Computational Finance

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November 13, 2022

Contents

1 Introduction

- **Forward contract**: a customized *binding* contract to buy/sell an asset at a specific price (**forward price**) on a *future* date.
- **American option**: gives the holder the *right* to exercise the option at any time before the date of expiration.
- **European option**: can only be exercised at the date of expiration.
- **call/put option**: gives the holder the *right* to buy/sell.
- **Derivative asset**: assets defined in terms of underlying financial asset.

2 The Binomial Model

2.1 The One Period Model

2.1.1 Definitions

- **Bond** price process is deterministic and given by

$$\begin{aligned} B_0 &= 1, \\ B_1 &= 1 + R. \end{aligned}$$

  where constant R is the spot rate for the period.

- **Stock** price process is stochastic and given by

$$S_0 = s,$$

$$S_1 = s \cdot Z$$

where Z is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u, \\ d, & \text{with probability } p_d. \end{cases}$$

with $p_u + p_d = 1$ and the assumption that $d < u$.

- **Portfolio** on the (B, S) market is a vector

$$h = (x, y).$$

with a deterministic market value at $t = 0$ and stochastic value at $t = 1$.

- **Value Process** of the portfolio h is defined as

$$V_t^h = xB_t + yS_t, \quad t = 0, 1,$$

or, in more detail,

$$\begin{aligned} V_0^h &= x + ys, \\ V_1^h &= x(1+R) + ysZ. \end{aligned}$$

- **Arbitrage** portfolio is an h with the properties

$$\begin{aligned} V_0^h &= 0, \\ V_1^h &> 0, \quad \textit{with probability 1} \end{aligned}$$

2.1.2 Contingent Claim Pricing

Proposition 1. The model h is free of arbitrage $\iff d \leq (1+R) \leq u$.

Comments The above proposition implies that $(1+R)$ is a convex combination of u and d, i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where q_u and q_d can be interpreted as probabilities for a new probability measure Q with $P(Z = u) = q_u$ and $P(Z = d) = q_d$. Denoting expectation w.r.t. this measure by E^Q and with the following calculation

$$\frac{1}{1+R} E^Q[S_1] = \frac{1}{1+R}[q_u su + q_d sd] = \frac{1}{1+R} \cdot s(1+R) = s,$$

we arrive at the following relation

$$s = \frac{1}{1+R} E^Q[S_1].$$

Definition 2. A probability measure Q is **martingale** if the following condition holds:

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Proposition 3. Arbitrage-free model $\iff \exists$ martingale measure Q.

Proposition 4. For the binomial model above, the martingale probabilities are

$$\begin{cases} q_u = \dfrac{(1+R) - d}{u - d}, \\ q_d = \dfrac{u - (1+R)}{u - d}. \end{cases}$$

Proof. Solve the system of equation

$$\begin{cases} 1 + R = q_u \cdot u + q_d \cdot d \\ q_u + q_d = 1. \end{cases}$$

□

Definition 5. A **contingent claim** (financial derivative) is any stchastic variable X of the form $X = \Phi(Z)$, where Z is the stochastic variable driving the stock price process above, Φ is the **contract function**.

Example 6. European call option on the stock with strike price K. Assuming that $sd < K < su$, we have

$$X = \begin{cases} su - K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

so the contract function is given by

$$\begin{aligned} \Phi(u) &= su - K, \\ \Phi(d) &= 0. \end{aligned}$$

Definition 7. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists h \text{ s.t. } V_1^h = X, \text{ with probability } 1.$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **complete**.

Proposition 8. If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is

$$\Pi_t[X] = V_t^h,\ t = 0, 1.$$

Here, "reasonable" means that $\Pi_0[X] \neq V_0^h \Rightarrow$ arbitrage possibility.

Proposition 9. The general binomial model is free of arbitrage $\Rightarrow$ it is complete.

Proof. Say a claim X has contract function Φ s.t.

$$V_1^h = \begin{cases} \Phi(u), & \text{if } Z = u \\ \Phi(d), & \text{if } Z = d, \end{cases}$$

we can obtain the following system of equations by expanding V_1^h

$$\begin{aligned} (1+R)x + xuy &= \Phi(u), \\ (1+R)x + xdy &= \Phi(d), \end{aligned}$$

and solve it to find out the replicating portfolio as

$$\begin{aligned} x &= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \\ y &= \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \end{aligned}$$

$\square$

Proposition 10. If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X],$$

where the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Proof. Using the results derived previously, we can find out that

$$\begin{aligned} \Pi_0[X] &= x + sy \\ &= \frac{1}{1+R} \left[\frac{(1+R)-d}{u-d} \cdot \Phi(u) + \frac{u-(1+R)}{u-d} \cdot \Phi(d) \right] \\ &= \frac{1}{1+R} \left[\Phi(u) q_u + \Phi(d) q_d \right] \\ &= \frac{1}{1+R} E^Q[X]. \end{aligned}$$

$\square$

2.2 The Multiperiod Model

2.2.1 Definitions

- Model
  - The bond price dynamics is given by

    $$B_{n+1} = (1+R)B_n, \quad B_0 = 1.$$

  - The stock price dynamics is given by

    $$S_{n+1} = S_n \cdot Z_n, \quad S_0 = s.$$

    where $Z_0, Z_1, \ldots, Z_{T-1}$ are assumed to be i.i.d. stochastic variables, with only two variables u and d and probabilities

    $$P(Z_n = u) = p_u, \quad P(Z_n = d) = p_d.$$

  - Illustrating the stock dynamics by means of a tree, we can see that it is **recombining**.

