MATH40011 Calculus for JMC

Demitrios PapaGeorgiou

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Contents

Chapter 1

Differentiation

1.1 Continuity and differentiability

Definition 1. $f(x)$ is said to be continuous on an interval $[a, b]$ if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \forall x_0 \in [a, b]$$

Definition 2. The function $f(x)$ is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(Newton quotient) exists. We call this $f'(x)$ the derivative of f at point x.
In other words, a function is differentiable at x if right and left derivatives exists *AND* are equal. i.e.

$$\lim_{t \to x^+} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x^-} \frac{f(t) - f(x)}{t - x}$$

Theorem 3. If $f(x)$ is differentiable at $x = x_0$ then it is also continuous there.

Proof.

$$\begin{aligned}
\lim_{x\to x_0}\Big(f(x)-f(x_0)\Big) &= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right)\\
&= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\right)\lim_{x\to x_0}(x-x_0)\\
&= f'(x_0)\cdot 0\\
&= 0
\end{aligned}$$

□

Theorem 4. Let f be a function which is defined and differentiable on the open interval (a,b). Let c be a number in the interval which is a maximum for the function. Then $f'(c)=0$. Same if c is a minimum of f.

Proof. $f(c)\geq f(c+h)\Rightarrow f(c+h)-f(c)\leq 0$, i.e.

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\leq 0$$

Similarly for the left limit,

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\geq 0$$

In order to be differentiable, $f'(c)$ can only be 0. □

Theorem 5. Let $f(x)$ be continuous on the closed interval $[a,b]$. Then $f(x)$ has a maximum and a minimum on this interval. i.e. $\exists c_1,c_2$ s.t. $f(c_1)\geq f(x)$ and $f(c_2)\leq f(x)$ $\forall x\in[a,b]$.

(Mean Value) Theorem 6. If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists a<c<b$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

1.2 Inverse functions

Definition 7. Let $y = f(x)$ be defined on some interval. Given any y_0 in the range of f, if we can find a *unique* value x_0 in its domain such that $f(x_0) = y_0$, then we can define the *inverse function*

$$x = g(y)(= f^{-1}(y))$$

Theorem 8. Let $f(x)$ be strictly increasing or strictly decreasing. Then the inverse function exists.

Theorem 9. let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0 \forall x \in (a, b)$. Then the inverse function exists and we have

$$g'(y)(= f^{-1}(y)) = \frac{1}{f(x)}$$

Chapter 2

Integration

2.1 Preliminaries

The *anti-derivative* or *integral* of a function $f(x)$

Given $f(x)$ defined over some interval, then if I can find a function $F(x)$ defined over the same interval s.t.

$$F'(x) = f(x)$$

Then $F(x)$ is the *indefinite integral* of f, $F = \int f(x)\mathrm{d}x$

This is not unique. Let G be another indefinite integral, i.e. $G'(x) = f(x)$. Then $\dfrac{\mathrm{d}(F-G)}{\mathrm{d}x} = 0 \Rightarrow F(x) = G(x) + K$, where K is a constant.

2.1.1 Area under a curve

Suppose $f(x) \geq 0$ in some given interval $[a, b]$ and it is also continuous on $[a, b]$. $(a < b)$

Define $F(x)$ to be the area under the curve between $x = a$ and some x. By definition $F(a) = 0$

Theorem 10. The function $F(x)$ is differentiable and its derivative is equal to $f(x)$. Another way to state this is

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)$$

That is, the rate of change of the area under curve at x is precisely $f(x)$. This is also known as *the first part of Fundamental Theorem of Calculus.*

Proof. Newton quotient

$$\frac{F(x+h)-F(x)}{h}$$

Suppose x and also $h > 0$. $F(x+h) - F(x)$ is the are under the graph between x and $x+h$.

Since $f(x)$ is continuous on $[x, x+h]$ and is defined in the interval, it must have a maximum at some point x_+ and minimum at some point x_-. Hence, $\forall t \in [x, x+h]$,

$$f(x_-) \leq f(t) \leq f(x_+)$$

Can also bound the area using the rectangles

$$h \cdot f(x_-) \leq F(x+h) - F(x) \leq h \cdot f(x_+),$$

i.e.

$$f(x_-) \leq \frac{F(x+h)-F(x)}{h} \leq f(x_+)$$

Since x_+ and x_- are contained in $[x, x+h]$, as $h \to 0$, $x_-, x_+ \to x$ and by the squeezing theorem, we have

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x),$$

i.e.

$$F'(x) = f(x)$$

Hence the antiderivative is connected to area under the curve. The constant is fixed by $F(a) = 0$.

In other words, if I can guess a function $G(x)$ whose derivative is $f(x)$ [e.g. Guess $\log x$ for the antiderivative of $\frac{1}{x}$.] Then since F and G differ by a constant, we have

$$F(x) = G(x) + K$$

But $F(a) = 0 \Rightarrow -G(a) = K \Rightarrow F(x) = G(x) - G(a)$. Hence

$$\int_a^b f(x)\mathrm{d}x = F(b) = G(b) - G(a)$$

This is the familiar *definite integral*. This is also known as *the second part of the Fundamental Theorem of Calculus*. □

Example 11.

$$\int_1^2 x^2\mathrm{d}x = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}\,.$$

Here $f(x) = x^2$, $G(x) = \frac{x^3}{3}$ is the guessed antiderivative.

2.1.2 Signed area definition

If $f(x) < 0$, then area is below the x-axis. Define $F(x)$ to be *minus* the area, leading to the definite integral.

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

2.2 The Riemann Sum

Given $f(x)$, $a \leq x \leq b$, take a *partition* of the interval $[a, b]$ to be

$$x_i = a + ih, i = 0, 1, 2, \ldots, n$$

$$h = \frac{b-a}{n}$$

<u>Note</u>: My partition has regular spacing. Can generalize this to have a partition defined by a sequence $\{x_k\}_{k=0,1,\ldots,n}$ and in the limit $\max\limits_k |x_k - x_{k-1}| \to 0$. I am avoiding this technical issue which is quite irrelevant to what we want to do!

Take any subinterval $[x_{i-1}, x_i]$ and let $x_i^* \in [x_{i-1}, x_i]$. Then the Riemann sum is $\sum_{i=1}^n f(x_i^*)h$.

Three particularly useful ways:

$$x_i^* = x_i - \text{“right-hand RS”}$$

$$x_i^* = x_{i-1} - \text{“left-hand RS”}$$
$$x_i^* = \frac{1}{2}(x_i + x_{i-1}) - \text{“midpoint RS”}$$

Now in the limit $n \to \infty$, $h \to 0$, we can prove

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)h = \int_a^b f(x)\mathrm{d}x$$

Sketch of the proof:

$$\text{(Lower Riemann Sum) } L_n := \sum_{i=1}^{n} \sup f([x_{i-1}, x_i])h$$

$$\text{(Upper Riemann Sum) } U_n := \sum_{i=1}^{n} \inf f([x_{i-1}, x_i])h$$

By geometry,

$$L_n \le \int_a^b f(x)\mathrm{d}x \le U_n$$

In the limit it gets squeezed, if the limit exists then it is the integral.

