DE condensed notes

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Contents

Chapter 1

Introduction

1.1 Ordinary Differential Equation (ODE) and Initial Value Problems (IVP)

Definition 1. Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. An equation of the form

$$\dot{x} = f(t, x)$$

is called a **d-dimensional (first-order) ODE**. A differentiable function $\lambda : I \to \mathbb{R}^d$ on an interval $I \subset \mathbb{R}$ is called a solution to the above differential equation if $(t, \lambda(t)) \in D$ and

$$\dot{\lambda}(t) = f(t, \lambda(t)) \quad \forall t \in I.$$

Specially, an ODE is **autonomous** if the equations is of the form

$$\dot{x} = f(x),$$

where $f : D \to \mathbb{R}^d$ for some open set $D \subset \mathbb{R}^d$.

Proposition 2. Consider an open set $D \subset \mathbb{R}^d$ and a function $f : D \to \mathbb{R}^d$, and consider the autonomous ODE

$$\dot{x} = f(x),$$

then $\exists \lambda : \mathbb{R} \to \mathbb{R}^d, a \in \mathbb{R}^d$ s.t. $\forall t \in \mathbb{R}, \lambda(t) = a$ iff $f(a) = 0$.

Definition 3. Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. The combination of the ODE

$$\dot{x} = f(t, x)$$

with an initial condition of the form

$$x(t_0) = x_0$$

where $(t_0, x_0) \in D$, is called an **Initial Value Problem (IVP)**.

1.2 Examples

The ODE ideally satisfies the following three conditions:

(i) A solution **exists** for every IVP.

(ii) The solution to each IVP is **unique**.

(iii) The solution to each IVP **exists globally**, i.e. can be defined on $I = \mathbb{R}$.

Now we look at examples which not all of the aforementioned properties are satisfied.

Example 4. Consider the 1-dimensional IVP

$$\dot{x} = f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}, \quad x(0) = 0$$

which has a discontinuous right hand side. Since $x(0) = 0$, we have $\dot{x}(0) = -1$. Also, $\exists \tau > 0$ s.t. $\forall 0 < t < \tau$, $x(t) < 0$ and $\dot{x} = 1$. This contradicts with MVT, that

$$\exists 0 < \mu < \frac{\tau}{2} \text{ s.t. } \dot{x}(\mu) = \frac{x(\frac{\tau}{2}) - x(0)}{\frac{\tau}{2}}.$$

Example 5. Consider the 1-dimensional IVP

$$\dot{x} = \sqrt{|x|}, \quad x(0) = 0.$$

Since $f(0) = 0$, Proposition 2 implies that there exists a constant solution with value 0. In addition, for any $b \geq 0$, the function $\lambda_b : \mathbb{R} \to \mathbb{R}$,

$$\lambda_b(t) = \begin{cases} 0, & t \leq b \\ \frac{1}{4}(t-b)^2, & t > b \end{cases}$$

is a solution to this IVP.

Example 6. Consider the IVP

$$\dot{x} = tx^2, \quad x(t_0) = x_0,$$

where $x_0 \neq 0$. Using the procedure of **separation of variables**, i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t)h(x) \Rightarrow \frac{\mathrm{x}}{h(x)} = g(t)\mathrm{d}t,\ x(t_0) = x_0$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{h(y)} \mathrm{d}y = \int_{t_0}^{t} g(s)\mathrm{d}s$$

and solving the above equation w.r.t. x to give the solution, we get

$$\frac{\mathrm{d}x}{x^2} = t\mathrm{d}t \Rightarrow \frac{1}{x_0} - \frac{1}{x} = \frac{t^2}{2} - \frac{t_0^2}{2},\ x(t_0) = x_0$$

$$\Rightarrow x = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}.$$

We can then see that all solutions with $x_0 > 0$ are defined on the bounded subinterval $(-\sqrt{2}, \sqrt{2}) \subset \mathbb{R}$ when $t_0 = 0$ and $x_0 = 1$.

1.3 Visualization

1.3.1 Solution Portrait

We consider a function $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and the corresponding ODE

$$\dot{x} = f(t, x),$$

a solution to this equation is a differentiable function $\lambda : I \to \mathbb{R}^d$ fulfilling $\dot{\lambda}(t) = f(t, \lambda(t))$ on the interval I. Then the graph of this solution, given by the so-called **solution curve**

$$G(\lambda) = \big\{(t, \lambda(t)) : t \in I\big\} \subset \mathbb{R} \times \mathbb{R}^d,$$

is a differentiable curve. The derivative of this curve in the point $t_0 \in I$ is given by $\frac{\mathrm{d}}{\mathrm{d}t}(t, \lambda(t))|_{t=t_0} = (1, \dot{\lambda}(t_0) = (1, f(t_0, \lambda(t_0)))$. This implies that in particular that the vector field

$$(t, x) \mapsto (1, f(t, x)),$$

defined on D is crucial for the shape of the solution curves.

A **solution portrait** is given by a visualization of several solution curves, in the (t, x)-space, the so-called **extended phase space**. The x-space is normally called the **phase space**.

1.3.2 Phase portrait

Proposition 7. Let $\lambda : I \to \mathbb{R}^d$ be a solution to the autonomous differential equation

$$\dot{x} = f(x).$$

Then $\forall \tau \in \mathbb{R}$, the function $\mu : \tilde{I} \to \mathbb{R}^d$, where $\tilde{I} := \{t \in \mathbb{R} : t + \tau \in I\}$ and

$$\mu(t) := \lambda(t + \tau) \quad \forall t \in \tilde{I},$$

is also a solution to this differential equation.

Proof. Since λ is a solution, we have $\dot{\lambda}(t) = f(\lambda(t))$ for $t \in I$. The chain rule implies that $\forall t \in \tilde{I}$, $\dot{\mu}(t) = \dot{\lambda}(t + \tau)$, and we get that $\forall t \in \tilde{I}$,

