

# Computational Finance

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## Contents

<b>1 Introduction</b>	<b>1</b>
<b>2 The Binomial Model</b>	<b>1</b>
2.1 The One Period Model . . . . .	1
2.1.1 Definitions . . . . .	1
2.1.2 Contingent Claim Pricing . . . . .	2
2.2 The Multiperiod Model . . . . .	3
2.2.1 Definitions . . . . .	3
2.2.2 Binomial Model Pricing Algorithm . . . . .	4
<b>3 Brownian Motion</b>	<b>5</b>
3.1 Definitions . . . . .	5
3.2 Path Properties . . . . .	5

## 1 Introduction

- **Forward contract:** a customized *binding* contract to buy/sell an asset at a specific price (**forward price**) on a *future* date.
- **American option:** gives the holder the *right* to exercise the option at any time before the date of expiration.
- **European option:** can only be exercised at the date of expiration.
- **call/put option:** gives the holder the *right* to buy/sell.
- **Derivative asset:** assets defined in terms of underlying financial asset.

## 2 The Binomial Model

### 2.1 The One Period Model

#### 2.1.1 Definitions

- **Bond** price process is deterministic and given by

$$\begin{aligned}B_0 &= 1, \\ B_1 &= 1 + R.\end{aligned}$$

where constant  $R$  is the spot rate for the period.

- **Stock** price process is stochastic and given by

$$S_0 = s,$$

$$S_1 = s \cdot Z$$

where  $Z$  is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u, \\ d, & \text{with probability } p_d. \end{cases}$$

with  $p_u + p_d = 1$  and the assumption that  $d < u$ .

- **Portfolio** on the  $(B, S)$  market is a vector

$$h = (x, y).$$

with a deterministic market value at  $t = 0$  and stochastic value at  $t = 1$ .

- **Value Process** of the portfolio  $h$  is defined as

$$V_t^h = xB_t + yS_t, \quad t = 0, 1,$$

or, in more detail,

$$V_0^h = x + ys,$$

$$V_1^h = x(1 + R) + yS_1.$$

- **Arbitrage** portfolio is an  $h$  with the properties

$$V_0^h = 0,$$

$$V_1^h > 0, \quad \text{with probability 1}$$

### 2.1.2 Contingent Claim Pricing

**Proposition 1.** The model  $h$  is free of arbitrage  $\iff d \leq (1 + R) \leq u$ .

**Comments** The above proposition implies that  $(1 + R)$  is a convex combination of  $u$  and  $d$ , i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where  $q_u$  and  $q_d$  can be interpreted as probabilities for a new probability measure  $Q$  with  $P(Z = u) = q_u$  and  $P(Z = d) = q_d$ . Denoting expectation w.r.t. this measure by  $E^Q$  and with the following calculation

$$\frac{1}{1 + R} E^Q[S_1] = \frac{1}{1 + R} [q_u su + q_d sd] = \frac{1}{1 + R} \cdot s(1 + R) = s,$$

we arrive at the following relation

$$s = \frac{1}{1 + R} E^Q[S_1].$$

**Definition 2.** A probability measure  $Q$  is **martingale** if the following condition holds:

$$S_0 = \frac{1}{1 + R} E^Q[S_1].$$

**Proposition 3.** Arbitrage-free model  $\iff \exists$  martingale measure  $Q$ .

**Proposition 4.** For the binomial model above, the martingale probabilities are

$$\begin{cases} q_u = \frac{(1 + R) - d}{u - d}, \\ q_d = \frac{u - (1 + R)}{u - d}. \end{cases}$$

*Proof.* Solve the system of equation

$$\begin{cases} 1 + R = q_u \cdot u + q_d \cdot d \\ q_u + q_d = 1. \end{cases}$$

□

**Definition 5.** A **contingent claim** (financial derivative) is any stochastic variable  $X$  of the form  $X = \Phi(Z)$ , where  $Z$  is the stochastic variable driving the stock price process above,  $\Phi$  is the **contract function**.