2.3 Properties of the definite integral

$$\int_a^b c\, f(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x \tag{2.1}$$

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x \tag{2.2}$$

If $c \in (a, b)$ (and here $a < b$), then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \tag{2.3}$$

If $f(x) \le g(x)$ for $x \in [a, b]$, then

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x \tag{2.4}$$

(Hence $\int_a^b f(x)\mathrm{d}x \le \int_a^b |f(x)|\mathrm{d}x$ and $|\int_a^b f(x)\mathrm{d}x| \le \int_a^b |f(x)|\mathrm{d}x$)

$$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x \tag{2.5}$$

2.3.1 Fundamental Theorem of Calculus

Suppose F is differentiable on $[a, b]$ and F' is integrable on $[a, b]$. Then

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

If f is integrable on $[a, b]$ and has *antiderivative* F, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Theorem 12.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t = f(g(x)) \cdot g'(x)$$

Proof. Let

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

Then $F'(x) = f(x)$ — already proved. Now $\int_a^{g(x)} f(t)\mathrm{d}t = F(g(x))$ by definition of F.

$$\begin{aligned} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) \\ &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x) \end{aligned}$$

□

Example 13.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{x^2} e^t\mathrm{d}t = e^{x^2} \cdot 2x$$

$$\text{or} \quad \int_a^{x^2} e^t\mathrm{d}t = e^t|_a^{x^2} = e^{x^2} - e^a \text{ (same as before)}$$

2.4 Improper Integrals

Definition 14. $\int_a^b f(x)\mathrm{d}x$ is an *improper integral* if

(i) $a = \infty$ and/or $b = \infty$

(ii) $f(x) \to \pm\infty$ in (a, b)

To find improper integrals, we take the limit of proper integrals. If the limit is finite, the integral *converges*; otherwise it *diverges*.

Example 15.

(i) $\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\mathrm{d}x = \lim_{b\to\infty}(-\frac{1}{b} + 1) = 1$

(ii) $\int_1^\infty \frac{\mathrm{d}x}{x} = \lim_{b\to\infty} \log b = \infty,$ i.e. diverges.

2.4.1 Comparison Test

Suppose f and g satisfy

(1) $f(x) \leq g(x)\ \forall x \geq a$

(2) $\exists \int_a^b f(x)\mathrm{d}x$ and $\int_a^b g(x)\mathrm{d}x\ \forall b > a$

Then

(i) If $\int_a^\infty g(x)\mathrm{d}x$ is convergent, so is $\int_a^\infty f(x)\mathrm{d}x$

(ii) If $\int_a^\infty f(x)\mathrm{d}x$ is divergent, so is $\int_a^\infty g(x)\mathrm{d}x$

Similarly for $\int_{-\infty}^b f(x)\mathrm{d}x$ and $\int_{-\infty}^\infty f(x)\mathrm{d}x$

Comparison test is useful if we cannot carry out the integral exactly. It will tell us if it exists, then we can find it numerically etc.

Common tricks for evaluation:

1. Substitute constants with unknowns, or the other way round. The most common variables are x and $\cos x$ or $\sin x$. Omitting/Adding certain variables may be better in some cases!

2. Substitute an expression with another expression which can be smaller/bigger for successful evaluation. Use derivative if necessary.

3. (*Special*) Use whatever means necessary (such as Integration by Parts) to generate x^p, where $p \leq -1$, so that the integral converges.

Example 16.

(1)
$$\int_0^\infty \frac{\sin x}{(1+x)^2} \mathrm{d}x \text{ converges}$$

First thing to quote is that $\int_0^\infty \frac{\mathrm{d}x}{(1+x)^2}$ converges by comparison to $\int_1^\infty \frac{\mathrm{d}x}{x^2}$. (Why? If $x \geq 1$, $\frac{1}{(1+x)^2} < \frac{1}{x^2}$.)

By the comparison test, it converges because

$$\frac{|\sin x|}{(1+x)^2} \leq \frac{1}{(1+x)^2} < \frac{1}{x^2} \text{ for } x \geq 1$$

(2)
$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}} \text{ diverges}$$

$$\int_1^b \frac{\mathrm{d}x}{\sqrt{1+x^2}} \geq \int_1^b \frac{\mathrm{d}x}{\sqrt{x^2+x^2}} = \int_1^b \frac{\mathrm{d}x}{\sqrt{2}x}$$

Now $\int_1^\infty \frac{\mathrm{d}x}{x}$ diverges $\Rightarrow$ So does $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}}$

(3)
$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}} \text{ diverges}$$

Here is a proof using the comparison theorem.

If $x \geq 1$,
$$\frac{1}{x} \leq \frac{1}{\sqrt{x}} \Rightarrow \int_1^b \frac{1}{x}\mathrm{d}x < \int_1^b \frac{\mathrm{d}x}{\sqrt{x}},\ b > 1$$

and $\int_1^\infty \frac{1}{x}\mathrm{d}x$ diverges.

2.4.2 Improper integrals of unbounded functions

WLOG, consider situation where $|f(x)| \to \infty$ as $x \to 0$. Again take limits of bounded integrals. E.g.

$$\int_0^1 \frac{1}{x^p}\mathrm{d}x \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

Proof. Exercise! □

Example 17.

(1)

$$\begin{aligned} \int_0^1 \log x \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \log x \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} (x \log x|_\varepsilon^1 - \int_\varepsilon^1 x \frac{1}{x} \mathrm{d}x) \\ &= \lim_{\varepsilon \to 0} (-\varepsilon \log \varepsilon - 1 + \varepsilon) \\ &= -1 \end{aligned}$$

(2) Show that the improper integral $I = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x$ converges.

Write $I = I_1 + I_2$ where $I_1 = \int_0^1 \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x$, $I_2 = \int_1^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x$.

$$I_1 = \int_0^1 \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x < \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x, \text{ which is convergent,}$$

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x < \int_1^\infty e^{-x}\mathrm{d}x, \text{ which is also convergent.}$$

2.4.3 Mean Value Theorem for Integrals

Given a function f that is integrable on $[a, b]$, we define its average $\langle f \rangle_{[a,b]}$ by the formula

$$\langle f(x) \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x$$

Since $\langle f \rangle_{[a,b]}$ is a number (constant), then we have

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \langle f \rangle_{[a,b]}\mathrm{d}x$$

Theorem 18. Let f be continuous on $[a, b]$. Then $\exists\, x_0 \in (a, b)$ s.t.

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Proof. Define $F(x) = \int_a^x f(t)\mathrm{d}t$. By the FTC we have

$$F'(x) = f(x)\ \forall x \in (a, b)$$

F is continuous at a and b (proof in exercises). By MVT we have

$$F'(x_0) = \frac{F(b) - F(a)}{b-a}$$

i.e.

$$f(x_0) = \frac{\int_a^b f(t)\mathrm{d}t - \int_a^a f(t)\mathrm{d}t}{b-a} = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$

□

Theorem 19. let f and g be continuous on $[a, b]$ with $g(x) \geq 0\, \forall x \in [a, b]$. Then $\exists\, c \in [a, b]$ with

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x$$

Proof. Since f is continuous on $[a, b]$ it must have maximum M and a minimum m on $[a, b]$, i.e. $m \leq f(x) \leq M$. Since $g(x) \geq 0$, we have $\forall\, x \in [a, b]$,

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

By the properties of integral, we have

$$m\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq M\int_a^b g(x)\mathrm{d}x$$

By IVT, $\exists\, c \in [a, b]$ s.t.

$$f(c)\int_a^b g(x)\mathrm{d}x = \int_a^b f(x)g(x)\mathrm{d}x$$

□

2.5 Applications of Integration

2.5.1 Arc Length

$$\begin{aligned} \text{Total length } L &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2} \end{aligned}$$