$$\dot{\mu}(t) = \dot{\lambda}(t + \tau) = f(\lambda(t + \tau)) = f(\mu(t))$$

$\square$

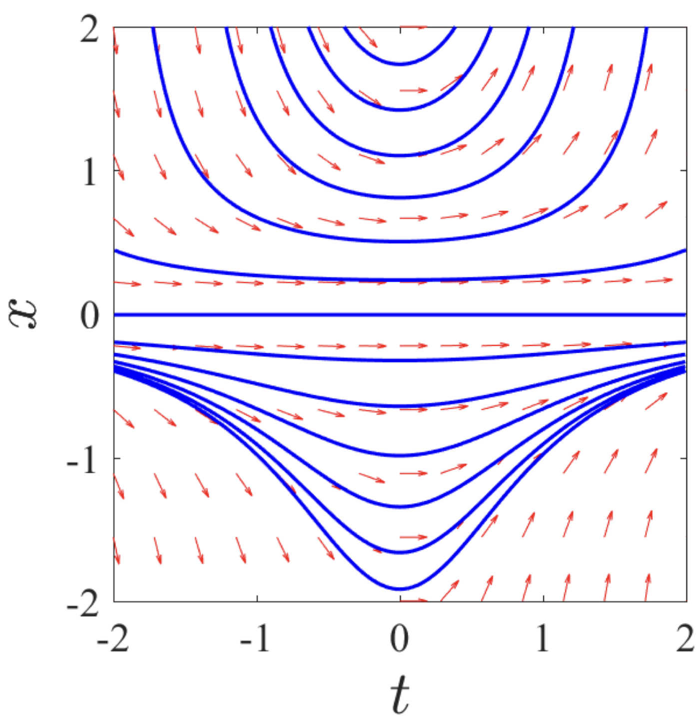

Figure 1.1: phase portrait for $\dot{x} = tx^2$

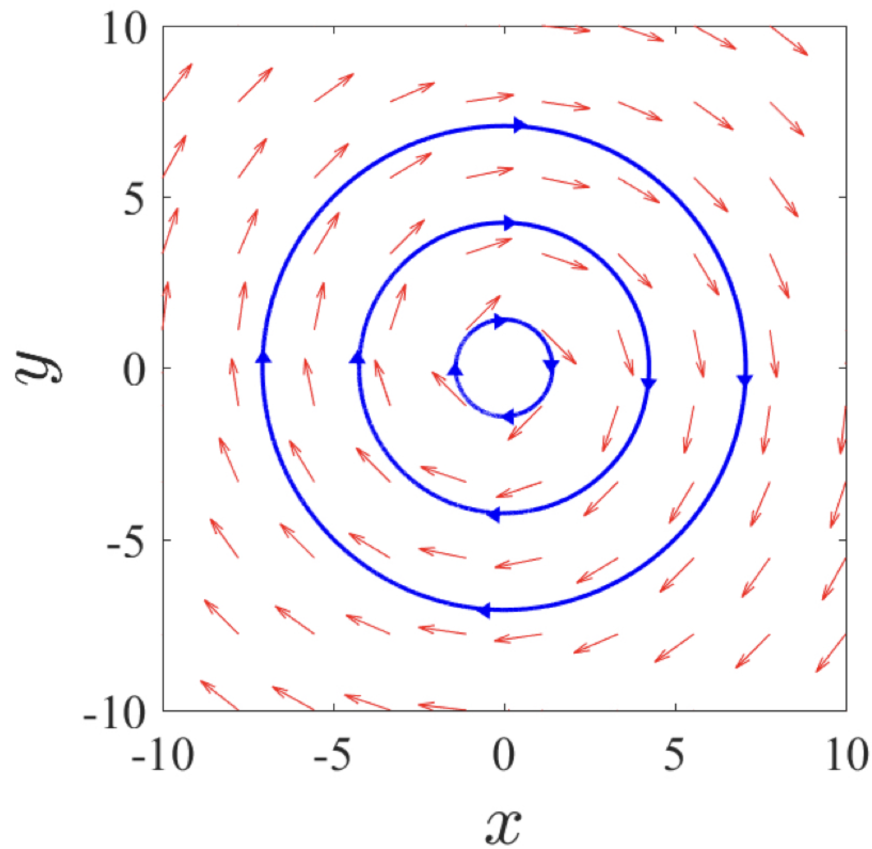

Figure 1.2: phase portrait for $\ddot{x} = -x$

Chapter 2

Existence and Uniqueness

2.1 Picard iterates

Proposition 8. Consider the IVP

$$\dot{x} = f(t,x), \quad x(t_0) = x_0,$$

where $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $(t_0, x_0) \in D$, and let $\lambda : I \to \mathbb{R}^d$ be a function on an interval I s.t. $t_0 \in I$ and $\big\{(t, \lambda(t)) : t \in I\big\} \subset D$. Then the following two statements are equivalent.

(i) λ solves the IVP, i.e.

$$\dot{\lambda}(t) = f(t, \lambda(t)) \quad \forall t \in I, \text{ and } \lambda(t_0) = x_0.$$

(ii) λ is continuous, and we have

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\mathrm{d}s \quad \forall t \in I.$$

Definition 9. Consider the IVP, and choose an interval J that contains t_0. We define an initial function

$$\lambda_0(t) \equiv x_0 \quad \forall t \in J,$$

and inductively, the Picard iterates

$$\lambda_{n+1}(t) := x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\mathrm{d}s \quad \forall t \in J \text{ and } n \in \mathbb{N}_0.$$

Example 10. We would like to compute the Picard iterates for the IVP

$$\dot{x} = ax, \quad x(t_0) = x_0,$$

where $a \in \mathbb{R}$ is fixed. The first three iterates are

$$\begin{aligned}
\lambda_0(t) &= x_0, \\
\lambda_1(t) &= x_0 + \int_{t_0}^{t} a x_0 \mathrm{d}s = x_0(1 + a(t - t_0)), \\
\lambda_2(t) &= x_0 + \int_{t_0}^{t} a x_0 (1 + a(s - t_0)) \mathrm{d}s = x_0\left(1 + a(t - t_0) + \frac{1}{2}a^2(t - t_0)^2\right). \\
&\vdots
\end{aligned}$$

2.2 Lipschitz continuity

Definition 11. A **norm** on a vector space V over the reals is a map $\|\cdot\| : V \to \mathbb{R}_0^+$ s.t.

(i) $\|x\| = 0 \iff x = 0$. (positive definiteness)

(ii) $\|ax\| = |a|\|x\|$ $\forall a \in \mathbb{R}$ and $x \in V$. (absolute homogeneity)

(iii) $\|x + y\| \leq \|x\| + \|y\|$ $\forall x, y \in V$. (triangle inequality)

A vector space with norm is called a **normed vector space**.

Definition 12. Let X be a subset of a normed vector space $(V, \|\cdot\|_V)$ and Y be a subset of a normed vector space $(W, \|\cdot\|_W)$. Then a function $f : X \to Y$ is called

(i) **continuous** if $\forall x \in X, \epsilon > 0$, $\exists \delta > 0$ s.t.

$$\|x - \overline{x}\|_V < \delta \implies \left\|f(x) - f(\overline{x})\right\|_W < \epsilon.$$

(ii) **Lipschitz continuous** if $\exists K > 0$ s.t.

$$\left\|f(x) - f(\overline{x})\right\|_W \leq K\|x - \overline{x}\|_V \quad \forall x, \overline{x} \in X.$$

The constant K is called a **Lipschitz constant**.