**Example 6.** European call option on the stock with strike price  $K$ . Assuming that  $sd < K < su$ , we have

$$X = \begin{cases} su - K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

so the contract function is given by

$$\begin{aligned} \Phi(u) &= su - K, \\ \Phi(d) &= 0. \end{aligned}$$

**Definition 7.** A given contingent claim  $X$  can be replicated / is reachable if

$$\exists h \text{ s.t. } V_1^h = X, \text{ with probability 1.}$$

We call such an  $h$  a **hedging/replicating** portfolio. If all claims can be replicated, then the market is complete.

**Proposition 8.** If a claim  $X$  is reachable with replicating portfolio  $h$ , then the only reasonable price process for  $X$  is

$$\Pi_t[X] = V_t^h, \quad t = 0, 1.$$

Here, “reasonable” means that  $\Pi_0[X] \neq V_0^h \Rightarrow$  arbitrage possibility.

**Proposition 9.** The general binomial model is free of arbitrage  $\Rightarrow$  it is complete.

*Proof.* Say a claim  $X$  has contract function  $\Phi$  s.t.

$$V_1^h = \begin{cases} \Phi(u), & \text{if } Z = u \\ \Phi(d), & \text{if } Z = d, \end{cases}$$

we can obtain the following system of equations by expanding  $V_1^h$

$$\begin{aligned} (1+R)x + xuy &= \Phi(u), \\ (1+R)x + xdy &= \Phi(d), \end{aligned}$$

and solve it to find out the replicating portfolio as

$$\begin{aligned} x &= \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \\ y &= \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \end{aligned}$$

□

**Proposition 10.** If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim  $X$  is

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X],$$

where the martingale measure  $Q$  is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

*Proof.* Using the results derived previously, we can find out that

$$\begin{aligned} \Pi_0[X] &= x + sy \\ &= \frac{1}{1+R} \left[ \frac{(1+R) - d}{u-d} \cdot \Phi(u) + \frac{u - (1+R)}{u-d} \cdot \Phi(d) \right] \\ &= \frac{1}{1+R} [\Phi(u)q_u + \Phi(d)q_d] \\ &= \frac{1}{1+R} E^Q[X]. \end{aligned}$$

□

## 2.2 The Multiperiod Model

### 2.2.1 Definitions

- Model

- The bond price dynamics is given by

$$B_{n+1} = (1+R)B_n, \quad B_0 = 1.$$

- The stock price dynamics is given by

$$S_{n+1} = S_n \cdot Z_n, \quad S_0 = s.$$

where  $Z_0, Z_1, \dots, Z_{T-1}$  are assumed to be i.i.d. stochastic variables, with only two variables  $u$  and  $d$  and probabilities

$$P(Z_n = u) = p_u, \quad P(Z_n = d) = p_d.$$

- Illustrating the stock dynamics by means of a tree, we can see that it is recombining.

- A **portfolio strategy** is a stochastic process

$$\{h_t = (x_t, y_t); t = 1, \dots, T\}$$

s.t.  $h_t$  is a function of  $S_0, S_1, \dots, S_{t-1}$ . By convention,  $h_0 = h_1$ .

- Interpretation of  $h_t$ : at  $t - 1$ ,  $x_t$  and  $y_t$  of bonds and stocks are bought and held until time  $t$ .

- The **value process** corresponding to the portfolio  $h$  is defined by

$$V_t^h = x_t(1 + R) + y_t S_t.$$

- A portfolio strategy  $h_t$  is said to be **self-financing** if  $\forall t = 0, \dots, T - 1$ ,

$$x_t(1 + R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

which says that the market value of the “old” portfolio  $h_t$  equals to the purchase value of the “new” portfolio  $h_{t+1}$ .