Now let $x_i - x_{i-1} = h = \frac{b-a}{n} := \Delta x$,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2} \\ &= \int_a^b \left(1 + (f'(x))^2\right)^{\frac{1}{2}} \mathrm{d}x \end{aligned}$$

In parametric form this is

$$L = \int_{t_0}^{t_1} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t} \right)^2 \right]^{\frac{1}{2}} \mathrm{d}t$$

2.5.2 Volumes and surface area of revolution

Given an area bounded by $x = a$, $x = b$, $y = f(x)$, $y = 0$, then the volume of the solid produced by revolving $y = f$ about the x-axis is given by

$$V = \int_a^b \pi (f(x))^2 \mathrm{d}x$$

Revolving the element about the y-axis gives a shell of volume

$$V = \int_a^b 2\pi x f(x) \mathrm{d}x$$

As we revolve about the x-axis, the area of the surface area swept out is a strip of length $\approx 2\pi f(x_i)$ and thickness

$$\begin{aligned} \Delta l_i &= [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{\frac{1}{2}} \\ &\approx [1 + (f'(x_i))^2]^{\frac{1}{2}} \Delta x \end{aligned}$$

$\Rightarrow$ in the limit, area S is

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\mathrm{d}x$$

2.5.3 Centre of mass

We must have a 0 total moment in each axis. For instance, in 2D:

$$\left.\begin{array}{ll}(1) & \sum m_i(\overline{x}-x_i)=0 \\ (2) & \sum m_i(\overline{y}-y_i)=0\end{array}\right\} \Rightarrow (\overline{x},\overline{y}) = \left(\frac{\sum m_i x_i}{\sum m_i}, \frac{\sum m_i y_i}{\sum m_i}\right)$$

General theory — divide into small rectangles, and so the moment about the whole plate A about the y-axis is

$$\iint_A x\rho(x,y)\mathrm{d}x\mathrm{d}y = \overline{x}\iint_A \rho(x,y)\mathrm{d}x\mathrm{d}y$$

Similarly, about the x-axis,

$$\iint_A y\rho(x,y)\mathrm{d}x\mathrm{d}y = \overline{y}\iint_A \rho(x,y)\mathrm{d}x\mathrm{d}y$$

In the case of having a region $\{(x,y) : a \le x \le b, 0 \le y \le f(x)\}$,

$$m = \int_a^b \rho f(x)\mathrm{d}x$$

$$M_y = \int_a^b \rho x f(x)\mathrm{d}x$$

$$M_x = \frac{1}{2}\int_a^b \rho(f(x))^2\mathrm{d}x$$

Then

$$\overline{x} = \frac{\int_a^b xf(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x} \qquad \overline{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}$$

In the case of having a region $\{(x,y) : a \le x \le b, g(x) \le y \le f(x)\}$, similarly

$$\overline{x} = \frac{\int_a^b x(f(x)-g(x))\mathrm{d}x}{\int_a^b f(x)-g(x)\mathrm{d}x} \qquad \overline{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2-(g(x))^2\mathrm{d}x}{\int_a^b f(x)-g(x)\mathrm{d}x}$$

2.5.4 Theorem of Pappus

Let R be a region that lies on one side of a line l.

- $A =$ area of R
- $V =$ Volume obtained by rotating about l
- $d =$ distance travelled by the centre of mass when R is rotated about l

Then
$$V = Ad$$

Example 20. Volume of a cylinder radius r
Take the function $y = r$, $0 \leq x \leq l$. Rotate about x-axis.

$$A = rl$$

since $\overline{y} = \frac{1}{2}r$ (symmetry),

$$\Rightarrow d = \frac{1}{2}r \cdot 2\pi = \pi r$$

$$V = rl \cdot \pi r = \pi r^2 l \text{ as known}$$

2.6 Length and area in polar coordinates

2.6.1 Length

Approach 1

Recall
$$L = \int_a^b \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \right]^{\frac{1}{2}}$$

for parametric curves $(x(t), y(t))$. Now in polar coordinates we have curves $r = f(\theta)$. So use θ as a parameter,

$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta$$

Substitute x and y into L above, we get

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \left[(f' \cos\theta - f \sin\theta)^2 + (f' \sin\theta + f\cos\theta)^2 \right]^{\frac{1}{2}} \mathrm{d}\theta \\ &= \int_{\alpha}^{\beta} \left[(f'(\theta))^2 + (f(\theta))^2 \right]^{\frac{1}{2}} \mathrm{d}\theta \\ &= \int_{\alpha}^{\beta} \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 + r^2 \right]^{\frac{1}{2}} \mathrm{d}\theta \end{aligned}$$

Approach 2

Use Pythagoras theorem to write infinitessimals,

$$(\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 = (\mathrm{d}s)^2, \quad \mathrm{d}s = \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 + r^2 \right]^{\frac{1}{2}}$$

$$\Rightarrow L = \int_{\alpha}^{\beta} \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 + r^2 \right]^{\frac{1}{2}} \mathrm{d}\theta$$

Example 21. Find the length of the cardioid $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta} \mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos(\theta)} \mathrm{d}\theta \\ &= \int_0^{\pi} 2\cos\frac{\theta}{2} \mathrm{d}\theta + \int_{\pi}^{2\pi} -2\cos\frac{\theta}{2} \mathrm{d}\theta \quad \text{Be careful with this step!} \\ &= 8 \end{aligned}$$

2.6.2 Area

Using segments of angles $\Delta\theta_i$,

$$\Delta A = \frac{1}{2}(f(\theta))^2 \Delta\theta_i$$

$$\Rightarrow A = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 \mathrm{d}\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$$

Example 22. Find the area enclosed by the four-petaled rose $r = \cos 2\theta$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 4\theta}{2} \mathrm{d}\theta \\
&= \left(\frac{1}{4}\right)\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{8}
\end{aligned}$$

Chapter 3

Power Series and Taylor's Theorem

3.1 Power Series

3.1.1 Convergence tests and radius of convergence

Definition 23. Let x be a real number (can extend to complex numbers also) and $\{a_n\}_{n\geq 0}$ be a sequence of numbers. Then we can form the *power series* $\sum_{n=0}^{\infty} a_n x^n$. The partial sums $S_N = \sum_{n=1}^{N} a_n x^n$ are degree N polynomials.

(Radius of Convergence) Theorem 24. $\exists R \in [0, \infty]$ s.t.

- $|z| < R \Rightarrow \sum a_n z^n$ is absolutely convergent, and
- $|z| > R \Rightarrow \sum a_n z^n$ is divergent

Proof. Let $S = \{|z| : a_n z^n \to 0\}$, nonempty since $0 \in S$. Then define

$$R = \begin{cases} \sup S & \text{if } S \text{ bounded,} \\ \infty & \text{if } S \text{ unbounded.} \end{cases}$$

Now suppose $|z| < R$. Since $|z|$ is not an upper bound for S, $\exists w$ s.t. $|w| > |z|$ and $a_w w^n \to 0$. In particular $|a_n w^n|$ is bounded by some A $\forall n$. Thus

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n$$

Therefore, by comparison with the convergent series $\sum \left|\frac{z}{w}\right|^n$ (recall $\left|\frac{z}{w}\right| < 1$) we find $\sum |a_n z^n|$ is convergent.