2.2.1 Lipschitz continuity and the mean value theorem (MVT)

Recall that the mean value theorem says that for any $x, y \in I$, $\exists \xi$ between x and y s.t.

$$f(x) - f(y) = f'(\xi)(x - y).$$

This implies that

$$\left|f(x) - f(y)\right| = |f'(\xi)||x - y|,$$

and it is clear that if the derivative f' is bounded on the interval I, then f is Lipschitz continuous.

2.2.2 Lipschitz continuity and the mean value inequality

Definition 13. For a given matrix $A \in \mathbb{R}^{m \times n}$, the **operator norm** of A is defined by

$$\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|},$$

which can be calculated by finding the square root of the largest eigenvalue of $A^T A$.

(Mean Value Inequality) Theorem 14. Consider an open set $D \subset \mathbb{R}^n$, and let $f : D \to \mathbb{R}^m$ be continuously differentiable. Then $\forall x, y \in D$ with $[x, y] \subset D$, $\exists \xi \in [x, y]$ s.t.

$$\left\| f(x) - f(y) \right\| \leq \left\| f'(\xi) \right\| \|x - y\|,$$

where for any $x, y \in \mathbb{R}^n$, $[x, y] := \{\alpha x + (1 - \alpha) y \in \mathbb{R}^n : \alpha \in [0, 1]\}$.

Lemma 15. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}^m$ be a continuous function. Then

$$\left\| \int_{t_0}^{t} f(s) \mathrm{d}s \right\| \leq \left| \int_{t_0}^{t} \left\| f(s) \right\| \mathrm{d}s \right| \quad \forall t, t_0 \in I.$$

Corollary 16. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be continuously differentiable. Then given a compact and convex set $C \subset U$, the restricted function $f|_C : C \to \mathbb{R}^m$ is Lipschitz continuous.

2.3 Picard-Lindelöf theorem

(Picard-Lindelöf theorem, global version) Theorem 17. Consider an ordinary differential equation

$$\dot{x} = f(t, x)$$

s.t. the function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies a global Lipschitz condition of the form

$$\left\| f(t, x) - f(t, y) \right\| \leq K \|x - y\| \quad \forall t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^d,$$

where $K > 0$ is a constant. Define $h := \frac{1}{2K}$. Then every IVP with $x(t_0) = x_0$ admits a unique solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$.

Definition 18. Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$.

(i) f is said to be **globally Lipschitz continuous** (w.r.t. x) if $\exists K > 0$ s.t.

$$\big\|f(t,x) - f(t,y)\big\| \le K\|x-y\| \quad \forall (t,x),(t,y) \in D.$$

(ii) f is said to be **locally Lipschitz continuous** (w.r.t. x) if $\forall (t_0, x_0) \in D$, there exists a neighbourhood $U \subset D$ of (t_0, x_0) and a constant $K > 0$ s.t.

$$\big\|f(t,x) - f(t,y)\big\| \le K\|x-y\| \quad \forall (t,x),(t,y) \in U.$$

(Picard-Lindelöf theorem, local version) Theorem 19. Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous w.r.t. x. Consider for a fixed $(t_0, x_0) \in D$ the IVP

$$\dot{x} = f(t,x), \quad x(t_0) = x_0.$$

Then the following two statements hold:

(i) Qualitative version. The IVP has locally a uniquely determined solution, i.e. $\exists h = h(t_0, x_0) > 0$ s.t. the above IVP has exactly one solution on $[t_0 - h, t_0 + h]$.

(ii) Quantitative version. Consider for some $\tau, \delta > 0$, the set

$$W^{\tau,\delta}(t_0, x_0) := [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)},$$

where $\overline{B_\delta(x_0)} := \big\{x \in \mathbb{R}^d : \|x - x_0\| \le \delta\big\}$, is the closed δ-neighbourhood of x_0. We assume that $W^{\tau,\delta}(t_0, x_0) \subset D$, and we suppose that $\exists K, M > 0$ s.t.

$$\big\|f(t,x) - f(t,y)\big\| \le K\|x-y\| \quad \forall (t,x),(t,y) \in W^{\tau,\delta}(t_0, x_0)$$

and

$$\big\|f(t,x)\big\| \le M \quad \forall (t,x) \in W^{\tau,\delta}(t_0, x_0).$$

Then the IVP has exactly one solution on $[t_0 - h, t_0 + h]$, where $h = h(t_0, x_0) := \min\{\tau, \frac{1}{2K}, \frac{\delta}{M}\}$.

Difference between global and local version of Picard-Lindelöf theorem

- Condition.

  global Lipschitz continuous v.s. continuously differentiable

- Value of h that affects the interval $[t_0 - h, t_0 + h]$.

  $\frac{1}{2K}$ v.s. $\min\Big\{\tau, \frac{1}{2K}, \frac{\delta}{M}\Big\}$.

Proposition 20. Consider an open set $D \subset \mathbb{R} \times \mathbb{R}^d$ and a continuously differentiable function $f : D \to \mathbb{R}^d$. Then f is locally Lipschitz continuous w.r.t. x, and thus, every IVP involving a differential equation with right hand side f can be solved locally uniquely.

Lemma 21. Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous w.r.t. x. Consider two solutions of

$$\dot{x} = f(t, x),$$

given by $\lambda : I \to \mathbb{R}^d$ and $\mu : J \to \mathbb{R}^d$, where I and J are intervals. Then either

$$\lambda(t) = \mu(t) \quad \forall t \in I \cap J$$

or

$$\lambda(t) \neq \mu(t) \quad \forall t \in I \cap J.$$

2.4 Maximal solutions

Definition 22. Consider the IVP and define

$$I_+(t_0, x_0) := \sup\left\{t_+ \geq t_0 : \exists \text{ a solution to IVP on } [t_0, t_+]\right\},$$
$$I_-(t_0, x_0) := \inf\left\{t_- \leq t_0 : \exists \text{ a solution to IVP on } [t_-, t_0]\right\},$$

and the interval

$$I_{\max}(t_0, x_0) := (I_-(t_0, x_0), I_+(t_0, x_0))$$

is called the **maximal existence interval** for the IVP.

Theorem 23. There exists a **maximal solution** $\lambda_{\max} : I_{\max}(t_0, x_0) \to \mathbb{R}^d$ to the IVP, i.e. any other solution to this IVP is defined on an interval that is a subset of $I_{\max}(t_0, x_0)$. The maximal solution has the following two properties:

(i) If $I_+(t_0, x_0)$ is finite, then either the maximal solution is unbounded for $t \geq t_0$, i.e.

$$\sup_{t \in (t_0, I_+(t_0, x_0))} \left\| \lambda_{\max}(t) \right\| = \infty,$$

or the boundary ∂D of D is nonempty, and we have

$$\lim_{t \nearrow I_+(t_0, x_0)} \operatorname{dist}\big((t, \lambda_{\max}(t)), \partial D\big) = 0.$$

In other words, the above equation implies that $\exists \lambda_{\max}$ and $\lambda_{\max}(t)$ is open.

