Introduction to Ito Calculus

Definition 0.1. A one-dimensional Brownian motion (or Wiener process) $\{B_t\}_{t\geq 0}$ is a stochastic process with the following properties:

1. $B_0 = 0$ almost surely.
2. For $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.
3. For $0 \leq s < t < u < v$, the increments $B_t - B_s$ and $B_v - B_u$ are independent.
4. The paths of B_t are almost surely continuous.

Remark 0.2. Brownian motion can be approximated by a scaled random walk with step size Δt and increments $\pm\sqrt{\Delta t}$ with equal probability. As $\Delta t \to 0$, the random walk converges to Brownian motion.

Proposition 0.3. *Let $\{B_t\}_{t\geq 0}$ be a one-dimensional Brownian motion. Then, for any partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$, the quadratic variation*

$$[B]_T = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 = T$$

almost surely, where $\|P\| = \max_{0\leq i<n}(t_{i+1} - t_i)$.

Theorem 0.4 (Ito's Lemma)**.** *Let $\{B_t\}_{t\geq 0}$ be a one-dimensional Brownian motion and let $f(t, x)$ be a twice continuously differentiable function. Then the stochastic process $X_t = f(t, B_t)$ satisfies the following stochastic differential equation:*

$$dX_t = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dB_t.$$

Example 0.5. A Geometric Brownian Motion (GBM) is defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where μ is the drift coefficient and σ is the volatility coefficient. Using Ito's Lemma with $f(t, x) = \ln(x)$, we find that

$$d(\ln S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t.$$

Integrating both sides gives

$$\ln S_t = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t,$$

or equivalently,

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right).$$

The distribution of $\ln S_t$ is normal with mean $\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$.

Applications to BS asset Pricing

In the Black-Scholes model, the stock price S_t follows a Geometric Brownian Motion with parameters μ and σ. The risk-neutral measure changes the drift from μ to the risk-free rate r, leading to the risk-neutral dynamics:

$$dS_t = rS_t dt + \sigma S_t dB_t^*,$$

where B_t^* is a Brownian motion under the risk-neutral measure.

Theorem 0.6 (Risk-Neutral Pricing Formula)**.** *The price of a European call option with strike price K and maturity T in the Black-Scholes model is given by*

$$C(0, S_0) = e^{-rT}\mathbb{E}^*\left[(S_T - K)^+\right],$$

where $\mathbb{E}^*$ denotes the expectation under the risk-neutral measure.

Theorem 0.7 (Black-Scholes PDE)**.** *Let $V(t, S_t)$ be the price of a European option with payoff function $h(S_T)$ at maturity T. Then $V(t, S_t)$ satisfies the Black-Scholes partial differential equation:*

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with terminal condition $V(T, S_T) = h(S_T)$.

Example 0.8. The price of a European call option with strike price K and maturity T is given by the Black-Scholes formula:

$$C(t, S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.