On the other hand, if $|z| > R$ then $a_n z^n \nrightarrow 0$ as $n \to \infty \Rightarrow \sum a_n z^n$ diverges. □

3.1.2 Differentiation and integration of power series

Theorem 25. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which converges absolutely for $|x| < R$.
Then for $|x| < R$, $f(x)$ is differentiable

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and integrable

$$\int f(x)\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

For a power series *within its radius of convergence*, we can differentiate or integrate term by term. (When we are differentiating a function, we are assuming that we are differentiating over points that are differentiable. Similarly, integrating a function assumes that we are integrating over a range that is integrable.)

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ can be differentiated an infinite number of times as long as $|x| < R$, and the derivatives will exist. The way to show this is to consider each differentiated series as a new power series. For example,

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum a_n x^n\right) &= \sum n(n-1)(n-2)\cdots(n-(k-1))x^{(n-k)}a_n \\ &= \sum \frac{n!}{(n-k)!} a_n x^{n-k}\end{aligned}$$

Using ratio test,

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1-k)!}\frac{(n-k)!}{n!}\left|\frac{a_{n+1}}{a_n}\right||x| = \lim_{n\to\infty}\left(\frac{n+1}{n+1-k}\right)\left|\frac{a_{n+1}}{a_n}\right||x| = L|x|$$

as for the undifferentiated power series,

$$\Rightarrow \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum a_n x^n\right) \text{ converges for } |x| < R$$

Example 26. Write down power series for $\frac{x}{1+x^2}$ and $\log(1+x^2)$

Recall geometric series $1 + r + r^2 + \cdots = \frac{1}{1-r}$ for $|r| < 1$. If $r = -x^2$ we have $x(1 - x^2 + x^4 - x^6 + \cdots) = \frac{1}{1+x^2} \cdot x$. Hence

$$\frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}, \qquad |x| < 1 \text{ for convergence}$$

Now for the $\log(1+x^2)$ we observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log(1+x^2) = \frac{2x}{1+x^2}$$

So

$$\log(1+x^2) = 2\int \frac{x}{1+x^2}\mathrm{d}x$$

If $|x| < 1$ we can integrate term by term. i.e.

$$\begin{aligned}\log(1+x^2) &= 2\int (x - x^3 + x^5 - \cdots)\mathrm{d}x \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\end{aligned}$$

Theorem 27. Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$ and $g(x) := \sum_{n=0}^{\infty} b_n x^n$, with radius of convergence being R_1 and R_2. Let $R := \min R_1, R_2$. Then for $|x| < R$,

(1) $f(x) + g(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$

(2) $cf(x) = \sum_{n=0}^{\infty} c\, a_n x^n$

(3) $f(x)g(x) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_m b_{n-m}\right)x^n$

3.2 Taylor Series

This is a power series that represents a function $f(x)$ by using its derivatives at a single point.

Assume the power series exists and identifies the coefficients. Take a fixed point $x = x_0$. If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges for $|x - x_0|$ small enough, we can find the coefficients as

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

This is the *Taylor series about the point* $x = x_0$. If $x_0 = 0$ we get the *Maclaurin series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In both formulas we have assumed that $f(x)$ is infinitely differentiable on some interval containig the point $x = x_0$. This of course can happen for many functions, e.g. $f(x) = \sin x$, $f(x) = e^x$, $f(x) = \cos x, \ldots$

3.2.1 Derivation of power series

1. By definition, differentiate required amount of times.

2. express known terms in power series and do operations on them instead

3. compare coefficients, given that an exact value is known

Some of the common Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\log(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

(Taylor Series) Theorem 28. Let f be a function defined on a closed interval between two numbers x_0 and x. Assume that the function has $n+1$ derivatives on the interval and that they are all continuous. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n$$

where the *remainder* R_n is given by

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t$$

Proof. Use integration by parts. From the FTC, we have

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\mathrm{d}t$$

Now $\int_{x_0}^{x} f'(t)\mathrm{d}t = \int_{x_0}^{x} f'(t)\mathrm{d}(-(x-t))$, and use integration by parts,

$$\begin{aligned}
\int_{x_0}^{x} f'(t)\mathrm{d}t &= +(t-x)f'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} (t-x)f''(t)\mathrm{d}t \\
&= (x-x_0)f'(x_0) + \int_{x_0}^{x} (x-t)f^{(2)}(t)\mathrm{d}t \\
&= (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2} f^{(2)}(x_0) + \int_{x_0}^{x} \frac{(x-t)^2}{2} f^{(3)}(t)\mathrm{d}t \\
&\vdots
\end{aligned}$$

Repeat n tiems to get the result. □

3.2.2 Estimation Accuracy

Alternative form of the remainder:

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t$$

Use the generalized MVT for integrals, since $x_0 \leq t \leq x$, $x - t \geq 0$, $g(t) := \frac{(x-t)^n}{n!} \geq 0$. Thus $\exists\, c \in [x_0, x]$ s.t.

$$R_n = f^{(n+1)}(c) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\mathrm{d}t$$

$$\Rightarrow R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Converges to $f(x)$ if $R_n \to 0$ as $n \to \infty$!

Here is a very useful alternative of Taylor's theorem that we use in Numerical Analysis. Put $x = x_0 + h$ (and after that $x_0 \to x$ if you want)

$$f(x_0+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x_0) + R_n(x_0, h)$$

$$\begin{aligned} R_n &= \int_{x_0}^{x_0+h} \frac{(x_0+h-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t \\ &= \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad \text{, where } c \in [x_0, x_0+h] \end{aligned}$$

In terms of *bounding the remainder*, for maclaurin series with $x_0 = 0$, for instance, use form

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

If $\left|f^{(n+1)}(c)\right| \leq M_{n+1}$, then

$$|R_n| \leq M_{n+1} \frac{|x|^{n+1}}{(n+1)!}$$

Example 29.

(1)
$$f(x) = \sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2n+3}$$
where
$$|R_{2n+1}| = \left| \frac{f^{(2n+3)}(x)}{(2n+3)!} x^{2n+3} \right| \leq \frac{|x|^{2n+3}}{(2n+3)!}$$
Hence
$$\sin(0.1) \approx 0.1 - \frac{10^{-3}}{3!} + \frac{10^{-5}}{5!}$$
with an error which is less than
$$\frac{0.1^7}{7!} = \frac{10^{-7}}{5040} < 10^{-10}$$

(2) Compute $\sin\left(\frac{\pi}{6} + 0.2\right)$ to an accuracy of 10^{-4}. Even though the power series of $\sin x$ will converge if we take enough terms, since $\frac{\pi}{6} + 0.2$ is not small, we will need *a lot of terms* to get to the required accuracy. It is *much* better to expand about $\frac{\pi}{6}$ using the formula
$$f(x_0 + h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x_0) + R_n(x_0, h)$$
Now $h = 0.2$, $f^{(n+1)}$ is sin or cos,
$$\Rightarrow |R_n| \leq \frac{0.2^{n+1}}{(n+1)!} \Rightarrow R_3 \leq \frac{0.2^4}{24} = \frac{16 \times 10^{-4}}{24} < 10^{-4}$$
$$\Rightarrow \sin\left(\frac{\pi}{6} + 0.2\right) \approx \sin\frac{\pi}{6} + 0.2\cos\frac{\pi}{6} + \frac{0.2^2}{2!}\left(-\sin\frac{\pi}{6}\right) + \frac{0.2^3}{3!}\left(-\cos\frac{\pi}{6}\right)$$