(ii) If $I_-(t_0, x_0)$ is finite, then either the maximal solution is unbounded for $t \leq t_0$, i.e.

$$\sup_{t \in (t_0, I_-(t_0, x_0))} \left\| \lambda_{\max}(t) \right\| = \infty,$$

or the boundary ∂D of D is nonempty, and we have

$$\lim_{t \searrow I_-(t_0, x_0)} \operatorname{dist}\big((t, \lambda_{\max}(t)), \partial D\big) = 0.$$

Here, for a given set $A \subset \mathbb{R}^n$, the function $\operatorname{dist}(\cdot, A) : \mathbb{R}^n \to \mathbb{R}_0^+$ is defined by

$$\operatorname{dist}(y, A) := \inf \big\{ \|y - a\| : a \in A \big\} \quad \forall y \in \mathbb{R}^n.$$

2.5 General solutions and flows

2.5.1 General solutions

Definition 24. Consider the nonautonomous differential equation, and define

$$\Omega := \Big\{ (t, t_0, x_0) \in \mathbb{R}^{1+1+d} : (t_0, x_0) \in D \text{ and } t \in I_{\max}(t_0, x_0) \Big\}.$$

Then the function $\lambda : \Omega \to \mathbb{R}^d$, defined by

$$\lambda(t, t_0, x_0) := \lambda_{\max}(t, t_0, x_0)$$

is called the **general solution** of the nonautonomous differential equation.

Proposition 25. Consider the nonautonomous differential equation, and let $(t_0, x_0) \in D$. Then $\forall x \in I_{\max}(t_0, x_0)$, we have

$$\begin{aligned} I_{\max}(s, \lambda(s, t_0, x_0)) &= I_{\max}(t_0, x_0), \\ \lambda(t_0, t_0, x_0) &= x_0, \\ \lambda(t, s, \lambda(s, t_0, x_0)) &= \lambda(t, t_0, x_0) \quad \forall t \in I_{\max}(t_0, x_0). \end{aligned}$$

The second one is called *initial value property*, while the third one is called the *cocycle property*.

2.5.2 Flows

Definition 26. Consider the autonomous differential equation, and define for any initial value $x_0 \in D$,

$$J_{\max}(x_0) := I_{\max}(0, x_0)$$

and

$$\varphi(t, x_0) = \lambda(t, 0, x_0) \quad \forall t \in J_{\max}(x_0).$$

The function $(t, x_0) \mapsto \varphi(t, x_0)$ is called the **flow** of the autonomous differential equation.

Proposition 27. Let φ be the flow of the autonomous differential equation. Then for any $x \in D$, the following statements hold.

$$\begin{aligned} J_{\max}(\varphi(t, x)) &= J_{\max}(x) - t & \forall t \in J_{\max}(x), \\ \varphi(0, x) &= x, \\ \varphi(t, \varphi(s, x)) &= \varphi(t + s, x) & \forall t, s \text{ with } s, t + s \in J_{\max}(x), \\ \varphi(-t, \varphi(t, x)) &= x & \forall t \in J_{\max}(x). \end{aligned}$$

The second one is called initial value condition, while the third one is called group property.

Interpretation of the first property: Given $\varphi(0, x) = x$, $\varphi(t, x)$ means to look at the value of x at time t, and then with the same x move *back* from time t to 0. This, in a way, "pushes back" the solution curve, which is what is written on the RHS.

Definition 28. Let φ be the flow of the autonomous differential equation. $\forall x \in D$, we call the set

$$O(x) := \left\{\varphi(t,x) \in D : t \in J_{\max}(x)\right\}$$

the **orbit** (or **trajectory**) through x. In addition, we call

$$O^+(x) := \left\{\varphi(t,x) \in D : t \in J_{\max}(x) \cap \mathbb{R}_0^+\right\}$$

the **positive half-orbit** through x, and

$$O^-(x) := \left\{\varphi(t,x) \in D : t \in J_{\max}(x) \cap \mathbb{R}_0^-\right\}$$

the **negative half-orbit** through x.

The geometric picture is that the domain D of the RHS f is partitioned into orbits of φ. There are essentially three different types of orbits $O(x)$ for $x \in D$.

(i) $O(x)$ is a singleton. This implies $f(x) = 0$, and $J_{\max}(x) = \mathbb{R}$. The point x is called **equilibrium**.

(ii) $O(x)$ is a closed curve, i.e. $\exists t > 0$ s.t. $\varphi(t,x) = x$, but $f(x) \neq 0$. This implies that $J_{\max} = \mathbb{R}$. The point x is called **periodic**, and $O(x)$ is called **periodic orbit**.

(iii) $O(x)$ is not a closed curve, i.e. the function $t \mapsto \varphi(t,x)$ is injective on $J_{\max}(x)$.

Proposition 29. Consider the autonomous differential equation, where $d = 1$. Then all solutions are monotone, and there do not exist periodic orbits. This means that a trajectory is either an equilibrium or a non-closed curve.

Chapter 3

Linear Systems

3.1 Matrix exponential function

We consider the linear differential equation

$$\dot{x} = Ax, \tag{3.1}$$

where $A \in \mathbb{R}^{d\times d}$. Since $\|Ax - Ay\| = \big\|A(x-y)\big\| \le \|A\|\|x-y\|$, this system is globally Lipschitz continuous with Lipschitz constant $\|A\|$, and due to the global version of the Picard-Lindelöf theorem, solutions to every IVP exist on $\mathbb{R}$ and are unique, and this generates a globally defined flow $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.

Lemma 30. For two matrices $B, C \in \mathbb{R}^{d\times d}$, we have

$$\|BC\| \le \|B\|\|C\|\,.$$

Proposition 31. Consider a matrix $B \in \mathbb{R}^{d\times d}$. Then its matrix exponential

$$e^B := \sum_{k=0}^{\infty} \frac{1}{k!} B^k$$

exists and is a matrix in $\mathbb{R}^{d\times d}$.

Theorem 32. Consider the autonomous linear differential equation (3.1) with coefficient matrix $A \in \mathbb{R}^{d\times d}$. Then the flow $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ generated by this differential equation is given by

$$\varphi(t, x) = e^{At} x \quad \forall t \in \mathbb{R}.$$

Proposition 33. Consider matrices $B, C, T \in \mathbb{R}^{d\times d}$ s.t. T is invertible. Then the following statements hold.

