Computational Finance

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Contents

1 Introduction

- **Forward contract**: a customized *binding* contract to buy/sell an asset at a specific price (**forward price**) on a *future* date.
- **American option**: gives the holder the *right* to exercise the option at any time before the date of expiration.
- **European option**: can only be exercised at the date of expiration.
- **call/put option**: gives the holder the *right* to buy/sell.
- **Derivative asset**: assets defined in terms of underlying financial asset.

2 The Binomial Model

2.1 The One Period Model

2.1.1 Definitions

- **Bond** price process is deterministic and given by

$$B_0 = 1,$$
$$B_1 = 1 + R.$$

where constant R is the spot rate for the period.

- **Stock** price process is stochastic and given by

$$S_0 = s,$$
$$S_1 = s \cdot Z$$

where Z is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u, \\ d, & \text{with probability } p_d. \end{cases}$$

with $p_u + p_d = 1$ and the assumption that $d < u$.

- **Portfolio** on the (B, S) market is a vector

$$h = (x, y).$$

with a deterministic market value at $t = 0$ and stochastic value at $t = 1$.

- **Value Process** of the portfolio h is defined as

$$V_t^h = xB_t + yS_t, \quad t = 0, 1,$$

or, in more detail,

$$V_0^h = x + ys,$$
$$V_1^h = x(1 + R) + ysZ.$$

- **Arbitrage** portfolio is an h with the properties

$$V_0^h = 0,$$
$$V_1^h > 0, \quad \textit{with probability 1}$$

2.1.2 Contingent Claim Pricing

Proposition 1. The model h is free of arbitrage $\iff d \leq (1+R) \leq u$.

Comments The above proposition implies that $(1 + R)$ is a convex combination of u and d, i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where q_u and q_d can be interpreted as probabilities for a new probability measure Q with $P(Z = u) = q_u$ and $P(Z = d) = q_d$. Denoting expectation w.r.t. this measure by E^Q and with the following calculation

$$\frac{1}{1+R}E^Q[S_1] = \frac{1}{1+R}[q_u su + q_d sd] = \frac{1}{1+R} \cdot s(1 + R) = s,$$

we arrive at the following relation

$$s = \frac{1}{1+R}E^Q[S_1].$$

Definition 2. A probability measure Q is **martingale** if the following condition holds:

$$S_0 = \frac{1}{1+R}E^Q[S_1].$$

Proposition 3. Arbitrage-free model $\iff$ $\exists$ martingale measure Q.

Proposition 4. For the binomial model above, the martingale probabilities are

$$\begin{cases} q_u = \dfrac{(1+R)-d}{u-d}, \\ q_d = \dfrac{u-(1+R)}{u-d}. \end{cases}$$

Proof. Solve the system of equation

$$\begin{cases} 1+R = q_u \cdot u + q_d \cdot d \\ q_u + q_d = 1. \end{cases}$$

□

Definition 5. A **contingent claim** (financial derivative) is any stchastic variable X of the form $X = \Phi(Z)$, where Z is the stochastic variable driving the stock price process above, Φ is the **contract function**.

Example 6. European call option on the stock with strike price K. Assuming that $sd < K < su$, we have

$$X = \begin{cases} su - K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

so the contract function is given by

$$\Phi(u) = su - K,$$
$$\Phi(d) = 0.$$

Definition 7. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists h \text{ s.t. } V_1^h = X, \text{ with probability } 1.$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **complete**.

Proposition 8. If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is

$$\Pi_t[X] = V_t^h, \ t = 0, 1.$$

Here, "reasonable" means that $\Pi_0[X] \neq V_0^h \Rightarrow$ arbitrage possibility.

Proposition 9. The general binomial model is free of arbitrage $\Rightarrow$ it is complete.

Proof. Say a claim X has contract function Φ s.t.

$$V_1^h = \begin{cases} \Phi(u), & \text{if } Z = u \\ \Phi(d), & \text{if } Z = d, \end{cases}$$

we can obtain the following system of equations by expanding V_1^h

$$(1+R)x + xuy = \Phi(u),$$
$$(1+R)x + xdy = \Phi(d),$$

and solve it to find out the replicating portfolio as

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d},$$
$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}.$$