(3) Compute e to 3 decimals.
$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$
If $x < 0$ then $c < 0$ and $|R_n| \leq \frac{|x|^{n+1}}{(n+1)!}$; if $x > 0$ and such that $x \leq b$ say, then $|R_n| \leq \frac{e^b}{(n+1)!}b^{n+1}$

Showed earlier (see chapter on logarithms) that $2 < e < 4$. Therefore

$$|R_n| \leq \frac{e}{(n+1)!} \leq \frac{4}{(n+1)!}$$

Need $\frac{4}{(n+1)!}$ to be less than 10^{-3}.

$$|R_5| = \frac{4}{6!} = \frac{1}{180} > 10^{-3}$$

$$|R_6| = \frac{4}{7!} = \frac{1}{1260} < 10^{-3}$$

So

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + R_6$$

(4) Compute $\log 1.1$ to 3 decimals.

$$|R_n| = \frac{f^{n+1}(0.1)}{(n+1)!} 0.1^{n+1} < \frac{0.1^{n+1}}{(n+1)!}$$

$$|R_1| < \frac{0.1^2}{2!} > 10^{-3}$$

$$|R_2| < \frac{0.1^3}{3!} < 10^{-3}$$

So

$$\log 1.1 \approx 0.1 - \frac{0.1^2}{2} + R_2$$

3.2.3 Alternative to L'Hôpital's rule

Sometimes it is easier to use Taylor's theorem instead of differentiating many times.

Example 30.

(1)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x - x}{\tan x - x} &= \lim_{x\to 0}\frac{\sin x\cos x - x\cos x}{\sin x - x\cos x} \\
&= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+\cdots\right)\left(1-\frac{x^2}{2}+\cdots\right)-x\left(1-\frac{x^2}{2}+\cdots\right)}{x-\frac{x^3}{6}+\cdots-x\left(1-\frac{x^2}{2}+\cdots\right)} \\
&= \lim_{x\to 0}\frac{-\frac{x^3}{6}+\cdots}{\frac{x^3}{3}} \\
&= -\frac{1}{2}
\end{aligned}$$

(2)

$$\begin{aligned}
\lim_{x\to 1}\frac{\log x}{e^x - e} &= \lim_{y\to 0}\frac{\log(1+y)}{e(e^y-1)} \qquad (\text{put } x = 1+y) \\
&= \lim_{y\to 0}\frac{y-\frac{y^2}{2}+\frac{y^3}{3}+\cdots}{e(1+y+\cdots-1)} \\
&= \frac{1}{e}
\end{aligned}$$

Chapter 4

Trigonometric Series — Fourier Series

4.1 Orthogonal and orthonormal function spaces

Definition 31. If f, g are real-valued functions that are Riemann integrable on $[a, b]$, then we define the inner product of f and g, denoted by (f, g), by

$$(f, g) := \int_a^b f(x)g(x)\mathrm{d}x$$

Note $(f, f)^{\frac{1}{2}} = \left(\int_a^b f^2 \mathrm{d}x\right)^{\frac{1}{2}} :=\| f \| \geq 0$

Definition 32. Let $S = \{\phi_0, \phi_1, \phi_2, \ldots\}$ be a collection of functions that are Riemann integrable on $[a, b]$. If $(\phi_n, \phi_m) = 0$ whenever $m \neq n$, then S is an *orthogonal* system on $[a, b]$.

If in addition $\|\phi_n\| = 1$, i.e. $\int_a^b \phi_n^2 \mathrm{d}x = 1$, then S is said to be *orthonormal* on $[a, b]$.

Note: Can easiily go from orthogonal to orthonomal by considering $\frac{\phi_n}{\|\phi_n\|}$

The orthonormal *trigonometric system* will be used, where

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \phi_{2n-1}(x) = \frac{\cos{(nx)}}{\sqrt{\pi}}, \phi_{2n}(x) = \frac{\sin{(nx)}}{\sqrt{\pi}} \qquad (n = 1, 2, \ldots)$$

i.e. The system is

$$S = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \dots \right\}$$

S defined above is orthonormal on any interval of length 2π, e.g. $[0, 2\pi]$, $[-\pi, \pi]$, etc.

4.2 Periodic functions and periodic extensions

A function $f(x)$ is periodic with period T if $\forall x$,

$$f(x+T) = f(x)$$

It follows that a T-periodic function is also mT-periodic $\forall m \in \mathbb{Z}$, i.e.

$$f(x \pm mT) = f(x)$$

Start with any continuous function $f(x)$ in an interval $a \leq x < b$. Can extend this periodically to have period $T = b - a$. Function on $[a, b)$ extended periodically and the new function is discontinuous at $x = a + mT$ $\forall m \in \mathbb{Z}$.

Definition 33. At points of discontinuity $x = \xi$, define

$$f(\xi) = \frac{1}{2}\left[f(\xi_+) + f(\xi_-)\right]$$

4.2.1 Integrals over a period

For a periodic function $f(x)$ of period T and for arbitrary values of a, we have

$$\int_{-a}^{T-a} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$$

In fact,

$$\int_\alpha^\beta f(x)\mathrm{d}x = \int_{\alpha+T}^{\beta+T} f(x)\mathrm{d}x$$

Proof.

$$\int_\alpha^\beta f(x)\mathrm{d}x = \int_{\alpha+T}^{\beta+T} f(y-T)\mathrm{d}y = \int_{\alpha+T}^{\beta+T} f(y)\mathrm{d}y = \int_{\alpha+T}^{\beta+T} f(x)\mathrm{d}x$$

□

4.3 Superposition of harmonics and trigonometric polynomials

Start with an oscillation $\sin(\omega x)$. Here ω is the frequency. The period is $T = \frac{2\pi}{\omega}$.

This is a "pure" harmonic oscillation. Signals — e.g. sound, electromagnetic waves, water waves, are not pure oscillations, they contain *higher harmonics.*

Lets add another oscillation of frequency 2ω, i.e. $\sin(2\omega x)$, whose period is $T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$. This is called the *1st harmonic.*

Signal could be

$$S_2(x) = A_1 \sin(\omega x) + B_2 \sin(2\omega x)$$

$S_2(x)$ has period $T = \frac{2\pi}{\omega}$ overall. 1st harmonic has period $\frac{T}{2}$.

Can add more and more higher frequencies and in fact can produce a wave (oscillation) that is a *trigonometric polynomial* defined by

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}[a_k \cos(k\omega x) + b_k \sin(k\omega x)]$$

The constant $\frac{1}{2}a_0$ is included. ($\frac{1}{2}$ is useful as we will see later)

Note: Went from $\omega_1 = \omega$ to $\omega_2 = 2\omega$ etc. All the frequencies have ratios that are rational. If $\frac{\omega_1}{\omega_2}$ is irrational, we get *quasi-periodic* oscillations.