(i) If $C = T^{-1}BT$, then $e^C = T^{-1}e^B T$.

(ii) $e^{-B} = (e^B)^{-1}$.

(iii) If $BC = CB$, then $e^{B+C} = e^B e^C$.

(iv) If B is a block diagonal matrix $B = \operatorname{diag}(B_1, \dots, B_p)$ with matrices $B_1, \dots, B_p$, then $e^B = \operatorname{diag}(e^{B_1}, \dots, e^{B_p})$.

3.2 Planar linear systems

See the previous notes for the solution in different cases.

It is worth noting that in short, say eigenvalue $\lambda = a + bi$ the rate of exponential growth is determined by a, while the rate of rotation is determined by b.

3.3 Jordan normal form

Theorem 34. Consider a matrix $A \in \mathbb{R}^{d\times d}$. Then $\exists T \in \mathbb{C}^{d\times d}$ so that under a basis transformation with the matrix T, we obtain the **complex Jordan normal** form

$$J := T^{-1}AT = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_p \end{pmatrix},$$

with the so-called **Jordan blocks**

$$J_j = \begin{pmatrix} \rho_j & 1 & & 0 & 0 \\ 0 & \rho_j & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \rho_j & 1 \\ 0 & 0 & \cdots & 0 & \rho_j \end{pmatrix} \quad \forall j \in \{1, \dots, p\}, \tag{3.2}$$

where the $\rho_j, j \in \{1, \dots, p\}$, are complex eigenvalues of the matrix A (some of which may be the same).

Note that if J_j is a 1×1 matrix, then $J_j = (\rho_j)$, and if J_j is a 2×2 matrix, then

$$J_j = \begin{pmatrix} \rho_j & 1 \\ 0 & \rho_j \end{pmatrix}.$$

Theorem 35. Consider a matrix $A \in \mathbb{R}^{d\times d}$. Then $\exists T \in \mathbb{R}^{d\times d}$ so that under a basis transformation with the matrix T, we obtain the **real Jordan normal** form

$$J := T^{-1}AT = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_p \end{pmatrix},$$

where the Jordan blocks J_j are either as in Theorem 34, i.e. given by (3.2), in case the eigenvalue ρ_j is real, or, in case the eigenvalue ρ_j is complex,

$$J_j = \begin{pmatrix} C_j & \mathrm{Id}_2 & & 0 & 0 \\ 0 & C_j & \mathrm{Id}_2 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & C_j & \mathrm{Id}_2 \\ 0 & 0 & \cdots & 0 & C_j \end{pmatrix}, \tag{3.3}$$

where $C_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$ with $\rho_j = a_j + ib_j$, and $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that if J_j is a 2×2 matrix, then

$$J_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}.$$

3.4 Explicit representation of the matrix exponential function

Proposition 36. Consider the matrix $A \in \mathbb{R}^{d\times d}$, and let J_j for $j \in \{1,\dots,p\}$ be the Jordan blocks for the real Jordan normal form with eigenvalues ρ_j. The matrix exponentials $e^{J_j t}$ for $t \in \mathbb{R}$ are then given as follows.

(i) If ρ_j is real, i.e. $J_j \in \mathbb{R}^{d_j\times d_j}$ is of the form 3.2, we obtain

$$e^{\begin{pmatrix} \rho_j & 1 & & 0 & 0 \\ 0 & \rho_j & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \rho_j & 1 \\ 0 & 0 & \cdots & 0 & \rho_j \end{pmatrix} t} = e^{\rho_j t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{d_j-1}}{(d_j-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ 0 & & & 1 & t \\ 0 & 0 & & 0 & 1 \end{pmatrix}.$$

(ii) If $\rho_j = a_j + ib_j \in \mathbb{C}$ is not real, i.e. $J_j \in \mathbb{R}^{2d_j\times 2d_j}$ is of the form 3.3, we obtain

$$e^{\begin{pmatrix} C_j & \mathrm{Id}_2 & & 0 & 0 \\ 0 & C_j & \mathrm{Id}_2 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & C_j & \mathrm{Id}_2 \\ 0 & 0 & \cdots & 0 & C_j \end{pmatrix} t} = e^{a_j t} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \cdots & \frac{t^{d_j-1}}{(d_j-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & & 0 & G(t) \end{pmatrix},$$

where $G(t) = \begin{pmatrix} \cos(b_j t) & \sin(b_j t) \\ -\sin(b_j t) & \cos(b_j t) \end{pmatrix} \ \forall t \in \mathbb{R}$.

3.5 Exponential growth behaviour

Proposition 37. Consider a matrix $A \in \mathbb{R}^{d\times d}$, and choose $\gamma \in \mathbb{R}$ s.t.

$$\gamma > \max\left\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\right\}.$$

If all eigenvalues ρ with $\operatorname{Re}\rho = \max\left\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\right\}$ are *semi-simple*, i.e. algebraic multiplicity = geometric multiplicity, we can use a smaller γ given by

$$\gamma := \max\left\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\right\}.$$

Then $\exists K > 0$ s.t.

$$\left\| e^{At} \right\| \leq K e^{\gamma t} \quad \forall t \geq 0.$$

3.6 Variation of constants formula

We are now interested in the general solution to the corresponding inhomogeneous equation

$$\dot{x} = Ax + g(t), \tag{3.4}$$

where $g : I \to \mathbb{R}^d$ is a continuous function on an interval $I \subset \mathbb{R}$.

Proposition 38. The general solution to (3.4) is given by

$$\lambda(t, t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)\mathrm{d}s \quad \forall t, t_0 \in I \text{ and } x_0 \in \mathbb{R}^d.$$

Chapter 4

Nonlinear systems

4.1 Stability

4.1.1 Basic definition

We introduce different notions of stability for an autonomous differential equation

$$\dot{x} = f(x), \tag{4.1}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. We denote the flow of this differential equation by φ.

Definition 39. Let x^* be an equilibrium of (4.1), i.e. $f(x^*) = 0$.

(i) x^* is called **stable** if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x \in B_\delta(x^*), t \geq 0$,

$$\left\|\varphi(t,x) - x^*\right\| < \epsilon.$$

(ii) x^* is called **unstable** if x^* is not stable.

(iii) x^* is called **attractive** if $\exists \delta > 0$ s.t. $\forall x \in B_\delta(x^*)$,

$$\lim_{t\to\infty} \varphi(t,x) = x^*.$$

(iv) x^* is called **asymptotically stable** if x^* is both stable and attractive.

(v) x^* is called **exponentially stable** if $\exists \delta > 0, K \geq 1, \gamma < 0$ s.t. $\forall x \in B_\delta(x^*), t \geq 0$,

$$\left\|\varphi(t,x) - x^*\right\| \leq K e^{\gamma t} \|x - x^*\|,$$

which could be thought of as "converges exponentially fast".