□

Proposition 10. If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is

$$\Pi_0[X] = \frac{1}{1+R} E^Q[X],$$

where the martingale measure Q is uniquely determined by the relation

$$S_0 = \frac{1}{1+R} E^Q[S_1].$$

Proof. Using the results derived previously, we can find out that

$$\begin{aligned}\Pi_0[X] &= x + sy \\ &= \frac{1}{1+R}\left[\frac{(1+R)-d}{u-d}\cdot\Phi(u) + \frac{u-(1+R)}{u-d}\cdot\Phi(d)\right] \\ &= \frac{1}{1+R}\left[\Phi(u)q_u + \Phi(d)q_d\right] \\ &= \frac{1}{1+R}E^Q[X].\end{aligned}$$

□

2.2 The Multiperiod Model

2.2.1 Definitions

- Model
  - The bond price dynamics is given by

    $$B_{n+1} = (1+R)B_n, \quad B_0 = 1.$$

  - The stock price dynamics is given by

    $$S_{n+1} = S_n \cdot Z_n, \quad S_0 = s.$$

    where $Z_0, Z_1, \ldots, Z_{T-1}$ are assumed to be i.i.d. stochastic variables, with only two variables u and d and probabilities

    $$P(Z_n = u) = p_u, \quad P(Z_n = d) = p_d.$$

  - Illustrating the stock dynamics by means of a tree, we can see that it is **recombining**.

- A **portfolio strategy** is a stochastic process

  $$\left\{h_t = (x_t, y_t);\ t = 1, \ldots, T\right\}$$

  s.t. h_t is a function of $S_0, S_1, \ldots, S_{t-1}$. By convention, $h_0 = h_1$.

  - Interpretation of h_t: at $t-1$, x_t and y_t of bonds and stocks are bought and held until time t.

- The **value process** corresponding to the portfolio h is defined by

  $$V_t^h = x_t(1+R) + y_tS_t.$$

- A portfolio strategy h_t is said to be **self-financing** if $\forall t = 0, \ldots, T-1$,

  $$x_t(1+R) + y_tS_t = x_{t+1} + y_{t+1}S_t$$

  which says that the market value of the "old" portfolio h_t equals to the purchase value of the "new" portfolio h_{t+1}.

- An **arbitrage** possibility is a self-financing portfolio h with the properties

  $$\begin{aligned}V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0.\end{aligned}$$

- The **martingale** probabilities q_u and q_d are defined as the probabilities for which the following relation holds

  $$s = \frac{1}{1+R}E^Q[S_{t+1}|S_t = s]$$

2.2.2 Binomial Model Pricing Algorithm

Proposition 11. The model is free of arbitrage $\Rightarrow d \leq (1+R) \leq u$.

Assumption 12. From now, assume that $d < u$ and $d \leq (1+R) \leq u$.

Definition 13. A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T),$$

where the **contract function** Φ is some given real valued function.

Definition 14. A given contingent claim X can be **replicated** / is **reachable** if

$$\exists \text{ self-financing } h \text{ s.t. } V_T^h = X, \text{ with probability } 1.$$

We call such an h a **hedging**/**replicating** portfolio. If all claims can be replicated, then the market is **(dynamically) complete**.

(Binomial Algorithm) Proposition 15. Consider a T-claim $X = \Phi(S_T)$, which could be replicated with a self-financing portfolio h. Let k be the number of up-moves occurred. So $V_t(k)$ can be computed recursively as

$$\begin{cases} V_t(k) &= \dfrac{1}{1+R}\left[q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\right], \\ V_T(k) &= \Phi(su^k d^{T-k}). \end{cases}$$

where the martingale probabilities q_u and q_d are

$$\begin{cases} q_u &= \dfrac{(1+R)-d}{u-d} \\ q_d &= \dfrac{u-(1+R)}{u-d}. \end{cases}$$

and the hedging portfolio is given by

$$\begin{cases} x_t(k) &= \dfrac{1}{1+R} \cdot \dfrac{uV_t(k) - dV_t(k+1)}{u-d}, \\ y_t(k) &= \dfrac{1}{S_{t-1}} \cdot \dfrac{V_t(k+1) - V_t(k)}{u-d}. \end{cases}$$

N.B. In Fig 2.9 in notes, (the figure might be confusing) $(-22.5, 5/8)$ is both h_0 and h_1, $(-42.5, 95/120)$ is $h_2(1)$, $(-2.5, 1/8)$ is $h_2(0)$, etc.

