If X_1 and X_2 are i.i.d. RVs,

$$\operatorname{Var}\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}\left(\operatorname{Var}(X_1) + \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1, X_2)\right)$$

so variance is reduced if $\operatorname{Cov}(X_1, X_2) < 0$.

Let $f(x)$ be monotonic. For instance, in the case of Gaussian variables X, $f(x)$ and $f(-x)$ are negatively correlated, i.e.

$$\operatorname{Cov}(f(X), f(-X)) < 0.$$

In the case of uniform distribution variables U,

$$\operatorname{Cov}(f(U), f(1-U)) < 0.$$

7.4.2 Control Variates

Suppose we want to estimate $\mathbb{E}X$ and know the mean of another random variable Y, $\mathbb{E}Y$, then

$$\mathbb{E}X = \mathbb{E}[X - Y + \mathbb{E}Y].$$

Y is called a control variate. While the expectation of

$$Z = X - Y + \mathbb{E}Y$$

is $\mathbb{E}X$ its variance will be

$$\operatorname{Var}(X - Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X, Y).$$

This idea works when X and Y are close.

7.4.3 Importance Sampling

We can write the expectation with respect to the probability density p as

$$\mathbb{E}^p g(X) = \int g(x)p(x)\mathrm{d}x.$$

Suppose that we had another density q that had the same support i.e. $p(x) > 0 \iff q(x) > 0$, then

$$\mathbb{E}^p g(x) = \int g(x)p(x)\mathrm{d}x = \int g(x)\frac{p(x)}{q(x)}q(x)\mathrm{d}x = \mathbb{E}^q g(x)\frac{p(x)}{q(x)}$$

Thus instead of calculating $\mathbb{E}^p g(x)$ using i.i.d. samples from p we calculate

$$\mathbb{E}^q g(x)\Lambda(x)$$

using samples from q, where $\Lambda(x)$ is the likelihood ratio.

7.5 Computation of Monte-Carlo Greeks

Monitoring the P&L of the hedging portfolio

$$\pi(t,s) = V(t,s) - \Delta S(t).$$