4.4 Complex Notation

Useful to use Euler's relation

$$\cos\theta + i\sin\theta = e^{i\theta}$$

and since $\cos\theta - i\sin\theta = e^{-i\theta}$,

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

So, can represent everything as complex and find real expressions by taking real or imaginary parts. e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x} a e^{i\omega(x-\phi)} = a i\omega e^{i\omega(x-\phi)}$$

Can also integrate

$$\int e^{inx}\mathrm{d}x = \int(\cos nx + i\sin nx)\mathrm{d}x = \left[\frac{\sin nx}{n} - i\frac{\cos nx}{n}\right] = \frac{e^{inx}}{in}$$

Orthogonality

$$\int_{-\pi}^{\pi} e^{inx}\mathrm{d}x = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}$$

For any integers m, n we have

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases}$$

4.4.1 Complex notation for trigonometric polynomials

Start with the polynomial (have set $\omega = 1$)

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

Use the relations found earlier

$$\begin{aligned} S_n(x) &= \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k\left(\frac{e^{ikx}+e^{-ikx}}{2}\right) + b_k\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^{n}\frac{1}{2}(a_k - ib_k)e^{ikx} + \sum_{k=1}^{n}\frac{1}{2}(a_k + ib_k)e^{-ikx} \end{aligned}$$

Can now write this as a single complex series as follows

$$S_n(x) = \sum_{k=-n}^{n} r_k e^{ikx} \tag{4.1}$$

where

$$\left.\begin{aligned} r_0 &= \frac{1}{2}a_0 \\ r_k &= \frac{1}{2}(a_k - ib_k) \\ r_{-k} &= \frac{1}{2}(a_k + ib_k) \end{aligned}\right\} k = 1, 2, \ldots, n$$

Notice that $r_k = r^*_{-k}$, where $*$ denotes complex conjugate. This is *not accidental.* $S_n(x) \in \mathbb{R}$ in equation(4.1). Hence it must be equal to its complex conjugate.

$$S_n(x) = \sum_{k=-n}^{n} r_k e^{ikx},\ S_n^*(x) = \sum_{k=-n}^{n} r_k^* e^{-ikx}$$

Change indexing, put $k = -l$, to find

$$S_n^*(x) = \sum_{l=n}^{-n} r^*_{-l} e^{ilx} = \sum_{l=-n}^{n} r^*_{-l} e^{ilx} = \sum_{k=-n}^{n} r^*_{-k} e^{ikx}$$

Comparing the two, we can see that they are equal iff

$$r_k = r^*_{-k} \quad \text{i.e.} \quad r_k^* = r_{-k}$$

Conversely, if we are given a complex form $f(x) = \sum_{k=-n}^{n} r_k e^{ikx}$, then $f(x) \in \mathbb{R} \iff r_k = r^*_{-k}$, i.e.

$$r_k + r_{-k} = r_k + r_k^* \quad \text{is real}$$

$$r_k - r_{-k} = r_k - r_k^* \quad \text{is imaginary}$$

Example 34. Take $S_n(x) = \cos x + \frac{1}{2}\sin x + 3\cos 2x$. Express as a complex trigonometric series.

$$\begin{aligned} S_n &= \frac{1}{2}(e^{ix} + e^{-ix}) - \frac{i}{4}(e^{ix} - e^{-ix}) + \frac{3}{2}(e^{2ix} + e^{-2ix}) \\ &= \left(\frac{1}{2} - \frac{i}{4}\right) e^{ix} + \left(\frac{1}{2} + \frac{i}{4}\right) e^{-ix} + \frac{3}{2}e^{2ix} + \frac{3}{2}e^{-2ix} \\ &= \sum_{k=-2}^{2} r_k e^{ikx} \end{aligned}$$

where $r_0 = 0, r_1 = \frac{1}{2} - \frac{i}{4}, r_{-1} = \frac{1}{2} + \frac{i}{4} = r_1^*, r_2 = \frac{3}{2}, r_{-2} = \frac{3}{2} = r_2^*$.

4.5 Fourier Series

Consider the trigonometric polynomial

$$f(x) = S_n(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \tag{4.2}$$

There are $2N+1$ coefficients to determine. Use orthogonality on the interval $[-\pi, \pi]$ for $\sin mx, \cos nx$, etc.

$$\int_{-\pi}^{\pi} \sin mx \cos nx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2}[\sin{(m+n)x} + \sin{(m-n)x}]\mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2}[\cos{(m-n)x} - \cos{(m+n)x}]\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \mathrm{d}x = \int_{-\pi}^{\pi} \frac{1}{2}[\cos{(m-n)x} + \cos{(m+n)x}]\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

The constant a_0 can be found immediately by integrating over $[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} f(x)\mathrm{d}x = 2\pi a_0 + \sum_{k=1}^{N} \int_{-\pi}^{\pi} (a_k \cos kx + b_k \sin kx)\mathrm{d}x = 2\pi a_0$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{d}x$$

i.e. a_0 is the average (or mean) value of the function over the domain.
Next, take any integer $m \geq 1$, multiple (4.2) by $\cos mx$ and integrate over $[-\pi, \pi]$. Using the *orthogonality* property, we see that only the term containing a_m in the sum will survive to give

$$\int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x = a_m \int_{-\pi}^{\pi} \cos^2 mx \mathrm{d}x = \pi a_m$$

$$\Rightarrow a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \mathrm{d}x$$

Similarly,

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \mathrm{d}x$$

(Fourier Series) Theorem 35. The fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

or

$$\sum_{n=-\infty}^{\infty} r_n e^{inx}$$

formed by the fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \mathrm{d}x \quad \& \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \mathrm{d}x$$

converges to the value $f(x)$ for any piecewise continuous $f(x)$ of period 2π which has piecewise continuous derivatives of first and second order.* At any discontinuities, the value of the function must be defined by $f(x) = \frac{1}{2}[f(x_+) + f(x_-)]$.

*Note: Can relax the assumption of the second derivative. It is enough to have $f'(x)$ be piecewise continuous, i.e. the functon is pieceswise smooth. If $f(x)$ is continuous, the convergence is *absolute* and *uniform*. If it is discontinuous, absolute and uniform convergence everywhere *except at the discontinuity.*

For the proof we will need some additional lemmas.

4.5.1 A trigonometric formula

We will prove the following — needed later

$$\begin{aligned} C_n(x) &= \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx \\ &= \frac{\sin{(n+1)x}}{2\sin\frac{1}{2}x} \end{aligned}$$

Clearly $\frac{1}{2}x \neq 0, \pm\pi, \pm 2\pi, \ldots$ i.e. $x \neq 0, \pm 2\pi, \pm 4\pi, \ldots$

If we definie $C_n(x)$ at these points by $(n+\frac{1}{2})$, then the function is continuous everywhere.

$$\lim_{x\to 2k\pi} \frac{\sin{(n+\frac{1}{2})x}}{2\sin\frac{1}{2}x} = \frac{\left(n+\frac{1}{2}\right)\cos{(n+\frac{1}{2})x}}{\cos\frac{1}{2}x} = n + \frac{1}{2}$$

Use $\cos kx = \frac{1}{2}(e^{ikx}+e^{-ikx})$ to re-write

$$C_n(x) = \frac{1}{2}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2}(e^{-inx} + e^{ix}e^{-inx} + \left(e^{ix}\right)^2 e^{-inx} + \cdots + e^{inx})$$

i.e. a geometric progression with ratio $r = e^{ix} = \cos x + i\sin x$.
Now $r = 1$ only if $x = 0, \pm 2\pi, \ldots$, i.e. the exceptional points that we excluded (treated separatedly). Sum it up to find

$$C_n(x) = \frac{1}{2}e^{-inx}\frac{1-r^{2n+1}}{1-r} = \frac{1}{2}\left[e^{-inx} - e^{i(n+1)x}\right]\frac{1}{1-e^{ix}}$$