Proposition 16. The arbitrage free price at $t = 0$ of a T-claim X is given by

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot E^Q[X],$$

where Q denotes the martingale measure, or more explicitly,

$$\Pi_0[X] = \frac{1}{(1+R)^T} \cdot \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi(su^k d^{T-k}).$$

Proposition 17. $d < (1+R) < u \iff$ free of arbitrage.

3 Brownian Motion

3.1 Definitions

- Let A, B be two random variables, we define **equal in distribution** as
$$A \overset{d}{=} B \iff P(A \in C) = P(B \in C) \ \forall \text{ possible sets } C.$$
- We say a random variable X has **stationary increments** if
$$X_t - X_s \overset{d}{=} X_{t+h} - X_{s+h} \quad \forall h > 0.$$
- A stochastic process B_t is called a **Brownian motion** if
  - $B_0 = 0$,
  - $B_t - B_s \sim N(0, t-s)\ \forall t \geq s \geq 0$, implying *stationary* increments,
  - it has *independent* increments.

  with
  - $\mathbb{E}[B_t] = 0$,
  - $\text{Cov}(B_t, B_s) = \min\{t, s\}$. Show this using (Practice!)
    - $\text{Cov}(B_t, B_s) = \mathbb{E}[(B_t - \mathbb{E}[B_t])(B_s - \mathbb{E}[B_s])]$
    - $\text{Cov}(B_t, B_s) = \mathbb{E}[B_t B_s] - \mathbb{E}[B_t]\mathbb{E}[B_s]$.

3.2 Path Properties

- Paths are continuous (no jumps).
- Nowhere differentiable.
- Paths are **self-similar**, i.e.
$$(T^H B_{t_1}, \ldots, T^H B_{t_n}) \overset{d}{=} (B_{Tt_1}, \ldots, B_{Tt_n})$$
  with $H = 0.5$ is the Hurst coefficient.

3.3 Brownian Motion with Drift

- A **Gaussian Process** is a stochastic process s.t. every finite collection of those random variables has a multivariate normal distribution.
- The process is
$$X_t = \mu t + \sigma B_t, \quad t \geq 0.$$
- It is a Gaussian process with the following properties:
  - $\mathbb{E}[X_t] = \mu t$,
  - $\mathrm{Cov}(X_t, X_s) = \sigma^2 \min\{t, s\}$. (Practice!)

3.4 Geometric Brownian Motion with Drift

- The process is
$$X_t = e^{\mu t + \sigma B_t}, \quad t \geq 0.$$
- It is *not* Gaussian, but log-normal instead, with the following properties:
  - $\mathbb{E}[X_t] = e^{\mu t + \frac{1}{2}\sigma^2 t}$, (Practice!)
  - $\mathrm{Cov}(X_t, X_s) = e^{\left(\mu + \frac{1}{2}\sigma^2\right)(t+s)}\left(e^{\sigma^2 s} - 1\right)$. (Practice!)

4 Stochastic Integration

4.1 Time Value of Money

- £x today is worth more than £x in the future.
- Compensation for postponed consumption
- Inflation: prices may rise so £x in the future will not have the same purchasing power
- Risk – money in the future may never be received.

4.2 Interest

4.2.1 Definitions

- $V(t)$: value of investment at time t
- r: interest rate $r \geq 0$
- t: time measured in years
- P: principal/initial investment (£)
- simple/annual compounding:
$$\begin{cases} V(0) &= P \\ V(t) &= (1 + tr)P \end{cases}$$

4.2.2 Compounding

- Periodic, e.g. monthly compounding:
$$V(t) = \left(1 + \frac{r}{12}\right)^t P$$
- Continuous:
$$\begin{aligned} V(t) = e^{tr}P \quad &\implies \quad \frac{\mathrm{d}V(t)}{\mathrm{d}t} = rV(t) \\ &\implies \quad V(t) = V(0) + \int_0^t rV(s)\mathrm{d}s. \end{aligned}$$

4.2.3 Stochastic Integral

Example:
$$I(T) = \int_0^T B(t)\mathrm{d}B(t)$$

Using left-hand point approximation, we can get

$$\begin{aligned} I(T) &= \lim_{L\to\infty} \sum_{i=0}^{L-1} B(t_i)\left[B(t_{i+1}) - B(t_i)\right] \\ &= \lim_{L\to\infty} \sum_{i=0}^{L-1} -\frac{1}{2}B(t_i)^2 - \frac{1}{2}[B(t_{i+1}) - B(t_i)]^2 + \frac{1}{2}B(t_{i+1})^2 \end{aligned}$$

$$= \frac{1}{2} \lim_{L\to\infty} \sum_{i=0}^{L-1} [B(t_{i+1})^2 - B(t_i)^2] - \delta B_i^2$$

where $\delta B_i = B(t_{i+1}) - B(t_i)$. Analyzing the above expression (exercise!), we can obtain

$$I(T) = \frac{1}{2} B(T)^2 - \frac{1}{2} T.$$

We can also show the following: (exercise!)