(vi) x^* is called **repulsive** if $\exists \delta > 0$ s.t. $\forall x \in B_\delta(x^*)$,

$$\lim_{t\to -\infty} \varphi(t,x) = x^*$$

Example 40. We study the trivial equilibrium $x^* = 0$ of the linear differential equation

$$\dot{x} = \alpha x,$$

where $\alpha \in \mathbb{R}$. This differential equation has the flow $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi(t, x) = xe^{\alpha t}$. Depending on the parameter α, the equilibrium x^* has different stability properties.

(i) x^* is stable for $\alpha \leq 0$. Choose $\delta := \epsilon$ for a given $\epsilon > 0$. Then $\forall t \geq 0, x \in (-\delta, \delta)$, we have $|\varphi(t, x) - x^*| = |x|e^{\alpha t} \leq |x| < \delta = \epsilon$.

(ii) x^* is unstable for $\alpha > 0$. We fix $\epsilon = 1$ and choose $\delta > 0$ arbitrarily. Then $|\varphi(t, \frac{\delta}{2})| = \frac{\delta}{2}e^{\alpha t} > \epsilon$ for some $t > 0$, since $e^{\alpha t} \to \infty$ as $t \to \infty$.

(iii) x^* is exponentially stable for $\alpha < 0$, since $|\varphi(t, x)| = |x|e^{\alpha t} = Ke^{\gamma t}|x|$ for $t \geq 0$ and $x \in B_\delta(0)$ with $\gamma := \alpha < 0$ and $K := \delta = 1$.

(iv) x^* is repulsive for $\alpha > 0$. Choose $\delta := 1$. Then $\forall x \in (-\delta, \delta)$, we have $|\varphi(t, x)| = |xe^{\alpha t}| = |x||e^{\alpha t}| < \delta|e^{\alpha t}| \to 0$ as $t \to -\infty$.

(Relation between stability and attractivity) Example 41. In general, the notions of stability and attractivity are not related.

(i) stable $\not\Rightarrow$ attractive. e.g.

$$\dot{x} = 0.$$

Every point is an equilibrium, and all equilibria are stable (choose $\delta := \epsilon$ for a given $\epsilon > 0$). It is clear that no equilibrium is attractive in this example.

(ii) attractive $\not\Rightarrow$ stable. e.g.

$$\dot{x} = x + xy - (x + y)\sqrt{x^2 + y^2},$$
$$\dot{y} = y - x^2 + (x - y)\sqrt{x^2 + y^2},$$

with phase portrait shown in 4.1. We note that this system has exactly two equilibria: (0, 0), which is an unstable knot with many tangents, and (1, 0), which admits a so-called homoclinic orbit, given by the unit circle. It can be proved that all orbits starting outside of the equilibrium (0, 0) converge to the other equilibrium (1, 0) in forward time: $\lim_{t\to\infty} \varphi(t, (x, y)) = (1, 0)\, \forall (x, y) \neq (0, 0)$. This can be shown easily using the Poincaré-Bendixson theory. It is also clear that it is not stable, since orbits starting in (x, y) on the unit circle very close to (1, 0), but with positive y take very long to complete the journey round the circle to come close to (1, 0) from below, and for $y \to 0$, this time converges to ∞. Hence $(-1, 0) \in \varphi(t, B_\delta(1, 0))\, \forall \delta > 0$ and t sufficiently large. This means that (1, 0) is unstable.

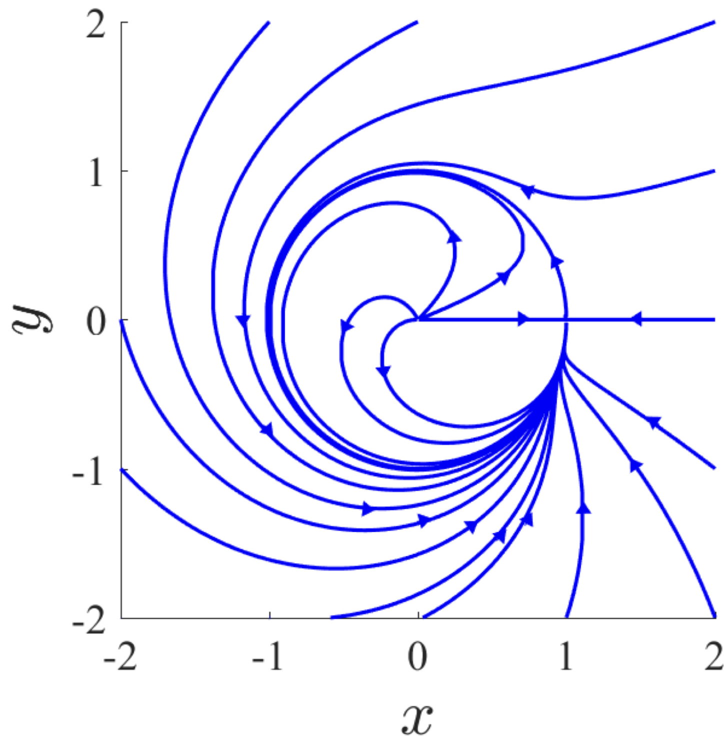

Figure 4.1: The attractive equilibrium (1,0) is not stable

Definition 42. Consider the differential equation (4.1) with associated flow φ.

(i) An orbit $O(x)$ for some $x \in D$ is called a **homoclinic orbit** if there exists an equilibrium $x^* \in D \backslash \{x\}$ s.t.

$$\lim_{t\to\infty} \varphi(t,x) = x^*, \quad \lim_{t\to-\infty} \varphi(t,x) = x^*.$$

(ii) An orbit $O(x)$ for some $x \in D$ is called a **heteroclinic orbit** if there exists two different equilibria $x_1^* \neq x_2^* \in D$ s.t.

$$\lim_{t\to\infty} \varphi(t,x) = x_1^*, \quad \lim_{t\to-\infty} \varphi(t,x) = x_2^*.$$

4.1.2 Stability of linear systems

Theorem 43. Consider the autonomous linear system

$$\dot{x} = Ax,$$

where $A \in \mathbb{R}^{d\times d}$. Then the trivial equilibrium $x^* = 0$ of this system is

(i) stable iif the following two statements hold:

  (a) the real part of all eigenvalues of A is non-positive, i.e. we have $\operatorname{Re}\rho \le 0$ for all eigenvalues ρ of A, and

  (b) the eigenvalue ρ is semi-simple for all eigenvalues ρ of A with $\operatorname{Re}\rho = 0$.

(ii) exponentially stable iif $\operatorname{Re}\rho < 0$ for all eigenvalues ρ of A.