Multiply top + bottom by $e^{-\frac{1}{2}ix}$

$$\begin{aligned} \Rightarrow C_n(x) &= \frac{1}{2}\frac{\left[e^{-i(n+\frac{1}{2})x} - e^{i(n+\frac{1}{2})x}\right]}{e^{-\frac{1}{2}ix} - e^{\frac{1}{2}ix}} \\ &= \frac{\sin{(n+\frac{1}{2})x}}{2\sin\frac{1}{2}x} \end{aligned}$$

Integrate from 0 to π we find

$$\int_0^\pi \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\mathrm{d}t = \int_0^\pi \left(\frac{1}{2} + \sum_{k=1}^{n}\cos kt\right)\mathrm{d}t = \frac{\pi}{2}$$

4.5.2 Lemmas

(Riemann-Lebesgue) Lemma 36. If the function $g(x)$ is integrable on $[a,b]$ (e.g. it is piecewise continuous), then

$$I_\lambda = \int_a^b g(x)\sin\lambda x\,\mathrm{d}x$$

tends to 0 as $\lambda\to\infty$

Proof. (Will do it when g' is also piecewise continuous. For the general case see HW problems.)
Using integration by parts,

$$\begin{aligned} I_\lambda &= \int_a^b g(x)\sin\lambda x\,\mathrm{d}x \\ &= \left[-\frac{\cos\lambda x}{\lambda}g(x)\right]_a^b + \int_a^b \frac{\cos\lambda x}{\lambda}g'(x)\mathrm{d}x \\ &= \frac{1}{\lambda}\left[g(a)\cos\lambda a - g(b)\cos\lambda b + \int_a^b \cos\lambda x g'(x)\mathrm{d}x\right] \end{aligned}$$

$$\Rightarrow |I_\lambda| \le \frac{1}{\lambda}M \quad \text{for some constant M}$$

And the result follows. □

Lemma 37.

$$\int_0^\infty \frac{\sin z}{z}\mathrm{d}z = \frac{\pi}{2}$$

Proof. Show improper integral exists, i.e.

$$\exists I = \lim_{M\to\infty}\int_0^M \frac{\sin z}{z}\mathrm{d}z$$

(Note $z = 0$ is not a problem.)
Consider $0 < M < N$ and calculate

$$\begin{aligned} I_N - I_M &= \int_M^N \frac{\sin z}{z}\mathrm{d}z \\ &= -\left.\frac{\cos z}{z}\right|_M^N - \int_M^N \frac{\cos z}{z^2}\mathrm{d}z \\ &= \frac{\cos M}{M} - \frac{\cos N}{N} - \int_M^N \frac{\cos z}{z^2}\mathrm{d}z \\ &\le \frac{1}{M} + \frac{1}{N} + \int_M^N \frac{\mathrm{d}z}{z^2} \\ &= \frac{2}{M} \end{aligned}$$

Hence convergence since $|I_N - I_M|$ can be made arbitrarily small (Cauchy). In fact letting $N \to \infty$, we see that $|I - I_M| \leq \frac{2}{M}$, so I_M approaches its limit algebraically.
Now take $p > 0$ arbitrarily and pick $M = \lambda p$.

$$I_M = I_{\lambda p} = \int_0^{\lambda p} \frac{\sin z}{z} \mathrm{d}z = \int_0^{p} \frac{\sin \lambda x}{\lambda x} \lambda \mathrm{d}x = \int_0^{p} \frac{\sin \lambda x}{x} \mathrm{d}x$$

where we have now *fixed* the integration range to $[0, p]$. As $M \to \infty$, $\lambda p \to \infty$, i.e. $\lambda \to \infty$, and by the estimate above

$$\left| I - \int_0^p \frac{\sin \lambda x}{x} \mathrm{d}x \right| \leq \frac{2}{M} = \frac{2}{\lambda p}$$

i.e.

$$\lim_{\lambda \to \infty} \int_0^p \frac{\sin \lambda x}{x} \mathrm{d}x = I \quad \forall p \text{ that are sufficiently big} \tag{4.3}$$

Cannot apply Riemann-Lebesgue directly. Consider the function

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2 \sin \frac{x}{2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Fact: $h(x)$ is continuous and also has a continuous first derivative for $0 \leq x < \pi$, since by using taylor series expansion, $\frac{1}{x} - \frac{1}{2 \sin \frac{x}{2}} = -\frac{x}{24} + Kx^3 + \cdots$.
Now use the Riemann-Lebesgue Lemma to see that for $0 \leq p < 2\pi$,

$$\int_0^p \sin \lambda x \left(\frac{1}{x} - \frac{1}{2 \sin \frac{x}{2}} \right) \mathrm{d}x \to 0 \quad \text{as} \quad \lambda \to \infty$$

Note: The convergence is uniform for $0 \leq p \leq \pi$ since $|h(x)|$ and $|h'(x)|$ are both bounded in this interval.
From (4.3) we have immediately

$$\lim_{\lambda \to \infty} \int_0^p \frac{\sin \lambda x}{2 \sin \frac{x}{2}} \mathrm{d}x = I$$

Pick $\lambda = n + \frac{1}{2}$ and $p = \pi$, we have shown already that $\int_0^\pi \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{x}{2}} \mathrm{d}x = \frac{\pi}{2}$ independent of n. Hence we have proved:

$$I = \int_0^\infty \frac{\sin z}{z} \mathrm{d}z = \frac{\pi}{2}$$

□

4.5.3 Proof of Fourier Series

Start with nth "Fourier polynomial"

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

and substitute the formulas for a_k b_k, change order of summation and integration (finite sum, so ok), to find

$$\begin{aligned}
S_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos kt \cos kx + \sin kt \sin kx)\right]\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right]\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\left[\left(n+\frac{1}{2}\right)(t-x)\right]}{2\sin\frac{1}{2}(t-x)}\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} \frac{f(x+\xi)\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}\xi}\mathrm{d}\xi \qquad \text{(substitute } \xi = t-x\text{)} \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\xi)\frac{\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}\xi}\mathrm{d}\xi
\end{aligned}$$

by using properties of integrals of periodic functions discussed earlier. Note that x is a fixed number.

We will prove that (and this proves Theorem 35)

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\xi)\frac{\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}\xi}\mathrm{d}\xi = f(x)$$

At all points $x \in [-\pi, \pi]$, even points of discontinuity, we have

$$f(x) = \frac{1}{2}[f(x_+) + f(x_-)]$$

We have proven already that $\int_0^{\pi}\frac{\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}t}\mathrm{d}t = \frac{\pi}{2}$, and by a change of variable $t = -t'$ we also find $\int_{-\pi}^{0}\frac{\sin(n+\frac{1}{2})t'}{2\sin\frac{1}{2}t'}\mathrm{d}t' = \frac{\pi}{2}$. Hence

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(x_+)\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\mathrm{d}t + \frac{1}{\pi}\int_{-\pi}^{0} f(x_-)\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\mathrm{d}t$$

Using this identity gives

$$\begin{aligned} S_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}[f(x+\xi) - f(x_+)]\frac{\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}\xi}\mathrm{d}\xi \\ +\frac{1}{\pi}\int_{-\pi}^{0}[f(x+\xi) - f(x_-)]\frac{\sin(n+\frac{1}{2})\xi}{2\sin\frac{1}{2}\xi}\mathrm{d}\xi \end{aligned}$$

What is left to do is to prove that

(i) $\frac{f(x+\xi)-f(x_+)}{\sin\frac{1}{2}\xi}$ is piecewise continuous and so is its 1st derivative, on $0 \leq \xi \leq \pi$.