- $\mathbb{E}[\delta B_i] = 0$
- $\mathrm{Var}(\delta B_i) = \delta t$
- $\mathbb{E}[\delta B_i^2] = \delta t$
- $\mathrm{Var}(\delta B_i^2) = 2(\delta t)^2$ (hint: $\mathbb{E}[Z^4] = 3$ for $Z \sim N(0,1)$)

5 Stochastic Calculus

5.1 Ito's Formula

5.1.1 Ito's Multiplication Rules

$$\begin{aligned} (\mathrm{d}t)^2 &= 0, \\ (\mathrm{d}t)(\mathrm{d}B(t)) &= 0, \\ (\mathrm{d}B(t))^2 &= \mathrm{d}t. \end{aligned}$$

5.1.2 Ito's Lemma

Consider an **Ito's process**

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

where

- μ_t (or just μ): drift process,
- σ_t (or just σ): diffusion process,

both can be either stochastic or deterministic. Using the multiplication rules, we can derive that

$$(\mathrm{d}X_t)^2 = \mu^2 (\mathrm{d}t)^2 + \sigma^2 (\mathrm{d}B_t)^2 + 2\mu\sigma(\mathrm{d}t)(\mathrm{d}B_t) = \sigma^2 \mathrm{d}t.$$

Let $Z(t) = f(t, X_t)$, then

$$\begin{aligned} \mathrm{d}Z(t) &= \frac{\partial f}{\partial X_t} \mathrm{d}X_t + \frac{\partial f}{\partial t} \mathrm{d}t + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (\mathrm{d}X_t)^2}_{\text{Itô correction}} \\ &= \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X_t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial X_t^2} \right) \mathrm{d}t + \frac{\partial f}{\partial X_t} \sigma \mathrm{d}B(t). \end{aligned}$$

5.1.3 Examples

1. $\mathrm{d}X_t = \mathrm{d}W_t$, $f(t,x) = te^{\alpha x}$. Since

$$\frac{\partial f}{\partial t} = e^{\alpha x}, \quad \frac{\partial f}{\partial x} = \alpha t e^{\alpha x}, \quad \frac{\partial^2 f}{\partial x^2} = \alpha^2 t e^{\alpha x},$$

and in this case $\mu = 0$, $\sigma = 1$, we have

$$\mathrm{d}f(t, W_t) = \left(e^{\alpha x} + + \frac{1}{2} \alpha^2 t e^{\alpha W_t} \right) \mathrm{d}t + \alpha t e^{\alpha W_t} \mathrm{d}W_t$$

2. $\mathrm{d}X = \mathrm{d}W$, $f(x) = x^2$.

$$\begin{aligned} \mathrm{d}f(t, W_t) &= \frac{\partial f}{\partial t} \mathrm{d}t + \frac{\partial f}{\partial x} \mathrm{d}x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \mathrm{d}x \\ &= 0 + 2W \mathrm{d}W + \frac{1}{2} 2\mathrm{d}t \\ &= \mathrm{d}t + 2W_t \mathrm{d}W_t \end{aligned}$$

3. Compute $\mathbb{E}\left[B_t^4\right]$. This could be transformed to the question

$$\mathrm{d}X_t = \mathrm{d}B_t, \quad Z_t = X_t^4.$$

Thus,

$$\mathrm{d}Z_t = 6B_t^2 \mathrm{d}t + 4B_t^3 \mathrm{d}B_t,$$

i.e.

$$Z_t = Z_0 + 6 \int_0^t B_s^2 \mathrm{d}s + 4 \int_0^t B_s^3 \mathrm{d}B_s$$

so

$$\begin{aligned}
\mathbb{E}[B_t^4] = \mathbb{E}Z_t &= 0 + 6\int_0^t \mathbb{E}[B_s^2]\mathrm{d}s + 4\int_0^t \mathbb{E}[B_s^3]\mathrm{d}B_s \\
&= 6\int_0^t s\mathrm{d}s + 0 \\
&= 3t^2
\end{aligned}$$