4.1.3 Hyperbolicity

Definition 44. A matrix $A \in \mathbb{R}^{d\times d}$ is called **hyperbolic** if all eigenvalues λ of A have non-zero part, i.e. $\operatorname{Re}\lambda \neq 0$. An equilibrium x^* of a differential equation

$$\dot{x} = f(x),$$

where $f : D \subset \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable, is called **hyperbolic** if the matrix $f'(x^*) \in \mathbb{R}^{d\times d}$ is hyperbolic.

Example 45. We consider the linear one-dimensional differential equation

$$\dot{x} = 0.$$

Then the trivial equilibrium $x^* = 0$ is stable, and the derivative $f'(x^*)$ of the RHS ($f(x) = 0\ \forall x \in \mathbb{R}$) is 0, and thus this equilibrium (and all other equilibria0 is non-hyperbolic. We consider the following nonlinear perturbations and discuss its effect on the stability of $x^* = 0$:

(i) $\dot{x} = x^2$: the equilibrium $x^* = 0$ becomes unstable, although it attracts all points starting in the negative half-line.

(ii) $\dot{x} = -x^2$: the equilibrium $x^* = 0$ becomes unstable, although it attracts all points starting in the negative half-line.

(iii) $\dot{x} = x^3$: the equilibrium $x^* = 0$ becomes unstable, and all points (except x^*) move away from x^* forward in time.

(iv) $\dot{x} = -x^3$: the equilibrium $x^* = 0$ becomes asymptotically stable, but it is not exponentially stable.

4.1.4 Linearised stability

(Gronwall lemma) Lemma 46. We consider a continuous function $u : [a, b] \to \mathbb{R}$ defined on an interval $[a, b]$, and let $c, d \geq 0$. We assume that the function u satisfies the implicit inequlity

$$0 \leq u(t) \leq c + d \int_a^t u(s)\mathrm{d}s \quad \forall t \in [a, b].$$

Then we have the explicit estimate

$$u(t) \leq ce^{d(t-a)} \quad \forall t \in [a, b].$$

Theorem 47. Let $D \subset \mathbb{R}^d$ be open and $f : D \to \mathbb{R}^d$ be continuously differentiable, and consider the autonomous differential equation

$$\dot{x} = f(x).$$

Assume that x^* is an equilibrium of the above equation (i.e. $f(x^*) = 0$) s.t. for all eigenvalues $\lambda \in \mathbb{C}$ of the linearisation $f'(x^*) \in \mathbb{R}^{d \times d}$, we have $\operatorname{Re} \lambda < 0$. Then the equilibrium x^* is exponentially stable.

Example 48. Consider a pendulum moving along a circle of radius $r > 0$, with a mass $m > 0$ and friction coefficient $k > 0$. Let x denote the angle from the vertical. The force tangential ot the circle depends on both the position x and the speed $\dot{x}$ of the pendulum, and is given by

$$F_{\tan}(x, \dot{x}) = -mg \sin(x) - kr\dot{x}.$$

Newton's law reads as $mr \because x = F_{\tan}(x, \dot{x})$, and thus we get the second-order one-dimensional differential equation

$$\ddot{x} = -\frac{g}{r} \sin(x) - \frac{k}{m}\dot{x},$$

which can be transformed into the first-order two-dimensional system

$$\dot{x} = y, \dot{y} = -\frac{g}{r} \sin(x) - \frac{k}{m}y.$$

The matrix of the linearisation at (0, 0) is

$$\begin{pmatrix} \frac{\mathrm{d}\dot{x}}{\mathrm{d}x} & \frac{\mathrm{d}\dot{x}}{\mathrm{d}y} \\ \frac{\mathrm{d}\dot{x}}{\mathrm{d}y} & \frac{\mathrm{d}\dot{y}}{\mathrm{d}y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{r} & -\frac{k}{m} \end{pmatrix}$$

which gives two eigenvalues $\lambda_\pm := \frac{1}{2}\left(-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{r}}\right)$. It follows that the real parts of both eigenvalues $\lambda_\pm$ are negative, and hence, Theorem 47 implies that the equilibria $(n\pi, 0)$ for n even are exponentially stable.

4.1.5 Stable and unstable sets, invariant sets

Definition 49. Consider the differential equation (4.1) with associated flow φ, and let x^* be an equilibrium. We define the **stable set** of x^* as

$$W^s(x^*) := \left\{ x \in D : \lim_{t\to\infty} \varphi(t,x) = x^* \right\},$$

and the **unstable set** of x^* is defined as

$$W^u(x^*)a := \left\{ x \in D : \lim_{t\to-\infty} \varphi(t,x) = x^* \right\}.$$

Note that if x^* is an attractive equilibrium, then $W^s(x^*)$ is called **domain of attraction**, and it follows from the definition of attractivity that this is sa neighbourhood of x^*. Moreover, $W^s(x^*)$ is an open set in this case.

Definition 50. Consider the differential equation (4.1). Then a set $M \subset D$ is called

(i) **positively invariant** if $\forall x \in M$, we have $O^+(x) \subset M$.

(ii) **negatively invariant** if $\forall x \in M$, we have $O^-(x) \subset M$.

(iii) **invariant** if $\forall x \in M$, we have $O(x) \subset M$.

Note that sets that consist of equilibria or periodic orbits are invariant. Stable and unstable sets are also invariant positively and negatively, and any union of orbits is invariant. Accordingly unions of half-orbits of the form $O^+(x)$ or $O^-(x)$ are positively invariant or negatively invariant, respectively.

4.2 Limit sets

We study an autonomous differential equation of the form

$$\dot{x} = f(x), \tag{4.2}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. We denote the flow of this differential equation by φ.

Definition 51. Consider the flow φ of the differential equation (4.2), and let $x \in D$.

(i) A point $x_\omega \in D$ is called **omega limit point** of x, if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} t_n = \infty$ and
$$x_\omega = \lim_{n\to\infty} \varphi(t_n, x).$$
We denote by $\omega(x)$ the set of all omega limit points of x. $\omega(x)$ is called **omega limit set** of x.

(ii) A point $x_\alpha \in D$ is called **alpha limit point** of x, if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{n\to\infty} t_n = -\infty$ and
$$x_\alpha = \lim_{n\to\infty} \varphi(t_n, x).$$
We denote by $\alpha(x)$ the set of all alpha limit points of x. $\alpha(x)$ is called **alpha limit set** of x.

Note that the omega limit set of a point x is empty if $\sup J_{\max}(x) < \infty$, and the alpha limit set to be nonempty requires $\inf J_{\max}(x) = -\infty$.