(ii) $\frac{f(x+\xi)-f(x_-)}{\sin\frac{1}{2}\xi}$ is piecewise continuous along with its 1st derivative on $-\pi \leq \xi \leq 0$.

Then by Riemann-Lebesgue Lemma $S_n \to f(x)$ as $n \to \infty$, i.e. convergence (uniform away from discontinuities).

4.5.4 Examples of Fourier Series

Will consider $f(x)$ to be 2π-periodic.

(i) If $f(x)$ is even, i.e. $f(-x) = f(x)$, then $f(x)$ has only a *cosine series.*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\mathrm{d}x$$

Similarly, if $f(x)$ is odd, $f(-x) = -f(x)$ then $f(x)$ has only a *sine series.*

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\mathrm{d}x$$

(ii) If a function is defined on $[0,\pi]$ by an expression $f(x)$, then it can be extended as an *even* or *odd* function on $[-\pi,\pi]$.

4.5.5 Exponential complex form of Fourier Series

Have already shown that for real $f(x)$,

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \\ &= \sum_{n=-\infty}^{\infty} r_n e^{inx} \quad -\pi < x < \pi \end{aligned}$$

where

$$\left.\begin{aligned} r_n &= \frac{1}{2}(a_n - ib_n) \\ r_{-n} &= \frac{1}{2}(a_n + ib_n) \end{aligned}\right\} \text{for } n = 1, 2, \ldots$$

$$\begin{aligned} r_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)\cos nx - if(x)\sin nx)\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x \end{aligned}$$

Similarly,

$$r_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\mathrm{d}x$$

(Clearly $r_n^* = r_{-n}$ since $f(x)$ is real)

Hence

$$f(x) = \sum_{n=-\infty}^{\infty} r_n e^{inx}, -\pi < x < \pi$$

where

$$r_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x, n = 0, \pm 1, \pm 2, \ldots$$

4.5.6 Fourier Series over $2L$ periodic intervals

The set of functions

$$\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\cos\left(n\frac{\pi x}{L}\right), \frac{1}{\sqrt{L}}\sin\left(n\frac{\pi x}{L}\right), \quad n = 1, 2, \ldots$$

are orthonormal on $[-L, L]$ (and in fact on any interval $[a, a+2L]$ since the function is periodic). In addition,

$$\int_{-L}^{L}\sin n\frac{\pi x}{L}\sin m\frac{\pi x}{L}\mathrm{d}x = \int_{-L}^{L}\cos n\frac{\pi x}{L}\cos m\frac{\pi x}{L}\mathrm{d}x = \begin{cases} L & m = n \\ 0 & n \neq n \end{cases}$$

and

$$\int_{-L}^{L}\sin n\frac{\pi x}{L}\cos m\frac{\pi x}{L}\mathrm{d}x = 0$$

Therefore the real form is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\frac{\pi x}{L} + b_n \sin n\frac{\pi x}{L}\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos n\frac{\pi x}{L}\mathrm{d}x \quad \& \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin n\frac{\pi x}{L}\mathrm{d}x$$

The complex form is

$$f(x) = \sum_{-\infty}^{\infty} r_n e^{in\frac{\pi x}{L}}, |x| \leq L$$

where

$$r_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\frac{\pi x}{L}}\mathrm{d}x, n = 0, \pm 1, \pm 2, \ldots$$

4.5.7 Parseval's Theorem

Theorem 38. If $f(x)$ is represented by its Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \; -\pi \leq x \leq \pi$$

then we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof. Easier to use complex notation

$$f(x) = \sum_{n=-\infty}^{\infty} r_n e^{-inx}$$

where

$$r_n = \frac{1}{2}(a_n - ib_n), \; r_{-n} = \frac{1}{2}(a_n + ib_n) = r_n^*, \; r_0 = \frac{1}{2}a_0$$

$$(f(x))^2 = \left(\sum_{n=-\infty}^{\infty} r_n e^{-inx}\right)\left(\sum_{m=-\infty}^{\infty} r_m e^{-imx}\right)$$

Integrate and use orthogonality,

$$\int_{-\pi}^{\pi} (f(x))^2 \mathrm{d}x = 2\pi \sum_{n=-\infty}^{\infty} r_n r_{-n} = 2\pi \sum_{n=-\infty}^{\infty} |r_n|^2$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 \mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

□

Example 39. Compute the Fourier series of $\cos\frac{x}{2}$ over $(-\pi,\pi]$. Use Parseval's theorem to deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}$$

Function is even, so

$$\cos\frac{x}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,\ -\pi \le x \le \pi$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\mathrm{d}x = \frac{4}{\pi}$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \cos\frac{x}{2}\cos nx \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos\left(n+\frac{1}{2}\right)x + \cos\left(n-\frac{1}{2}\right)x\right]\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\sin(n+\frac{1}{2})\pi}{n+\frac{1}{2}} + \frac{\sin(n-\frac{1}{2})\pi}{n-\frac{1}{2}}\right] \\
&= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+\frac{1}{2}} - \frac{\cos n\pi}{n-\frac{1}{2}}\right] \\
&= \frac{(-1)^n}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right] \\
&= \frac{(-1)^n}{\pi}\frac{-4}{4n^2-1}
\end{aligned}$$

By Parseval's theorem,

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2\frac{x}{2}\mathrm{d}x &= \frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2\\ &= \frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\end{aligned}$$

$LHS = 1$, therefore

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\frac{\pi^2-8}{16}$$

4.5.8 Fourier Transform as limit of Fourier Series

We discussed 2π-periodic functions in detail. Consider now $f(x)$ being periodic on $[-L, L]$ with L being arbitrary. We have shown that

$$f(x)=\sum_{n=-\infty}^{\infty}r_n e^{in\frac{\pi x}{L}},\ -L\leq x\leq L$$

where

$$r_n=\frac{1}{2L}\int_{-L}^{L}f(t)e^{-in\frac{\pi t}{L}}\mathrm{d}t,\ n=0,\pm1,\pm2,\dots$$

Put r_n into the sum to find

$$f(x)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L}f(t)e^{-in\frac{\pi t}{L}}\mathrm{d}t\right)e^{in\frac{\pi x}{L}}$$

This is exact. We want to send $L\to\infty$.

$$f(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}h\left(\int_{-L}^{L}f(t)e^{-inht}\mathrm{d}t\right)e^{inhx}$$

where $h=\frac{\pi}{L}$. In the limit $L\to\infty$, $h\to\infty$, but $nh:=\omega_n=0(1)$.

$$f(x)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}h\left(\int_{-L}^{L}f(t)e^{-i\omega_n t}\mathrm{d}t\right)e^{i\omega_n x}$$

This is of the form $\sum_{n=-\infty}^{\infty} G(\omega_n)h$. Now $h = \omega_{n+1} - \omega_n = (n+1)h - nh := \delta\omega$,

$$\text{Riemann sum} \quad \sum_{n=-\infty}^{\infty} G(\omega_n)\delta\omega \to \int_{-\infty}^{\infty} G(\omega_n)\mathrm{d}\omega$$

This gives, sending $L \to \infty$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t \right) e^{i\omega x} \mathrm{d}\omega$$

where $f(x)$ is defined on $\mathbb{R}$. This gives the *Fourier Transform pair*:

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \mathrm{d}k \\ \hat{f}(x) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x \end{aligned}}$$

This is very useful in many applications. You will use them a lot to solve differential equations.