- A **portfolio strategy** is a stochastic process

$$\{h_t = (x_t, y_t);\ t = 1, \ldots, T\}$$

s.t. h_t is a function of $S_0, S_1, \ldots, S_{t-1}$. By convention, $h_0 = h_1$.

  - Interpretation of h_t: at $t-1$, x_t and y_t of bonds and stocks are bought and held until time t.

- The **value process** corresponding to the portfolio h is defined by

$$V_t^h = x_t(1+R) + y_t S_t.$$

- A portfolio strategy h_t is said to be **self-financing** if $\forall t = 0, \ldots, T-1$,

$$x_t(1+R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

which says that the market value of the "old" portfolio h_t equals to the purchase value of the "new" portfolio h_{t+1}.

- An **arbitrage** possibility is a self-financing portfolio h with the properties

$$V_0^h = 0,$$
$$P(V_T^h \geq 0) = 1,$$
$$P(V_T^h > 0) > 0.$$

- The **martingale** probabilities q_u and q_d are defined as the probabilities for which the following relation holds

$$s = \frac{1}{1+R} E^Q[S_{t+1} | S_t = s]$$

2.2.2 Binomial Model Pricing Algorithm

Proposition 11. The model is free of arbitrage $\Rightarrow d \leq (1+R) \leq u$.

Assumption 12. From now, assume that $d < u$ and $d \leq (1+R) \leq u$.

Definition 13. A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T),$$

where the **contract function** Φ is some given real valued function.

Definition 14. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists \text{ self-financing } h \text{ s.t. } V_T^h = X, \text{ with probability } 1.$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **(dynamically) complete**.

(Binomial Algorithm) Proposition 15. Consider a T-claim $X = \Phi(S_T)$, which could be replicated with a self-financing portfolio h. Let k be the number of up-moves occurred. So $V_t(k)$ can be computed recursively as

$$\begin{cases} V_t(k) &= \dfrac{1}{1+R}\left[q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\right], \\ V_T(k) &= \Phi(s u^k d^{T-k}). \end{cases}$$

where the martingale probabilities q_u and q_d are

$$\begin{cases} q_u &= \dfrac{(1+R)-d}{u-d} \\ q_d &= \dfrac{u-(1+R)}{u-d}. \end{cases}$$

and the hedging portfolio is given by

$$\begin{cases} x_t(k) &= \dfrac{1}{1+R} \cdot \dfrac{uV_t(k) - dV_t(k+1)}{u-d}, \\ y_t(k) &= \dfrac{1}{S_{t-1}} \cdot \dfrac{V_t(k+1) - V_t(k)}{u-d}. \end{cases}$$

N.B. In Fig 2.9 in notes, (the figure might be confusing) $(-22.5, 5/8)$ is both h_0 and h_1, $(-42.5, 95/120)$ is $h_2(1)$, $(-2.5, 1/8)$ is $h_2(0)$, etc.

Proposition 16. The arbitrage free price at $t = 0$ of a T-claim X is given by

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot E^Q[X],$$

where Q denotes the martingale measure, or more explicitly,

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi(su^k d^{T-k}).$$

Proposition 17. $d < (1+R) < u \iff$ free of arbitrage.

3 Brownian Motion

3.1 Definitions

- Let A, B be two random variables, we define **equal in distribution** as
$$A \overset{d}{=} B \iff P(A \in C) = P(B \in C) \ \forall \text{ possible sets } C.$$
- We say a random variable X has **stationary increments** if
$$X_t - X_s \overset{d}{=} X_{t+h} - X_{s+h} \quad \forall h > 0.$$
- A stochastic process B_t is called a **Brownian motion** if
  - $B_0 = 0$,
  - $B_t - B_s \sim N(0, t-s) \ \forall t \geq s \geq 0$, implying *stationary* increments,
  - it has *independent* increments.

  with
  - $\mathbb{E}[B_t] = 0$,
  - $\mathrm{Cov}(B_t, B_s) = \min\{t, s\}$. Show this using (Practice!)
    - $\mathrm{Cov}(B_t, B_s) = \mathbb{E}[(B_t - \mathbb{E}[B_t])(B_s - \mathbb{E}[B_s])]$
    - $\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_t B_s] - \mathbb{E}[B_t]\mathbb{E}[B_s]$.

3.2 Path Properties

- Paths are continuous (no jumps).
- Nowhere differentiable.
- Paths are **self-similar**, i.e.
$$(T^H B_{t_1}, \ldots, T^H B_{t_n}) \overset{d}{=} (B_{Tt_1}, \ldots, B_{Tt_n})$$
with $H = 0.5$ is the Hurst coefficient.

3.3 Brownian Motion with Drift

- A **Gaussian Process** is a stochastic process s.t. every finite collection of those random variables has a multivariate normal distribution.
- The process is
$$X_t = \mu t + \sigma B_t, \quad t \geq 0.$$
- It is a Gaussian process with the following properties:
  - $\mathbb{E}[X_t] = \mu t$,
  - $\mathrm{Cov}(X_t, X_s) = \sigma^2 \min\{t, s\}$. (Practice!)

3.4 Geometric Brownian Motion with Drift

- The process is
$$X_t = e^{\mu t + \sigma B_t}, \quad t \geq 0.$$
- It is *not* Gaussian, but log-normal instead, with the following properties:
  - $\mathbb{E}[X_t] = e^{\mu t + \frac{1}{2}\sigma^2 t}$, (Practice!)
  - $\mathrm{Cov}(X_t, X_s) = e^{\left(\mu + \frac{1}{2}\sigma^2\right)(t+s)} \left(e^{\sigma^2 s} - 1\right)$. (Practice!)

4 Stochastic Integration