Proposition 52. Consider the flow φ of the differential equation (4.2), and let $x \in D$. Then we have
$$\omega(x) = \bigcap_{t\geq 0} \overline{O^+(\varphi(t,x))}, \alpha(x) = \bigcap_{t\leq 0} \overline{O^-(\varphi(t,x))}.$$

Proposition 53. Consider the differential equation (4.2), and let $x \in D$. Then the following statements hold.

(i) The omega limit set $\omega(x)$ is invariant. In addition, if $O^+(x)$ is bounded and $\overline{O^+(x)} \subset D$, then $\omega(x)$ is non-empty and compact.

(ii) The alpha limit set $\alpha(x)$ is invariant. In addition, if $O^-(x)$ is bounded and $\overline{O^-(x)} \subset D$, then $\alpha(x)$ is non-empty and compact.

4.3 Lyapunov functions

Definition 54. Consider the differential equation (4.2), and let $V : D \to \mathbb{R}$ be a continuously differentiable function. Then the **orbital derivative** $\dot{V}$ of the function V is defined by

$$\dot{V}(x) := V'(x) \cdot f(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(x) f_i(x),$$

which describes the derivative of V along solutions $\mu : I \to D$ of (4.2). This follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mu(t)) = V'(\mu(t)) \cdot \dot{\mu}(t) = \dot{V}(\mu(t)) \quad \forall t \in I.$$

Definition 55. Consider the differential equation (4.2), and let $V : D \to \mathbb{R}$ be a continuously differentiable function. Then V is called a **Lyapunov function** if

$$\dot{V}(x) \leq 0 \quad \forall x \in D,$$

which decreases along solutions, i.e.

$$V(\varphi(t, x)) \leq V(x) \quad \forall t \in [0, \sup J_{\max}(x)).$$

Proposition 56. Consider the differential equation (4.2) with a Lyapunov function $V : D \to \mathbb{R}$. Then any sublevel set of the form

$$S_c := \left\{ x \in D : V(x) \leq c \right\},$$

where $c \in \mathbb{R}$, is positively invariant.

Theorem 57. Consider the differential equation (4.2) with an equilibrium $x^* \in D$, and let $V : D \to \mathbb{R}$ be a Lyapunov function s.t.

$$V(x^*) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in D \backslash \{x^*\}.$$

Then the equilibrium x^* is stable.

Example 58. We consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= -y - xy^2 \\ \dot{y} &= x - yx^2. \end{aligned}$$

The only equilibrium of this system is the trivial equilibrium (0, 0). The linearisation in (0, 0) is given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and thus the system is non-hyperbolic and stability cannot be deduced from Theorem 47.

We first show that the trivial equilibrium is stable by considering the quadratic function $V(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$. Then

$$\dot{V}(x,y) = V'(x,y) \cdot f(x,y) = (2x, 2y) \begin{pmatrix} -y - xy^2 \\ x - yx^2 \end{pmatrix} = -4x^2y^2 \leq 0,$$

so (0, 0) is stable.

Theorem 59. Consider the differential equation (4.2) with a Lyapunov function $V : D \to \mathbb{R}$. Then

$$\omega(x) \subset \left\{ y \in D : \dot{V}(y) = 0 \right\} \quad \forall x \in D.$$

Corollary 60. Consider the differential equation (4.2) with a Lyapunov function $V : D \to \mathbb{R}$. Then for any $x \in D$, $\omega(x)$ is contained in the largest invariant subset of $\left\{ y \in D : \dot{V}(y) = 0 \right\}$. Here the largest invariant subset is given by the union of invariant subsets of $\left\{ y \in D : \dot{V}(y) = 0 \right\}$.

Theorem 61. Consider the differential equation (4.2) with an equilibrium $x^* \in D$, and let $V : D \to \mathbb{R}$ be a Lyapunov function s.t.

$$\begin{array}{lcl} V(x^*) = 0 & \text{and} & V(x) > 0 \quad \forall x \in D \backslash \{x^*\}, \\ \dot{V}(x^*) = 0 & \text{and} & \dot{V}(x) < 0 \quad \forall x \in D \backslash \{x^*\}. \end{array}$$

Then the equilibrium x^* is asymptotically stable.

Corollary 62. Under the assumptions of Theorem 61, we consider the sublevel sets of the Lyapunov function V, which are of the form

$$S_c := \left\{ x \in D : V(x) \leq c \right\},$$

where $c > 0$. Then S_c is a subset of the domain of attraction $W^s(x^*)$ if $S_c \subset D$ is compact.

4.4 Poincaré-Bendixson theorem

(Poincaré-Bendixson theorem) Theorem 63. Consider the differential equation (4.2), and assume that for some $x \in D$, $O^+(x)$ lies in a compact subset K of D, which contains not more than finitely many equilibria. Then one of the following three statements holds for $\omega(x)$.

(i) $\omega(x)$ is a singleton consisting of an equilibrium.

(ii) $\omega(x)$ is a periodic orbit.

(iii) $\omega(x)$ consists of equilibria and non-closed orbits. The non-closed orbits in $\omega(x)$ converge forward and backward in time to equilibria in $\omega(x)$, so they are either homoclinic or heteroclinic orbits.

Corollary 64. Consider the differential equation (4.2), and assume that for some $x \in D$, $O^+(x)$ lies in a compact subset K of D that does not contain an equilibrium. Then $\omega(x)$ is a periodic orbit.

Example 65. We consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned}$$

We first show that $M := \left\{(x, y) \in \mathbb{R}^2 : \frac{1}{3} \le x^2 + y^2 \le 2\right\}$ is positively invariant. We show that the vector field of the RHS points inwards at the boundary of M. More precisely, we consider the scalar-valued function $V(x, y) = x^2 + y^2$ and show that the orbital derivative $\dot{V}$ satisfies $\dot{V}(x, y) < 0$ for $x^2 + y^2 = 2$ and $\dot{V}(x, y) > 0$ for $x^2 + y^2 = \frac{1}{3}$. Firstly, the orbital derivative reads as

$$\dot{V}(x, y) = (2x, 2y) \begin{pmatrix} y \\ -x + y(1 - x^2 - 2y^2) \end{pmatrix} = 2y^2(1 - x^2 - 2y^2).$$

For $x^2 + y^2 = 2$, we have $\dot{V}(x, y) = 2y^2(-1 - y^2) < 0$ and for $x^2 + y^2 = \frac{1}{3}$, we have $\dot{V}(x, y) = 2y^2(\frac{2}{3} - y^2) \ge 0$. This shows the positive invariance of M. Note that the only equilibrium is clearly given by $(0, 0) \notin M$. We apply the corollary to the Poincaré-Bendixson theorem (previous corollary) and conclude that the positively invariant set M contains a periodic orbit.