Condensed Notes for Maths40010

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Contents

Chapter 1

Numbers

1.1 Countability

Definition 1. A set S is *countable* iff $\exists$ bijection $f : \mathbb{N} \to S$.

Theorem 2. Suppose $S \subset \mathbb{N}$ is infinite. Then S is *countable*.

Theorem 3. $\mathbb{Z}$ is countable.

Theorem 4. $\mathbb{Q}$ is countable.

Theorem 5. $\mathbb{R}$ is uncountable.

1.2 The Completeness Axiom

Definition 6. $\varnothing \neq S \subset \mathbb{R}$ is *bounded above* if

$$\exists M \in \mathbb{R} \,\text{s.t.}\, \forall x \in S, x \leq M$$

Such an M is called an *upper bound* for S. In addition, we say $x \in \mathbb{R}$ is a *least upper bound* for S or **supremum** of S iff

- x is an upper bound for S (i.e. $x \geq s \; \forall s \in S$)
- $x \leq y \; \forall$ upper bounds y of S (i.e. $y \geq s \; \forall s \in S \Rightarrow y \geq x$)

Theorem 7. $\varnothing \neq S \subset \mathbb{R}$ is *bounded below* if

$$\exists M \in \mathbb{R} \,\text{s.t.}\, \forall x \in S, x \geq M$$

Such an M is called an *lower bound* for S. In addition, we say $x \in \mathbb{R}$ is a *greatest lower bound* for S or **infimum** of S iff

- x is a lower bound for S (i.e. $x \leq s \; \forall s \in S$)
- $x \geq y \; \forall$ lower bounds y of S (i.e. $y \leq s \; \forall s \in S \Rightarrow y \leq x$)

Theorem 8. Suppose $S \subseteq \mathbb{R}$ is nonempty, bounded above, then $\exists \sup S \in \mathbb{R}$

1.3 Dedekind cuts

Definition 9. We say a nonempty subset $s \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfy

(i) $\forall s \in S$, $[t < s \Rightarrow t \in S]$, i.e. S is a semi-infinite interval to the left.

(ii) S has an upper bound but <u>no maximum</u>

Definition 10. New Definition of $\mathbb{R}$:

$$\mathbb{R} := \{\text{Dedekind cuts } S \subset \mathbb{Q}\}$$

1.4 triangle inequalities

Theorem 11. $\forall a, b \in \mathbb{R}$, we have

$$|a+b| \leq |a| + |b|$$

$$|a+b| \geq \Big| |a| - |b| \Big|$$

$$|a| \leq |b| + |a-b|$$

$$|a| \geq |b| - |a-b|$$

$$|a-b| \leq |a-c| + |b-c|$$

Chapter 2

Sequences

Definition 12. A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$

2.1 convergence of sequences

Definition 13. We say that $a_n \to a$ as $n \to \infty$ iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |a_n - a| < \varepsilon \ \forall n > N$$

Definition 14. We say that a_n *converges* iff $\exists a \in \mathbb{R}$ s.t. $a_n \to a$, i.e. a_n converges iff

$$\exists a \in \mathbb{R} \ s.t. \ \forall \varepsilon > 0, \exists N \in \mathbb{N} \ s.t. \ \forall n \geq N, |a_n - a| < \varepsilon$$

Definition 15. We say a_n *diverges* iff it does not converge (to any $a \in \mathbb{R}$), i.e.

$$\forall a \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \geq N \ s.t. \ |a_n - a| \geq \varepsilon$$

Definition 16. We say $a_n \to +\infty$ iff

$$\forall R > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, a_n > R$$

Definition 17. Let $a_n \in \mathbb{C}, \forall \geq 1$. We say $a_n \to a \in \mathbb{C}$ iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |a_n - a| < \varepsilon$$

Theorem 18. Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$.

Theorem 19. if $a_n \to a$ and $b_n \to b$ then:

1. $a_n + b_n \to a + b$
2. $a_n b_n \to ab$
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$.

Theorem 20. If (a_n) is *bounded above* and *monotonically increasing* then a_n converges to $a := \sup \{a_i : i \in \mathbb{N}\}$. We write $a_n \uparrow a$.

2.2 Cauchy Sequences

Definition 21. $(a_n)_{n \geq 1}$ is called a *Cauchy* sequence iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, |a_n - a_m| < \varepsilon$$

Theorem 22. (a_n) is Cauchy $\Rightarrow$ (a_n) is bounded.

Theorem 23. (a_n) is Cauchy $\iff$ (a_n) is convergent.

2.3 Subsequences

Definition 24. A *subsequence* of a_n is a new sequence $b_i = a_{n(i)} \forall i \in \mathbb{N}$, where $n(1) < n(2) < \cdots < n(i) < \ldots \forall i$.

(Bolzano-Weiestrass) Theorem 25. If (a_n) is a *bounded* sequence of real numbers, then it has a convergent subsequence.

Theorem 26. If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.

2.4 Series

Definition 27. An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \ldots,$$

where $(a_i)_{i \geq 1}$ is a sequence.