- An **arbitrage** possibility is a self-financing portfolio  $h$  with the properties

$$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

- The **martingale** probabilities  $q_u$  and  $q_d$  are defined as the probabilities for which the following relation holds

$$s = \frac{1}{1 + R} E^Q[S_{t+1} | S_t = s]$$

### 2.2.2 Binomial Model Pricing Algorithm

**Proposition 11.** The model is free of arbitrage  $\Rightarrow d \leq (1 + R) \leq u$ .

**Assumption 12.** From now, assume that  $d < u$  and  $d \leq (1 + R) \leq u$ .

**Definition 13.** A **contingent claim** is a stochastic variable  $X$  of the form

$$X = \Phi(S_T),$$

where the **contract function**  $\Phi$  is some given real valued function.

**Definition 14.** A given contingent claim  $X$  can be **replicated** / is **reachable** if

$$\exists \text{ self-financing } h \text{ s.t. } V_T^h = X, \text{ with probability 1.}$$

We call such an  $h$  a **hedging/replicating** portfolio. If all claims can be replicated, then the market is **(dynamically) complete**.

**(Binomial Algorithm) Proposition 15.** Consider a  $T$ -claim  $X = \Phi(S_T)$ , which could be replicated with a self-financing portfolio  $h$ . Let  $k$  be the number of up-moves occurred. So  $V_t(k)$  can be computed recursively as

$$\begin{cases} V_t(k) &= \frac{1}{1 + R} [q_u V_{t+1}(k + 1) + q_d V_{t+1}(k)], \\ V_T(k) &= \Phi(su^k d^{T-k}). \end{cases}$$

where the martingale probabilities  $q_u$  and  $q_d$  are

$$\begin{cases} q_u &= \frac{(1 + R) - d}{u - d} \\ q_d &= \frac{u - (1 + R)}{u - d}. \end{cases}$$

and the hedging portfolio is given by

$$\begin{cases} x_t(k) &= \frac{1}{1 + R} \cdot \frac{u V_t(k) - d V_t(k + 1)}{u - d}, \\ y_t(k) &= \frac{1}{S_{t-1}} \cdot \frac{V_t(k + 1) - V_t(k)}{u - d}. \end{cases}$$

**N.B.** In Fig 2.9 in notes, (the figure might be confusing)  $(-22.5, 5/8)$  is both  $h_0$  and  $h_1$ ,  $(-42.5, 95/120)$  is  $h_2(1)$ ,  $(-2.5, 1/8)$  is  $h_2(0)$ , etc.

**Proposition 16.** The arbitrage free price at  $t = 0$  of a  $T$ -claim  $X$  is given by

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot E^Q[X],$$

where  $Q$  denotes the martingale measure, or more explicitly,

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^T \binom{T}{k} q_u^k q_d^{T-k} \Phi(su^k d^{T-k}).$$

**Proposition 17.**  $d < (1+R) < u \iff$  free of arbitrage.

### 3 Brownian Motion

#### 3.1 Definitions

- Let  $A, B$  be two random variables, we define equal in distribution as

$$A \stackrel{d}{=} B \iff P(A \in C) = P(B \in C) \forall \text{ possible sets } C.$$

- We say a random variable  $X$  has stationary increments if

$$X_t - X_s \stackrel{d}{=} X_{t+h} - X_{s+h} \quad \forall h > 0.$$

- A stochastic process  $B_t$  is called a Brownian motion if

- $B_0 = 0$ ,
- $B_t - B_s \sim N(0, t-s) \forall t \geq s \geq 0$ , implying *stationary* increments,
- it has *independent* increments.

with

- $\mathbb{E}[B_t] = 0$ ,
- $\text{Cov}(t, s) = \min\{t, s\}$ .

#### 3.2 Path Properties

- Paths are continuous (no jumps).
- Nowhere differentiable.
- Paths are self-similar, i.e.

$$(T^H B_{t_1}, \dots, T^H B_{t_n}) \stackrel{d}{=} (B_{T t_1}, \dots, B_{T t_n})$$

with  $H = 0.5$  is the Hurst coefficient.