Definition 28. n^{th} partial sum is

$$S_n := \sum_{i=1}^{n} a_i \in \mathbb{R}$$

2.5 Convergence of Series

Definition 29. We say that the series $\sum a_n$ "converges to $A \in \mathbb{R}$" iff the sequence (S_n) of partial sums converges to A:

$$\sum_{n=1}^{\infty} a_n = A \in \mathbb{R} \iff S_n \to A \text{ as } n \to \infty$$

Theorem 30. $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow a_n \to 0$. In other words, $a_n \nrightarrow 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 31. Suppose $a_n \geq 0\ \forall n$ ($\iff$ $S_n = \sum_{i=1}^{n} a_i$ monotonically increasing), then $S_\infty = \sum_{n=1}^{\infty} a_n$ convergent $\iff$ (S_n) bounded above. Similary, $\sum_{n=1}^{\infty} a_n \to +\infty \iff (S_n)$ is unbounded.

Theorem 32. if $\sum a_n$, $\sum b_n$ are convergent then so is $\sum(\lambda a_n + \mu b_n)$, to

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n$$

2.6 Absolute convergence

Definition 33. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* iff the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem 34. If $\sum a_n$ is absolutely convergent, then it is convergent.

2.7 Tests for convergence

Theorem 35. if $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum a_n$ convergent and $0 \leq \sum a_n \leq \sum b_n$.

Theorem 36. If $c_n \leq a_n \leq b_n\ \forall n$ and $\sum c_n$, $\sum b_n$ both convergent, then $\sum a_n$ convergent and $\sum c_n \leq \sum a_n \leq \sum b_n$.

Theorem 37. If $\frac{a_n}{b_n} \to L \in \mathbb{R}(b_n \neq 0\ \forall n)$, then if $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

Theorem 38. If (a_n) is alternating and $|a_n| \downarrow 0$, then $\sum a_n$ is convergent.

Theorem 39. If $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Theorem 40. If $|a_n|^{\frac{1}{n}} \to r < 1$, then $\sum a_n$ is absolutely convergent.

2.8 Rearangement of series

Definition 41. Given a bijection $n : \mathbb{N} \to \mathbb{N}$, define $b_i := a_{n(i)}$. Then $(b_i)_{i\geq 1}$ is a *rearrangement* or *reordering* of $(a_n)_{n\geq 1}$.

Theorem 42. $\sum a_n$ is absolutely convergent $\iff$ (1)+(2) $\Rightarrow$ (3)+(4), where

1. $\sum_{a_n\geq 0} a_n$ is convergent (to A say),
2. $\sum_{a_n<0} a_n$ is convergent (to B say),
3. $\sum a_n = A + B$,
4. $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n).

2.9 Power Series

Theorem 43. Fix a real complex series (a_n) an consider the series $\sum a_n z^n$ for $z \in \mathbb{C}$. Then $\exists R \in [0, \infty]$ s.t.

- $|z| < R \Rightarrow \sum a_n z^n$ is absolutely convergent, and
- $|z| > R \Rightarrow \sum a_n z^n$ is divergent.

2.9.1 Products of Series

Definition 44. Given series $\sum a_n$, $\sum b_n$, their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^{n} a_i b_{n-i}$.

Theorem 45. If $\sum a_n$, $\sum b_n$ are absolutely convergent, then their Cauchy Product $\sum c_n$ is absolutely convergent to $(\sum a_n) \cdot (\sum b_n)$.

2.10 Exponential Power Series

Definition 46. For any $z \in \mathbb{C}$ set

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Theorem 47. $E(x)$ has the following properties for $x \in \mathbb{R}$.

1. $E(x) > 0 \; \forall x \in \mathbb{R}$
2. $x \geq 0 \Rightarrow E(x) \geq 1$ and $x > 0 \Rightarrow E(x) > 1$
3. $E(x)$ is strictly increasing for $x \in \mathbb{R}$
4. $|E(x) - 1| < \frac{|x|}{1-|x|} \; \forall |x| < 1$
5. $x \mapsto E(x)$ is a continuous bijection $\mathbb{R} \to (0, \infty)$

Chapter 3

Continuity

3.1 Limits

Definition 48. Fix a function $f : \mathbb{R} \to \mathbb{R}$ and points $a, b \in \mathbb{R}$.
We say that $f(x) \to b$ as $x \to a$ (or "$\lim_{x\to a} f(x) = b$") iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$$

3.2 Continuity

Definition 49. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *continuous* at $a \in \mathbb{R}$ iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \text{ that are } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

We say that f is continuous on $\mathbb{R}$ (or just "continuous") if it is continuous at all $a \in \mathbb{R}$.

Definition 50. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *discontinuous* at $a \in \mathbb{R}$ iff

$$\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \text{ with } |x - a| < \delta \Rightarrow |f(x) - f(a)| \geq \varepsilon$$

Theorem 51. $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f(x_n) \to f(a) \quad \forall$ sequences (x_n) which tends to a.

In other words, $f : \mathbb{R} \to \mathbb{R}$ is *not* continuous at $a \in \mathbb{R} \iff f(x_n) \nrightarrow f(a) \quad \forall$ sequences (x_n) which tends to a.